# Commuting Isometries and Invariant Subspaces in Several Variables

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# Commuting Isometries and Invariant Subspaces in Several Variables

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*Dedicated to my Parents*

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# Contents

# Notations & Abbreviations

| | |
|---|---|
| $\mathbb{N}$ | Set of all Natural numbers. |
| $\mathbb{Z}_+$ | $\mathbb{N} \cup \{0\}$. |
| $\mathbb{N}^n$ | $\{\boldsymbol{k} = (k_1, \ldots, k_n) : k_i \in \mathbb{N}, i = 1, \ldots, n\}$. |
| $\mathbb{Z}_+^n$ | $\{\boldsymbol{t} = (t_1, \ldots, t_n) : t_i \in \mathbb{Z}_+, i = 1, \ldots, n\}$. |
| $\boldsymbol{z}$ | $(z_1, \ldots, z_n) \in \mathbb{C}^n$. |
| $\boldsymbol{z}^{\boldsymbol{k}}$ | $z_1^{k_1} \ldots z_n^{k_n}$. |
| $\lvert\boldsymbol{k}\rvert$ | $k_1 + \ldots + k_n$. |
| $(T_1, \ldots, T_n)$ | n-tuple of commuting operators on Hilbert spaces. |
| $T^{\boldsymbol{k}}$ | $T_1^{k_1} \ldots T_n^{k_n}$. |
| $\mathbb{D}^n$ | $\{\boldsymbol{z} : \lvert z_i\rvert < 1, i = 1, \ldots, n\}$. |
| $\mathbb{B}^n$ | $\{\boldsymbol{z} : \sum_{i=1}^n \lvert z_i\rvert^2 < 1\}$. |
| $\mathcal{E}, \mathcal{E}_*$ | Hilbert spaces. |
| $\mathcal{O}(\Omega, \mathcal{E})$ | The set of all holomorphic functions on $\Omega \subseteq \mathbb{C}^n$ to $\mathcal{E}$. |
| $\mathcal{O}(\mathbb{B}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ | The set of all $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued holomorphic functions on $\mathbb{B}^n$. |
| $A(\mathbb{B}^n)$ | Ball algebra. |
| $H^\infty(\mathbb{D}^n)$ | The set of all bounded analytic functions on $\mathbb{D}^n$. |

# Introduction

A very general and fundamental problem in the theory of bounded linear operators on Hilbert spaces is to find invariants and representations of commuting families of isometries.

In the case of single isometries this question has a complete and explicit answer: If $V$ is an isometry on a Hilbert space $\mathcal{H}$, then there exists a Hilbert space $\mathcal{H}_u$ and a unitary operator $U$ on $\mathcal{H}_u$ such that $V$ on $\mathcal{H}$ and

$$\begin{bmatrix} S \otimes I_{\mathcal{W}} & 0 \\ 0 & U \end{bmatrix} \in \mathcal{B}((l^2(\mathbb{Z}_+) \otimes \mathcal{W}) \oplus \mathcal{H}_u),$$

are unitarily equivalent, where

$$\mathcal{W} = \ker V^*,$$

is the wandering subspace for $V$ and $S$ is the shift operator on $l^2(\mathbb{Z}_+)$ [66]. This fundamental result is due to J. von Neumann [81] and H. Wold [110] (see Theorem 1.2.1 for more details).

In one hand, unitary operators are completely determined by the representing spectral measure. And, on the other hand, given $n \in \mathbb{N} \cup \{\infty\}$, there exists precisely one Hilbert space $\mathcal{E}$, up to unitary equivalence, of dimension $n$ (here all Hilbert spaces are assumed to be separable), and given a Hilbert space $\mathcal{E}$, there exists precisely one shift operator, up to unitary equivalence, of multiplicity $\dim \mathcal{E}$ on some Hilbert space $\mathcal{H}$. Therefore, multiplicity is the only (numerical) invariant of a shift operator. Note that shift operators are special class of isometries, and moreover, the defect operator of a shift determines the multiplicity of the shift.

Now we turn to tuples of commuting isometries on Hilbert spaces. It is remarkable that tractable invariants (whatever it means including the possibilities of numerical and analytical invariants) of commuting pairs of isometries are largely unknown. We stress on the fact that the case of pairs of commuting isometries itself is more subtle, and is directly related to the commutant lifting theorem [51] (in terms of an explicit, and then unique solution), invariant subspace problem [70] and representations of contractions on Hilbert spaces in function Hilbert spaces [79]. For instance:

(a) Let $\mathcal{S}$ be a closed joint $(M_{z_1}, M_{z_2})$-invariant subspace of $H^2(\mathbb{D}^2)$, the Hardy space over the bidisc $\mathbb{D}^2$. Then $(M_{z_1}|_{\mathcal{S}}, M_{z_2}|_{\mathcal{S}})$ on $\mathcal{S}$ is a pure (see Chapter 3) pair of commuting isometries. Classification of such pairs of isometries is largely unknown (see Rudin [94, 93]).

(b) Let $T$ be a contraction on a Hilbert space $\mathcal{H}$. Then there exists a pair of commuting isometries $(V_1, V_2)$ on a Hilbert space $\mathcal{K}$ such that $T$ and $P_{\ker V_2^*} V_1|_{\ker V_2^*}$ are unitarily equivalent (see Bercovici, Douglas and Foias [18]).

(c) The celebrated Ando dilation theorem (see Ando [9]) states that a commuting pair of contractions dilates to a commuting pair of isometries. Again, the structure of Ando's pairs of commuting isometries is largely unknown.

(d) Contrary to the simpler structure of shift invariant subspaces of the one variable Hardy space, structure of invariant subspaces for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$, $n > 1$, is quite complicated. For example (see Rudin [94, 93]): There exist invariant subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ for $(M_{z_1}, M_{z_2})$ on $H^2(\mathbb{D}^2)$ such that (i) $\mathcal{S}_1$ is not finitely generated, and (ii) $\mathcal{S}_2 \cap H^\infty(\mathbb{D}^2) = \{0\}$.

In this thesis, we aim at exploring the structure of tuples of commuting isometries. We present a number of results concerning tuples of commuting isometries. The main contributions of this thesis are:

1. Berger, Coburn and Lebow pairs: An explicit version of Berger, Coburn and Lebow's classification result for pure pairs of commuting isometries in the sense of an explicit recipe for constructing pairs of commuting isometric multipliers with precise coefficients. We describe a complete set of (joint) unitary invariants and compare the Berger, Coburn and Lebow's representations with other natural analytic representations of pure pairs of commuting isometries. We also study the defect operators of pairs of commuting isometries.

2. Invariant subspaces of shift operators on the Hardy space over the unit polydisc: We give a complete characterization of invariant subspaces for $(M_{z_1}, \ldots, M_{z_n})$ on the Hardy space $H^2(\mathbb{D}^n)$ over the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, $n > 1$. In particular, this yields a complete set of unitary invariants for invariant subspaces for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$. As a consequence, we classify a large class of $n$-tuples of commuting isometries.

3. Pairs of projections and commuting isometries: It is known that a commuting Berger, Coburn and Lebow pair of isometries $(V_1, V_2)$ on a Hilbert space $\mathcal{H}$ is uniquely associated to an orthogonal projection $P$ and a unitary $U$ on a Hilbert space $\mathcal{E}$ (and vice versa). In this case, the "defect operator" of $(V_1, V_2)$, say $T$, is given by the difference of orthogonal projections on $\mathcal{E}$:

$$T = UPU^* - P.$$

> Here, we aim to determine whether irreducible commuting pairs of isometries $(V_1, V_2)$ can be built up from compact operators $T$ on $\mathcal{E}$ such that $T$ is a difference of two orthogonal projections. The answer to this question is sometimes in the affirmative and sometimes in the negative.
>
> The range of constructions of $(V_1, V_2)$ presented here also yields examples of a number of concrete pairs of commuting isometries.

Let us now explain the setting and the content of this thesis in more detail. We begin with the construction of the classical Wold-von Neumann decomposition of isometric operators on Hilbert spaces. Here our presentation is more algebraic and geared towards the main theme of the thesis. First, recall that an isometry $V$ on a Hilbert space $\mathcal{H}$ is said to be *pure*, or a *shift*, if it has no unitary direct summand, or equivalently, if $\lim\limits_{m\to\infty} V^{*m} = 0$ in the strong operator topology (see Halmos [66]).

Let $V$ be an isometry on a Hilbert space $\mathcal{H}$, and let $\mathcal{W}(V)$ be the *wandering subspace* [66] for $V$, that is,

$$\mathcal{W}(V) = \mathcal{H} \ominus V\mathcal{H}.$$

The classical Wold-von Neumann decomposition states the following: Let $V$ be an isometry on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ decomposes as a direct sum of $V$-reducing subspaces $\mathcal{H}_s(V) = \bigoplus\limits_{m=0}^{\infty} V^m \mathcal{W}(V)$ and $\mathcal{H}_u(V) = \mathcal{H} \ominus \mathcal{H}_s(V)$ and

$$V = \begin{bmatrix} V_s & 0 \\ 0 & V_u \end{bmatrix} \in \mathcal{B}(\mathcal{H}_s(V) \oplus \mathcal{H}_u(V)), \tag{0.0.1}$$

where $V_s = V|_{\mathcal{H}_s(V)}$ is a shift operator and $V_u = V|_{\mathcal{H}_u(V)}$ is a unitary operator.

We will refer to this decomposition as the *Wold-von Neumann orthogonal decomposition* of $V$. For any Hilbert space $\mathcal{E}$, the $\mathcal{E}$-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ is canonically identified with the tensor product Hilbert space $H^2(\mathbb{D}) \otimes \mathcal{E}$. To simplify the notation, we often identify $H^2(\mathbb{D}) \otimes \mathcal{E}$ with the $\mathcal{E}$-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$. The space of $\mathcal{B}(\mathcal{E})$-valued bounded holomorphic functions on $\mathbb{D}$ will be denoted by $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$. Finally, let $M_z^{\mathcal{E}}$ (or simply $M_z$, if $\mathcal{E}$ is clear from the context) denote the multiplication operator by the coordinate function $z$ on $H^2_{\mathcal{E}}(\mathbb{D})$. Then $M_z^{\mathcal{E}}$ is a shift operator and

$$\mathcal{W}(M_z^{\mathcal{E}}) = \mathcal{E}.$$

Let $V$ be an isometry on $\mathcal{H}$, and let $\mathcal{H} = \mathcal{H}_s(V) \oplus \mathcal{H}_u(V)$ be the Wold-von Neumann orthogonal decomposition of $V$. Then (0.0.1) implies the existence of a (canonical) unitary $\Pi_V : \mathcal{H}_s(V) \oplus \mathcal{H}_u(V) \to H^2_{\mathcal{W}(V)}(\mathbb{D}) \oplus \mathcal{H}_u(V)$ such that

$$\Pi_V \begin{bmatrix} V_s & 0 \\ 0 & V_u \end{bmatrix} = \begin{bmatrix} M_z^{\mathcal{W}(V)} & 0 \\ 0 & V_u \end{bmatrix} \Pi_V.$$

In particular, this implies that $V$ is a shift operator if and only if $V$ is unitarily equivalent to $M_z^{\mathcal{E}}$ on $H^2_{\mathcal{E}}(\mathbb{D})$, where $\dim \mathcal{E} = \dim \mathcal{W}(V)$. In the sequel we denote by $(\Pi_V, M_z^{\mathcal{W}(V)})$, or simply by $(\Pi_V, M_z)$, the Wold-von Neumann decomposition of the pure isometry $V$ in the above sense.

With these preparations, we are now ready to explain the main contribution of this thesis.

**Chapter 2:** After a preliminary chapter on the basic notions of operator theory and function theory, in Chapter 2, we first characterize and present an analytic description of commutators of shift operators. Recall that if $C$ is a bounded linear operator on $H^2_{\mathcal{E}}(\mathbb{D})$ for some Hilbert space $\mathcal{E}$, then $C \in \{M_z\}'$, that is, $CM_z = M_zC$, if and only if (cf. [79])

$$C = M_\Theta$$

for some $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ and $(M_\Theta f)(w) = \Theta(w)f(w)$ for all $f \in H^2_{\mathcal{E}}(\mathbb{D})$ and $w \in \mathbb{D}$.

Now let $V$ be a pure isometry, and let $C \in \{V\}'$. Let $(\Pi_V, M_z)$ be the Wold-von Neumann decomposition of $V$, and let $\mathcal{W} = \mathcal{W}(V)$. Since $\Pi_V C \Pi_V^*$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is the representation of $C$ on $\mathcal{H}$ and $(\Pi_V C \Pi_V^*) M_z = M_z (\Pi_V C \Pi_V^*)$, it follows that

$$\Pi_V C \Pi_V^* = M_\Theta,$$

for some $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$. From this point of view, we prove:

**Theorem 0.0.1.** *Let $V$ be a pure isometry on $\mathcal{H}$, and let $C$ be a bounded operator on $\mathcal{H}$. Let $(\Pi_V, M_z)$ be the Wold-von Neumann decomposition of $V$. Set $\mathcal{W} = \mathcal{W}(V)$, $M = \Pi_V C \Pi_V^*$ and let*

$$\Theta(w) = P_{\mathcal{W}}(I_{\mathcal{H}} - wV^*)^{-1} C|_{\mathcal{W}} \qquad (w \in \mathbb{D}).$$

*Then $CV = VC$ if and only if $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ and*

$$M = M_\Theta.$$

Note that $\|wV^*\| = |w|\|V\| < 1$ for all $w \in \mathbb{D}$, and so it follows that the function $\Theta$ defined above is a $\mathcal{B}(\mathcal{W})$-valued holomorphic function in the unit disc $\mathbb{D}$. However, what is not guaranteed in general here is that the function $\Theta$ is in $H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$. The above theorem says that this is so if $CV = VC$.

Then we move to study a class of pairs of commuting isometries, namely, Berger, Coburn and Lebow pairs of commuting isometries.

A pair of commuting isometries $(V_1, V_2)$ on $\mathcal{H}$ is said to be *pure* if $V := V_1V_2$ is a shift (that is, a pure isometry). By a *BCL triple* (after Berger, Coburn and Lebow [20]) we mean an ordered triple $(\mathcal{E}, U, P)$ which consists of a Hilbert space $\mathcal{E}$, a unitary operator $U$ and an orthogonal projection $P$ on $\mathcal{E}$. By a *BCL pair* (again, after Berger, Coburn and Lebow [20]) we mean a commuting pair of isometries $(V_1, V_2)$ on some Hilbert space $\mathcal{H}$ such that $V_1V_2$ is a shift operator.

In [20], Berger, Coburn, and Lebow established the following characterization: A pair of commuting isometries $(V_1, V_2)$ on a Hilbert space $\mathcal{H}$ is a BCL pair if and only if there exists a BCL triple $(\mathcal{E}, U, P)$ such that $(V_1, V_2)$ and $(M_{\Phi_1}, M_{\Phi_2})$ on $H^2_{\mathcal{E}}(\mathbb{D})$ are unitarily equivalent, where

$$\Phi_1(z) = U^*(P + zP^\perp) \quad \text{and} \quad \Phi_2(z) = (P^\perp + zP)U,$$

for all $z \in \mathbb{D}$ and $P^\perp$ denotes the orthogonal projection $I - P$.

Note that the representations of $V_1$ and $V_2$ on $H^2_{\mathcal{E}}(\mathbb{D})$ are analytic Toeplitz operators corresponding to one degree operator-valued polynomials. We prove the following explicit representations of BCL pairs.

**Theorem 0.0.2.** *Let $(V_1, V_2)$ be a BCL pair on $\mathcal{H}$. Suppose $\mathcal{W} = \mathcal{H} \ominus V_1V_2\mathcal{H}$ and $\mathcal{W}_j = \mathcal{W}(V_j) = \mathcal{H} \ominus V_j\mathcal{H}$, $j = 1, 2$. Then the BCL representation of $(V_1, V_2)$ is given by $(M_{\Phi_1}, M_{\Phi_2})$ on $H^2_{\mathcal{W}}(\mathbb{D})$, where*

$$\Phi_1(z) = U^*(P_{\mathcal{W}_2} + zP^\perp_{\mathcal{W}_2}) \quad \text{and} \quad \Phi_2(z) = (P^\perp_{\mathcal{W}_2} + zP_{\mathcal{W}_2})U,$$

*and*

$$U = \begin{bmatrix} V_2|_{\mathcal{W}_1} & 0 \\ 0 & V_1^*|_{V_1\mathcal{W}_2} \end{bmatrix} : \begin{matrix} \mathcal{W}_1 \\ \oplus \\ V_1\mathcal{W}_2 \end{matrix} \to \begin{matrix} V_2\mathcal{W}_1 \\ \oplus \\ \mathcal{W}_2 \end{matrix},$$

*is a unitary operator on $\mathcal{W}$.*

Note that the above result yields an explicit representations of the auxiliary operators $U$ and $P$. Moreover, we prove that:

**Theorem 0.0.3.** *Let $(V_1, V_2)$ and $(\tilde{V}_1, \tilde{V}_2)$ be two pure pairs of commuting isometries on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively. Then $(V_1, V_2)$ and $(\tilde{V}_1, \tilde{V}_2)$ are unitarily equivalent if and only if $(V_1|_{\mathcal{W}_2}, V_2^*|_{V_2\mathcal{W}_1})$ and $(\tilde{V}_1|_{\tilde{\mathcal{W}}_2}, \tilde{V}_2^*|_{\tilde{V}_2\tilde{\mathcal{W}}_1})$ are unitarily equivalent.*

In other words, the pair $\{V_1|_{\mathcal{W}_2}, V_2^*|_{V_2\mathcal{W}_1}\}$ is a complete set of unitary invariants of BCL pairs.

Then we turn to analytic representations of those pairs of commuting isometries $(V_1, V_2)$ for which both $V_1$ and $V_2$ are shift operators. Given such a pair $(V_1, V_2)$ on some Hilbert space $\mathcal{H}$, let $(\Pi_V, M_z)$ denote the Wold-von Neumann decomposition of $V = V_1V_2$. Then $\Pi_V V_i = M_{\Phi_i}\Pi_V$ for all $i = 1, 2$. Now applying Theorem 0.0.1 to $V_1 \in \{V_2\}'$, we find unitary operator $\Pi_{V_1} : \mathcal{H} \to H^2_{\mathcal{W}_1}(\mathbb{D})$ such that $\Pi_{V_1}V_2 = M_{\Theta_{V_2}}\Pi_{V_1}$, where $\Theta_{V_2} \in H^\infty_{\mathcal{B}(\mathcal{W}_1)}(\mathbb{D})$ is an inner multiplier and

$$\Theta_{V_2}(z) = P_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}V_2|_{\mathcal{W}_1} \qquad (z \in \mathbb{D}).$$

Similarly, we have unitary map $\Pi_{V_2} : \mathcal{H} \to H^2_{\mathcal{W}_2}(\mathbb{D})$ and inner multiplier $\Theta_{V_1} \in H^\infty_{\mathcal{B}(\mathcal{W}_2)}(\mathbb{D})$. We prove the following:

**Theorem 0.0.4.** *Let $(V_1, V_2)$ be a pair of commuting isometries on $\mathcal{H}$. Let $i, j \in \{1, 2\}$ and $i \neq j$. If $V_i$ is a pure isometry, then*

$$\tilde{\Pi}_i = \Pi_{V_i}\Pi_V^* \in \mathcal{B}(H^2_{\mathcal{W}}(\mathbb{D}), H^2_{\mathcal{W}_i}(\mathbb{D})),$$

*is a unitary operator,*

$$\tilde{\Pi}_i M_z^{\mathcal{W}} = M_{z\Theta_{V_j}}\tilde{\Pi}_i,\ \tilde{\Pi}_i^* M_z^{\mathcal{W}_i} = M_{\Phi_i}\tilde{\Pi}_i^*,$$

*and*

$$\tilde{\Pi}_i(\mathbb{S}(\cdot, w)\eta) = (I_{\mathcal{W}_i} - \bar{w}z\Theta_{V_j}(z))^{-1}P_{\mathcal{W}_i}[I_{\mathcal{H}} + z(I - zV_i^*)^{-1}V_i^*]\eta,$$

*for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$, where*

$$\Theta_{V_j}(z) = P_{\mathcal{W}_i}(I_{\mathcal{H}} - zV_i^*)^{-1}V_j|_{\mathcal{W}_i}$$

*for all $z \in \mathbb{D}$. Moreover*

$$\tilde{\Pi}_i^*(\mathbb{S}(\cdot, w)\eta_i) = (I_{\mathcal{W}} - \Phi_i(z)\bar{w})^{-1}\eta_i,$$

*for all $w \in \mathbb{D}$ and $\eta_i \in \mathcal{W}_i$.*

And, as a corollary, we have:

**Corollary 0.0.5.** *Let $(V_1, V_2)$ be a BCL pair on a Hilbert space $\mathcal{H}$. If $(M_{\Phi_1}, M_{\Phi_2})$ is the BCL representation of $(V_1, V_2)$, then $M_{\Phi_1}$ and $M_{\Phi_2}$ are pure isometries,*

$$\tilde{\Pi}_1 M_{\Phi_2} = M_{\Theta_{V_2}}\tilde{\Pi}_1,\ \tilde{\Pi}_2 M_{\Phi_1} = M_{\Theta_{V_1}}\tilde{\Pi}_2,$$

*$\tilde{\Pi} = \tilde{\Pi}_2\tilde{\Pi}_1^* : H^2_{\mathcal{W}_1}(\mathbb{D}) \to H^2_{\mathcal{W}_2}(\mathbb{D})$ is a unitary operator, and*

$$\tilde{\Pi} M_z^{\mathcal{W}_1} = M_{\Theta_{V_1}} \quad \textit{and} \quad M_{\Theta_{V_2}} = M_z^{\mathcal{W}_2}\tilde{\Pi}.$$

*Moreover, for each $w \in \mathbb{D}$ and $\eta_j \in \mathcal{W}_j$, $j = 1, 2$,*

$$\tilde{\Pi}(\mathbb{S}(\cdot, w)\eta_1) = (I_{\mathcal{W}_2} - \bar{w}\Theta_{V_1}(z))^{-1}P_{\mathcal{W}_2}(I_{\mathcal{H}} - zV_2^*)^{-1}\eta_1,$$

*and*

$$\tilde{\Pi}^*(\mathbb{S}(\cdot, w)\eta_2) = (I_{\mathcal{W}_1} - \bar{w}\Theta_{V_2}(z))^{-1}P_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}\eta_2.$$

The final section of Chapter 2 concerns some basic observation about defect operators of pairs of commuting isometries. Recall that the *defect operator* $C(V_1, V_2)$ of a pair of commuting isometries $(V_1, V_2)$ is the following self-adjoint operator

$$C(V_1, V_2) = I - V_1V_1{}^* - V_2V_2{}^* + V_1V_2V_1{}^*V_2{}^*.$$

We prove that:

**Theorem 0.0.6.** *Let $(V_1, V_2)$ be a pair of commuting isometries on $\mathcal{H}$. Then the following are equivalent:*

*(a)* $C(V_1, V_2) \geq 0$.

*(b)* $V_2\mathcal{W}_1 \subseteq \mathcal{W}_1$.

*(c)* $(V_1, V_2)$ *is doubly commuting.*

*(d)* $C(V_1, V_2)$ *is a projection.*

*(e) The fringe operator* $F_2$ *is an isometry.*

We prove a pair of definite results concerning negative defect operators:

**Theorem 0.0.7.** *Let $(V_1, V_2)$ be a pair of commuting isometries on $\mathcal{H}$. Suppose that $V_1$ or $V_2$ is pure. Then $C(V_1, V_2) \leq 0$ if and only if $C(V_1, V_2) = 0$.*

**Theorem 0.0.8.** *Let $(V_1, V_2)$ be a pair of commuting isometries on $\mathcal{H}$. Suppose that $dim\mathcal{W}_j < \infty$ for some $j \in \{1, 2\}$ . Then $C(V_1, V_2) \leq 0$ if and only if $C(V_1, V_2) = 0$.*

**Chapter 3:** Let $\mathcal{E}$ be a Hilbert space, $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$, $n \geq 1$, denotes the $\mathcal{E}$-valued Hardy space over the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, and let $(M_{z_1}, \ldots, M_{z_{n+1}})$ denotes the commuting tuple of multiplication operators by the coordinate functions on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$. Here we present a complete characterization of invariant subspaces for $(M_{z_1}, \ldots, M_{z_{n+1}})$. Given a pair of Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, we will denote by $H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ (or simply $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ if $\mathcal{E} = \mathcal{E}_*$) the Banach algebra of $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued bounded analytic functions on $\mathbb{D}$.

We first use the doubly commutativity property of the multiplication tuple on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ to reduce the invariant subspace problem in one variable as follows:

**Theorem 0.0.9.** *Let $\mathcal{E}$ be a Hilbert space. Then $(M_{z_1}, M_{z_2} \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent, where*

$$\mathcal{E}_n = H^2(\mathbb{D}^n) \otimes \mathcal{E},$$

*and $\kappa_i \in H^\infty_{\mathcal{B}(\mathcal{E}_n)}(\mathbb{D})$ is the constant function*

$$\kappa_i(w) = M_{z_i} \in \mathcal{B}(\mathcal{E}_n),$$

*for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$.*

In the light of above reduction, we present the following classification of invariant subspaces:

**Theorem 0.0.10.** *Let $\mathcal{E}$ be a Hilbert space, $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed subspace, and let $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$. Then $\mathcal{S}$ is invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ if and only if $(M_{\Phi_1}, \ldots, M_{\Phi_n})$*

*is an $n$-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$ and there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

*and*

$$\kappa_i \Theta = \Theta \Phi_i,$$

*where*

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wP_{\mathcal{S}}M_z^*)^{-1}M_{\kappa_i}|_{\mathcal{W}},$$

*for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$.*

Furthermore, the multiplier $\Phi_i$ can be represented as

$$\Phi_i(w) = P_{\mathcal{W}}M_\Theta(I_{H^2_{\mathcal{W}}(\mathbb{D})} - wM_z^*)^{-1}M_\Theta^* M_{\kappa_i}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$.

A well known consequence of the Beurling, Lax and Halmos theorem (cf. page 239, Foias and Frazho [51]) implies that a closed subspace $\mathcal{S} \subseteq H^2_{\mathcal{E}}(\mathbb{D})$ is invariant for $M_z$ if and only if $\mathcal{S} \cong H^2_{\mathcal{F}}(\mathbb{D})$ for some Hilbert space $\mathcal{F}$ with

$$\dim \mathcal{F} \leq \dim \mathcal{E}.$$

More specifically, if $\mathcal{S}$ is a closed invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ and if $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$, then the pure isometry $M_z|_{\mathcal{S}}$ on $\mathcal{S}$ and $M_z$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent, and $\dim \mathcal{W} \leq \dim \mathcal{E}$. The above theorem sets the stage for a similar result.

**Corollary 0.0.11.** *Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$, and*

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wP_{\mathcal{S}}M_z^*)^{-1}M_{\kappa_i}|_{\mathcal{W}} \qquad (w \in \mathbb{D}),$$

*for all $i = 1, \ldots, n$. Then $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$ and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent.*

We also prove that the representation of a invariant subspace, as in Theorem 0.0.10, is unique:

**Theorem 0.0.12.** *In the setting of Theorem 0.0.10, if $\mathcal{S} = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ and $\kappa_i \tilde{\Theta} = \tilde{\Theta}\tilde{\Phi}_i$ for some Hilbert space $\tilde{\mathcal{W}}$, inner function $\tilde{\Theta} \in H^\infty_{\mathcal{B}(\tilde{\mathcal{W}})}(\mathbb{D})$ and shift $M_{\tilde{\Phi}_i}$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$, $i = 1, \ldots, n$, then there exists a unitary operator (constant in $z$) $\tau : \mathcal{W} \to \tilde{\mathcal{W}}$ such that*

$$\Theta = \tilde{\Theta}\tau,$$

*and*

$$\tau \Phi_i = \tilde{\Phi}_i \tau,$$

*for all $i = 1, \dots, n$.*

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hilbert spaces, and let $\mathcal{E}_n = H^2(\mathbb{D}^n) \otimes \mathcal{E}$ and $\tilde{\mathcal{E}_n} = H^2(\mathbb{D}^n) \otimes \tilde{\mathcal{E}}$. Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be closed invariant subspaces of the multiplication tuples on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $H^2_{\tilde{\mathcal{E}_n}}(\mathbb{D})$, respectively. We say that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are *unitarily equivalent*, and write $\mathcal{S} \cong \tilde{\mathcal{S}}$, if there exists a unitary map $U : \mathcal{S} \to \tilde{\mathcal{S}}$ such that

$$UM_z|_{\mathcal{S}} = M_z|_{\tilde{\mathcal{S}}} U \quad \text{and} \quad UM_{\kappa_i}|_{\mathcal{S}} = M_{\kappa_i}|_{\tilde{\mathcal{S}}} U,$$

for all $i = 1, \dots, n$. We prove that the multipliers $\{\Phi_i\}_{i=1}^n$ is a complete set of unitary invariants of invariant subspaces:

In the final section of this chapter we present a geometric proof of the following dimensional inequality:

**Theorem 0.0.13.** *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces and let $X : H^2_{\mathcal{E}_1}(\mathbb{D}^n) \to H^2_{\mathcal{E}_2}(\mathbb{D}^n)$ be an isometry. If*

$$XM_{z_i}^{\mathcal{E}_1} = M_{z_i}^{\mathcal{E}_2} X,$$

*for all $i = 1, \dots, n$, then*

$$\dim \mathcal{E}_1 \leq \dim \mathcal{E}_2.$$

We believe that the above result (possibly) follows from the boundary behavior of bounded analytic functions following the classical case $n = 1$. Here, however, we take a shorter approach than generalizing the classical theory of bounded analytic functions on the unit polydisc.

**Chapter 4:** In this chapter, we return to the idea of defect operators of pairs of commuting isometries. Consider the BCL pair

$$\begin{aligned} V_1 &= (I_{H^2(\mathbb{D})} \otimes P + M_z \otimes P^\perp)(I_{H^2(\mathbb{D})} \otimes U^*), \\ V_2 &= (I_{H^2(\mathbb{D})} \otimes U)(M_z \otimes P + I_{H^2(\mathbb{D})} \otimes P^\perp). \end{aligned}$$

An easy computation reveals that the defect operator of $(V_1, V_2)$ is given by

$$C(V_1, V_2) = P_{\mathbb{C}} \otimes (UPU^* - P) = P_{\mathbb{C}} \otimes (P^\perp - UP^\perp U^*),$$

and hence,

$$C(V_1, V_2)|_{zH^2(\mathbb{D}) \otimes \mathcal{E}} = 0 \quad \text{and} \quad \overline{ran\ C(V_1, V_2)} \subseteq \mathbb{C} \otimes \mathcal{E}.$$

Thus it suffices to study $C(V_1, V_2)$ only on $(zH^2(\mathbb{D}) \otimes \mathcal{E})^\perp = \mathbb{C} \otimes \mathcal{E}$. In summary, if $(V_1, V_2)$ is a BCL pair on $H^2_{\mathcal{E}}(\mathbb{D})$, then the block matrix of $C(V_1, V_2)$ with respect to the orthogonal decomposition $H^2_{\mathcal{E}}(\mathbb{D}) = zH^2_{\mathcal{E}}(\mathbb{D}) \oplus \mathcal{E}$ is given by

$$C(V_1, V_2) = \begin{bmatrix} 0 & 0 \\ 0 & P^\perp - UP^\perp U^* \end{bmatrix}.$$

If $(V_1, V_2)$ is clear from the context, then we define

$$C := C(V_1, V_2)|_{\mathcal{E}} = P^{\perp} - UP^{\perp}U^*.$$

Note that $C$, being the difference of a pair of projections, is a self-adjoint contraction. In addition, if it is compact, then clearly its spectrum lies in $[-1, 1]$ and the non-zero elements of the spectrum are precisely the non-zero eigen values of $C$. In this case, for each eigen value $\lambda$ of $C$, we denote by $E_\lambda$ the eigen space corresponding to $\lambda$, that is

$$E_\lambda = \ker(C - \lambda I_{\mathcal{E}}).$$

Moreover, we have (see [69, Lemma 4.2]): If $C$ is compact, then for each non-zero eigen value $\lambda$ of $C$ in $(-1, 1)$, $-\lambda$ is also an eigen value of $C$ and

$$\dim E_\lambda = \dim E_{-\lambda}.$$

Consequently, one can decompose $(\ker C)^{\perp}$ as

$$(\ker C)^{\perp} = E_1 \oplus (\bigoplus_\lambda E_\lambda) \oplus E_{-1} \oplus (\bigoplus_\lambda E_{-\lambda}),$$

where $\lambda$ runs over the set of positive eigen values of $C$ lying in $(0, 1)$. With respect to the above decomposition of $(\ker C)^{\perp}$, the non-zero part of $C$, that is, $C|_{(\ker C)^{\perp}}$, the restriction of $C$ to $(\ker C)^{\perp}$, has the following block diagonal operator matrix form

$$C|_{(\ker C)^{\perp}} = \begin{bmatrix} I_{E_1} & 0 & 0 & 0 \\ 0 & \bigoplus\limits_\lambda \lambda I_{E_\lambda} & 0 & 0 \\ 0 & 0 & -I_{E_{-1}} & 0 \\ 0 & 0 & 0 & \bigoplus\limits_\lambda (-\lambda) I_{E_{-\lambda}} \end{bmatrix}$$

and consequently, the matrix representation of $C|_{(\ker C)^{\perp}}$, with respect to a chosen orthonormal basis of $(\ker C)^{\perp}$, is unitarily equivalent to the diagonal matrix given by

$$[C|_{(\ker C)^{\perp}}] = \begin{bmatrix} I_{l_1} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{l_1'} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}$$

where $l_1 = \dim E_1$, $l_1' = \dim E_{-1}$, $D = \bigoplus\limits_\lambda \lambda I_{k_\lambda}$, $I_k$ denotes the $k \times k$ identity matrix for any positive integer $k$ and

$$k_\lambda = \dim E_\lambda = \dim E_{-\lambda}.$$

Summarising the foregoing observations, one obtains the following [69, Theorem 4.3]:

**Theorem 0.0.14.** *With the notations as above, if the defect operator $C(V_1, V_2)$ is compact, then its non-zero part is unitarily equivalent to the diagonal block matrix*

$$\begin{bmatrix} I_{l_1} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{l'_1} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix} \tag{0.0.2}$$

This chapter concerns the reverse direction of Theorem 0.0.14: Given an operator $T$ on $\mathcal{E}$ of the form (0.0.2), construct, if possible, a BCL pair $(V_1, V_2)$ such that $C|_{(\ker C)^\perp}$, the non-zero part of $C(V_1, V_2)$, is unitarily equivalent to $T$.

Now we note that in view of the constructions of simple blocks in [69, Section 6], one can always construct a reducible BCL pair $(V_1, V_2)$ such that the non-zero part of $C(V_1, V_2)$ is equal to $T$ (see [69, Theorem 6.7]). This consideration leads us to raise the following natural question:

*Question* 1. *Given a compact block operator $T \in B(\mathcal{E})$ of the form (0.0.2), does there exist an irreducible BCL pair $(V_1, V_2)$ on the Hilbert space $H^2_{\mathcal{E}}(\mathbb{D})$ such that the non-zero part of the defect operator $C(V_1, V_2)$ is equal to $T$ (that is, $\overline{ranC(V_1, V_2)} = \mathcal{E}$ and $C(V_1, V_2)|_{\mathcal{E}} = T$)?*

The above question also has been framed in [69, page 18].

We first prove that the answer to the above question is not necessarily always in the affirmative:

**Theorem 0.0.15.** *Let $\mathcal{E}$ be a finite-dimensional Hilbert space and let $T$ on $\mathcal{E}$ be a compact block matrix of the form (0.0.2), that is,*

$$T = \begin{bmatrix} I_{dim\, E_1(T)} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{dim\, E_{-1}(T)} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}.$$

*If*

$$dim\, E_1(T) \neq dim\, E_{-1}(T),$$

*then it is not possible to find a BCL pair $(V_1, V_2)$ (reducible or irreducible) on $H^2_{\mathcal{E}}(\mathbb{D})$ such that the non-zero part of the defect operator $C(V_1, V_2)$ is equal to $T$.*

This result motivated us to investigate the cases where the answer to the aforementioned question, *Question* 1, is in the affirmative. To this end, we prove that:

**Theorem 0.0.16.** *Let $\mathcal{E}$ be a finite-dimensional Hilbert space, and let $T \in B(\mathcal{E})$ be of the form* (0.0.2)*, that is,*

$$T = \begin{bmatrix} I_{dim\,E_1(T)} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{dim\,E_{-1}(T)} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}.$$

*Assume that $dimE_1(T) = dimE_{-1}(T)$. Then, in each of the following two cases, there exists an irreducible BCL pair $(V_1, V_2)$ on $H^2_{\mathcal{E}}(\mathbb{D})$ such that the non-zero part of the defect operator $C(V_1, V_2)$ is given by $T$.*

*(i) $T$ has at least two distinct positive eigen values,*

*(ii) $T$ has only one positive eigen value lying in $(0, 1)$ with dimension of the corresponding eigen space being at least two.*

*Moreover, (iii) if $1$ is the only positive eigen value of $T$, then it is not possible to construct such an irreducible pair $(V_1, V_2)$ unless $dimE_1(T) = 1$.*

We also deal with the case when $\mathcal{E}$ is infinite-dimensional: If $\mathcal{E}$ is infinite dimensional Hilbert space, then *Question* 1 is in the affirmative in the case when

$$\dim E_1(T) = \dim E_{-1}(T),$$

or

$$\dim E_1(T) = \dim E_{-1}(T) \pm 1.$$

The second and third chapters of this thesis is based on the published papers [75] and [74], respectively. The fourth chapter is based on the preprint [39].

# Chapter 1

# Preliminaries

In this chapter we introduce the necessary notation, set up definitions and recall some classical results.

## 1.1 Hardy space

We begin with a brief introduction of Hardy space. Our presentation is motivated by [96]. The *Hardy space* $H^2(\mathbb{D})$ over $\mathbb{D}$ is the set of all power series

$$f = \sum_{m=0}^{\infty} a_m z^m, \qquad (a_m \in \mathbb{C}),$$

such that

$$\|f\|_{H^2(\mathbb{D})} := (\sum_{m=0}^{\infty} |a_m|^2)^{\frac{1}{2}} < \infty.$$

Let $f = \sum\limits_{m=0}^{\infty} a_m z^m \in H^2(\mathbb{D})$. It is obvious that $\sum\limits_{m=0}^{\infty} |w|^m < \infty$ for each $w \in \mathbb{D}$. This and $\sum\limits_{m=0}^{\infty} |a_m|^2 < \infty$ readily implies that $\sum\limits_{m=0}^{\infty} a_m w^m$ converges absolutely for each $w \in \mathbb{D}$. In other words, $f = \sum_{m=0}^{\infty} a_m z^m$ is in $H^2(\mathbb{D})$ if and only if $f$ is a square summable holomorphic function on $\mathbb{D}$.

Now, for each $w \in \mathbb{D}$ one can define a complex-valued function $\mathbb{S}(\cdot, w) : \mathbb{D} \to \mathbb{C}$ by

$$(\mathbb{S}(\cdot, w))(z) = \sum_{m=0}^{\infty} \bar{w}^m z^m. \qquad (z \in \mathbb{D}),$$

Since

$$\sum_{m=0}^{\infty} |\bar{w}^m|^2 = \sum_{m=0}^{\infty} (|w|^2)^m = \frac{1}{1 - |w|^2},$$

it follows that $\mathbb{S}(\cdot, w) \in H^2(\mathbb{D})$ for all $w \in \mathbb{D}$ and

$$\|\mathbb{S}(\cdot, w)\|_{H^2(\mathbb{D})} = \frac{1}{(1-|w|^2)^{\frac{1}{2}}} \qquad (w \in \mathbb{D}).$$

Moreover, if $f = \sum_{m=0}^{\infty} a_m z^m \in H^2(\mathbb{D})$ and $w \in \mathbb{D}$, then

$$f(w) = \sum_{m=0}^{\infty} a_m w^m = \langle \sum_{m=0}^{\infty} a_m z^m, \sum_{m=0}^{\infty} \bar{w}^m z^m \rangle_{H^2(\mathbb{D})} = \langle f, \mathbb{S}(\cdot, w) \rangle_{H^2(\mathbb{D})}.$$

Therefore, the vector $\mathbb{S}(\cdot, w) \in H^2(\mathbb{D})$ *reproduces* the value of $f \in H^2(\mathbb{D})$ at $w \in \mathbb{D}$. In particular,

$$(\mathbb{S}(\cdot, w))(z) = \langle \mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z) \rangle_{H^2(\mathbb{D})} = \sum_{m=0}^{\infty} z^m \bar{w}^m = (1 - z\bar{w})^{-1} \qquad (z, w \in \mathbb{D}).$$

The function $\mathbb{S} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ defined by

$$\mathbb{S}(z, w) = (1 - z\bar{w})^{-1}, \qquad (z, w \in \mathbb{D})$$

is called the *Szegő* or *Cauchy-Szegő* kernel of $\mathbb{D}$. Consequently, $H^2(\mathbb{D})$ is a *reproducing kernel Hilbert space* with kernel function $\mathbb{S}$.

The next goal is to show that the set $\{\mathbb{S}(\cdot, w) : w \in \mathbb{D}\}$ is *total* in $H^2(\mathbb{D})$, that is,

$$\overline{\text{span}}\{\mathbb{S}(\cdot, w) : w \in \mathbb{D}\} = H^2(\mathbb{D}).$$

To see this notice that the reproducing property of the Szegő kernel yields

$$f(w) = \langle f, \mathbb{S}(\cdot, w) \rangle_{H^2(\mathbb{D})},$$

for all $f \in H^2(\mathbb{D})$ and $w \in \mathbb{D}$. Now the result follows from the fact that $f \perp \mathbb{S}(\cdot, w)$ for $f \in H^2(\mathbb{D})$ and for all $w \in \mathbb{D}$ if and only if $f = 0$. It also follows that for each $w \in \mathbb{D}$, the *evaluation map* $ev_w : H^2(\mathbb{D}) \to \mathbb{C}$ defined by

$$ev_w(f) = f(w), \qquad (f \in H^2(\mathbb{D}))$$

is continuous.

Now we recall some of the most elementary properties of $M_z$ on $H^2(\mathbb{D})$. Observe first that

$$\langle z(z^k), z(z^l) \rangle_{H^2(\mathbb{D})} = \langle z^{k+1}, z^{l+1} \rangle_{H^2(\mathbb{D})} = \delta_{k,l} = \langle z^k, z^l \rangle_{H^2(\mathbb{D})}. \qquad (k, l \in \mathbb{N})$$

Using the fact that the set $\{z^m : m \in \mathbb{N}\}$ is total in $H^2(\mathbb{D})$, the previous equality implies that the multiplication operator $M_z$ on $H^2(\mathbb{D})$ defined by

$$(M_z f)(w) = wf(w), \qquad (f \in H^2(\mathbb{D}), w \in \mathbb{D})$$

is an isometric operator, that is,

$$M_z^* M_z = I_{H^2(\mathbb{D})}.$$

Moreover

$$\langle M_z^* z^k, z^l \rangle = \langle z^k, z^{l+1} \rangle = \delta_{k,l+1} = \delta_{k-1,l} = \langle z^{k-1}, z^l \rangle,$$

for all $k \geq 1$ and $l \in \mathbb{N}$. Also it follows that $\langle M_z^* 1, z^l \rangle_{H^2(\mathbb{D})} = 0$. Consequently,

$$M_z^* z^k = \begin{cases} z^{k-1} & \text{if } k \geq 1; \\ 0 & \text{if } k = 0. \end{cases}$$

It also follows that

$$\begin{aligned} \langle (I_{H^2(\mathbb{D})} - M_z M_z^*) \mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z) \rangle &= \langle \mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z) \rangle - \langle M_z^* \mathbb{S}(\cdot, w), M_z^* \mathbb{S}(\cdot, z) \rangle \\ &= \mathbb{S}(z, w) - z\bar{w}\mathbb{S}(z, w) = 1 \\ &= \langle P_{\mathbb{C}} \mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z) \rangle, \end{aligned}$$

where $P_{\mathbb{C}}$ is the orthogonal projection of $H^2(\mathbb{D})$ onto the one-dimensional subspace of all constant functions on $\mathbb{D}$. Therefore,

$$I_{H^2(\mathbb{D})} - M_z M_z^* = P_{\mathbb{C}}.$$

To compute the kernel, $\ker(M_z - wI_{H^2(\mathbb{D})})^*$ for $w \in \mathbb{D}$, note that

$$\begin{aligned} M_z^* \mathbb{S}(\cdot, w) &= M_z^*(1 + \bar{w}z + \bar{w}^2 z^2 + \cdots) = \bar{w} + \bar{w}^2 z + \bar{w}^3 z^2 + \cdots = \bar{w}(1 + \bar{w}z + \bar{w}^2 z^2 + \cdots) \\ &= \bar{w}\mathbb{S}(\cdot, w). \end{aligned}$$

On the other hand, if $M_z^* f = \bar{w} f$ for some $f \in H^2(\mathbb{D})$ then

$$f(0) = P_{\mathbb{C}} f = (I_{H^2(\mathbb{D})} - M_z M_z^*) f = (1 - z\bar{w}) f,$$

that is, $f = f(0)\mathbb{S}(\cdot, w)$. Consequently, $M_z^* f = \bar{w} f$ if and only if $f = \lambda \mathbb{S}(\cdot, w)$ for some $\lambda \in \mathbb{C}$. That is,

$$\ker(M_z - wI_{H^2(\mathbb{D})})^* = \{\lambda \mathbb{S}(\cdot, w) : \lambda \in \mathbb{C}\}.$$

In particular,

$$\bigvee_{w \in \mathbb{D}} \ker(M_z - wI_{H^2(\mathbb{D})})^* = H^2(\mathbb{D}).$$

The following theorem summarizes the above observations.

**Theorem 1.1.1.** *Let $H^2(\mathbb{D})$ denote the Hardy space over $\mathbb{D}$ and $M_z$ denote the multiplication operator by the coordinate function $z$ on $H^2(\mathbb{D})$. Then, the following properties hold:*

*(i) The set $\{\mathbb{S}(\cdot, w) : w \in \mathbb{D}\}$ is total in $H^2(\mathbb{D})$.*

*(ii) The evaluation map $ev_w : H^2(\mathbb{D}) \to \mathbb{C}$ defined by $ev_w(f) = f(w)$ is continuous for each $w \in \mathbb{D}$.*

*(iii) $\sigma_p(M_z^*) = \mathbb{D}$ and $ker(M_z - wI_{H^2(\mathbb{D})})^* = \{\lambda \mathbb{S}(\cdot, w) : \lambda \in \mathbb{C}\}$.*

*(iv) $f(w) = \langle f, \mathbb{S}(\cdot, w)\rangle_{H^2(\mathbb{D})}$ for all $f \in H^2(\mathbb{D})$ and $w \in \mathbb{D}$.*

*(v)$I_{H^2(\mathbb{D})} - M_z M_z^* = P_{\mathbb{C}}$.*

*(vi) $\bigvee_{w\in\mathbb{D}} ker(M_z - wI_{H^2(\mathbb{D})})^* = H^2(\mathbb{D})$.*

Finally, let $\mathcal{E}$ be a Hilbert space. In what follows, $H^2_{\mathcal{E}}(\mathbb{D})$ stands for the Hardy space of $\mathcal{E}$-valued analytic functions on $\mathbb{D}$. Moreover, by virtue of the unitary $U : H^2_{\mathcal{E}}(\mathbb{D}) \to H^2(\mathbb{D}) \otimes \mathcal{E}$ defined by

$$z^m\eta \mapsto z^m \otimes \eta, \quad (\eta \in \mathcal{E}, m \in \mathbb{N})$$

the vector valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ will be identified with the Hilbert space tensor product $H^2(\mathbb{D}) \otimes \mathcal{E}$. The reproducing kernel of $H^2_{\mathcal{E}}(\mathbb{D})$ is given by

$$(z, w) \to \mathbb{S}(z, w)I_{\mathcal{E}} \qquad (z, w \in \mathbb{D}).$$

Note that

$$UM_z^{\mathcal{E}} = (M_z \otimes I_{\mathcal{E}})U,$$

where $M_z^{\mathcal{E}}$ denotes the multiplication operator by the coordinate function $z$ on $H^2_{\mathcal{E}}(\mathbb{D})$, that is

$$(M_z^{\mathcal{E}} f)(w) = wf(w) \qquad (f \in H^2_{\mathcal{E}}(\mathbb{D}), w \in \mathbb{D}).$$

Therefore, $M_z^{\mathcal{E}}$ on $H^2_{\mathcal{E}}(\mathbb{D})$ and $M_z \otimes I_{\mathcal{E}}$ on $H^2(\mathbb{D}) \otimes \mathcal{E}$ are unitarily equivalent. If $\mathcal{E}$ is clear from the context, then we will denote $M_z^{\mathcal{E}}$ simply by $M_z$.

For a more extensive treatment of the Hardy space and related topics, the reader is referred to the books by Sz.-Nagy and Foias [79], Rosenblum and Rovnyak [92], Radjavi and Rosenthal [89] and Halmos [64].

## 1.2 Isometries and shift operators

Let $V$ be an isometry on a Hilbert space $\mathcal{H}$, that is, $V^*V = I_{\mathcal{H}}$. A closed subspace $\mathcal{W} \subseteq \mathcal{H}$ is said to be *wandering subspace* for $V$ if $V^k\mathcal{W} \perp V^l\mathcal{W}$ for all $k, l \in \mathbb{N}$ with $k \neq l$, or equivalently, if $V^m\mathcal{W} \perp \mathcal{W}$ for all $m \geq 1$. An isometry $V$ on $\mathcal{H}$ is said to be a *unilateral shift* or *shift* if

$$\mathcal{H} = \bigoplus_{m\geq 0} V^m\mathcal{W},$$

for some wandering subspace $\mathcal{W}$ for $V$.

For a shift $V$ on $\mathcal{H}$ with a wandering subspace $\mathcal{W}$ we have

$$\mathcal{H} \ominus V\mathcal{H} = \Big(\bigoplus_{m\geq 0} V^m\mathcal{W}\Big) \ominus \Big(V(\bigoplus_{m\geq 0} V^m\mathcal{W})\Big) = \Big(\bigoplus_{m\geq 0} V^m\mathcal{W}\Big) \ominus \Big(\bigoplus_{m\geq 1} V^m\mathcal{W}\Big) = \mathcal{W}.$$

In other words, the wandering subspace of a shift is unique and is given by

$$\mathcal{W} = \ker V^* = \mathcal{H} \ominus V\mathcal{H}.$$

The dimension of the wandering subspace of a shift is called the *multiplicity* of the shift.

The classical Wold-von Neumann decomposition theorem ([110], see also page 3 in [79]) states that every isometry on a Hilbert space is either a shift, or a unitary, or a direct sum of shift and unitary:

**Theorem 1.2.1.** *(Wold-von Neumann decomposition) Let $V$ be an isometry on $\mathcal{H}$. Then $\mathcal{H}$ admits a unique decomposition $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$, where $\mathcal{H}_s$ and $\mathcal{H}_u$ are $V$-reducing subspaces of $\mathcal{H}$ and $V|_{\mathcal{H}_s}$ is a shift and $V|_{\mathcal{H}_u}$ is unitary. Moreover,*

$$\mathcal{H}_s = \bigoplus_{m=0}^{\infty} V^m\mathcal{W} \qquad \textit{and} \qquad \mathcal{H}_u = \bigcap_{m=0}^{\infty} V^m\mathcal{H},$$

*where $\mathcal{W} = ran(I - VV^*) = \ker V^*$ is the wandering subspace for $V$.*

*Proof.* Let $\mathcal{W} = \mathrm{ran}(I - VV^*)$ be the wandering subspace for $V$ and

$$\mathcal{H}_s := \bigoplus_{m=0}^{\infty} V^m\mathcal{W}.$$

Consequently, $\mathcal{H}_s$ is a $V$-reducing subspace of $\mathcal{H}$ and that $V|_{\mathcal{H}_s}$ is an isometry. Furthermore

$$\mathcal{H}_u := \mathcal{H}_s^{\perp} = (\bigoplus_{m=0}^{\infty} V^m\mathcal{W})^{\perp} = \bigcap_{m=0}^{\infty} (V^m\mathcal{W})^{\perp}.$$

We observe now that $I - VV^*$ is an orthogonal projection, hence $V^l(I - VV^*)V^{*l}$ is also an orthogonal projection and

$$V^l(I - VV^*)V^{*l} = (V^l(I - VV^*))(V^l(I - VV^*))^*,$$

for all $l \geq 0$. Thus we obtain

$$\mathrm{ran}V^l(I - VV^*) = \mathrm{ran}\big((V^l(I - VV^*))(V^l(I - VV^*))^*\big) = \mathrm{ran}V^l(I - VV^*)V^{*l},$$

and hence

$$
\begin{aligned}
(V^l\mathcal{W})^\perp &= (V^l \mathrm{ran}(I - VV^*))^\perp = (\mathrm{ran}V^l(I - VV^*))^\perp \\
&= (\mathrm{ran}V^l(I - VV^*)V^{*l})^\perp = \mathrm{ran}(I - V^l(I - VV^*)V^{*l}) \\
&= \mathrm{ran}[(I - V^lV^{*l}) \oplus V^{l+1}V^{*\,l+1}] = \mathrm{ran}(I - V^lV^{*l}) \oplus \mathrm{ran}V^{l+1} \\
&= (V^l\mathcal{H})^\perp \oplus V^{l+1}\mathcal{H} = \mathrm{ker}V^{*l} \oplus V^{l+1}\mathcal{H},
\end{aligned}
$$

for all $l \geq 0$. Consequently, we have

$$
\mathcal{H}_u = \bigcap_{m=0}^{\infty} \left(\mathrm{ker}V^{*m} \oplus V^{m+1}\mathcal{H}\right) = \bigcap_{m=0}^{\infty} V^m\mathcal{H}.
$$

Uniqueness of the decomposition readily follows from the uniqueness of the wandering subspace $\mathcal{W}$ for $V$. This completes the proof. □

Note that $V$ is a shift if and only if $\mathcal{H}_s = \mathcal{H}$, which is equivalent to the fact that

$$
SOT - \lim_{k\to\infty} V^{*k} = 0.
$$

Therefore, an isometry $V$ is shift if and only if $SOT - \lim\limits_{k\to\infty} V^{*k} = 0$. We will sometimes call a shift as *pure* isometry.

We now prove that shift operators are simply the multiplication operators $M_z$ on vector-valued Hardy spaces. Let $V$ be an isometry on $\mathcal{H}$, and let $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$ be the Wold-von Neumann orthogonal decomposition of $V$. Define

$$
\Pi_V : \mathcal{H}_s \oplus \mathcal{H}_u \to H^2_{\mathcal{W}}(\mathbb{D}) \oplus \mathcal{H}_u
$$

by

$$
\Pi_V(V^m\eta \oplus f) = z^m\eta \oplus f \qquad (m \geq 0, f \in \mathcal{H}_u).
$$

Then $\Pi_V$ is a unitary and

$$
\Pi_V \begin{bmatrix} V_s & 0 \\ 0 & V_u \end{bmatrix} = \begin{bmatrix} M_z^{\mathcal{W}(V)} & 0 \\ 0 & V_u \end{bmatrix} \Pi_V,
$$

that is, $V$ on $\mathcal{H}$ and $\begin{bmatrix} M_z^{\mathcal{W}} & 0 \\ 0 & V_u \end{bmatrix}$ on $H^2_{\mathcal{W}}(\mathbb{D}) \oplus \mathcal{H}_u$ are unitarily equivalent. In particular, if $V$ is a shift, then $\mathcal{H}_u = \{0\}$ and hence

$$
\Pi_V V = M_z^{\mathcal{W}} \Pi_V.
$$

Therefore, an isometry $V$ on $\mathcal{H}$ is a shift operator if and only if $V$ is unitarily equivalent to $M_z^{\mathcal{W}}$ on $H^2_{\mathcal{W}}(\mathbb{D})$. Moreover, we note that $\dim \mathcal{W} = \dim(\mathcal{H} \ominus V\mathcal{H})$ is the only (numerical) unitary invariant of $V$ (or $M_z^{\mathcal{W}}$).

## 1.3 Multipliers and invariant subspaces

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Hilbert spaces. We will denote by $H^\infty_{\mathcal{B}(\mathcal{E}_1,\mathcal{E}_2)}(\mathbb{D})$ the set of all maps $\Theta : \mathbb{D} \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ such that

$$\Theta H^2_{\mathcal{E}_1}(\mathbb{D}) \subseteq H^2_{\mathcal{E}_2}(\mathbb{D}).$$

Elements of $H^\infty_{\mathcal{B}(\mathcal{E}_1,\mathcal{E}_2)}(\mathbb{D})$ are called *multipliers*.

The following characterization is well known and classical. However, the proof presented below, borrowed from [97], seems new and short.

**Theorem 1.3.1.** *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Hilbert spaces and let $X \in \mathcal{B}(H^2(\mathbb{D})\otimes\mathcal{E}_1, H^2(\mathbb{D})\otimes\mathcal{E}_2)$. Then*

$$X(M_z \otimes I_{\mathcal{E}_1}) = (M_z \otimes I_{\mathcal{E}_2})X,$$

*if and only if $X = M_\Theta$ for some $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_1,\mathcal{E}_2)}(\mathbb{D})$.*

*Proof.* Let $X \in \mathcal{B}(H^2(\mathbb{D}) \otimes \mathcal{E}_1, H^2(\mathbb{D}) \otimes \mathcal{E}_2)$ and $X(M_z \otimes I_{\mathcal{E}_1}) = (M_z \otimes I_{\mathcal{E}_2})X$. If $\zeta \in \mathcal{E}_2$ and $w \in \mathbb{D}$ then

$$(M_z \otimes I_{\mathcal{E}_1})^*[X^*(\mathbb{S}(\cdot, w) \otimes \zeta)] = X^*(M_z \otimes I_{\mathcal{E}_2})^*(\mathbb{S}(\cdot, w) \otimes \zeta) = \bar{w}[X^*(\mathbb{S}(\cdot, w) \otimes \zeta)],$$

that is,

$$X^*(\mathbb{S}(\cdot, w) \otimes \zeta) \in \ker(M_z \otimes I_{\mathcal{E}_1} - wI_{H^2(\mathbb{D})\otimes\mathcal{E}_1})^*.$$

This and the fact that $\ker(M_z - wI_{H^2(\mathbb{D})})^* =< \mathbb{S}(\cdot, w) >$ readily implies that

$$X^*(\mathbb{S}(\cdot, w) \otimes \zeta) = \mathbb{S}(\cdot, w) \otimes X(w)\zeta, \qquad (w \in \mathbb{D}, \zeta \in \mathcal{E}_2)$$

for some linear map $X(w) : \mathcal{E}_2 \to \mathcal{E}_1$. Moreover,

$$\|X(w)\zeta\|_{\mathcal{E}_1} = \frac{1}{\|\mathbb{S}(\cdot, w)\|_{H^2(\mathbb{D})}}\|X^*(\mathbb{S}(\cdot, w) \otimes \zeta)\|_{H^2(\mathbb{D})\otimes\mathcal{E}_1} \leq \frac{\|\mathbb{S}(\cdot, w)\|_{H^2(\mathbb{D})}}{\|\mathbb{S}(\cdot, w)\|_{H^2(\mathbb{D})}}\|X\|\|\zeta\|_{\mathcal{E}_2},$$

for all $w \in \mathbb{D}$ and $\zeta \in \mathcal{E}_2$. Therefore $X(w)$ is bounded and $\Theta(w) := X(w)^* \in \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ for each $w \in \mathbb{D}$. Thus

$$X^*(\mathbb{S}(\cdot, w) \otimes \zeta) = \mathbb{S}(\cdot, w) \otimes \Theta(w)^*\zeta \qquad (w \in \mathbb{D},\ \zeta \in \mathcal{E}_2).$$

In order to prove that $\Theta(w)$ is holomorphic we compute

$$\begin{aligned}\langle\Theta(w)\eta, \zeta\rangle_{\mathcal{E}_2} &= \langle\eta, \Theta(w)^*\zeta\rangle_{\mathcal{E}_1} = \langle\mathbb{S}(\cdot, 0) \otimes \eta, \mathbb{S}(\cdot, w) \otimes \Theta(w)^*\zeta\rangle_{H^2(\mathbb{D})\otimes\mathcal{E}_1}\\ &= \langle X(\mathbb{S}(\cdot, 0) \otimes \eta), \mathbb{S}(\cdot, w) \otimes \zeta\rangle_{H^2(\mathbb{D})\otimes\mathcal{E}_2}. \qquad (\eta \in \mathcal{E}_1,\ \zeta \in \mathcal{E}_2)\end{aligned}$$

Since $w \mapsto \mathbb{S}(\cdot, w)$ is anti-holomorphic, we conclude that $w \mapsto \Theta(w)$ is holomorphic. Hence $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_1,\mathcal{E}_2)}(\mathbb{D})$ and $X = M_\Theta$.

Conversely, let $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_1,\mathcal{E}_2)}(\mathbb{D})$. For $f \in H^2(\mathbb{D}) \otimes \mathcal{E}_1$ and $w \in \mathbb{D}$ this implies that

$$(z\Theta f)(w) = w\Theta(w)f(w) = \Theta(w)wf(w) = (\Theta zf)(w).$$

So $M_\Theta$ intertwines the multiplication operators which completes the proof. $\square$

As an application of the Neumann-Wold decomposition theorem and the above characterization of multipliers, we now prove the classical Beurling-Lax-Halmos Theorem.

**Theorem 1.3.2.** *(Beurling-Lax-Halmos Theorem) Let $\mathcal{S}$ be an $M_z$ invariant subspace of Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$. Then there exists a Hilbert space $\mathcal{F}$ and a unitary operator $U : H^2_{\mathcal{F}}(\mathbb{D}) \to \mathcal{S}$ such that*

$$UM_z = (M_z|_{\mathcal{S}})U.$$

*Moreover, $dim\mathcal{F} \leq dim\mathcal{E}$ and there exists an inner multiplier $\Theta \in H^\infty_{B(\mathcal{F},\mathcal{E})}(\mathbb{D})$ such that $M_\Theta : H^2_{\mathcal{F}}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D})$ is an isometric multiplier and $\mathcal{S} = \Theta H^2_{\mathcal{F}}(\mathbb{D})$. The inner multiplier $\Theta$ is unique upto a unitary right factor, that is, if $\mathcal{S} = \tilde{\Theta} H^2_{\tilde{\mathcal{F}}}(\mathbb{D})$ for some Hilbert space $\tilde{\mathcal{F}}$ and an inner function $\tilde{\Theta} \in H^\infty_{B(\tilde{\mathcal{F}},\mathcal{E})}(\mathbb{D})$, then $\Theta = \tilde{\Theta}\tau$ for some unitary operator $\tau$ in $B(\mathcal{F},\tilde{\mathcal{F}})$.*

*Proof.* Let $V = M_z|_{\mathcal{S}}$. Clearly, $V$ is an isometry on $\mathcal{S}$ and

$$\bigcap_{n=0}^{\infty} V^n\mathcal{S} \subseteq \bigcap_{n=0}^{\infty} V^n\mathcal{H} = \{0\},$$

which implies that $V$ is a shift on $\mathcal{S}$. By Theorem 1.2.1, it follows that

$$\mathcal{S} = \bigoplus_{m=0}^{\infty} V^m\mathcal{F},$$

where $\mathcal{F} = ran(I - VV^*)$. Then

$$U : H^2_{\mathcal{F}}(\mathbb{D}) \to \mathcal{S} = \bigoplus_{n=0}^{\infty} V^n\mathcal{F},$$

defined by $U(z^k\eta) = V^k\eta$, for all $\eta \in \mathcal{F}$ and $k \geq 0$, is the desired unitary. Now assume that $i_{\mathcal{S}} : \mathcal{S} \to H^2_{\mathcal{E}}(\mathbb{D})$ is the natuarl inclusion map. Then

$$\tilde{U} := i_{\mathcal{S}} \circ U : H^2_{\mathcal{F}}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D}),$$

defines an isometry. Moreover

$$\text{ran}\tilde{U} = \text{ran} i_{\mathcal{S}} = \mathcal{S},$$

and $\tilde{U}M_z^{\mathcal{F}} = M_z^{\mathcal{E}}\tilde{U}$. By Theorem 1.3.1, it then follows that $\tilde{U} = M_\Theta$ for some inner multiplier $\Theta \in H^\infty_{B(\mathcal{F},\mathcal{E})}(\mathbb{D})$. The dimension inequality follows from the well known boundary behaviour of bounded analytic functions (or see Chapter 3 Theorem 3.6.1 for an independent and geometric proof). The uniqueness part of $\Theta$ is left to the reader. $\square$

## 1.4 Hardy space over the polydisc

Let $n \geq 1$, and let $\mathbb{D}^n$ be the open unit polydisc in $\mathbb{C}^n$. The *Hardy space* $H^2(\mathbb{D}^n)$ over $\mathbb{D}^n$ is the Hilbert space of all holomorphic functions $f$ on $\mathbb{D}^n$ such that

$$\|f\|_{H^2(\mathbb{D}^n)} = \left( \sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(re^{i\theta_1}, \ldots, re^{i\theta_n})|^2 \, d\theta \right)^{\frac{1}{2}} < \infty,$$

where $d\theta$ is the normalized Lebesgue measure on the torus $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}^n$. It is well known that $H^2(\mathbb{D}^n)$ is a reproducing kernel Hilbert space corresponding to the Szegö kernel $\mathbb{S}_n$ on $\mathbb{D}^n$, where

$$\mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} (1 - z_i \bar{w}_i)^{-1} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

Clearly

$$\mathbb{S}_n^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{0 \leq |\boldsymbol{k}| \leq n} (-1)^{|\boldsymbol{k}|} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{k}},$$

where $|\boldsymbol{k}| = \sum_{i=1}^n k_i$ and $0 \leq k_i \leq 1$ for all $i = 1, \ldots, n$. Here we use the notation $\boldsymbol{z}$ for the $n$-tuple $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$. Also for any multi-index $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ and $\boldsymbol{z} \in \mathbb{C}^n$, we write $\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}$.

Let $\mathcal{E}$ be a Hilbert space, and let $H^2_{\mathcal{E}}(\mathbb{D}^n)$ denote the $\mathcal{E}$-valued Hardy space over $\mathbb{D}^n$. Then $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is the $\mathcal{E}$-valued reproducing kernel Hilbert space with the $\mathcal{B}(\mathcal{E})$-valued kernel function

$$(\boldsymbol{z}, \boldsymbol{w}) \mapsto \mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) I_{\mathcal{E}} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

Like the one variable Hardy space, in the sequel, by virtue of the canonical unitary $U$ from $H^2_{\mathcal{E}}(\mathbb{D}^n)$ to $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ defined by

$$U(\boldsymbol{z}^{\boldsymbol{k}} \eta) = \boldsymbol{z}^{\boldsymbol{k}} \otimes \eta \qquad (\boldsymbol{k} \in \mathbb{Z}_+^n, \eta \in \mathcal{E}),$$

we will identify the vector valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D}^n)$ with the tenor product Hilbert space $H^2(\mathbb{D}^n) \otimes \mathcal{E}$. Let $(M_{z_1}, \ldots, M_{z_n})$ denote the $n$-tuple of multiplication operators on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ by the coordinate functions $\{z_i\}_{i=1}^n$, that is,

$$(M_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}),$$

for all $f \in H^2_{\mathcal{E}}(\mathbb{D}^n)$, $\boldsymbol{w} \in \mathbb{D}^n$ and $i = 1, \ldots, n$. It is well known and easy to check that

$$\|M_{z_i} f\| = \|f\|,$$

and
$$\|M_{z_i}^{*m} f\| \to 0,$$
as $m \to \infty$ and for all $f \in H^2_{\mathcal{E}}(\mathbb{D}^n)$, that is, $M_{z_i}$ defines a shift (see the definition of shift below) on $H^2_{\mathcal{E}}(\mathbb{D}^n)$, $i = 1, \ldots, n$. If $n > 1$, then it also follows easily that
$$M_{z_i} M_{z_j} = M_{z_j} M_{z_i},$$
and
$$M_{z_i}^* M_{z_j} = M_{z_j} M_{z_i}^*,$$
for all $1 \leq i < j \leq n$. Therefore, $(M_{z_1}, \ldots, M_{z_n})$ is an $n$-tuple of *doubly commuting* shifts on $H^2_{\mathcal{E}}(\mathbb{D}^n)$.

Note that
$$U(z_1^{k_1} \cdots z_n^{k_n}) = z^{k_1} \otimes \cdots \otimes z^{k_n},$$
for all $k_1, \ldots, k_n \in \mathbb{Z}_+$, defines a unitary map $\tilde{U}$ from $H^2(\mathbb{D}^n)$ to $H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$, the $n$-fold Hilbert space tensor product of $H^2(\mathbb{D})$. Moreover
$$UM_{z_i} = \Big(I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})} \otimes \underbrace{M_z}_{i^{th}\text{place}} \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}\Big) U,$$
for all $i = 1, \ldots, n$. One can now easily verify all the above mentioned properties of $M_{z_i}$, $i = 1, \ldots, n$. This along with the other canonical identification of $M_{z_i}$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ with $M_{z_i} \otimes I_{\mathcal{E}}$ on $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ will be used throughout the rest of the thesis.

# Chapter 2

# Pairs of Commuting Isometries

## 2.1 Introduction

The main purpose of this chapter is to explore and relate various natural representations of a large class of pairs of commuting isometries on Hilbert spaces. The geometry of Hilbert spaces, the classical Wold-von Neumann decomposition for isometries, the analytic structure of the commutator of the unilateral shift, and the Berger, Coburn and Lebow [20] representations of pure pairs of commuting isometries are the main guiding principles for our study. The Berger, Coburn and Lebow theorem states that: Let $(V_1, V_2)$ be a pair of commuting isometries on a Hilbert space $\mathcal{H}$, and let $V = V_1V_2$ be a shift (or, a pure isometry - see Section 2). Then there exist a Hilbert space $\mathcal{W}$, an orthogonal projection $P$ and a unitary operator $U$ on $\mathcal{W}$ such that

$$\Phi_1(z) = U^*(P + zP^{\perp}) \quad \text{and} \quad \Phi_2(z) = (P^{\perp} + zP)U \qquad (z \in \mathbb{D}),$$

are commuting isometric multipliers in $H^{\infty}_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$, and $(V_1, V_2, V)$ on $\mathcal{H}$ and $(M_{\Phi_1}, M_{\Phi_2}, M_z)$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent (see Bercovici, Douglas and Foias [18] for an elegant proof).

Recall that, given a Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$. We also set

$$P_{\mathcal{S}}^{\perp} = I_{\mathcal{H}} - P_{\mathcal{S}}.$$

In this chapter we give a new and more concrete treatment, in the sense of explicit representations and analytic descriptions, to the structure of pure pairs of commuting isometries. More specifically, we provide an explicit recipe for constructing the isometric multipliers $(\Phi_1(z), \Phi_2(z))$, and the operators $U$ and $P$ involved in the coefficients of $\Phi_1$ and $\Phi_2$ (see Theorems 2.3.2 and 2.3.3). Then we compare the Berger, Coburn and Lebow representations with other possible analytic representations of pairs of commuting isometries.

In Section 6, we analyze defect operators for (not necessarily pure) pairs of commuting isometries. We provide a list of characterizations of pairs of commuting isometries with positive defect operators. Our results hold in a more general setting with somewhat simpler proofs (see Theorem 2.6.5 for instance) than the one considered by He, Qin and Yang [69]. Moreover, we prove that for a large class of pure pairs of commuting isometries the defect operator is negative if and only if the defect operator is the zero operator.

The chapter is organized as follows. In Section 2 we prove a representation theorem for commutators of shifts. In Section 3 we discuss some basic relationships between wandering subspaces for commuting isometries, followed by a new and explicit proof of the Berger, Coburn and Lebow characterizations of pure pairs of commuting isometries. Section 4 is devoted to a short discussion about joint unitary invariants of pure pairs of commuting isometries. Section 5 ties together the explicit Berger, Coburn and Lebow representation and other possible analytic representations of a pair of commuting isometries. Then, in Section 6, we present a general theory for pairs of commuting isometries and analyze the defect operators. Concluding remarks, future directions and a close connection of our consideration with the Sz.-Nagy and Foias characteristic functions for contractions are discussed in Section 7.

This chapter is based on the published paper [75].

## 2.2 Commutators of shifts

Let $V$ be an isometry on $\mathcal{H}$, and let $\mathcal{H} = \mathcal{H}_s(V) \oplus \mathcal{H}_u(V)$ be the Wold-von Neumann orthogonal decomposition of $V$ (see Chapter 1, Theorem 1.2.1). Define

$$\Pi_V : \mathcal{H}_s(V) \oplus \mathcal{H}_u(V) \to H^2_{\mathcal{W}(V)}(\mathbb{D}) \oplus \mathcal{H}_u(V)$$

by

$$\Pi_V(V^m \eta \oplus f) = z^m \eta \oplus f \qquad (m \geq 0, \eta \in \mathcal{W}(V), f \in \mathcal{H}_u(V)).$$

Then $\Pi_V$ is a unitary and

$$\Pi_V \begin{bmatrix} V_s & 0 \\ 0 & V_u \end{bmatrix} = \begin{bmatrix} M_z^{\mathcal{W}(V)} & 0 \\ 0 & V_u \end{bmatrix} \Pi_V.$$

In particular, if $V$ is a shift, then $\mathcal{H}_u(V) = \{0\}$ and hence

$$\Pi_V V = M_z^{\mathcal{W}(V)} \Pi_V.$$

Therefore, an isometry $V$ on $\mathcal{H}$ is a shift operator if and only if $V$ is unitarily equivalent to $M_z^{\mathcal{E}}$ on $H^2_{\mathcal{E}}(\mathbb{D})$, where $\dim \mathcal{E} = \dim \mathcal{W}(V)$.

In the sequel we denote by $(\Pi_V, M_z^{\mathcal{W}(V)})$, or simply by $(\Pi_V, M_z)$, the Wold-von Neumann decomposition of the pure isometry $V$ in the above sense.

Let $\mathcal{E}$ be a Hilbert space, and let $C$ be a bounded linear operator on $H^2_{\mathcal{E}}(\mathbb{D})$. Then $C \in \{M_z\}'$, that is, $CM_z = M_zC$, if and only if (cf. [79])

$$C = M_\Theta$$

for some $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ and $(M_\Theta f)(w) = \Theta(w)f(w)$ for all $f \in H^2_{\mathcal{E}}(\mathbb{D})$ and $w \in \mathbb{D}$.

Now let $V$ be a pure isometry, and let $C \in \{V\}'$. Let $(\Pi_V, M_z)$ be the Wold-von Neumann decomposition of $V$, and let $\mathcal{W} = \mathcal{W}(V)$. Since $\Pi_V C \Pi_V^*$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is the representation of $C$ on $\mathcal{H}$ and $(\Pi_V C \Pi_V^*) M_z = M_z (\Pi_V C \Pi_V^*)$, it follows that

$$\Pi_V C \Pi_V^* = M_\Theta,$$

for some $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$. The main result of this section is the following explicit representation of $\Theta$.

**Theorem 2.2.1.** *Let $V$ be a pure isometry on $\mathcal{H}$, and let $C$ be a bounded operator on $\mathcal{H}$. Let $(\Pi_V, M_z)$ be the Wold-von Neumann decomposition of $V$. Set $\mathcal{W} = \mathcal{W}(V)$, $M = \Pi_V C \Pi_V^*$ and let*

$$\Theta(w) = P_{\mathcal{W}}(I_{\mathcal{H}} - wV^*)^{-1} C|_{\mathcal{W}} \qquad (w \in \mathbb{D}).$$

*Then*

$$CV = VC,$$

*if and only if $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ and*

$$M = M_\Theta.$$

*Proof.* Let $h \in \mathcal{H}$. One can express $h$ as $h = \sum_{m=0}^{\infty} V^m \eta_m$, for some $\eta_m \in \mathcal{W}$, $m \geq 0$ (as $\mathcal{H} = \bigoplus_{m=0}^{\infty} V^m \mathcal{W}$). Applying $P_{\mathcal{W}} V^{*l}$ to both sides and using the fact that $\mathcal{W} = \mathcal{W}(V) = \ker V^*$, we obtain $\eta_l = P_{\mathcal{W}} V^{*l} h$ for all $l \geq 0$. This implies, for any $h \in \mathcal{H}$,

$$h = \sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m} h. \tag{2.2.1}$$

Now let $CV = VC$. Then there exists a bounded analytic function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ such that $\Pi_V C \Pi_V^* = M_\Theta$. For each $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$ we have

$$\begin{aligned}\Theta(w)\eta &= (M_\Theta \eta)(w)\\ &= (\Pi_V C \Pi_V^* \eta)(w)\\ &= (\Pi_V C\eta)(w),\end{aligned}$$

as $\Pi_V^* \eta = \eta$. Since in view of (2.2.1)

$$C\eta = \sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m} C\eta,$$

it follows that

$$\begin{aligned}\Theta(w)\eta &= (\Pi_V(\sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m} C\eta))(w)\\ &= (\sum_{m=0}^{\infty} M_z^m (P_{\mathcal{W}} V^{*m} C\eta))(w)\\ &= \sum_{m=0}^{\infty} w^m (P_{\mathcal{W}} V^{*m} C\eta)\\ &= P_{\mathcal{W}}(I_{\mathcal{H}} - wV^*)^{-1} C\eta.\end{aligned}$$

Therefore

$$\Theta(w) = P_{\mathcal{W}}(I_{\mathcal{H}} - wV^*)^{-1} C|_{\mathcal{W}} \qquad (w \in \mathbb{D}),$$

as required. Finally, since the sufficient part is trivial, the proof is complete. $\square$

Note that in the above proof we have used the standard projection formula (see, for example, Rosenblum and Rovnyak [92]) $I_{\mathcal{H}} = \text{SOT} - \sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m}$. It may also be observed that $\|wV^*\| = |w|\|V\| < 1$ for all $w \in \mathbb{D}$, and so it follows that the function $\Theta$ defined in Theorem 2.2.1 is a $\mathcal{B}(\mathcal{W})$-valued holomorphic function in the unit disc $\mathbb{D}$. However, what is not guaranteed in general here is that the function $\Theta$ is in $H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$. The above theorem says that this is so if $CV = VC$.

## 2.3 Berger, Coburn and Lebow representations

This section is devoted to a detailed study of Berger, Coburn and Lebow's representation of pure pairs of commuting isometries. Our approach is different and yields sharper results, along with new proofs, in terms of explicit coefficients of the one variable polynomials associated with the class of pure pairs of commuting isometries. Before dealing more specifically with pure pairs of commuting isometries we begin with some general observations about pairs of commuting isometries.

Let $(V_1, V_2)$ be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. In the sequel, we will adopt the following notations:

$$V = V_1 V_2,$$

$$\mathcal{W} = \mathcal{W}(V) = \mathcal{W}(V_1 V_2) = \mathcal{H} \ominus V_1 V_2 \mathcal{H},$$

and

$$\mathcal{W}_j = \mathcal{W}(V_j) = \mathcal{H} \ominus V_j\mathcal{H} \qquad (j = 1, 2).$$

A pair of commuting isometries $(V_1, V_2)$ on $\mathcal{H}$ is said to be *pure* if $V$ is a pure isometry.

The following useful lemma on wandering subspaces for commuting isometries is simple.

**Lemma 2.3.1.** *Let $(V_1, V_2)$ be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then*

$$\mathcal{W} = \mathcal{W}_1 \oplus V_1\mathcal{W}_2 = V_2\mathcal{W}_1 \oplus \mathcal{W}_2,$$

*and the operator $U$ on $\mathcal{W}$ defined by*

$$U(\eta_1 \oplus V_1\eta_2) = V_2\eta_1 \oplus \eta_2,$$

*for $\eta_1 \in \mathcal{W}_1$ and $\eta_2 \in \mathcal{W}_2$, is a unitary operator. Moreover,*

$$P_{\mathcal{W}}V_i = V_iP_{\mathcal{W}_j} \qquad (i \neq j).$$

*Proof.* The first equality follows from

$$I - VV^* = (I - V_1V_1^*) \oplus V_1(I - V_2V_2^*)V_1^* = V_2(I - V_1V_1^*)V_2^* \oplus (I - V_2V_2^*).$$

The second part directly follows from the first part, and the last claim follows from $(I - VV^*)V_i = V_i(I - V_jV_j^*)$ for all $i \neq j$. This concludes the proof of the lemma. □

Let $(V_1, V_2)$ be a pure pair of commuting isometries on a Hilbert space $\mathcal{H}$, and let $(\Pi_V, M_z)$ be the Wold-von Neumann decomposition of $V$. Since

$$VV_i = V_iV \qquad (i = 1, 2),$$

there exist isometric multipliers (that is, inner functions [79]) $\Phi_1$ and $\Phi_2$ in $H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ such that

$$\Pi_V V_i = M_{\Phi_i}\Pi_V \qquad (i = 1, 2).$$

In other words, $(M_{\Phi_1}, M_{\Phi_2})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is the representation of $(V_1, V_2)$ on $\mathcal{H}$. Following Berger, Coburn and Lebow [20], we say that $(M_{\Phi_1}, M_{\Phi_2})$ is the *BCL representation* of $(V_1, V_2)$, or simply the *BCL pair* corresponding to $(V_1, V_2)$.

We now present an explicit description of the BCL pair $(M_{\Phi_1}, M_{\Phi_2})$.

**Theorem 2.3.2.** *Let $(V_1, V_2)$ be a pure pair of commuting isometries on a Hilbert space $\mathcal{H}$, and let $(M_{\Phi_1}, M_{\Phi_2})$ be the BCL representation of $(V_1, V_2)$. Then*

$$\Phi_1(z) = V_1|_{\mathcal{W}_2} \oplus V_2^*|_{V_2\mathcal{W}_1}z, \;\; \Phi_2(z) = V_2|_{\mathcal{W}_1} \oplus V_1^*|_{V_1\mathcal{W}_2}z,$$

*for all $z \in \mathbb{D}$.*

*Proof.* Let $\eta$ in $\mathcal{W} = V_2\mathcal{W}_1 \oplus \mathcal{W}_2$, and let $w \in \mathbb{D}$. Then there exist $\eta_1 \in \mathcal{W}_1$ and $\eta_2 \in \mathcal{W}_2$ such that $\eta = V_2\eta_1 \oplus \eta_2$. Then $V_1\eta = V\eta_1 + V_1\eta_2$, and hence

$$\Phi_1(w)\eta = (M_{\Phi_1}\eta)(w) = (\Pi_V V_1 \Pi_V^* \eta)(w) = (\Pi_V V_1\eta)(w) = (\Pi_V V\eta_1 + \Pi_V V_1\eta_2)(w).$$

This along with the fact that $V_1\eta_2 \in \mathcal{W}$ (see Lemma 2.3.1) gives

$$\begin{aligned}\Phi_1(w)\eta &= (M_z\Pi_V\eta_1 + V_1\eta_2)(w)\\ &= (M_z\eta_1 + V_1\eta_2)(w)\\ &= w\eta_1 + V_1\eta_2\\ &= wV_2^*\eta + V_1\eta_2,\end{aligned}$$

for all $w \in \mathbb{D}$. Therefore

$$\Phi_1(z) = V_1|_{\mathcal{W}_2} \oplus V_2^*|_{V_2\mathcal{W}_1} z,$$

for all $z \in \mathbb{D}$, as $\mathcal{W}_2 = Ker(V_2^*)$. The representation of $\Phi_2$ follows similarly. □

In the following, we present Berger, Coburn and Lebow's version of representations of pure pairs of commuting isometries. This yields an explicit representations of the auxiliary operators $U$ and $P$ (see Section 1). The proof readily follows from Lemma 2.3.1 and Theorem 2.3.2.

**Theorem 2.3.3.** *Let $(V_1, V_2)$ be a pure pair of commuting isometries on $\mathcal{H}$. Then the BCL pair $(M_{\Phi_1}, M_{\Phi_2})$ corresponding to $(V_1, V_2)$ is given by*

$$\Phi_1(z) = U^*(P_{\mathcal{W}_2} + zP_{\mathcal{W}_2}^{\perp}),$$

*and*

$$\Phi_2(z) = (P_{\mathcal{W}_2}^{\perp} + zP_{\mathcal{W}_2})U,$$

*where*

$$U = \begin{bmatrix} V_2|_{\mathcal{W}_1} & 0 \\ 0 & V_1^*|_{V_1\mathcal{W}_2} \end{bmatrix} : \begin{matrix} \mathcal{W}_1 \\ \oplus \\ V_1\mathcal{W}_2 \end{matrix} \to \begin{matrix} V_2\mathcal{W}_1 \\ \oplus \\ \mathcal{W}_2 \end{matrix},$$

*is a unitary operator on $\mathcal{W}$.*

Therefore, $(V_1, V_2, V_1V_2)$ on $\mathcal{H}$ and $(M_{\Phi_1}, M_{\Phi_2}, M_z^{\mathcal{W}})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{W}$ is the wandering subspace for $V = V_1V_2$.

## 2.4 Unitary invariants

In this short section we present a complete set of joint unitary invariants for pure pairs of commuting isometries. Recall that two commuting pairs $(T_1, T_2)$ and $(\tilde{T}_1, \tilde{T}_2)$ on $\mathcal{H}$

and $\tilde{\mathcal{H}}$, respectively, are said to be (jointly) unitarily equivalent if there exists a unitary operator $U : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $UT_j = \tilde{T}_j U$ for all $j = 1, 2$.

First we note that, by virtue of Theorem 2.9 of [18], the orthogonal projection $P_{\mathcal{W}_2}$ and the unitary operator $U$ on $\mathcal{W}$, as in Theorem 2.3.3, form a complete set of (joint) unitary invariants of pure pairs of commuting isometries. More specifically: Let $(V_1, V_2)$ and $(\tilde{V}_1, \tilde{V}_2)$ be two pure pairs of commuting isometries on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively. Let $\tilde{\mathcal{W}}_j$ be the wandering subspace for $\tilde{V}_j$, $j = 1, 2$. Then $(V_1, V_2)$ and $(\tilde{V}_1, \tilde{V}_2)$ are unitarily equivalent if and only if

$$\left(\begin{bmatrix} V_2|_{\mathcal{W}_1} & 0 \\ 0 & V_1^*|_{V_1\mathcal{W}_2} \end{bmatrix}, P_{\mathcal{W}_2}\right) \quad \text{and} \quad \left(\begin{bmatrix} \tilde{V}_2|_{\tilde{\mathcal{W}}_1} & 0 \\ 0 & \tilde{V}_1^*|_{\tilde{V}_1\tilde{\mathcal{W}}_2} \end{bmatrix}, P_{\tilde{\mathcal{W}}_2}\right)$$

are unitarily equivalent.

In addition to the above, the following unitary invariants are also explicit. The proof is an easy consequence of Theorem 2.3.2. Here we will make use of the identifications of $A$ on $H^2_{\mathcal{W}}(\mathbb{D})$ and $AM_z$ on $H^2_{\mathcal{W}}(\mathbb{D})$ with $I_{H^2(\mathbb{D})} \otimes A$ on $H^2(\mathbb{D}) \otimes \mathcal{W}$ and $M_z \otimes A$ on $H^2(\mathbb{D}) \otimes \mathcal{W}$, respectively, where $A \in \mathcal{B}(\mathcal{W})$ (see Section 2).

**Theorem 2.4.1.** *Let $(V_1, V_2)$ and $(\tilde{V}_1, \tilde{V}_2)$ be two pure pairs of commuting isometries on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively. Then $(V_1, V_2)$ and $(\tilde{V}_1, \tilde{V}_2)$ are unitarily equivalent if and only if $(V_1|_{\mathcal{W}_2}, V_2^*|_{V_2\mathcal{W}_1})$ and $(\tilde{V}_1|_{\tilde{\mathcal{W}}_2}, \tilde{V}_2^*|_{\tilde{V}_2\tilde{\mathcal{W}}_1})$ are unitarily equivalent.*

*Proof.* Let $(M_{\Phi_1}, M_{\Phi_2})$ and $(M_{\tilde{\Phi}_1}, M_{\tilde{\Phi}_2})$ be the BCL pairs corresponding to $(V_1, V_2)$ and $(\tilde{V}_1, \tilde{V}_2)$, respectively, as in Theorem 2.3.2. Let $C_1 = V_1|_{\mathcal{W}_2}$ and $C_2 = V_2^*|_{V_2\mathcal{W}_1}$ be the coefficients of $\Phi_1$. Similarly, let $\tilde{C}_1$ and $\tilde{C}_2$ be the coefficients of $\tilde{\Phi}_1$.

Now let $Z : \mathcal{W} \to \tilde{\mathcal{W}}$ be a unitary such that $ZC_j = \tilde{C}_j Z$, $j = 1, 2$. Then

$$\begin{aligned} M_{\Phi_1} &= I_{H^2(\mathbb{D})} \otimes C_1 + M_z \otimes C_2 \\ &= I_{H^2(\mathbb{D})} \otimes Z^*\tilde{C}_1 Z + M_z \otimes Z^*\tilde{C}_2 Z \\ &= (I_{H^2(\mathbb{D})} \otimes Z^*)(I_{H^2(\mathbb{D})} \otimes \tilde{C}_1 + M_z \otimes \tilde{C}_2)(I_{H^2(\mathbb{D})} \otimes Z) \\ &= (I_{H^2(\mathbb{D})} \otimes Z^*) M_{\tilde{\Phi}_1} (I_{H^2(\mathbb{D})} \otimes Z). \end{aligned}$$

Because $M_{\Phi_2} = M_{\Phi_1}^* M_z$ and $M_{\tilde{\Phi}_2} = M_{\tilde{\Phi}_1}^* M_z$, it follows that $(M_{\Phi_1}, M_{\Phi_2})$ and $(M_{\tilde{\Phi}_1}, M_{\tilde{\Phi}_2})$ are unitarily equivalent, that is, $(V_1, V_2)$ and $(\tilde{V}_1, \tilde{V}_2)$ are unitarily equivalent.

To prove the necessary part, let $(M_{\Phi_1}, M_{\Phi_2})$ and $(M_{\tilde{\Phi}_1}, M_{\tilde{\Phi}_2})$ are unitarily equivalent. Then there exists a unitary operator $X : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ [92] such that

$$XM_{\Phi_j} = M_{\tilde{\Phi}_j} X \qquad (j = 1, 2).$$

Since

$$XM_z^{\mathcal{W}} = XM_{\Phi_1} M_{\Phi_2} = M_{\tilde{\Phi}_1} XX^* M_{\tilde{\Phi}_2} X = M_{\tilde{\Phi}_1} M_{\tilde{\Phi}_2} X = M_z^{\tilde{\mathcal{W}}} X,$$

there exists a unitary operator $Z : \mathcal{W} \to \tilde{\mathcal{W}}$ such that

$$X = I_{H^2(\mathbb{D})} \otimes Z.$$

This and $XM_{\Phi_1} = M_{\tilde{\Phi}_1} X$ implies that

$$(I_{H^2(\mathbb{D})} \otimes Z)(I_{H^2(\mathbb{D})} \otimes C_1 + M_z \otimes C_2) = (I_{H^2(\mathbb{D})} \otimes \tilde{C}_1 + M_z \otimes \tilde{C}_2)(I_{H^2(\mathbb{D})} \otimes Z).$$

Hence $(C_1, C_2)$ and $(\tilde{C}_1, \tilde{C}_2)$ are unitarily equivalent. This completes the proof of the theorem. □

Observe that the set of joint unitary invariants $\{V_1|_{\mathcal{W}_2}, V_2^*|_{V_2\mathcal{W}_1}\}$, as above, is associated with the coefficients of $\Phi_1$ of the BCL pair $(M_{\Phi_1}, M_{\Phi_2})$ corresponding to $(V_1, V_2)$. Clearly, by duality, a similar statement holds for the coefficients of $\Phi_2$ as well: $\{V_2|_{\mathcal{W}_1}, V_1^*|_{V_1\mathcal{W}_2}\}$ is a complete set of joint unitary invariants for pure pairs of commuting isometries.

## 2.5 Pure isometries

In this section we will analyze pairs of commuting isometries $(V_1, V_2)$ such that either $V_1$ or $V_2$ is a pure isometry, or both $V_1$ and $V_2$ are pure isometries. We begin with a concrete example which illustrates this particular class and also exhibits its complex structure.

Let $\mathcal{S}$ be a joint $(M_{z_1}, M_{z_2})$-invariant closed subspace of $H^2(\mathbb{D}^2)$, that is, $M_{z_j}\mathcal{S} \subseteq \mathcal{S}$. Set

$$V_j = M_{z_j}|_{\mathcal{S}} \qquad (j = 1, 2).$$

It follows immediately that $V_j$ is a pure isometry and $V_1V_2 = V_2V_1$, and hence $(V_1, V_2)$ is a pair of commuting pure isometries on $\mathcal{S}$.

If we assume, in addition, that $(V_1, V_2)$ is doubly commuting (that is, $V_1^*V_2 = V_2V_1^*$), then it follows that $(V_1, V_2)$ on $\mathcal{S}$ and $(M_{z_1}, M_{z_2})$ on $H^2(\mathbb{D}^2)$ are unitarily equivalent. See Slocinski [106] for more details. In general, however, the classification of pairs of commuting isometries, up to unitary equivalence, is complicated and very little seems to be known. For instance, see Rudin [93] for a list of pathological examples (also see Qin and Yang [88]).

We now turn our attention to the general problem. Let $(V_1, V_2)$ be a pair of commuting isometries on $\mathcal{H}$, and let $V_1$ be a pure isometry. Then, in particular, $V = V_1V_2$ is a pure isometry, and hence $(V_1, V_2)$ is a pure pair of commuting isometries. Since $V_1V_2 = V_2V_1$, by Theorem 2.2.1, it follows that

$$\Pi_{V_1} V_2 = M_{\Theta_{V_2}} \Pi_{V_1}, \tag{2.5.1}$$

where $\Theta_{V_2} \in H^\infty_{\mathcal{B}(\mathcal{W}_1)}(\mathbb{D})$ is an inner multiplier and

$$\Theta_{V_2}(z) = P_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}V_2|_{\mathcal{W}_1} \qquad (z \in \mathbb{D}).$$

Let $(M_{\Phi_1}, M_{\Phi_2})$ be the BCL pair (see Theorem 2.3.3) corresponding to $(V_1, V_2)$, that is, $\Pi_V V_i = M_{\Phi_i}\Pi_V$ for all $i = 1, 2$. Set

$$\tilde{\Pi}_1 = \Pi_{V_1}\Pi_V^*.$$

Then $\tilde{\Pi}_1 : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\mathcal{W}_1}(\mathbb{D})$ is a unitary operator such that $\tilde{\Pi}_1 M_{\Phi_1} = M_z^{\mathcal{W}_1}\tilde{\Pi}_1$ and $\tilde{\Pi}_1 M_{\Phi_2} = M_{\Theta_{V_2}}\tilde{\Pi}_1$. Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\Pi_V"] \arrow[dr, "\Pi_{V_1}"'] & H^2_{\mathcal{W}}(\mathbb{D}) \arrow[d, "\tilde{\Pi}_1"] \\
& H^2_{\mathcal{W}_1}(\mathbb{D})
\end{tikzcd}$$

where $(M_{\Phi_1}, M_{\Phi_2})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ and $(M_z^{\mathcal{W}_1}, M_{\Theta_{V_2}})$ on $H^2_{\mathcal{W}_1}(\mathbb{D})$ are the representations of $(V_1, V_2)$ on $\mathcal{H}$.

We now proceed to settle the non-trivial part of this consideration: An analytic description of the unitary map $\tilde{\Pi}_1$. To this end, observe first that since $\Pi_{V_1}V_1 = M_z^{\mathcal{W}_1}\Pi_{V_1}$, (2.5.1) gives

$$\Pi_{V_1}V = M_z^{\mathcal{W}_1}M_{\Theta_{V_2}}\Pi_{V_1}.$$

Then

$$\tilde{\Pi}_1 M_z^{\mathcal{W}} = \Pi_{V_1}V\Pi_V^* = M_z^{\mathcal{W}_1}M_{\Theta_{V_2}}\Pi_{V_1}\Pi_V^*,$$

that is,

$$\tilde{\Pi}_1 M_z^{\mathcal{W}} = (M_z^{\mathcal{W}_1}M_{\Theta_{V_2}})\tilde{\Pi}_1. \tag{2.5.2}$$

Let $\eta \in \mathcal{W}$. By Equation (2.2.1) we can write $\eta = \sum_{m=0}^{\infty} V_1^m P_{\mathcal{W}_1} V_1^{*m}\eta$. Therefore

$$\begin{aligned}
(\Pi_{V_1}\eta)(w) &= (\sum_{m=0}^{\infty} \Pi_{V_1}V_1^m P_{\mathcal{W}_1}V_1^{*m}\eta)(w)\\
&= (\sum_{m=0}^{\infty} M_z^m P_{\mathcal{W}_1}V_1^{*m}\eta)(w)\\
&= \sum_{m=0}^{\infty} w^m (P_{\mathcal{W}_1}V_1^{*m}\eta),
\end{aligned}$$

which yields

$$\tilde{\Pi}_1\eta = \Pi_{V_1}\Pi_V^*\eta = \Pi_{V_1}\eta = \sum_{m=0}^{\infty} z^m (P_{\mathcal{W}_1}V_1^{*m}\eta),$$

that is

$$\tilde{\Pi}_1 \eta = P_{\mathcal{W}_1}[I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]\eta,$$

for all $\eta \in \mathcal{W}$. It now follows from (2.5.2) that

$$\tilde{\Pi}_1(z^m \eta) = (z\Theta_{V_2}(z))^m P_{\mathcal{W}_1}[I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]\eta,$$

for all $m \geq 0$, and so, by $\mathbb{S}(\cdot, w)\eta = \sum\limits_{m=0}^{\infty} z^m \bar{w}^m \eta$, it follows that

$$\begin{aligned}\tilde{\Pi}_1(\mathbb{S}(\cdot, w)\eta) &= \tilde{\Pi}_1(\sum_{m=0}^{\infty} z^m \bar{w}^m \eta)\\ &= (I_{\mathcal{W}_1} - \bar{w}z\Theta_{V_2}(z))^{-1} P_{\mathcal{W}_1}[I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]\eta,\end{aligned}$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$. Finally, from $\tilde{\Pi}_1^* M_z^{\mathcal{W}_1} = M_{\Phi_1}\tilde{\Pi}_1^*$ and $\tilde{\Pi}_1^*\eta_1 = \eta_1$ for all $\eta_1 \in \mathcal{W}_1$, it follows that $\tilde{\Pi}_1^*(z^m\eta_1) = M_{\Phi_1}^m \eta_1$ for all $m \geq 0$, and hence

$$\tilde{\Pi}_1^*(\mathbb{S}(\cdot, w)\eta_1) = (I_{\mathcal{W}} - \Phi_1(z)\bar{w})^{-1}\eta_1,$$

for all $w \in \mathbb{D}$ and $\eta_1 \in \mathcal{W}_1$.

We summarize the above observations in the following theorem.

**Theorem 2.5.1.** *Let $(V_1, V_2)$ be a pair of commuting isometries on $\mathcal{H}$. Let $i, j \in \{1, 2\}$ and $i \neq j$. If $V_i$ is a pure isometry, then*

$$\tilde{\Pi}_i = \Pi_{V_i}\Pi_V^* \in \mathcal{B}(H^2_{\mathcal{W}}(\mathbb{D}), H^2_{\mathcal{W}_i}(\mathbb{D})),$$

*is a unitary operator,*

$$\tilde{\Pi}_i M_z^{\mathcal{W}} = M_{z\Theta_{V_j}}\tilde{\Pi}_i, \ \tilde{\Pi}_i^* M_z^{\mathcal{W}_i} = M_{\Phi_i}\tilde{\Pi}_i^*,$$

*and*

$$\tilde{\Pi}_i(\mathbb{S}(\cdot, w)\eta) = (I_{\mathcal{W}_i} - \bar{w}z\Theta_{V_j}(z))^{-1} P_{\mathcal{W}_i}[I_{\mathcal{H}} + z(I - zV_i^*)^{-1}V_i^*]\eta,$$

*for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$, where*

$$\Theta_{V_j}(z) = P_{\mathcal{W}_i}(I_{\mathcal{H}} - zV_i^*)^{-1}V_j|_{\mathcal{W}_i}$$

*for all $z \in \mathbb{D}$. Moreover*

$$\tilde{\Pi}_i^*(\mathbb{S}(\cdot, w)\eta_i) = (I_{\mathcal{W}} - \Phi_i(z)\bar{w})^{-1}\eta_i,$$

*for all $w \in \mathbb{D}$ and $\eta_i \in \mathcal{W}_i$.*

Note that the inner multipliers $\Theta_{V_i} \in H^\infty_{\mathcal{B}(\mathcal{W}_j)}(\mathbb{D})$ above satisfy the following equalities:

$$\Pi_{V_j} V_i = M_{\Theta_{V_i}} \Pi_{V_j}.$$

Now let $(V_1, V_2)$ be a pair of commuting isometries such that both $V_1$ and $V_2$ are pure isometries. The above result leads to an analytic representation of such pairs.

**Corollary 2.5.2.** *Let $(V_1, V_2)$ be a pair of commuting pure isometries on a Hilbert space $\mathcal{H}$. If $(M_{\Phi_1}, M_{\Phi_2})$ is the BCL representation corresponding to $(V_1, V_2)$, then $M_{\Phi_1}$ and $M_{\Phi_2}$ are pure isometries,*

$$\tilde{\Pi}_1 M_{\Phi_2} = M_{\Theta_{V_2}} \tilde{\Pi}_1, \; \tilde{\Pi}_2 M_{\Phi_1} = M_{\Theta_{V_1}} \tilde{\Pi}_2,$$

$\tilde{\Pi} = \tilde{\Pi}_2 \tilde{\Pi}_1^* : H^2_{\mathcal{W}_1}(\mathbb{D}) \to H^2_{\mathcal{W}_2}(\mathbb{D})$ *is a unitary operator, and*

$$\tilde{\Pi} M_z^{\mathcal{W}_1} = M_{\Theta_{V_1}} \tilde{\Pi} \text{ and } \tilde{\Pi} M_{\Theta_{V_2}} = M_z^{\mathcal{W}_2} \tilde{\Pi}.$$

*Moreover, for each $w \in \mathbb{D}$ and $\eta_j \in \mathcal{W}_j$, $j = 1, 2$,*

$$\tilde{\Pi}(\mathbb{S}(\cdot, w)\eta_1) = (I_{\mathcal{W}_2} - \bar{w}\Theta_{V_1}(z))^{-1} P_{\mathcal{W}_2} (I_{\mathcal{H}} - z V_2^*)^{-1} \eta_1,$$

*and*

$$\tilde{\Pi}^*(\mathbb{S}(\cdot, w)\eta_2) = (I_{\mathcal{W}_1} - \bar{w}\Theta_{V_2}(z))^{-1} P_{\mathcal{W}_1} (I_{\mathcal{H}} - z V_1^*)^{-1} \eta_2.$$

*Proof.* A repeated application of Theorem 2.5.1 yields

$$\begin{aligned}
\tilde{\Pi}_1 M_{\Phi_2} &= \tilde{\Pi}_1 M^*_{\Phi_1} (M_{\Phi_1} M_{\Phi_2}) \\
&= \tilde{\Pi}_1 M^*_{\Phi_1} M_z^{\mathcal{W}} \\
&= (M_z^{\mathcal{W}_1})^* \tilde{\Pi}_1 M_z^{\mathcal{W}} \\
&= (M_z^{\mathcal{W}_1})^* M_{z\Theta_{V_2}} \tilde{\Pi}_1,
\end{aligned}$$

that is, $\tilde{\Pi}_1 M_{\Phi_2} = M_{\Theta_{V_2}} \tilde{\Pi}_1$ and similarly $\tilde{\Pi}_2 M_{\Phi_1} = M_{\Theta_{V_1}} \tilde{\Pi}_2$. For $\eta_1 \in \mathcal{W}_1$, we have $\Pi_{V_2} \eta_1 = P_{\mathcal{W}_2} (I_{\mathcal{H}} - z V_2^*)^{-1} \eta_1$. Since $\tilde{\Pi}_1^* \eta_1 = \eta_1$ and $\Pi_V^* \eta_1 = \eta_1$, it follows that

$$\tilde{\Pi} \eta_1 = \tilde{\Pi}_2 \eta_1 = \Pi_{V_2} \Pi_V^* \eta_1 = \Pi_{V_2} \eta_1,$$

that is $\tilde{\Pi} \eta_1 = P_{\mathcal{W}_2} (I_{\mathcal{H}} - z V_2^*)^{-1} \eta_1$. Now using the identity $\tilde{\Pi}(z \eta_1) = M_{\Theta_{V_1}} \tilde{\Pi} \eta_1$, we have

$$\tilde{\Pi}(z^m \eta_1) = \Theta_{V_1}(z)^m P_{\mathcal{W}_2} (I_{\mathcal{H}} - z V_2^*)^{-1} \eta_1,$$

for all $m \geq 0$ and $\eta_1 \in \mathcal{W}_1$. Finally $\mathbb{S}(\cdot, w)\eta_1 = \sum_{m=0}^{\infty} \bar{w}^m z^m \eta_1$ gives

$$\tilde{\Pi}(\mathbb{S}(\cdot, w)\eta_1) = (I_{\mathcal{W}_2} - \bar{w}\Theta_{V_1}(z))^{-1} P_{\mathcal{W}_2} (I_{\mathcal{H}} - z V_2^*)^{-1} \eta_1.$$

The final equality of the corollary follows from the equality

$$\tilde{\Pi}^*(z^m\eta_2) = \Theta_{V_2}(z)^m(\tilde{\Pi}^*\eta_2) = \Theta_{V_2}(z)^m P_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}\eta_2,$$

for all $m \geq 0$ and $\eta_2 \in \mathcal{W}_2$. This concludes the proof. □

In the final section, we will connect the analytic descriptions of $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ as in Theorem 2.5.1 with the classical notion of the Sz.-Nagy and Foias characteristic functions of contractions on Hilbert spaces [79].

## 2.6 Defect Operators

Throughout this section, we will mostly work on general (not necessarily pure) pairs of commuting isometries. Let $(V_1, V_2)$ be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. The *defect operator* $C(V_1, V_2)$ of $(V_1, V_2)$ (cf. [63, 69]) is defined as the self-adjoint operator

$$C(V_1, V_2) = I - V_1{V_1}^* - V_2{V_2}^* + V_1V_2{V_1}^*{V_2}^*.$$

Recall from Section 3 that given a pair of commuting isometries $(V_1, V_2)$, we write $V = V_1V_2$, and denote by

$$\mathcal{W}_j = \mathcal{W}(V_j) = \ker V_j^* = \mathcal{H} \ominus V_j\mathcal{H},$$

the wandering subspace for $V_j$, $j = 1, 2$. The wandering subspace for $V$ is denoted by $\mathcal{W}$. Finally, we recall that (see Lemma 2.3.1) $\mathcal{W} = \mathcal{W}_1 \oplus V_1\mathcal{W}_2 = V_2\mathcal{W}_1 \oplus \mathcal{W}_2$. This readily implies

$$P_{\mathcal{W}} = P_{\mathcal{W}_1} \oplus P_{V_1\mathcal{W}_2} = P_{V_2\mathcal{W}_1} \oplus P_{\mathcal{W}_2}. \tag{2.6.1}$$

The following lemma is well known to the experts, but for the sake of completeness we provide a proof of the statement.

**Lemma 2.6.1.** *Let $(V_1, V_2)$ be a commuting pair of isometries on $\mathcal{H}$. Then $\mathcal{H}_s(V)$ and $\mathcal{H}_u(V)$ are $V_j$-reducing subspaces,*

$$\mathcal{H}_s(V_j) \subseteq \mathcal{H}_s(V), \text{ and } \mathcal{H}_u(V_j) \supseteq \mathcal{H}_u(V),$$

*for all $j = 1, 2$.*

*Proof.* For the first part we only need to prove that $\mathcal{H}_s(V)$ is a $V_1$-reducing subspace. Note that since (see Lemma 2.3.1) $V_1\mathcal{W} \subseteq \mathcal{W} \oplus V\mathcal{W}$, it follows that

$$V_1V^m\mathcal{W} \subseteq V^m(\mathcal{W} \oplus V\mathcal{W}) \subseteq \mathcal{H}_s(V),$$

for all $m \geq 0$. This clearly implies that $V_1\mathcal{H}_s(V) \subseteq \mathcal{H}_s(V)$. On the other hand, since $V_1^*\mathcal{W} = \mathcal{W}_2 \subseteq \mathcal{W}$ and

$$V_1^*V^m\mathcal{W} = V^{m-1}(V_2\mathcal{W}) \subseteq V^{m-1}(\mathcal{W} \oplus V\mathcal{W}),$$

it follows that $V_1^*\mathcal{H}_s(V) \subseteq \mathcal{H}_s(V)$. To prove the second part of the statement, it is enough to observe that

$$V^m\mathcal{H} = V_1^m(V_2^m\mathcal{H}) = V_2^m(V_1^m\mathcal{H}) \subseteq V_1^m\mathcal{H}, V_2^m\mathcal{H},$$

for all $m \geq 0$, and as $n \to \infty$

$$V_1^{*n}h \to 0, \text{ or } V_2^{*n}h \to 0 \Rightarrow V^{*n}h \to 0,$$

for any $h \in \mathcal{H}$. This concludes the proof of the lemma. $\square$

The following characterizations of doubly commuting isometries will prove important in the sequel.

**Lemma 2.6.2.** *Let $(V_1, V_2)$ be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

*(i) $(V_1, V_2)$ is doubly commuting.*

*(ii) $V_2\mathcal{W}_1 \subseteq \mathcal{W}_1$.*

*(iii) $V_1\mathcal{W}_2 \subseteq \mathcal{W}_2$.*

*Proof.* Since (i) implies (ii) and (iii), by symmetry we only need to show that (ii) implies (i). Let $V_2\mathcal{W}_1 \subseteq \mathcal{W}_1$. Let $\mathcal{H} = \mathcal{H}_s(V) \oplus \mathcal{H}_u(V)$ be the Wold-von Neumann orthogonal decomposition of $V$ (see Theorem 1.2.1). Then $\mathcal{H}_s(V)$ and $\mathcal{H}_u(V)$ are joint $(V_1, V_2)$-reducing subspaces, and the pair $(V_1|_{\mathcal{H}_u(V)}, V_2|_{\mathcal{H}_u(V)})$ on $\mathcal{H}_u$ is doubly commuting, because $V_j|_{\mathcal{H}_u(V)}$, $j = 1, 2$, are unitary operators, by Lemma 2.6.1. Now it only remains to prove that $V_1^*V_2 = V_2V_1^*$ on $\mathcal{H}_s(V)$. Since

$$(V_1^*V_2 - V_2V_1^*)V^m = V_1^*V^mV_2 - V_2V_1^*V^m = V^{m-1}V_2^2 - V_2^2V^{m-1} = 0,$$

it follows that $V_1^*V_2 - V_2V_1^* = 0$ on $V^m\mathcal{W}$ for all $m \geq 1$. In order to complete the proof we must show that $V_1^*V_2 = V_2V_1^*$ on $\mathcal{W}$. To this end, let $\eta = \eta_1 \oplus V_1\eta_2 \in \mathcal{W}$ for some $\eta_1 \in \mathcal{W}_1$ and $\eta_2 \in \mathcal{W}_2$. Then

$$V_1^*V_2(\eta_1 \oplus V_1\eta_2) = V_1^*V_2\eta_1 + V_1^*V_2V_1\eta_2 = V_2\eta_2,$$

as $V_2\mathcal{W}_1 \subseteq \mathcal{W}_1$, and on the other hand

$$V_2V_1^*(\eta_1 \oplus V_1\eta_2) = V_2V_1^*\eta_1 + V_2V_1^*V_1\eta_2 = V_2\eta_2.$$

This completes the proof. □

The key of our geometric approach is the following simple representation of defect operators.

**Lemma 2.6.3.** $C(V_1, V_2) = P_{\mathcal{W}_1} - P_{V_2\mathcal{W}_1} = P_{\mathcal{W}_2} - P_{V_1\mathcal{W}_2}$.

*Proof.* The result readily follows from (2.6.1) and

$$\begin{aligned} C(V_1, V_2) &= (I - V_1 V_1{}^*) + (I - V_2 V_2{}^*) - (I - VV^*) \\ &= P_{\mathcal{W}_1} + P_{\mathcal{W}_2} - P_{\mathcal{W}}. \end{aligned}$$

□

The final ingredient to our analysis is the fringe operator $F_2$. The notion of fringe operators plays a significant role in the study of joint shift-invariant closed subspaces of the Hardy space over $\mathbb{D}^2$ (see the discussion at the beginning of Section 5). Given a pair of commuting isometries $(V_1, V_2)$ on $\mathcal{H}$, the *fringe operators* $F_1 \in \mathcal{B}(\mathcal{W}_2)$ and $F_2 \in \mathcal{B}(\mathcal{W}_1)$ are defined by

$$F_j = P_{\mathcal{W}_i} V_j|_{\mathcal{W}_i} \qquad (i \neq j).$$

Of particular interest to us are the isometric fringe operators. Note that

$$F_2^* F_2 = P_{\mathcal{W}_1} V_2^* P_{\mathcal{W}_1} V_2|_{\mathcal{W}_1}.$$

**Lemma 2.6.4.** *The fringe operator $F_2$ on $\mathcal{W}_1$ is an isometry if and only if $V_2\mathcal{W}_1 \subseteq \mathcal{W}_1$.*

*Proof.* As $I_{\mathcal{W}_1} - F_2^* F_2 = I_{\mathcal{W}_1} - P_{\mathcal{W}_1} V_2^* P_{\mathcal{W}_1} V_2|_{\mathcal{W}_1}$, (2.6.1) implies that

$$I_{\mathcal{W}_1} - F_2^* F_2 = P_{\mathcal{W}_1} V_2^* P_{V_1\mathcal{W}_2} V_2|_{\mathcal{W}_1}.$$

Then $F_2^* F_2 = I_{\mathcal{W}_1}$ if and only if $P_{V_1\mathcal{W}_2} V_2|_{\mathcal{W}_1} = 0$, or, equivalently, if and only if $V_2\mathcal{W}_1 \perp V_1\mathcal{W}_2 = \mathcal{W}_1^{\perp}$, by Lemma 2.3.1. This completes the proof. □

Therefore, the fringe operator $F_2$ is an isometry if and only if the pair $(V_1, V_2)$ is doubly commuting.

We are now ready to formulate a generalization of Theorem 3.4 in [69] by He, Qin and Yang. Here we do not assume that $(V_1, V_2)$ is pure.

**Theorem 2.6.5.** *Let $(V_1, V_2)$ be a pair of commuting isometries on $\mathcal{H}$. Then the following are equivalent:*

*(a)* $C(V_1, V_2) \geq 0$.

*(b)* $V_2\mathcal{W}_1 \subseteq \mathcal{W}_1$.

*(c)* $(V_1, V_2)$ *is doubly commuting.*

*(d)* $C(V_1, V_2)$ *is a projection.*

*(e) The fringe operator* $F_2$ *is an isometry.*

*Proof.* The equivalences of (a) and (b), (b) and (c), and (b) and (e) are given in Lemma 2.6.3, Lemma 2.6.2 and Lemma 2.6.4, respectively. The implication (c) implies (d) follows from

$$C(V_1, V_2) = P_{\mathcal{W}_1} P_{\mathcal{W}_2} = P_{\mathcal{W}_2} P_{\mathcal{W}_1}.$$

Clearly (d) implies (a). This completes the proof. □

We now prove that for a large class of pairs of commuting isometries negative defect operator always implies the zero defect operator.

**Theorem 2.6.6.** *Let* $(V_1, V_2)$ *be a pair of commuting isometries on* $\mathcal{H}$*. Suppose that* $V_1$ *or* $V_2$ *is pure. Then* $C(V_1, V_2) \leq 0$ *if and only if* $C(V_1, V_2) = 0$*.*

*Proof.* With out loss of generality assume that $V_2$ is pure. If $C(V_1, V_2) \leq 0$, then by Lemma 2.6.3, we have $P_{\mathcal{W}_1} \leq P_{V_2 \mathcal{W}_1}$, or, equivalently

$$\mathcal{W}_1 \subseteq V_2 \mathcal{W}_1,$$

and hence

$$\mathcal{W}_1 \subseteq {V_2}^m \mathcal{W}_1 \subseteq {V_2}^m \mathcal{H},$$

for all $m \geq 0$. Therefore

$$\mathcal{W}_1 = \bigcap_{m=0}^{\infty} {V_2}^m \mathcal{W}_1 \subseteq \bigcap_{m=0}^{\infty} {V_2}^m \mathcal{H} = \{0\},$$

as $V_2$ is pure. Hence $\mathcal{W}_1 = \{0\}$ and $V_2 \mathcal{W}_1 = \{0\}$. This gives $C(V_1, V_2) = P_{\mathcal{W}_1} - P_{V_2 \mathcal{W}_1} = 0$. □

The same conclusion holds if we allow $\dim \mathcal{W}_j < \infty$ for some $j \in \{1, 2\}$.

**Theorem 2.6.7.** *Let* $(V_1, V_2)$ *be a pair of commuting isometries on* $\mathcal{H}$*. Suppose that* $\dim \mathcal{W}_j < \infty$ *for some* $j \in \{1, 2\}$*. Then* $C(V_1, V_2) \leq 0$ *if and only if* $C(V_1, V_2) = 0$*.*

*Proof.* We may suppose that $\dim \mathcal{W}_1 < \infty$. Let $C(V_1, V_2) \leq 0$. Since $\mathcal{W}_1 \subseteq V_2 \mathcal{W}_1$ and $V_2$ is an isometry, it follows that

$$\mathcal{W}_1 = V_2 \mathcal{W}_1.$$

Hence $C(V_1, V_2) = P_{\mathcal{W}_1} - P_{V_2 \mathcal{W}_1} = 0$. This completes the prove. □

The same conclusion also holds for positive defect operators.

## 2.7 Concluding Remarks

As pointed out in the introduction, a general theory for pairs of commuting isometries is mostly unknown and unexplored (however, see Popovici [86]). In comparison, we would like to add that a great deal is known about the structure of pairs (and even of $n$-tuples) of commuting isometries with finite rank defect operators (see [29], [27], [28]). A complete classification result is also known for $n$-tuples of doubly commuting isometries (cf. [53], [106], [96]). It is now natural to ask whether the present results for pure pairs of commuting isometries can be extended to arbitrary pairs of commuting isometries (see [43] and [52] for closely related results). Another relevant question is to analyze the joint shift invariant subspaces of the Hardy space over the unit bidisc [3] from our analytic and geometric point of views.

We conclude this chapter by inspecting a connection between the Sz.-Nagy and Foias characteristic functions of contractions on Hilbert spaces [79] and the analytic representations of $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ as described in Theorem 2.5.1.

Let $T$ be a contraction on a Hilbert space $\mathcal{H}$. The *defect operators* of $T$, denoted by $D_{T^*}$ and $D_T$, are defined by

$$D_{T^*} = (I - TT^*)^{1/2}, \;\; D_T = (I - T^*T)^{1/2}.$$

The defect spaces, denoted by $\mathcal{D}_{T^*}$ and $\mathcal{D}_T$, are the closure of the ranges of $D_{T^*}$ and $D_T$, respectively. The *characteristic function* [79] of the contraction $T$ is defined by

$$\theta_T(z) = [-T + zD_{T^*}(I - zT^*)^{-1}D_T]|_{\mathcal{D}_T} \qquad (z \in \mathbb{D}).$$

It follows that $\theta_T \in H^\infty_{\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})}(\mathbb{D})$ [79]. The characteristic function is a complete unitary invariant for the class of completely non-unitary contractions. This function is also closely related to the Beurling-Lax-Halmos inner functions for shift invariant subspaces of vector-valued Hardy spaces. For a more detailed discussion of the theory and applications of characteristic functions we refer to the monograph by Sz.-Nagy and Foias [79].

Now let us return to the study of pairs of commuting isometries. Let $(V_1, V_2)$ be a pair of commuting isometries on $\mathcal{H}$. We compute

$$\begin{aligned} P_{\mathcal{W}_1}[I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_{\mathcal{W}} &= [P_{\mathcal{W}_1} + zP_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_{\mathcal{W}} \\ &= [I_{\mathcal{H}} - V_1V_1^* + zP_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_{\mathcal{W}} \\ &= I_{\mathcal{W}} + [-V_1 + zP_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}]V_1^*|_{\mathcal{W}}. \end{aligned}$$

Since $V_1^*\mathcal{W} = \mathcal{W}_2$, it follows that

$$[-V_1 + zP_{\mathcal{W}_1}(I_{\mathcal{H}} - zV_1^*)^{-1}]V_1^*|_{\mathcal{W}} = [-V_1 + zD_{V_1^*}(I_{\mathcal{H}} - zV_1^*)^{-1}D_{V_2^*}]|_{\mathcal{D}_{V_2^*}}(V_1^*|_{\mathcal{W}}).$$

Therefore, setting

$$\theta_{V_1,V_2}(z) = [-V_1 + zD_{V_1^*}(I_{\mathcal{H}} - zV_1^*)^{-1}D_{V_2^*}]|_{\mathcal{D}_{V_2^*}}, \tag{2.7.1}$$

for $z \in \mathbb{D}$, we have

$$P_{\mathcal{W}_1}[I_{\mathcal{H}} + z(I_{\mathcal{H}} - zV_1^*)^{-1}V_1^*]|_{\mathcal{W}} = I_{\mathcal{W}} + \theta_{V_1,V_2}(z)V_1^*|_{\mathcal{W}},$$

for all $z \in \mathbb{D}$. Therefore, if $V_1$ is a pure isometry, then the formula for $\tilde{\Pi}_1$ in Theorem 2.5.1(i) can be expressed as

$$\tilde{\Pi}_1(\mathbb{S}(\cdot, w)\eta) = (I_{\mathcal{W}_1} - \bar{w}\Theta_{V_2}(z))^{-1}P_{\mathcal{W}_1}[I_{\mathcal{W}} + \theta_{V_1,V_2}(z)V_1^*|_{\mathcal{W}}]\eta.$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$. Similarly, if $V_2$ is a pure isometry, then the formula for $\tilde{\Pi}_2$ in Theorem 2.5.1 (ii) can be expressed as

$$\tilde{\Pi}_2(\mathbb{S}(\cdot, w)\eta) = (I_{\mathcal{W}_2} - \bar{w}\Theta_{V_1}(z))^{-1}P_{\mathcal{W}_2}[I_{\mathcal{W}} + \theta_{V_2,V_1}(z)V_2^*|_{\mathcal{W}}]\eta,$$

for all $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$, where

$$\theta_{V_2,V_1}(z) = [-V_2 + zD_{V_2^*}(I_{\mathcal{H}} - zV_2^*)^{-1}D_{V_1^*}]|_{\mathcal{D}_{V_1^*}}, \tag{2.7.2}$$

for all $z \in \mathbb{D}$.

It is easy to see that $\theta_{V_i,V_j}(z) \in \mathcal{B}(\mathcal{W}_j, \mathcal{W})$ for all $z \in \mathbb{D}$ and $i \neq j$.

Note that since the defect operator $D_{V_j} = 0$, the characteristic function $\theta_{V_j}$ of $V_j$, $j = 1, 2$, is the zero function. From this point of view, it is expected that the pair of analytic invariants $\{\theta_{V_i,V_j} : i \neq j\}$ will provide more information about pairs of commuting isometries.

Subsequent theory for pairs of commuting contractions and a more detailed connection between pairs of commuting pure isometries $(V_1, V_2)$ and the analytic invariants $\{\theta_{V_i,V_j} : i \neq j\}$ as defined in (2.7.1) and (2.7.2) will be exhibited in more details in future occasion.

# Chapter 3

# Characterization of Invariant subspaces in the polydisc

## 3.1 Introduction

An important problem in multivariable operator theory and function theory of several complex variables is the question of a Beurling type representations of joint invariant subspaces for $(M_{z_1}, \dots, M_{z_n})$ on the Hardy space $H^2(\mathbb{D}^n)$, $n > 1$. The main obstacle here seems to be the subtleties of the theory of holomorphic functions in several complex variables. This problem is compounded by another difficulty associated with the complex (and mostly unknown) structure of $n$-tuples, $n > 1$, of commuting isometries on Hilbert spaces.

In this chapter, we answer the above question by providing a complete list of natural conditions on closed subspaces of $H^2(\mathbb{D}^n)$. Here we use the analytic representations of shift invariant subspaces, representations of Toeplitz operators on the unit disc, geometry of tensor product of Hilbert spaces and identification of bounded linear operators under unitary equivalence to overcome such difficulties.

As motivation, recall that if $n = 1$, then the celebrated Beurling theorem [21] (also see Theorem 1.3.2) says that a non-zero closed subspace $\mathcal{S}$ of $H^2(\mathbb{D})$ is invariant for $M_z$ if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that

$$\mathcal{S} = \theta H^2(\mathbb{D}).$$

Note also that it follows (or the other way around) in particular from the above representation of $\mathcal{S}$ that

$$\mathcal{S} \ominus z\mathcal{S} = \theta\mathbb{C},$$

and so

$$\mathcal{S} = \bigoplus_{m=0}^{\infty} z^m (\mathcal{S} \ominus z\mathcal{S}).$$

One may now ask whether an analogous characterization holds for invariant subspaces for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$, $n > 1$. However, Rudin's pathological examples (see Rudin [93], page 70) indicates that the above Beurling type properties does not hold in general for invariant subspaces for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$, $n > 1$: There exist invariant subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ for $(M_{z_1}, M_{z_2})$ on $H^2(\mathbb{D}^2)$ such that

(1) $\mathcal{S}_1$ is not finitely generated, and

(2) $\mathcal{S}_2 \cap H^\infty(\mathbb{D}^2) = \{0\}$.

In fact, Beurling type invariant subspaces for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$, $n > 1$, are rare. They are closely connected with the tensor product structure of the Hardy space (or the product domain $\mathbb{D}^n$).

Therefore, the structure of invariant subspaces for

$$(M_{z_1}, \ldots, M_{z_n}) \text{ on } H^2(\mathbb{D}^n),\ n > 1,$$

is quite complicated. The list of important works in this area include the papers by Agrawal, Clark, and Douglas [3], Ahern and Clark [6], Douglas and Yan [47], Douglas, Paulsen, Sah and Yan [45], Guo [59, 58], Fang [49], Guo, Sun, Zheng and Zhong [61], Rudin [94], Guo and Yang [63], Izuchi [71], Mandrekar [77] etc. (also see the references therein).

In this paper, first, we represent $H^2(\mathbb{D}^{n+1})$, $n \geq 1$, by the $H^2(\mathbb{D}^n)$-valued Hardy space $H^2_{H^2(\mathbb{D}^n)}(\mathbb{D})$. Under this identification, we prove that

$$(M_{z_1}, M_{z_2}, \ldots, M_{z_{n+1}}) \text{ on } H^2(\mathbb{D}^{n+1}),$$

corresponds to

$$(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n}) \text{ on } H^2_{H^2(\mathbb{D}^n)}(\mathbb{D}),$$

where $\kappa_i \in H^\infty_{\mathcal{B}(H^2(\mathbb{D}^n))}(\mathbb{D})$, $i = 1, \ldots, n$, is a constant as well as simple and explicit $\mathcal{B}(H^2(\mathbb{D}^n))$-valued analytic function (see Theorem 3.2.1, or part (i) of Theorem 3.1.1 below). Then we prove that a closed subspace $\mathcal{S} \subseteq H^2_{H^2(\mathbb{D}^n)}(\mathbb{D})$ is invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ if and only if $\mathcal{S}$ is of Beurling [21], Lax [73] and Halmos [66] type and the corresponding Beurling, Lax and Halmos inner function solves, in an appropriate sense, $n$ operator equations explicitly and uniquely.

Recall that two $m$-tuples, $m \geq 1$, of commuting operators $(A_1, \ldots, A_m)$ on $\mathcal{H}$ and $(B_1, \ldots, B_m)$ on $\mathcal{K}$ are said to be *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $UA_i = B_iU$ for all $i = 1, \ldots, m$.

We now summarize the main contents, namely, Theorems 3.2.1 and 3.2.2 restricted to the scalar-valued Hardy space case, of this paper in the following statement.

**Theorem 3.1.1.** *Let $n$ be a natural number, and let $H_n = H^2(\mathbb{D}^n)$. Let $\kappa_i \in H^\infty_{\mathcal{B}(H_n)}(\mathbb{D})$ denote the $\mathcal{B}(H_n)$-valued constant function on $\mathbb{D}$ defined by*

$$\kappa_i(w) = M_{z_i} \in \mathcal{B}(H_n),$$

*for all $w \in \mathbb{D}$, and let $M_{\kappa_i}$ denote the multiplication operator on $H^2_{H_n}(\mathbb{D})$ defined by*

$$M_{\kappa_i} f = \kappa_i f,$$

*for all $f \in H^2_{H_n}(\mathbb{D})$ and $i = 1, \ldots, n$. Then the following statements hold true:*

*(i) $(M_{z_1}, M_{z_2} \ldots, M_{z_{n+1}})$ on $H^2(\mathbb{D}^{n+1})$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{H_n}(\mathbb{D})$ are unitarily equivalent.*

*(ii) Let $\mathcal{S}$ be a closed subspace of $H^2_{H_n}(\mathbb{D})$, and let $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$. Then $\mathcal{S}$ is invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ if and only if $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is an $n$-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$ and there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, H_n)}(\mathbb{D})$ such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

*and*

$$\kappa_i \Theta = \Theta \Phi_i,$$

*where*

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - w P_{\mathcal{S}} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}},$$

*for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$*

The representation of $\mathcal{S}$, in terms of $\mathcal{W}$, $\Theta$ and $\{M_{\Phi_i}\}_{i=1}^n$, in part (ii) above is unique in an appropriate sense (see Theorem 3.4.2). Furthermore, the multiplier $\Phi_i$ can be represented as

$$\Phi_i(w) = P_{\mathcal{W}} M_\Theta (I_{H^2_{\mathcal{W}}(\mathbb{D})} - w M_z^*)^{-1} M_\Theta^* M_{\kappa_i}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$. For a more detailed discussion on the analytic functions $\{\Phi_i\}_{i=1}^n$ on $\mathbb{D}$ we refer to Remarks 3.2.1 and 3.2.3.

As an immediate application of Theorem 3.1.1 we have (see Corollary 3.2.3): If $\mathcal{S} \subseteq H^2_{H_n}(\mathbb{D})$ is a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$, then the tuples $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$ and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ and

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - w P_{\mathcal{S}} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$. Our approach also yields a complete set of unitary invariants for invariant subspaces: The $n$-tuples of commuting shifts $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is a complete set of unitary invariants for invariant subspaces for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{H_n}(\mathbb{D})$ (see Theorem 3.5.1 for more details).

We also contribute to the classification problem of commuting tuples of isometries on Hilbert spaces. On the one hand, $n$-tuples of commuting isometries play a central role in multivariable operator theory and function theory, whereas, on the other hand, the structure of $n$-tuples, $n > 1$, of commuting isometries on Hilbert spaces is complicated. In Corollary 3.2.3, as a byproduct of our analysis, we completely classify $n$-tuples of commuting isometries of the form $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \dots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$, where $\mathcal{S}$ is a closed invariant subspace for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$.

This chapter is organized as follows. In Section 2, we prove the central result of this chapter - representations of invariant subspaces of vector-valued Hardy spaces over polydisc. In Section 3 we study and analyze the model tuples of commuting isometries. Section 4 complements the main results on representations of invariant subspaces and deals with the uniqueness part. In Section 5 we give some applications related to the main theorems. The final section of this chapter is devoted to a dimension inequality which is relevant to the present context and of independent interest.

This chapter is based on the published paper [74].

## 3.2 Main results

Let $\mathcal{E}$ be a Hilbert space, and consider the vector-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$. Our strategy here is to identify $M_{z_1}$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ with the multiplication operator $M_z$ on the $H^2_{\mathcal{E}}(\mathbb{D}^n)$-valued Hardy space on the disc $\mathbb{D}$. Then we show that under this identification, the remaining operators $\{M_{z_2}, \dots, M_{z_{n+1}}\}$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ can be represented as the multiplication operators by $n$ simple and constant $\mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}^n))$-valued functions on $\mathbb{D}$. For this we need a few more notations.

For each Hilbert space $\mathcal{L}$, for the sake of notational ease, define

$$\mathcal{L}_n = H^2(\mathbb{D}^n) \otimes \mathcal{L}.$$

When $\mathcal{L} = \mathbb{C}$, we simply write $\mathcal{L}_n = H_n$, that is,

$$H_n = H^2(\mathbb{D}^n).$$

Also, for each $i = 1, \dots, n$, we define

$$\kappa_{\mathcal{L},i}(w) = M_{z_i} \otimes I_{\mathcal{L}},$$

for all $w \in \mathbb{D}$, and write

$$\kappa_{\mathcal{L},i} = \kappa_i,$$

when $\mathcal{L}$ is clear from the context. It is evident that $\kappa_i \in H^\infty_{\mathcal{B}(\mathcal{L}_n)}(\mathbb{D})$ is a constant function and $M_{\kappa_i}$ on $H^2_{\mathcal{L}_n}(\mathbb{D})$, defined by

$$M_{\kappa_i} f = \kappa_i f \qquad (f \in H^2_{\mathcal{L}_n}(\mathbb{D})),$$

is a shift on $H^2_{\mathcal{L}_n}(\mathbb{D})$ for all $i = 1, \ldots, n$.

Now we return to the invariant subspaces of $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$. First we identify $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ with $H^2(\mathbb{D}) \otimes \mathcal{E}_n$ by the natural unitary map $\hat{U} : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2(\mathbb{D}) \otimes \mathcal{E}_n$ defined by

$$\hat{U}(z_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}} \eta) = z^{k_1} \otimes (z_1^{k_2} \cdots z_n^{k_{n+1}} \eta),$$

for all $k_1, \ldots, k_{n+1} \geq 0$ and $\eta \in \mathcal{E}$. Then it is clear that

$$\hat{U} M_{z_1} = (M_z \otimes I_{\mathcal{E}_n}) \hat{U}.$$

Moreover, a simple computation shows that

$$\hat{U} M_{z_{1+i}} = (I_{H^2(\mathbb{D})} \otimes K_i) \hat{U},$$

where $K_i$ is the multiplicational operator $M_{z_i}$ on $\mathcal{E}_n$, that is

$$K_i = M_{z_i},$$

for all $i = 1, \ldots, n$. Therefore, the tuples $(M_{z_1}, M_{z_2}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ and $(M_z \otimes I_{\mathcal{E}_n}, I_{H^2(\mathbb{D})} \otimes K_1, \ldots, I_{H^2(\mathbb{D})} \otimes K_n)$ on $H^2(\mathbb{D}) \otimes \mathcal{E}_n$ are unitarily equivalent. We further identify $H^2(\mathbb{D}) \otimes \mathcal{E}_n$ with the $\mathcal{E}_n$-valued Hardy space $H^2_{\mathcal{E}_n}(\mathbb{D})$ by the canonical unitary map $\tilde{U} : H^2(\mathbb{D}) \otimes \mathcal{E}_n \to H^2_{\mathcal{E}_n}(\mathbb{D})$ defined by

$$\tilde{U}(z^k \otimes \eta) = z^k \eta,$$

for all $k \geq 0$ and $\eta \in \mathcal{E}_n$. Clearly

$$\tilde{U}(M_z \otimes I_{\mathcal{E}_n}) = M_z \tilde{U}.$$

Now for each $i = 1, \ldots, n$, define the constant $\mathcal{B}(\mathcal{E}_n)$-valued (analytic) function on $\mathbb{D}$ by

$$\kappa_i(z) = K_i,$$

for all $z \in \mathbb{D}$. Then $\kappa_i \in H^\infty_{\mathcal{B}(\mathcal{E}_n)}(\mathbb{D})$, and the multiplication operator $M_{\kappa_i}$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$, defined by

$$(M_{\kappa_i}(z^m \eta))(w) = w^m (K_i \eta),$$

for all $m \geq 0$, $\eta \in \mathcal{E}_n$ and $w \in \mathbb{D}$, is a shift on $H^2_{\mathcal{E}_n}(\mathbb{D})$. It is now easy to see that

$$\tilde{U}(I_{H^2(\mathbb{D})} \otimes K_i) = M_{\kappa_i} \tilde{U}.$$

for all $i = 1, \ldots, n$. Finally, by setting

$$U = \tilde{U}\hat{U},$$

it follows that $U : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{E}_n}(\mathbb{D})$ is a unitary operator and

$$UM_{z_1} = M_z U,$$

and

$$UM_{z_{1+i}} = M_{\kappa_i} U,$$

for all $i = 1, \ldots, n$. This proves the vector-valued version of the first half of the statement of Theorem 3.1.1:

**Theorem 3.2.1.** *Let $\mathcal{E}$ be a Hilbert space. Then $(M_{z_1}, M_{z_2} \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent, where $\kappa_i \in H^\infty_{\mathcal{B}(\mathcal{E}_n)}(\mathbb{D})$ is the constant function*

$$\kappa_i(w) = M_{z_i} \in \mathcal{B}(\mathcal{E}_n),$$

*for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$.*

Now we proceed to prove the remaining half of Theorem 3.1.1 in the vector-valued Hardy space setting. Let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Set

$$V = M_z|_{\mathcal{S}},$$

and

$$V_i = M_{\kappa_i}|_{\mathcal{S}},$$

for all $i = 1, \ldots, n$. Clearly, $(V, V_1, \ldots, V_n)$ is a commuting tuple of isometries on $\mathcal{S}$. Note that if $f \in \mathcal{S}$, then

$$\begin{aligned}\|V_i^{*m} f\|_{\mathcal{S}} &= \|P_{\mathcal{S}} M_{\kappa_i}^{*m} f\|\|_{\mathcal{S}} \\ &\leq \|M_{\kappa_i}^{*m} f\|\|_{H^2_{\mathcal{E}_n}(\mathbb{D})},\end{aligned}$$

that is, $V_i$, $i = 1, \ldots, n$, is a shift on $\mathcal{S}$, and similarly $V$ is also a shift on $\mathcal{S}$. Let $\mathcal{W} = \mathcal{S} \ominus V\mathcal{S}$ denote the wandering subspace for $V$, that is

$$\begin{aligned}\mathcal{W} &= \ker V^* \\ &= \ker P_{\mathcal{S}} M_z^*,\end{aligned}$$

and let $\Pi_V : \mathcal{S} \to H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of $V$ on $\mathcal{S}$ (see Section 2). Then $\Pi_V$ is a unitary operator and

$$\Pi_V V = M_z \Pi_V.$$

Since

$$VV_i = V_iV,$$

applying Theorem 2.2.1 in Chapter 2 to $V_i$, we obtain

$$\Pi_V V_i = M_{\Phi_i}\Pi_V,$$

where

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wV^*)^{-1}V_i|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$, $\Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$, $M_{\Phi_i}$ is a shift on $H^2_{\mathcal{W}}(\mathbb{D})$ since $V_i$ is a shift on $\mathcal{S}$ and $i = 1, \ldots, n$. Now since $\Pi_V$ is unitary, we obtain that

$$\Pi_V^* M_z = V\Pi_V^*,$$

and

$$\Pi_V^* V_i = M_{\Phi_i}\Pi_V^*,$$

for all $i = 1, \ldots, n$. Finally, if we let $i_{\mathcal{S}}$ denote the inclusion map $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow H^2_{\mathcal{E}_n}(\mathbb{D})$, then $\Pi_{\mathcal{S}} : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\mathcal{E}_n}(\mathbb{D})$ is an isometry, where

$$\Pi_{\mathcal{S}} = i_{\mathcal{S}} \circ \Pi_V^*.$$

Clearly $\Pi_{\mathcal{S}}\Pi_{\mathcal{S}}^* = i_{\mathcal{S}}i_{\mathcal{S}}^*$. This implies that

$$\text{ran } \Pi_{\mathcal{S}} = \text{ran } i_{\mathcal{S}},$$

and so

$$\text{ran } \Pi_{\mathcal{S}} = \mathcal{S}.$$

Now, using $i_{\mathcal{S}}V = M_z i_{\mathcal{S}}$ and $i_{\mathcal{S}}V_j = M_{\kappa_j} i_{\mathcal{S}}$, we have

$$\Pi_{\mathcal{S}} M_z = M_z \Pi_{\mathcal{S}},$$

and

$$\Pi_{\mathcal{S}} M_{\Phi_i} = M_{\kappa_i}\Pi_{\mathcal{S}},$$

for all $i = 1, \ldots, n$. From the first equality it follows that there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ such that

$$\Pi_{\mathcal{S}} = M_\Theta.$$

This and the second equality implies that

$$\kappa_i\Theta = \Theta\Phi_i,$$

for all $i = 1, \ldots, n$. Moreover, ran $\Pi_{\mathcal{S}} = \mathcal{S}$ yields

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}).$$

To prove that $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is a commuting tuple, observe that

$$\begin{aligned} M_{\Phi_i} M_{\Phi_j} \Pi_V &= M_{\Phi_i} \Pi_V V_j \\ &= \Pi_V V_i V_j \\ &= \Pi_V V_j V_i \\ &= M_{\Phi_j} M_{\Phi_i} \Pi_V, \end{aligned}$$

and so

$$M_{\Phi_i} M_{\Phi_j} = M_{\Phi_j} M_{\Phi_i},$$

for all $i, j = 1, \ldots, n$. For the converse, let us begin by observing that if $\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$ for some inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, \mathcal{E}_n)}(\mathbb{D})$, then $\mathcal{S}$ is invariant for $M_z$ and

$$P_{\mathcal{S}} M_z^* P_{\mathcal{S}} = P_{\mathcal{S}} M_z^*.$$

In particular

$$P_{\mathcal{S}} M_z^*|_{\mathcal{S}} = P_{\mathcal{S}} M_z^* \in \mathcal{B}(\mathcal{S}),$$

and so $\{\Phi_1, \ldots, \Phi_n\}$ is a well-defined set of $\mathcal{B}(\mathcal{W})$-valued analytic functions on $\mathbb{D}$. Furthermore, if $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is an $n$-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$ (so, in particular, $\Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ for all $i = 1, \ldots, n$. See Remark 3.2.1) and $\kappa_i \Theta = \Theta \Phi_i$, then it follows obviously that $\kappa_i \mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \ldots, n$, that is, $\mathcal{S}$ is invariant for $(M_{\kappa_1}, \ldots, M_{\kappa_n})$. This proves the last part of Theorem 3.1.1 in the vector-valued Hardy space setting:

**Theorem 3.2.2.** *Let $\mathcal{E}$ be a Hilbert space, $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed subspace, and let $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$. Then $\mathcal{S}$ is invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ if and only if $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is an $n$-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$ and there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, \mathcal{E}_n)}(\mathbb{D})$ such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

*and*

$$\kappa_i \Theta = \Theta \Phi_i,$$

*where*

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - w P_{\mathcal{S}} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}},$$

*for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$.*

A few remarks are in order.

**Remark 3.2.1.** *Note that since $\|w P_{\mathcal{S}} M_z^*\| < 1$ for all $w \in \mathbb{D}$, the $\mathcal{B}(\mathcal{W})$-valued function $\Phi_i$, as defined in the above theorem, is analytic on $\mathbb{D}$. Here the boundedness condition (or the shift condition) on $M_{\Phi_i}$ on $H^2_{\mathcal{W}}(\mathbb{D})$ assures that $\Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ for all $i = 1, \ldots, n$.*

**Remark 3.2.2.** *Clearly, one obvious necessary condition for a closed subspace $\mathcal{S}$ of $H^2_{\mathcal{E}_n}(\mathbb{D})$ to be invariant for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ is that $\mathcal{S}$ is invariant for $M_z$, and, consequently*

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

*is the classical Beurling, Lax and Halmos representation of $\mathcal{S}$, where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ is the wandering subspace for $M_z|_{\mathcal{S}}$ and $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ is the (unique up to a unitary constant right factor; see Section 4) Beurling, Lax and Halmos inner function. Moreover, since $\kappa_i \mathcal{S} \subseteq \mathcal{S}$, another condition which is evidently necessary (by Douglas's range inclusion theorem) is that*

$$\kappa_i \Theta = \Theta \Gamma_i,$$

*for some $\Gamma_i \in \mathcal{B}(H^2_{\mathcal{W}}(\mathbb{D}))$, $i = 1, \dots, n$. In the above theorem, we prove that $\Gamma_i$ is explicit, that is*

$$\Gamma_i = \Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D}),$$

*for all $i = 1, \dots, n$, and $(\Gamma_1, \dots, \Gamma_n)$ is an $n$-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$. This is probably the most non-trivial part of our treatment to the invariant subspace problem in the present setting.*

**Remark 3.2.3.** *Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{W}$, $\Theta$ and*

$$\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D}),$$

*be as in Theorem 3.2.2. Now it follows from $P_{\mathcal{S}} = M_\Theta M_\Theta^*$ that*

$$P_{\mathcal{S}} M_z^{*m} = M_\Theta M_z^{*m} M_\Theta^*,$$

*for all $m \geq 0$. Hence the equality*

$$(I_{\mathcal{S}} - w P_{\mathcal{S}} M_z^*)^{-1} = \sum_{m=0}^{\infty} w^m P_{\mathcal{S}} M_z^{*m},$$

*yields*

$$(I_{\mathcal{S}} - w P_{\mathcal{S}} M_z^*)^{-1} = M_\Theta (I_{H^2_{\mathcal{W}}(\mathbb{D})} - w M_z^*)^{-1} M_\Theta^*,$$

*so that*

$$\Phi_i(w) = P_{\mathcal{W}} M_\Theta (I_{H^2_{\mathcal{W}}(\mathbb{D})} - w M_z^*)^{-1} M_\Theta^* M_{\kappa_i}|_{\mathcal{W}},$$

*for all $w \in \mathbb{D}$ and $i = 1, \dots, n$.*

A well known consequence of the Beurling, Lax and Halmos theorem (cf. page 239, Foias and Frazho [51]) implies that a closed subspace $\mathcal{S} \subseteq H^2_{\mathcal{E}}(\mathbb{D})$ is invariant for $M_z$ if and only if $\mathcal{S} \cong H^2_{\mathcal{F}}(\mathbb{D})$ for some Hilbert space $\mathcal{F}$ with

$$\dim \mathcal{F} \leq \dim \mathcal{E}.$$

More specifically, if $\mathcal{S}$ is a closed invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ and if $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$, then the pure isometry $M_z|_{\mathcal{S}}$ on $\mathcal{S}$ and $M_z$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent, and dim $\mathcal{W} \leq$ dim $\mathcal{E}$. The above theorem sets the stage for a similar result.

**Corollary 3.2.3.** *Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$, and*

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wP_{\mathcal{S}}M_z^*)^{-1}M_{\kappa_i}|_{\mathcal{W}} \qquad (w \in \mathbb{D}),$$

*for all $i = 1, \ldots, n$. Then $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$ and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent.*

*Proof.* Let $\mathcal{W}$, $\Theta$ and $\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ be as in Theorem 3.2.2. Then it follows that

$$X : H^2_{\mathcal{W}}(\mathbb{D}) \to \Theta H^2_{\mathcal{W}}(\mathbb{D}) = \mathcal{S},$$

is a unitary operator, where

$$X = M_\Theta.$$

It is now clear that $X$ intertwines $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ and

$$(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}}),$$

on $\mathcal{S}$. This completes the proof of the corollary. □

Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be an invariant subspace for $M_z$. Then $\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$, where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ and $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ is the Beurling, Lax and Halmos inner function. A natural question arises in connection with Remark 3.2.2: Under what additional condition(s) on $\Theta$ is $\mathcal{S}$ also invariant for $(M_{\kappa_1}, \ldots, M_{\kappa_n})$? An answer to this question directly follows, with appropriate reformulation, from Theorem 3.2.2 and Remark 3.2.3:

**Theorem 3.2.4.** *Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be an invariant subspace for $M_z$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$, where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ and $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ is the Beurling Lax and Halmos inner function. Set*

$$\Phi_i(w) = P_{\mathcal{W}}M_\Theta(I_{H^2_{\mathcal{W}}(\mathbb{D})} - wM_z^*)^{-1}M_\Theta^*M_{\kappa_i}|_{\mathcal{W}},$$

*for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$. Then $\mathcal{S}$ is invariant for $(M_{\kappa_1}, \ldots, M_{\kappa_n})$ if and only if $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is an $n$-tuple of commuting shifts, and*

$$\kappa_i\Theta = \Theta\Phi_i,$$

*for all $i = 1, \ldots, n$. Moreover, in this case, $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$ and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent.*

Thus the $n$-tuples of commuting shifts

$$(M_{\Phi_1}, \ldots, M_{\Phi_n}) \text{ on } H^2_{\mathcal{L}}(\mathbb{D}),$$

for Hilbert spaces $\mathcal{L}$ and inner multipliers $\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{L})}(\mathbb{D})$, yielding invariant subspaces of vector-valued Hardy spaces over $\mathbb{D}^{n+1}$ are distinguished among the general $n$-tuples of commuting shifts by the fact that

$$\Phi_i(w) = P_{\mathcal{L}}(I_{\mathcal{S}} - wP_{\mathcal{S}}M_z^*)^{-1}M_{\kappa_i}|_{\mathcal{L}} \qquad (w \in \mathbb{D}),$$

where $\mathcal{S} = \Theta H^2_{\mathcal{L}}(\mathbb{D})$ for some inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{L},\mathcal{E}_n)}(\mathbb{D})$, and

$$\kappa_i\Theta = \Theta\Phi_i,$$

for all $i = 1, \ldots, n$. Moreover, in view of Remark 3.2.3, the above condition is equivalent to the condition that

$$\Phi_i(w) = P_{\mathcal{W}}M_\Theta(I_{H^2_{\mathcal{L}}(\mathbb{D})} - wM_z^*)^{-1}M_\Theta^*M_{\kappa_i}|_{\mathcal{W}},$$

for some inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{L},\mathcal{E}_n)}(\mathbb{D})$ such that

$$\kappa_i\Theta = \Theta\Phi_i,$$

for all $i = 1, \ldots, n$.

## 3.3 Representations of model isometries

In connection with Theorem 3.2.1 (or part (i) of Theorem 3.1.1), a natural question arises: Given a Hilbert space $\mathcal{E}$, how to identify Hilbert spaces $\mathcal{F}$ and $\mathcal{B}(\mathcal{F})$-valued multipliers $\{\Psi\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{F})}(\mathbb{D})$ such that $(M_z, M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent. More generally, given a Hilbert space $\mathcal{E}$, characterize $(n+1)$-tuples of commuting shifts on Hilbert spaces that are unitarily equivalent to $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$.

This question has a simple answer, although a rigorous proof of it involves some technicalities. More specifically, the answer to this question is related to a numerical invariant, the rank of an operator associated with the Szegö kernel on $\mathbb{D}^{n+1}$. First, however, we need a few more definitions.

Let $(T_1, \ldots, T_m)$ be an $m$-tuple of commuting contractions on a Hilbert space $\mathcal{H}$. Define the *defect operator* [63] corresponding to $(T_1, \ldots, T_m)$ as

$$\mathbb{S}_m^{-1}(T_1, \ldots, T_m) = \sum_{0 \le |\boldsymbol{k}| \le m} (-1)^{|\boldsymbol{k}|} T_1^{k_1} \cdots T_m^{k_m} T_1^{*k_1} \cdots T_m^{*k_m},$$

where $0 \leq k_i \leq 1$, $i = 1, \ldots, m$. This definition is motivated by the representation of the Szegö kernel on the polydisc $\mathbb{D}^m$ (see Chapter 2). We say that $(T_1, \ldots, T_m)$ is of *rank* $p$ ($p \in \mathbb{N} \cup \{\infty\}$) if

$$\text{rank } [\mathbb{S}_m^{-1}(T_1, \ldots, T_m)] = p,$$

and we write

$$\text{rank } (T_1, \ldots, T_m) = p.$$

The defect operators plays an important role in multivariable operator theory (cf. [58, 63] and also see Chapter 2 and Chapter 4). For instance, if $\mathcal{E}$ is a Hilbert space, then the defect operator of the multiplication operator tuple $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is given by

$$\mathbb{S}_n^{-1}(M_{z_1}, \ldots, M_{z_n}) = P_{H^2_c(\mathbb{D}^n)} \otimes I_{\mathcal{E}},$$

where $P_{H^2_c(\mathbb{D}^n)}$ denotes the orthogonal projection of $H^2(\mathbb{D}^n)$ onto the one dimensional space of constant functions. Furthermore, as is evident from the definition (and also see the proof of Theorem 3.2.1), the defect operator for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ is given by

$$\mathbb{S}_{n+1}^{-1}(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n}) = P_{H^2_c(\mathbb{D})} \otimes P_{H^2_c(\mathbb{D}^n)} \otimes I_{\mathcal{E}}.$$

In particular,

$$\dim \mathcal{E} = \text{rank } (M_z, M_{\kappa_1}, \ldots, M_{\kappa_n}) = \text{rank } (M_{z_1}, \ldots, M_{z_n}).$$

Now let $\mathcal{E}$ and $\mathcal{K}$ be Hilbert spaces, and let $(V, V_1 \ldots, V_n)$ be an $(n+1)$-tuple of commuting shifts on $\mathcal{K}$. Suppose that $(V, V_1 \ldots, V_n)$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $\mathcal{K}$ and $H^2_{\mathcal{E}_n}(\mathbb{D})$, respectively, are unitarily equivalent. In this case, it is necessary that $M_z$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $V$ on $\mathcal{K}$ are unitarily equivalent. As $VV_i = V_i V$ and $V_i V_j = V_j V_i$ for all $i, j = 1, \ldots, n$, Theorem 2.2.1 implies that $(V, V_1, \ldots, V_n)$ and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{W} = \mathcal{K} \ominus V\mathcal{K}$, and

$$\Phi_i(z) = P_{\mathcal{W}}(I_{\mathcal{K}} - zV^*)^{-1} V_i|_{\mathcal{W}},$$

for all $z \in \mathbb{D}$ and $i = 1, \ldots, n$. Since $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ is doubly commuting, another necessary condition is that $(V, V_1, \ldots, V_n)$ is doubly commuting. In particular, $V^* V_i = V_i V^*$, and so

$$V^{*m} V_i = V_i V^{*m},$$

for all $m \geq 0$ and $i = 1, \ldots, n$. Using $V^{*m}|_{\mathcal{W}} = 0$ for all $m \geq 1$, this implies that $\Phi_i(z) = P_{\mathcal{W}} V_i|_{\mathcal{W}}$ for all $z \in \mathbb{D}$. Again using $VV_i^* = V_i^* V$, we have

$$V_i(I - VV^*) = (I - VV^*)V_i,$$

for all $i = 1, \ldots, n$. This implies that $\mathcal{W}$ is a reducing subspace for $V_i$, and hence we obtain

$$\Phi_i(z) = V_i|_{\mathcal{W}},$$

that is, $\Phi_i$ is a constant shift-valued function on $\mathbb{D}$ for all $i = 1, \ldots, n$. This observation leads to the following proposition:

**Proposition 3.3.1.** *Let $(V, V_1, \ldots, V_n)$ be an $(n+1)$-tuple of doubly commuting shifts on some Hilbert space $\mathcal{H}$. Let $\mathcal{W} = \mathcal{H} \ominus V\mathcal{H}$, and let*

$$\Phi_i(z) = V_i|_{\mathcal{W}} \qquad (i = 1, \ldots, n),$$

*for all $z \in \mathbb{D}$. Then $\mathcal{W}$ is reducing for $V_i$, $i = 1, \ldots, n$, and $(V, V_1, \ldots, V_n)$ and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent.*

In particular, if $\mathcal{L}$ is a Hilbert space and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{L}}(\mathbb{D})$, for some $\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{L})}(\mathbb{D})$, is a tuple of doubly commuting shifts, then

$$\Phi_i(z) = \Phi_i(0) \qquad (z \in \mathbb{D}),$$

that is, $\Phi$ is a constant function for all $i = 1, \ldots, n$.

Now we return to $(V, V_1 \ldots, V_n)$, which in turn is an $(n+1)$-tuple of doubly commuting shifts on $\mathcal{H}$. For simplicity of notation, set $U_1 = V$, $U_{i+1} = V_i$ for all $i = 1, \ldots, n$, and let

$$\mathcal{D} = \text{ran } \mathbb{S}^{-1}_{n+1}(V, V_1, \ldots, V_n) = \bigcap_{i=1}^{n+1} \ker U_i^*,$$

is the wandering subspace for $(V, V_1, \ldots, V_n)$ (cf. [96]). From here, one can use the fact that (cf. Theorem 3.3 in [96])

$$\mathcal{H} = \bigoplus_{\boldsymbol{k} \in \mathbb{Z}_+^{n+1}} U^{\boldsymbol{k}} \mathcal{D},$$

to prove that the map $\Gamma : \mathcal{H} \to H^2_{\mathcal{D}}(\mathbb{D}^{n+1})$ defined by

$$\Gamma(U^{\boldsymbol{k}}\eta) = \boldsymbol{z}^{\boldsymbol{k}}\eta \qquad (\boldsymbol{k} \in \mathbb{Z}_+^{n+1}, \eta \in \mathcal{D}),$$

is a unitary and

$$\Gamma U_i = M_{z_i}\Gamma,$$

for all $i = 1, \ldots, n+1$. Therefore, $(V, V_1, \ldots, V_n)$ on $\mathcal{H}$ and $(M_{z_1}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{D}}(\mathbb{D}^{n+1})$ are unitarily equivalent. In addition, if $\mathcal{E}$ is a Hilbert space, and

$$\dim \mathcal{E} = \text{rank } \ (V, V_1, \ldots, V_n) \ \ (= \dim \mathcal{D}),$$

then it follows that (see the equivalence of (ii) and (v) of Theorem 3.3 in [96]) $(M_{z_1}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{D}}(\mathbb{D}^{n+1})$ and $(M_{z_1}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ are unitarily equivalent. But then Theorem 3.2.1 yields immediately that $(M_{z_1}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{D}}(\mathbb{D}^{n+1})$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent. This gives the following:

**Theorem 3.3.2.** *In the setting of Proposition 3.3.1 the following hold:* $(V, V_1, \ldots, V_n)$ *on* $\mathcal{H}$, $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ *on* $H^2_{\mathcal{W}}(\mathbb{D})$, *and* $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ *on* $H^2_{\mathcal{E}_n}(\mathbb{D})$ *are unitarily equivalent, where* $\mathcal{E}$ *is a Hilbert space and*

$$dim\ \mathcal{E} = rank\ (V, V_1, \ldots, V_n).$$

Therefore, an $(n+1)$-tuple of doubly commuting shift operators

$$(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n}),$$

is completely determined by the numerical invariant rank $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$:

**Corollary 3.3.3.** *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be Hilbert spaces. Let* $(M_z, M_{\Psi_1}, \ldots, M_{\Psi_n})$ *be an* $(n+1)$*-tuple of commuting shifts on* $H^2_{\mathcal{F}}(\mathbb{D})$. *Then* $(M_z, M_{\Psi_1}, \ldots, M_{\Psi_n})$ *on* $H^2_{\mathcal{F}}(\mathbb{D})$ *and* $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ *on* $H^2_{\mathcal{E}_n}(\mathbb{D})$ *are unitarily equivalent if and only if*

$$(M_z, M_{\Psi_1}, \ldots, M_{\Psi_n})$$

*is doubly commuting and*

$$dim\ \mathcal{E} = rank\ (M_z, M_{\Psi_1}, \ldots, M_{\Psi_n}).$$

The above corollary should be compared with the uniqueness of the multiplicity of shift operators on Hilbert spaces [66].

## 3.4 Nested invariant subspaces and uniqueness

Now we proceed to the description of nested invariant subspaces of $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two closed invariant subspaces for

$$(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n}) \text{ on } H^2_{\mathcal{E}_n}(\mathbb{D}).$$

Let $\mathcal{W}_j = \mathcal{S}_j \ominus z\mathcal{S}_j$, and let

$$\Phi_{j,i}(w) = P_{\mathcal{W}_j}(I_{\mathcal{S}_j} - wP_{\mathcal{S}_j}M_z^*)^{-1}M_{\kappa_i}|_{\mathcal{W}_j},$$

for all $w \in \mathbb{D}$, $j = 1, 2$, and $i = 1, \ldots, n$. Hence by Theorem 3.2.2 there exists an inner function $\Theta_j \in H^\infty_{\mathcal{B}(\mathcal{W}_j, \mathcal{E}_n)}(\mathbb{D})$ such that

$$\mathcal{S}_j = \Theta_j H^2_{\mathcal{W}_j}(\mathbb{D}),$$

and

$$\kappa_i \Theta_j = \Theta_j \Phi_{j,i}, \tag{3.4.1}$$

for all $j = 1, 2$, and $i = 1, \dots, n$. Now, let

$$\mathcal{S}_1 \subseteq \mathcal{S}_2,$$

that is

$$\Theta_1 H^2_{\mathcal{W}_1}(\mathbb{D}) \subseteq \Theta_2 H^2_{\mathcal{W}_2}(\mathbb{D}).$$

Then there exists an inner multiplier $\Psi \in H^\infty_{\mathcal{B}(\mathcal{W}_1, \mathcal{W}_2)}(\mathbb{D})$ [51] such that

$$\Theta_1 = \Theta_2 \Psi.$$

Using this in (3.4.1), we get

$$\begin{aligned} \Theta_2 \Psi \Phi_{1,i} &= \Theta_1 \Phi_{1,i} \\ &= \kappa_i \Theta_1 \\ &= \kappa_i \Theta_2 \Psi \\ &= \Theta_2 \Phi_{2,i} \Psi, \end{aligned}$$

and so

$$\Psi \Phi_{1,i} = \Phi_{2,i} \Psi,$$

for all $i = 1, \dots, n$. On the other hand, given two invariant subspaces $\mathcal{S}_j = \Theta_j H^2_{\mathcal{W}_j}(\mathbb{D})$, $j = 1, 2$, for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ described as above, if there exists an inner multiplier $\Psi \in H^\infty_{\mathcal{B}(\mathcal{W}_1, \mathcal{W}_2)}(\mathbb{D})$ such that $\Theta_1 = \Theta_2 \Psi$, then it readily follows that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. We state this in the following theorem:

**Theorem 3.4.1.** *Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S}_1 = \Theta_1 H^2_{\mathcal{W}_1}(\mathbb{D})$ and $\mathcal{S}_2 = \Theta_2 H^2_{\mathcal{W}_2}(\mathbb{D})$ be two invariant subspaces for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let*

$$\Phi_{j,i}(w) = P_{\mathcal{W}_j}(I_{\mathcal{S}_j} - w P_{\mathcal{S}_j} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}_j},$$

*for all $w \in \mathbb{D}$, $j = 1, 2$, and $i = 1, \dots, n$. Then $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if there exists an inner multiplier $\Psi \in H^\infty_{\mathcal{B}(\mathcal{W}_1, \mathcal{W}_2)}(\mathbb{D})$ such that $\Theta_1 = \Theta_2 \Psi$ and $\Psi \Phi_{1,i} = \Phi_{2,i} \Psi$ for all $i = 1, \dots, n$.*

We now proceed to prove the uniqueness of the representations of invariant subspaces as described in Theorem 3.2.2. Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S}$ be an invariant subspace for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$ and

$$\kappa_i \Theta = \Theta \Phi_i \qquad (i = 1, \dots, n),$$

in the notation of Theorem 3.2.2. Now assume that $\tilde{\Theta} \in H^\infty_{\mathcal{B}(\tilde{\mathcal{W}})}(\mathbb{D})$ is an inner function, for some Hilbert space $\tilde{\mathcal{W}}$, and

$$\mathcal{S} = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D}).$$

Also assume that

$$\kappa_i\tilde{\Theta} = \tilde{\Theta}\tilde{\Phi}_i,$$

for some shift $M_{\tilde{\Phi}_i}$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ and $i = 1, \ldots, n$. Then as an application of the uniqueness of the Beurling, Lax and Halmos inner functions (cf. Theorem 2.1 in page 239 [51] and also Theorem 1.3.2 in Chapter 1) to

$$\Theta H^2_{\mathcal{W}}(\mathbb{D}) = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D}),$$

we get

$$\Theta = \tilde{\Theta}\tau,$$

for some unitary operator (constant in $z$) $\tau : \mathcal{W} \to \tilde{\mathcal{W}}$. Then, the previous line of argument shows that

$$\tau\Phi_i = \tilde{\Phi}_i\tau,$$

for all $i = 1, \ldots, n$. This proves the uniqueness of the representations of invariant subspaces in Theorem 3.2.2.

**Theorem 3.4.2.** *In the setting of Theorem 3.2.2, if* $\mathcal{S} = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ *and* $\kappa_i\tilde{\Theta} = \tilde{\Theta}\tilde{\Phi}_i$ *for some Hilbert space* $\tilde{\mathcal{W}}$*, inner function* $\tilde{\Theta} \in H^\infty_{\mathcal{B}(\tilde{\mathcal{W}})}(\mathbb{D})$ *and shift* $M_{\tilde{\Phi}_i}$ *on* $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$*,* $i = 1, \ldots, n$*, then there exists a unitary operator (constant in $z$)* $\tau : \mathcal{W} \to \tilde{\mathcal{W}}$ *such that*

$$\Theta = \tilde{\Theta}\tau,$$

*and*

$$\tau\Phi_i = \tilde{\Phi}_i\tau,$$

*for all* $i = 1, \ldots, n$.

## 3.5 Applications

In this section, first, we explore a natural connection between the intertwining maps on vector-valued Hardy space over $\mathbb{D}$ and the commutators of the multiplication operators on the Hardy space over $\mathbb{D}^{n+1}$. Then, as a noteworthy added benefit to our approach, we compute a complete set of unitary invariants for invariant subspaces of vector-valued Hardy space over $\mathbb{D}^{n+1}$. We also test our main results on invariant subspaces unitarily equivalent to $H^2_{\mathcal{E}_n}(\mathbb{D})$. As a by-product, we obtain some useful results about the structure of invariant subspaces for the Hardy space. We begin with the following definition.

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be two Hilbert spaces. Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be invariant subspaces for the $(n+1)$-tuples of multiplication operators on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$, respectively. We say that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are *unitarily equivalent*, and write $\mathcal{S} \cong \tilde{\mathcal{S}}$, if there is a unitary map $U : \mathcal{S} \to \tilde{\mathcal{S}}$ such that

$$UM_z|_{\mathcal{S}} = M_z|_{\tilde{\mathcal{S}}}U \quad \text{and} \quad UM_{\kappa_i}|_{\mathcal{S}} = M_{\kappa_i}|_{\tilde{\mathcal{S}}}U,$$

for all $i = 1, \dots, n$.

### 3.5.1 Intertwining maps

Recall that, given a Hilbert space $\mathcal{E}$, there exists a unitary operator $U_{\mathcal{E}} : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{E}_n}(\mathbb{D})$ (see Section 2) such that

$$U_{\mathcal{E}} M_{z_1} = M_z U_{\mathcal{E}},$$

and

$$U_{\mathcal{E}} M_{z_{i+1}} = M_{\kappa_i} U_{\mathcal{E}},$$

for all $i = 1, \dots, n$. Let $\mathcal{F}$ be another Hilbert space, and let $X : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{F}}(\mathbb{D}^{n+1})$ be a bounded linear operator such that

$$X M_{z_i} = M_{z_i} X, \tag{3.5.1}$$

for all $i = 1, \dots, n+1$. Set

$$X_n = U_{\mathcal{F}} X U_{\mathcal{E}}^*.$$

Then $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ is bounded and

$$X_n M_z = M_z X_n \quad \text{and} \quad X_n M_{\kappa_i} = M_{\kappa_i} X_n, \tag{3.5.2}$$

for all $i = 1, \dots, n$. Conversely, a bounded linear operator $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ satisfying (3.5.2) yields a canonical bounded linear map $X : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{F}}(\mathbb{D}^{n+1})$, namely

$$X = U_{\mathcal{F}}^* X_n U_{\mathcal{E}}$$

such that (3.5.1) holds. Moreover, this construction shows that

$$X \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}^{n+1}), H^2_{\mathcal{F}}(\mathbb{D}^{n+1}))$$

is a contraction (respectively, isometry, unitary, etc.) if and only if

$$X_n \in \mathcal{B}(H^2_{\mathcal{E}_n}(\mathbb{D}), H^2_{\mathcal{F}_n}(\mathbb{D}))$$

is a contraction (respectively, isometry, unitary, etc.).

For brevity, any map satisfying (3.5.2) will be referred to *module maps*.

### 3.5.2 A complete set of unitary invariants

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hilbert spaces, and let $\{\Psi_1, \dots, \Psi_n\} \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ and $\{\tilde{\Psi}_1, \dots, \tilde{\Psi}_n\} \subseteq H^\infty_{\mathcal{B}(\tilde{\mathcal{E}})}(\mathbb{D})$. We say that $\{\Psi_1, \dots, \Psi_n\}$ and $\{\tilde{\Psi}_1, \dots, \tilde{\Psi}_n\}$ coincide if there exists a unitary operator $\tau : \mathcal{E} \to \tilde{\mathcal{E}}$ such that

$$\tau \Psi_i(z) = \tilde{\Psi}_i(z)\tau,$$

for all $z \in \mathbb{D}$ and $i = 1, \ldots, n$.

Now let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ and $\tilde{\mathcal{S}} \subseteq H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$ be invariant subspaces for

$$(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$$

on $H^2_{\mathcal{E}_n}(\mathbb{D})$, and $H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$, respectively. Let $\mathcal{S} \cong \tilde{\mathcal{S}}$. By Theorem 3.2.4, this implies that

$$(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n}) \text{ on } H^2_{\mathcal{W}}(\mathbb{D}),$$

and $(M_z, M_{\tilde{\Phi}_1}, \ldots, M_{\tilde{\Phi}_n})$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$, $\tilde{\mathcal{W}} = \tilde{\mathcal{S}} \ominus z\tilde{\mathcal{S}}$ and

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wP_{\mathcal{S}}M_z^*)^{-1}M_{\kappa_i}|_{\mathcal{W}},$$

and

$$\tilde{\Phi}_i(w) = P_{\tilde{\mathcal{W}}}(I_{\tilde{\mathcal{S}}} - wP_{\tilde{\mathcal{S}}}M_z^*)^{-1}M_{\kappa_i}|_{\tilde{\mathcal{W}}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$. Let $U : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ be a unitary map such that

$$UM_z = M_zU,$$

and

$$UM_{\Phi_i} = M_{\tilde{\Phi}_i}U,$$

for all $i = 1, \ldots, n$. The former condition implies that

$$U = I_{H^2(\mathbb{D})} \otimes \tau,$$

for some unitary operator $\tau : \mathcal{W} \to \tilde{\mathcal{W}}$, and so the latter condition implies that

$$\tau \Phi_i(z) = \tilde{\Phi}_i(z)\tau,$$

for all $z \in \mathbb{D}$ and $i = 1, \ldots, n$. Therefore $\{\Phi_1, \ldots, \Phi_n\}$ and $\{\tilde{\Phi}_1, \ldots, \tilde{\Phi}_n\}$ coincide. To prove the converse, assume now that the above equality holds for a given unitary operator $\tau : \mathcal{W} \to \tilde{\mathcal{W}}$. Obviously $U = I_{H^2(\mathbb{D})} \otimes \tau$ is a unitary from $H^2_{\mathcal{W}}(\mathbb{D})$ to $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$. Clearly $UM_z = M_zU$ and $UM_{\Phi_i} = M_{\tilde{\Phi}_i}U$ for all $i = 1, \ldots, n$. So we have the following theorem on a complete set of unitary invariants for invariant subspaces:

**Theorem 3.5.1.** *Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hilbert spaces. Let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ and $\tilde{\mathcal{S}} \subseteq H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$ be invariant subspaces for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$, respectively. Then $\mathcal{S} \cong \tilde{\mathcal{S}}$ if and only if $\{\Phi_1, \ldots, \Phi_n\}$ and $\{\tilde{\Phi}_1, \ldots, \tilde{\Phi}_n\}$ coincide.*

Now, if we consider the Beurling, Lax and Halmos representations of the given invariant subspaces $\mathcal{S}$ and $\tilde{\mathcal{S}}$ as

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

and

$$\tilde{\mathcal{S}} = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D}),$$

where $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ and $\tilde{\Theta} \in H^\infty_{\mathcal{B}(\tilde{\mathcal{W}},\tilde{\mathcal{E}}_n)}(\mathbb{D})$, then, in view of Remark 3.2.3, the multipliers in Theorem 3.5.1 can be represented as

$$\Phi_i(w) = P_{\mathcal{W}} M_\Theta (I_{H^2_{\mathcal{W}}(\mathbb{D})} - w M_z^*)^{-1} M_\Theta^* M_{\kappa_i}|_{\mathcal{W}},$$

and

$$\tilde{\Phi}_i(w) = P_{\tilde{\mathcal{W}}} M_{\tilde{\Theta}} (I_{H^2_{\tilde{\mathcal{W}}}(\mathbb{D})} - w M_z^*)^{-1} M_{\tilde{\Theta}}^* M_{\kappa_i}|_{\tilde{\mathcal{W}}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$.

### 3.5.3 Unitarily equivalent invariant subspaces

Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces, and let $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ be a module map. If $X_n$ is an isometry, then the closed subspace $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ defined by

$$\mathcal{S} = X_n(H^2_{\mathcal{E}_n}(\mathbb{D})),$$

is invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$ and $\mathcal{S} \cong H^2_{\mathcal{E}_n}(\mathbb{D})$. In other words, the tuples $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent. Conversely, let $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$, and let $\mathcal{S} \cong H^2_{\mathcal{E}_n}(\mathbb{D})$ for some Hilbert space $\mathcal{E}$. Let $\tilde{X}_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to \mathcal{S}$ be the unitary map which intertwines $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$. Suppose that $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow H^2_{\mathcal{F}_n}(\mathbb{D})$ is the inclusion map. Then

$$X_n = i_{\mathcal{S}} \circ \tilde{X}_n,$$

is an isometry from $H^2_{\mathcal{E}_n}(\mathbb{D})$ to $H^2_{\mathcal{F}_n}(\mathbb{D})$, $X_n M_z = M_z X_n$, $X_n M_{\kappa_i} = M_{\kappa_i} X_n$ for all $i = 1, \ldots, n$, and

$$\text{ran } X_n = \mathcal{S}.$$

Therefore, if $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ is a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$, then $\mathcal{S} \cong H^2_{\mathcal{E}_n}(\mathbb{D})$, for some Hilbert space $\mathcal{E}$, if and only if there exists an isometric module map $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ such that $\mathcal{S} = X_n(H^2_{\mathcal{E}_n}(\mathbb{D}))$. Now, it also follows from the discussion at the beginning of this section that $X : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{F}}(\mathbb{D}^{n+1})$ (corresponding to the module map $X_n$) is an isometry and $X M_{z_i} = M_{z_i} X$ for all $i = 1, \ldots, n$. Then Theorem 3.6.1 tells us that

$$\dim \mathcal{E} \leq \dim \mathcal{F}.$$

Therefore, we have the following theorem:

**Theorem 3.5.2.** *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces, and let $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$. Then $\mathcal{S} \cong H^2_{\mathcal{E}_n}(\mathbb{D})$ if and only* *if there exists an isometric module map* $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ *such that*

$$\mathcal{S} = X_n H^2_{\mathcal{E}_n}(\mathbb{D}).$$

*Moreover, in this case*

$$dim\ \mathcal{E} \leq dim\ \mathcal{F}.$$

Of particular interest is the case when $\mathcal{F} = \mathbb{C}$. In this case (see Section 2) the tensor product Hilbert space $\mathcal{F}_n = H^2(\mathbb{D}^n) \otimes \mathbb{C}$ is denoted by $H_n$, that is, $H_n = H^2(\mathbb{D}^n)$.

**Corollary 3.5.3.** *Let* $\mathcal{S} \subseteq H^2_{H_n}(\mathbb{D})$ *be a closed invariant subspace for* $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ *on* $H^2_{H_n}(\mathbb{D})$. *Then* $\mathcal{S} \cong H^2_{H_n}(\mathbb{D})$ *if and only if there exists an isometric module map* $X_n : H^2_{H_n}(\mathbb{D}) \to H^2_{H_n}(\mathbb{D})$ *such that*

$$\mathcal{S} = X_n(H^2_{H_n}(\mathbb{D})).$$

The above result, in the polydisc setting, was first observed by Agrawal, Clark and Douglas (see Corollary 1 in [3]). Also see Mandrekar [77].

We now proceed to analyze doubly commuting invariant subspaces. Let $\mathcal{F}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$. Set

$$V = M_z|_{\mathcal{S}},$$

and

$$V_i = M_{\kappa_i}|_{\mathcal{S}},$$

for all $i = 1, \dots, n$. We say that $\mathcal{S}$ is *doubly commuting* if $V_i^* V_j = V_j V_i^*$ for all $1 \leq i < j \leq n$.

Now let $\mathcal{E}$ be a Hilbert space, and suppose that $H^2_{\mathcal{E}_n}(\mathbb{D}) \cong \mathcal{S}$. In view of Theorem 3.5.2 this implies that $(V, V_1, \dots, V_n)$ on $\mathcal{S}$ and $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent. Because $H^2_{\mathcal{E}_n}(\mathbb{D})$ is doubly commuting this immediately implies that $\mathcal{S}$ is doubly commuting.

Conversely, let $\mathcal{S}$ be doubly commuting. From Theorem 3.2.4 we readily conclude that $(M_z, M_{\Phi_1}, \dots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ and $(V, V_1, \dots, V_n)$ on $\mathcal{S}$ are unitarily equivalent.

Applying Theorem 3.3.2 with $(M_z, M_{\Phi_1}, \dots, M_{\Phi_n})$ in place of

$$(M_z, M_{\Psi_1}, \dots, M_{\Psi_n}),$$

we see that $(V, V_1, \dots, V_n)$ on $\mathcal{S}$ and $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{E}$ is a Hilbert space. Now, proceeding as in the proof of the necessary part of Theorem 3.5.2 one checks that there exists a module isometry $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ such that

$$\text{ran } X_n = \mathcal{S}.$$

This proves the following variant of Theorem 3.5.2:

**Theorem 3.5.4.** *Let $\mathcal{F}$ be a Hilbert space. An invariant subspace $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ is doubly commuting if and only if there exists a Hilbert space $\mathcal{E}$ and an isometric module map $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ such that*

$$\mathcal{S} = X_n H^2_{\mathcal{E}_n}(\mathbb{D}).$$

*Moreover, in this case*

$$dim\ \mathcal{E} \leq dim\ \mathcal{F}.$$

The above result, in the polydisc setting, was first observed by Mandrekar [77]. Also this should be compared with the discussion prior to Corollary 3.2.3 on the application of the classical Beurling, Lax and Halmos theorem to invariant subspaces of the Hardy space over the unit disc.

## 3.6 An inequality on fibre dimensions

Given a Hilbert space $\mathcal{E}$, the $n$-tuple of multiplication operators by the coordinate functions $z_i$, $i = 1, \ldots, n$, on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is denoted by $(M^{\mathcal{E}}_{z_1}, \ldots, M^{\mathcal{E}}_{z_n})$. Whenever $\mathcal{E}$ is clear from the context, we will omit the superscript $\mathcal{E}$. Clearly, one can regard $\mathcal{E}$ as a closed subspace of $H^2_{\mathcal{E}}(\mathbb{D}^n)$ by identifying $\mathcal{E}$ with the constant $\mathcal{E}$-valued functions on $\mathbb{D}^n$.

In this Section, we aim to prove the following result:

**Theorem 3.6.1.** *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces and let $X : H^2_{\mathcal{E}_1}(\mathbb{D}^n) \to H^2_{\mathcal{E}_2}(\mathbb{D}^n)$ be an isometry. If*

$$X M^{\mathcal{E}_1}_{z_i} = M^{\mathcal{E}_2}_{z_i} X,$$

*for all $i = 1, \ldots, n$, then*

$$\dim \mathcal{E}_1 \leq \dim \mathcal{E}_2.$$

We believe that the above result (possibly) follows from the boundary behavior of bounded analytic functions following the classical case $n = 1$ (See end of this section). Here, however, we take a shorter approach than generalizing the classical theory of bounded analytic functions on the unit polydisc. We first prove the $L^2$-version of the above statement.

**Theorem 3.6.2.** *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces and let $\tilde{X} : L^2_{\mathcal{E}_1}(\mathbb{T}^n) \to L^2_{\mathcal{E}_2}(\mathbb{T}^n)$ be an isometry. If*

$$\tilde{X} M_{e^{i\theta_j}} = M_{e^{i\theta_j}} \tilde{X},$$

*for all $j = 1, \ldots, n$, then*

$$\dim \mathcal{E}_1 \leq \dim \mathcal{E}_2.$$

*Proof.* By the triviality, we can assume that

$$m := \dim \mathcal{E}_2 < \infty.$$

Let $\{\eta_j\}_{j=1}^m$ be an orthonormal basis for $\mathcal{E}_2$. Since $\{e_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{Z}^n\}$, where

$$e_{\boldsymbol{k}} = \prod_{j=1}^{n} e^{ik_j\theta_j} \qquad (\boldsymbol{k} \in \mathbb{Z}^n),$$

is an orthonormal basis for $L^2(\mathbb{T}^n)$, this implies that $\{e_{\boldsymbol{k}}\eta_j : \boldsymbol{k} \in \mathbb{Z}^n, j = 1, \ldots, n\}$ is an orthonormal basis for $L^2_{\mathcal{E}_2}(\mathbb{T}^n)$. Let $\{f_j : j \in J\}$ be an orthonormal basis for $\tilde{X}(\mathcal{E}_1)$, where $J$ is a subset of $\mathbb{Z}_+$. In view of the intertwining property of $\tilde{X}$, this implies that $\{e_{\boldsymbol{k}} f_j : \boldsymbol{k} \in \mathbb{Z}^n, j \in J\}$ is an orthonormal basis for

$$\widetilde{X}(L^2_{\mathcal{E}_1}(\mathbb{T}^n)) \subseteq L^2_{\mathcal{E}_2}(\mathbb{T}^n),$$

and so, an orthonormal set in $L^2_{\mathcal{E}_2}(\mathbb{T}^n)$. It follows from the Parseval's identity that

$$\begin{aligned}
\dim \mathcal{E}_1 &= \dim(\tilde{X}\mathcal{E}_1)\\
&= \sum_{j\in J} \|f_j\|^2\\
&= \sum_{j\in J} \sum_{l=1}^{m} \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\langle M_{e^{i\theta}}^{\boldsymbol{k}} \eta_l, f_j\rangle|^2\\
&= \sum_{j\in J} \sum_{l=1}^{m} \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\langle \eta_l, M_{e^{i\theta}}^{\boldsymbol{k}} f_j\rangle|^2\\
&= \sum_{j\in J} \sum_{l=1}^{m} \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\langle \eta_l, e_{\boldsymbol{k}} f_j\rangle|^2,
\end{aligned}$$

on the one hand, and on the other, by Bessel's Inequality,

$$\begin{aligned}
m &= \sum_{l=1}^{m} \|\eta_l\|^2\\
&\geq \sum_{l=1}^{m} \sum_{j\in J} \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\langle \eta_l, e_{\boldsymbol{k}} f_j\rangle|^2.
\end{aligned}$$

This proves $\dim \mathcal{E}_1 \leq m$ and completes the proof of the theorem. □

*Proof of Theorem 3.6.1:* Define $\tilde{X}$ on $\{e_{\boldsymbol{k}}\eta : \boldsymbol{k} \in \mathbb{Z}^n, \eta \in \mathcal{E}_1\}$ by

$$\tilde{X}(e_{\boldsymbol{k}}\eta) = e_{\boldsymbol{k}} X\eta,$$

for all $\boldsymbol{k} \in \mathbb{Z}^n$ and $\eta \in \mathcal{E}_1$. The intertwining property of the isometry $X$ then gives

$$\langle \tilde{X}(e_{\boldsymbol{k}}\eta), \tilde{X}(e_{\boldsymbol{l}}\zeta)\rangle_{L^2_{\mathcal{E}_2}(\mathbb{T}^n)} = \langle e_{\boldsymbol{k}}\eta, e_{\boldsymbol{l}}\zeta\rangle_{L^2_{\mathcal{E}_1}(\mathbb{T}^n)},$$

for all $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{Z}^n$ and $\eta, \zeta \in \mathcal{E}_1$. Therefore this map extends uniquely to an isometry, denoted again by $\tilde{X}$ from $L^2_{\mathcal{E}_1}(\mathbb{T}^n)$ to $L^2_{\mathcal{E}_2}(\mathbb{T}^n)$, such that

$$\tilde{X} M_{e^{i\theta_j}} = M_{e^{i\theta_j}} \tilde{X},$$

for all $j = 1, \ldots, n$. The result then easily follows from Theorem 3.6.2. $\square$

If $X : H^2_{\mathcal{E}_1}(\mathbb{D}^n) \to H^2_{\mathcal{E}_2}(\mathbb{D}^n)$ is an isometry, and if $XM_{z_i} = M_{z_i}X$ for all $i = 1, \ldots, n$, then it is easy to see that

$$X = M_\Theta,$$

for some isometric multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)}(\mathbb{D}^n)$ (that is, $M_\Theta : H^2_{\mathcal{E}_1}(\mathbb{D}^n) \to H^2_{\mathcal{E}_2}(\mathbb{D}^n)$ is an isometry). In the case $n = 1$, the conclusion of Theorem 3.6.1 follows from the boundary behavior of bounded analytic functions on the open unit disc: $M_\Theta$ is an isometry if and only if $\Theta(e^{i\theta})$ is isometry a.e. on $\mathbb{T}$ (cf. [79]). Unlike the proof of the classical case $n = 1$, our proof does not use the boundary behavior of $\Theta$.

# Chapter 4

# Pairs of projections and commuting isometries

## 4.1 Introduction

Given $n \in \mathbb{N} \cup \{\infty\}$, there exists precisely one Hilbert space $\mathcal{E}$, up to unitary equivalence, of dimension $n$ (here all Hilbert spaces are assumed to be separable), and given a Hilbert space $\mathcal{E}$, there exists precisely one shift operator, up to unitary equivalence, of multiplicity $\dim \mathcal{E}$ on some Hilbert space $\mathcal{H}$. Therefore, multiplicity is the only (numerical) invariant of a shift operator. Note that shift operators are special class of isometries, and moreover, the defect operator of a shift determines the multiplicity of the shift.

Now we turn to commuting pairs of isometries. It is remarkable that tractable invariants (whatever it means including the possibilities of numerical and analytical invariants) of commuting pairs of isometries are largely unknown. However, in one hand, the notion of defect operator associated with commuting pairs of isometries has some resemblance to multiplicities (and hence defect operators) of shift operators. On the other hand, the defect operator of a general pair of commuting isometries is fairly complex and not completely helpful in dealing with the complicated structure of pair of isometries.

In this chapter we will restrict pairs of commuting isometries to Berger, Coburn and Lebow pairs of isometries (which we call *BCL pairs*) resulting in somewhat more tractable defect operators (see Section 4). Indeed, each BCL pair $(V_1, V_2)$ is uniquely associated with a triple $(\mathcal{E}, U, P)$, where $\mathcal{E}$ is a Hilbert space and $U$ is a unitary and $P$ is a projection (throughout, projection will always mean orthogonal projection) on $\mathcal{E}$. Moreover, in this case, the defect operator of $(V_1, V_2)$ is given by (see (4.1.4))

$$C(V_1, V_2) = UPU^* - P. \tag{4.1.1}$$

Clearly, $(UPU^*, P)$ is a pair of orthogonal projections on $\mathcal{E}$ and hence, $C(V_1, V_2)$ is a self-adjoint contraction.

In summary, given a BCL pair $(V_1, V_2)$, up to unitary equivalence, there exists precisely one triple $(\mathcal{E}, U, P)$, and given a triple $(\mathcal{E}, U, P)$, there exists a pair of projections $(UPU^*, P)$ such that the defect operator of $(V_1, V_2)$, denoted by $C(V_1, V_2)$, is the difference of the projections $UPU^*$ and $P$ as in (4.1.1). In particular, the defect operator is a self-adjoint contraction. If, in addition, the defect operator $C(V_1, V_2)$ is compact, then $C(V_1, V_2)|_{(\ker C(V_1,V_2))^\perp}$ admits the following decomposition

$$\begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_2 & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}, \tag{4.1.2}$$

where $I_1$ and $I_2$ are the identity operators and $D$ is a positive contractive diagonal operator. The goal of this chapter, largely, is to suggest the (missing) link between compact differences of pairs of projections and BLC pairs. More specifically, given a self-adjoint compact contraction $T$ of the form (4.1.2) on a Hilbert space $\mathcal{E}$, we are interested in computing irreducible (that is, non-reducing - in an appropriate sense, see Definition 4.1.3) BCL pairs $(V_1, V_2)$ such that $C(V_1, V_2)|_{(\ker C(V_1,V_2))^\perp}$ is equal (or unitarily equivalent) to $T$. The complication involved in the range of our answers for self-adjoint compact contractions will further indicate the delicate structure of BCL pairs (let alone the general class of pairs of commuting isometries).

It is worthwhile to note that the geometric examples of concrete pairs of commuting isometries out of our construction might be of independent interest. Indeed, despite its importance, little is known about the structure of pairs of commuting isometries.

Our main motivation comes from the work of Berger, Coburn and Lebow [20] and a question of He, Qin and Yang [69]. Moreover, one of the key tools applied here is a projection formulae of Shi, Ji and Du [102] (more specifically, see Theorem 4.2.2).

Furthermore, we note, from a general point of view, that the concept of difference of two projections on Hilbert spaces is an important tool in the theory of linear operators (both finite and infinite dimensional Hilbert spaces). In this context, we refer to [11] on products of orthogonal projections, [23, 54, 55] on isometries of Grassmann spaces, [90] on $C^*$-algebras generated by pairs of projections, [7] invariant subspaces of pairs of projections, [87] on differences of spectral projections and [13] on index of pairs of projections. We refer the reader to [25] for a nice account on pairs of projections. Also see [8, 10, 37, 64, 104, 105].

Let us now explain the setting and the content of this chapter in more detail. Let $\mathcal{H}$ be a Hilbert space and let $V$ be an isometry on $\mathcal{H}$. The *multiplicity* of $V$ is the number

$$\operatorname{rank}(I_{\mathcal{H}} - VV^*) \in \mathbb{N} \cup \{\infty\}.$$

The projection $I_{\mathcal{H}} - VV^*$ is known as the *defect operator* associated with $V$ which we denote by

$$C(V) = I_{\mathcal{H}} - VV^*.$$

Recall that the defect operator of $M_z$ on $H^2(\mathbb{D})$ is given by

$$C(M_z) = P_{\mathbb{C}},$$

where $P_{\mathbb{C}}$ denotes the projection of $H^2(\mathbb{D})$ onto $\mathbb{C}$, the one dimensional subspace of constant functions of $H^2(\mathbb{D})$. Consequently, for any Hilbert space $\mathcal{E}$, the fact that

$$C(M_z \otimes I_{\mathcal{E}}) = P_{\mathbb{C}} \otimes I_{\mathcal{E}},$$

implies that the multiplicity of the shift $M_z \otimes I_{\mathcal{E}}$ on $H^2(\mathbb{D}) \otimes \mathcal{E}$ is given by $\dim\mathcal{E}$. Moreover, if $V$ is a shift on a Hilbert space $\mathcal{H}$, then $V$ on $\mathcal{H}$ and $M_z \otimes I_{\mathcal{W}}$ on $H^2(\mathbb{D}) \otimes \mathcal{W}$ are unitarily equivalent, where

$$\mathcal{W} = \mathcal{H} \ominus V\mathcal{H} = \operatorname{ran} C(V).$$

In particular, for Hilbert spaces $\mathcal{E}$ and $\tilde{\mathcal{E}}$, $M_z \otimes I_{\mathcal{E}}$ on $H^2(\mathbb{D}) \otimes \mathcal{E}$ and $M_z \otimes I_{\tilde{\mathcal{E}}}$ on $H^2(\mathbb{D}) \otimes \tilde{\mathcal{E}}$ are unitarily equivalent if and only if

$$\dim\mathcal{E} = \dim\tilde{\mathcal{E}}.$$

This also follows, in particular, from the fact that $C(M_z \otimes I_{\mathcal{E}}) = P_{\mathbb{C}} \otimes I_{\mathcal{E}}$.

By a *BCL triple* (after Berger, Coburn and Lebow [20]) we mean an ordered triple $(\mathcal{E}, U, P)$ which consists of a Hilbert space $\mathcal{E}$, a unitary operator $U$ and an orthogonal projection $P$ on $\mathcal{E}$.

Now, let $(V_1, V_2)$ be a pair of commuting isometries acting on the Hilbert space $\mathcal{H}$. We say that $(V_1, V_2)$ is *pure* if $V := V_1V_2$ is a shift. In [20], Berger, Coburn, and Lebow established the following model for pure pair of commuting isometries (also see Chapter 2):

Let $(\mathcal{E}, U, P)$ be a BCL triple and suppose

$$\begin{aligned} V_1 &= (I_{H^2(\mathbb{D})} \otimes P + M_z \otimes P^{\perp})(I_{H^2(\mathbb{D})} \otimes U^*), \\ V_2 &= (I_{H^2(\mathbb{D})} \otimes U)(M_z \otimes P + I_{H^2(\mathbb{D})} \otimes P^{\perp}). \end{aligned} \tag{4.1.3}$$

One can easily check that

$$V_1V_2 = V_2V_1 = M_z \otimes I_{\mathcal{E}},$$

that is, $(V_1, V_2)$ is a commuting pair of pure isometries. Conversely, it is proved in [20] that a pure pair of commuting isometries, up to unitary equivalence, is of the form (4.1.3) for some BCL triple $(\mathcal{E}, U, P)$.

We shall call $(V_1, V_2)$, as given in (4.1.3), the *BCL pair* associated with the BCL triple $(\mathcal{E}, U, P)$. Often we shall not explicitly distinguish between BCL pair $(V_1, V_2)$, as given in (4.1.3), and the corresponding BCL triple $(\mathcal{E}, U, P)$.

The *defect operator* of a BCL pair $(V_1, V_2)$ (or, a general pair of commuting isometries), denoted $C(V_1, V_2)$, is defined by

$$C(V_1, V_2) = I_{H^2_{\mathcal{E}}(\mathbb{D})} - V_1V_1^* - V_2V_2^* + V_1V_2V_1^*V_2^*.$$

An easy computation reveals that

$$C(V_1, V_2) = P_{\mathbb{C}} \otimes (UPU^* - P) = P_{\mathbb{C}} \otimes (P^{\perp} - UP^{\perp}U^*), \tag{4.1.4}$$

and hence,

$$C(V_1, V_2)|_{zH^2(\mathbb{D})\otimes\mathcal{E}} = 0 \quad \text{and} \quad \overline{ran\ C(V_1, V_2)} \subseteq \mathbb{C} \otimes \mathcal{E}.$$

Thus it suffices to study $C(V_1, V_2)$ only on $(zH^2(\mathbb{D}) \otimes \mathcal{E})^{\perp} = \mathbb{C} \otimes \mathcal{E}$. In summary, if $(V_1, V_2)$ is a BCL pair on $H^2_{\mathcal{E}}(\mathbb{D})$, then the block matrix of $C(V_1, V_2)$ with respect to the orthogonal decomposition $H^2_{\mathcal{E}}(\mathbb{D}) = zH^2_{\mathcal{E}}(\mathbb{D}) \oplus \mathcal{E}$ is given by

$$C(V_1, V_2) = \begin{bmatrix} 0 & 0 \\ 0 & P^{\perp} - UP^{\perp}U^* \end{bmatrix}.$$

If $(V_1, V_2)$ is clear from the context, then we define

$$C := C(V_1, V_2)|_{\mathcal{E}} = P^{\perp} - UP^{\perp}U^*.$$

Note that $C$, being the difference of a pair of projections, is a self-adjoint contraction. In addition, if it is compact, then clearly its spectrum lies in $[-1, 1]$ and the non-zero elements of the spectrum are precisely the non-zero eigen values of $C$. In this case, for each eigen value $\lambda$ of $C$, we denote by $E_{\lambda}$ the eigen space corresponding to $\lambda$, that is

$$E_{\lambda} = \ker(C - \lambda I_{\mathcal{E}}).$$

The following useful lemma is due to He, Qin and Yang [69, Lemma 4.2]:

**Lemma 4.1.1.** *If $C$ is compact, then for each non-zero eigen value $\lambda$ of $C$ in $(-1, 1)$, $-\lambda$ is also an eigen value of $C$ and*

$$dim E_{\lambda} = dim E_{-\lambda}.$$

Consequently, one can decompose $(\ker C)^{\perp}$ as

$$(\ker C)^{\perp} = E_1 \oplus (\bigoplus_{\lambda} E_{\lambda}) \oplus E_{-1} \oplus (\bigoplus_{\lambda} E_{-\lambda}), \tag{4.1.5}$$

where $\lambda$ runs over the set of positive eigen values of $C$ lying in $(0, 1)$. With respect to the above decomposition of $(\ker C)^{\perp}$, the non-zero part of $C$, that is, $C|_{(\ker C)^{\perp}}$, the restriction of $C$ to $(\ker C)^\perp$, has the following block diagonal operator matrix form

$$C|_{(\ker C)^\perp} = \begin{bmatrix} I_{E_1} & 0 & 0 & 0 \\ 0 & \bigoplus\limits_{\lambda} \lambda I_{E_\lambda} & 0 & 0 \\ 0 & 0 & -I_{E_{-1}} & 0 \\ 0 & 0 & 0 & \bigoplus\limits_{\lambda} (-\lambda) I_{E_{-\lambda}} \end{bmatrix} \tag{4.1.6}$$

and consequently, the matrix representation of $C|_{(\ker C)^\perp}$, with respect to a chosen orthonormal basis of $(\ker C)^\perp$, is unitarily equivalent to the diagonal matrix given by

$$[C|_{(\ker C)^\perp}] = \begin{bmatrix} I_{l_1} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{l'_1} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}$$

where $l_1 = \dim E_1$, $l'_1 = \dim E_{-1}$, $D = \bigoplus\limits_{\lambda} \lambda I_{k_\lambda}$, $I_k$ denotes the $k \times k$ identity matrix for any positive integer $k$ and

$$k_\lambda = \dim E_\lambda = \dim E_{-\lambda}.$$

Summarising the foregoing observations, one obtains the following [69, Theorem 4.3]:

**Theorem 4.1.2.** *With the notations as above, if the defect operator $C(V_1, V_2)$ is compact, then its non-zero part is unitarily equivalent to the diagonal block matrix*

$$\begin{bmatrix} I_{l_1} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{l'_1} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix} \tag{4.1.7}$$

**Remark 4.1.1.** *(Word of caution) At this point we make it clear that throughout this article, whenever we say "let $T \in B(\mathcal{E})$ be of the form (4.1.7)", or we write*

$$\text{“}T = \begin{bmatrix} I_{l_1} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{l'_1} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix} \in B(\mathcal{E})'',$$

*we always mean that $T$ is a compact self-adjoint operator on $\mathcal{E}$ such that the orthogonal decomposition of $\mathcal{E}$ into eigen spaces of $T$ is as given by (4.1.5), so that with respect to this decomposition of $\mathcal{E}$, $T$ is represented by the block diagonal operator matrix form as given by (4.1.6) and consequently, the matrix representation of $T$ (with respect to an ordered orthonormal basis of $\mathcal{E}$) is unitarily equivalent to the diagonal matrix as given by (4.1.7).*

This chapter concerns the reverse direction of Theorem 4.1.2: Given an operator $T$ on $\mathcal{E}$ of the form (4.1.7), construct, if possible, a BCL pair $(V_1, V_2)$ such that $C|_{(\ker C)^\perp}$, the non-zero part of $C(V_1, V_2)$, is unitarily equivalent to $T$. The following definition will make the discussion more concise (in this context, see Lemma 4.2.1).

**Definition 4.1.3.** *A BCL pair $(V_1, V_2)$ corresponding to the BCL triple $(\mathcal{E}, U, P)$ is said to be irreducible if there is no non-trivial joint reducing subspace of $U$ and $P$.*

Now we note that in view of the constructions of simple blocks in [69, Section 6], one can always construct a reducible BCL pair $(V_1, V_2)$ such that the non-zero part of $C(V_1, V_2)$ is equal to $T$ (see [69, Theorem 6.7]). This consideration leads us to raise the following natural question:

*Question* 1. *Given a compact block operator $T \in B(\mathcal{E})$ of the form (4.1.7), does there exist an irreducible BCL pair $(V_1, V_2)$ on the Hilbert space $H^2_{\mathcal{E}}(\mathbb{D})$ such that the non-zero part of the defect operator $C(V_1, V_2)$ is equal to $T$ (that is, $\overline{ran\ C(V_1, V_2)} = \mathcal{E}$ and $C(V_1, V_2)|_{\mathcal{E}} = T$)?*

The above question also has been framed in [69, page 18]. The purpose of this paper is to shed some light on this question through some concrete constructions of BCL pairs.

We observe in Section 4.2 that the answer to the above question is not necessarily always in the affirmative. In fact we show in Theorem 4.2.4 that given an operator $T$ on a finite-dimensional Hilbert space $\mathcal{E}$ of the form (4.1.7) with

$$\dim E_1(T) \neq \dim E_{-1}(T),$$

it is not possible to find any (reducible or irreducible) BCL pair on $H^2_{\mathcal{E}}(\mathbb{D})$ with the desired properties. This result motivated us to investigate the cases where the answer to the aforementioned question, *Question* 1, is in the affirmative. Our first result to this end is Theorem 4.3.2 in Section 4.3: Let $\mathcal{E}$ be a finite-dimensional Hilbert space, $T \in B(\mathcal{E})$ is of the form (4.1.7), and let

$$\dim E_1(T) = \dim E_{-1}(T).$$

If $T$ has either at least two distinct positive eigen values or only one positive eigen value lying in $(0, 1)$ with dimension of the corresponding eigen space being at least two, then it is always possible to construct such an irreducible BCL pair. On the other hand, if 1 is the only positive eigen value of $T$, then it is not possible to construct such an irreducible pair $(V_1, V_2)$ unless $\dim E_1(T) = 1$.

Finally, in Section 4.9 we deal with the case when $\mathcal{E}$ is infinite-dimensional. Our main results of this section are Theorem 4.9.1 and Theorem 4.9.2. In Theorem 4.9.1 we answer the *Question* 1 above in the affirmative in the case when

$$\dim E_1(T) = \dim E_{-1}(T),$$

whereas Theorem 4.9.2 provides an affirmative answer to the *Question* 1 in the case when

$$\dim E_1(T) = \dim E_{-1}(T) \pm 1.$$

What deserves special attention is that Theorem 4.9.2 points out a crucial difference between the finite and infinite-dimensional cases: If $T \in B(\mathcal{E})$ is of the form (4.1.7), then the equality $\dim E_1(T) = \dim E_{-1}(T)$ is a necessary condition for the existence of an irreducible BCL pair $(V_1, V_2)$ such that the non-zero part of $C(V_1, V_2)$ is given by $T$, only when $\mathcal{E}$ is finite-dimensional.

This chapter is based on the preprint [39].

## 4.2 Question 1 is not affirmative

We begin by characterising joint reducing subspaces of BCL pairs.

**Lemma 4.2.1.** *Let $(V_1, V_2)$ be a BCL pair corresponding to the BCL triple $(\mathcal{E}, U, P)$ and let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{E}}(\mathbb{D})$. Then $\mathcal{S}$ is a joint reducing subspace for $(V_1, V_2)$ if and only if there exists a closed subspace $\tilde{\mathcal{E}}$ of $\mathcal{E}$ such that $\tilde{\mathcal{E}}$ is reducing for both $U$ and $P$ and $\mathcal{S} = H^2_{\tilde{\mathcal{E}}}(\mathbb{D})$.*

*Proof.* Let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ that is reducing for both $V_1$ and $V_2$. Then $\mathcal{S}$ is reducing for $M_z$, and hence, there exists a closed subspace $\tilde{\mathcal{E}}$ of $\mathcal{E}$ such that $S = H^2_{\tilde{\mathcal{E}}}(\mathbb{D})$. Thus, it just remains to show that $\tilde{\mathcal{E}}$ is reducing for both $U$ and $P$. Given $\eta \in \tilde{\mathcal{E}}$, it follows from the definitions of $V_1$ and $V_2$ as given by (4.1.3) that

$$V_1\eta = PU^*\eta + (P^{\perp}U^*\eta)z \quad \text{and} \quad V_2\eta = UP^{\perp}\eta + (UP\eta)z.$$

As $S = H^2_{\tilde{\mathcal{E}}}(\mathbb{D})$ is invariant under $V_1$ and $V_2$, we must have that

$$PU^*\eta, P^{\perp}U^*\eta, UP^{\perp}\eta, UP\eta \in \tilde{\mathcal{E}}.$$

Now $PU^*\eta \in \tilde{\mathcal{E}}$ and $P^{\perp}U^*\eta \in \tilde{\mathcal{E}}$ together imply that

$$U^*(\eta) = PU^*\eta + P^{\perp}U^*\eta \in \tilde{\mathcal{E}},$$

so that $\tilde{\mathcal{E}}$ invariant under $U^*$. Similarly, $UP^{\perp}\eta \in \tilde{\mathcal{E}}$ and $UP\eta \in \tilde{\mathcal{E}}$ together imply that $\tilde{\mathcal{E}}$ invariant under $U$, showing that $\tilde{\mathcal{E}}$ is reducing for $U$. Since $PU^*$ and $UP$ leave $\tilde{\mathcal{E}}$ invariant, $P(= (PU^*)(UP))$ leaves $\tilde{\mathcal{E}}$ invariant. Thus $\tilde{\mathcal{E}}$ is reducing for $P$ also, completing the proof. $\square$

Now we set one of the key tools on pairs of projections for our consideration. In [102] the authors analysed self-adjoint contractions on Hilbert spaces which are difference of pairs of projections. Let $A \in B(\mathcal{H})$ be a self-adjoint contraction. Then $\ker A$, $\ker(A - I)$

and $\ker(A+I)$ are reducing subspaces of $A$ and hence, $\mathcal{H}$ admits the following direct sum decomposition:

$$\mathcal{H} = \ker A \oplus \ker(A-I) \oplus \ker(A+I) \oplus \mathcal{H}_0.$$

Recall that, if $\ker A = \ker(A-I) = \ker(A+I) = \{0\}$, then $A$ is said to be in the *generic position* (see Halmos [64]). Now assume that

$$\mathcal{H}_0 = \mathcal{K} \oplus \mathcal{K},$$

for some Hilbert space $\mathcal{K}$ and suppose that with respect to the orthogonal decomposition

$$\mathcal{H} = \ker A \oplus \ker(A-I) \oplus \ker(A+I) \oplus \mathcal{K} \oplus \mathcal{K},$$

the operator $A$ has the following block diagonal form

$$A = \begin{bmatrix} 0 & & & & \\ & I & & & \\ & & -I & & \\ & & & D & \\ & & & & -D \end{bmatrix} \tag{4.2.1}$$

where $D \in B(\mathcal{K})$ is a positive contraction and without any confusion, we denote by $I$ the identity on any Hilbert space. In [102, Theorem 3.2] the authors proved that:

**Theorem 4.2.2.** *With notations as above, $A$, as given by (4.2.1), is a difference of two projections and moreover, if $(P,Q)$ is a pair of projections such that $A = P - Q$, then $P, Q$ must be of the form*

$$P = E \oplus I \oplus 0 \oplus P_U \quad \text{and} \quad Q = E \oplus 0 \oplus I \oplus Q_U$$

*where $E$ is a projection on $\ker A$ and $P_U$ and $Q_U$ are projections in $B(\mathcal{K} \oplus \mathcal{K})$ of the form*

$$P_U = \frac{1}{2}\begin{bmatrix} I+D & U(I-D^2)^{\frac{1}{2}} \\ U^*(I-D^2)^{\frac{1}{2}} & I-D \end{bmatrix}$$

*and*

$$Q_U = \frac{1}{2}\begin{bmatrix} I-D & U(I-D^2)^{\frac{1}{2}} \\ U^*(I-D^2)^{\frac{1}{2}} & I+D \end{bmatrix}$$

*where $U \in B(\mathcal{K})$ is a unitary commuting with $D$.*

In what follows, in the setting of the above theorem, we will be interested in the case when $\ker A = \{0\}$. Hence, the projections in the above theorem will be of the form $P = I \oplus 0 \oplus P_U$ and $Q = 0 \oplus I \oplus Q_U$. Moreover, with notations as above, we note that if $D \in B(\mathcal{K})$ is a positive scalar contraction, that is, $D = \lambda I_{\mathcal{K}}$ for some $\lambda$ in $(0,1)$, then

$P_U$ takes the form

$$P_U = \begin{bmatrix} \frac{1+\lambda}{2} I_{\mathcal{K}} & \frac{\sqrt{1-\lambda^2}}{2} U \\ \frac{\sqrt{1-\lambda^2}}{2} U^* & \frac{1-\lambda}{2} I_{\mathcal{K}} \end{bmatrix}. \tag{4.2.2}$$

Projections of this form will play a crucial role in the forthcoming considerations. Our next lemma determines an orthonormal basis of the range of projections of slightly more general type.

**Lemma 4.2.3.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $U : \mathcal{H} \to \mathcal{K}$ be a unitary operator. For each $\lambda \in (0,1)$, define the projection $P : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ by*

$$P = \begin{bmatrix} \frac{1+\lambda}{2} I_{\mathcal{H}} & \frac{\sqrt{1-\lambda^2}}{2} U^* \\ \frac{\sqrt{1-\lambda^2}}{2} U & \frac{1-\lambda}{2} I_{\mathcal{K}} \end{bmatrix}.$$

*If $\{e_i : i \in \Lambda\}$ is an orthonormal basis of $\mathcal{H}$, then*

$$\Big\{ \sqrt{\frac{1+\lambda}{2}} e_i \oplus \sqrt{\frac{1-\lambda}{2}} U e_i : i \in \Lambda \Big\}$$

*is an orthonormal basis of $ranP$.*

*Proof.* Note that if $x \in \mathcal{H}$ and $y \in \mathcal{K}$, then

$$P(x \oplus 0) = \frac{1+\lambda}{2} x \oplus \frac{\sqrt{1-\lambda^2}}{2} U x = P\Big(0 \oplus \sqrt{\frac{1+\lambda}{1-\lambda}} U x\Big),$$

and hence, by duality

$$P(0 \oplus y) = \frac{\sqrt{1-\lambda^2}}{2} U^* y \oplus \frac{1-\lambda}{2} y = P\Big(\sqrt{\frac{1-\lambda}{1+\lambda}} U^* y \oplus 0\Big).$$

Therefore

$$\mathrm{ran} P = \{P(x \oplus 0) : x \in \mathcal{H}\} = \{P(0 \oplus y) : y \in \mathcal{K}\}. \tag{4.2.3}$$

If $\{e_i : i \in \Lambda\}$ is an orthonormal basis of $\mathcal{H}$, then

$$\|P(e_i \oplus 0)\| = \sqrt{\frac{1+\lambda}{2}},$$

for all $i \in \Lambda$. A straightforward computation then shows that

$$\Big\{ \sqrt{\frac{1+\lambda}{2}} e_i \oplus \sqrt{\frac{1-\lambda}{2}} U e_i : i \in \Lambda \Big\},$$

is an orthonormal basis of $ranP$. □

With this terminology and notation in hand, we are now ready to state the main result of this section, which shows that the answer to the *Question* 1 is not necessarily always in the affirmative.

**Theorem 4.2.4.** *Let $\mathcal{E}$ be a finite-dimensional Hilbert space and let $T$ on $\mathcal{E}$ be a compact block matrix of the form (4.1.7), that is,*

$$T = \begin{bmatrix} I_{dim\, E_1(T)} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{dim\, E_{-1}(T)} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}. \tag{4.2.4}$$

*If*

$$dim\, E_1(T) \neq dim\, E_{-1}(T),$$

*then it is not possible to find a BCL pair $(V_1, V_2)$ on $H^2_{\mathcal{E}}(\mathbb{D})$ such that the non-zero part of the defect operator $C(V_1, V_2)$ is equal to $T$.*

*Proof.* Suppose that there exists a BCL triple $(\mathcal{E}, U, P)$ such that the non-zero part of the defect operator $C = C(V_1, V_2)$ of the corresponding BCL pair $(V_1, V_2)$ is equal to $T \in B(\mathcal{E})$, where $T$ is as in (4.2.4). That is,

$$ran\, C = \mathcal{E}, \text{ and } C|_{\mathcal{E}} = T.$$

Then, since $C|_{\mathcal{E}} = P^{\perp} - UP^{\perp}U^*$, it follows that

$$T = P^{\perp} - UP^{\perp}U^*.$$

Let $\Lambda = \{\lambda_i : 1 \leq i \leq m\}$ denote the (possibly empty) set of eigen values of $T$ lying in $(0, 1)$. Now for each $i = 1, \ldots, m$, choose a unitary $V_i : E_{-\lambda_i}(T) \to E_{\lambda_i}(T)$ and combine these to construct a unitary

$$U := \overset{m}{\underset{i=1}{\oplus}} V_i : \Big( \overset{m}{\underset{i=1}{\oplus}} E_{-\lambda_i}(T) \Big) \to \Big( \overset{m}{\underset{i=1}{\oplus}} E_{\lambda_i}(T) \Big).$$

Also note that

$$\mathcal{E} = E_1(T) \oplus \Big( \overset{m}{\underset{i=1}{\oplus}} E_{\lambda_i}(T) \Big) \oplus E_{-1}(T) \oplus \Big( \overset{m}{\underset{i=1}{\oplus}} E_{-\lambda_i}(T) \Big).$$

Then, if we set

$$\tilde{\mathcal{E}} := E_1(T) \oplus E_{-1}(T) \oplus \mathcal{K} \oplus \mathcal{K},$$

where

$$\mathcal{K} = \overset{m}{\underset{i=1}{\oplus}} E_{\lambda_i}(T),$$

we obtain a unitary $W : \mathcal{E} \to \tilde{\mathcal{E}}$ defined by

$$W = \begin{bmatrix} I_{E_1(T)} & 0 & 0 & 0 \\ 0 & 0 & I_{E_{-1}(T)} & 0 \\ 0 & I_{\mathcal{K}} & 0 & 0 \\ 0 & 0 & 0 & U \end{bmatrix}.$$

Next, we set $\tilde{T} := WTW^*$, $P_1 = WP^{\perp}W^*$ and $P_2 = W(UP^{\perp}U^*)W^*$. A simple computation shows that

$$\tilde{T} = \operatorname{diag}\left[ I_{E_1(T)} \quad -I_{E_{-1}(T)} \quad \bigoplus_{i=1}^{m} \lambda_i I_{E_{\lambda_i}(T)} \quad -\Big(\bigoplus_{i=1}^{m} \lambda_i I_{E_{\lambda_i}(T)}\Big) \right].$$

Moreover, $P_1$ and $P_2$ are projections on $\tilde{\mathcal{E}}$ and

$$P_1 - P_2 = W(P^{\perp} - UP^{\perp}U^*)W^* = WTW^* = \tilde{T}.$$

Now an appeal to Theorem 4.2.2 shows that there is a unitary $V$ on $\mathcal{K}$ commuting with $\bigoplus_{i=1}^{m} \lambda_i I_{E_{\lambda_i}(T)}$ such that

$$P_1 = I_{E_1(T)} \oplus 0_{E_{-1}(T)} \oplus P_V, \;\; P_2 = 0_{E_1(T)} \oplus I_{E_{-1}(T)} \oplus Q_V$$

where the projections $P_V$ and $Q_V$ are given by

$$P_V = \begin{bmatrix} \bigoplus_{i=1}^{m} \Big(\frac{1+\lambda_i}{2} I_{E_{\lambda_i}(T)}\Big) & V\Big[\bigoplus_{i=1}^{m} \Big(\frac{(1-\lambda_i^2)^{\frac{1}{2}}}{2} I_{E_{\lambda_i}(T)}\Big)\Big] \\ V^*\Big[\bigoplus_{i=1}^{m} \Big(\frac{(1-\lambda_i^2)^{\frac{1}{2}}}{2} I_{E_{\lambda_i}(T)}\Big)\Big] & \bigoplus_{i=1}^{m} \Big(\frac{1-\lambda_i}{2} I_{E_{\lambda_i}(T)}\Big) \end{bmatrix}$$

and

$$Q_V = \begin{bmatrix} \bigoplus_{i=1}^{m} \Big(\frac{1-\lambda_i}{2} I_{E_{\lambda_i}(T)}\Big) & V\Big[\bigoplus_{i=1}^{m} \Big(\frac{(1-\lambda_i^2)^{\frac{1}{2}}}{2} I_{E_{\lambda_i}(T)}\Big)\Big] \\ V^*\Big[\bigoplus_{i=1}^{m} \Big(\frac{(1-\lambda_i^2)^{\frac{1}{2}}}{2} I_{E_{\lambda_i}(T)}\Big)\Big] & \bigoplus_{i=1}^{m} \Big(\frac{1+\lambda_i}{2} I_{E_{\lambda_i}(T)}\Big) \end{bmatrix}.$$

We claim that $P_V$ and $Q_V$ have the same rank. Indeed, a similar calculation, as in (4.2.3), shows that

$$ranP_V = \{P_V(x \oplus 0) : x \in \mathcal{K}\} \quad \text{and} \quad ran\,Q_V = \{Q_V(0 \oplus x) : x \in \mathcal{K}\}.$$

On the other hand, we can verify easily

$$ranP_V \ni P_V(x \oplus 0) \mapsto Q_V(0 \oplus x) \in ranQ_V,$$

is a linear isomorphism and hence, ranks of $P_V$ and $Q_V$ are the same. Note that

$$\text{rank}P_1 = \text{rank}P^{\perp} = \text{rank}(UP^{\perp}U^*) = \text{rank}P_2.$$

Now since

$$\text{rank}P_1 = \text{dim}E_1(T) + \text{rank}P_V,$$

and

$$\text{rank}P_2 = \text{dim}E_{-1}(T) + \text{rank}Q_V,$$

we must have that $\text{dim}E_1(T) = \dim E_{-1}(T)$. Hence the proof follows. □

## 4.3 $\mathcal{E}$ is finite-dimensional

In this section we deal with *Question* 1 and the case when $\mathcal{E}$ is finite-dimensional. Note that, in view of Lemma 4.2.4, it is natural to ask that in case $\text{dim}E_1(T) = \text{dim}E_{-1}(T)$, whether it is always possible to construct an irreducible BCL pair $(V_1, V_2)$ such that the non-zero part of the defect operator of $C(V_1, V_2)$ is exactly $T$. Theorem 4.3.2, the main result of this section, settles the *Question* 1 completely.

We first introduce (following Shields [103]) the notion of *weighted shift* type operators. Let $\mathcal{H}$ be a Hilbert space (finite or infinite-dimensional). If $\mathcal{H}$ is finite-dimensional, say $dim\ \mathcal{H} = n$, we let $\{e_i : 1 \leq i \leq n\}$ be an orthonormal basis of $\mathcal{H}$ and if $\mathcal{H}$ is infinite-dimensional, we let $\{e_i : i \in \mathbb{Z}\}$ be an orthonormal basis of $\mathcal{H}$. Let $S$ be a bounded linear operator on $\mathcal{H}$ defined by

$$Se_i = \lambda_i e_{i+1} \qquad (i \in \mathbb{Z}),$$

if $\mathcal{H}$ is infinite-dimensional, and

$$Se_i = \begin{cases} \lambda_i e_{i+1} & \text{if } 1 \leq i < n \\ \lambda_n e_1 & \text{if } i = n, \end{cases}$$

in case $\mathcal{H}$ is finite-dimensional, where all the $\lambda_i$'s are non-zero complex numbers. We call such operators (or matrices of such operators) as operators (respectively, matrices) of *weighted shift* type. If $\mathcal{H}$ is finite-dimensional, note that the matrix of $S$ with respect to the orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ is a generalised permutation matrix (that is, a square matrix whose each row and each column has only one non-zero element) whose only non-zero elements are the subdiagonal entries and the first entry of the last column, that is the $(1, n)$-th entry.

**Lemma 4.3.1.** *With the notations as above, for any $i \in \{1, 2, \cdots, n\}$, $\{e_i\}$ is a cyclic vector for $S$ if $\mathcal{H}$ is finite-dimensional and if $\mathcal{H}$ is infinite-dimensional, for any $i \in \mathbb{Z}$, $e_i$ is a star-cyclic vector for $S$ (that is, the linear span of $\{S^n e_i, S^{*n} e_i : n \geq 0\}$ is dense in $\mathcal{H}$).*

*Proof.* It is easy to check directly that

$$S^n = (\prod_{j=1}^{n} \lambda_j) I_{\mathcal{H}},$$

if $\mathcal{H}$ is finite-dimensional, and

$$SS^*e_j = |\lambda_{j-1}|^2 e_j \qquad (j \in \mathbb{Z}),$$

if $\mathcal{H}$ is infinite-dimensional. Clearly this yields the desired result. □

After these preparations we are ready to state and prove the main result of this section. Before we proceed to state the theorem, it is necessary to point out at this moment that if $\mathcal{E} = \mathbb{C}^2$, the two-dimensional complex space and if $T \in B(\mathcal{E})$ is of the form (4.1.7) such that $T$ has two eigen values $\lambda$ and $-\lambda$ where $0 < \lambda \leq 1$, then [69, Example 6.6] constructs an irreducible BCL pair $(V_1, V_2)$ on $H^2_{\mathcal{E}}(\mathbb{D})$ such that the non-zero part of $C(V_1, V_2)$ is given by $T$, thus answering the *Question* 1 in the affirmative in this case. The following theorem analyses all the remaining cases, thus settling the *Question* 1 completely in the finite-dimensional case.

**Theorem 4.3.2.** *Let $\mathcal{E}$ be a finite-dimensional Hilbert space, and let $T \in B(\mathcal{E})$ be of the form (4.1.7), that is,*

$$T = \begin{bmatrix} I_{dim\,E_1(T)} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{dim\,E_{-1}(T)} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}.$$

*Assume that $dim E_1(T) = dim E_{-1}(T)$. Then, in each of the following two cases, there exists an irreducible BCL pair $(V_1, V_2)$ on $H^2_{\mathcal{E}}(\mathbb{D})$ such that the non-zero part of the defect operator $C(V_1, V_2)$ is given by $T$.*

*(i) $T$ has at least two distinct positive eigen values,*

*(ii) $T$ has only one positive eigen value lying in $(0, 1)$ with dimension of the corresponding eigen space being at least two.*

*Moreover, (iii) if $1$ is the only positive eigen value of $T$, then it is not possible to construct such an irreducible pair $(V_1, V_2)$ unless $dim E_1(T) = 1$.*

The proof is divided in several steps (subsections) to detach the independent ideas and constructions. Some of the constructions of these steps are also of independent interest. We note that the above result also includes the case where $\dim E_1(T) = 0$.

First, we note that in order to construct an irreducible BCL pair $(V_1, V_2)$ on $H^2_{\mathcal{E}}$ such that $V_1V_2 = M_z$ and the non-zero part of the defect operator $C(V_1, V_2)$ is given by $T$, it suffices, by an appeal to Lemma 4.2.1 and the discussion preceding Lemma 4.1.1, to construct a unitary $U \in B(\mathcal{E})$ and a projection $P \in B(\mathcal{E})$ such that $U$ and $P$ do not have any common non-trivial reducing subspace and $P^\perp - UP^\perp U^* = T$.

Let $\{\lambda_i : i \in \Lambda\}$ denote the set of positive eigen values of $T$, where $\Lambda$ is a finite indexing set, say, $\Lambda = \{1, 2, \cdots, n\}$ with $n \in \mathbb{N}$. Then the set of eigen values of $T$ is given by

$$\sigma(T) = \{\pm\lambda_i : i \in \Lambda\}.$$

## 4.4 Orthonormal bases and the projection $P$

For each $i \in \Lambda$, let

$$k_i := \dim E_{\lambda_i}(T) = \dim E_{-\lambda_i}(T),$$

and let $U_i$ be a unitary from $E_{\lambda_i}(T)$ to $E_{-\lambda_i}(T)$ which exists since $\dim E_{\lambda_i}(T) = \dim E_{-\lambda_i}(T)$. Let $\{e_t^i : 1 \le t \le k_i\}$ be an orthonormal basis of $E_{\lambda_i}(T)$, $i \in \Lambda$. Then

$$\{U_i e_t^i : t = 1, \dots, k_i\},$$

is an orthonormal basis of $E_{-\lambda_i}(T)$, $i \in \Lambda$. It is evident that $\mathcal{E}$ has the following orthogonal decomposition

$$\mathcal{E} = \bigoplus_{i \in \Lambda} \Big( E_{\lambda_i}(T) \oplus E_{-\lambda_i}(T) \Big).$$

For each $i \in \Lambda$, define the projection $Q_i \in B(E_{\lambda_i}(T) \oplus E_{-\lambda_i}(T))$ by

$$Q_i = \begin{bmatrix} \frac{1+\lambda_i}{2} I_{E_{\lambda_i}(T)} & \frac{\sqrt{1-\lambda_i^2}}{2} U_i^* \\ \frac{\sqrt{1-\lambda_i^2}}{2} U_i & \frac{1-\lambda_i}{2} I_{E_{-\lambda_i}(T)} \end{bmatrix}.$$

It follows from Lemma 4.2.3 that $\{f_t^i : t = 1, \dots, k_i\}$ is an orthonormal basis of $ran\, Q_i$, where

$$f_t^i := \sqrt{\frac{1+\lambda_i}{2}} e_t^i \oplus \sqrt{\frac{1-\lambda_i}{2}} U_i e_t^i,$$

for all $t = 1, \dots, k_i$. Similarly, Lemma 4.2.3 applied to $I - Q_i$ yields an orthonormal basis $\{\tilde{f}_t^i : t = 1, \dots, k_i\}$ of $ran\, Q_i^{\perp}$, where

$$\tilde{f}_t^i := \sqrt{\frac{1-\lambda_i}{2}} e_t^i \oplus \Big( -\sqrt{\frac{1+\lambda_i}{2}} \Big) U_i e_t^i,$$

for all $t = 1, \dots, k_i$. Consider the projection $Q \in B(\mathcal{E})$ given by

$$Q = \bigoplus_{i \in \Lambda} Q_i$$

and set

$$P = Q^{\perp} \in B(\mathcal{E}).$$

Therefore, from the definition of $P$, it follows that

$$\bigcup_{i\in\Lambda}\{\tilde{f}_t^i : t=1,\dots,k_i\} \quad \text{and} \quad \bigcup_{i\in\Lambda}\{f_t^i : t=1,\dots,k_i\}, \tag{4.4.1}$$

are orthonormal bases of $ranP$ and $ranP^{\perp}$, respectively. Then, clearly

$$\{f_t^i, \tilde{f}_t^i : t=1,\dots,k_i\},$$

is an orthonormal basis of $E_{\lambda_i}(T)\oplus E_{-\lambda_i}(T)$, and hence, a simple computation, by changing $\lambda_i$ to $-\lambda_i$, shows that

$$\Big\{\sqrt{\frac{1-\lambda_i}{2}}\; e_t^i \oplus \sqrt{\frac{1+\lambda_i}{2}}\; U_i e_t^i,\; \sqrt{\frac{1+\lambda_i}{2}}\; e_t^i \oplus \Big(-\sqrt{\frac{1-\lambda_i}{2}}\Big)\; U_i e_t^i : t=1,\dots,k_i\Big\},$$

is also an orthonormal basis of $E_{\lambda_i}(T)\oplus E_{-\lambda_i}(T), i\in\Lambda$. Since

$$\sqrt{\frac{1-\lambda_i}{2}}\; e_t^i + \sqrt{\frac{1+\lambda_i}{2}}\; U_i e_t^i \; = \sqrt{1-\lambda_i^2} f_t^i - \lambda_i \tilde{f}_t^i,$$

and

$$\sqrt{\frac{1+\lambda_i}{2}}\; e_t^i - \sqrt{\frac{1-\lambda_i}{2}}\; U_i e_t^i \; = \lambda_i f_t^i + \sqrt{1-\lambda_i^2}\tilde{f}_t^i,$$

for all $i$ and $t$, it follows that

$$\bigcup_{i\in\Lambda}\{\sqrt{1-\lambda_i^2} f_t^i \oplus \Big(-\lambda_i\Big)\tilde{f}_t^i,\; \lambda_i f_t^i \oplus \sqrt{1-\lambda_i^2}\tilde{f}_t^i : t=1,\dots,k_i\}, \tag{4.4.2}$$

is an orthonormal basis of $\mathcal{E}$.

*In summary, the sets in* (4.4.1) *are orthonormal bases of* $ranP$ *and* $ranP^{\perp}$, *respectively and the set in* (4.4.2) *is that of* $\mathcal{E}$.

## 4.5 The unitary $U$ for part (i)

We now proceed to construct the unitary $U \in B(\mathcal{E})$ of the BCL triple $(\mathcal{E}, U, P)$. Here we assume that $n \geq 2$, that is $T$ has at least two positive eigen values. In this case, we construct $U$ on $\mathcal{E}$ as follows:

Define $U$ on $ranP^{\perp}$ by

$$U f_t^i = \sqrt{1-\lambda_i^2} f_t^i \oplus (-\lambda_i)\tilde{f}_t^i,$$

for all $t = 1, \dots, k_i$ and $i = 1, \dots, n$, and define $U$ on $ran\, P$ by

$$U\tilde{f}_t^i = \begin{cases} \lambda_i f_{t+1}^i \oplus \left(\sqrt{1-\lambda_i^2}\right)\tilde{f}_{t+1}^i & \text{if } 1 \le t < k_i \text{ and } 1 \le i \le n, \\ \lambda_{i+1} f_1^{i+1} \oplus \left(\sqrt{1-\lambda_{i+1}^2}\right)\tilde{f}_1^{i+1} & \text{if } t = k_i \text{ and } 1 \le i < n, \\ \\ \lambda_1 f_1^1 \oplus \left(\sqrt{1-\lambda_1^2}\right)\tilde{f}_1^1 & \text{if } t = k_n \text{ and } i = n. \end{cases}$$

The fact that $U$ is unitary can easily be deduced from the definition of $U$ itself by observing that $U$ carries an orthonormal basis of $\mathcal{E}$ to an orthonormal basis of $\mathcal{E}$. With respect to the decomposition $\mathcal{E} = ranP^\perp \oplus ranP$, let

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}.$$

Then, with respect to the ordered orthonormal bases $\bigcup_{i=1}^n \{f_t^i : 1 \le t \le k_i\}$ of $ranP^\perp$ and $\bigcup_{i=1}^n \{\tilde{f}_t^i : 1 \le t \le k_i\}$ of $ranP$, a simple computation yields the following:

- $U_{11} : ranP^\perp \to ranP^\perp$ is represented by the diagonal matrix

$$\operatorname{diag}\Big(\underbrace{\sqrt{1-\lambda_1^2}, \dots, \sqrt{1-\lambda_1^2}}_{k_1 \text{ times}}, \underbrace{\sqrt{1-\lambda_2^2}, \cdots, \sqrt{1-\lambda_2^2}}_{k_2 \text{ times}}, \dots, \underbrace{\sqrt{1-\lambda_n^2}, \dots, \sqrt{1-\lambda_n^2}}_{k_n \text{ times}}\Big).$$

- $U_{21} : ranP^\perp \to ranP$ is represented by the invertible diagonal matrix

$$\operatorname{diag}\Big(\underbrace{-\lambda_1, \dots, -\lambda_1}_{k_1 \text{ times}}, \underbrace{-\lambda_2, \dots, -\lambda_2}_{k_2 \text{ times}}, \underbrace{-\lambda_n, \dots, -\lambda_n}_{k_n \text{ times}}\Big).$$

- Both $U_{12} : ranP \to ranP^\perp$ and $U_{22} : ranP \to ranP$ are represented by matrices of *weighted shift* type (whose only non-zero elements are the subdiagonal entries and the first entry of the last column, that is, the $(1, \dim(ranP))$-th entry). One can easily verify that the $(1, \dim(ranP))$-th entry of $U_{12}$ equals $\lambda_1$ and the subdiagonal of $U_{12}$ is given by

$$\underbrace{\lambda_1, \dots, \lambda_1}_{k_1 - 1 \text{ times}}, \underbrace{\lambda_2, \dots, \lambda_2}_{k_2 \text{ times}}, \underbrace{\lambda_n, \dots, \lambda_n}_{k_n \text{ times}},$$

  whereas the $(1, \dim(ran\, P))$-th entry of $U_{22}$ equals $\sqrt{1-\lambda_1^2}$ and the subdiagonal of $U_{22}$ is given by

$$\underbrace{\sqrt{1-\lambda_1^2}, \cdots, \sqrt{1-\lambda_1^2}}_{k_1 - 1 \text{ times}}, \underbrace{\sqrt{1-\lambda_2^2}, \cdots, \sqrt{1-\lambda_2^2}}_{k_2 \text{ times}}, \cdots, \underbrace{\sqrt{1-\lambda_n^2}, \cdots, \sqrt{1-\lambda_n^2}}_{k_n \text{ times}}.$$

## 4.6 The remaining details of part (i)

We first verify that $P^\perp - UP^\perp U^* = T$. With respect to the decomposition $\mathcal{E} = ranP^\perp \oplus ranP$, let

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$

Note that verification of the fact $P^\perp - UP^\perp U^* = T$ amounts to verifying the following set of equations

$$\begin{cases} T_{11} = I_{ranP^\perp} - U_{11}U_{11}^*, \\ T_{12} = -U_{11}U_{21}^*, \\ T_{21} = -U_{21}U_{11}^*, \\ T_{22} = -U_{21}U_{21}^*. \end{cases} \tag{4.6.1}$$

Indeed, a simple computation shows that for each $i$, $1 \le i \le n$,

$$Tf_t^i = \lambda_i^2 f_t^i + \lambda_i\sqrt{1-\lambda_i^2}\,\tilde{f}_t^i,$$

and

$$T\tilde{f}_t^i = \lambda_i\sqrt{1-\lambda_i^2}\,f_t^i - \lambda_i^2\tilde{f}_t^i \text{ for } 1 \le t \le k_i,$$

from which it is now evident that with respect to the ordered orthonormal bases $\bigcup_{i=1}^{n}\{f_t^i : 1 \le t \le k_i\}$ of $ranP^\perp$ and $\bigcup_{i=1}^{n}\{\tilde{f}_t^i : 1 \le t \le k_i\}$ of $ranP$, all the operators $T_{ij}$, $i,j = 1,2$, are represented by diagonal matrices. In fact, we have the following equalities

$$T_{11} = \text{diag}\Big(\underbrace{\lambda_1^2,\dots,\lambda_1^2}_{k_1 \text{ times}},\underbrace{\lambda_2^2,\dots,\lambda_2^2}_{k_2 \text{ times}},\dots,\underbrace{\lambda_n^2,\cdots,\lambda_n^2}_{k_n \text{ times}}\Big),$$

and

$$T_{12} = T_{21} = \text{diag}\Big(\underbrace{\lambda_1\sqrt{1-\lambda_1^2},\dots,\lambda_1\sqrt{1-\lambda_1^2}}_{k_1 \text{ times}},\dots,\underbrace{\lambda_n\sqrt{1-\lambda_n^2},\cdots,\lambda_n\sqrt{1-\lambda_n^2}}_{k_n \text{ times}}\Big),$$

and finally,

$$T_{22} = \text{diag}\Big(\underbrace{-\lambda_1^2,\dots,-\lambda_1^2}_{k_1 \text{ times}},\underbrace{-\lambda_2^2,\dots,-\lambda_2^2}_{k_2 \text{ times}},\dots,\underbrace{-\lambda_n^2,\cdots,-\lambda_n^2}_{k_n \text{ times}}\Big).$$

One can now easily verify the equations of (4.6.1), proving that $P^\perp - UP^\perp U^* = T$.

We now show that the BCL pair $(V_1, V_2)$ corresponding to the BCL triple $(\mathcal{E}, U, P)$ is irreducible, that is, we prove that there is no non-trivial joint $(U,P)$-reducing subspace of $\mathcal{E}$. Let $\mathcal{S}$ be a non-zero joint $(U,P)$-reducing subspace of $\mathcal{E}$. We show that $\mathcal{S} = \mathcal{E}$.

First notice that

$$\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2,$$

where $\mathcal{S}_1 = P^{\perp}\mathcal{S}$ and $\mathcal{S}_2 = P\mathcal{S}$. Since $\mathcal{S}$ is non-zero, one of the spaces $\mathcal{S}_1$, $\mathcal{S}_2$ must be non-zero. Assume that $\mathcal{S}_1$ is non-zero. We assert that in order to prove that $\mathcal{S} = \mathcal{E}$, it suffices to show that $\mathcal{S}_1 = ranP^{\perp}$ for if $\mathcal{S}_1 = ranP^{\perp}$, then the observation that

$$U_{21}(\mathcal{S}_1) \subseteq \mathcal{S}_2,$$

and the fact that $U_{21}$ is a linear isomorphism of $ranP^{\perp}$ onto $ran\,P$ together imply that

$$U_{21}(\mathcal{S}_1) = U_{21}(ranP^{\perp}) = ranP \subseteq \mathcal{S}_2,$$

so that $\mathcal{S}_2 = ranP$ and consequently, $\mathcal{S} = \mathcal{E}$. It follows easily from the definitions of the operators $U_{12}$ and $U_{21}$ that the operator $U_{12}U_{21}$ is indeed an operator of *weighted shift* type on $ranP^{\perp}$ (with respect to the ordered orthonormal basis $\bigcup_{i=1}^{n}\{f_t^i : t = 1, \ldots, k_i\}$ of $ran\,P^{\perp}$). As $U_{12}U_{21}$ leaves $\mathcal{S}_1$ invariant, in order to prove that $\mathcal{S}_1 = ran\,P^{\perp}$, it suffices to prove, by virtue of Lemma 4.3.1, that some $f_t^i$ belongs to $\mathcal{S}_1$.

The fact that $\mathcal{S}$ is invariant under $U$ immediately implies that $\mathcal{S}_1$ is invariant under $U_{11}$. Since $U_{11}$ is a diagonalizable operator on $ranP^{\perp}$ with eigen values $\{\sqrt{1-\lambda_i^2} : 1 \leq i \leq n\}$, we have

$$ranP^{\perp} = \bigoplus_{i=1}^{n}\Big(E_{\sqrt{1-\lambda_i^2}}(U_{11})\Big),$$

and we also observe that $\{f_t^i : t = 1, \ldots, k_i\}$ is a basis of $E_{\sqrt{1-\lambda_i^2}}(U_{11})$. Let $x \in \mathcal{S}_1$ be a non-zero element. Then

$$x = \sum_{i=1}^{n} x_i$$

with $x_i \in E_{\sqrt{1-\lambda_i^2}}(U_{11})$. Now the fact that $\mathcal{S}_1$ is invariant under $U_{11}$ implies that $x_i$ indeed lies in $\mathcal{S}_1$ for each $i$. Choose $j \in \{1, \ldots, n\}$ such that $x_j \neq 0$. Note that

$$x_j = \sum_{t=1}^{k_j} \alpha_t f_t^j,$$

where $\alpha_t^j, 1 \leq t \leq k_j$, are all scalars. Let $t_0$ be the largest value of $t$, $1 \leq t \leq k_j$, such that $\alpha_{t_0} \neq 0$. A little computation, using the definition of $U_{12}$ and $U_{21}$, yields that

$$(U_{12}U_{21})^{k_j - t_0 + 1}(f_s^j) \in E_{\sqrt{1-\lambda_j^2}}(U_{11}),$$

for $s < t_0$ and $(U_{12}U_{21})^{k_j - t_0 + 1}(f_{t_0}^j)$ is a non-zero scalar multiple of $f_1^{j+1}$ or $f_1^1$ according as $j < n$ or $j = n$. Consequently

$$(U_{12}U_{21})^{k_j - t_0 + 1}(x_j) = y + z,$$

with

$$y \in E_{\sqrt{1-\lambda_j^2}}(U_{11}),$$

and $z(\neq 0)$ is a scalar multiple of $f_1^{j+1}$ or $f_1^1$ according as $j < n$ or $j = n$. Thus

$$z \in E_{\sqrt{1-\lambda_{j+1}^2}}(U_{11}) \text{ or } E_{\sqrt{1-\lambda_1^2}}(U_{11}),$$

according as $j < n$ or $j = n$. Since $U_{12}U_{21}$ leaves $\mathcal{S}_1$ invariant and $x_j \in \mathcal{S}_1$, it follows that $y + z \in \mathcal{S}_1$ and since $\mathcal{S}_1$ is invariant under $U_{11}$, we conclude that both $y, z \in \mathcal{S}_1$. Note that $z \in \mathcal{S}_1$ is equivalent to saying that exactly one of $f_1^{j+1}$ and $f_1^1$ belongs to $\mathcal{S}_1$. Thus it follows that $\mathcal{S}_1 = ran\, P^\perp$ and hence, $\mathcal{S} = \mathcal{E}$. A similar proof shows that if $\mathcal{S}_2 \neq 0$, then also $\mathcal{S} = \mathcal{E}$. Thus there is no non-trivial joint $(U, P)$-reducing subspace of $\mathcal{E}$, completing the proof.

## 4.7 Proof of part (ii)

We now study the case when $T$ has only one positive eigen value lying in $(0, 1)$ such that the dimension of the corresponding eigen space is at least 2. Thus, in this case, the set of eigen values of $T$ is given by

$$\sigma(T) = \{\pm\lambda_1\},$$

with $0 < \lambda_1 < 1$ and

$$\dim E_{\lambda_1}(T) = \dim E_{-\lambda_1}(T) \geq 2.$$

Let $\alpha \neq 1$ be a complex number with $|\alpha| = 1$. Construct a unitary $U : \mathcal{E} \to \mathcal{E}$ as follows: Define $U$ on $ranP^\perp$ by

$$Uf_t^1 = \begin{cases} \alpha\Big((\sqrt{1-\lambda_1^2})f_1^1 \oplus (-\lambda_1)\tilde{f}_1^1\Big) & \text{if } t = 1, \\ (\sqrt{1-\lambda_1^2})f_t^1 \oplus (-\lambda_1)\tilde{f}_t^1 & \text{if } 2 \leq t \leq k_1, \end{cases}$$

and on $ranP$ by

$$U\tilde{f}_t^1 = \begin{cases} \lambda_1 f_{t+1}^1 \oplus (\sqrt{1-\lambda_1^2})\tilde{f}_{t+1}^1 & \text{if } 1 \leq t < k_1, \\ \lambda_1 f_1^1 \oplus (\sqrt{1-\lambda_1^2})\tilde{f}_1^1 & \text{if } t = k_1. \end{cases}$$

As before, with respect to the decomposition $\mathcal{E} = ran\, P^\perp \oplus ran\, P$, let

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}.$$

With respect to the ordered orthonormal bases $\{f_t^1 : 1 \leq t \leq k_1\}$ of $ranP^\perp$ and $\{\tilde{f}_t^1 : 1 \leq t \leq k_1\}$ of $ranP$, it follows easily from the definition of $U$ that

- $U_{11} : ranP^{\perp} \to ranP^{\perp}$ is represented by the diagonal matrix

$$\mathrm{diag}\Big(\alpha\sqrt{1-\lambda_1^2}, \underbrace{\sqrt{1-\lambda_1^2}, \ldots, \sqrt{1-\lambda_1^2}}_{k_1 - 1 \text{ times}}\Big),$$

- $U_{21} : ranP^{\perp} \to ranP$ is represented by the invertible diagonal matrix

$$\mathrm{diag}\Big(-\alpha\lambda_1, \underbrace{-\lambda_1, \ldots, -\lambda_1}_{k_1 - 1 \text{ times}}\Big),$$

- both $U_{12} : ranP \to ranP^{\perp}$ and $U_{22} : ranP \to ranP$ are represented by matrices of *weighted shift* type and one can easily verify that the $(1, \dim(ranP))$-th entry of $U_{12}$ equals $\lambda_1$ and the subdiagonal of $U_{12}$ is given by

$$\underbrace{\lambda_1, \cdots, \lambda_1}_{k_1 - 1 \text{ times}},$$

whereas the $(1, \dim(ranP))$-th entry of $U_{22}$ equals $\sqrt{1-\lambda_1^2}$ and the subdiagonal of $U_{22}$ is given by

$$\underbrace{\sqrt{1-\lambda_1^2}, \ldots, \sqrt{1-\lambda_1^2}}_{k_1 - 1 \text{ times}}.$$

Proceeding along the same line of argument as in Subsection 4.6, one can easily see that in this case also there is no non-trivial joint $(U, P)$-reducing subspace of $\mathcal{E}$.

## 4.8 Proof of part (iii)

Finally, we deal with the case when 1 is the only positive eigen value of $T$. Then, with respect to the decomposition

$$\mathcal{E} = E_1(T) \oplus E_{-1}(T),$$

the operator $T$ admits the following diagonal representation

$$T = \begin{bmatrix} I_{E_1(T)} & 0 \\ 0 & -I_{E_{-1}(T)} \end{bmatrix}.$$

Suppose $U$ is a unitary on $\mathcal{E}$ and $P$ is a projection on $\mathcal{E}$ such that

$$P^{\perp} - UP^{\perp}U^* = T.$$

An appeal to Theorem 4.2.2 immediately implies that with respect to the decomposition $\mathcal{E} = E_1(T) \oplus E_{-1}(T)$, $P^\perp$ and $UP^\perp U^*$ must be of the form

$$P^\perp = \begin{bmatrix} I_{E_1(T)} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } UP^\perp U^* = \begin{bmatrix} 0 & 0 \\ 0 & I_{E_{-1}(T)} \end{bmatrix}.$$

It is clear from the forms of $P^\perp$ and $UP^\perp U^*$ that $U$ carries $E_1(T)$ (resp., $E_{-1}(T)$) onto $E_{-1}(T)$ (resp., $E_1(T)$). Thus, $U$ has the block operator matrix form

$$U = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix},$$

where $A : E_{-1}(T) \to E_1(T)$ and $B : E_1(T) \to E_{-1}(T)$ are unitaries. Thus, if

$$\dim E_1(T) = \dim E_{-1}(T) = 1,$$

then there is no non-trivial joint $(U, P)$-reducing subspace of $\mathcal{E}$. Now assume that

$$\dim E_1(T) = \dim E_{-1}(T) \geq 2.$$

Let $v \in \mathcal{E}$ be an eigen vector of $U$ and let $Uv = \alpha v$ where, $\alpha$, of course, has modulus one. Write $v = v_1 + v_2$ with $v_1 \in E_1(T), v_2 \in E_{-1}(T)$. It then follows from $Uv = \alpha v$ that $Av_2 = \alpha v_1, Bv_1 = \alpha v_2$. Consider the subspace

$$W = \text{span}\{v_1\} \oplus \text{span}\{v_2\}.$$

One can easily verify that $W$ is reducing for $U$ also. Thus, $W$ is a non-zero proper joint $(U, P)$-reducing subspace of $\mathcal{E}$. This completes the proof of part (iii) of Theorem 4.3.2.

## 4.9 $\mathcal{E}$ is infinite-dimensional

This section deals with the case when $\mathcal{E}$ is infinite-dimensional. We aim to show that given an operator $T \in B(\mathcal{E})$ of the form (4.1.7) such that either

$$\dim E_1(T) = \dim E_{-1}(T) \text{ (may be zero also)},$$

or

$$\dim E_1(T) = \dim E_{-1}(T) \pm 1,$$

then one can construct an irreducible BCL pair on $H^2_{\mathcal{E}}(\mathbb{D})$ with the desired properties. Our first result, namely Theorem 4.9.1, treats the case when $\dim E_1(T) = \dim E_{-1}(T)$.

**Theorem 4.9.1.** *Let $\mathcal{E}$ be an infinite-dimensional Hilbert space and let $T \in B(\mathcal{E})$ be of the form (4.1.7), that is,*

$$T = \begin{bmatrix} I_{\dim E_1(T)} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{\dim E_{-1}(T)} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}.$$

*Suppose that $\dim E_1(T) = \dim E_{-1}(T)$. Then there exists an irreducible BCL pair $(V_1, V_2)$ on $H^2_{\mathcal{E}}(\mathbb{D})$ such that the non-zero part of $C(V_1, V_2)$ is equal to $T$.*

*Proof.* The proof proceeds, to some extent, along the line of argument as that of Theorem 4.3.2. However, at any rate, some detail is necessary. Let

$$\sigma(T) = \{\lambda_n : n \in \mathbb{N}\}.$$

Choose a bijection $g : \mathbb{Z} \to \mathbb{N}$ so that the set of eigen values of $T$ is expressed as $\{\lambda_{g(n)} : n \in \mathbb{Z}\}$. Define

$$k_n := \dim E_{\lambda_n}(T) = \dim E_{-\lambda_n}(T).$$

Let $U_n$ denote a unitary from $E_{\lambda_n}(T)$ to $E_{-\lambda_n}(T)$, and let $\{e^n_t : 1 \le t \le k_n\}$ be an orthonormal basis of $E_{\lambda_n}(T)$, $n \in \mathbb{N}$. Then $\{U_n e^n_t : 1 \le t \le k_n\}$ is an orthonormal basis of $E_{-\lambda_n}(T)$, $n \in \mathbb{N}$. Clearly $\mathcal{E}$ has the orthogonal decomposition

$$\begin{aligned} \mathcal{E} &= \bigoplus_{n\in\mathbb{N}} \Big( E_{\lambda_n}(T) \oplus E_{-\lambda_n}(T) \Big) \\ &= \bigoplus_{n\in\mathbb{Z}} \Big( E_{\lambda_{g(n)}}(T) \oplus E_{-\lambda_{g(n)}}(T) \Big). \end{aligned}$$

Let $n \in \mathbb{N}$. As in the proof of Theorem 4.3.2, define a projection $Q_n \in B(E_{\lambda_n}(T) \oplus E_{-\lambda_n}(T))$ by

$$Q_n = \begin{bmatrix} \frac{1+\lambda_n}{2} I_{E_{\lambda_n}(T)} & \frac{\sqrt{1-\lambda_n^2}}{2} U_n^* \\ \frac{\sqrt{1-\lambda_n^2}}{2} U_n & \frac{1-\lambda_n}{2} I_{E_{-\lambda_n}(T)} \end{bmatrix}.$$

Then $\{f^n_t : 1 \le t \le k_n\}$ and $\{\tilde{f}^n_t : 1 \le t \le k_n\}$ are orthonormal bases of $ran\, Q_n$ and $ran\, Q_n^\perp$, respectively, where

$$f^n_t := \sqrt{\frac{1+\lambda_n}{2}} e^n_t \oplus \sqrt{\frac{1-\lambda_n}{2}} U_n e^n_t$$

and

$$\tilde{f}^n_t := \sqrt{\frac{1-\lambda_n}{2}} e^n_t \oplus \Big( -\sqrt{\frac{1+\lambda_n}{2}} \Big) U_n e^n_t.$$

Finally, consider the projection $Q \in B(\mathcal{E})$ given by

$$Q = \bigoplus_{n\in\mathbb{N}} Q_n$$

and set $P = Q^{\perp}$. It follows immediately from the definition of $P$ that

$$\bigcup_{n\in\mathbb{N}}\{f_t^n : 1 \leq t \leq k_n\} \quad \text{and} \quad \bigcup_{n\in\mathbb{N}}\{\tilde{f}_t^n : 1 \leq t \leq k_n\},$$

are orthonormal bases for $ranP^{\perp}$ and $ranP$, respectively. Define the unitary $U : \mathcal{E} \to \mathcal{E}$ by specifying

$$U(f_t^{g(n)}) = \sqrt{1-\lambda_{g(n)}^2}\, f_t^{g(n)} \oplus \Big(-\lambda_{g(n)}\Big)\tilde{f}_t^{g(n)},$$

for all $1 \leq t \leq k_{g(n)}$ and

$$U(\tilde{f}_t^{g(n)}) = \begin{cases} \lambda_{g(n)} f_{t+1}^{g(n)} \oplus \sqrt{1-\lambda_{g(n)}^2}\,\tilde{f}_{t+1}^{g(n)} & \text{if } 1 \leq t < k_{g(n)}, \\ \lambda_{g(n+1)} f_1^{g(n+1)} \oplus \sqrt{1-\lambda_{g(n+1)}^2}\,\tilde{f}_1^{g(n+1)} & \text{if } t = k_{g(n)}, \end{cases}$$

where $n \in \mathbb{Z}$. With respect to the decomposition $\mathcal{E} = ranP^{\perp} \oplus ranP$, let

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}.$$

With respect to the ordered orthonormal bases $\bigcup_{n\in\mathbb{Z}}\{f_t^{g(n)} : 1 \leq t \leq k_{g(n)}\}$ of ran$P^{\perp}$ and $\bigcup_{n\in\mathbb{Z}}\{\tilde{f}_t^{g(n)} : 1 \leq t \leq k_{g(n)}\}$ of ran$P$, it is clear from the definition of $U$ that $U_{11}$ as well as $U_{21}$ are represented by diagonal matrices whereas $U_{12}U_{21}$ is an operator of the *weighted shift* type.

Now let $\mathcal{S}$ be a non-zero joint $(U, P)$-reducing subspace of $\mathcal{E}$. Decompose $\mathcal{S}$ as

$$\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2,$$

where $\mathcal{S}_1 = P^{\perp}(\mathcal{S})$ and $\mathcal{S}_2 = P(\mathcal{S})$. Assume, without loss of generality, that $\mathcal{S}_1$ is non-zero. Similar argument as in the proof of Theorem 4.3.2 in Subsection 4.6 shows that in order to prove that $\mathcal{S} = \mathcal{E}$, it suffices to show that $\mathcal{S}_1 = ranP^{\perp}$. Since $\mathcal{S}_1$ reducing for $U_{12}U_{21}$, to prove that $\mathcal{S}_1 = ranP^{\perp}$, it is enough to show, by an appeal to Lemma 4.3.1, that some basis vector $f_t^{g(n)}$ belongs to $\mathcal{S}_1$.

Note that for each $n \in \mathbb{Z}$, $\sqrt{1-\lambda_{g(n)}^2}$ is an eigen value of $U_{11}$ with $\{f_t^{g(n)} : 1 \leq t \leq k_{g(n)}\}$ being an orthonormal basis for $E_{\sqrt{1-\lambda_{g(n)}^2}}(U_{11})$ and hence, $ranP^{\perp}$ has the following orthogonal decomposition

$$ranP^{\perp} = \oplus_{n\in\mathbb{Z}} E_{\sqrt{1-\lambda_{g(n)}^2}}(U_{11}).$$

Let $0 \neq x \in \mathcal{S}_1$. Then

$$x = \sum_{n \in \mathbb{Z}} x_{g(n)}$$

with $x_{g(n)} \in E_{\sqrt{1-\lambda_{g(n)}^2}}(U_{11})$. Since $\mathcal{S}_1$ is reducing for $U_{11}$, an appeal to the spectral theorem immediately yields that $x_{g(n)}$ indeed lies in $\mathcal{S}_1$ for each $n$. Choose $n$ such that $x_{g(n)} \neq 0$ and let

$$x_{g(n)} = \sum_{t=1}^{k_{g(n)}} \alpha_t f_t^{g(n)},$$

where $\alpha_t, 1 \leq t \leq k_{g(n)}$, are all scalars. If $t_0$ is the largest value of $t$, $1 \leq t \leq k_{g(n)}$, such that $\alpha_{t_0} \neq 0$, similar argument as in the proof of Theorem 4.3.2 in Subsection 4.6 shows that

$$(U_{12}U_{21})^{k_{g(n)}-t_0+1}(f_s^{g(n)}) \in E_{\sqrt{1-\lambda_{g(n)}^2}}(U_{11}),$$

for $s < t_0$ and

$$(U_{12}U_{21})^{k_{g(n)}-t_0+1}(f_{t_0}^{g(n)}),$$

is a non-zero scalar multiple of $f_1^{g(n+1)}$ from which we conclude, proceeding again along the same line of argument as in the proof of Theorem 4.3.2 in Subsection 4.6, that $f_1^{g(n+1)} \in \mathcal{S}_1$, completing the proof. $\square$

The next theorem analyses the case when $\dim E_1(T) = \dim E_{-1}(T) \pm 1$.

**Theorem 4.9.2.** *Let $\mathcal{E}$ be an infinite-dimensional Hilbert space and let $T \in B(\mathcal{E})$ be of the form* (4.1.7)*, that is,*

$$T = \begin{bmatrix} I_{\dim E_1(T)} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{\dim E_{-1}(T)} & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}.$$

*Suppose that*

$$\dim E_1(T) = \dim E_{-1}(T) \pm 1.$$

*Then there exists an irreducible BCL pair $(V_1, V_2)$ on $H^2_{\mathcal{E}}(\mathbb{D})$ such that the non-zero part of $C(V_1, V_2)$ is equal to $T$.*

*Proof.* Assume, without loss of generality, that $\dim E_{-1}(T) = \dim E_1(T) + 1$. Further, assume that $\dim E_1(T) > 0$, that is, 1 is an eigen value of $T$. Set $\lambda_0 = 1$ and let the set of positive eigen values of $T$ lying in $(0,1)$ be given by $\{\lambda_n : n \in \mathbb{N}\}$. Then the set of eigen values of $T$ is given by $\{\pm\lambda_n : n \geq 0\}$. We use the same notations as in the proof of Theorem 4.9.1 so that for each $n \in \mathbb{N}$, $k_n = \dim E_{\lambda_n}(T)$, $\{e_t^n : 1 \leq t \leq k_n\}$ represents an orthonormal basis of $E_{\lambda_n}(T)$ and for $1 \le t \le k_n$, $f_t^n$ and $\tilde{f}_t^n$ are defined by

$$f_t^n = \sqrt{\frac{1+\lambda_n}{2}}e_t^n + \sqrt{\frac{1-\lambda_n}{2}}U_n e_t^n \quad \text{and} \quad \tilde{f}_t^n = \sqrt{\frac{1-\lambda_n}{2}}e_t^n - \sqrt{\frac{1+\lambda_n}{2}}U_n e_t^n,$$

where $U_n$ denotes a unitary operator from $E_{\lambda_n}(T)$ to $E_{-\lambda_n}(T)$. Finally, let $k_0 = \dim E_1(T)$ so that $\dim E_{-1}(T) = k_0 + 1$ and let

$$\{f_t^0 : 1 \le t \le k_0\} \quad \text{and} \quad \{\tilde{f}_t^0 : 1 \le t \le k_0 + 1\}$$

denote orthonormal bases of $E_1(T)$ and $E_{-1}(T)$, respectively. This implies that

$$\Big\{f_t^0 : 1 \le t \le k_0\Big\} \bigcup \Big\{\tilde{f}_t^0 : 1 \le t \le k_0 + 1\Big\} \bigcup_{n \in \mathbb{N}} \Big\{f_t^n, \tilde{f}_t^n : 1 \le t \le k_n\Big\}$$

is an orthonormal basis of $\mathcal{E}$. As usual, our goal is to construct a projection $P$ and a unitary $U$ on $\mathcal{E}$ such that $P^\perp - UP^\perp U^* = T$ and there is no non-trivial joint $(U,P)$-reducing subspace of $\mathcal{E}$. Consider the orthogonal projection $P \in B(\mathcal{E})$ such that an orthonormal basis of $ran\, P$ is given by

$$\Big\{\tilde{f}_t^0 : 1 \le t \le k_0 + 1\Big\} \bigcup_{n \in \mathbb{N}} \Big\{\tilde{f}_t^n : 1 \le t \le k_n\Big\}.$$

Consequently,

$$\Big\{f_t^0 : 1 \le t \le k_0\Big\} \bigcup_{n \in \mathbb{N}} \Big\{f_t^n : 1 \le t \le k_n\Big\}$$

is an orthonormal basis of $ranP^\perp$. Let us consider the unitary $U : \mathcal{E} \to \mathcal{E}$ defined as follows: For each $n \ge 1$, define

$$Uf_t^n = \begin{cases} \sqrt{1-\lambda_n^2} f_{t+1}^n \oplus \Big(-\lambda_n\Big)\tilde{f}_{t+1}^n & \text{if } 1 \le t < k_n; \\ \sqrt{1-\lambda_{n-1}^2} f_1^{n-1} \oplus \Big(-\lambda_{n-1}\Big)\tilde{f}_1^{n-1} & \text{if } t = k_n; \end{cases}$$

and

$$U\tilde{f}_t^n = \begin{cases} \lambda_n f_{t+1}^n \oplus \sqrt{1-\lambda_n^2}\tilde{f}_{t+1}^n & \text{if } 1 \le t < k_n; \\ \lambda_{n+1} f_1^{n+1} \oplus \sqrt{1-\lambda_{n+1}^2}\tilde{f}_1^{n+1} & \text{if } t = k_n; \end{cases}$$

and finally,

$$\begin{aligned} U(f_t^0) &= \tilde{f}_{t+1}^0 && \text{if } 1 \le t \le k_0; \\ U(\tilde{f}_t^0) &= f_t^0 && \text{if } 1 \le t \le k_0; \\ U(\tilde{f}_{k_0+1}^0) &= \lambda_1 f_1^1 + \sqrt{1-\lambda_1^2}\tilde{f}_1^1. \end{aligned}$$

Then it is easy to check that $P^{\perp} - UP^{\perp}U^* = T$. Now, with respect to the decomposition $\mathcal{E} = ranP^{\perp} \oplus ranP$, let

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}.$$

It follows from the definition of $U$ that

$$\begin{aligned}
&U_{11}(f_t^0) = 0, \text{ for } 1 \leq t \leq k_0;\\
&U_{11}(f_t^n) = (\sqrt{1-\lambda_n^2})f_{t+1}^n, \text{ for } n \geq 1, 1 \leq t < k_n;\\
&U_{11}(f_{k_n}^n) = (\sqrt{1-\lambda_{n-1}^2})f_1^{n-1}, \text{ for } n \geq 1.
\end{aligned}$$

A little computation shows that

$$\begin{aligned}
&U_{11}^*(f_t^0) = 0, \text{ for } 1 \leq t \leq k_0;\\
&U_{11}^*(f_{t+1}^n) = (\sqrt{1-\lambda_n^2})f_t^n, \text{ for } n \geq 1, 1 \leq t < k_n;\\
&U_{11}^*(f_1^n) = (\sqrt{1-\lambda_n^2})f_{k_{n+1}}^{n+1}, \text{ for } n \geq 1;
\end{aligned}$$

and consequently,

$$\begin{aligned}
&U_{11}^*U_{11}(f_t^0) = 0, \text{ for } 1 \leq t \leq k_0;\\
&U_{11}^*U_{11}(f_t^n) = (1-\lambda_n^2)f_t^n, \text{ for } n \geq 1, 1 \leq t < k_n;\\
&U_{11}^*U_{11}(f_{k_n}^n) = (1-\lambda_{n-1}^2)f_{k_n}^n, \text{ for } n \geq 1.
\end{aligned}$$

Thus, we see that $U_{11}^*U_{11}$ is a digonalizable operator on $ran\, P^{\perp}$ with eigen values $\{1-\lambda_n^2 : n \geq 0\}$. Clearly,

$$\left\{f_t^0 : 1 \leq t \leq k_0\right\} \bigcup \left\{f_{k_1}^1\right\},$$

is an orthonormal basis for $E_0(U_{11}^*U_{11}) = E_{1-\lambda_0^2}(U_{11}^*U_{11})$, and

$$\left\{f_t^n : 1 \leq t < k_n\right\} \bigcup \left\{f_{k_{n+1}}^{n+1}\right\},$$

is an orthonormal basis of $E_{1-\lambda_n^2}(U_{11}^*U_{11})$ for all $n \geq 1$.

Let $\mathcal{S}$ be a non-trivial joint $(U, P)$-reducing subspace of $\mathcal{E}$. Then

$$\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2,$$

where $\mathcal{S}_1 = P^{\perp}(\mathcal{S})$ and $\mathcal{S}_2 = P(\mathcal{S})$. Assume, without loss of generality, that $\mathcal{S}_1$ is non-zero and let $0 \neq x \in \mathcal{S}_1$. Let

$$x = \bigoplus_{n \geq 0} x_n,$$

where $x_n \in E_{1-\lambda_n^2}(U_{11}^*U_{11})$ for all $n \geq 0$. Since $\mathcal{S}_1$ is reducing for $U_{11}^*U_{11}$, we must have that $x_n \in \mathcal{S}_1$ for each $n \geq 0$. Let $n_0$ be the smallest non-negative integer such that

$x_{n_0} \neq 0$. First assume that $n_0 \geq 1$ and let

$$x_{n_0} = \sum_{t=1}^{k_{n_0}-1} \alpha_t^{n_0} f_t^{n_0} + \beta f_{k_{n_0}+1}^{n_0+1},$$

where $\beta, \alpha_t^{n_0}(1 \leq t < k_{n_0})$ are all scalars. If $\alpha_t^{n_0} = 0$ for all $t, 1 \leq t < k_{n_0}$, then clearly $\beta \neq 0$ and hence, $f_{k_{n_0}+1}^{n_0+1} \in \mathcal{S}_1$. If $\alpha_t^{n_0}$ are not all zero, let $t_0$ be the maximum value of $t, 1 \leq t < k_{n_0}$, such that $\alpha_t^{n_0} \neq 0$. Then one can easily see that

$$\begin{aligned} \mathcal{S}_1 \ni U_{11}^{k_{n_0}-t_0}(x_{n_0}) &= \text{an element in span}\{f_1^{n_0}, \cdots, f_{k_{n_0}-1}^{n_0}\} \oplus \alpha_{t_0}^{n_0}(1-\lambda_{n_0}^2)^{\frac{k_{n_0}-t_0}{2}} f_{k_{n_0}}^{n_0} \\ &\in E_{1-\lambda_{n_0}^2}(U_{11}^* U_{11}) \oplus E_{1-\lambda_{n_0-1}^2}(U_{11}^* U_{11}), \end{aligned}$$

and consequently, $f_{k_{n_0}}^{n_0} \in \mathcal{S}_1$. Now assume that $n_0 = 0$ and let

$$x_0 = \sum_{t=1}^{k_0} \alpha_t^0 f_t^0 + \beta f_{k_1}^1,$$

where $\beta$ and $\alpha_t^0$, $1 \leq t \leq k_0$, are all scalars. Note that if $\beta \neq 0$, then $U_{11}^* x_0 \neq 0$ and

$$U_{11}^* x_0 = \beta\sqrt{1-\lambda_1^2} f_{k_1-1}^1 \text{ or } \beta\sqrt{1-\lambda_1^2} f_{k_2}^2,$$

depending on whether $k_1 > 1$ or $k_1 = 1$ and thus, $\mathcal{S}_1$ contains either $f_{k_1-1}^1$ or $f_{k_2}^2$ according as $k_1 > 1$ or $k_1 = 1$. Suppose now that $\beta = 0$ and let $t_0 = \max\{t : \alpha_t^0 \neq 0\}$. A simple computation shows that

$$\mathcal{S} \ni U^{2(k_0-t_0)+2}(x_0) = \text{an element in span}\{f_1^0, \cdots, f_{k_0}^0\} + \alpha_{t_0}^0(\lambda_1 f_1^1 + \sqrt{1-\lambda_1^2}\tilde{f}_1^1)$$

and hence, $\tilde{f}_1^1 \in \mathcal{S}_2$. Since

$$U(\tilde{f}_1^1) = \lambda_1 f_2^1 + \sqrt{1-\lambda_1^2}\tilde{f}_2^1 \quad \text{or} \quad \lambda_2 f_1^2 + \sqrt{1-\lambda_2^2}\tilde{f}_1^2,$$

according as $k_1 > 1$ or $k_1 = 1$, we have that either $f_2^1$ or $f_1^2$ belongs to $\mathcal{S}_1$. Thus we conclude that $f_t^n \in \mathcal{S}_1$ for some $n \geq 1$ and $1 \leq t \leq k_n$. It is easy to see that

$$U(P^\perp U)^{(k_n-t)+k_{n-1}+\cdots+k_2+k_1}(f_t^n) = \text{ a non-zero scalar multiple of } \tilde{f}_1^0$$

and hence, $\tilde{f}_1^0 \in \mathcal{S}$. Since $\mathcal{S}$ is invariant under $U$, applying $U$ repeatedly on $\tilde{f}_1^0$ we see that

$$\{f_t^0 : 1 \leq t \leq k_0\} \bigcup \{\tilde{f}_t^0 : 1 \leq t \leq k_0 + 1\}$$

is contained in $\mathcal{S}$. Again, using the definition of $U$ and $P$, a simple computation shows that

$$\begin{aligned}
&(PU)^t(\tilde{f}^0_{k_0+1}) = \text{a non-zero scalar multiple of } \tilde{f}^1_t \text{ for } 1 \le t \le k_1,\\
&(PU)^{k_1+k_2+\cdots+k_{n-1}+t}(\tilde{f}^0_{k_0+1}) = \text{a non-zero scalar multiple of } \tilde{f}^n_t \text{ for } n > 1, 1 \le t \le k_n,
\end{aligned}$$

and since $\mathcal{S}$ is reducing for both $U$ and $P$, it follows immediately that $\mathcal{S}$ contains

$$\bigcup_{n\in\mathbb{N}} \{\tilde{f}^n_t : 1 \le t \le k_n\}.$$

Finally, we observe that

$$\begin{aligned}
&(P^{\perp}U)(\tilde{f}^0_{k_0+1}) = \text{a non-zero scalar multiple of } f^1_1,\\
&(P^{\perp}U)(\tilde{f}^n_t) = \text{a non-zero scalar multiple of } f^n_{t+1} \text{ for } 1 \le t < k_n, n \ge 1,\\
&(P^{\perp}U)(\tilde{f}^n_{k_n}) = \text{a non-zero scalar multiple of } f^{n+1}_1 \text{ for } n \ge 1.
\end{aligned}$$

Since $\mathcal{S}$ is $(U,P)$-reducing and contains $\{\tilde{f}^0_{k_0+1}\}\bigcup_{n\in\mathbb{N}}\{\tilde{f}^n_t : 1 \le t \le k_n\}$, it follows that $\mathcal{S}$ contains

$$\bigcup_{n\in\mathbb{N}} \{f^n_t : 1 \le t \le k_n\}.$$

As an immediate consequence of all these observations, we conclude that $\mathcal{S}$ indeed contains the orthonormal basis of $\mathcal{E}$ given by

$$\{f^0_t : 1 \le t \le k_0\} \bigcup \{\tilde{f}^0_t : 1 \le t \le k_0 + 1\} \bigcup_{n\in\mathbb{N}} \{f^n_t, \tilde{f}^n_t : 1 \le t \le k_n\},$$

and hence, $\mathcal{S} = \mathcal{E}$, finishing the proof of this case. The proof for the case when 1 is not an eigen value of $T$, that is, $k_0 = 0$, works in the same way. □

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## List of Publications

1. Amit Maji, Jaydeb Sarkar, Sankar TR,
   Pairs Of Commuting Isometries 1,
   *Studia Mathematica*, 87 (2017), 225–244.

2. Amit Maji, Aneesh Mundayan, Jaydeb Sarkar, Sankar TR,
   Characterization of Invariant Subspaces in the Polydisc ,
   *Journal of Operator Theory*, 322 (2017), 186–200.

3. Sandipan De, Shankar P, Jaydeb Sarkar, Sankar TR,
   Pairs Of Projections and Commuting Isometries,
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