On the Inertia Conjecture and its generalizations

Soumyadip Das

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On the Inertia Conjecture and its generalizations

Author:
SOUMYADIP DAS

Supervisor:
MANISH KUMAR

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Statistics & Mathematics Unit

Indian Statistical Institute, Bangalore Centre

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To
my parents

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Chapter 1

Introduction

This thesis concerns problems related to the ramification behaviour of the branched Galois covers of smooth projective connected curves defined over an algebraically closed field of positive characteristic. Our first main problem is the Inertia Conjecture proposed by Abhyankar in 2001. We will show several new evidence for this conjecture. We also formulate a certain generalization of it which is our second problem, and we provide evidence for it. We give a brief overview of these problems in this introduction and reserve the details for Chapter 4.

Let k be an algebraically closed field, and U be a smooth connected affine k-curve. Let $U \subset X$ be the smooth projective completion. An interesting and challenging problem is to understand the étale fundamental group $\pi_1(U)$. We only consider this as a profinite group up to isomorphism, and so the base point is ignored. When k has characteristic 0, it is well known that this group is the profinite completion of the topological fundamental group. In particular, it is a free profinite group, topologically generated by $2g + r - 1$ elements where g is the genus of X, and r is the number of points in $X - U$. But when k has prime characteristic $p > 0$, these statements are no longer true. The full structure of $\pi_1(U)$ is not known in this case. Now onward, assume that k has characteristic $p > 0$. By the definition of $\pi_1(U)$, the set $\pi_A(U)$ of isomorphic classes of finite (continuous) group quotients of $\pi_1(U)$ is in bijective correspondence with the finite Galois étale covers of U. For a finite group G, let $p(G)$ denote the subgroup of G generated by all its Sylow p-subgroups. In 1957 Abhyankar conjectured on what groups can occur in the set $\pi_A(U)$.

Conjecture 1.1 (Abhyankar's Conjecture on the affine curves; [2, Section 4.2])**.** *Let X be a smooth projective connected curve of genus g over an algebraically closed field k of characteristic $p > 0$. Let $r \geq 1$, and B be a finite set of closed points in X. Set $U := X - B$. Then a finite group G occurs as the Galois group of an étale cover $V \to U$ of smooth connected k-curves if and only if $G/p(G)$ is generated by at most $2g + r - 1$ elements.*

The above conjecture is now a Theorem due to the works of Serre, Raynaud and Harbater ([27], [25], [12]). Since $\pi_1(U)$ is not a topologically finitely generated group, it is important to note that to understand the structure of $\pi_1(U)$, it is merely not enough to know the set $\pi_A(U)$, rather one also needs to know how these groups fit together in the inverse system defining $\pi_1(U)$.

Conjecture 1.1 says that when U is the affine k-line $\mathbb{A}^1$, $\pi_A(\mathbb{A}^1)$ is the set of isomorphic classes of the quasi p-groups G (a finite group G is said to be a *quasi p-group* if G is generated by all its Sylow p-subgroups, i.e. $G = p(G)$). So for a quasi p group G, the question is to understand the inertia groups over ∞ in a connected G-Galois étale cover of $\mathbb{A}^1$. In contrast to characteristic 0 where all the inertia groups are cyclic of order equal to the ramification index, these inertia groups have a complicated structure in general. By studying the branched covers of $\mathbb{P}^1$ given by explicit equations Abhyankar posed the following conjecture, now known as the Inertia Conjecture (IC).

Conjecture 1.2 (IC, [4, Section 16])**.** *Let G be a finite quasi p-group. Let I be a subgroup of G which is an extension of a p-group P by a cyclic group of order prime-to-p. Then there is a connected G-Galois cover of $\mathbb{P}^1$ étale away from ∞ such that I occurs as an inertia group at a point over ∞ if and only if the conjugates of P in G generate G.*

The special case of the above conjecture when I is a p-group is known as the Purely Wild Inertia Conjecture (PWIC).

Conjecture 1.3 (PWIC)**.** *Let G be a finite quasi p-group. A p-subgroup P of G occurs as the inertia group at a point above ∞ in a connected G-Galois cover of $\mathbb{P}^1$ branched only at ∞ if and only if the conjugates of P generate G.*

One can see that the condition on the inertia groups is necessary. The other direction, namely whether one can realize each possible subgroup I of G as an inertia group at a point above ∞ in a connected G-Galois cover of $\mathbb{P}^1$ branched only at ∞, remains wide open at the moment. We only know the evidence for this conjecture in a few cases (see [8], [23], [22], [20], [17] and Section 4.2 for more details).

Our results from [10] and [9] give new evidence for the Inertia Conjecture, specially for the Alternating groups. We discuss the main results in Section 4.4. In [10] we also showed that the 'minimal jump problem' (Question 4.7) for the Alternating groups has an affirmative answer when the Sylow p-subgroups of the inertia groups are generated by p-cycles.

Our second problem, introduced in [9], asks a general question motivated by the Inertia Conjecture. Set $B := X - U$. This question concerns the kind of inertia groups which occur over a point in B in a Galois étale cover of U with a fixed Galois group. As the tame fundamental group of U is also not well understood at the moment, the question assumes the existence of a tower of a certain tame Galois covers, and asks about the existence of an appropriate Galois cover dominating the tower so that the necessary conditions are satisfied. In [9] this problem was called $Q[r, X, B, G]$ (see Section 4.3). Here we state two special cases of this question for which we have obtained several positive results.

Question 1.4 (Question 4.10; $Q[r, X, B, G]$)**.** *Let $r \geq 1$ be an integer, X be a smooth projective connected k-curve, and let $B := \{x_1, \cdots, x_r\}$ be a set of closed points in X. Let G be a finite group, $P_1, \cdots, P_r$ be (possibly trivial except for P_1) p-subgroups of G. Set $H := \langle P_i^G | 1 \leq i \leq r\rangle$. Assume that either of the following holds.*

1. $G = \langle P_1^G, \cdots, P_r^G\rangle$;
2. $H = \langle P_1^H, \cdots, P_r^H\rangle$.

For $1 \leq i \leq r$, let I_i be a subgroup of G which is an extension of the p-group P_i by a cyclic group of order prime-to-p such that there is a connected G/H-Galois cover ψ of X étale away from B and $I_i/I_i \cap H$ occurs as an inertia group above x_i, $1 \leq i \leq r$. Does there exist a connected G-Galois cover ϕ of X étale away from B dominating the cover ψ such that I_i occurs as an inertia group above the point $x_i \in B$?

Using formal patching and studying the branched covers of curves given by the explicit equations, we obtain some affirmative answers to the above question which are listed in Section 4.4. One special case of the above question is when $X = \mathbb{P}^1$ which we pose as the Generalized Inertia Conjecture (GIC, Conjecture 4.15). Even more specializing to the case when the inertia groups are the p-groups, we pose the Generalized Purely Wild Inertia Conjecture (GPWIC, Conjecture 4.18). We see that for the groups for which the PWIC is already established, the GPWIC is also true.

The structure of the thesis is as follows. Chapter 2 contains the essential notation and some definition which are used throughout this thesis. Chapter 3 contains the basics of the ramification theory, its application to the theory of covers of curves, the definition of the étale fundamental group and its related quotients, and the basics of formal patching. We describe our main problems of this thesis along with a detailed motivation for them and the statements of our main results in Chapter 4. The auxiliary results which pave the way to prove our main results is the content of Chapter 5. Among them, Section 5.1 and Section 5.2 contain the results which are related to the understanding of the ramification behaviour of covers. In Section 5.3 we recall some well known results and techniques which are helpful to reduce the inertia groups. One of the most important sections of this thesis is Chapter 6 which concerns the construction of the Galois covers using some explicit equations and the formal patching technique. In Section 6.1 we follow [9] to construct the two point branched Galois covers of $\mathbb{P}^1$ with Alternating or Symmetric Galois groups. Moreover, we show that the Galois étale covers of the affine line with these groups which were studied by Abhyankar are the special cases of our construction. Section 6.2 contains the results on the construction of the Galois covers using the formal patching technique. Chapter 7 contains the proofs of our main results. In Section 7.1 we give an overview of the strategies of the proofs. In Section 7.2

we prove that the PWIC (Conjecture 4.4) is true for any product of certain Alternating groups. Section 7.3 contains the proof of the IC (Conjecture 4.3) for certain Alternating groups. The Generalized Purely Wild Inertia Conjecture (Conjecture 4.18) is the object of study in Section 7.4. The last section concerns some evidence towards the general question (Question 4.10).

Chapter 2

Notation and Convention

- All rings in this thesis are commutative rings with the identity element and all algebras are unitary.

- For an integral domain A, denote its quotient field or field of fractions by $QF(A)$. For any prime ideal $\mathfrak{p}$ in A, let $A_{\mathfrak{p}}$ denote the localization of A with respect to the multiplicative set $A - \mathfrak{p}$. We denote the completion of the local ring $A_{\mathfrak{p}}$ at its maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ by $\widehat{A}_{\mathfrak{p}}$. If $K = QF(A)$, $\widehat{K}_{\mathfrak{p}}$ denotes the quotient field of $\widehat{A}_{\mathfrak{p}}$. Denote the residue field of A at $\mathfrak{p}$ by $k(\mathfrak{p})$. So $k(\mathfrak{p}) = QF(A/\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \widehat{A}_{\mathfrak{p}}/\mathfrak{p}\widehat{A}_{\mathfrak{p}}$.

- For a scheme X and a point $x \in X$, let $O_{X,x}$ denote the local ring at x with the maximal ideal $\mathfrak{m}_x$. The complete local ring at x, namely the completion of the local ring $O_{X,x}$ with respect to $\mathfrak{m}_x$ is denoted by $\widehat{O}_{X,x}$. When x is an integral point, $K_{X,x}$ denotes the fraction field of $\widehat{O}_{X,x}$.

- Throughout this thesis, p denotes a prime, k denotes an algebraically closed field of characteristic p, and all the k-curves considered will be smooth connected curves, unless otherwise specified. Chapter 5 onward we will require p to be an odd prime.

- All the Alternating and Symmetric groups considered will be of degree ≥ 5.

- For a finite group G, $p(G)$ denotes the subgroup of G generated by all the Sylow p-subgroups of G. It is a characteristic subgroup of G which is also the maximal normal subgroup of index coprime to p.

- For a finite group G and a subgroup I of G, we say that the pair (G, I) is *realizable* if there exists a connected G-Galois cover of $\mathbb{P}^1$ branched only at ∞ such that I occurs as an inertia group above ∞. In this case G is necessarily a quasi p-group i.e. $G = p(G)$.

- For a finite group G and a subgroup H of G, we let H^G denote the set of conjugates of H in G. $\langle H^G \rangle$ denotes the subgroup of G generated by all the conjugates of H in G.

Chapter 3

Preliminaries

3.1 Local Ramification Theory

The ramification theory for a Noetherian normal domain is standard in the literature (see [26], [6]). In this section, we briefly recall the notion and basic definitions related to the ramification theory that will be used throughout this thesis. Notation from Chapter 2 will be frequently used without further mention.

Let $A \subset B$ be any ring extension. Let $\mathfrak{q}$ be a prime ideal in B. Let $\mathfrak{p} = \mathfrak{p} \cap A$. This is equivalent to $\mathfrak{q}$ containing the ideal $\mathfrak{p}B$, and we say that the prime ideal $\mathfrak{q}$ in B *lies* over the prime ideal $\mathfrak{p}$ in A, denoted by $\mathfrak{q}|\mathfrak{p}$. So $B_{\mathfrak{q}}$ is naturally an $A_{\mathfrak{p}}$-algebra.

Definition 3.1. Let A be an integral domain, and B be a reduced ring. Let $Q(B)$ denotes the total ring of fractions of B. An extension $A \subset B$ of rings is said to be *generically separable* if the corresponding extension $Q(B)/QF(A)$ is a separable extension, and no non-zero element of A become a zero divisor in B.

Definition 3.2. Let $A \subset B$ be an extension of rings.

1. Let $\mathfrak{q}$ be a prime ideal in B and $\mathfrak{p} = \mathfrak{q} \cap A$. Then $\mathfrak{q}$ is said to be *unramified* in the extension B/A if $\mathfrak{p}B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$, and $B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$ is a separable field extension of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. In this case we also say that $B_{\mathfrak{q}}/A_{\mathfrak{p}}$ is an unramified extension.

2. The extension B/A is said to be *unramified* if every prime ideal of B is unramified in B/A and for any prime ideal $\mathfrak{p}$ of A, there are only finitely many prime ideals $\mathfrak{q}$ of B with $\mathfrak{p} = \mathfrak{q} \cap A$.

Note that when A is an integral domain and B is a reduced ring, $\mathfrak{q}$ is the nilradical in B if and only if $\mathfrak{p} = A \cap \mathfrak{q}$ is the zero ideal in A. In this case, $A_{\mathfrak{p}} = QF(A)$ and $B_{\mathfrak{q}} = Q(B)$ with $\mathfrak{p}Q(B) = \mathfrak{q}Q(B)$. So $\mathfrak{q}$ is unramified over $\mathfrak{p}$ if and only if the extension $A \subset B$ is generically separable.

Consider the following setup. Let A be a Noetherian normal domain with quotient field K. Let L be a finite separable field extension of K. Let B be the integral closure of A in L. Then B is a finitely generated A-module, hence is a Noetherian normal domain with quotient field L. Also B is generically separable over A. For the rest of this section, we consider the following assumption.

Let $\mathfrak{q}$ be a non-zero prime ideal in B with $\mathfrak{p} = \mathfrak{q} \cap A$, and the finite field extension $k(\mathfrak{q})/k(\mathfrak{p})$ is a separable extension.

Consider the complete local rings $\widehat{B_{\mathfrak{q}}}$ and $\widehat{A_{\mathfrak{p}}}$ which are the completions at their respective maximal ideals. Let $\widehat{K_{\mathfrak{p}}} = QF(\widehat{A_{\mathfrak{p}}})$ and $\widehat{L_{\mathfrak{q}}} = QF(\widehat{B_{\mathfrak{q}}})$. As L/K is a finite separable extension, by [26, Page 31, Chapter II, Section 3, Corollary 3], $\widehat{L_{\mathfrak{q}}}/\widehat{K_{\mathfrak{p}}}$ is also a separable field extension. Since the completion homomorphism is flat, $B_{\mathfrak{q}}/A_{\mathfrak{p}}$ is an unramified extension if and only if $\widehat{B_{\mathfrak{q}}}/\widehat{A_{\mathfrak{p}}}$ is an unramified extension. We also recall the definitions of the decomposition and the inertia groups.

Definition 3.3. Let A be a Noetherian normal domain with quotient field K. Let L/K be a Galois field extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{q}$ be a prime idea in B and $\mathfrak{p} = \mathfrak{q} \cap A$.

1. The *decomposition group* at $\mathfrak{q}$ is defined to be the subgroup $D_{\mathfrak{q}} := \{g \in G | g(\mathfrak{q}) \subset \mathfrak{q}\}$ of G. It is the setwise stabilizer of the prime $\mathfrak{q}$ in G.

2. The *inertia group* at $\mathfrak{q}$ is defined to be the subgroup $I_{\mathfrak{q}} := \{g \in G | g(b) - b \in \mathfrak{q} \text{ for all } b \in \mathfrak{q}\}$ of $D_{\mathfrak{q}}$.

Note that in the above situation, the separable field extension $\widehat{L}_{\mathfrak{q}}/\widehat{K}_{\mathfrak{p}}$ is also Galois with Galois group $D_{\mathfrak{q}}$. If $\mathfrak{q}'$ is another prime ideal of B lying over $\mathfrak{p}$, then $D_{\mathfrak{q}}$ and $D_{\mathfrak{q}'}$ are two conjugate subgroups of G.

A more detailed understanding of the above definitions can be obtained when A is a Dedekind domain, i.e. A is an integrally closed Noetherian domain in which every non-zero prime ideal is maximal.

In what follows, we assume that A is a Dedekind domain.

Then B is also a Dedekind domain. Consider the unique decomposition of the ideal $\mathfrak{p}B$ in B given by

$$\mathfrak{p}B = \prod_{\mathfrak{q}|\mathfrak{p}} \mathfrak{q}^{e_{\mathfrak{q}}}. \tag{3.1.1}$$

For $\mathfrak{q}$ lying over $\mathfrak{p}$, the integer $e_{\mathfrak{q}} \geq 1$ is called the *ramification index* of $\mathfrak{q}$ in the extension L/K. By Definition 3.2, $\mathfrak{q}$ is unramified if and only if $e_{\mathfrak{q}} = 1$.

Definition 3.4. The prime $\mathfrak{q}$ is said to be *ramified* in the extension L/K if $e_{\mathfrak{q}} > 1$.

Since B is a Dedekind domain and $\mathfrak{q}$ is a non-zero ideal, $B_{\mathfrak{q}}$ is a discrete valuation ring. Note that the $e_{\mathfrak{q}}$ is the valuation of the ideal $\mathfrak{p}B$. The *residue degree* $f_{\mathfrak{q}}$ of $\mathfrak{q}$ in the extension L/K is defined to be the degree of the extension $k(\mathfrak{q})/k(\mathfrak{p})$. By [26, Page 14, Chapter I, Section 4, Proposition 10], the ring $B/\mathfrak{p}B$ is an $A/\mathfrak{p}$-algebra isomorphic to $\prod_{\mathfrak{q}|\mathfrak{p}} B/\mathfrak{q}^{e_{\mathfrak{q}}}$, and

$$\deg(L/K) = \sum_{\mathfrak{q}|\mathfrak{p}} e_{\mathfrak{q}} f_{\mathfrak{q}}. \tag{3.1.2}$$

For the rest of this section, we additionally assume that L/K is a finite Galois extension, that is, consider the following situation.

- *Let A be a Dedekind domain with quotient field K. Let L/K be a finite Galois field extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{q}$ be a* *non-zero prime ideal in B with $\mathfrak{p} = \mathfrak{q} \cap A$, and the finite field extension $k(\mathfrak{q})/k(\mathfrak{p})$ is a separable extension.*

So the residue extension $k(\mathfrak{q})/k(\mathfrak{p})$ is also a normal extension, and hence it is a Galois extension. For any non-zero prime $\mathfrak{p}$ of A, G acts transitively on the set of primes $\mathfrak{q}$ in B lying over $\mathfrak{p}$. So the integers $e_\mathfrak{q}$ and $f_\mathfrak{q}$ depend only on $\mathfrak{p}$, and we denote them by $e_\mathfrak{p}$ and $f_\mathfrak{p}$, respectively. If there are $g_\mathfrak{p}$ many primes of B lying over $\mathfrak{p}$, Equation (3.1.2) becomes

$$\deg(L/K) = e_\mathfrak{p} f_\mathfrak{p} g_\mathfrak{p}. \tag{3.1.3}$$

Since the decomposition group $D_\mathfrak{q}$ is the stabilizer subgroup in G of $\mathfrak{q}$, by the Orbit-Stabilizer Theorem, $g_\mathfrak{p}$ is the index of $D_\mathfrak{q}$ in G. Consider the epimorphism of groups $\epsilon\colon D_\mathfrak{q} \to \mathrm{Aut}(k(\mathfrak{q})/k(\mathfrak{p}))$ given as follows. For $g \in D_\mathfrak{q}$, $b \in B$ with image $\bar{b} \in k(\mathfrak{q})$, $\epsilon(g)(\bar{b}) = \overline{gb}$. Then the inertia group $I_\mathfrak{q}$ is the kernel of the homomorphism ϵ. In particular, we have the following.

Remark 3.5. *If the residue degree is* 1*,* $D_\mathfrak{q} = I_\mathfrak{q}$.

Finally, note that $\mathfrak{q}$ is unramified if and only if $e_\mathfrak{q} = 1$. This is equivalent to the $\mathfrak{q}$-adic valuation of the ideal $\mathfrak{p}B_\mathfrak{q}$ being 1. This in turn is equivalent to the group $I_\mathfrak{q}$ being trivial.

Another insight on the ramification criterion is given in terms of the different ideal. Consider the trace map $\mathrm{Tr}\colon L \to K$ which is a surjective K-linear homomorphism. The *codifferent* $\mathfrak{C}_{L/K}$ is defined as the fractional ideal $\{x \in L | \mathrm{Tr}(xB) \subset A\}$. Then $B \subset \mathfrak{C}_{L/K}$ and it is the maximal sub-B-module C in L such that $\mathrm{Tr}(C) \subset A$. Since A is a Dedekind domain, so is B. Also $\mathfrak{C}_{L/K}$ is non-zero. So $\mathfrak{C}_{L/K}$ has an inverse in the multiplicative group of non-zero fractional ideals of B. The *different* $\mathfrak{D}_{L/K}$ is defined to be the inverse fractional ideal $\{x \in L | x\mathfrak{C}_{L/K} \subset B\}$. Then $\mathfrak{D}_{L/K}$ is an ideal of B. By [26, Page 53, Chapter III, Section 5, Theorem 1], we have the following useful result which determines precisely when a prime of B is ramified.

Theorem 3.6 ([26, Page 53, Chapter III, Section 5, Theorem 1])**.** *Under the above notation, let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q} \cap A$. Then $\mathfrak{q}$ is unramified in the extension L/K if and only if $\mathfrak{q}$ does not contain the different ideal $\mathfrak{D}_{L/K}$.*

We recall the definition of the higher ramification groups. Let v denote the $\mathfrak{q}$-adic valuation which is the discrete valuation on $\widehat{B}_{\mathfrak{q}}$ with respect to which it is a complete local ring. As noted earlier, the residue field extension $k(\mathfrak{q})/k(\mathfrak{p})$ is a finite Galois extension, and $\widehat{L}_{\mathfrak{q}}/\widehat{K}_{\mathfrak{p}}$ is a Galois extension with $D_{\mathfrak{q}}$ as the Galois group.

Definition 3.7. Let π be a local parameter of the discrete valuation ring $\widehat{B}_{\mathfrak{q}}$. The *lower indexed ramification filtration* $\{G_i\}_{i\geq -1}$ at $\mathfrak{q}$ is defined to be

$$G_i := \{g \in D_{\mathfrak{q}} | g(\pi) \equiv \pi \pmod{\pi^{i+1}}\}.$$

Equivalently, $G_i = \{g \in D_{\mathfrak{q}} | v(g(\pi) - \pi) \geq i + 1\}$ is the subgroup of $D_{\mathfrak{q}}$ which acts trivially on the quotient ring $\widehat{B}_{\mathfrak{q}}/\pi^{i+1}$. Note that $G_{-1} = D_{\mathfrak{q}}$, $G_0 = I_{\mathfrak{q}}$, and the filtration is a ([26, Page 63, Chapter IV, Section 1, Proposition 1]) decreasing filtration by normal subgroups such that G_i is the trivial group for large i. We also refer to the groups G_i as the *higher ramification groups*. Using the above definition, we can compute the valuation of the different ideal $\mathfrak{D}_{L/K}$ at $\mathfrak{q}$ as follows.

Proposition 3.8 (Hilbert's Different formula, [26, Page 64, Chapter IV, Section 1, Proposition 4])**.** *Let $\mathfrak{D}_{L/K}$ be the different ideal of the extension L/K. Then*

$$v(\mathfrak{D}_{L/K}) = \Sigma_{i=0}^{\infty}(|G_i| - 1).$$

Remark 3.9. *The above result shows that $v(\mathfrak{D}_{L/K}) = 0$ if and only if the prime $\mathfrak{q}$ does not contain the ideal $\mathfrak{D}_{L/K}$, which by Theorem 3.6, is equivalent to $\mathfrak{q}$ being unramified in the extension L/K.*

We summarize some more useful properties of the higher ramification groups following [26, Chapter IV, Section 2]. The multiplicative group $U := \widehat{B}_{\mathfrak{q}} - \mathfrak{q}\widehat{B}_{\mathfrak{q}}$ of invertible elements of $\widehat{B}_{\mathfrak{q}}$ has a filtration given by $U^{(0)} := U$, $U^{(i)} := 1 + \mathfrak{q}^i$ for $i \geq 1$. By [26, Page 66, Chapter IV, Section 2, Proposition 6], the quotient $U^{(0)}/U^{(1)}$ is the same as the multiplicative group $k(\mathfrak{q})^{\times}$ of $k(\mathfrak{q})$, and for $i \geq 1$ the group $U^{(i)}/U^{(i+1)}$ is (non-canonically) isomorphic to the additive group of $k(\mathfrak{q})$. For $i \geq 0$, define a group homomorphism

$$\Theta_i \colon G_i/G_{i+1} \to U^i/U^{i+1}$$

given by $\Theta_i(gG_{i+1}) = g(\pi)/\pi U^{i+1}$, where π is a uniformizer of $B_\mathfrak{q}$. By [26, Page 67, Chapter IV, Section 2, Proposition 7], for each $i \geq 0$, Θ_i is a monomorphism. In particular, G_0/G_1 is a cyclic group isomorphic to a subgroup of roots of unity contained in $k(\mathfrak{q})$ under Θ_0, and so its order is coprime to the characteristic of $k(\mathfrak{q})$. The following determines the structure of the inertia groups which is of utmost importance to us.

Theorem 3.10. *Under the above notation the following hold.*

1. *Let $k(\mathfrak{q})$ be of characteristic 0. Then G_1 is the trivial group, and G_0 is a cyclic group.*

2. *Let $k(\mathfrak{q})$ be of characteristic $p > 0$. Then the quotient G_0/G_1 is a cyclic group of order prime-to-p, and for each $i \geq 1$, G_i/G_{i+1} is an abelian group, which is either trivial or is an elementary abelian p-group. In particular, G_1 is a p-group, and $G_0 = G_1 \rtimes (G_0/G_1)$.*

The following yields a useful conjugation result.

Proposition 3.11 ([26, Page 69, Chapter IV, Section 2, Proposition 9])**.** *For $\alpha \in G_0$, $\tau \in G_i/G_{i+1}$ and $i \geq 0$, we have*

$$\Theta_i(\alpha\tau\alpha^{-1}) = \Theta_0(\alpha^i) \cdot \Theta_i(\tau).$$

We end this section by by recalling the definition of the *upper indexed ramification filtration* of the decomposition group $D_\mathfrak{q}$. First extend the definition of G_i as follows. For any real number $u \geq -1$, set $G_u := G_i$ where i is the smallest integer ≥ 1. Then we have a continuous, piece wise linear, increasing function

$$\alpha(u) = \int_0^u \frac{dt}{[G_0 : G_t]}. \tag{3.1.4}$$

It can be seen that this function α is a homeomorphism from $[-1, \infty)$ to itself. Let β be the inverse map. Define the upper numbering of the ramification groups as

$$G^v := G_{\beta(v)}. \tag{3.1.5}$$

Definition 3.12. A *lower jump* at $\mathfrak{q}$ is defined to be an integer $i \geq 0$ such that $G_{i+1} \neq G_i$. An *upper jump* at $\mathfrak{q}$ is defined to be a real number v such that $G^v \neq G^{v+\epsilon}$ for all $\epsilon > 0$.

3.2 Covers of Curves and Ramification Theory

In this section, we recall the basic notion and properties of the covers of a curve. We will also see the associated ramification theoretic properties.

A morphism $\psi\colon Y \to X$ of schemes is said to be *generically separable* if there is an open affine cover $\{U_i = \operatorname{Spec}(R_i)\}$ of X such that each extension $R_i \subset \mathcal{O}_Y(\psi^{-1}(U_i))$ of domains is generically separable (see Definition 3.1).

Definition 3.13. Let X be a scheme. A *cover* of X is defined to be a finite generically separable surjective morphism $Y \to X$ of schemes. We say that the cover is *connected* (respectively, *normal*) if X and Y are both connected (respectively, a normal scheme).

In practice, we will mostly consider the covers of regular integral curves, and hence the covers will be normal.

Definition 3.14. Let $\psi\colon Y \to X$ be a normal cover. We say that a closed point $y \in Y$ is *unramified* over $\psi(x)$ if extension $\widehat{\mathcal{O}}_{Y,y}/\widehat{\mathcal{O}}_{X,\psi(x)}$ of complete local rings is unramified (cf. Definition 3.2). We say that the cover ψ is *unramified* if every closed point $y \in Y$ is unramified.

Note that since ψ is generically separable, the generic points are always unramified in the sense of Definition 3.2. Recall that an étale morphism of a scheme is defined to be a flat, unramified morphism of finite type. We will use étale cover to mean that the morphism is a finite, étale morphism. The following well known result shows that when X is either a regular curve or a surface and when Y is normal, any finite morphism $Y \to X$ is flat (for example, see the proof of [11, Proposition 4(b)]).

Lemma 3.15. *Let $A \subset B$ be an extension of Noetherian domains, either A is of dimension 1 or both A and B are of dimension 2. Let A be a regular ring, and B be a normal domain which is finitely generated as an A-module. Then B is flat over A.*

Proof. When A is of dimension 1, it is also normal being regular. So A is a Dedekind domain. Since B is also an integral domain, it is torsion-free over A and hence is flat. Now assume that A and B are of dimension 2. Consider a maximal ideal $\mathfrak{m}$ of A. Set $C := B \otimes_A A_{\mathfrak{m}}$. By [5, Theorem 3.7], $\text{projdim}_{A_{\mathfrak{m}}}(C) + \text{depth}_{A_{\mathfrak{m}}}(C) = \text{depth}(A_{\mathfrak{m}})$. Since C is normal of dimension 2, it is Cohen-Macaulay, and hence both the depths are equal to 2. So C is free over $A_{\mathfrak{m}}$ and hence B is flat over A. □

Now we define a Galois cover of a scheme. The *automorphism group* $\text{Aut}(Y/X)$ of a cover $\psi\colon Y \to X$ is the group of automorphisms σ of Y such that $\phi \circ \sigma = \phi$.

Definition 3.16. Let G be a finite group. A *G-Galois cover* is a cover $Y \to X$ of schemes together with an inclusion $\rho\colon G \hookrightarrow \text{Aut}(Y/X)$ such that G acts simply transitively on each generic geometric fibre.

When X is a connected integral scheme, the inclusion ρ is necessarily an isomorphism. If the inclusion ρ is not specified, we simply say that the cover $Y \to X$ is Galois with group G. We will see shortly some examples (Example 3.23, 3.24) of Galois covers of the projective line.

Now onward, let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective k-curve, and $k(X)$ be its function field. Let $\psi\colon Y \to X$ be a cover of smooth connected projective k-curves. Note that for any affine open subset $U \subset X$, $\mathcal{O}_X(U)$ is a Dedekind domain.

By Definition 3.13, the extension $k(Y)/k(X)$ of function fields is a finite separable extension. Let $x \in X$ be a closed point, and let $y \in \psi^{-1}(x) \subset Y$. In this context we say that the point y *lies* over x. Let $\mathfrak{m}_x$ and $\mathfrak{m}_y$ denote the unique maximal ideal of the complete discrete valuation rings $\widehat{\mathcal{O}}_{X,x}$ and of $\widehat{\mathcal{O}}_{Y,y}$, respectively. Then $\widehat{\mathcal{O}}_{Y,y}$ is the integral closure of the ring $\widehat{\mathcal{O}}_{X,x}$ in $K_{Y,y}$. Also note that $K_{Y,y}$ is equal to the compositum $k(Y) \cdot K_{X,x}$ of fields. So the field extension $K_{Y,y}/K_{X,x}$ is a finite separable extension. Since X is a curve over the field k, by Lemma 3.15, the covering morphism ψ is flat. So y is unramified over x is equivalent to the finite morphism ψ being étale at y. The *ramification index* of y over x is defined to be the integer $e(y|x) := e_{\mathfrak{m}_y}$ (see Equation (3.1.1)). By

Definition 3.4 and by the discussion in Section 3.1, y is unramified over x if $e(y|x) = 1$ and is *ramified* when $e(y|x) > 1$.

Definition 3.17. Under the above notation, if y is ramified over x, we also say the x is a *branched point*. The *branch locus* of the cover ϕ is defined to be the set of branched points in X. We also say that the cover ϕ is *étale away from a subset* B in X if the branch locus of the cover ϕ is contained in the set B.

Since $k(Y)/k(X)$ is an separable extension, the branch locus of the cover ϕ is a finite set of closed points in X. Since k is algebraically closed, $k(\mathfrak{m}_y) = k = k(\mathfrak{m}_x)$. By Equation (3.1.2),

$$\deg(k(Y)/k(X)) = \sum_{y \in \phi^{-1}(x)} e(y|x),$$

and $e(y|x)$ is equal to the degree of the field extension $K_{Y,y}/K_{X,x}$.

Now we deal with the Galois covers of smooth projective k-curves.

Definition 3.18. Let $\psi\colon Y \to X$ be a cover of smooth projective connected k-curves. Let $L/k(X)$ be the Galois closure of the finite extension $k(Y)/k(X)$ of function fields, and Z be the smooth projective connected k-curve with function field L. The *Galois closure* of ψ is defined to be the connected Galois cover $Z \to X$, the normalization of X in L.

Let $\phi\colon Z \to X$ be the Galois closure of ψ with Galois group G. It is well-known ([22, Lemma 4.2]) that the branch locus for the covers ψ and ϕ are the same. We denote it by B. Let $x \in B$. Then G acts transitively on the set $\phi^{-1}(x)$, the set of points in Z lying over x. In view of Definition 3.3, we have the following.

Definition 3.19. Let $z \in Z$ be a point lying above x. Let π_z be a local parameter of the discrete valuation ring $\mathcal{O}_{Z,z}$. The *inertia group* I_z at z in the cover ϕ is the subgroup of G which acts by the identity on $\mathcal{O}_{Z,z}/\pi_z \cong \widehat{\mathcal{O}}_{Z,z}/\pi_z$.

Since k is an algebraically closed field, the residue degree is one. By Remark 3.5, the decomposition group $D_{\mathfrak{m}_z} = I_z$. For two points z and z' in Z lying above x, the groups I_z and $I_{z'}$ are conjugates in G. So up to conjugacy, we can talk about the *inertia group* above x which we denote by I_x. By Theorem 3.10, the inertia group I_x is of the form $P \rtimes \mathbb{Z}/m$ for some p-group P and m coprime to p. As in Definition 3.7, there are lower indexed and upper indexed filtrations of the inertia group I_z, denoted by $\{I_{z,i}\}_{i\geq 0}$ and $\{I_z^a\}_{a\in[-1,\infty)}$, respectively. So if π_z be a uniformizer of $\mathcal{O}_{Z,z}$, $I_{z,i} = \{g \in I_z | g(\pi_z) \equiv \pi_z \mod \pi_z^{i+1}\}$ and $I_z^a := I_{z,\beta(a)}$, where β is the the inverse of the map α given by Equation (3.1.4). By Theorem 3.10, $I_{z,1}$ is the p-group $p(I_z)$.

Definition 3.20. Under the above notation we define the following.

1. The *purely wild part* of I_z is defined to be the subgroup $I_{z,1} = p(I_z)$.
2. If $I_z = p(I_z)$ is non-trivial, we say that the ramification at z is *purely wild*.
3. We say that the ramification at z is *tame* if $(|I_z|, p) = 1$, and is *wild* otherwise.
4. A lower (respectively, upper) jump at z is defined to be a lower (respectively, upper) jump at the prime $\mathfrak{m}_z$ (cf. Definition 3.12).
5. The *conductor* is defined to be the minimal integer $h \geq 1$ such that $I_{z,h+1}$ is the trivial group. If $|I_x| = pm$, $p \nmid m$, the *upper jump* is defined to be $\sigma := h/m$.

Note that the conductor is the highest jump in the lower indexed ramification filtration. The lower and the upper jumps behave as follows with respect to the formation of subgroups and quotients.

Proposition 3.21 ([26, Chapter IV])**.** *Let $z \in \phi^{-1}(x) \subset Z$ and let I denote the inertia group at z. Let J be a subgroup of I. Then $J_i = I_i$ for all $i \geq 0$. If J is a normal subgroup of I, then $(I/J)^v = I_v J/J$ for all $v \geq -1$. Moreover, if $J = I_j$ for some $j \geq 0$, $(I/J)_i = I_i/J$ for $i \leq j$ and is trivial otherwise.*

The *ramification divisor* associated to the cover ϕ is defined to be the Weil divisor $D_\phi = \Sigma_{z\in Z} v_z(\mathfrak{D}_{z/x})z$ on Z, where $\mathfrak{D}_{z/x}$ is the different ideal $\mathfrak{D}_{K_{Y,y}/K_{X,x}}$ and v_z is the $\mathfrak{m}_z$-adic valuation on $\widehat{\mathcal{O}}_{Y,y}$. Using the Hilbert's different formula (Proposition 3.8), the Riemann Hurwitz formula associates the genus of the curves with the degree of the ramification divisor.

Proposition 3.22 (Riemann Hurwitz formula)**.** *Let $\phi\colon Z \to X$ be a G-Galois cover of smooth projective connected k-curves. Let Z has genus g_Z and X has genus g_X. For a point $z \in Z$, let $\{I_{z,i}\}_{i\geq 0}$ be the lower indexed ramification filtration of the inertia group at z. Then*

$$2g_Z - 2 = |G|(2g_X - 2) + deg(D_\phi) = |G|(2g_X - 2) + \Sigma_{z\in Z}\Sigma_{i=0}^{\infty}(|I_{z,i}| - 1).$$

The following well known examples of Galois covers of $\mathbb{P}^1$ shows the ramification theoretic properties discussed above.

Example 3.23. (Artin-Schrier Covers) Let p be a prime, h be coprime to p. Let k be an algebraically closed field of characteristic p. Let $f(x) \in k[x]$ be a polynomial of degree h, and α be a root of the polynomial $g(x, y) = y^p - y - f(x)$ in a splitting field over $k(x)$. Then the $\mathbb{Z}/p$-Galois field extension $L = k(x)[\alpha]/k(x)$ corresponds to a $\mathbb{Z}/p$-Galois cover $\phi\colon Y \to \mathbb{P}^1$ of smooth projective connected k-curves, where $k(Y) = L$. For a choice τ of a generator of $\mathbb{Z}/p$, the $\mathbb{Z}/p$-action is given by $\tau(\alpha) = \alpha + 1$. The y-derivative of the polynomial $g(x, y)$ is $-1 \neq 0$. So the cover ϕ is unramified over $\mathbb{A}^1 = \mathrm{Spec}(k[x])$. From the equation $g(x, y) = 0$ it follows that $v_{x=\infty}(y^{-1}) = h$. So ϕ is totally ramified over ∞. Let π be a uniformizer of $\widehat{O}_{Y,(y=\infty)}$, and let v_∞ denote the corresponding valuation. After possibly a change of variable, we have $\pi = y^{-1/h}$ and $\widehat{O}_{Y,(y=\infty)} = k[[y^{-1/h}]]$. The τ action on y^{-1} is given by

$$\tau(y^{-1}) = \frac{1}{y+1} = \frac{y^{-1}}{1+y^{-1}} = \frac{\pi^h}{1+\pi^h}.$$

So $v_\infty(\tau(\pi) - \pi) = v_\infty(-\frac{1}{h}\pi^{h+1} + \cdots) = h + 1$, and hence the conductor at ∞ is equal to h. Furthermore, using The Riemann Hurwitz formula (Proposition 3.22), Y has genus $\frac{(h-1)(p-1)}{2}$.

Example 3.24. (Kummer Cover) Let p be a prime, and n be coprime to p. Consider the cover $\phi\colon Y \to \mathbb{P}^1_x$ corresponding to the extension

$$k(x) \hookrightarrow k(x)[y]/(y^n - x).$$

Since x is given in terms of y, $Y \cong \mathbb{P}^1_y$. Then ϕ is a connected $\mathbb{Z}/n = \langle\sigma\rangle$-Galois cover with the action given by $\sigma(y) = \zeta_n y$, where ζ_n is an n^{th} root of unity in k. The y-derivative of $g(x, y) := y^n - x$ is given by ny^{n-1}. Thus $g(x, y)$ and its y-derivative have a common root if and only $(x, y) = (0, 0)$. So ϕ is étale away from $\{0, \infty\}$. When $n = 1$, ϕ is a the identity map of $\mathbb{P}^1$. So let $n > 1$. Since $v_x(y) = n = v_{x^{-1}}(y^{-1})$, ϕ is totally ramified over the points 0 and ∞.

We end this section with the following useful definition mentioned in Chapter 2.

Definition 3.25. For a finite group G and a subgroup I of G, we say that the pair (G, I) is *realizable* if there exists a connected G-Galois cover of $\mathbb{P}^1$ branched only at ∞ such that I occurs as an inertia group above ∞.

In this case, G is necessarily a quasi p-group, that is, $G = p(G)$ (see Chapter 2 for notation).

3.3 Étale Fundamental Group

In this short section, we recall the notion of the étale fundamental group of a connected scheme and some of its important quotients. Let U be a connected scheme, and Ω be an algebraically closed field. Let $\bar{u}\colon \mathrm{Spec}(\Omega) \to U$ be a geometric point. Consider the functor

$$F_{\bar{u}}\colon (\text{ finite étale covers of U}) \to \text{ finite sets}$$

which to any finite étale cover $f\colon V \to U$ of schemes associates the set of geometric points $\bar{v}\colon \mathrm{Spec}(\Omega) \to V$ such that $f \circ \bar{v} = \bar{u}$.

Definition 3.26. Let U be a connected scheme, and let $\bar{u}\colon \mathrm{Spec}(\Omega) \to U$ be a geometric point. The *étale fundamental group* $\pi_1(U, \bar{u})$ is defined to be the automorphism group of the functor $F_{\bar{u}}$.

It follows that $\pi_1(U, \bar{u})$ is a profinite group, and for any finite étale cover $V \to U$, the set $F_{\bar{u}}(V)$ is a finite set with a continuous $\pi_1(U, \bar{u})$-action. So the functor $F_{\bar{u}}$ factors as

$$F_{\bar{u}}\colon (\text{ finite étale covers of U}) \to \text{ finite } \pi_1(U, \bar{u}) - \text{sets} \to \text{ finite sets,}$$

where the later functor is the forgetful functor. One important result is that $F_{\bar{u}}$ induces an equivalence of categories between the category of finite étale covers of U and the category of finite sets equipped with a continuous $\pi_1(U, \bar{u})$-action. For a different choice of the geometric point $\bar{u}'$, the groups $\pi_1(U, \bar{u})$ and $\pi_1(U, \bar{u}')$ are isomorphic up to an inner automorphism of either of the groups. So we can talk of the group $\pi_1(U)$, up to its isomorphism class as a profinite group, without considering the base point. It can also be shown that the profinite group $\pi_1(U)$ is the inverse limit

$$\pi_1(U) = \varprojlim \operatorname{Aut}(V/U)$$

where V varies over all connected finite Galois étale covers $V \to U$. So any continuous finite quotient G of $\pi_1(U)$ corresponds to a connected Galois étale cover of U with group G. When the surjective homomorphism $\pi_1(U) \twoheadrightarrow G$ is specified, there is an isomorphism $G \to \operatorname{Aut}(V/U)$ associated to the étale Galois cover.

We are interested in the case when U is a smooth affine connected curve over an algebraically closed field k. There is a unique smooth projective connected k-curve X containing U. Also any étale cover $V \to U$ extends uniquely to a cover $Y \to X$ of smooth connected projective k-curves. The cover $Y \to X$ need not be étale, and its branch locus is contained in $X - U$. Conversely, given any cover $\phi\colon Y \to X$ of smooth connected projective k-curves which is étale away from a finite set B of closed points in X, the cover $\phi\colon \phi^{-1}(X - B) \to X - B$ is a connected étale cover of curves. With this in view, we recall some important quotients of the étale fundamental group in this setup.

Definition 3.27. Let U be a smooth connected curve over an algebraically closed field k of characteristic $p > 0$. Let X be the smooth projective completion of U. Set $B := X - U$.

1. The *tame fundamental group* $\pi_1^t(U)$ is defined to be the inverse limit

$$\pi_1^t(U) := \varprojlim \mathrm{Aut}(\phi^{-1}(U)/U)$$

where ϕ varies over all the connected finite Galois covers $\phi\colon Y \to X$ which are étale away from B and are tamely ramified over B.

2. The *prime-to-p fundamental group* $\pi_1^{p'}(U)$ is defined to be the inverse limit

$$\pi_1^{p'}(U) := \varprojlim \mathrm{Aut}(V/U)$$

where V varies over all connected finite Galois étale covers $V \to U$ whose Galois group has order prime-to-p.

3. The *pro-p fundamental group* is defined to be the inverse limit $\varprojlim \mathrm{Aut}(V/U)$ where V varies over all connected finite Galois étale covers $V \to U$ whose Galois group is a p-group.

Note that $\pi_1^{p'}(U)$ is the maximal prime-to-p quotient of the group $\pi_1(U)$. It is clear from the definition that $\pi_1^t(U)$, $\pi_1^{p'}(U)$ and $\pi_1^p(U)$ are quotients of $\pi_1(U)$, and furthermore, $\pi_1^{p'}(U)$ is a quotient of $\pi_1^t(U)$.

3.4 Formal Patching

In this section, we recall the notion of formal patching for covers of schemes. As a consequence of the Riemann Existence Theorem ([1, page 332, Exposé XII, Theorem 5.1]), covers of smooth connected complex algebraic curves can be constructed by a cut and paste method in analytic topology. However, this method is not applicable over a more general field (for example, over an algebraically closed field k of characteristic $p > 0$) because the Zariski topology is too week to patch covers defined over its vast open sets. To this end, one works with schemes over $k[[t]]$, and uses formal geometry. One such patching result is [11, Theorem 1]. Using this technique together with a Lefschetz type result we can eventually construct Galois covers of k-curves with the required type of Galois group and inertia groups. Although this technique works with a certain generality, we only consider patching over the base ring $k[[t]]$ for a field k. We start by recalling this patching technique following [18]. Consider the following notation.

Notation 3.28. Let X be an integral scheme. Let $\mathcal{M}(X)$ denote the category of coherent sheaves of O_X-modules, and $\mathcal{P}(X)$ denote the subcategory of $\mathcal{M}(X)$ consisting of projective coherent O_X-modules. Let $\mathcal{A}(X)$ and $\mathcal{AP}(X)$ be the category of coherent O_X-algebras which are the full subcategories of $\mathcal{M}(X)$ and $\mathcal{P}(X)$, respectively. By $\mathcal{S}(X)$ and $\mathcal{SP}(X)$ we denote the categories of generically separable (cf. Section 3.2) locally free sheaves of O_X-algebras which lie in $\mathcal{M}(X)$ and in $\mathcal{P}(X)$, respectively, as O_X-modules. For any finite group G, $G(X)$ and $G\mathcal{P}(X)$ denote the corresponding subcategories of G-Galois O_X-algebras.

Let k be any field. Consider the power series ring $R := k[[t]]$ in one variable over k. An R-curve is defined to be a scheme X together with a flat proper morphism $X \to \operatorname{Spec}(R)$ whose geometric fibres are reduced connected curves. We work with the following setup.

Let T^* be a regular irreducible projective R-curve with the closed fibre T^0. Let $S^0 \subset T^0$ be a non-empty finite set of closed points that contains all the singular points of T^0.

For any affine open subset U of the regular affine curve $T^0 - S^0$, consider an affine open subset $\widetilde{U}$ of T^* whose closed fibre is U. Set U^* to be the t-adic completion of $\widetilde{U}$. The definition of U^* is independent of the choice of $\widetilde{U}$. For a point $s \in S^0$, let $K_{T^*,s}$ denote the function field of $\widehat{O}_{T^*,s}$. So $\operatorname{Spec}(K_{T^*,s})$ is the the t-adic completion of $\operatorname{Spec}(\widehat{O}_{T^*,s}) - s$.

Definition 3.29. The module patching problem $\bar{M}$ for the pair (T^*, S^0) consists of the following data.

1. for every irreducible component U of $T^0 - S^0$, a finite O_{U^*}-module M_U;

2. for every point $s \in S^0$, a finite $\widehat{O}_{T^*,s}$-module M_s;

3. (patching data) for each point $s \in S^0$ and each irreducible component U of $T^0 - S^0$ whose closure in T^* contains s, a $K_{T^*,s}$-module isomorphism

$$\mu_{s,U} \colon M_U \otimes_{O_{U^*}} K_{T^*,s} \to M_s \otimes_{\widehat{O}_{T^*,s}} K_{T^*,s}.$$

A module patching problem $\bar{M}$ for (T^*, S^0) as above can be summarized with the following diagram.

$$\begin{array}{ccccccc} O_{U^*} & \longrightarrow & K_{T^*,s} & = & K_{T^*,s} & \longleftarrow & \widehat{O}_{T^*,s} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_U & \longrightarrow & M_U \otimes_{O_{U^*}} K_{T^*,s} & \xrightarrow[\mu_{s,U}]{\sim} & M_s \otimes_{\widehat{O}_{T^*,s}} K_{T^*,s} & \longleftarrow & M_s \end{array}$$

Definition 3.30. A morphism between patching problems $\bar{M} = (\{M_U\}, \{M_s\}, \{\mu_{U,s}\})$ and $\bar{M}' = (\{M'_U\}, \{M'_s\}, \{\mu'_{U,s}\})$ is defined to be a collection of morphisms $M_U \to M'_U$ and $M_s \to M'_s$ of $\widehat{O}_{U^*}$ and $\widehat{O}_{T^*,s}$-modules respectively, which are compatible with the $K_{T^*,s}$-module isomorphisms.

Now consider the category $\mathcal{M}(T^*, S^0)$ of module patching problems. Similarly we define the categories $\mathcal{P}(T^*, S^0)$, $\mathcal{A}(T^*, S^0)$ and so on. There is a natural base change functor

$$\beta_{\mathcal{M},S^0} \colon \mathcal{M}(T^*) \to \mathcal{M}(T^*, S^0).$$

Similarly there are base change functors for $\mathcal{P}(T^*, S^0)$, $\mathcal{A}(T^*, S^0)$ and so on, which we denote by replacing the subscript $\mathcal{M}$ in β accordingly. The following are the main results of formal patching.

Theorem 3.31. *Let T^* be a regular irreducible projective $R = k[[t]]$-curve with the closed fibre T^0. Let $S^0 \subset T^0$ be a non-empty finite set of closed points containing all the singular points of T^0. Then the following hold.*

1. ([16, Theorem 3.2.12]) *The base change functor $\beta_{\mathcal{M},S^0} \colon \mathcal{M}(T^*) \to \mathcal{M}(T^*, S^0)$ is an equivalence of categories. The same result holds for the functors $\beta_{\mathcal{A},S^0}$ and β_{G,S^0} for any finite group G.*

2. ([18, Theorem 1]) *The base change functors $\beta_{\mathcal{P},S^0}$, $\beta_{\mathcal{AP},S^0}$ and $\beta_{G\mathcal{P},S^0}$ for any finite group G are equivalences of categories.*

Corollary 3.32 ([18, Corollary to Theorem 1]). *Under the hypothesis of Theorem 3.31 with $U = T^0 - S^0$, suppose that $Y_U \to U^*$ and for each $s \in S^0$, $Y_s \to Spec(\widehat{O}_{T^*,s})$ are normal (respectively, normal G-Galois for a finite group G) covers. Also assume that for each $s \in S^0$, there is an isomorphism (respectively, isomorphism as G-Galois covers) $Y_U \times_{U^*} Spec(K_{T^*,s}) \cong Y_s \times_{Spec(\widehat{O}_{T^*,s})} Spec(K_{T^*,s})$. Then there is a unique normal cover (respectively, a normal G-Galois cover) $Y \to T^*$ the induces the given covers, compatible with the given isomorphisms.*

Remark 3.33. *Note that by Lemma 3.15, the covers $Y_U \to U^*$ and $Y_s \to Spec(\widehat{O}_{T^*,s})$ corresponds to elements in $\mathcal{AP}(U^*)$ and in $\mathcal{AP}(Spec(\widehat{O}_{T^*,s}))$ respectively.*

Corollary 3.32 generalizes the following result from [11].

Corollary 3.34 ([11, Proposition 4(b)]). *Let X be a smooth projective connected curve over a field k and $x \in X$ be a closed point. Let $U = X - x = Spec(A)$. Suppose that $Y_1 \to Spec(A[[t]])$ and $Y_2 \to Spec(\widehat{O}_{X,x}[[t]])$ are normal (respectively, normal G-Galois for a finite group G) covers. Assume that there is an isomorphism (G-Galois equivariant isomorphism) between the induced covers over $Spec(K_{X,x}[[t]])$. Then there is a unique normal cover (respectively, a normal G-Galois cover) $Y \to X \times_k k[[t]]$ the induces Y_1 and Y_2 compatibly with the given isomorphism over $Spec(K_{X,x}[[t]])$.*

Using formal patching, Harbater showed that the wild part of the inertia groups of a given cover can be increased.

Theorem 3.35 ([15, Theorem 3.6], [11, Theorem 2]). *Let G be a finite group, H be a subgroup of G, and let $\psi\colon Y \to X$ be an H-Galois cover of smooth connected projective curves over an algebraically closed field k of characteristic $p > 0$ which is étale away from a finite non-empty set $B = \{x_1, \cdots, x_r\}$ of closed points in X. For $1 \le i \le r$, let $I_i = P_i \rtimes E_i$ occurs as an inertia group above x_i for a p-group P_i and a cyclic group E_i of order prime-to-p. For each i, suppose that $I_i' = P_i' \rtimes E_i$ is a subgroup of G such that P_i' is a p-group containing P_i. Then there is a normal G-Galois cover $\phi\colon Z \to X$ étale* *away from B such that I_i' occurs as an inertia group above x_i and ϕ dominates the cover ψ. Moreover, if $G = \langle H, I_1', \cdots, I_r' \rangle$, the cover ϕ is a connected cover.*

We end this section by an explicit construction of T^* which will be used later.

Remark 3.36. *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective connected k-curve. Let T be the blow-up of $X \times \mathbb{P}^1_t$ at the point $(x, t = 0)$. The total transform T^0 of the zero locus of $(t = 0)$ consists of two irreducible components, a copy of X and the exceptional divisor $\mathbb{P}^1_y$ meeting at the point τ. Let T^* be the formal completion of T along T^0. So T^* is an irreducible projective $k[[t]]$-curve whose closed fibre is T^0. The rational functions $x + y$ and t define morphisms $T \to \mathbb{P}^1$ which we denote by $\pi(x + y)$ and $\pi(t)$, respectively. Consider the finite generically separable map $(\pi(x+y), \pi(t))\colon T \to \mathbb{P}^1_z \times \mathbb{P}^1_t$. So we have a cover $T^* \to \mathbb{P}^1_z \times Spec(k[[t]])$. Also T in a neighborhood of τ is given by the equation $t = xy$. So $\widehat{T}^* = Spec(\widehat{O}_{T^*,\tau}) = Spec(k[[x, y]][t]/(t - xy)) = Spec(k[[x, y]])$.*

Chapter 4

Main Problems

4.1 Motivating Problems

For a detail historic motivation behind the conjectures posed by Abhyankar, see [17]. In this section, we present some of these conjectures and results which motivate our main problems. These are related to the structure of the étale fundamental group of a smooth connected affine curve.

Let k be an algebraically closed field, and U be a smooth connected affine k-curve. Let $U \subset X$ be the smooth projective completion. Suppose that X has genus g, and $B := X - U$ consists of $r \geq 1$ closed points. We consider the étale fundamental group $\pi_1(U)$ (see Definition 3.26) as a profinite group, ignoring the base point. First assume that k is a field of characteristic 0. Then it is known (using localization and Lefschetz theorems) that the structure of $\pi_1(U)$ is independent of the base field, and $\pi_1(U)$ is isomorphic to the profinite completion of the topological fundamental group $\Pi := \pi_1^{\text{top}}(U(\mathbb{C}))$ of the topological space of the $\mathbb{C}$-points of U. More precisely, by [1, XIII, Corollary 2.12], $\pi_1(U)$ is the profinite group $\widehat{\Pi}$ on $2g + r$ generators $a_1, \cdots, a_g, b_1, \cdots, b_g, c_1, \cdots, c_r$ which satisfy $\prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} c_j = 1$. Thus $\pi_1(U)$ is a free profinite group which is topologically finitely generated by $2g + r - 1$ elements. So $\pi_1(U)$ is determined by its finite quotients. A finite group G occurs as a Galois group of a connected étale cover of U if and only if it is generated by at $2g + r - 1$ elements. In particular, $\pi_1(\mathbb{A}^1_{\mathbb{C}}) = \{1\}$.

For the rest of the chapter assume that k has characteristic $p > 0$.

It can be seen from Example 3.23 that the above results are no longer true. In fact, by the Artin Schreier theory, there are infinitely many linearly disjoint connected $\mathbb{Z}/p$-Galois étale covers of U, and consequently, the étale fundamental group of a smooth connected affine curve in positive characteristic is never finitely generated. One of the fundamental problem in arithmetic geometry is to understand the structure of the group $\pi_1(U)$. The full structure of $\pi_1(U)$ is not known in this case. [1, XIII, Corollary 2.12] describes the following results for certain quotients (see Definition 3.27) of $\pi_1(U)$ given as follows.

1. $\pi_1^t(U)$ is a quotient of $\widehat{\Pi}$ and hence is a finitely generated profinite group.
2. $\pi_1^{p'}(U)$ is isomorphic to the maximal prime-to-p quotient of the group $\widehat{\Pi}$.

We note that although the tame fundamental group $\pi_1^t(U)$ is a finitely generated group, its structure is not understood in general. More precisely, we do not know all the finite groups that occur as the Galois groups of tamely ramified connected Galois covers of U.

Another aspect of the study of $\pi_1(U)$ is to understand the set $\pi_A(U)$ of isomorphism classes of finite (continuous) group quotients of $\pi_1(U)$, or equivalently, the groups which occur as the Galois groups of connected étale covers of U. Although this set does not determine the full structure of $\pi_1(U)$ (this group not being finitely generated), it gives some idea of the possible Galois étale covers of U. As before, for any finite group G we denote the maximal quasi p-subgroup of G by $p(G)$. It is generated by all elements of p-power order. In 1957 Abhyankar conjectured (known as Abhyankar's Conjecture on the affine curves, [2]) on what groups can occur in the set $\pi_A(U)$, which is now a Theorem. The forward direction of the conjecture follows from the work of Grothendieck. The backward direction was shown to be true by Serre (for solvable groups and U as the affine line, [27]), Raynaud (for general groups and $U = \mathbb{A}^1$, [25]), Harbater (general case, [12, Theorem 6.2]).

Theorem 4.1 (Abhyankar's Conjecture on the affine curves; [2, Section 4.2]). *Let X be a smooth projective connected curve of genus g over an algebraically field k of* *characteristic $p > 0$. Let $r \geq 1$ and B be a finite set of closed points in X. Set $U := X - B$. Then a finite group G occurs as the Galois group of an étale cover $V \to U$ of smooth connected k-curves if and only if $G/p(G)$ is generated by at most $2g + r - 1$ elements.*

The following is a special case of the above result.

Theorem 4.2 (Abhyankar's Conjecture on the affine line; [25, Corollary 2.2.2])**.** *Over an algebraically closed field k of characteristic $p > 0$, there is a Galois cover $Y \to \mathbb{P}^1$ of smooth connected projective k-curves branched only at ∞ with group G if and only if G is a quasi p-group, i.e. $G = p(G)$.*

4.2 The Inertia Conjecture

In this section, we state one of our main problems, the Inertia Conjecture, which is a more refined statement about the local ramification behaviour of the connected Galois étale covers of $\mathbb{A}^1$.

As before, let k be an algebraically closed field of characteristic $p > 0$. Theorem 4.2 provides a classification of all the finite groups that occur as the Galois groups of the connected étale covers of the affine line, namely the quasi p-groups. The next natural question is that given a quasi p-group G, what are the inertia groups that occur over ∞ in a G-Galois covers of $\mathbb{P}^1$ étale away from ∞? Abhyankar's study of the branched covers of $\mathbb{P}^1$ given by explicit equations led him to the following conjecture, known as the Inertia Conjecture (IC).

Conjecture 4.3 (IC, [4, Section 16])**.** *Let G be a finite quasi p-group. Let $I \subset G$ be an extension of a p-group P by a cyclic group of order prime-to-p. Then there is a connected G-Galois cover of $\mathbb{P}^1$ étale away from ∞ such that I occurs as an inertia group at a point over ∞ if and only if the conjugates of P in G generate G.*

A special case of the above conjecture is when I is a p-group. This is known as the Purely Wild Inertia Conjecture (PWIC).

Conjecture 4.4 (PWIC). *Let G be a finite quasi p-group. A p-subgroup P of G occurs as the inertia group at a point above ∞ in a connected G-Galois cover of $\mathbb{P}^1$ branched only at ∞ if and only if the conjugates of P generate G.*

The forward direction of the IC (Conjecture 4.3), namely, if such a cover exists, then the conjugates of the p-subgroup P in G generate G, can be solved using the fact that the tame fundamental group of the affine k-line is trivial. So the question remains whether for each possible subgroup I of G which satisfy the necessary conditions of Conjecture 4.3, the pair (G, I) is realizable (cf. Definition 3.25).

Using a formal patching technique, Harbater has shown (Theorem 3.35) that the purely wild part of the inertia groups can be enlarged. In particular, the PWIC is true when P is a Sylow p-subgroup of G. As a consequence, the PWIC is true for any quasi p-group whose order is strictly divisible by p.

In general, the IC has been proved to be true only in a few cases (see [8], [23], [22], [20], [10] and [17] for more details) and even the PWIC remains wide open at this moment. Previously the following affirmative results were known.

Theorem 4.5. *The IC is true for the following groups G.*

1. ([8, Theorem 1.2]) $p \geq 5$ *and G is A_p or $PSL_2(p)$.*

2. ([22, Theorem 1.2]) $p \equiv 2 \pmod 3$ *is an odd prime and $G = A_{p+2}$.*

The following is the first existence result for covers of the affine line whose inertia groups are strictly contained in a Sylow p-subgroup.

Theorem 4.6 ([23, Corollary 3.6]). *Let $p \geq 7$. Suppose l is a prime such that $PSL_2(l)$ has order divisible by p. If I is cyclic of order p^r or dihedral of order $2p^r$ with $1 \leq r \leq v_p(|PSL_2(l)|)$, the pair (G, I) is realizable.*

Other examples of the realization of the inertia groups can be found in [20, Section 4].

Another related open problem is the 'minimal upper jump problem' ([7]). Let G be a quasi p, transitive permutation group of degree d, and suppose that $\phi\colon Y \to \mathbb{P}^1$ is a G-Galois cover of smooth connected projective k-curves branched only at ∞ such that $P \rtimes \mathbb{Z}/m$ occurs as an inertia group over ∞ where $P \cong \mathbb{Z}/p$ and $(p, m) = 1$. If the upper jump for this cover over ∞ is σ, by [24, Theorem 2.2.2], for any $i \geq 1$ with $p \nmid (\sigma + i)m$, there is a connected G-Galois étale cover of the affine line such that $P \rtimes \mathbb{Z}/m$ occurs as an inertia group above ∞ and the upper jump is $\sigma + i$. One can see that for a cover ϕ as above to exist, σ has a certain form that will be made explicit in Theorem 5.9(3). This imposes a lower bound on the possible upper jumps for the above kind of covers when G and P are fixed. The "minimal possible upper jump" problem asks whether for G and P as above (i.e. $G = \langle P^G \rangle$) this lower bound is attained for a suitable G-Galois étale cover of the affine line with the inertia group above ∞ of the form $P \rtimes \mathbb{Z}/m$ for some m coprime to p.

Question 4.7. *Let G be a quasi p group which is a transitive permutation group of degree $d = p + t$, $t \geq 1$, and $P \cong \mathbb{Z}/p$ be a p-cyclic subgroup of G such that the pair (G, P) is realizable. Does there exist a connected G-Galois étale cover of the affine line which realizes the minimal possible upper jump?*

Such questions were studied and answered for a few cases in [7].

Remark 4.8. *For $G = A_{p+t}$ with $P = \langle (1, \cdots, p) \rangle$, the above question combined with Theorem 5.9(3) asks the existence of a connected A_{p+t}-Galois étale cover of the affine line such that P is the Sylow p-subgroup of an inertia group above ∞ and the upper jump over ∞ is given by*

$$\sigma = \begin{cases} \frac{d+1}{p-1} & \text{if } p|t \\ \frac{d}{p-1} & \text{otherwise.} \end{cases}$$

By [7, Corollary 2.2], the above question has an affirmative answer for $p + 2 \leq d < 2p$ with $(p, d) \neq (7, 9)$.

4.3 Generalizations of the Inertia Conjecture

In this section, we pose certain questions which generalize the Inertia Conjecture (Conjecture 4.3). Although we will study these generalizations for $p > 2$, in this section, let p be any prime. We consider the following notation for the rest of this section.

Notation 4.9. Let $r \geq 1$, X be a smooth projective connected k-curve, G be a finite group. Let $B = \{x_1, \cdots, x_r\} \subset X$ be a set of closed points in X. Let $P_1, \cdots, P_r$ be p-subgroups of G, possibly trivial. Define a subnormal series $\{H_j\}_{j\geq 0}$ of G inductively as follows.

$$H_0 \coloneqq G,\ H_{j+1} \coloneqq \langle P_i^{H_j} | 1 \leq i \leq r \rangle \subset G.$$

Then each H_j is a normal subgroup (normal quasi p-subgroup if P_i is non-trivial for some i) of H_{j-1} containing all the P_i's. Since G is a finite group, there is a minimal non-negative integer l such that $H_j = H_l$ for all $j \geq l$.

Under the above notation, let $\phi\colon Z \to X$ be a connected G-Galois cover étale away from B such that I_i occurs as an inertia group above x_i, $1 \leq i \leq r$. Set $Y_0 \coloneqq X$. Since H_1 is a normal subgroup of G, the cover ϕ factors through a connected G/H_1-Galois cover $\psi_1\colon Y_1 \to X$ étale away from B such that $I_i/I_i \cap H_1$ occurs as an inertia group above x_i, $1 \leq i \leq r$. Inductively for $1 \leq j \leq l$, the H_{j-1}-Galois cover $Z \to Y_{j-1}$ factors through a connected H_{j-1}/H_j-Galois cover $\psi_j\colon Y_j \to Y_{j-1}$ étale away from B. Moreover, if $y_{i,j}$ is a point of Y_j lying above x_i, then $I_i \cap H_{j-1}/I_i \cap H_j$ occurs as an inertia group above $y_{i,j-1}$ in the cover ψ_j, $1 \leq i \leq r$. So ϕ is the composition of a tower

$$Z \longrightarrow Y_l \xrightarrow{\psi_l} Y_{l-1} \xrightarrow{\psi_{l-1}} \cdots \xrightarrow{\psi_2} Y_1 \xrightarrow{\psi_1} Y_0 = X,$$

where $\psi_j\colon Y_j \to Y_{j-1}$ is an H_{j-1}/H_j-Galois cover of smooth projective connected k-curves for $1 \leq j \leq l$. Also note that l is the minimal non-negative integer such that $H_l = \langle P_i^{H_l} | 1 \leq i \leq r \rangle$.

We ask whether the converse is also true.

Question 4.10 ($Q[r, X, B, G]$)**.** *Let r, X, G and $B = \{x_1, \cdots, x_r\} \subset X$ be as in Notation 4.9. For $1 \leq i \leq r$, let $I_i \subset G$ be an extension of a p-group P_i (possibly trivial) by a* *cyclic group of order prime-to-p such that there is a tower*

$$Y_l \xrightarrow{\psi_l} Y_{l-1} \xrightarrow{\psi_{l-1}} \cdots \xrightarrow{\psi_2} Y_1 \xrightarrow{\psi_1} Y_0 := X$$

of covers of smooth projective connected k-curves with l and H_j as in Notation 4.9, $\psi_j \colon Y_j \to Y_{j-1}$ is an H_{j-1}/H_j-Galois cover, and if $y_{i,j}$ is a point of Y_j lying above x_i, then $I_i \cap H_{j-1}/I_i \cap H_j$ occurs as an inertia group above $y_{i,j-1}$ in the cover ψ_j, $1 \leq i \leq r$. Let $\psi \colon Y_l \to X$ denote the composite morphism.

Does there exist a connected G-Galois cover ϕ of X, étale away from B, dominating the cover ψ such that I_i occurs as an inertia group above the point x_i for $1 \leq i \leq r$?

First consider the following two examples that will help us make some observations on the necessity of the conditions we impose in the above question.

Example 4.11. Take $r = 1$, X as an elliptic curve with origin 0, $B = \{0\}$, P_1 as the trivial group, $G = I_1 = \mathbb{Z}/m$ where m is coprime to p. Then H_1 is the trivial group. Since there is a connected $\mathbb{Z}/m$-Galois étale cover of X, we have $G/H = \mathbb{Z}/m \in \pi_A^t(X - B)$. From the Riemann Hurwitz formula (Proposition 3.22) we obtain

$$2g_Y - 2 = m(2-2) + \frac{m}{m}(m-1).$$

So m must be an odd integer. Thus if m is an even integer, there is no $\mathbb{Z}/m$-Galois connected cover $Y \to X$ étale away from $\{0\}$ over which it is totally ramified.

Example 4.12. Let H be the elementary abelian group $H = \langle \tau_1 \rangle \times \cdots \times \langle \tau_p \rangle \cong \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$ of p-exponent p. Then $\langle \sigma \rangle \cong \mathbb{Z}/p$ acts on H via the action of σ on the set $\{1, \cdots, p\}$ as an element of S_p. Consider the wreath product $G = H \rtimes \langle \sigma \rangle \cong \mathbb{Z}/p \wr \mathbb{Z}/p$. Take X to be an an ordinary elliptic curve E with origin 0. Take $r = 1$, $P = I = \langle \tau_1 \rangle$. Then $H_1 = H$, and $H_2 = P$. There is a unique connected $G/H \cong \mathbb{Z}/p$-Galois étale cover $\psi_1 \colon Y_1 := E \to E$. Now assume that there is a connected G-Galois cover $Y \to E$ dominating ψ_1 such that P occurs as an inertia group over 0. As observed before, the connected H_1-Galois cover $Y \to Y_1$ factors via an étale Galois cover $Y_2 \to Y_1 = E$ with group H_1/H_2 which is an elementary abelian group of order p^{p-1}, a contradiction since the pro-p étale fundamental group $\pi_1^p(E)$ of E is a free profinite group on one generator.

We have the following preliminary observations.

Remark 4.13. *We note the following in Question 4.10.*

1. *Since $p(I_i) = P_i \subset H_l$ for all i, the cover ψ and each of the Galois covers ψ_i are tamely ramified.*

2. *For $1 \leq j \leq l$, let $B_{j-1} \subset Y_{j-1}$ be the brunch locus of the cover ψ_j, $1 \leq j \leq l$. Then $H_{j-1}/H_j \in \pi_A^t(Y_{j-1} - B_{j-1})$. But only assuming this, we get a negative answer to the above question as seen from Example 4.11. So we need the hypothesis about the existence of the cover ψ_j's with the given ramification behavior.*

3. *Example 4.12 shows that only assuming the existence of ψ_j's for $j < l$, we get a negative answer to Question 4.10.*

4. *When all the p-groups P_i are trivial, taking $\phi = \psi$ gives the affirmative answer to the question.*

In what follows, we assume that P_1 is non-trivial.

So H_j is a non-trivial normal quasi p-subgroup of H_{j-1} for $1 \leq j \leq l$. We make the following observations when $X = \mathbb{P}^1$ and $l \in \{0, \infty\}$.

Remark 4.14. *Let $X = \mathbb{P}^1$. If each $I_i \subset H_l$, then ψ is the identity map $\mathbb{P}^1 \to \mathbb{P}^1$ as there is no non-trivial étale cover of $\mathbb{P}^1$. So $l = 0$, i.e. $G = H_0 = \langle P_i^G | 1 \leq i \leq r \rangle$, and G must be a quasi p-group. This holds in particular, when all the I_i's are p-groups. Also note that if $l = 1$ and ψ_1 is a two point branched Galois cover of $\mathbb{P}^1$ (so in particular, $r = 2$), by the Riemann Hurwitz formula, ψ is the $\mathbb{Z}/n$-Galois Kummer cover of $\mathbb{P}^1$ branched at $\{x_1, x_2\}$ which is totally ramified above these points.*

With the above remark, a special case of Question 4.10 (i.e. $Q[r, \mathbb{P}^1, B, G]$ with each $I_i \subset H_l$ or equivalently $l = 0$) is the following which we pose as the Generalized Inertia Conjecture (GIC).

Conjecture 4.15 (GIC)**.** *Let $r \geq 1$, G be a finite quasi p-group. For $1 \leq i \leq r$, let $I_i \subset G$ be an extension of a p-group P_i by a cyclic group of order prime-to-p such that*

$G = \langle P_1^G, \cdots, P_r^G \rangle$. Let $B = \{x_1, \cdots, x_r\}$ be a set of closed points in $\mathbb{P}^1$. Then there is a connected G-Galois cover of $\mathbb{P}^1$ étale away from B such that I_i occurs as an inertia group above the point x_i for $1 \leq i \leq r$.

Remark 4.16. *As in the case of Question 4.10, we again allow P_i's to be trivial for $2 \leq i \leq r$. Note that $r = 1$ is the unsolved direction of the IC (Conjecture 4.3).*

Remark 4.17. *Under the notation and the hypothesis of Question 4.10, we can ask the following weaker questions.*

1. *($Q[r, X, B, G]_{\text{weak}}$) Does there exist a connected G-Galois cover ϕ of X étale away from B such that I_i occurs as an inertia group above x_i, $1 \leq i \leq r$? (Here we drop the condition on ϕ to dominate the cover ψ).*

2. *($Q[r, X, G]$) Do there exist a set $B' = \{x'_1, \cdots, x'_r\}$ of closed points in X and a connected G-Galois cover ϕ of X étale away from B' such that I_i occurs as an inertia group above x'_i, $1 \leq i \leq r$?*

Note that when the ψ_j's in Question 4.10 are uniquely determined, $Q[r, X, B, G]_{\text{weak}}$ is equivalent to $Q[r, X, B, G]$. This holds for the two cases in Example 4.11. Later we will see some partial answer to Question 4.10 for $l \in \{0, 1\}$ as the application of the formal patching technique and by constructing covers given by the explicit equations. When the inertia groups I_i are p-groups P_i we have the following special case of the GIC which we see as a generalization of the PWIC (Conjecture 4.4).

Conjecture 4.18 (GPWIC or Generalized Purely Wild Inertia Conjecture)**.** *Let $r \geq 1$, and G be a finite quasi p-group. Let $P_1, \cdots, P_r$ be non-trivial p-subgroup of G such that $G = \langle P_1^G, \cdots, P_r^G \rangle$. Let $B = \{x_1, \cdots, x_r\}$ be a set of closed points in $\mathbb{P}^1$. Then there is a connected G-Galois cover of $\mathbb{P}^1$ étale away from B such that P_i occurs as an inertia group above the point x_i for $1 \leq i \leq r$.*

4.4 Main Results

In this section, we briefly mention our main results. As before, p denotes an odd prime. Our first result from [10] is the following evidence for the PWIC.

Theorem 4.19 ([10, Corollary 7.11]). *Let $u \geq 1$ be an integer. For $1 \leq i \leq u$, let $d_i = p$ or $d_i > p$ be coprime to p. Then the PWIC (Conjecture 1.3) is true for the group $A_{d_1} \times \cdots \times A_{d_u}$. Moreover, for any p-group P, the PWIC is true for $A_{d_1} \times \cdots \times A_{d_u} \times P$.*

This is a consequence of realizing the inertia group generated by a p-cycle for an étale Alternating group cover of the affine line (Corollary 7.3) together with the following result.

Theorem 4.20 ([10, Theorem 7.6]). *Let G_1, G_2 be two perfect quasi p-groups. Let $\tau \in G_1$ and $\sigma \in G_2$ be of order p and p^r for some r respectively. Let $P_1 = \langle \tau \rangle \leq G_1$, $P_2 = \langle \sigma \rangle \leq G_2$. Assume that the pairs (G_1, P_1) and (G_2, P_2) are realizable (cf. Definition 3.25). Then there exists $1 \leq a \leq p-1$ such that for $I := \langle (\tau^a, \sigma) \rangle \leq G_1 \times G_2$, the pair $(G_1 \times G_2, I)$ is also realizable.*

By Remark 7.12, a similar result to Theorem 4.19 is also true for any product of simple non-abelian quasi p-groups. Moreover, using explicit covers we show that we have affirmative answer to the minimal jump problem (Question 4.7) for $G = A_d$ and $P = \langle (1, \cdots, p) \rangle$ (see Corollary 6.15). These results are embodied in Section 6.1, Section 6.2 and Section 7.2.

In [9], we introduced Question 4.10. In order to study this question for covers of the projective line with Alternating or Symmetric group as Galois groups, we studied the covers given by explicit affine equations and obtained crutial results using formal patching. These are the contents of Section 6.1 and Section 6.2. Using these technique, we can produce evidence towards the IC (Conjecture 4.3) in the following cases (see Section 7.3).

Theorem 4.21 (Theorem 7.15, 7.16, 7.17, 7.19). *For an odd prime $p \equiv 2 \pmod 3$, the IC is true for the groups A_{p+1}, A_{p+3} and A_{p+4}. When $p \equiv 2 \pmod 3$, $4 \nmid (p+1)$ and $p \geq 11$, the IC is true for A_{p+5}.*

The result for A_{p+1} is of special interest since before [9], there was no example of an A_{p+1}-Galois étale cover of the affine line such that the tame part of the inertia group at a point above ∞ is non-trivial.

In Section 7.4 we show that for the groups for which the PWIC is already known to be true, the GPWIC is also true. Namely, we prove the following result.

Theorem 4.22 (Corollary 7.26)**.** *Let G be a quasi p-group, $P_1, \cdots, P_r$ are p-subgroups of G for some $r \geq 1$ such that $G = \langle P_1^G, \cdots, P_r^G \rangle$. Let $B := \{x_1, \cdots, x_r\}$ be a set of closed points in $\mathbb{P}^1$. There is a connected G-Galois cover of $\mathbb{P}^1$ étale away from B such that P_i occurs as an inertia group above x_i where G is one of the following groups.*

1. *G is a p-group;*
2. *G has order strictly divisible by p;*
3. *$G = G_1 \times \cdots \times G_u$ where each G_i is either a simple Alternating group of degree $d \geq p$, where $d = p$ or $(d, p) = 1$ or $PSL_2(p)$ or a p-group or a simple non-abelian group of order strictly divisible by p.*

We also show that the GPWIC holds for a certain product of groups if it holds for individual groups. This generalizes [20, Corollary 4.6].

Theorem 4.23 (Theorem 7.24)**.** *Let G_1 and G_2 be two finite quasi p-groups such that they have no non-trivial quotient in common. If the GPWIC is true for the groups G_1 and G_2, then the GPWIC is also true for $G_1 \times G_2$.*

Finally, in Section 7.5 we study $Q[r, X, B, G]$ (Question 4.10) and the generalizations of the Inertia Conjecture. We obtain the following result towards $Q[2, \mathbb{P}^1, \{0, \infty\}, G]$.

Theorem 4.24 (Theorem 7.36, 7.39, 7.40)**.** *For $r = 2$, $X = \mathbb{P}^1$, Question 4.10 has an affirmative answer when G is an extension of a p-group by a cyclic group of order prime-to-p or when $p \equiv 2 \pmod 3$, $G = S_p$ or S_{p+1} and both P_1 and P_2 are non-trivial.*

We also have the similar results for the groups S_{p+2} and S_{p+3} with more restrictions on p. We obtain the following result for $r = 1$ and X any curve of genus ≥ 1, an affirmative result towards $Q[1, X, \{*\}, S_d]$.

Theorem 4.25 (Corollary 7.34)**.** *Let p be an odd prime and let X be any smooth projective k-curve of genus ≥ 1. Then for $r = 1$, Question 4.10 has an affirmative answer for the group S_p and when $p \equiv 2 \pmod 3$ for the groups S_{p+1}, S_{p+2}, S_{p+3}, S_{p+4}.*

The above theorem is a corollary to Proposition 7.30 which suggests that the answer to Question 4.10 which is related to the understanding of the structure of the group $\pi_1(X-B)$ has a close connection with the IC, the tame fundamental group $\pi_1^t(X \setminus B)$ and the group theoretic behavior of the Galois groups.

In Section 6.2 we also obtain some generalizations of the previously known results. In particular, Corollary 6.22 generalizes a patching result [25, Theorem 2.2.3] of Raynaud which was an important step towards proving Abhyankar's conjecture on the affine line (Theorem 4.2). Another such generalization is Theorem 6.26 of [15, Theorem 2.1, Theorem 4.1] which solved the split quasi p embedding problem.

Chapter 5

Useful Results towards the Main Problems

5.1 Group Theoretic Results

Our main problems concern the elements of the normalizer of a certain (inertia) subgroup inside our (Galois) group of interest. To understand these, we need some group theoretic results. The following results are useful to decide which subgroups of A_d or of a product of Alternating groups are potentially inertia subgroups. A special case ($r = 1$ and $d < 2p$) of the following proposition can be found in [22, Lemma 4.13, Lemma 4.14].

Proposition 5.1 ([10, Proposition 2.1])**.** *Let p be a prime, $d \geq p$. Let τ be an element of order p in the Symmetric group S_d. Let $\tau = \Pi_{i=1}^{r}\tau_i$ be the disjoint cycle decomposition of τ with $\tau_1, \cdots, \tau_r$ disjoint p-cycles and $rp \leq d$. For $\sigma \in S_d$ let $Supp(\sigma)$ denote the support of σ. Then there exists an element $\theta \in Sym(Supp(\tau)) \cap N_{S_d}(\langle\tau\rangle)$ of order $p-1$ such that conjugation by θ is a generator of $Aut(\langle\tau\rangle)$. Moreover, let $H' := \{\sigma \in Sym(Supp(\tau)) | \sigma\tau_i\sigma^{-1} = \tau_j \text{ for } 1 \leq i, j \leq r\}$. Then $N_{S_d}(\langle\tau\rangle) = \langle\theta, H'\rangle \times H$, where H is the Symmetric group on the set $\{1, \cdots, d\} \setminus Supp(\tau)$.*

In particular, if $\beta \in N_{S_d}(\langle(1, \cdots, p)\rangle)$ has order prime-to-p, then $\beta = \theta^i\omega$ for some integer $1 \leq i \leq p-1$ and an element $\omega \in H = Sym(\{p+1, \cdots, d\})$.

Proof. For a p-cycle τ' in S_p, the normalizer $N_{S_p}(\langle\tau'\rangle)$ is the affine general linear group $AGL(1, p)$ of order $p(p-1)$ and has an element of order $p-1$. Diagonally embedding S_p in Sym(Supp(τ)) we obtain an element θ of order $p-1$ in Sym(Supp(τ)). Then θ normalizes $\langle\tau\rangle$ and the conjugation by θ has order $p-1$. Since τ is of order p, the full automorphism group of $\langle\tau\rangle$ is generated by θ. So the natural homomorphism $N_{\mathrm{Sym(Supp}(\tau))}(\langle\tau\rangle) \to \mathrm{Aut}(\langle\tau\rangle)$ is a surjection whose kernel is the centralizer of $\langle\tau\rangle$ in Sym(Supp(τ)). Observe that the centralizer of $\langle\tau\rangle$ in Sym(Supp(τ)) is H'. So $N_{S_d}(\langle\tau\rangle) = \langle\theta, H'\rangle \times H$. □

Notation 5.2. Let $u \geq 1$, and $G = G_1 \times \cdots \times G_u$ where each G_i is a finite group. For $g \in G$, let $g = (g^{(i)})_{1\leq i\leq u}$, and set $S(g) = \{1 \leq i \leq u | g^{(i)} \neq 1\}$. Set $l(g) = |S(g)|$. For $\lambda \subseteq \{1, \cdots, u\}$, let $H_\lambda := \Pi_{i\in\lambda} G_i$ and let $\pi_\lambda \colon G \twoheadrightarrow H_\lambda$ be the projection.

Lemma 5.3 ([10, Lemma 2.3]). *Let $u \geq 1$ be an integer, $G = G_1 \times \cdots \times G_u$, where each G_i is a simple non-abelian quasi p-group. Let Q be a p-subgroup whose conjugates generate G. Then there are elements $g_1, \cdots, g_r$ in Q for some $r \geq 1$ satisfying the following properties. Set $S_{\leq j} := \cup_{i=1}^{j} S(g_i)$, $1 \leq j \leq r$.*

1. *$S_{\leq r} = \{1, \cdots, u\}$, $l(g_1) \geq \cdots \geq l(g_r)$ and for all $1 \leq i, j \leq r$, $S(g_i) \cap S(g_j) \neq \emptyset$.*
2. *For all $1 \leq i \leq r$, $H_{S_{\leq i}}$ is generated by the conjugates of $\langle g_1, \cdots, g_i\rangle$.*
3. *For any subset $\{i_1, \cdots, i_t\} \subset \{1, \cdots, r\}$ and any integers $1 \leq a_{i_1}, \cdots, a_{i_t} \leq p-1$, $l(g_{i_1}^{a_{i_1}} \cdots g_{i_t}^{a_{i_t}}) \leq max_{1\leq j\leq t} l(g_{i_j})$.*
4. *For each $1 \leq i \leq r$, $ord(g_i^{(j)}) = p$ for some $1 \leq j \leq u$.*

Proof. Choose $g_1 \in Q$ such that $l(g_1) \geq l(g)$ for all $g \in Q$. If $l(g_1) = u$ then set $r = 1$, and note that conditions (1) and (3) are trivially satisfied. Condition (2) holds because G_j's are simple groups. Inductively define g_i as follows. For $i \geq 1$, if $S_{\leq i} \neq \{1, \cdots, u\}$, choose g_{i+1} among $g \in Q$ with $S(g) \cap (S_{\leq i})^c \neq \emptyset$ such that $l(g_{i+1})$ is maximal. The inclusion $S_{\leq i} \subset S_{\leq i+1}$ implies $l(g_i) \geq l(g_{i+1})$. Since G_j's are all simple non-abelian groups, the conjugates of $\langle g_1, \cdots, g_i\rangle$ generate $H_{S_{\leq i}}$. For $j \leq i-1$, $l(g_i g_j) \leq l(g_j)$ and hence $S(g_i) \cap S(g_j) \neq \emptyset$. Condition (3) follows by the choice of g_j's to maximize $l(g_j)$'s.

Finally for condition (4), note that if p^{k+1} is the least order of $g_i^{(j)}$ for various j's among the non-trivial $g_i^{(j)}$, then replacing g_i by $g_i^{p^k}$ we obtain the result. □

Remark 5.4. *In the above lemma, if* $1 \leq u \leq p$*, then* $r = 1$*, i.e. there exist* $g \in Q$ *such that* $S(g) = \{1, \cdots, u\}$*. Also note that we can take each* $l(g_i) \geq p$*.*

The following lemma follows from the fact that the abelianization of a quasi p-group is a p-group.

Lemma 5.5 ([10, Lemma 2.5])**.** *Let* G *be a finite quasi* p*-group. Then the following are equivalent.*

1. *G is perfect (i.e. G equals its commutator subgroup).*
2. *There is no nontrivial homomorphism from G to $\mathbb{Z}/p$.*
3. *There is no nontrivial homomorphism from G to any p-group.*

5.2 Ramification Theory for some special type of Covers

In this section, we obtain more precise ramification properties of the Galois covers of the projective line when the group is a transitive subgroup of a Symmetric group. Let $d \geq p$ be an integer. Let G be a transitive subgroup of S_d. Let $\phi\colon Z \to X$ be a G-Galois cover of smooth projective connected k-curves. Consider the subgroup S_{d-1} of S_d fixing the element 1. Set $H := G \cap S_{d-1}$. Then ϕ factors as a composite of covers

$$Z \longrightarrow Y \xrightarrow{\psi} X$$

where $Z \to Y$ is a connected H-Galois cover, and $\psi\colon Y \to X$ is a degree-d cover of smooth connected projective k-curves. Note that ϕ is the Galois closure of the cover ψ. Let $x \in X$ be a branched point. Let $I = P \rtimes \mathbb{Z}/m$ occurs as an inertia group above x where P is a p-group and $(m, p) = 1$. We will work with the following setup.

P is a p-cyclic group generated by the p-cycle $\tau := (1, \cdots, p)$.

Let $\beta \in \mathbb{Z}/m$ be an element of order m. We have $I = N_I(\langle\tau\rangle) \subseteq N_G(\langle\tau\rangle) = N_{S_d}(\langle\tau\rangle) \cap G$. By Proposition 5.1, $\beta = \theta^i\omega$ for an element $\theta \in \mathrm{Sym}(\mathrm{Supp}(\tau)) = S_p$ of order $p-1$, $\omega \in N_{S_d}(\langle\tau\rangle) \cap \mathrm{Sym}(\{p+1,\cdots,d\})$ and $1 \leq i \leq p-1$. We study some invariants of the action of the group $\langle\beta\rangle$ on the p-cyclic group P.

Consider the group homomorphism

$$g\colon \langle\beta\rangle \to \mathrm{Aut}\,(\langle\tau\rangle), \quad g(\beta)\colon \tau \mapsto \beta\tau\beta^{-1}.$$

Let m' be the order of $\ker(g) = \{\beta^j | \beta^j \text{ commutes with } \tau\}$. This is the prime-to-$p$ part of the center of I, known as *the central part of the tame ramification*. Set $m'' := \frac{m}{m'}$. Then $\ker(g) = \langle\beta^{m''}\rangle$ is the subgroup acting trivially on $\langle\tau\rangle$. Also observe that $\mathrm{Im}(g)$ has order m'' in $\mathrm{Aut}(\langle\tau\rangle)$, and m'' is called *the faithful part of the tame ramification*. Let h be the conductor for the inertia group I. Applying the conjugation relation from Proposition 3.11 and from the structure of the group I, we obtain the following result.

Lemma 5.6 ([10, Lemma 2.6])**.** *Under the above notation, $m' = (h,m)$ and $ord(\theta^i) = m''$.*

Proof. Set $\gamma := (h,m)$. Applying Proposition 3.11 to the conjugation of τ by $\beta^{\frac{m}{\gamma}}$ and $\beta^{m''}$, we get that $\beta^{\frac{m}{\gamma}}\tau\beta^{-\frac{m}{\gamma}} = \tau$ and $\beta^{m''h} = 1$. So m'' divides $\frac{m}{\gamma} = \frac{m'm''}{\gamma}$ and $m'm''$ divides $m''h$. So γ divides m' and m' divides $(h,m) = \gamma$. Thus $\gamma = m'$.

Note that the conjugation by θ has order $p-1$ (Proposition 5.1). Since θ also has order $p-1$, θ^j commutes with τ if only if θ^j is the identity element. Now, ω commutes with τ. So the conjugation by $\beta = \theta^i\omega$ is the same as the conjugation by θ^i. Hence $(\theta^i)^k$ does not commute with τ for $0 < k < m''$, and $(\theta^i)^{m''}$ commutes with τ. So m'' is the order of θ^i. □

By an analogous argument as in the proof of [17, Proposition 4.16], one obtains the following result.

Lemma 5.7 ([9, Lemma 3.1])**.** *Let $d \geq p$, and G be a transitive subgroup of S_d. Let $\phi\colon Z \to X$ be a G-Galois cover of smooth connected projective k-curves, and $x \in X$ be* *a closed point. Let I occurs as an inertia group over x. Consider the subgroup S_{d-1} of S_d fixing the element 1. Set $H := G \cap S_{d-1}$. Let $\psi\colon Y \to X$ be the connected degree d cover via which ϕ factors. Then the set $\psi^{-1}(x)$ in Y is in a bijective correspondence with the set of orbits of the action of I on $\{1, \cdots, d\}$. Moreover, for a point $y \in \psi^{-1}(x)$, the ramification index of y over x is given by the length of the corresponding orbit.*

Proof. As ψ has degree d, its fibres can be indexed by the numbers $1, \cdots, d$. After labeling a point of Z in $\phi^{-1}(x)$ to correspond to the identity coset of G/I, there is a bijection between the sets G/I and $\phi^{-1}(x)$. The left action of G on $\phi^{-1}(x)$ is given by left multiplication. The points of Y in $\psi^{-1}(x)$ correspond to the orbits of H on G/I (via left action), which correspond to double cosets $H\backslash G/I$. Since G is a transitive subgroup of S_d, the set of cosets $H\backslash G$ is in a bijection with $\{1, \cdots, d\}$ via the image of 1. So these double cosets are given by the orbits of I acting on the right on $\{1, \cdots, d\}$, which is the first statement. For the moreover part, fix a point $y \in \psi^{-1}(x) \subset Y$. Then y corresponds to an orbit Iy of I acting on the right on $\{1, \cdots, d\}$. By the Orbit-Stabilizer Theorem, we have $|Iy| = |I/\mathrm{Stab}_I(y)|$. But the later quantity is just the ramification index of y over ∞. So the lengths of the orbits correspond to the ramification indices. □

Lemma 5.8 ([9, Lemma 3.2])**.** *Under the hypothesis of Lemma 5.7, we have the following.*

1. *If $\psi^{-1}(x)$ consists of s points with the ramification indices $n_1, \cdots, n_s$, where each n_i is coprime to p and $\sum_{i=1}^{s} n_i = d$, then ϕ is tamely ramified over 0. If γ is a generator of I, then the disjoint cycle decomposition of γ in S_d consists of s cycles of length $n_1, \cdots, n_s$.*

2. *If $d = p$ and $\psi^{-1}(x)$ consists of a unique point with the ramification index p, then I is of the form $I = \langle\tau\rangle \rtimes \langle\theta^i\rangle$ for a p-cycle τ and some $1 \leq i \leq p-1$. If $d \geq p+1$ and $\psi^{-1}(x)$ consists of $r+1$ points with the ramification indices $p, m_1, \cdots, m_r$, where each m_l is coprime to p and $\sum_{l=1}^{r} m_l = d - p$, then I is of the form $I = \langle\tau\rangle \rtimes \langle\theta^i\omega\rangle$ for a p-cycle τ, $1 \leq i \leq p-1$, and an $\omega \in Sym(\{p+1, \cdots, d\})$ having a disjoint cycle decomposition consisting of r cycles of length $m_1, \cdots, m_r$. Here θ is as in Proposition 5.1.*

Proof.

1. By Lemma 5.7, it is enough to show that $p \nmid |I|$. Assume that p divides $|I|$. Let $\tau \in p(I)$ be an element of order p. Then $p(I) \subset H$. Since ψ is tamely ramified over x, for any $g \in G$, $g^{-1}\tau g \in g^{-1}p(I)g \subset H$. But since G acts transitively on $\{1, \cdots, d\}$, for any $i \in \mathrm{Supp}(\tau)$, there is a $g \in G$ such that $g^{-1}\tau g$ does not fix 1, a contradiction.

2. Let $z \in \phi^{-1}(x) \subset Z$ having image $y \in Y$ such that I is the inertia group at z. We first claim that $p^2 \nmid |I|$. Assume on the contrary. Let $\tau \in p(I) \cap H$ be an element of order p. Since for any point $y' \in \psi^{-1}(x)$, $y' \neq y$, $e(y'|x)$ is coprime to p, we have $g^{-1}\tau g \in H$ for all $g \in G$. Again as in (1), we obtain a contradiction to our assumption, and the claim follows.

   Since $|I|$ is divisible by p, $p(I)$ is a p-cyclic group generated by an element $\tau = \tau_1 \cdots \tau_a$ of order p, where τ_i are disjoint p-cycles in S_d. By Proposition 5.1, $I = \langle \tau, \theta^i, \sigma \rangle \times \langle \omega \rangle$ for some $1 \leq i \leq p-1$, $\omega \in \mathrm{Sym}(\{1, \cdots, d\} - \mathrm{Supp}(\tau))$ and $\sigma \in \mathrm{Sym}(\mathrm{Supp}(\tau))$ of order prime-to-p. By Lemma 5.7, the fibre $\psi^{-1}(x)$ consists of points with the ramification indices $a_1 p, \cdots, a_t p, u_1, \cdots, u_{t'}$, where a_ν and u_η are coprime to p.

   So if $d = p$ and $\psi^{-1}(x)$ consists of a unique point with the ramification index p, then τ must be a p-cycle and I is of the form $I = \langle \tau \rangle \rtimes \langle \theta^i \rangle$ for some $1 \leq i \leq p-1$. In the second case, τ is again a p-cycle, $I = \langle \tau \rangle \rtimes \langle \theta^i \omega \rangle$ for some $1 \leq i \leq p-1$, and the disjoint cycle decomposition of ω in $\mathrm{Sym}(\{p+1, \cdots, d\})$ consists of r cycles length $m_1, \cdots, m_r$.

□

Using the above results and a technique used in [7, Proposition 1.3] (when p strictly divides the order of G), we obtain the following result for a certain type of two point branched Galois cover of the projective line $\mathbb{P}^1$. In Section 6.1 we will encounter such covers. This result is important in deciphering their local ramification behaviour.

Theorem 5.9 ([9, Proposition 3.3]). *Let p be a prime, $t \geq 0$, $d := p + t$. Let G be a transitive subgroup of S_d. Let $\phi\colon Z \to \mathbb{P}^1$ be a G-Galois cover of smooth projective connected k-curves. Consider the degree-d cover $\psi\colon Y := Z/(G \cap S_{d-1}) \to \mathbb{P}^1$ of smooth projective connected k-curves where S_{d-1} is the subgroup of elements in S_d fixing 1. Assume that the following hold.*

(i) There are exactly s points in $\psi^{-1}(0)$ with the ramification indices $n_1, \cdots, n_s$ such that each n_i is coprime to p and $\sum_{i=1}^{s} n_i = d$;

(ii) when $d = p$, there is a unique point in Y lying over ∞ with ramification index p. When $d > p$, there are exactly $r+1$ points in $\psi^{-1}(\infty)$ with the ramification indices $p, m_1, \cdots, m_r$ such that each m_j is coprime to p and $\sum_{j=1}^{r} m_j = t$.

Then ϕ is tamely ramified over 0, and $I = \langle(1, \cdots, p)\rangle \rtimes \langle\theta^i\omega\rangle$ occurs as an inertia group over ∞ (θ, ω are as in Proposition 5.1, $0 \leq i \leq p-1$). If γ is a generator of an inertia group over 0, the disjoint cycle decomposition of γ in S_d consists of s cycles of length $n_1, \cdots, n_s$. If $t = 0$, ω is the trivial permutation. If $t \geq 1$, the disjoint cycle decomposition of ω in $Sym(p+1, \cdots, d)$ consists of r cycles length $m_1, \cdots, m_r$.

Moreover, if the cover ϕ is étale away from $\{0, \infty\}$, we have the following.

1. *$ord(\theta^i) = m'' = \frac{p-1}{(p-1, 2g_Y+s+r-1)}$ where g_Y is the genus of Y;*
2. *the invariant m' associated to the local extension above ∞ is given by*

$$m' = \begin{cases} 1 & \text{if } t = 0 \\ \frac{l.c.m.\ (m_1, \cdots, m_r)}{g.c.d(\ l.c.m.\ (m_1, \cdots, m_r), m'')} & \text{if } t \geq 1; \end{cases}$$

3. *the upper jump for any local extension above ∞ is $\frac{2g_Y+s+r-1}{p-1}$.*

Furthermore, $|I| = pm$, where $m = m'm'' = lcm(m'', ord(\omega))$.

Proof. The structure of the inertia groups are the consequence of Lemma 5.8. Now suppose that the cover ϕ is étale away from $\{0, \infty\}$.

Assume that $t \geq 1$. We use the Riemann Hurwitz formula (Proposition 3.22) for the two Galois covers ϕ and $Z \to Y$. Let g_Z be the genus of Z. For any local I-Galois extension over ∞ let h be the conductor and set $|I| = pm$. Also set $N := \mathrm{lcm}(n_1, \cdots, n_s)$. So for the G-Galois cover ϕ we have

$$2g_Z - 2 = |G|(-2) + \frac{|G|}{N}(N-1) + \frac{|G|}{pm}(pm - 1 + h(p-1)).$$

Now let $z \in Z$ is a point lying over $y \in Y$ such that $\phi(z) = \infty$. By Proposition 3.21, $e(z|y) = \frac{pm}{m_j}$ with conductor for the local extension $K_{Z,z}/K_{Y,y}$ being h when $e(y|\infty) = m_j$, and $e(z|y) = m$ when $e(y|\infty) = p$. Also for $1 \leq i \leq s$, let $z_i \in Z$ is a point lying over $y_i \in Y$ such that $\phi(z_i) = 0$. If $e(y_i|0) = n_i$, $e(z_i|y_i) = \frac{N}{n_i}$. So for the $G \cap S_{d-1}$-Galois cover $Z \to Y$ we have

$$2g_Z - 2 = \frac{|G|}{d}(2g_Y - 2) + \sum_{i=1}^{s} \frac{|G|n_i}{dN}(\frac{N}{n_i} - 1) + \frac{|G|}{dm}(m-1) + \sum_{j=1}^{r} \frac{|G|m_j}{dpm}(\frac{pm}{m_j} - 1 + h(p-1)).$$

Equating the above two equations, we obtain the upper jump $\sigma = \frac{h}{m} = \frac{2g_Y+s+r-1}{p-1}$ at ∞. By Lemma 5.6, m'' is the smallest positive integer such that $m''\sigma$ is again an integer. Thus $m'' = \frac{p-1}{(p-1, 2g_Y+r+s-1)}$. As the order of θ^i is m'' and $m'm'' = m =$ l.c.m. (ord (θ^i), ord (ω)), (2) follows. A similar calculation shows the case for $t = 0$. □

Remark 5.10. *Note that for any G-Galois cover of smooth projective connected k-curves with G a transitive permutation group, the local behaviour at a tamely ramified point or when the Sylow p-subgroup of an inertia group is generated by a p-cycle, a similar structural result for the inertia groups holds, which is essentially a consequence of Lemma 5.7. In this general setup, one argues as in the proof of Theorem 5.9 to obtain the ramification invariants.*

5.3 Reduction of Inertia Groups

In this section, we will see some results related to the normalized pullback of a Galois cover. We recall the following result.

Theorem 5.11 ([21]). *Let $Y_1 \to X$ and $Y_2 \to X$ be Galois covers of smooth projective connected k-curves. Let $x \in X$ be a closed point, and $y_1 \in Y_1$, $y_2 \in Y_2$ be closed points lying above x in the respective covers. Let W be the dominant component of the normalization of the fibre product $Y_1 \times_X Y_2$ containing the point $w := (y_1, y_2)$. Then $W \to X$ is a connected Galois cover with $k(W)$ being the compositum $k(Y_1)k(Y_2)$ of fields over $k(X)$, and the inertia group at the point w is the Galois group of the field extension $K_{W,w} = K_{Y_1,y_1}K_{Y_2,y_2}/K_{X,x}$. Here all the fields are considered as subfields of $K_{W,w}$. Moreover, if I is the inertia group at y_1 over x and N is a normal subgroup of I such that the fixed field $K^N_{Y_1,y_1}$ is the same as the field extension K_{Y_2,y_2} over $K_{X,x}$, the cover $W \to Y_2$ is a connected $Gal(k(W)/k(Y_2))$-Galois cover and the inertia group at w over y_2 is the group N.*

Proof. The statement about the Galois cover $W \to X$ is [21, Theorem 3.4], and the moreover part is [21, Theorem 3.5]. □

As a consequence, the main result of [21] states that for a Galois cover with a perfect Galois group (see Lemma 5.5) one can reduce the inertia group under certain condition.

Theorem 5.12 ([21, Theorem 3.7]). *Let G be a perfect group, $X \to \mathbb{P}^1$ be a G-Galois cover of smooth projective connected k-curve étale away from ∞ such that I occurs as an inertia group above ∞. Assume that P be a p-subgroup of I such that $I_1 \subset P \subset I_2$, where I_i are the lower indexed higher ramification groups for I. Then there is a G-Galois cover $W \to \mathbb{P}^1$ of smooth projective connected k-curves étale away from ∞ such that P occurs as an inertia group above ∞.*

A similar well known result for the tame part of an inertia group is known as Abhyankar's Lemma ([1, page 279, Expose X, Lemma 3.6]). Before recalling one of its refinements, let us set the following definitions.

For an integer n coprime to p, there is a unique connected $\mathbb{Z}/n$-Galois cover $\psi \colon Z \cong \mathbb{P}^1 \to \mathbb{P}^1$ étale away from $\{0, \infty\}$ over which the cover is totally ramified, given by Example 3.24. We will call this cover *the* $[n]$*-Kummer cover*.

Definition 5.13. Let n be coprime to p. Let $\phi\colon Y \to \mathbb{P}^1$ be a connected cover of smooth projective connected k-curves. Let W be a dominant component in the normalization of $Y \times_{\mathbb{P}^1} Z$. We say that the cover $W \to Z$ *is obtained by a pullback by the* $[n]$*-Kummer cover.*

Theorem 5.14 (Refined Abhyankar's Lemma, [22, Lemma 4.1])**.** *Let m, r_1, r_2 be coprime to p. Let $\psi\colon Y \to \mathbb{P}^1$ be a G-Galois cover of smooth projective connected k-curves étale away from $\{0,\infty\}$ such that ψ has ramification index r_1 above 0 and $I \cong \mathbb{Z}/p \rtimes \mathbb{Z}/m$ occurs as an inertia group above ∞ with conductor h. Assume that ψ and the $[r_2]$-Kummer cover are linearly disjoint. Then the pullback of ϕ by the $[r_2]$-Kummer cover is a G-Galois cover of $\mathbb{P}^1$ étale away from $\{0,\infty\}$ such that $\frac{r_1}{(r_1,r_2)}$ is the ramification index above 0 and $I' \subset I$ of order $\frac{pm}{(m,r_2)}$occurs as an inertia group above ∞ with conductor $\frac{hr_2}{(m,r_2)}$. Moreover, if σ and σ' are the upper jumps at ∞ for the covers ψ and ϕ, respectively, then $\sigma' = r_2\sigma$.*

The statement about the pullback of ϕ by the $[r_2]$-Kummer cover is a the classical statement of Abhyankar's Lemma, which is a direct consequence of Theorem 5.11. Using a similar argument we have the following result which will be used later.

Corollary 5.15. *Let $G \in \{S_d, A_d\}$ for $d \geq p$, and n be coprime to p. Let $\phi \coloneqq Y \to \mathbb{P}^1$ be a G-Galois cover of smooth projective connected k-curves étale away from $\{0,\infty\}$ such that ϕ has ramification index n above 0 and $I = P \rtimes \langle\beta\rangle$ occurs as an inertia group above ∞ for some p-subgroup P and β of order prime-to-p. Then the $[n]$-Kummer pullback of ϕ is a connected A_d-Galois cover of $\mathbb{P}^1$ étale away from ∞ such that $P \rtimes \langle\beta^n\rangle$ occurs as an inertia group above ∞.*

Chapter 6

Construction of Covers via different methods

The proof of the Abhyankar's conjecture on the affine curves (Theorem 4.1) and the attempts to study the Inertia Conjecture has so far produced several methods to construct branched Galois covers of curves. One of such methods is to construct covers by considering explicit affine equations. In a series of publications, Abhyankar constructed several such covers with the desired Galois groups by using the classification results from the theory of finite groups. But the ramification behaviour of such covers are mostly not well studied when the Galois groups have large order. In Section 6.1 we continue the study of covers given by the explicit equations and are able to control the ramification data much better using the local ramification theory together with group theoretic results. Another method to construct covers is to use the formal patching techniques introduced in Section 3.4. Starting with a certain Galois cover of a curve and by constructing a family of local covers, one obtains a cover with a bigger Galois group and potentially bigger inertia groups. In this line of study, the control over the local extensions and the inertia groups is important. There are also other ways to obtain covers of curves, for example, using the reduction of a cover defined over a characteristic 0 field (see [8], [25]). As our main results do not use such constructions, we will not discuss these methods in this thesis.

Throughout this chapter, p denotes an odd prime.

6.1 Construction of Covers by Explicit Equations

In this section, we construct the Galois covers of the projective line by considering the Galois closures of the finite covers given by some explicit affine equations. Our primary interest is when the Galois groups are Alternating or Symmetric groups. In [3, Section 11], Abhyankar introduced some degree-d covers and showed that the Galois group of the Galois closure of these covers are the Alternating groups A_d. But for $d \geq 2p$, the ramification behaviour remained not well studied. Our goal is to understand these local behaviours completely. We completed this study in [10], and as a consequence, proved the PWIC for any product of Alternating groups. In this thesis, we get these results as some special cases of our study of the two point branched covers of the projective line, done in [9].

More precisely, we consider the Galois closure of a finite degree-d ($d \geq p$) cover $\mathbb{P}^1 \to \mathbb{P}^1$ étale away from $\{0, \infty\}$. Any finite cover $\mathbb{P}^1_y \to \mathbb{P}^1_x$ is given by an affine equation of the form $xf(y) - g(y) = 0$ for some polynomials $f(y)$ and $g(y)$ in $k[y]$ such that they do not have any common zero. We impose a certain conditions on these polynomials so that the resulting cover has the desired branch locus, inertia and Galois groups.

First consider the case $d = p$. We will consider the following assumption.

Assumption 6.1. *Let $s \geq 2$ be an integer. Let $n_1, \cdots, n_s$ be coprime to p such that $\Sigma_{i=1}^{s} n_i = p$. Let $\alpha_1 = 0$ and assume that there exist non-zero distinct elements $\alpha_2, \cdots, \alpha_s$ in k so that the polynomial*

$$\sum_{i=1}^{s} n_i \prod_{j \neq i, 1 \leq j \leq s} (y - \alpha_j) \in k[y] \tag{6.1.1}$$

is a non-zero constant in k. In terms of the coefficients of this polynomial, the assumption is equivalent to the existence of the non-zero distinct elements α_i's in k, $2 \leq i \leq s$, *such that for each* $1 \le \nu \le s-2$,

$$\sum_{2\le i_1<\cdots<i_{s-\nu-1}\le s}\left(\sum_{i\in\{i_1,\cdots,i_{s-\nu-1}\}} n_i\right)\alpha_{i_1}\cdots\alpha_{i_{s-\nu-1}} = 0.$$

Remark 6.2. *Note that Assumption 6.1 is satisfied when* $s = 2$ *or* $s = 3$. *It is immediate when* $s = 2$. *When* $s = 3$, *the assumption holds for the pair* $(\alpha_2, \alpha_3) = (1, -\frac{n_2}{n_3})$.

Now we construct some degree-p covers of $\mathbb{P}^1$.

Proposition 6.3 ([9, Proposition 3.6])**.** *Let* $p \ge 5$ *be a prime and* $s \ge 2$ *be an integer. Let* $n_1, \cdots, n_s$ *be coprime to* p *such that* $\sum_{i=1}^{s} n_i = p$. *Let* $\alpha_1, \cdots, \alpha_s$ *be distinct elements in* k. *Let* $\psi\colon Y \to \mathbb{P}^1$ *be the degree-*p *cover given by the affine equation* $\bar{f}(x,y) = 0$ *where*

$$\bar{f}(x,y) := \prod_{i=1}^{s}(y-\alpha_i)^{n_i} - x. \tag{6.1.2}$$

Let $\phi\colon Z \to \mathbb{P}^1$ *be the Galois closure of* ψ *with group* G. *Then the following hold.*

1. G *is a primitive subgroup of* S_p;
2. ϕ *is tamely ramified with cyclic inertia group of order l.c.m.*$\{n_1, \cdots, n_s\}$ *over* 0. *If* γ *is one of its generators, then* γ *has a disjoint cycle decomposition in* S_p *with cycle lengths* $n_1, \cdots, n_s$;
3. *over* ∞, *the inertia group is of the form* $I = \langle(1, \cdots, p)\rangle \rtimes \langle\theta^i\rangle$ *for some* $1 \le i \le p-1$ *and where* θ *is as in Proposition 5.1.*

Additionally, if α_i*'s satisfy Assumption 6.1, then the cover* ϕ *is étale away from* $\{0,\infty\}$. *Also* ord$(\theta^i) = \frac{p-1}{(p-1,s-1)}$, $|I| = \frac{p(p-1)}{(p-1,s-1)}$, *and the upper jump for any I-Galois local extension over* ∞ *is given by* $\frac{s-1}{p-1}$.

Moreover, if there is a positive integer j *such that* γ^j *is a non-trivial cycle fixing* ≥ 3 *points in* $\{1, \cdots, p\}$ *or if* $p \ne 11, 23$, $p \notin \{\frac{q^n-1}{q-1} | q$ *prime power* $, n \ge 2\}$, *and* γ *is not a conjugate of* θ^i *for any* $1 \le i \le p-1$,

$$G = \begin{cases} A_p, & \text{if } \gamma \text{ is an even permutation} \\ S_p, & \text{if } \gamma \text{ is an odd permutation.} \end{cases}$$

Proof. The polynomial $\bar{f}(x, y)$ is linear and monic in x. So it is irreducible in $k[y][x]$ and hence in $k(x)[y]$. So G is a transitive subgroup of S_p and hence it is a primitive subgroup of S_p. This proves (1).

From the equation $\bar{f}(x, y) = 0$ it follows that $v_{(y-\alpha_i)}(x) = n_i$ for $1 \leq i \leq s$ and $v_{(y^{-1})}(x^{-1}) = p$. Since $\Sigma_i n_i = p$, we see that the fibre $\psi^{-1}(0)$ consists of s points in Y and the ramification index at the point $(y = \alpha_i)$ is given by n_i. Also there is a unique point in Y lying above ∞ at which the ramification index is p. Then (2) and (3) follow from Theorem 5.9.

Now suppose that α_i's satisfy Assumption 6.1. The y-derivative of $\bar{f}(x, y)$ is given by

$$\bar{f}_y(x, y) = \prod_{i=1}^{s}(y - \alpha_i)^{n_i - 1}\left(\sum_{i=1}^{s} n_i \prod_{j \neq i, 1 \leq j \leq s}(y - \alpha_j)\right).$$

Let (a, b) be a common zero of $\bar{f}$ and $\bar{f}_y$. Then $a = 0$ if $n_i > 1$ for some i and there is no common zero if $n_i = 1$ for all $1 \leq i \leq s$. So the cover ψ, and hence ϕ is étale away from $\{0, \infty\}$. By Theorem 5.9, the upper jump is $\frac{s-1}{p-1}$, $\mathrm{ord}(\theta^i) = \frac{p-1}{(p-1,s-1)}$. So $|I| = \frac{p(p-1)}{(p-1,s-1)}$.

Since G is a primitive subgroup of S_p containing a p-cycle, under the additional hypothesis on p and γ, G contains A_p by [19, Theorem 1.2]. So if γ is an odd permutation, $G = S_p$. Now let γ be an even permutation and assume that $G = S_p$. Then the connected $\mathbb{Z}/2$-Galois cover $Z/A_p \to \mathbb{P}^1$ is étale away from ∞ and is tamely ramified above ∞, a contradiction to the fact that $\pi_1^t(\mathbb{A}^1)$ is the trivial group. So if γ is an even permutation, $G = A_p$. □

Remark 6.4. *In [3, Section 20], Abhyankar introduced the following cover and calculated its Galois group. Consider the degree-p cover of $\mathbb{P}^1$ given by the affine equation $\tilde{f} = 0$ where $\tilde{f}(x, y) = y^p - y^t + x$. Consider its Galois closure $\widetilde{Y} \to \mathbb{P}^1$ with group G. Abhyankar showed that for $2 \leq t \leq p - 3$,*

$$G = \begin{cases} S_p, & \text{if } t \text{ is even,} \\ A_p, & \text{if } t \text{ is odd.} \end{cases}$$

This is a special case of Proposition 6.3.

Now we construct covers of degree $d \geq p+1$. Similar to the previous case we consider the following assumption.

Assumption 6.5. *Let p be an odd prime, $t \geq 1$ be coprime to p such that $d := p+t \geq 5$. Let r and s be two positive integers. Let $n_1, \cdots, n_s, m_1, \cdots, m_r$ be coprime to p integers such that $\Sigma_{i=1}^{s} n_i = p+t$, $\Sigma_{l=1}^{r} m_l = t$. Assume that there exist distinct elements $\alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_r$ in k such that the polynomial*

$$g(y) := \prod_{l=1}^{r}(y-\beta_l)\left(\sum_{i=1}^{s} n_i \prod_{j\neq i, 1\leq j\leq s}(y-\alpha_j)\right) - \prod_{i=1}^{s}(y-\alpha_i)\left(\sum_{l=1}^{r} m_l \prod_{u\neq l, 1\leq u\leq r}(y-\beta_u)\right) \in k[y] \tag{6.1.3}$$

is a non-zero constant in k.

Remark 6.6. *In particular, if $r = 1$, setting $\beta_1 = 0$, Assumption 6.5 says that there are non-zero distinct elements α_i in k, $1 \leq i \leq s$, such that the polynomial*

$$g(y) = y\left(\sum_{i=1}^{s} n_i \prod_{j\neq i, 1\leq j\leq s}(y-\alpha_j)\right) - t\prod_{i=1}^{s}(y-\alpha_i) \in k[y] \tag{6.1.4}$$

is a non-zero constant in k. In terms of coefficients, we need the α_i's to satisfy the following condition for each $1 \leq v \leq s-1$.

$$\sum_{1\leq i_1<\cdots<i_{s-v}\leq s} (n_{i_1} + \cdots + n_{i_{s-v}})\, \alpha_{i_1} \cdots \alpha_{i_{s-v}} = 0. \tag{6.1.5}$$

When $s = 1$, setting $\alpha_1 = 0$, we also get a similar condition on the choice of β_l's.

Before proceeding to the construction of the covers, let us see some of the cases where Assumption 6.5 is satisfied.

Lemma 6.7 ([9, Lemma 3.10])**.** *Assumption 6.5 holds with a choice of distinct α_i's and β_l's in the following cases.*

1. *$s = 1 = r$ with $(\alpha_1, \beta_1) = (1, 0)$;*
2. *$s = 2$, $r = 1$ with $(\alpha_1, \alpha_2, \beta_1) = (1, -n_1/n_2, 0)$;*
3. *$s = 1$, $r = 2$ with $(\alpha_1, \beta_1, \beta_2) = (0, 1, -m_1/m_2)$;*

4. $s = 3$, $r = 1$ *with* $(\alpha_1, \alpha_2, \alpha_3, \beta_1) = (\frac{t+2}{4}, -\frac{t-2}{4}, 1, 0)$ *where* $(p, t+2) = 1 = (p, t-2)$ *and* $n_1 = p - 2$, $n_2 = 2$, $n_3 = t$;

5. $r = s = 2$ *with* $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (1, \frac{n_2 - n_1}{2n_2}, 0, \frac{t}{2n_2})$ *where* $n_i \equiv m_i \bmod p$ *and* $n_1 \neq n_2$ *in* k.

6. $r = 1$, $s = d$ *with* $\beta_1 = 0$ *and* α_i*'s such that* $1 + y^d = \prod_{i=1}^{d}(y - \alpha_i)$.

Proof. It is easy to see that in each of the cases $(1) - (5)$, the assigned values of α_i's and β_l's are all distinct and they satisfy Assumption 6.5. Consider $r = 1$ and $s = d$. Since $y^d + 1$ is a separable polynomial over $k[y]$, it has d distinct roots $\alpha_1, \cdots, \alpha_d$ which are also non-zero. Also each $n_i = 1$, and the y-derivative of $1 + y^d$ is equal to $ty^{d-1} = \sum_{i=1}^{s} \prod_{j \neq i, 1 \leq j \leq s}(y - \alpha_j))$. From Equation (6.1.3),

$$g(y) = y \sum_i \prod_{j \neq i}(y - \alpha_j) - t \prod_i (y - \alpha_i) = ty^d - t - ty^d = -t \neq 0$$

in k. By Remark 6.6, Assumption 6.5 is satisfied. □

The following result produces our main example of the S_d-Galois or A_d-Galois two point branched covers of $\mathbb{P}^1$.

Proposition 6.8 ([9, Proposition 3.11])**.** *Let p be an odd prime such that $d := p + t \geq 5$. Let r and s be two positive integers. Let $n_1, \cdots, n_s, m_1, \cdots, m_r$ be coprime to p such that $\sum_{i=1}^{s} n_i = p + t$, $\sum_{l=1}^{r} m_l = t$. Let $\alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_r$ are distinct elements in k. Let $\psi \colon Y \to \mathbb{P}^1$ be the degree-d cover given by the affine equation $f(x, y) = 0$ where*

$$f(x, y) = \prod_{i=1}^{s}(y - \alpha_i)^{n_i} - x \prod_{l=1}^{r}(y - \beta_l)^{m_l}. \tag{6.1.6}$$

Let $\phi \colon Z \to \mathbb{P}^1$ be its Galois closure with group G.

1. *Then G is a transitive subgroup of S_d;*

2. *the cover ϕ is tamely ramified with cyclic inertia group generated by an element $\gamma \in G$ of order l.c.m.$\{n_1, \cdots, n_s\}$ over 0, whose disjoint cycle decomposition in S_d consists of s disjoint cycles of length $n_1, \cdots, n_s$;.*

3. *Over ∞, the inertia group is of the form $I = \langle(1,\cdots,p)\rangle \rtimes \langle\theta^i\omega\rangle$ for some $1 \leq i \leq p-1$, where θ is as in Proposition 5.1 and $\omega \in Sym(\{p+1,\cdots,d\})$ is a product of r disjoint cycles of length $m_1, \cdots, m_r$.*

Additionally, if $(\alpha_1,\cdots,\alpha_s,\beta_1,\cdots,\beta_r)$ satisfies Assumption 6.5, the cover ϕ is étale away from $\{0,\infty\}$. Also $ord(\theta^i) = \frac{p-1}{(p-1,r+s-1)}$, $|I| = p \times l.c.m\{ord(\theta^i), ord(\omega)\}$, and the upper jump for any I-Galois local extension over ∞ is given by $\frac{r+s-1}{p-1}$. Moreover, if $t < p$, G is a primitive subgroup of S_d. Furthermore, if either there is a positive integer j such that γ^j is a non-trivial cycle fixing ≥ 3 points in $\{1,\cdots,d\}$ or if $3 \leq t \leq p-1$,

$$G = \begin{cases} A_d, & \text{if } \gamma \text{ is an even permutation} \\ S_d, & \text{if } \gamma \text{ is an odd permutation.} \end{cases}$$

Proof. Since α_i's are distinct from β_l's by our assumption and the polynomial $f(x,y)$ is linear in x, it is irreducible in $k(x)[y]$. So G is a transitive subgroup of S_d, proving (1).

From the equation $f(x,y) = 0$ we have $v_{(y-\alpha_i)}(x) = n_i$ for $1 \leq i \leq s$, $v_{(y-\beta_l)}(x^{-1}) = m_l$ for $1 \leq l \leq r$ and $v_{(y^{-1})}(x^{-1}) = p$. Since $\sum_i n_i = p + t$ the fibre $\psi^{-1}(0)$ consists of s points in Y with the ramification index at the point $(y = \alpha_i)$ given by n_i, and also since $\sum_l m_l = t$, there are exactly $r+1$ points in Y lying above ∞ with ramification indices given by p, m_1, $\cdots$, m_r. Then the description of the inertia groups above 0 and ∞ follows from Theorem 5.9.

Now suppose that Assumption 6.5 holds. The y-derivative of the polynomial $f(x,y)$ is given by

$$f_y(x,y) = \prod_{i=1}^{s}(y-\alpha_i)^{n_i-1}\left(\sum_i n_i \prod_{j\neq i}(y-\alpha_j)\right) - x\prod_{l=1}^{r}(y-\beta_l)^{m_l-1}\left(\sum_l m_l \prod_{u\neq l}(y-\beta_u)\right).$$

Let (a,b) be a common zero of f and f_y. Then $0 = f_y(a,b)\prod_{l=1}^{r}(b-\beta_l) = g(b)\prod_i(b-\alpha_i)^{n_i-1}$ (where g is the polynomial given by Equation (6.1.3)). Then $a = 0$ if $n_i \geq 2$ for some i and there is no such common zero otherwise. So the cover ψ, and hence ϕ is étale away from $\{0,\infty\}$. Again by Theorem 5.9, the upper jump is $\frac{r+s-1}{p-1}$ and $\mathrm{ord}(\theta^i) = \frac{p-1}{(p-1,r+s-1)}$.

Since G is a transitive subgroup of S_d containing the p-cycle τ which fixes t points in $\{1,\cdots,d\}$ and $t < \frac{p+t}{2}$, by [19, Remark 1.6], G is primitive.

Finally if some power of γ is a non-trivial cycle fixing ≥ 3 points or if $3 \leq t \leq p-1$, by [19, Theorem 1.2], G contains A_d. The rest follows as in Proposition 6.3. □

From the above proposition, we deduce the following results which will be used later.

Corollary 6.9 ([9, Corollary 3.12]). *Let p be an odd prime, $3 \leq t \leq p-2$ be an odd integer and $d = p+t$. Then there is a connected A_d-Galois étale cover of the affine line such that $I := \langle (1,\cdots,p) \rangle \rtimes \langle \theta^2(p+1,\cdots,d) \rangle$ occurs as an inertia group at a point above ∞.*

Proof. Take $s = 2$, $r = 1$, $n_1 = p+t-1$, $n_2 = 1$ in Proposition 6.8. By Lemma 6.7(2), Assumption 6.5 is satisfied. Since t is an odd integer, a $(p+t-1)$-cycle is an even permutation. By Proposition 6.8, there is a connected A_d-Galois cover of $\mathbb{P}^1$ branched only at 0 and ∞, over 0 the inertia groups are generated by conjugates of a $(p+t-1)$-cycle in A_d and I occurs as an inertia group above ∞. By Abhyankar's Lemma ([1, page 279, Expose X, Lemma 3.6]), we obtain a connected A_d-Galois cover $\phi\colon Y \to \mathbb{P}^1$ étale away from ∞. Since t is odd, we have $(p+t-1, p-1) = 1 = (p+t-1, t)$. So I occurs as an inertia group in the cover ϕ above ∞. □

Corollary 6.10 ([9, Corollary 3.13]). *Let p be an odd prime, $4 \leq t \leq p-1$ an integer such that $(t+1, p-1) = 1 = (t-1, p+1)$ and $d = p+t$. Then there is a connected A_d-Galois étale cover of the affine line such that $I := \langle (1,\cdots,p) \rangle \rtimes \langle \theta^2(p+1,\cdots,d-1) \rangle$ occurs as an inertia group at a point above ∞.*

Corollary 6.11 ([9, Corollary 3.13]). *Let $p \equiv 2 \pmod 3$ be an odd prime. Let $3 \leq t \leq p-1$ be an integer, $d = p+t$. Then there is a connected A_d-Galois étale cover of the affine line such that $I := \langle (1,\cdots,p) \rangle \rtimes \langle \theta^{(t,p-1)}(p+1,\cdots,d-1) \rangle$ occurs as an inertia group at a point above ∞.*

Proof. Take $s = 2$, $r = 2$, $n_1 = p + t - 1$, $n_2 = 1$, $m_1 = t - 1$, $m_2 = 1$ in Proposition 6.8. By Lemma 6.7(5), Assumption 6.5 is satisfied. A $(p+t-1)$-cycle is an odd permutation if and only if t is an even integer. By Proposition 6.8, there is a connected G-Galois cover of $\mathbb{P}^1$ branched only at 0 and ∞, where $G = A_d$ if t is odd and $G = S_d$ if t is even. Over 0 the inertia groups are generated by conjugates of a $(p + t - 1)$-cycle in G and I occurs as an inertia group above ∞. After a pullback via the $[p + t - 1]$-Kummer cover (cf. Definition 5.13) we obtain a connected A_d-Galois cover $\phi\colon Y \to \mathbb{P}^1$ étale away from ∞ such that I occurs as an inertia group in the cover ϕ above ∞. □

Also as a consequence of Proposition 6.8, we can study the ramification behaviour of the Alternating group covers introduced by Abhyankar. This was proved in [22, Theorem 4.9] using a different method when $t < p$.

Proposition 6.12 ([10, Proposition 4.3])**.** *Let p be an odd prime. Let $d = p + t \geq p + 2$ such that $(p, t) \neq (7, 2)$ and t be coprime to p. Consider the degree-d cover $\psi\colon Y \to \mathbb{P}^1$ given by the affine Equation $f(x, y) = 0$ where*

$$f(x, y) = y^d + 1 - xy^t. \tag{6.1.7}$$

Let $\phi\colon Z \to \mathbb{P}^1$ be its Galois closure. Then ϕ is an A_d-Galois cover of smooth projective connected k-curves branched only at ∞ such that $I = \langle(1, \cdots, p)\rangle \rtimes \langle\theta^i\omega\rangle$ (θ as in Proposition 5.1) occurs as an inertia group above ∞ where ω is the t-cycle $(p + 1, \cdots, p + t)$ and $ord(\theta^i) = \frac{p-1}{(p-1,t+1)}$. Also $|I| = p \times l.c.m.\{\frac{p-1}{(p-1,t+1)}, t\}$ and the upper jump for any I-Galois local extension over ∞ is given by $\frac{d}{p-1}$.

Proof. By [3, Section 11], $G = A_d$ and the cover ϕ is branched only at ∞. So the fibre $\psi^{-1}(0)$ consists of d points all having ramification index one, and $\psi^{-1}(\infty)$ consists of two points with ramification indices p and t. Now the structure of an inertia group above ∞ follows from Theorem 5.9. □

Remark 6.13. *Note that the cover obtained in the above proposition has upper jump $\frac{d}{p-1}$ which is the minimal possible as described in Remark 4.8. So this gives an affirmative answer to Question 4.7 for A_d, with $d \geq 2p + 1$ coprime to p (the case $d < 2p$ being already done in [7, Corollary 2.2]) and $P = \langle(1, \cdots, p)\rangle$.*

In view of the above remark, we introduce the following A_d-cover with $d \geq 2p$ divisible by p.

Proposition 6.14 ([9, Proposition 6.1])**.** *Let p be an odd prime, $a \geq 2$ be an integer and $d = ap$. Let $1 \leq s \leq d-p-1$ be coprime to p. Let $\psi\colon Y \to \mathbb{P}^1$ be the degree-d cover given by the affine equation $f(x,y) = 0$ where*

$$f(x,y) = 1 + y^{d-s}(y+1)^s - xy^{d-p-s}(y+1)^s. \tag{6.1.8}$$

Let $\phi\colon Z \to \mathbb{P}^1$ be its Galois closure. Then ϕ is an A_d-Galois cover of $\mathbb{P}^1$ branched only at ∞ such that $I = \langle\tau\rangle \rtimes \langle\beta\rangle$ occurs as an inertia group at a point above ∞, where τ is a p-cycle and β has order l.c.m.$(\frac{p-1}{(p-1,d+1)}, s, d-p-s)$. Furthermore, it has upper jump $\frac{h}{m} = \frac{d+1}{p-1}$.

Proof. Let G be the Galois group of the cover ϕ. Since the polynomial $f(x,y)$ is linear in x, and the coefficient of x does not have any common factor with the terms devoid of x, it is irreducible in $k[y][x]$ and hence in $k(x)[y]$. So G is a transitive subgroup of S_d.

Set $h(y) := 1 + y^{d-s}(y+1)^s$. Then the y-derivative of $f(x,y)$ is given by

$$f_y(x,y) = h'(y) + sxy^{d-p-s-1}(y+1)^{s-1}.$$

Assume that f and f_y have a common zero (a,b). So $f_y(a,b) = 0$ which implies that $h'(b) = -sab^{d-p-s-1}(b+1)^{s-1}$. Also since $f(a,b) = 0$, we have $sh(b) + b(b+1)h'(b) = 0$. But we see that $sh(y) + y(y+1)h'(y) = s \neq 0$ in $k[y]$, showing that f and f_y cannot have a common zero. Since the cover ψ is non-trivial, it is branched only above $x = \infty$. So the fibre $\psi^{-1}(0)$ consists of d points all having ramification index one.

We can rewrite $f(x,y) = 0$ as

$$\begin{aligned} x^{-1} &= \frac{y^{d-p-s}(y+1)^s}{1 + y^{d-s}(y+1)^s}; \\ x^{-1} &= \frac{((y+1)-1)^{d-p-s}(y+1)^s}{1 + ((y+1)-1)^{d-s}(y+1)^s}; \\ x^{-1} &= \frac{y^{-p}(y^{-1}+1)^s}{y^{-d} + (y^{-1}+1)^s}. \end{aligned}$$

Thus $v_{(y)}(x^{-1}) = d - p - s$, $v_{(y+1)}(x^{-1}) = s$, and $v_{(y^{-1})}(x^{-1}) = p$. So there are exactly three points in Y in the fibre of ψ above $x = \infty$ with ramification indices p, s and $d - p - s$. Now apply Theorem 5.9 to obtain the result about the inertia groups above ∞.

The cover ϕ factors via a connected $G/\langle\langle\tau\rangle^G\rangle$-Galois cover of $\mathbb{P}^1$ which is étale away from ∞ and is tamely ramified over ∞. Since $\pi_1^t(\mathbb{A}^1)$ is trivial, G is generated by the conjugates of τ. By [28, Lemma 4.4.4.], G is a primitive quasi p-subgroup of S_d. Also the p-cycle τ in G fixes $(a-1)p \geq 3$ points in $\{1, \cdots, d\}$. So by Jordan's Theorem [3, pg 111], $G = A_d$. The ramification behaviour follows from Theorem 5.9. □

The following is a positive result towards the minimal jump problem (Question 4.7).

Corollary 6.15. *For any $d \geq p + 2$ with $(p, d) \neq (7, 9)$, Question 4.7 has an affirmative answer for the group A_d with $P = \langle(1, \cdots, p)\rangle$.*

Proof. For $d < 2p$, this is [7, Corollary 2.2]. Let $d \geq 2p$. In view of Remark 4.8 we want to prove that there is a connected A_d-Galois étale cover of the affine line such that P is a Sylow p-group of an inertia group above ∞ and the upper jump is

$$\sigma = \begin{cases} \frac{d+1}{p-1} & \text{if } p|t \\ \frac{d}{p-1} & \text{otherwise.} \end{cases}$$

Now the result follows from Proposition 6.12, Proposition 6.14 with $s = d - p - 1$. □

6.2 Construction of Covers by Formal Patching

In this section, we construct covers using the formal patching techniques (see Section 3.4). Fix the following notation for the rest of this section.

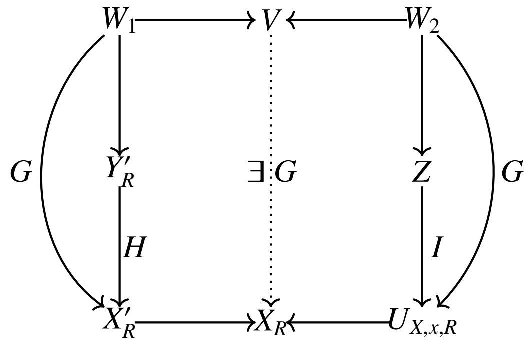
Notation 6.16. Let $R = k[[t]]$, $\widetilde{K} = k((t))$, $U = \text{Spec}(k[[x^{-1}]])$. Let b be the closed point of U. For any k-algebra A with $K = QF(A)$ and any k-scheme W, let $W_A := W \times_k A$, $W_K := W_A \times_A K$, and for any closed point $w \in W$, $w_A := w \times_k A$, $w_K := w_A \times_A K$. For an R-scheme V, let V^0 denote the closed fiber of $V \to \text{Spec}(R)$.

The following two results ([10, Lemma 3.2, Lemma 3.3]) are of great importance. Given a Galois cover $\phi\colon Y \to X$ of smooth projective connected k-curves with inertia group I at a point $y \in Y$ and $f(y) = x$, we deform the local I-Galois cover at x to obtain a new cover with desired local properties. We use Corollary 3.34 to first get a cover over X_R and then use a Lefschetz type principle to obtain Galois covers of X, again with desired local behavior.

Lemma 6.17 ([10, Lemma 3.2]). *Let G be a finite group, and X be an irreducible smooth projective k-curve. Let $I_1 \subset I$ and H be subgroups of G such that $G = \langle H, I\rangle$. Assume that there is an irreducible H-Galois cover $\phi\colon Y \to X$ of connected smooth projective k-curves branched only at a point $x \in X$, with inertia group I_1 at a point y in Y above x. Let $U_{X,x} = \operatorname{Spec}(\widehat{O}_{X,x})$, $K_{X,x} = QF(\widehat{O}_{X,x})$ and $L = QF(\widehat{O}_{Y,y})$. Let b_x denote the closed point of $U_{X,x}$. Let $Z \to U_{X,x,R}$ be an irreducible I-Galois cover of integral R-schemes totally ramified over $b_{x,R}$ such that the normalization of the pullback of $Z^0 \to U_{X,x}$ to $\operatorname{Spec}(K_{X,x})$ is isomorphic to $\operatorname{Ind}_{I_1}^{I}\operatorname{Spec}(L)$ as the I-Galois covers of $\operatorname{Spec}(K_{X,x})$. Then there is a normal G-Galois cover $V \to X_R$ of irreducible R-curves such that the following hold.*

1. *$V \to X_R$ is étale away from x_R with inertia group I above x_R;*
2. *$V \times_{X_R} U_{X,x,R} \cong \operatorname{Ind}_I^G Z$ and if $X' = X \setminus \{x\}$ then $V \times_{X_R} X'_R \cong \operatorname{Ind}_H^G Y'_R$ where $Y' = \phi^{-1}(X')$;*
3. *the closed fibre V^0 is connected and the normalization of the pullback of $V^0 \to X$ to $\operatorname{Spec}(K_{X,x})$ is isomorphic to $\operatorname{Ind}_{I_1}^G \operatorname{Spec}(L)$ as G-Galois cover of $\operatorname{Spec}(K_{X,x})$.*

Proof. Consider the trivial deformation $Y'_R \to X'_R$ of the H-Galois cover $Y' \to X'$. Taking a disjoint union of $[G : H]$-copies of Y'_R we obtain a (disconnected) normal G-Galois cover $W_1 := \operatorname{Ind}_H^G Y'_R \to X'_R$, in which the stabilizers of the components are the conjugates of H in G. Now taking a disjoint union of $[G : I]$-copies of Z, we obtain a (disconnected) normal G-Galois branched cover $W_2 := \operatorname{Ind}_I^G Z \to U_{X,x,R}$, in which the stabilizers of the components are the conjugates of I in G. We want to show that there exists a G-Galois cover $V \to X_R$ such that the following diagram commutes.



The fibre of W_1^0 over $\mathrm{Spec}(K_{X,x})$ is given by $\mathrm{Ind}_H^G \mathrm{Ind}_{I_1}^H \mathrm{Spec}(L)$ and that of W_2^0 over $\mathrm{Spec}(K_{X,x})$ is given by $\mathrm{Ind}_I^G \mathrm{Ind}_{I_1}^I \mathrm{Spec}(L)$. Since both of these covers are indexed by the left cosets of I_1 in G, we can choose an isomorphism between these fibers that is compatible with the indexing, and hence with the G-action. Since the pullbacks of $W_1 \to X'_R$ and $W_2 \to U_{X,x,R}$ to $\mathrm{Spec}(K_{X,x,R})$ are étale, by [1, I, Corollary 6.2], they are trivial deformations of the pullbacks of $W_1^0 \to X'$ and $W_2^0 \to U_{X,x}$ to $\mathrm{Spec}(K_{X,x})$. So the above isomorphism lifts uniquely to a G-isomorphism $W_1 \times_{X'_R} K_{X,x,R} \cong W_2 \times_{U_{X,x,R}} K_{X,x,R}$ over $\mathrm{Spec}(K_{X,x,R})$. By Corollary 3.34, there exists a unique normal G-Galois cover $V \to X_R$ such that $V \times_{X_R} X'_R \cong W_1$ and $V \times_{X_R} U_{X,x,R} \cong W_2$ as covers of X'_R and $U_{X,x,R}$, respectively. Since W_2 has branch locus x_R and W_1 is étale, $V \to X_R$ has branch locus $\{x_R\}$. Also since $Z \to U_{X,x,R}$ is totally ramified above $b_{x,R}$, we have $\mathrm{Gal}(V \times_{X_R} (K_{X,x} \otimes_k \widetilde{K}))/(K_{X,x} \otimes_k \widetilde{K})) = \mathrm{Gal}(Z \times_{U_{X,x,R}} (K_{X,x} \otimes_k \widetilde{K}))/(K_{X,x} \otimes_k \widetilde{K})) = I$. So the inertia group above x_R is I. Finally, since the the stabilizers of the identity components of $W_1 \to X'_R$ and of $W_2 \to U_{X,x,R}$ are H and I, respectively, the stabilizer of the identity component of $V \to X_R$ is $\langle H, I\rangle = G$. So V is irreducible. □

Lemma 6.18 ([10, Lemma 3.3])**.** *Let X be an irreducible smooth projective k-curve, and $x \in X$ be a closed point. Let $U_{X,x} = \mathrm{Spec}(\widehat{\mathcal{O}}_{X,x})$ and $K_{X,x} = QF(\widehat{\mathcal{O}}_{X,x})$. Let b_x denote the closed point of $U_{X,x}$. For $1 \le i \le r$, let G_i be a finite group, and I_i be a subgroup of G_i. Assume that for each $1 \le i \le r$, there is an I_i-Galois cover $\phi_i \colon \widetilde{S}_i \to U_{X,x,k[t]}$ of integral schemes with branch locus $b_{x,k[t]}$ over which it is totally ramified. For those closed points $\beta \in \mathbb{A}^1_t$ where the fiber $\widetilde{S}_{i,\beta}$ of $\widetilde{S}_i \to \mathbb{A}^1_t$ is integral, let $M^{(i)}_\beta/K_{X,x}$ denote the field extension corresponding to the cover $\widetilde{S}_{i,\beta} \to U_{X,x}$ at $t = \beta$. For $1 \le i \le r$, let $Z_i := \widetilde{S}_i \times_{U_{X,x,k[t]}} U_{X,x,R}$, and $V_i \to X_R$ be a normal G_i-Galois cover of irreducible R-curves étale away from x_R with inertia group I_i above x_R such that $V_i \times_{X_R} U_{X,x,R} \cong \mathrm{Ind}_{I_i}^{G_i} Z_i$. Then* *there is an open dense subset $\mathcal{V}$ of $\mathbb{A}^1_t$ such that for all closed points $(t = \beta)$ in $\mathcal{V}$ the following holds. For each $1 \le i \le r$, there is a G_i-Galois cover $W_i \to X$ branched only at x with inertia group I_i at a point in W_i above x such that the local I_i-Galois extension corresponding to the formal neighbourhood at that point is $M^{(i)}_\beta / K_{X,x}$.*

Proof. Let $1 \le i \le r$. Consider the G_i-Galois covers $f^{(i)} \colon V_i \times_{X_R} U_{X,x,R} \to U_{X,x,R}$ and $g^{(i)} \colon \operatorname{Ind}^{G_i}_{I_i} Z_i \to U_{X,x,R}$. Since $f^{(i)}$ and $g^{(i)}$ are finite morphisms and are G_i-Galois covers, the isomorphism $V_i \times_{X_R} U_{X,x,R} \cong \operatorname{Ind}^{G_i}_{I_i} Z_i$ is equivalent to a G_i-equivariant isomorphism of coherent sheaves $f^{(i)}_*(\mathcal{O}_{V_i \times_{X_R} U_{X,x,R}})$ and $g^{(i)}_*(\mathcal{O}_{\operatorname{Ind}^{G_i}_{I_i} Z_i})$ over $U_{X,x,R}$, and hence it is defined locally by matrices involving only finitely many functions over $U_{X,x,R}$. So there exists a finite type $k[t]$-algebra $A \subset R$ having smooth connected spectrum $E = \operatorname{Spec}(A)$ and for each $1 \le i \le r$, an irreducible G_i-Galois cover $\pi_i \colon F_i \to X_A$ branched only over x_E with inertia group I_i above x_E, together with an isomorphism $F_i \times_{X_A} U_{X,x,A} \cong \operatorname{Ind}^{G_i}_{I_i} \widetilde{S}_i \times_{U_{X,x,k[t]}} U_{X,x,A}$ such that $F_i \times_A R \cong V_i$, and the fibre over each closed point of E is irreducible and non-empty. So for each point $e \in E$ and for each $1 \le i \le r$, $\widetilde{F_{i,e}} \to X \times_k \{e\} \cong X$ is a G_i-Galois cover étale away from x with inertia group I_i above x, where $\widetilde{F_{i,e}}$ is the normalization of the fibre $F_{i,e} = \pi_i^{-1}(X \times_k \{e\})$. Since the ring map $k[t] \to A$ is also injective, the finite type map $E \to \mathbb{A}^1_t$ is flat and dominant. So the image of E in $\mathbb{A}^1_t$ is an open dense set which is our $\mathcal{V}$. Now for every point $\beta \in \mathcal{V}$ with preimage $e_\beta \in E$ and for each $1 \le i \le r$, the corresponding fibre $W_i := \widetilde{F}_{i,e_\beta} \to X$ is a G_i-Galois cover branched only at x, and the I_i-Galois extension corresponding to the formal neighborhood of a point in W_i lying above x is $M^{(i)}_\beta / K_{X,x}$. □

Remark 6.19. *Note that Lemma 6.17 and Lemma 6.18 can be easily generalized to the case with branch locus containing more than one point.*

The following result gives a deformation of local extensions with pre-assigned special fibres.

Lemma 6.20 ([9, Lemma 4.2])**.** *Let I be an extension of a p-group by a cyclic group of order m, $p \nmid m$. Let I_1 and I_2 be two subgroups of I such that $|I_1|/|p(I_1)| = m = |I_2|/|p(I_2)|$. For $i = 1, 2$, assume that $V_i \to S$ be a connected I_i-Galois cover which is totally ramified over s. Then there is a connected I-Galois cover $V \to S \times \mathbb{A}^1_u$ of* *integral schemes with branch locus* $s \times \mathbb{A}^1_u$ *over which it is totally ramified such that* $V \times_{S \times \mathbb{A}^1_u} S \times \{u = 1\} \cong \mathrm{Ind}^I_{I_1} V_1$ *and* $V \times_{S \times \mathbb{A}^1_u} S \times \{u = -1\} \cong \mathrm{Ind}^I_{I_2} V_2$ *as I-Galois covers over* S.

Proof. Let $i \in \{1, 2\}$. Then $V_i = \mathrm{Spec}(A_i)$ for a complete discrete valuation ring A_i with residue field k and let L_i be the field of fractions of A_i. After a change of variable we assume that $L_i^{p(I)} = K' = k((T))$ where the extension K'/K is given by $T^m = t$. The trivial deformation of the cover $\mathrm{Spec}(K') \to \mathrm{Spec}(K)$ is the connected $\mathbb{Z}/m$-Galois cover $\mathrm{Spec}(K'[u]) \to \mathrm{Spec}(K[u])$ of integral K-curves. Now the compositions

$$\mathrm{Ind}^I_{I_1}\mathrm{Spec}(L_1) \to \mathrm{Spec}(K') \to \mathrm{Spec}(K) = \mathrm{Spec}(K[u]) \times_k (u = 1),$$

$$\mathrm{Ind}^I_{I_2}\mathrm{Spec}(L_2) \to \mathrm{Spec}(K') \to \mathrm{Spec}(K) = \mathrm{Spec}(K[u]) \times_k (u = -1)$$

are (possibly disconnected) I-Galois covers. By applying [14, Proposition 3.11] with $\Gamma = I$, $G = \mu_m$, X as the affine u-line over K and X' as the closed subset of X consisting of the points $(u = 1)$ and $(u = -1)$, we obtain a connected $p(I)$-Galois cover $V' \to \mathbb{A}^1_{K'}$ of integral curves such that the composition $f : V' \to \mathbb{A}^1_K$ is a connected I-Galois cover and such that $V' \times_{\mathbb{A}^1_u} (u = 1) \cong \mathrm{Ind}^I_{I_1}\mathrm{Spec}(L_1)$ and $V' \times_{\mathbb{A}^1_u} (u = -1) \cong \mathrm{Ind}^I_{I_2}\mathrm{Spec}(L_2)$ as I-Galois covers of $\mathrm{Spec}(K)$. Take V to be the normalization of $S \times \mathbb{A}^1_u$ in the function field of V'. Then the cover $V \to S \times \mathbb{A}^1_u$ satisfies the stated properties. □

Using the above lemma and formal patching technique, we obtain the following result which produces Galois covers with bigger Galois and inertia groups starting from a certain Galois covers with a compatibility on the tame parts of the inertia groups.

Theorem 6.21 ([9, Theorem 4.3]). *Let* G *be a finite group,* $I \subset G$ *be an extension of a* p*-group by a cyclic group of order* m, $p \nmid m$. *Let* G_1, G_2 *be subgroups of* G. *Let* X *be a smooth projective connected* k*-curve, and* $f_1 : Y_1 \to X$ *be a connected* G_1*-Galois cover étale away from* $B_1 \subset X$. *Assume that there is a closed point* $x \in B_1$ *such that an inertia group* I_1 *over* x *is contained in the group* I *above. For* $x_1 \neq x$ *in* B_1, *let* J_{x_1} *occurs as an inertia group above* x_1. *Also assume the existence of a connected* G_2*-Galois cover* $f_2 : Y_2 \to \mathbb{P}^1$ *étale away from a set* $B_2 \subset \mathbb{P}^1$ *such that* $0 \in B_2$ *and an* *inertia group I_2 above 0 is contained in I. For points $y \neq 0$ in B_2, let I'_y occurs as an inertia group above y. Assume further that the subgroups I_1 and I_2 of I are such that $|I_1/p(I_1)| = m = |I_2/p(I_2)|$. If $G = \langle G_1, G_2, I\rangle$, then there is subset $B_0 \subset X$ disjoint from B_1 together with a bijection $\eta\colon B_0 \xrightarrow{\sim} B_2 - \{0\}$ and a connected G-Galois cover of X étale away from $B_1 \sqcup B_0$ such that I occurs as an inertia group above x, for $x_1 \neq x$ in B_1, J_{x_1} occurs as an inertia group above x_1 and for x_0 in B_0, $I'_{\eta(x_0)}$ occurs as an inertia group above x_0.*

Proof. Let $y_1 \in Y_1$ with $f_1(y_1) = x$ such that I_1 is the Galois group of the field extension $K_{Y_1,y_1}/K_{X,x}$. Also let $y_2 \in Y_2$ be a point lying above ∞ such that I_2 is the Galois group of the field extension $K_{Y_2,y_2}/k((x))$. We identify $\widehat{\mathcal{O}}_{X,x}$ with $k[[x]]$. Then taking $V_1 = \mathrm{Spec}(\widehat{\mathcal{O}}_{Y_1,y_1}) \to \mathrm{Spec}(k[[x]])$ and $V_2 = \mathrm{Spec}(\widehat{\mathcal{O}}_{Y_2,y_2}) \to \mathrm{Spec}(k[[x]])$ in Lemma 6.20 we obtain a connected I-Galois cover $g\colon V \to \mathrm{Spec}(k[[x]][u])$ of integral schemes with branch locus $x \times \mathbb{A}^1_u$ over which it is totally ramified and such that $V \times_{\mathrm{Spec}(k[[x]][u])} \mathrm{Spec}(k[[x]]) \times \{u = 1\} \cong \mathrm{Ind}^I_{I_1} V_1$ and $V \times_{\mathrm{Spec}(k[[x]][u]} \mathrm{Spec}(k[[x]]) \times \{u = -1\} \cong \mathrm{Ind}^I_{I_2} V_2$ as I-Galois covers over $\mathrm{Spec}(k[[x]])$. Then the cover f_1 and the I-Galois cover g satisfy the hypothesis of Lemma 6.17 and by Lemma 6.18, there is an open dense set $\mathcal{W}_1 \subset \mathbb{A}^1_u$ such that for all closed point $(u = \beta)$ in $\mathcal{W}_1$ the following holds. There is a connected $H_1 := \langle G_1, I\rangle$-Galois cover $h_1\colon Z_1 \to X$ étale away from B_1 such that there is a point $z_1 \in Z_1$ above x for which $\mathrm{Spec}(\widehat{\mathcal{O}}_{Z_1,z_1})$ is equal to the fibre of the cover $g\colon V \to \mathrm{Spec}(k[[x]][u])$ over $(u = \beta)$ as I-Galois covers over $\mathrm{Spec}(k[[x]])$, and for any point $x_1 \neq x$ in B_1, J_{x_1} occurs as an inertia group above x_1. Similarly, the covers f_2 and g satisfy the hypothesis of Lemma 6.17 and by Lemma 6.18, there is an open dense set $\mathcal{W}_2 \subset \mathbb{A}^1_u$ such that for all closed point $(u = \alpha)$ in $\mathcal{W}_2$ the following holds. There is a connected $H_2 := \langle G_2, I\rangle$-Galois cover $h_2\colon Z_2 \to \mathbb{P}^1$ étale away from B_2 such that there is a point $z_2 \in Z_2$ above 0 for which $\mathrm{Spec}(\widehat{\mathcal{O}}_{Z_2,z_2})$ as an I-Galois cover of $\mathrm{Spec}(k[[x]])$ is equal to the fibre of g over $(u = \alpha)$. We fix a closed point $(u = a)$ in $\mathcal{W}_1 \cap \mathcal{W}_2$ and consider the corresponding covers $h_1\colon Z_1 \to X$ and $h_2\colon Z_2 \to \mathbb{P}^1$ as above. Then $\mathrm{Spec}(\widehat{\mathcal{O}}_{Z_1,z_1})$ and $\mathrm{Spec}(\widehat{\mathcal{O}}_{Z_2,z_2})$ are isomorphic I-Galois covers of $\mathrm{Spec}(k[[x]])$.

Let T^* (see Remark 3.36 for an explicit construction of T^*) be a regular irreducible projective R-curve with generic fibre X_K together with a cover $T^* \to \mathbb{P}^1_R$ and whose closed fibre T' is the union of two irreducible components X and $\mathbb{P}^1_y$ meeting at a point η and such that the complete local ring of T^* at η is given by $\widehat{O}_{T^*,\tau} = k[[x,y]][t]/(t-xy) \cong k[[x,y]]$. Now consider the trivial deformation $\mathrm{Spec}(\widehat{O}_{Z_2,z_2}[y]) \to \mathrm{Spec}(k[[x]][y])$. Let $\widehat{N}^* := \mathrm{Spec}(\widehat{O}_{Z_2,z_2}[y]) \times_{\mathrm{Spec}(k[[x]][y])} \mathrm{Spec}(k[[x,y]])$. Let $X - x = \mathrm{Spec}(A)$, $Z_1 - z_1 = \mathrm{Spec}(B_1)$ and $Z_2 - z_2 = \mathrm{Spec}(B_2)$. Then the hypothesis of [12, Proposition 2.3] is satisfied with the covers $W_1'^* = \mathrm{Spec}(B_1[[t]]) \to X_1'^* = \mathrm{Spec} A[[t]]$ and $W_2'^* = \mathrm{Spec}(B_2[[t]]) \to X_2'^* = \mathrm{Spec}(k[y^{-1}][[t]])$ induced by h_1 and h_2, respectively, and with the isomorphisms

$$\widehat{N}^* \times_{\mathrm{Spec}(k[[x,y]])} \mathrm{Spec}(K_{X,x}[[t]]) \cong \mathrm{Spec}(K_{Z_1,z_1}) \text{ and}$$

$$\widehat{N}^* \times_{\mathrm{Spec}(k[[x,y]])} \mathrm{Spec}(k((y))[[t]]) = \mathrm{Spec}(K_{Z_2,z_2}).$$

By [12, Proposition 2.3], there is an irreducible normal G-Galois cover $h^*\colon V^* \to T^*$ such that $V^* \times_{T^*} X_1'^* \cong \mathrm{Ind}_{H_1}^G W_1'^*$, $V^* \times_{T^*} X_2'^* \cong \mathrm{Ind}_{H_2}^G W_2'^*$ and $V^* \times_{T^*} \widehat{O}_{T^*,\eta} \cong \mathrm{Ind}_I^G \widehat{N}^*$ as G-Galois covers. Consider the generic fibre $h^0\colon V^0 \to X_K$ of the cover h^*. Then there is a set $B_0 \in X$ disjoint from B_1 together with a set bijection $\eta\colon B_0 \xrightarrow{\sim} B_2 - \{0\}$ such that h^0 is étale away from $\{x'_K | x' \in B_1 \sqcup B_0\}$, I occurs as an inertia group above x_K, for $x_1 \neq x$ in B_1, J_{x_1} occurs as an inertia group above $x_{1,K}$ and for x_0 in B_0, $I'_{\eta(x_0)}$ occurs as an inertia group above $x_{0,K}$. Since $Z_1 \times_R K$, $Z_2 \times_R K$ and $T \times_R K$ are smooth over K, V_0 is also smooth over K. Also since T' is generically smooth and the cover $V^* \to T^*$ is generically unramified, the closed fibre $V^* \times_{T^*} T' \to T'$ of h^* is generically smooth. Now the result follows by [12, Corollary 2.7]. □

By induction on n and using the above theorem, we obtain the following result which generalizes a patching result by Raynaud ([25, Theorem 2.2.3]). This result allows us to construct Galois covers with certain control on the inertia groups from covers with smaller Galois groups. As an application (Lemma 7.14) of it, we will see that we can restrict ourselves to a fewer cases of proving the IC.

Corollary 6.22 ([9, Corollary 4.4])**.** *Let $n \geq 2$ be an integer. Let G be a finite group, $I \subset G$ be an extension of a p-group by a cyclic group of order m, $p \nmid m$. For $1 \leq i \leq n$, let G_i be a subgroup of G, $I_i \subset G_i$ be a subgroup of I with $|I_i/p(I_i)| = m$ and such that the following hold.*

1. *X is a smooth projective connected k-curve with a connected G_1-Galois cover $f\colon Y \to X$ étale away from $B \subset X$. Assume that I_1 occurs as an inertia group above a point $x \in B$. For $x' \neq x$ in B let $J_{x'}$ occurs as an inertia group above x'.*

2. *For each $2 \leq i \leq n$, the pair (G_i, I_i) is realizable.*

If $G = \langle G_1, \cdots, G_n, I\rangle$, there is a connected G-Galois cover $Z \to X$ étale away from B such that I occurs as an inertia group above x, and for $x' \neq x$ in B, $J_{x'}$ occurs as an inertia group above x'.

For our next result, let us fix the following notation.

Notation 6.23. Let G_1, G_2 be two finite groups, X a smooth projective connected k-curve. Let $B \subset X$ be a finite set of closed points. Assume that for $i = 1, 2$, there is a connected G_i-Galois cover $f_i\colon Y_i \to X$ étale away from B such that a p-group (possibly trivial) $P_{x,i}$ occurs as the inertia group above $x \in B$. For each $x \in B$, let Q_x be a p-group (possibly trivial), and for $i = 1, 2$, let $N_{x,i}$ be a normal subgroup of $P_{x,i}$ such that $P_{x,i}/N_{x,i} \cong Q_x$.

The following result shows that in the setup of the above notation, certain kind of field extensions can be realized as local extensions by the Galois covers for both the groups G_1 and G_2. This will be used to show that the GPWIC is true for certain product of groups.

Lemma 6.24 ([9, Lemma 4.6])**.** *Assume that Notation 6.23 hold. Then for $i = 1, 2$, there is a connected G_i-Galois cover $Z_i \to X$ étale away from B such that $P_{x,i}$ occurs as an inertia group above $x \in B$ and such that there is a point $z_i \in Z_i$ over x with $K_{Z_1,z_1}^{N_{x,1}}/K_{X,x} \cong K_{Z_2,z_2}^{N_{x,2}}/K_{X,x}$ as Q_x-Galois extensions of $K_{X,x}$.*

Proof. Let $x \in B$, $i = 1, 2$. Without loss of generality, we may assume that all the groups $P_{x,i}$ and Q_x are non-trivial. Let $y_i \in Y_i$ with $f_i(y_i) = x$ such that $P_{x,i}$ is the Galois group of the field extension $K_{Y_i,y_i}/K_{X,x}$. Take $I_1 = I_2 = I = Q_x$ and $V_i = \mathrm{Spec}(\widehat{\mathcal{O}}_{Y_i,y_i}^{N_{x,i}})$ in Lemma 6.20. Then we obtain a connected Q_x-Galois cover $V \to \mathrm{Spec}(\widehat{\mathcal{O}}_{X,x}) \times \mathbb{A}^1_u$ of integral schemes with branch locus $x \times \mathbb{A}^1_u$ over which it is totally ramified such that

$V \times_{\mathbb{A}^1_u} \{u = 1\} \cong V_1$ and $V \times_{\mathbb{A}^1_u} \{u = -1\} \cong V_2$ as Q_x-Galois covers over $\mathrm{Spec}(\widehat{O}_{X,x})$. By [14, Theorem 3.11], there is a connected $P_{x,i}$-Galois cover $g_i \colon W_i \to \mathrm{Spec}(\widehat{O}_{X,x}) \times \mathbb{A}^1_u$ of integral schemes dominating $V \to \mathrm{Spec}(\widehat{O}_{X,x}) \times \mathbb{A}^1_u$ such that the fibre of g_1 over $(u = 1)$ is $\mathrm{Spec}(\widehat{O}_{Y_1,y_1})$ and the fibre of g_2 over $(u = -1)$ is $\mathrm{Spec}(\widehat{O}_{Y_2,y_2})$. Then by Lemma 6.17, there are connected G_i-Galois covers of X_R satisfying the hypothesis of Lemma 6.18. Using Remark 6.19, for each $x \in B$, $i \in \{1, 2\}$, choose a dense open set $\mathcal{W}_{x,i} \subset \mathbb{A}^1_u$ and fix a closed point in $\mathcal{W}_{x,1} \cap \mathcal{W}_{x,2}$. Then for $i = 1, 2$, there is a connected G_i-Galois cover $Z_i \to X$ étale away from B such that $P_{x,i}$ occurs as an inertia group above $x \in B$ and such that there is a point $z_i \in Z_i$ over x with $K^{N_{x,1}}_{Z_1,z_1}/K_{X,x} \cong K^{N_{x,2}}_{Z_2,z_2}/K_{X,x}$ as Q_x-Galois extensions. □

We recall and restate the following theorem due to Harbater which will be used throughout the sections.

Theorem 6.25 ([13, Corollary to Patching Theorem])**.** *Let $r \geq 1$ be an integer. Let G be a finite group, G_1 and G_2 be two subgroups of G such that $G = \langle G_1, G_2 \rangle$. Let X be a smooth projective connected k-curve. Let B be a finite set of closed points of X containing a point x_0. Let $B' := \{\eta_0, \cdots, \eta_r\}$ be a set of distinct points of $\mathbb{P}^1$. Let $a \in G_1 \cap G_2$ be an element of order prime-to-p. Assume that*

1. *there is a connected G_1-Galois cover $f \colon Y \to X$ étale away from B such that I_x occurs as an inertia group above $x \in B$ and $I_{x_0} = \langle a \rangle$;*
2. *there is a connected G_2-Galois cover $g \colon W \to \mathbb{P}^1$ étale away from B' such that J_i occurs as an inertia group above η_i, $1 \leq i \leq r$, and such that $J_0 = \langle a^{-1} \rangle$.*

Then there is a set $B'' = \{x_1, \cdots, x_{r-1}\}$ of closed points of X disjoint from B and a connected G-Galois cover of X étale away from $B \sqcup B''$ such that I_x occurs as an inertia group above $x \in B \setminus \{x_0\}$, J_r occurs as an inertia group above x_0 and J_i occurs as an inertia group above x_i, $1 \leq i \leq r - 1$.

In [15], a special case of the following result appeared that also solves the split quasi p embedding problem.

Theorem 6.26 ([9, Theorem 4.8])**.** *Let G be a finite group, $\psi\colon Y \to X$ be a smooth connected G-Galois cover. Let $x_0 \in X$ be a closed point. Let Γ be a finite group generated by G and a quasi p-group H such that there is a p-subgroup P of H which is normalized by G and such that the pair (H, P) is realizable. Then there is a Γ-Galois cover $\phi\colon Z \to X$ of smooth projective connected k-curves dominating the cover ψ such that the following hold.*

1. *For a closed point $x \neq x_0$, if I_x occurs as an inertia group at a point over x for the cover ψ, then I_x also occurs as an inertia group at a point over x for the cover ϕ;*
2. *if I_0 occurs as an inertia group at a point above x_0 for the cover ψ, I_0P occurs as an inertia group at a point above x_0 for the cover ϕ;*
3. *the covers $Z/\langle H^\Gamma\rangle \to X$ and $Y/(G \cap \langle H^\Gamma\rangle) \to X$ are isomorphic as $\Gamma/\langle H^\Gamma\rangle$-Galois covers of X.*

Proof. By [15, Theorem 2.1, Theorem 4.1], the above conclusion holds when P is replaced by a Sylow p-subgroup of H which is normalized by G. But the same proof works under our hypothesis with the additional assumption that there is a connected H-Galois étale cover of the affine line such that P occurs as an inertia group above ∞. □

Chapter 7

Proofs of the Main Results

7.1 Strategy of the proofs

In Section 7.2 one of the main results is Theorem 7.6. This theorem proves that given two prefect quasi p-groups G_1 and G_2, $\langle \tau_i \rangle = P_i \subset G_i$ where τ_1 has order p and τ_2 has order p^a for some $a \geq 1$ such that the pairs (G_1, P_1) and (G_2, P_2) are realizable, the pair $(G_1 \times G_2, \langle (\tau_1^b, \tau_2) \rangle)$ is also realizable for some $1 \leq b \leq p-1$. To prove this, we first deform both the G_1-Galois and G_2-Galois étale covers of the affine line suitably to obtain respective covers whose compositum is a $G_1 \times G_2$-Galois étale cover of the affine line such that $P_1 \times P_2$ occurs as an inertia group over ∞ and there is an upper jump at 1 in the upper indexed ramification filtration. Then we use Theorem 5.12 to reduce inertia. We also obtain several connected A_d-Galois covers of the affine line given by the explicit equations from Section 6.1. Applying the above result to these covers together with a patching result of Raynaud ([25, Theorem 2.2.3]), we obtain in Corollary 7.11 that the PWIC is true for any product of Alternating groups, each of degree p or coprime to p.

In Section 7.3 we show evidence towards the IC (Conjecture 4.3). For each A_d, $p+1 \leq d \leq 2p-1$, we figure out the potential candidates I for the inertia groups using a group theoretic result (Proposition 5.1). Then we show that it is enough to prove that only a fewer suitable pairs (A_d, I) are needed to be realized in order to prove the IC.

This is done by using a formal patching result (Corollary 6.22). Then we treat each individual d and use results from Section 6.1 to construct covers given by the explicit affine equations.

The results supporting GPWIC in Section 7.4 are proved using formal patching technique developed in Section 6.2 together with induction. In the last section affirmative answer to the general question is provided in some special cases using the construction of covers via explicit equations from Section 6.1, formal patching technique from Section 6.2, and results from Section 7.4 and Section 7.3.

7.2 Purely Wild Inertia Conjecture for Product of Alternating Groups

The content of this section is in the paper [10]. Fix an odd prime p throughout this section. Our objective is to prove the PWIC (Conjecture 4.4) for any finite product of simple quasi p Alternating groups A_d. Note that A_d is quasi p if and only if $d \geq p$. Let P be a p-subgroup in A_d whose conjugates in A_d generate A_d. Since we only consider the simple groups A_d, for any element $\tau \in P$, the conjugates of $\langle \tau \rangle$ in A_d generate A_d. In view of Theorem 3.35, to prove the PWIC for A_d, it is enough to prove that for any element τ in A_d of order p, the pair $(A_d, \langle \tau \rangle)$ is realizable (Definition 3.25). Since for a Galois cover $Z \to \mathbb{P}^1$ branched only at ∞, the inertia groups that occur over ∞ are conjugate to each other, it suffices to consider the elements τ up to conjugation. Without loss of generality, any such element is of the form $\tau = \tau_1 \cdots \tau_a$ for some $a \geq 1$ with $ap \leq d$ and for $1 \leq i \leq a$, τ_i is the p-cycle $((i-1)p+1, \cdots, ip)$. We first prove that for $d \geq p$ and for any p-group in A_d containing the p-cycle $(1, \cdots, p)$, the pair (A_d, P) is realizable (Corollary 7.3). This is done by studying the covers introduced by Abhyankar (Proposition 6.12). In particular, this proves that the PWIC is true for the groups A_d, $p \leq d \leq 2p-1$, which is well known due to an application of Abhyankar's Lemma (Theorem 5.14) to Raynaud's proof of the Abhyankar's Conjecture on the affine line (Theorem 4.2). Using this result together with a result about a product of Alternating groups obtained using formal patching technique, we will prove for any $a \geq 1$ and

$1 \leq r \leq a$ the existence of a certain A_{ap+1}-Galois étale cover of the affine line such that $\langle \tau_1 \cdots \tau_r \rangle$ occurs as an inertia group above ∞. Finally, another application of the formal patching shows that the PWIC is true for any product of Alternating groups of degree p or of degree coprime to p.

In the following, we prove that for $d \geq p$, the pair $(A_d, \langle \tau \rangle)$ is realizable where τ is a p-cycle. Since the PWIC holds for $d \leq 2p$, we may assume that $d \geq 2p$.

Theorem 7.1 ([10, Corollary 4.4])**.** *Let $d = p + t \geq 2p$, $p \nmid t$. Then for any p-cycle τ the pair $(A_d, \langle \tau \rangle)$ is realizable.*

Proof. Consider the degree d cover $\psi \colon Y \to \mathbb{P}^1$ given by the affine Equation $f(x, y) = 0$ where $f(x, y)$ is given by Equation (6.1.7). Let $\phi \colon Z \to \mathbb{P}^1$ be its Galois closure. By Proposition 6.12, ϕ is an A_d-Galois étale connected cover of $\mathbb{P}^1$ branched only at ∞ such that the Sylow p-subgroup of an inertia group above ∞ is generated by a p-cycle. Now the result follows by applying Abhyankar's Lemma (Theorem 5.14) to this cover. □

Theorem 7.2 ([10, Corollary 4.5])**.** *Let $d = ap$ for some integer $a \geq 1$. Then for any p-cycle τ the pair $(A_d, \langle \tau \rangle)$ is realizable.*

Proof. We may assume that $a \geq 2$. Consider the A_{d+1}-Galois cover $\phi \colon Z \to \mathbb{P}^1$ branched only at ∞, which is the Galois closure of the degree-d cover $\psi \colon Y \to \mathbb{P}^1$ given by the affine Equation $f(x, y) = 0$ where $f(x, y) = y^{d+1} + 1 - xy^{d-p+1}$. By Proposition 6.12, $\langle (1, \cdots, p) \rangle \rtimes \langle \beta \rangle$ occurs as an inertia group above ∞ where $\beta = \theta^i \omega$ (cf. Proposition 5.1) has order $m = \text{l.c.m.}(\frac{p-1}{(p-1, d+1)}, d - p + 1)$. Consider the A_d-Galois cover $\epsilon \colon Z \to Y \cong \mathbb{P}^1$. By Lemma 5.7, this cover has branch locus $\{0, \infty\}$ with respective inertia groups of order $\frac{pm}{d-p+1}$ and m above them. Also the Sylow p-subgroup of an inertia group above $y = 0$ is generated by a p-cycle. By Corollary 5.15, the pullback of ϵ under the $[m]$-Kummer cover is a connected A_d-Galois cover of $\mathbb{P}^1$ branched only at ∞ such that $\langle (1, \cdots, p) \rangle$ occurs as an inertia group above ∞. □

Applying Theorem 3.35 to the above results we obtain the following.

Corollary 7.3 ([10, Corollary 4.7])**.** *Let P be a p-subgroup of A_d containing a p-cycle. Then the pair (A_d, P) is realizable.*

The following result shows that to prove the PWIC for A_d, $d \geq p$, it is enough to prove the conjecture for the cases $d \equiv 0$ and $1 \pmod{p}$.

Proposition 7.4 ([10, Proposition 4.9]). *Let $r \geq 2$ be an integer. Assume that the pair $(A_{rp+1}, \langle \tau \rangle)$ is realizable where τ is the product of r disjoint p-cycles in A_{rp+1}. Then for any $d \geq rp+1$, the pair $(A_d, \langle \tau \rangle)$ is realizable.*

Proof. This is an immediate consequence of Raynaud's result [25, Theorem 2.2.3] with $G_i = \mathrm{Alt}(\mathrm{Supp}(\tau) \cup \{i\})$ and $Q = Q_i = \langle \tau \rangle$ for $i \in \{1, \cdots, d\} \setminus \mathrm{Supp}(\tau)$. □

The next two results use formal patching results from Section 6.2 to construct a $G_1 \times G_2$-Galois cover of $\mathbb{P}^1$ from the given G_1-Galois and G_2-Galois covers such that the inertia group over ∞ is smaller than the one obtained from the fiber product of the two covers. This will be used to construct a product of Alternating group covers with a certain cyclic p-group as the inertia group. In view of Lemma 5.3 and Theorem 3.35, this is exactly what we need.

Lemma 7.5 ([10, Lemma 5.1]). *Let G_1, G_2 be two quasi p-groups, P_1 and P_2 be p-subgroups of G_1 and G_2 respectively. Assume that the pairs (G_1, P_1) and (G_2, P_2) are realizable, and let Q_1 and Q_2 be index-p subgroups of P_1 and P_2 respectively. Assume that the local P_1/Q_1 and P_2/Q_2 Galois extensions are given by the Artin-Schreier polynomials $f_0 = Z_1^p - Z_1 - f(x_0)$ and $g_0 = Z_2^p - Z_2 - g(x_0)$ respectively, where x_0 is the local parameter of $\mathbb{P}^1$ at ∞. Assume that $\mathrm{ord}_{x_0}(g)$ and $\mathrm{ord}_{x_0}(f)$ are different and not multiples of p. For $\alpha \in k$, let*

$$f_\alpha = Z^p - Z - (1-\alpha)f(x_0) - \alpha g(x_0) - \alpha x_0^{-1} \text{ and } g_\alpha = Z^p - Z - (1+\alpha)g(x_0) + \alpha f(x_0)$$

be polynomials over $k((x_0))$. Let $M_\alpha/k((x_0))$ and $N_\alpha/k((x_0))$ be the corresponding $\mathbb{Z}/p\mathbb{Z}$-Galois extensions. Then there is a dense open subset $\mathcal{V}$ of $\mathbb{A}^1_t$ such that for all closed points $(t = \alpha)$ in $\mathcal{V}$, there exist a P_1-Galois extension $\widetilde{M}_\alpha/k((x_0))$ and a P_2-Galois extension $\widetilde{N}_\alpha/k((x_0))$ which are realized by the pairs (G_1, P_1) and (G_2, P_2) respectively. Moreover $M_\alpha = \widetilde{M}_\alpha^{Q_1}$, $N_\alpha = \widetilde{M}_\alpha^{Q_2}$ for all $\alpha \in \mathcal{V}$.

Proof. Let $U = \mathrm{Spec}(k[[x_0]])$, $R = k[[t]]$. Let $A := k((x_0))[t][Z]/(f_t)$, where $f_t(Z) = Z^p - Z - (1-t)f(x_0) - tg(x_0) - tx_0^{-1} \in k((x_0))[t][Z]$ and $B := k((x_0))[t][Z]/(g_t)$, where $g_t(Z) = Z^p - Z - (1+t)g(x_0) + tf(x_0) \in k((x_0))[t][Z]$. We have the maps $\phi'_{\mathbb{A}^1_t} \colon S' := \mathrm{Spec}(A) \to \mathrm{Spec}(k((x_0))[t])$ and $\psi'_{\mathbb{A}^1_t} \colon T' := \mathrm{Spec}(B) \to \mathrm{Spec}(k((x_0))[t])$. Let $\phi_{\mathbb{A}^1_t} \colon S \to U \times_k \mathbb{A}^1_t$ and $\psi_{\mathbb{A}^1_t} \colon T \to U \times_k \mathbb{A}^1_t$ be the normalization maps in A and B respectively. Let $\phi_R \colon S_R \to U_R$ and $\psi_R \colon T_R \to U_R$ be their pullbacks under the map $U_R \to U \times_k \mathbb{A}^1_t$. Then the normalization of the closed fibre of ϕ_R and ψ_R correspond to the field extensions $M_0/k((x_0))$ and $N_0/k((x_0))$ respectively. The covers ϕ_R and ψ_R are branched only at the R-valued point $x_0 = 0$ since it is the only pole of the functions f_t and g_t in $R[[x_0]]$. By [14, Theorem 3.11], there exist connected P_1-Galois and P_2-Galois étale covers $\Phi'_{\mathbb{A}^1_t} \colon \widetilde{S}' \to \mathrm{Spec}(k((x_0))[t])$ and $\Psi'_{\mathbb{A}^1_t} \colon \widetilde{T}' \to \mathrm{Spec}(k((x_0))[t])$ dominating $\phi'_{\mathbb{A}^1_t}$ and $\psi'_{\mathbb{A}^1_t}$ respectively. Taking normalization of $U \times_k \mathbb{A}^1_t$ in the function fields of $\widetilde{S}'$ and $\widetilde{T}'$, we obtain P_1-Galois and P_2-Galois covers $\Phi_{\mathbb{A}^1_t} \colon \widetilde{S} \to U \times_k \mathbb{A}^1_t$ and $\Psi_{\mathbb{A}^1_t} \colon \widetilde{T} \to U \times_k \mathbb{A}^1_t$ dominating $\phi_{\mathbb{A}^1_t}$ and $\psi_{\mathbb{A}^1_t}$ respectively. One also obtains Φ_R and Ψ_R which dominate ϕ_R and ψ_R by pull backs.

So by Lemma 6.17, there are G_1-Galois and G_2-Galois covers of $\mathbb{P}^1_R$ satisfying the hypothesis of Lemma 6.18. So there is a dense open subset $\mathcal{V}$ of $\mathbb{A}^1_t$ such that for all points $(t = \alpha)$ in $\mathcal{V}$, the extension $\widetilde{M}_\alpha/k((x_0))$ is realized by the pair (G_1, P_1), and the extension $\widetilde{N}_\alpha/k((x_0))$ is realized by the pair (G_2, P_2). $\square$

Theorem 7.6 ([10, Theorem 5.2])**.** *Let G_1, G_2 be two perfect quasi p-groups. Let $\tau \in G_1$ and $\sigma \in G_2$ be of order p and p^r for some r respectively. Let $P_1 = \langle \tau \rangle \leq G_1$, $P_2 = \langle \sigma \rangle \leq G_2$. Assume that the pairs (G_1, P_1) and (G_2, P_2) are realizable. Then there exists $1 \leq a \leq p-1$ such that for $I := \langle (\tau^a, \sigma) \rangle \leq G_1 \times G_2$, the pair $(G_1 \times G_2, I)$ is also realizable.*

Proof. Let $\phi_i \colon Y_i \to \mathbb{P}^1$ be a G_i-Galois cover of $\mathbb{P}^1$ branched only at ∞ with inertia group P_i above ∞ and the first lower jump h_i above ∞, for $i = 1, 2$. As usual it can be arranged that $h_2 < h_1$ and the first upper jump of the P_2-extension is at least 2 ([24, Theorem 2.2.2]). Let N_2 be the index p subgroup of P_2. Let the local P_1-Galois and P_2/N_2-Galois extensions be given by the Artin-Schreier polynomials $f_0 = Z_1^p - Z_1 - f(x_0) \in k((x_0))[Z_1]$ and $g_0 = Z_2^p - Z_2 - g(x_0) \in k((x_0))[Z_2]$ respectively where x_0 is a local parameter at ∞. By

Lemma 7.5, there is a dense open subset $\mathcal{V}$ of $\mathbb{A}^1_t$ such that for all points $(t = \alpha)$ in $\mathcal{V}$, the extension $M_\alpha/k((x_0))$ given by the polynomial $f_\alpha = Z^p - Z - (1-\alpha)f(x_0) - \alpha g(x_0) - \alpha x_0^{-1}$ is realized by the pair (G_1, P_1), and the extension $N_\alpha/k((x_0))$ given by the polynomial $g_\alpha = Z^p - Z - (1+\alpha)g(x_0) + \alpha f(x_0)$ is dominated by a P_2-Galois extension $\widetilde{N}_\alpha/k((x_0))$ which is realized by the pair (G_2, P_2). So there is an $\alpha \neq 0, 1$ such that the points $(t = \alpha)$ and $(t = \alpha - 1)$ both lie in $\mathcal{V}$. Let $X_i \to \mathbb{P}^1$ be the corresponding G_i-Galois covers of the affine line with inertia groups P_i above ∞, and η_i be points in X_i over ∞ such that $K_{X_1,\eta_1} = M_\alpha$ and $K_{X_2,\eta_2} = \widetilde{N}_{\alpha-1}$.

Let $a_1 \in M_\alpha$ be a root of f_α and $a_2 \in N_{\alpha-1}$ be a root of $g_{\alpha-1}$. So $a_1 - a_2$ is a root of the Artin-Schreier equation $Z^p - Z + \alpha x_0^{-1} = 0$. Since $-v_{x_0}(((1-\alpha)f(x_0) + \alpha g(x_0) + \alpha x_0^{-1}) - (\alpha g(x_0) - (\alpha-1)f(x_0))) = 1$, by Theorem [20, Proposition 3.1], the compositum $M = M_\alpha \widetilde{N}_\alpha$ is a $Q = \mathbb{Z}/p \times \mathbb{Z}/p^r$-Galois extension with the first lower jump at 1, and $Q_1 = Q$, $Q_2 = \mathrm{Gal}(M/k((x_0))(a_1 - a_2)) \cong \mathbb{Z}/p^r$.

Let X be the dominant connected component of the normalization of $X_1 \times_{\mathbb{P}^1} X_2$ containing the point $\eta := (\eta_1, \eta_2)$. Then $\Theta\colon X \to \mathbb{P}^1$ is a $G_1 \times G_2$-Galois cover branched only above ∞ with the local extension $K_{X,\eta}/K_{\mathbb{P}^1,\infty}$ is given by $M/k((x_0))$, and the inertia subgroup Q has a lower jump at 1. Since G_1 and G_2 are perfect, $G_1 \times G_2$ is also perfect, and so by [21, Theorem 3.7], there is a $G_1 \times G_2$-Galois cover of the affine line with inertia group above ∞ given by $I = \mathrm{Gal}(M/k((x_0))(a_1 - a_2))$ in G. Now the projection maps from $G_1 \times G_2$ restricted to I surjects onto $\mathrm{Gal}(M_\alpha/k((x_0))) = P_1 = \langle \tau \rangle$ and $\mathrm{Gal}(\widetilde{N}_{\alpha-1}/k((x_0))) = P_2 = \langle \sigma \rangle$. So I is of the form $\langle (\tau^a, \sigma) \rangle$ for some $1 \leq a \leq p-1$. □

Remark 7.7. *Note that as in Lemma 7.5, in the above theorem we may take P_1 and P_2 to be arbitrary p-subgroups of G_1 and G_2, and the same proof shows that $(G_1 \times G_2, I)$ is realizable for a subgroup I of $P_1 \times P_2$ of index p. Also observe that a finite product of groups preserves the properties of being quasi p and perfect. In particular, if $G = A_{d_i}$, $d_i \geq p$, $i = 1, 2$, then the hypothesis of Theorem 7.6 is satisfied.*

We are now ready to prove the wild part of the Inertia Conjecture for the Alternating groups of degree d where $d = p$ or $d \geq p+1$ is coprime to p.

Theorem 7.8 ([10, Theorem 5.4])**.** *Let $a \geq 1$ be an integer and $1 \leq r \leq a$. For $1 \leq i \leq r$ consider the p-cycle $\tau_i := ((i-1)p+1, \cdots, ip)$ and set $\tau := \tau_1 \cdots \tau_r \in A_{ap+1}$. Then there* *is a connected A_{ap+1}-Galois étale cover of the affine line such that the p-cyclic group $\langle\tau\rangle$ occurs as an inertia group above ∞.*

Proof. For $1 \le i \le a$ set $S_i := \{(i-1)p+1, \cdots, ip\}$. For $1 \le i \le r$ and $1 \le j \le r-1$ set

$$H_{ij} := \begin{cases} \mathrm{Alt}(S_j) \cong A_p, & \text{if } i \neq j \\ \mathrm{Alt}(S_i \cup \{ap+1\}) \cong A_{p+1}, & \text{if } i = j. \end{cases}$$

Also for $1 \le i \le r$ set

$$H_{ir} := \begin{cases} \mathrm{Alt}(\{(r-1)p+1, \cdots, ap\}) \cong A_{(a-r+1)p}, & \text{if } 1 \le i \le r-1 \\ \mathrm{Alt}(\{(r-1)p+1, \cdots, ap+1\}) \cong A_{(a-r+1)p+1}, & \text{if } i = r. \end{cases}$$

For $1 \le i \le r$ set $G_i := H_{i1} \times \cdots \times H_{ir} \subset A_{ap+1}$. Now by Corollary 7.3, each of the pairs $(H_{ij}, \langle\tau_j\rangle)$ is realizable. Since any two p-cycles in an Alternating group are conjugates, by Theorem 7.6, for each $1 \le i \le r$, the pair $(G_i, \langle(\tau_1, \cdots, \tau_r)\rangle)$ is also realizable. Note that $G_i \cap \langle\tau\rangle = \langle(\tau_1, \cdots, \tau_r)\rangle$ for each i. Set $G := \langle\{G_i\}_{1\le i\le r}\rangle \subset A_{ap+1}$. Then G is a transitive subgroup of A_{ap+1} by construction. Since each H_{ij} is generated by p-cycles, so is G. Also G contains the 3-cycle $(1,2,3) \in H_{11}$. So by [28, Lemma 4.4.4], $G = A_{ap+1}$. So by [25, Theorem 2.2.3] there is a connected A_{ap+1}-Galois étale cover of the affine line such that $\langle\tau\rangle$ occurs as an inertia group above ∞. □

Applying Proposition 7.4 to Theorem 7.8, we obtain the following results.

Corollary 7.9 ([10, Corollary 5.5])**.** *Let $d = p$ or $d \ge p+1$ be coprime to p. Then the wild part of the Inertia Conjecture holds for A_d.*

Corollary 7.10 ([10, Corollary 5.6])**.** *Let $d = ap$, $a \ge 2$, τ_i's be as in Theorem 7.8 for $1 \le i \le r$ with $1 \le r \le a-1$. Then the pair $(A_d, \langle\tau_1 \cdots \tau_r\rangle)$ is realizable.*

Corollary 7.11 ([10, Corollary 5.7])**.** *Let $u \ge 1$ be an integer. For $1 \le i \le u$, let $d_i = p$ or $d_i > p$ be coprime to p. Then the wild part of the Inertia Conjecture is true for the* *group $G := A_{d_1} \times \cdots \times A_{d_u}$. Moreover, if Q is any p-group, the wild part of the Inertia Conjecture is true for $G \times Q$.*

Proof. Let P be a p-subgroup of G whose conjugates generate G. There exist $g_1, \cdots, g_r$ in P satisfying conditions (1)-(4) of Lemma 5.3. The case $r = 1$ will be proved by the induction on u. Let $\pi_i \colon G \twoheadrightarrow A_{d_i}$ and $\pi \colon G \to A_{d_2} \times \cdots \times A_{d_u}$ be the projection maps. So there exists a cyclic subgroup $P' = \langle g_1 \rangle$ of P such that $\pi_1(P') = \langle \tau \rangle \leq A_{d_1}$ is a cyclic group of order p, $\pi(P')$ is a cyclic p-group with generator σ, say, and $\pi_i(P')$ are nontrivial subgroups for all i. By the induction hypothesis on u the pair $(A_{d_2} \times \cdots \times A_{d_u}, \pi(P'))$ is realizable and $(A_{d_1}, \langle \tau \rangle)$ is realizable by Corollary 7.9. Moreover $P' = \langle (\tau, \sigma) \rangle$ and τ is of order p. Now by Theorem 7.6, $(A_{d_1} \times \cdots \times A_{d_u}, I)$ is realizable where $I = \langle (\tau^a, \sigma) \rangle$ for some $1 \leq a \leq p-1$. But there is an automorphism of A_{d_1} which sends τ to τ^a. Hence $(A_{d_1} \times \cdots \times A_{d_u}, P')$ is realizable. Finally by Harbater's result [11, Theorem 2], $(A_{d_1} \times \cdots \times A_{d_u}, P)$ is realizable. Now for $r \geq 2$, in the Notation 5.2 of Lemma 5.3, let $H_i = H_{S(g_i)}$. By $r = 1$ case, the pairs $(H_i, \langle g_i \rangle)$ are realizable for $1 \leq i \leq r$. Now the result follows from [25, Theorem 2.2.3]. The moreover part follows from [20, Corollary 4.6]. □

Remark 7.12. *For $1 \leq i \leq u$, let G_i be a non-abelian simple quasi p-group whose order is strictly divisible by p. Then the same argument of Corollary 7.11 shows that that the wild part of the Inertia Conjecture is also true for the group $G = G_1 \times \cdots \times G_u$.*

7.3 The Inertia Conjecture for Alternating Groups

In this section, we prove the IC (Conjecture 4.3) for some Alternating groups following [9]. Recall that the IC was proved to be true for A_p ([8, Theorem 1.2]) and when $p \equiv 2 \pmod 3$ for A_{p+2}([22, Theorem 1.2]). We show that when $p \equiv 2 \pmod 3$ the IC is true for the groups A_{p+1}, A_{p+3} and A_{p+4}, and with some extra condition on p the IC is also true for A_{p+5}. The covers will be constructed using the results and techniques from Section 6.1.

Throughout this section τ denote the p-cycle $(1, \cdots, p)$ in S_p.

In view of Proposition 5.1, to prove the IC for A_d, $p < d < 2p$, we need to prove that there is a connected A_d-Galois étale cover of the affine line such that $I = \langle\tau\rangle \rtimes \langle\theta^i\omega\rangle$ occurs as an inertia group above ∞ for every $1 \le i \le p-1$ and $\omega \in \mathrm{Sym}(\{p+1,\cdots,d\})$ such that $\theta^i\omega$ is an even permutation. Note that for $\theta^i\omega \in A_d$, i an even integer if and only if ω is an even permutation. Now we make some observations which reduces the proof of the IC to the realization of the pair (A_d, I) to a fewer cases.

Remark 7.13. *Since $\langle\theta^i\omega\rangle = \langle\theta^{(i,p-1)}\omega\rangle$, it is enough to consider the i's dividing $p-1$. Also using Abhyankar's Lemma ([1, XIII, Proposition 5.2]), it is enough to prove for the cases where I is a maximal inertia group in the sense of [17, Section 4.9].*

In fact, the following result shows that it is enough to consider the more restricted cases when ω acts on the set $\{p+1,\cdots,d\}$ either with one fixed point or without any fixed point, provided the IC is true for the Alternating groups of lower degree.

Lemma 7.14 ([9, Lemma 5.2])**.** *Let p be an odd prime, $p+1 \le d \le 2p-1$. Assume that the pair (A_d, I) is realizable for a subgroup $I \subset A_d$ which fixes ≥ 1 points in $\{1,\cdots,d\}$. Then for all $d' \ge d$, the pair $(A_{d'}, I)$ is also realizable.*

In particular, to prove the IC for A_d, $p+2 \le d \le 2p-1$, it is enough to prove that the IC is true for each A_u, $p \le u \le d-2$, and that the pair (A_d, I) is realizable for each $I \subset N_{A_d}(\langle\tau\rangle)$ such that I fixes 0 or 1 point in the set $\{1,\cdots,d\}$.

Proof. This is a direct consequence of Corollary 6.22 with $G_i = \mathrm{Alt}(\mathrm{Supp}(I) \cup S_i)$ for every set $S_i \subset \{1,\cdots,d'\} \setminus \mathrm{Supp}(I)$ of size $d - |\mathrm{Supp}(I)|$. The second statement follows from the first one. □

We are now ready to prove that the IC is true for certain Alternating groups.

Theorem 7.15 ([9, Theorem 5.3])**.** *Let $p \equiv 2 \pmod 3$ be an odd prime. Then the IC is true for the group A_{p+1}.*

Proof. By Remark 7.13, it is enough to prove that the pair (A_{p+1}, I) is realizable for $I = \langle\tau\rangle \rtimes \langle\theta^2\rangle$. Set $s = 3$, $r = 1$, $n_1 = p-2$, $n_2 = 2$, $n_3 = 1$. By Lemma 6.7(4), Assumption 6.5 holds for the choice $(\alpha_1, \alpha_2, \alpha_3, \beta_1) = (3/4, 1/4, 1, 0)$. Let $\psi\colon Y \to \mathbb{P}^1$ be the degree-$(p+1)$ cover given by the affine equation

$$\Pi_{i=1}^{3}(y-\alpha_i)^{n_i} - xy = 0. \tag{7.3.1}$$

Let $\phi\colon Z \to \mathbb{P}^1$ be the Galois closure of the cover ψ with Galois group G. Then by Proposition 6.8, G is a primitive subgroup of S_{p+1} and the cover ϕ is étale away from $\{0, \infty\}$ such that $\langle(1, \cdots, p-2)(p-1, p)\rangle$ occurs as an inertia group over 0 and $\langle\tau\rangle \rtimes \langle\theta\rangle$ occurs as an inertia group over ∞. Since $p-2$ is odd, G contains the transposition $(p-1, p)$ and since G also contains the p-cycle τ, by [28, Lemma 4.4.3], $G = S_{p+1}$. After the pullback of ϕ under the $[2(p-2)]$-Kummer cover we obtain a connected A_{p+1}-Galois étale cover of the affine line such that I occurs as an inertia group above ∞. □

Theorem 7.16 ([9, Theorem 5.3])**.** *Let $p \equiv 2 \pmod 3$ be an odd prime. Then the IC is true for the group A_{p+3}.*

Proof. When $p \equiv 2 \pmod 3$, by Abhyankar's Lemma it is enough to prove that there is a connected A_{p+3}-Galois cover of $\mathbb{P}^1$ étale away from ∞ such that $I = \langle\tau\rangle \rtimes \langle\beta\rangle$ occurs as an inertia group at a point above ∞ where β is of the form $\beta = \theta^2(p+1, p+2, p+3)$ or $\beta = \theta(p+1, p+2)$. These are immediate from Corollary 6.9 and Corollary 6.11. □

Theorem 7.17 ([9, Theorem 5.4])**.** *Let $p \equiv 2 \pmod 3$ be an odd prime. Then the IC is true for the group A_{p+4}.*

Proof. Let $p \equiv 2 \pmod 3$ be an odd prime. By Abhyankar's Lemma, to prove the IC for A_{p+4} it is enough to prove that there is a connected A_{p+4}-Galois cover of $\mathbb{P}^1$ étale away from ∞ such that $I = \langle\tau\rangle \rtimes \langle\beta\rangle$ occurs as an inertia group at a point above ∞ where β is of the form $\beta = \theta(p+1, p+2)$ or $\beta = \theta^2(p+1, p+2, p+3)$ or $\beta = \theta(p+1, p+2, p+3, p+4)$.

Since the IC is true for A_{p+2}, by Lemma 7.14, the pair $(A_{p+4}, \langle\tau\rangle \rtimes \langle\theta(p+1, p+2)\rangle)$ is realizable.

Now fix an element $w_3 \in k$ such that $w_3^2 = 3$. Then for $s = 2, r = 2, n_1 = p+2, n_2 = 2, m_1 = 3, m_2 = 1$, Assumption 6.5 holds for the choice $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (\frac{1+w_3}{4}, \frac{1-w_3}{4}, 0, 1)$.

So we can apply Proposition 6.8 to obtain a connected S_{p+4}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle(1, \cdots, p+2)(p+3, p+4)\rangle$ occurs as an inertia group above 0 and $\langle\tau\rangle \rtimes \langle\theta(p+1, p+2, p+3)\rangle$ occurs as an inertia group above ∞. After a $[2(p+2)]$-Kummer pullback we obtain a connected A_{p+4}-Galois étale cover of the affine line such that $\langle\tau\rangle \rtimes \langle\theta^2(p+1, p+2, p+3)\rangle$ occurs as an inertia group above ∞.

In the last case, fix an element $w_2 \in k$ such that $w_2^2 = 2$. Then for $s = 3$, $r = 1$, $n_1 = p-2$, $n_2 = n_3 = 3$, Assumption 6.5 holds for the choice $(\alpha_1, \alpha_2, \alpha_3, \beta_1) = (1, \frac{1+w_2}{3}, \frac{1-w_2}{3}, 0)$. By Proposition 6.8, there is a connected A_{p+4}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle(1, \cdots, p-2)(p-1, p, p+1)(p+2, p+3, p+4)\rangle$ occurs as an inertia group above 0 and $\langle\tau\rangle \rtimes \langle\theta(p+1, p+2, p+3, p+4)\rangle$ occurs as an inertia group above ∞. So by Abhyankar's Lemma, the pair $(A_{p+4}, \langle\tau\rangle \rtimes \langle\theta(p+1, p+2, p+3, p+4)\rangle)$ is realizable. □

Lemma 7.18 ([9, Lemma 5.6])**.** *When $p \equiv 2 \pmod 3$ is a prime > 5, the pair (A_{p+5}, I_i) is realizable for $2 \leq i \leq 5$, where $I_2 = \langle\tau\rangle \rtimes \langle\theta(p+1, p+2)\rangle$, $I_3 = \langle\tau\rangle \rtimes \langle\theta^2(p+1, p+2, p+3)\rangle$, $I_4 = \langle\tau\rangle \rtimes \langle\theta(p+1, p+2, p+3, p+4)\rangle$, $I_5 = \langle\tau\rangle \rtimes \langle\theta^2(p+1, p+2, p+3, p+4, p+5)\rangle$. Additionally if $4 \nmid (p+1)$, the pair $(A_{p+5}, \langle\tau\rangle \rtimes \langle\theta(p+1, p+2)(p+3, p+4, p+5)\rangle)$ is also realizable.*

Proof. Let $p \equiv 2 \pmod 3$ be a prime > 5. Since the IC is true for A_{p+2} ([22, Theorem 1.2]) and for A_{p+3} (Theorem 7.16), by Lemma 7.14 the first two cases follow.

Now we consider the realization of I_4 as an inertia group. Fix $w_{2/3} \in k$ such that $w_{2/3}^2 = 2/3$. Then for $s = 2$, $r = 2$, $n_1 = p+2$, $n_2 = 3$, $m_1 = 4$, $m_2 = 1$, Assumption 6.5 holds for the choice $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (\frac{1-3w_{2/3}}{5}, \frac{1+2w_{2/3}}{5}, 0, 1)$. So we can apply Proposition 6.8 to obtain a connected A_{p+5}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle(1, \cdots, p+2)(p+3, p+4, p+5)\rangle$ occurs as an inertia group above 0 and $\langle\tau\rangle \rtimes \langle\theta(p+1, p+2, p+3, p+4)\rangle$ occurs as an inertia group above ∞. By Abhyankar's Lemma, the pair (A_{p+5}, I_4) is realizable.

For the next case, when $(5, p-1) = 1$, the pair (A_{p+5}, I_5) is realizable by Corollary 6.9. So let $(5, p-1) = 5$. Fix $w_7 \in k$ such that $w_7^2 = 7$. Then for $s = 3$, $r = 1$, $n_1 = p-2$, $n_2 = 6$, $n_3 = 1$, $m_1 = 5$, Assumption 6.5 holds for the choice $(\alpha_1, \alpha_2, \alpha_3, \beta_1) =$

$(\frac{w_7+1}{2}, \frac{w_7-1}{6}, 2, 0)$. By Proposition 6.8, there is a connected S_{p+5}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle(1, \cdots, p-2)(p-1, p, p+1, p+2, p+3, p+4)\rangle$ occurs as an inertia group above 0 and $\langle\tau\rangle \rtimes \langle\theta(p+1, p+2, p+3, p+4, p+5)\rangle$ occurs as an inertia group above ∞. Since $(p-1, 5) = 5$, we have $(p-2, 5) = 1$. So after a $[6(p-2)]$-Kummer pullback we obtain a connected A_{p+5}-Galois étale cover of the affine line such that I_5 occurs as an inertia group above ∞.

Now we consider the last case with the additional assumption $(p+1, 4) = 2$. Set $s = 2 = r$, $m_1 = 2$, $m_2 = 3$, $n_1 = n_2 = \frac{p+5}{2}$. Choose an element $w_3 \in k$ such that $w^2 = 3$. Then for $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (\frac{3+2w_3}{5}, \frac{3-2w_3}{5}, 0, 1)$, Assumption 6.5 holds. So we can apply Proposition 6.8 to obtain a connected A_{p+5}-Galois cover of $\mathbb{P}^1$ branched only at 0 and ∞ such that $\langle(1, \cdots, \frac{p+5}{2})(\frac{p+5}{2}+1, \cdots, p+5)\rangle$ occurs as an inertia groups above 0 and $I = \langle\tau\rangle \rtimes \langle\theta(p+1, p+2)(p+3, p+4, p+5)\rangle\rangle$ occurs as an inertia group at a point over ∞. Then using Abhyankar's Lemma we see that the pair (A_{p+5}, I) is realizable. □

Using the above lemma and Abhyankar's Lemma ([1, XIII, Proposition 5.2]), we conclude the following result.

Theorem 7.19 ([9, Theorem 5.7]). *Let $p \equiv 2 \pmod 3$ be a prime ≥ 17 such that $(p+1, 4) = 2$. Then the IC is true for the group A_{p+5}.*

7.4 Towards the Generalized Purely Wild Inertia Conjecture

In this section, we see some affirmative results for the GPWIC (Conjecture 4.18) and some of the weaker cases following [9]. Let G be a quasi p-group, $P_1, \cdots, P_r$ be non-trivial p-subgroups of G for some $r \geq 1$ such that $G = \langle P_1^G, \cdots, P_r^G\rangle$. Let $B = \{x_1, \cdots, x_r\}$ be a set of closed points in $\mathbb{P}^1$. Then the GPWIC says that there is a connected G-Galois cover of $\mathbb{P}^1$ étale away from B such that P_i occurs as an inertia group above the point x_i. We use these notation throughout this section. We will use results from Section 6.2 for the proofs. Let us start with the following group theoretic observation.

Lemma 7.20. *Let G be a p-group, $r \geq 1$ be an integer. Let $P_1, \cdots, P_r$ be p-subgroups of G such that $G = \langle P_1^G, \cdots, P_r^G \rangle$. Then $G = \langle P_1, \cdots, P_r \rangle$.*

Proof. The result holds for an abelian p-group G. So assume that G is non-abelian. Then the Frattini subgroup $\Phi(G)$ of G is non-trivial. Consider the Frattini quotient $G \twoheadrightarrow G/\Phi(G)$. Under this epimorphism, P_i has image $P_i/P_i \cap \Phi(G)$, and $G/\Phi(G) = \langle (P_i/P_i \cap \Phi(G))^{G/\Phi(G)} | 1 \leq i \leq r \rangle$. Since $G/\Phi(G)$ is elementary abelian, $G/\Phi(G) = \langle P_1/P_1 \cap \Phi(G), \cdots, P_r/P_r \cap \Phi(G) \rangle$. Thus $G = \langle P_1, \cdots, P_r, \Phi(G) \rangle = \langle P_1, \cdots, P_r \rangle$ since $\Phi(G)$ is the set of non-generators of G. □

Theorem 7.21 ([9, Theorem 7.2])**.** *The GPWIC holds for every p-group.*

Proof. By Lemma 7.20, $G = \langle P_1, \cdots, P_r \rangle$. We proceed via induction on r. The pair (P_r, P_r) is realizable. By the induction hypothesis there is a connected $G_1 := \langle P_1, \cdots, P_{r-1} \rangle$-Galois cover of $\mathbb{P}^1$ étale away from $\{x_1, \cdots, x_{r-1}\}$ such that P_i occurs as an inertia group above the point x_i for $1 \leq i \leq r-1$. Now the result follows from Theorem 6.25. □

Theorem 7.22 ([9, Theorem 7.3])**.** *The GPWIC holds for any quasi p-group G whose order is strictly divisible by p. In particular, it holds for Alternating groups A_d with $p \leq d \leq 2p-1$ and for $PSL_2(p)$ when p is an odd prime ≥ 5.*

Proof. Since $p^2 \nmid |G|$, each P_i is a p-cyclic Sylow p-subgroup of G. Since G is a quasi p-group, for each i, $G = \langle P_i^G \rangle$. We proceed by induction on r. By Raynaud's proof of the Abhyankar's Conjecture on the affine line ([25, Corollary 2.2.2]), the pair (G, P_r) is realizable. Now we argue as in the proof of Theorem 7.21. □

Theorem 7.23 ([9, Theorem 7.4])**.** *Let $u \geq 1$ be an integer. Let $G = G_1 \times \cdots \times G_u$ where each G_i is either a non-abelian simple quasi p-group of order strictly divisible by p or a simple Alternating group of degree coprime to p. Then the GPWIC is true for G.*

Proof. For any subset $\Lambda \subseteq \{1, \cdots, u\}$, set $H_\Lambda := \Pi_{\lambda \in \Lambda} G_\lambda$ and let $\pi_\Lambda \colon G \to H_\Lambda$ be the projection. For each $1 \leq i \leq r$, consider the set α_i consisting of $j \in \{1, \cdots, u\}$ such that $\pi_j(P_i)$ is non-trivial. Since G_i's are simple non-abelian groups, the conjugates of $\pi_{\alpha_i}(P_i)$

in H_{α_i} generate H_{α_i}. We proceed via induction on r. If $r = 1$, by [10, Remark 5.8, Corollary 5.4], the pair (G, P_1) is realizable. So let $r \geq 2$. By the induction hypothesis there is a connected $H_{\cup_{i=1}^{r-1}\alpha_i}$-Galois cover of $\mathbb{P}^1$ étale away from $\{x_1, \cdots, x_{r-1}\}$ such that P_i occurs as an inertia group above the point x_i for $1 \leq i \leq r-1$. If $H_{\cup_{i=1}^{r-1}\alpha_i} = G$, the result follows by [11, Theorem 2]. Otherwise, set $S := \{1, \cdots, u\} \setminus \cup_{i=1}^{r-1}\alpha_i$. Since $\cup_{1 \leq i \leq r}\alpha_i = \{1, \cdots, u\}$, we must have $S \subset \alpha_r$. Again by [10, Remark 5.8, Corollary 5.7], the pair (H_{α_r}, P_r) is realizable. Then the result follows by applying Theorem 6.25 with the groups $H_{\cup_{i=1}^{r-1}\alpha_i}$ and H_{α_r}. □

In the following, we show that the GPWIC is true for certain product of groups if it is true for the individual groups.

Theorem 7.24 ([9, Theorem 7.5]). *Let G_1 and G_2 be two finite quasi p-groups such that they have no non-trivial quotient in common. If the GPWIC is true for the groups G_1 and G_2, then it is also true for $G_1 \times G_2$.*

Proof. Set $G := G_1 \times G_2$ and let $\pi_j \colon G \twoheadrightarrow G_j$ denote the projections for $j \in \{1, 2\}$. Let $r \geq 1$, $P_1, \cdots, P_r$ be non-trivial p-subgroups of G such that $G = \langle P_1^G, \cdots, P_r^G \rangle$. Then for each $1 \leq i \leq r$, both of the groups $\pi_1(P_i)$ and $\pi_2(P_i)$ cannot be trivial. For $1 \leq i \leq r$, by the Goursat's lemma, there is p-group Q_i (possibly trivial), normal subgroups $N_{j,i}$ of $\pi_j(P_i)$ such that $\pi_j(P_i)/N_{j,i} \cong Q_i$ and $P_i \cong \pi_1(P_i) \times_{Q_i} \pi_2(P_i)$. Let Q'_i be a common quotient of $\pi_1(P_i)$ and $\pi_2(P_i)$ such that Q_i is a quotient of Q'_i. Then $\pi_1(P_i) \times_{Q'_i} \pi_2(P_i) \subset \pi_1(P_i) \times_{Q_i} \pi_2(P_i)$ and so by [11, Theorem 2], it is enough to consider Q_i to be a maximal common quotient of $\pi_1(P_i)$ and $\pi_2(P_i)$. Now $G_j = \langle \pi_j(P_1)^{G_j}, \cdots, \pi_j(P_r)^{G_j} \rangle$. Let $B = \{x_1, \cdots, x_r\}$ be a sets of closed points in $\mathbb{P}^1$. By the hypothesis, for $j = 1, 2$, there is a connected G_j-Galois cover $f_j \colon Y_j \to \mathbb{P}^1$ étale away from B such that $\pi_j(P_i)$ occurs as an inertia group above the point $x_i \in B$. By Lemma 6.24, for $j = 1$ and 2, there is a connected G_j-Galois cover $Z_j \to \mathbb{P}^1$ étale away from B such that $\pi_j(P_i)$ occurs as an inertia group above x_i, $1 \leq i \leq r$, and such that there is a point $z_{j,i} \in Z_j$ over x_i with $K_{Z_1,z_{1,i}}^{N_{1,i}}/K_{\mathbb{P}^1,x_i} \cong K_{Z_2,z_{2,i}}^{N_{2,i}}/K_{\mathbb{P}^1,x_i}$ as Q_i-Galois extensions. Since Q_i is a maximal common quotient of $\pi_1(P_i)$ and $\pi_2(P_i)$, the extensions $K_{Z_1,z_{1,i}}/K_{\mathbb{P}^1,x_i}$ and $K_{Z_2,z_{2,i}}/K_{\mathbb{P}^1,x_i}$ are linearly disjoint over $K_{Z_1,z_{1,i}}^{N_{1,i}} \cong K_{Z_2,z_{2,i}}^{N_{2,i}}/K_{\mathbb{P}^1,x_i}$. Let W be a dominant connected component of the normalization of $Z_1 \times_{\mathbb{P}^1} Z_2$. Since G_1 and G_2 have no common non-trivial quotient in common, the cover $\Psi\colon W \to \mathbb{P}^1$ is a connected $G_1 \times G_2$-Galois cover. For a point $w = (z'_1, z'_2)$ in W with $f_j(z'_j) = x$ the extension $K_{W,w}/K_{\mathbb{P}^1,x}$ is the compositum of the extension $K_{Z_1,z'_1}/K_{\mathbb{P}^1,x}$ with the extension $K_{Z_2,z'_2}/K_{\mathbb{P}^1,x}$. So the cover Ψ is étale away from B and $\pi_1(P_i) \times_{Q_i} \pi_2(P_2) = P_i$ occurs as an inertia group above x_i. □

Remark 7.25. *The above theorem generalizes [20, Corollary 4.6] where the result was proved for a perfect group G_1 and a p-group G_2.*

In the following, we summarize the results of this section.

Corollary 7.26 ([9, Theorem 1.7])**.** *The GPWIC (Conjecture 4.18) is true for the following quasi p-groups G.*

1. *G is a p-group;*
2. *G has order strictly divisible by p;*
3. *$G = G_1 \times \cdots \times G_u$ where each G_i is either a simple Alternating group of degree $d \geq p$, where $d = p$ or $(d, p) = 1$ or a p-group or a simple non-abelian group of order strictly divisible by p.*

Now we will see some weaker results towards the GPWIC. Namely, when we allow the branch locus sufficiently large or if we allow bigger inertia groups, there are suitable covers with the prescribed ramification. We also show that when the group is A_d, it is enough to add only one more branch point. Using Theorem 3.35, one can increase the wild part of the inertia groups of a cover. In particular, let G be a quasi p-group, P_1, $\cdots$, P_r be p-subgroups of G. Let Q_1 be a Sylow p-subgroup of G containing P_1. Let $B = \{x_1, \cdots, x_r\}$ be a set of closed points of $\mathbb{P}^1$. Then there is a connected G-Galois cover of $\mathbb{P}^1$ étale away from B such that Q_1 occurs as an inertia group above x_1, and P_i occurs as an inertia group above x_i for $2 \leq i \leq r$. In the following, we show that with the hypothesis of the GPWIC, the inertia groups can be taken as the Sylow p-subgroups of the normal quasi p-groups $\langle P_i^G \rangle$.

Proposition 7.27 ([9, Proposition 8.1]). *Under the notation and hypothesis of Conjecture 4.18, for each i, there is a p-subgroup $Q_i \supset P_i$ in $\langle P_i^G \rangle$ such that there is a connected G-Galois cover of $\mathbb{P}^1$ étale away from B and Q_i occurs as an inertia group above the point x_i for $1 \leq i \leq r$.*

Proof. We proceed by induction on r. When $r = 1$, it is the consequence of [11, Theorem 2]. So let $r \geq 2$. For $1 \leq i \leq r$, let Q_i be a Sylow p-subgroup of $H_i = \langle P_i^G \rangle$ containing P_i. Since H_i is a quasi p-group, $H_i = \langle Q_i^{H_i} \rangle$ and by the $r = 1$ case, the pair (H_i, Q_i) is realizable. By the induction hypothesis, we may assume that there is a connected $G_1 := \langle H_1, \cdots, H_{r-1} \rangle$-Galois cover of $\mathbb{P}^1$ étale away from $\{x_1, \cdots, x_{r-1}\}$ and Q_i occurs as an inertia group above x_i. If $G_1 = G$, apply [11, Theorem 2]. Otherwise the result follows by Theorem 6.25 with $G_2 = H_r$. □

Proposition 7.28 ([9, Proposition 8.2]). *Assume that the hypothesis and notation of Conjecture 4.18 hold. For each $1 \leq i \leq r$, let $C_i := \{P_{i,j}\}_{1 \leq j \leq t_i}$ be the set of all conjugates of P_i in G. Let $\emptyset \neq A_i \subset C_i$ such that $G = \langle P | P \in A_i, 1 \leq i \leq r \rangle$. Let $l := \Sigma_i |A_i|$ and let $B = \{x_{i,j} | 1 \leq i \leq r, P_{i,j} \in A_i\}$ be a set of l closed points in $\mathbb{P}^1$ such that $x_{i,1} = x_i$ for each i. Then there is a connected G-Galois cover of $\mathbb{P}^1$ étale away from B and P_i occurs as an inertia group above the point $x_{i,j} \in B$.*

Proof. Since the inertia groups above a point in a connected Galois cover are conjugates, it is enough to prove that there is a connected G-Galois cover of $\mathbb{P}^1$ étale away from B such that $P_{i,j}$ occurs as an inertia group above the point $x_{i,j} \in B$. We proceed via induction on l. If $l = 1$ then $G = P_1$ and the result follows. Let $l \geq 2$. Fix an i_0, $1 \leq i_0 \leq r$, and element P_{i_0,j_0} in A_{i_0}. Then the pair $(P_{i_0,j_0}, P_{i_0,j_0})$ is realizable. By the induction hypothesis, we may assume that there is a connected $G_1 := \langle \{P_{i,j} | 1 \leq i \leq r, j \in A_i\} - \{P_{i_0,j_0}\} \rangle$-Galois cover of $\mathbb{P}^1$ étale away from $B - \{x_{i_0,j_0}\}$ and $P_{i,j}$ occurs as an inertia group above the point $x_{i,j} \in B$ for $(i, j) \neq (i_0, j_0)$. Now use Theorem 6.25 with $G_2 = P_{i_0,j_0}$. □

By the above proposition, if we allow enough number of branch points we can obtain covers with the desired purely wild ramification. By Corollary 7.9, when $d = p$ or when

$d \geq p+1$ is coprime to p, the GPWIC holds for A_d. So assume that $a \geq 2$ and $d = ap$. The following result shows that in this case we only need one extra branched point.

Proposition 7.29 ([9, Proposition 8.3])**.** *Let p be an odd prime, $a \geq 2$ be an integer, $d = ap$. Let $r \geq 1$ be an integer and $P_1, \cdots, P_r$ be non-trivial p-subgroups of A_d. Let $B = \{x_1, \cdots, x_r\}$ be a set of closed points in $\mathbb{P}^1$ and let $x_0 \in \mathbb{P}^1$ be a closed point outside B. Fix $1 \leq i_0 \leq r$. Then there is a connected A_d-Galois cover of $\mathbb{P}^1$ étale away from $B \sqcup \{x_0\}$ such that P_i occurs as an inertia group above x_i and P_{i_0} occurs as an inertia group above x_0.*

Proof. We may assume that $i_0 = 1$. By Theorem 3.10, it is enough to consider the case $r = 1$ and when $P_1 = \langle \tau \rangle$ for an element τ of order p. Without loss of generality we may assume that $\tau = \tau_1 \cdots \tau_v$ where τ_i is a the p-cycle $((i-1)p+1, \cdots, ip)$, $1 \leq i \leq v$. By Corollary 7.10, we may assume that $v = a$.

For $1 \leq i \leq a$, set $H_{i1} := \mathrm{Alt}(\{(i-1)p+1, \cdots, ip\})$. For $1 \leq j \leq a-1$ set $H_{j2} := \mathrm{Alt}(\{(j-1)p+2, \cdots, jp+1\})$ and let $H_{a2} := \mathrm{Alt}(\{(a-1)p+2, \cdots, ap, 1\})$. For $i = 1, 2$, set $G_i := H_{1i} \times \cdots H_{ai} \subset A_d$. For $1 \leq j \leq a$, consider the p-cycle σ_j given by $\sigma_j := ((j-1)p+2, \cdots, jp+1)$ for $1 \leq j \leq a-1$, $\sigma_a := ((a-1)p+2, \cdots, ap, 1)$. Consider the element $\sigma := \sigma_1 \cdots \sigma_a$ in A_d of order p. By Theorem 7.23, the pairs $(G_1, \langle(\tau_1, \cdots, \tau_a)\rangle)$ and $(G_2, \langle(\sigma_1, \cdots, \sigma_a)\rangle)$ are realizable. Set $G := \langle G_1, G_2 \rangle \subset A_d$. Since each H_{ij} are generated by p-cycles, so is G. Also the 3-cycle $(1,2,3) \in H_{11}$ is contained in G. So by [28, Lemma 4.4.4], $G = A_d$. Since σ is a conjugate of τ in A_d, by Theorem 6.25, there is a connected A_d-Galois cover of $\mathbb{P}^1$ étale away from $\{x_0, x_1\}$ such that $\langle \tau \rangle$ occurs as an inertia group over x_0 and over x_1. □

7.5 Towards the General Question

In this section, we see some evidence for $Q[r, X, B, G]$ (Question 4.10). The following result is a consequence of the formal patching technique (Theorem 6.25).

Proposition 7.30 ([9, Proposition 9.1])**.** *Let $r \geq 1$, X be a smooth projective connected k-curve, G be a finite group. Let $B = \{x_1, \cdots, x_r\} \subset X$ be a set of closed points in X.*

For $1 \leq i \leq r$, let I_i be a subgroup of G which is an extension of a p-group P_i by a cyclic group of order prime-to-p, and set $H := \langle P_i^G | 1 \leq i \leq r \rangle$. Assume that H has a complement H' in G. Suppose that the following hold.

1. *There is a connected H'-Galois étale cover $\psi\colon Z \to X$.*

2. *There is a connected H-Galois cover of $\mathbb{P}^1$ étale away from a set $\{\eta_1, \cdots, \eta_r\}$ of r-distinct points such that I_i occurs as an inertia group above the point η_i for $1 \leq i \leq r$.*

Then there is a connected G-Galois cover of X étale away from a set $B' = \{x'_1, \cdots, x'_r\}$ of closed points such that I_i occurs as an inertia group above the point x'_i, $1 \leq i \leq r$. Moreover, we can choose an i such that $x'_i = x_i$. Furthermore, if each (H, I_i) is realizable, we can take $x'_i = x_i$ for all $1 \leq i \leq r$.

Proof. Since ψ is an unramified cover, each $I_i \subset H$. By the hypothesis, G/H is a subgroup of G which together with H generates G. By Theorem 6.25, there is a connected G-Galois cover of X étale away from B' such that I_i occurs as an inertia group above the point x'_i for $1 \leq i \leq r$ and we can choose one $1 \leq i \leq r$ such that $x'_i = x_i$. For the last assertion we use Theorem 6.25 inductively for r. □

Remark 7.31. *Note that when $(|G/H|, p) = 1$, by the Schur-Zassenhaus Theorem, the group H always has a complement in G.*

The hypotheses of the above result assumes the existence of the unramified cover ψ. In general, the unramified Galois covers of X are not well understood. But they are known when the Galois group either has order prime-to-p or when the Galois group is a p-group. Using these structures, we have the following result.

Corollary 7.32 ([9, Corollary 9.3])**.** *Assume that the hypotheses of Question 4.10 hold. Further suppose that one of the following holds.*

1. *$(|G/H|, p) = 1$ and the cover ψ is an étale G/H-Galois cover of X;*

2. *$G = H \rtimes H'$ for some p-group H' of rank s and X has p-rank $\geq s$ and ψ is an étale H'-Galois cover of X.*

Assume that each $I_i \subset \langle P_i^G \rangle$ and each pair $(\langle P_i^G \rangle, I_i)$ is realizable. Then for any set $B = \{x_1, \cdots, x_r\}$ of closed points in X, there is a connected G-Galois cover of X étale away from B such that I_i occurs as an inertia group above the point x_i for $1 \leq i \leq r$.

Remark 7.33. *Note that the above results can be applied to the Question $Q[r, X, B, G]$ if there exists $0 \leq j \leq l$ such that H_j is normal in G which has a complement in G, the composite cover $\psi_j \circ \cdots \circ \psi_1$ is étale, and each pair $(\langle P_i^G \rangle, I_i)$ is realizable.*

Using the IC for the Alternating groups proved in Section 7.3, the above Corollary implies the following result.

Corollary 7.34 ($Q[1, X, \{*\}, S_d]$, [9, Corollary 9.5])**.** *Let p be an odd prime and X be any smooth projective k-curve of genus ≥ 1. Then for $r = 1$, Question 4.10 has an affirmative answer for the group S_p and when $p \equiv 2 \pmod 3$ for the groups S_{p+1}, S_{p+2}, S_{p+3}, S_{p+4}.*

Proof. Let $d = p$ or when $p \equiv 2 \pmod 3$, $d \in \{p+1, p+2, p+3, p+4\}$. Set $G = S_d$. Let $x \in X$ be a closed point, $I \subset G$ be an extension of a p-group P by a cyclic group of order prime-to-p. Then $\langle P^G \rangle = A_d$. Let $\psi \colon Y \to X$ be a connected $\mathbb{Z}/2$-Galois cover of X étale away from x. By the Riemann-Hurwitz formula, the ramification index above x must be an odd integer. So ψ is étale everywhere and $I \subset A_d$. Now the result follows from Corollary 7.32 applied to [8, Theorem 1.2], [22, Theorem 1.2] and Theorem 7.15–Theorem 7.17. □

Again using the IC for the Alternating groups proved in Section 7.3 together with formal patching technique, we have the following result towards the GIC (Conjecture 4.15).

Proposition 7.35 ([9, Proposition 9.6])**.** *Let $p \geq 5$ be a prime number. Let $d = p$ or when $p \equiv 2 \pmod 3$, $d \in \{p+1, p+2, p+3, p+4\}$. Let $r \geq 1$ be an integer. For $1 \leq i \leq r$, let I_i be a subgroup of A_d which is an extension of a p-group P_i (possibly* *empty) by a cyclic group of order prime-to-p such that P_1 is non-trivial, and if P_i is trivial for some i, there is a $2 \leq j \leq r$, $j \neq i$, such that $I_i = I_j$. Then there exists a connected A_d-Galois cover of $\mathbb{P}^1$ étale away from a set $B = \{x_1, \cdots, x_r\}$ of closed points in $\mathbb{P}^1$ such that I_i occurs as an inertia group above x_i, $1 \leq i \leq r$. Moreover, if all the I_i are equal whenever P_i is trivial, the set B can be chosen arbitrarily.*

Proof. For $P_i = \{1\}$ set $A_i := \{2 \leq j \leq r | I_i = I_j\}$. For i such that $P_i = \{1\}$, let β be a generator of I_i and by the Kummer theory there is a connected $\langle\beta\rangle$-Galois cover of $\mathbb{P}^1$ étale away from A_i which is totally ramified over each x_j, $j \in A_i$. Now the result follows by inductively applying Theorem 6.25 to the above covers and the covers obtained from [8, Theorem 1.2], [22, Theorem 1.2] and Theorem 7.15–Theorem 7.17. □

Now onward, we study the general question for $X = \mathbb{P}^1$ and $r = 2$. We have the following result when $G = P \rtimes \mathbb{Z}/n$ for a p-group P and n coprime to p.

Theorem 7.36 ($Q[2, \mathbb{P}^1, \{0, \infty\}, P \rtimes \mathbb{Z}/n]$, [9, Theorem 9.7])**.** *Let $G = P \rtimes \mathbb{Z}/n$ for a p-group P and $(p, n) = 1$. Then Question 4.10 has an affirmative answer for G, $\mathbb{P}^1$ and $r = 2$.*

Proof. In view of Theorem 7.21 we may assume $n \geq 2$. Let P_1, P_2 be two p-subgroups of P where P_1 is non-trivial and P_2 is possible trivial and such that $\langle P_1^G, P_2^G\rangle = P$. For $i = 1, 2$, let $I_i = P_i \rtimes \mathbb{Z}/m_i$. Let $\psi\colon Y \to \mathbb{P}^1$ be a connected $\mathbb{Z}/n$-Galois cover étale away from $\{0, \infty\}$ such that $\mathbb{Z}/m_1$ occurs as an inertia group above 0 and $\mathbb{Z}/m_2$ occurs as an inertia group above ∞. By the Riemann-Hurwitz formula, we have $m_1 = m_2 = n$ and ψ is the $\mathbb{Z}/n$-Galois Kummer cover totally ramified over 0 and ∞. By Remark 4.14, it is enough to show that Question (A) has an affirmative answer. Note that since I_i normalizes P_i, $\mathbb{Z}/n$ also normalizes P_i in G and so we have $\langle P_1^P, P_2^P\rangle = P$. By Lemma 7.20, $P = \langle P_1, P_2\rangle$. Consider the connected $P_1 \rtimes \mathbb{Z}/n$-Galois HKG cover ψ of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ which is totally ramified above ∞ and such that $\mathbb{Z}/n$ occurs as an inertia group above 0 (Theorem 6.26). If P_2 is the trivial group, $G = P_1 \rtimes \mathbb{Z}/n$. Otherwise apply [15, Theorem 3.6] to this cover to obtain the result. □

In the rest of this thesis, we study some S_d-Galois covers with $X = \mathbb{P}^1$ and $r = 2$. These realization results are the evidence for the Question $Q[2, \mathbb{P}^1, \{0, \infty\}, S_d]$. When P_2 is the trivial group, we have seen some of the cases that can occur from studying explicit equations (Section 6.1). The following result shows the existence of another such cover with $P_2 = \{1\}$ and with the same tame part of the inertia groups over both points as an application of Theorem 6.26 to [10, Corollary 5.5].

Corollary 7.37 ([9, Corollary 9.8])**.** *Let p be an odd prime, $d \geq p$ be an integer such that either $d = p$ or $(d, p) = 1$. Let I be subgroup of S_d which is an extension of a p-subgroup P by a cyclic group of order prime-to-p whose generator is an odd permutation γ in S_d. Then there is a connected S_d-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle\gamma\rangle$ occurs as an inertia group at a point over 0 and I occurs as an inertia group at a point over ∞.*

Proof. Set $n := \mathrm{ord}(\gamma)$. Consider the $\mathbb{Z}/n$-Galois Kummer cover $\psi \colon \mathbb{P}^1 \to \mathbb{P}^1$ totally ramified over $\{0, \infty\}$ and étale everywhere else. By [10, Corollary 5.5], the pair (A_d, P) is realizable. Now the result follows from Theorem 6.26 by taking $\Gamma = S_d = \langle A_d, \langle\gamma\rangle\rangle$. □

Now onward, we consider the cases where P_1 and P_2 are both non-trivial subgroups of S_d, $p \leq d \leq 2p - 1$. Without loss of generality, we may assume that $P_i = \langle\tau\rangle$ for $i = 1, 2$, where τ is the p-cycle $(1, \cdots, p)$. We prove that for $1 \leq j_i \leq p - 1$, $\omega_i \in \mathrm{Sym}\{p + 1, \cdots, d\}$ such that $\theta^{j_i}\omega_i$ is an odd permutation, there is a connected S_d-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle\tau\rangle \rtimes \langle\theta^{j_1}\omega_1\rangle$ occurs as an inertia group above 0 and $\langle\tau\rangle \rtimes \langle\theta^{j_2}\omega_2\rangle$ occurs as an inertia group above ∞. Similar to the case of proving the IC for the Alternating groups, we make the following reduction steps.

Remark 7.38. *Since $\langle\theta^i\rangle = \langle\theta^{(i,p-1)}\rangle$, it is enough to consider $i|(p - 1)$. Also since two elements in S_d are conjugate if and only if they have the same cycle structure, by Theorem 6.25, it is enough to show that there exists a $\gamma \in S_d$ so that the following holds. For each $1 \leq i \leq p - 1$ dividing $p - 1$ and $\omega \in \mathrm{Sym}\{p + 1, \cdots, d\}$ with $\theta^i\omega$ an odd permutation, there is a connected S_d-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle\gamma\rangle$ occurs as an inertia group above 0 and $\langle\tau\rangle \rtimes \langle\theta^i\omega\rangle$ occurs as an inertia group above ∞.*

Theorem 7.39 ([9, Theorem 9.10]). *Let $p \geq 5$ be a prime. Let $I_1 := \langle\tau\rangle \rtimes \langle\theta^i\rangle$, $I_2 := \langle\tau\rangle \rtimes \langle\theta^j\rangle$ be two subgroups of S_p for some $1 \leq i, j \leq p-1$ odd integers. Then there is a connected S_p-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that I_1 occurs as an inertia group above 0 and I_2 occurs as an inertia group above ∞.*

Proof. By Remark 7.38, we need to show that for some odd permutation $\gamma \in S_p$, for each odd divisor i of $p-1$, there is a connected S_p-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle\gamma\rangle$ occurs as an inertia group above 0 and $\langle\tau\rangle \rtimes \langle\theta^i\omega\rangle$ occurs as an inertia group above ∞.

Consider the degree p cover $\psi\colon Y \to \mathbb{P}^1$ given by the affine equation $f(x,y) = 0$ where

$$f(x,y) = y^p - y^2 - x = 0.$$

Let $\phi\colon \widetilde{Y} \to \mathbb{P}^1$ be its Galois closure. By Remark 6.4, ϕ is a connected S_p-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that the inertia groups over 0 are 2-cyclic groups generated by transpositions and $\langle\tau\rangle \rtimes \langle\theta\rangle$ occurs as an inertia group above ∞. Since i is odd, after the $[i]$-Kummer pullback of ϕ, we obtain a connected S_p-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that the inertia groups over 0 are 2-cyclic groups generated by transpositions and $\langle\tau\rangle \rtimes \langle\theta^i\rangle$ occurs as an inertia group above ∞. $\square$

Theorem 7.40 ([9, Theorem 9.11]). *Let $p \equiv 2 \pmod 3$ be an odd prime. Let $I_1 := \langle\tau\rangle \rtimes \langle\theta^i\rangle$, $I_2 := \langle\tau\rangle \rtimes \langle\theta^j\rangle$ be two subgroups of S_{p+1} for some $1 \leq i, j \leq p-1$ odd integers. Then there is a connected S_{p+1}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that I_1 occurs as an inertia group above 0 and I_2 occurs as an inertia group above ∞.*

Proof. Consider the S_{p+1}-Galois cover $\phi\colon \widetilde{Y} \to \mathbb{P}^1$ which is the Galois closure of the degree-$(p+1)$ cover of $\mathbb{P}^1$ given by Equation (7.3.1). Then as in the proof of Theorem 7.15 the cover ϕ is étale away from $\{0, \infty\}$ such that $\langle(1, \cdots, p-2)(p-1, p)\rangle$ occurs as an inertia group above 0 and $\langle\tau\rangle \rtimes \langle\theta\rangle$ occurs as an inertia group above ∞. After a $[p-2]$-Kummer pullback we obtain a connected S_{p+2}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle(p-1, p)\rangle$ occurs as an inertia group above 0 and $\langle\tau\rangle \rtimes \langle\theta\rangle$ occurs as an inertia group above ∞. Now we can argue as in the proof of Theorem 7.39. $\square$

Theorem 7.41 ([9, Theorem 9.12])**.** *Let $p \equiv 11 \pmod{12}$ be a prime such that p is not of the form $l+1$ for any prime $l \geq 5$. For $i = 1, 2$ let $I_i := \langle \tau \rangle \rtimes \langle \theta^{j_i}\omega_i \rangle$ be a subgroup of S_{p+2} for some $1 \leq j_i \leq p-1$ and $\omega_i \in Sym\{p+1, p+2\}$ such that $\theta^{j_i}\omega_i$ is an odd permutation. Then there is a connected S_{p+2}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that I_1 occurs as an inertia group above 0 and I_2 occurs as an inertia group above ∞.*

Proof. Let $p \equiv 11 \pmod{12}$ or equivalently, p satisfies $p \equiv 2 \pmod 3$ and $p \equiv 3 \pmod 4$. Let γ be the $(p+1)$-cycle $(1, \cdots, p+1)$ in S_{p+2}. By Remark 7.38, it is enough to show that the following cases hold.

1. For each odd $1 \leq i \leq p-1$ dividing $p-1$ there is a connected S_{p+2}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle \gamma \rangle$ occurs as an inertia group above 0 and $\langle \tau \rangle \rtimes \langle \theta^i \rangle$ occurs as an inertia group above ∞;
2. For each even $1 \leq j \leq p-1$ dividing $p-1$ there is a connected S_{p+2}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle \gamma \rangle$ occurs as an inertia group above 0 and $\langle \tau \rangle \rtimes \langle \theta^j (p+1, p+2) \rangle$ occurs as an inertia group above ∞.

First set $s = 2 = r$, $n_1 = p+1$, $n_2 = 1$, $m_1 = 1$, $m_2 = 1$. Then by Lemma 6.7(3), Assumption 6.5 holds. Consider the Galois cover ϕ_1 of $\mathbb{P}^1$ which is the Galois closure of the degree-$(p+2)$ cover of $\mathbb{P}^1$ given by the affine equation (6.1.6). By Proposition 6.8, ϕ_1 is a connected Galois cover of $\mathbb{P}^1$ with group G_1, a primitive subgroup of S_{p+2}, which is étale away from $\{0, \infty\}$ such that $\langle \gamma \rangle$ occurs as an inertia group above 0 and since $p \equiv 2 \pmod 3$, $\langle \tau \rangle \rtimes \langle \theta \rangle$ occurs as an inertia group above ∞.

Now let $s = 2$, $r = 1$, $n_1 = p+1$, $n_2 = 1$. Then by Lemma 6.7(1) Assumption 6.5 holds. Consider the Galois cover ϕ_2 of $\mathbb{P}^1$ which is the Galois closure of the degree-$(p+2)$ cover of $\mathbb{P}^1$ given by Equation (6.1.6). By Proposition 6.8, ϕ_2 is a connected Galois cover of $\mathbb{P}^1$ with group G_2, a primitive subgroup of S_{p+2}, which is étale away from $\{0, \infty\}$ such that $\langle \gamma \rangle$ occurs as an inertia group above 0 and $\langle \tau \rangle \rtimes \langle \theta^2 (p+1, p+2) \rangle$ occurs as an inertia group above ∞.

The Galois groups G_1 and G_2 are primitive subgroup of S_{p+2} containing a p-cycle which fixes 2 points in $\{1, \cdots, p+2\}$ and a $(p+1)$-cycle which fixes 1 point. Since p is not of the form $l+1$ for a prime $l \geq 5$, by [Theorem 1.2][19], both G_1 and G_2 contain A_{p+2}. Since γ is an odd permutation, $G_i = S_{p+2}$ for $i = 1, 2$. Since i is an odd divisor of $p-1$, $(p+1, i) = 1$. After an $[i]$-Kummer pullback of ϕ_1 we obtain a connected S_{p+2}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle\gamma\rangle$ occurs as an inertia group over 0 and $\langle\tau\rangle \rtimes \langle\theta^i\rangle$ occurs as an inertia group above ∞. Since $4 \nmid p$, $j/2$ is odd. After a $[j/2]$-Kummer pullback of ϕ_2 we obtain a connected S_{p+2}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle\gamma\rangle$ occurs as an inertia group over 0 and $\langle\tau\rangle \rtimes \langle\theta^j(p+1, p+2)\rangle$ occurs as an inertia group above ∞. □

Theorem 7.42 ([9, Theorem 9.13])**.** *Let p be a prime such that $p \equiv 11 \pmod{12}$. For $i = 1, 2$ let $I_i := \langle\tau\rangle \rtimes \langle\theta^{j_i}\omega_i\rangle$ be a subgroup of S_{p+3} for some $1 \leq j_i \leq p-1$ and $\omega_i \in Sym\{p+1, p+2, p+3\}$ such that $\theta^{j_i}\omega_i$ is an odd permutation. Then there is a connected S_{p+3}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that I_1 occurs as an inertia group above 0 and I_2 occurs as an inertia group above ∞.*

Proof. Let $p \equiv 11 \pmod{12}$. Consider the $(p+3)$-cycle $\gamma := (1, \cdots, p+3)$ in S_{p+3}. In view of Remark 7.38 we show that there is a connected S_{p+3}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle\gamma\rangle$ occurs as an inertia groups above 0 and $I = \langle\tau\rangle \rtimes \langle\beta\rangle$ occurs as an inertia group above ∞ where β is of the following: $\beta = \theta^{i_1}$ for all odd integer $i_1|(p-1)$, $\beta = \theta^{i_2}(p+1, p+2)$ for any even integer $i_2|(p-1)$ and $\beta = \theta^{i_3}(p+1, p+2, p+3)$ for all odd integer $i_3|(p-1)$.

First set $s = 1$, $r = 3$, $m_1 = m_2 = m_3 = 1$. Choose an element $w \in k$ such that $w^2 = p-3$. Then with the choice $(\alpha_1, \beta_1, \beta_2, \beta_3) = (0, 1, \frac{w-1}{2}, -\frac{w+1}{2})$, Assumption 6.5 is satisfied. By Proposition 6.8 there is a connected S_{p+3}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle\gamma\rangle$ occurs as an inertia groups above 0 and since $p \equiv 2 \pmod{3}$, $I = \langle\tau\rangle \rtimes \langle\theta\rangle$ occurs as an inertia group above ∞. Since $i_1|(p-1)$ is odd and $(p-1, p+3) = (p-1, 4) = 2$, after an $[i]$-Kummer pullback we obtain the required cover with $\beta = \theta^{i_1}$.

Now take $s = 1$, $r = 2$, $m_1 = 2$, $m_2 = 1$. Then by Lemma 6.7 for the choice $(\alpha_1, \beta_1, \beta_2) = (0, 1, -2)$ Assumption 6.5 is satisfied. By Proposition 6.8 there is a connected S_{p+3}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle\gamma\rangle$ occurs as an inertia groups above 0 and $I = \langle\tau\rangle \rtimes \langle\theta^2(p + 1, p + 2)\rangle$ occurs as an inertia group above ∞. As $p \equiv 3 \pmod 4$ and $i_2|(p-1)$ is even, $(p+3, i_2/2) = 1$. After an $[i_2/2]$-Kummer pullback we obtain the required cover with $\beta = \theta^{i_2}(p + 1, p + 2)$.

Finally, take $s = 1 = r$. Then by Lemma 6.7 Assumption 6.5 holds and by Proposition 6.8 there is a connected S_{p+3}-Galois cover of $\mathbb{P}^1$ étale away from $\{0, \infty\}$ such that $\langle\gamma\rangle$ occurs as an inertia groups above 0 and $I = \langle\tau\rangle \rtimes \langle\theta(p+1, p+2, p+3)\rangle$ occurs as an inertia group above ∞. Since i_3 is an odd divisor of $p - 1$, via an $[i_3]$-Kummer pullback we obtain the required cover with $\beta = \theta^{i_3}(p + 1, p + 2, p + 3)$. □

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List of Publications

1. **On the Inertia Conjecture for Alternating group covers**.
   (With Manish Kumar).
   Journal of Pure and Applied Algebra, Volume 224, Issue 9, September 2020, https://doi.org/10.1016/j.jpaa.2020.106363

2. **Local Oort groups and the isolated differential data criterion**.
   (With Huy Dang, Kostas Kanagiannis, Andrew Obus and Vaidehee Thatte).
   Preprint, 2019. arXiv:1912.12797 (Submitted)

3. **On the Inertia Conjecture and its generalizations**.
   Preprint, 2020. arXiv:2002.04934 (Submitted)