Some Topics involving Derived Categories over Noetherian Formal Schemes

Saurabh Singh

Indian Statistical Institute

September 28, 2019

INDIAN STATISTICAL INSTITUTE

DOCTORAL THESIS

Some Topics involving Derived Categories over Noetherian Formal Schemes

---

Author:
SAURABH SINGH

Supervisor:
DR. SURESH NAYAK

A thesis submitted to the Indian Statistical Institute
in partial fulfilment of the requirements for
the degree of
Doctor of Philosophy (in Mathematics)

Statistics & Mathematics Unit

Indian Statistical Institute, Bangalore Centre

September 28, 2019

Dedicated to my Father and Mother

Acknowledgements

I would like to acknowledge my indebtedness and render my warmest thanks to my supervisor, Professor Suresh Nayak, for providing me with the opportunity to work with him. His friendly guidance, immense patience, and continuous support have been invaluable throughout all the stages of this work.

I would also wish to express my gratitude to Professor Jishnu Biswas for his lectures and problem solving sessions which have been immensely beneficial during the course of this PhD.

I am also indebted to my colleagues and seniors with whom I have had enriching informal discussions. In this vein I would like to thank Vaibhav, Souradeep, Kalyan, Pratik, Soumyadip, Srijan and Hrishikesh.

During the course of this PhD I have had the pleasure enjoying the company of numerous friends who have contributed in making my stay at ISI a memorable one. I regret that I can't name each of them personally in this acknowledgement but I express my gratitude to all of them.

I would like to thank my mother and father for their love and support and their acknowledgement and appreciation for my career and life choices.

Special thanks are due to my wife Coralie for being a source of positive energy and constant encouragement.

Finally, I would like to thank Indian Statistical Institute for providing excellent living and working conditions and for its financial support during the entire course of this PhD.

Saurabh Singh

Contents

Acknowledgements **vii**

Contents **ix**

1 Introduction **1**

I Duality Pseudofunctor over the Composites of Smooth and Pseudoproper Morphisms of Noetherian Formal Schemes **9**

2 Notation and Preliminaries **13**
2.0.1 Sign Convention 14
2.4.2 The torsion functor 15

3 The pseudofunctors $(-)^{\times}_t$ and $(-)^{s}_t$ **17**
3.0.1 Right-adjoint to the derived direct image functor for pseudoproper maps 17
3.3.2 Projection formula and flat-base-change isomorphism 18
3.10.3 Differential forms and smoothness over formal schemes 20
3.13.4 Pseudofunctors for smooth maps. 22

4 Smooth Base Change Isomorphism **25**

5 Fundamental Local Isomorphism **35**
5.0.1 Twisted Inverse Image for Closed Immersions 35

6 Tor-independent Base Change Isomorphism **45**

7 Identity Factorization **55**

8 The Output **67**
8.0 Extension to Non-torsion version 68

II Reconstruction of Formal Schemes using their Derived Categories **71**

9 Introduction and Preliminaries **75**
9.0 Spectrum for Tensor Triangulated Categories 76

9.3.1 Triangular presheaves 78

10 Localizing subcategories of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ **79**
10.0.1 Torsion functors 79
10.7.2 Localizing subcategories 81
10.9.3 Tensor-compatible localizations 82

11 The topological space $\mathrm{Spc}(\mathbf{D}_{\mathrm{qct}}(-))$ **85**

12 Reconstruction of the structure sheaf $\mathscr{O}_{\mathfrak{X}}$ **93**
12.2.1 Presheaf of rings 94

Bibliography **101**

Chapter 1

Introduction

There are two parts to this thesis and both the parts involve working with derived categories over noetherian formal schemes. Beyond this there is no overlap between them and we discuss them separately.

(I)

The first part concerns Grothendieck duality on noetherian formal schemes.

Grothendieck duality is a vast generalisation of Serre duality in algebraic geometry. The main statements in this theory are expressed in the language of derived categories. We begin with an important special case.

Let $f\colon X \to Y$ be a proper map of noetherian schemes which is smooth of relative dimension n. For any $\mathcal{G} \in \mathbf{D}^+_{\mathrm{qc}}(Y)$ (where $\mathbf{D}^+_{\mathrm{qc}}(-)$ denotes the derived category of bounded-below complexes with quasi-coherent homology), set $f^s(\mathcal{G}) := f^*\mathcal{G} \otimes \omega_f[n]$. Then for any $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qc}}(X)$ there is a natural bi-functorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}^+_{\mathrm{qc}}(X)}(\mathcal{F}, f^s\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^+_{\mathrm{qc}}(Y)}(\boldsymbol{R}f_*\mathcal{F}, \mathcal{G}) \tag{1.1}$$

where ω_f is the top exterior power of the sheaf of relative differential forms for X over Y.

In other words, $f^s\colon \mathbf{D}^+_{\mathrm{qc}}(Y) \to \mathbf{D}^+_{\mathrm{qc}}(X)$ is a right adjoint to $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathrm{qc}}(X) \to \mathbf{D}^+_{\mathrm{qc}}(Y)$.

In particular, if X is a smooth projective variety of dimension n over an algebraically closed field k and $\mathcal{F}$ is a locally free sheaf on X, we recover Serre duality by plugging in $Y = \mathrm{Spec}(k)$ and $\mathcal{G} = k$ so that $f_* = \Gamma(X, -)$ the global sections functor.

The right-adjointness of f^s however does not hold in general if the properness or the smoothness assumption on f is dropped. But it turns out we do have the following:

For any separated finite-type map $f\colon X \to Y$ of noetherian schemes, the functor $\boldsymbol{R}f_*$ has a right adjoint, i.e., there is a functor $f^\times\colon \mathbf{D}^+_{\mathrm{qc}}(Y) \to \mathbf{D}^+_{\mathrm{qc}}(X)$ such that for any $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qc}}(X)$ and $\mathcal{G} \in \mathbf{D}^+_{\mathrm{qc}}(Y)$, there is a natural bi-functorial isomorphism

$$\mathrm{Hom}_{D(X)}(\mathcal{F}, f^\times\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^+_{\mathrm{qc}}(Y)}(\boldsymbol{R}f_*\mathcal{F}, \mathcal{G}). \tag{1.2}$$

Thus if f is both proper and smooth, f^s and $f^\times$ agree, but not in general. The adjointness property of $f^\times$, in effect, gives a duality statement and under properness assumption this is of considerable interest since $\mathbf{R}f_*$ then preserves coherence of homology. It is interesting that the restriction of $(-)^\times$ to the category of proper maps appears to blend seamlessly with the restriction of $(-)^s$ to the category of smooth maps and so gives rise to a family of functors $(-)^!$ which forms the central object of study in Grothendieck duality.

Thus Grothendieck duality mainly concerns constructing a theory of $(-)^!$ primarily determined by the following conditions. For now, let $\mathcal{C}$ denote the category of finite-type separated map of noetherian schemes.

1. When $f\colon X \to Y$ is a smooth map in $\mathcal{C}$ of relative dimension n, then we must have $f^! \xrightarrow{\sim} f^s$.
2. When $f\colon X \to Y$ is a proper map in $\mathcal{C}$, then we must have $f^! \xrightarrow{\sim} f^\times$.
3. The theory of $(-)^!$ should behave well with respect to compositions, in the sense that for maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, there is a canonical comparison isomorphism $f^!g^! \xrightarrow{\sim} (gf)^!$ which is moreover associative vis-a-vis composition of three or more maps in $\mathcal{C}$. This is expressed by saying that $(-)^!$ forms a pseudofunctor.

These conditions remain the guiding principles even as we extend $\mathcal{C}$ to include formal schemes or essentially finite-type maps or stacks, etc.

The actual construction of $(-)^!$ satisfying (i), (ii) and (iii) is a non-trivial task which remains unfinished in some situations. Over noetherian schemes, where the theory is essentially complete, we refer to the books [**H**], [**Co**] and [**L**] for a detailed exposition of the construction, see also, [**Y**] and [**YJ**]. Broadly there are two approaches to it. Gluing (i) and (ii) via (iii) necessarily gives rise to many compatibility issues which are often subtle in nature, hence a natural starting point is to use lesser data than that prescribed by (i) and (ii). This means one starts with a smaller category in at least one of (i) or (ii), e.g., that of open immersions or of etale maps in (i) or that of closed immersions or of finite maps in (ii). The formula for $(-)^!$ being already prescribed in these cases, one then tries to factor an arbitrary map as a composite of these two cases and now (iii) provides a guide for handling compositions. Finally, one has to come back and verify that the full strength of (i) and (ii) is satisfied by the construction.

Thus, in the more explicit approach of [**H**] or [**Co**] one glues $(-)^s$ on smooth maps with $(-)^\times$ on closed immersions while in the more abstract approach of [**L**] one glues $(-)^\times$ on proper maps with $(-)^s$ on open immersions (where it is isomorphic to the inverse image $(-)^*$). One advantage in the latter case is that via Nagata's compactification theorem, every map f in $\mathcal{C}$ factors as an open immersion followed by a proper map and so in this approach we can define $(-)^!$ over all of $\mathcal{C}$. In the former case, there is no such global factorisation result and so for dealing with maps that are neither smooth nor finite, the construction goes through a considerable detour involving dualizing complexes and the final theory still involves restrictions on the maps of schemes and homology of the complexes.

A lot of the work discussed above has been extended to the category of (noetherian) formal schemes. Once the appropriate analog of the maps, complexes, functors, etc have been worked out, one follows a similar outline as above though the details are considerably more technical. However, one crucial aspect is the lack of a suitable compactification result over formal schemes where it is not expected to hold. So while there is a theory of $(-)^!$ for composites of etale and (pseudo)-proper maps, we do not know if this theory applies to an arbitrary smooth map.

In this thesis we take a direct approach of gluing (i) and (ii) in the category of noetherian formal schemes (whose morphisms are the pseudo-finite-type separated maps). Thus our construction extends the existing one of $(-)^!$ and in particular includes the case of all smooth maps. Even over ordinary schemes, this gives a slightly different perspective on why the cases (i) and (ii) glue and gives a streamlined view of the compatibilities needed to put them together. We use suitable modifications of many of the technical results used in earlier constructions. The basic method for gluing comes from [**Nk**] whose main result is informally recalled below.

Let $\mathbb{F}$ be the category of separated pseudofinite-type maps of noetherian formal schemes and $\mathcal{P}$ and $\mathcal{S}$ be two subcategories of $\mathbb{F}$ closed under isomorphisms, composition and fibered products. Let $\mathcal{C}$ be the full subcategory of composites of $\mathcal{P}$-maps and $\mathcal{S}$-maps. For the input for the pasting result, assume that there are pseudofunctors $(-)^\times$ and $(-)^{\mathrm{s}}$ on $\mathcal{P}$ and $\mathcal{S}$ respectively. Furthermore, assume that for every fibered square originating as a fibered product of $\mathcal{P}$-map with a $\mathcal{S}$-map, there is an associated *base-change* isomorphism β^{s} and that for every factorization of the identity map into a $\mathcal{P}$-map followed by a $\mathcal{S}$-map, there is an associated *fundamental isomorphism* $\phi_{-,-}$. The comparison maps of $(-)^\times$ and $(-)^{\mathrm{s}}$, the base change isomorphism β^{s} and the fundamental isomorphism $\phi_{-,-}$ are assumed to satisfy certain compatibility conditions (see chapter 4 and chapter 7 for compatibility conditions). With these input conditions, Nayak's pasting result gives a pseudofunctor $(-)^!$ over the subcategory $\mathcal{C}$ such that the restriction of $(-)^!$ to the subcategory of $\mathcal{P}$-maps is isomorphic to $(-)^\times$ and the restriction of $(-)^!$ to the subactegory of $\mathcal{S}$-maps is isomorphic to $(-)^{\mathrm{s}}$.

We begin by establishing the first input condition in chapter 3. We define the functor $(-)^\times_t$ on the category of pseudoproper morphisms of noetherian formal schemes and the functor $(-)^{\mathrm{s}}_t$ on the category of smooth morphisms. We will go on to show that these are in fact pseudofunctors on their respective subcategories.

In chapter 4 we define and prove the *smooth-base-change isomorphism* for fibered diagram of noetherian formal schemes. For its proof we shall use the *flat-base-change isomorphism*, see [**AJL**, Section 7]. We will then show that the smooth-base-change isomorphism is horizontally and vertically transitive. In chapter (6) we also define and prove a special case of *tor-independent* base change isomorphism of fibered squares of pseudoproper and local complete intersection morphisms.

In chapter 5 we give a concrete description of the functor $f_t^{\times}$, when f is a local complete intersection. We denote this concrete version of $f_t^{\times}$ by $f^{\natural}$ and establish a smooth-base-change isomorphism for fibered squares involving smooth morphisms and local complete intersection morphisms and prove that this is compatible with the smooth-base-change isomorphism β^{s} defined in chapter 3.

In chapter 6 we give a fundamental isomorphism for smooth and pseudoproper maps factoring identity. Consider the sequence, $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{u} \mathfrak{X}$, where u is smooth separated map (and hence f is locally a complete intersection), such that $uf = 1$. For such a factorization of identity map, we want to define a canonical isomorphism $\phi_{f,u}\colon f_t^{\times} u_t^{\mathrm{s}} \xrightarrow{\sim} 1_{\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})}$. Moreover, we want this isomorphism to behave well with smooth or pseudo-proper base change on $\mathfrak{X}$.

Using the results in the chapters 1-6 we have established the input conditions given in [**Nk**] required to paste the pseudofunctors $(-)_t^{\times}$ and $(-)_t^{\mathrm{s}}$ and hence as output we obtain a $\mathbf{D}^+_{\mathrm{qct}}$-valued pseudofunctor $(-)^!$ on $\mathcal{C}$, the category of composites of smooth and pseudoproper morphisms of noetherian formal schemes. We then have the following theorem.

Theorem 1.1. *On the category of noetherian formal schemes where morphisms are composites (any number of factors) of smooth and pseudo-proper morphisms, there exists a $\mathbf{D}^+_{\mathrm{qct}}$-valued pseudofunctor $(-)^!$ with the following properties.*

1. *If $f : \mathfrak{X} \to \mathfrak{Y}$ is smooth, then $f^!(-) = \boldsymbol{R}\varGamma'_{\mathfrak{X}}(f^*(-) \otimes \widehat{\omega}_f[n])$, where n is the relative dimension of f and $\widehat{\omega}_f$ is the top exterior power of the complete module of differentials of f.*

2. *If $f : \mathfrak{X} \to \mathfrak{Y}$ is pseudoproper, then $f^!(-) = f^{\times}(-)$, the right adjoint to $\boldsymbol{R}f_*$.*

We also introduce the derived category $\tilde{\mathbf{D}}(-)$ and its associated derived subcategories $\tilde{\mathbf{D}}_{\mathrm{qc}}(-)$ and $\tilde{\mathbf{D}}^+_{\mathrm{qc}}(-)$ and state a *non-torsion* version of the Theorem 1.1. Using the right adjoint $\Lambda_{\mathfrak{X}}$ of the derived torsion functor $\mathbf{R}\varGamma'_{\mathfrak{X}}$, we obtain the following theorem.

Theorem 1.2. *On the category $\mathcal{C}$ of noetherian formal schemes whose morphisms are separated finite-type composites of smooth and pseudoproper morphisms, there is a $\tilde{\mathbf{D}}^+_{\mathrm{qc}}(-)$-valued pre-pseudofunctor $f^{\tilde{!}}$ with the following properties.*

1. *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a pseudoproper morphism, then $f^{\tilde{!}} = \Lambda_{\mathfrak{X}} f_t^{\times} \boldsymbol{R}\varGamma'_{\mathfrak{Y}}$ and is right adjoint to $\boldsymbol{R}f_*\boldsymbol{R}\varGamma'_{\mathfrak{X}}$.*

2. *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a separated smooth morphism, then $f^{\tilde{!}} \simeq \Lambda_{\mathfrak{X}} f_t^{\mathrm{s}} \boldsymbol{R}\varGamma'_{\mathfrak{Y}}$ and $f^{\tilde{!}} \simeq \Lambda_{\mathfrak{X}}(f^* \otimes \omega_f[n])$.*

3. *Finally, for any $f\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathcal{C}$, $f^{\tilde{!}}(\mathbf{D}^+_{\mathrm{c}}(\mathfrak{Y})) \subset \mathbf{D}^+_{\mathrm{c}}(\mathfrak{X})$, and thus $f^{\tilde{!}}|_{\mathbf{D}^+_{\mathrm{c}}}$ is a pseudofunctor such that if f is smooth of relative dimension n, then $f^{\tilde{!}} \simeq f^*\otimes\omega_f[n]$.*

(II)

The second part of this thesis deals with the idea of reconstruction of formal schemes from their derived categories.

Balmer in [**B1**] defines the spectrum $\mathrm{Spc}(\mathcal{K})$ of a *tensor triangulated category* $\mathcal{K}$ and shows that it has a structure of a locally ringed space. He proves that for $\mathcal{K} = \mathbf{D}^{\mathrm{perf}}(X)$, where X is a topologically noetherian scheme, $\mathrm{Spc}(\mathcal{K}) \simeq X$. The topology on $\mathrm{Spc}(\mathcal{K})$ is obtained using Thomason's classification of $\otimes$-thick subcategories of a triangulated category, see [**T1**]. Later, AJS in [**AJS**], give a correspondence between $\otimes$-compatible localizing subcategories of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and the specialization closed subsets of $\mathfrak{X}$.

In part two of the thesis we use classification of $\otimes$-*compatible localizing subcategories* of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ given by [**AJS**] to construct the Balmer spectrum $\mathrm{Spc}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and show that it is homeomorphic to $\mathfrak{X}$. Furthermore, we define the *triangular presheaf structure* on $\mathrm{Spc}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$. Here we defer from Balmer's construction of *triangular presheaf on* $\mathrm{Spc}(\mathbf{D}^{\mathrm{perf}}(X)$ as we did not have the equivalent version of [**TT**][Proposition 5.2.4(a)] for $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. Hence we make use of direct computation to prove the equivalent version of [**B1**][Theorem 8.4]. Finally we prove that for any open subset $\mathscr{U} \subset \mathfrak{X}$, there is a natural map

$$\mathrm{End}_{\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})}(\mathscr{O}_{\mathscr{U}}) \simeq \mathscr{O}_{\mathfrak{X}}(\mathscr{U}) \tag{1.3}$$

which is an isomorphism. Using the isomorphism (1.3) we recover the structure sheaf $\mathscr{O}_{\mathfrak{X}}$ along with the *adic*-structure. The result is summarized in the following theorem.

Theorem 1.3. *Consider the functor $\mathcal{D}\colon \mathbb{F} \to \mathcal{T}$ from the category $\mathbb{F}$ of noetherian formal schemes which are either separated or of finite Krull dimension to the category $\mathcal{T}$ of tensor triangulated categories with unit, given by $\mathfrak{X} \mapsto (\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}), \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}}, \boldsymbol{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}})$. Then this functor is faithful and takes isomorphisms to isomorphisms. Moreover, over the subcategory $\mathcal{T}' \subset \mathcal{T}$ comprising tensor triangulated categories satisfying Properties 1, 2 and 3 of Definition 11.1, there exists a functor from $\mathcal{T}$ into ringed spaces such that its pre-composition with the functor $\mathcal{D}$ yields the natural inclusion of $\mathbb{F}$ into the ringed spaces.*

Part I

Duality Pseudofunctor over the Composites of Smooth and Pseudoproper Morphisms of Noetherian Formal Schemes

Chapter 2

Notation and Preliminaries

We begin by setting up some notation and recall a few definitions and results which we shall be using in this article. All our formal scheme maps have been assumed to be separated unless stated otherwise, and all the formal schemes are *a priori* assumed to be noetherian. We begin by recalling a few notions about formal schemes and derived categories of sheaves on formal schemes.

A homomorphism $f : A \to B$ of noetherian adic rings is *(essentially) of pseudo-finite type* if it is continuous and the composition $A \to B \to B/\mathfrak{b}$, with $\mathfrak{b} \subset B$ a defining ideal, is (essentially) of finite type, i.e., $B/\mathfrak{b}$ is a localization of a finite-type A-algebra.

A map of ordinary schemes $f : X \to Y$ is essentially of finite type if every $y \in Y$ has an affine open neighbourhood $V = \mathrm{Spec}(A)$ such that $f^{-1}V$ can be covered by finitely many affine open $U_i = \mathrm{Spec}(C_i)$ such that the corresponding maps $A \to C_i$ are essentially of finite type. For any morphism of formal schemes $f : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$, there exist ideals of definition $\mathscr{I} \subset \mathscr{O}_{\mathfrak{X}}$ and $\mathscr{J} \subset \mathscr{O}_{\mathfrak{Y}}$ satisfying $\mathscr{J}\mathscr{O}_{\mathfrak{X}} \subset \mathscr{I}$; and correspondingly there is an induced map of ordinary schemes $f_n : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}^{n+1}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{J}^{n+1})$. We say f is *separated* (resp. *(essentially) of pseudo-finite type*, resp. *pseudo-finite*, resp. *affine*, resp. *pseudo-proper*) if for some (hence for all) n, f_n is separated (resp. (essentially) of finite type, resp. finite, resp. affine, resp. proper).

We denote by $\mathcal{A}(\mathfrak{X})$ the abelian category of sheaves of modules over the formal scheme $\mathfrak{X}$. We then denote by $\mathcal{A}_{\mathrm{qc}}(\mathfrak{X})$ the full subcategory of $\mathcal{A}(\mathfrak{X})$ of quasi-coherent sheaves of $\mathscr{O}_{\mathfrak{X}}$-modules. Similarly, we denote by $\mathcal{A}_{\mathrm{c}}(\mathfrak{X})$ the full subcategory of $\mathcal{A}(\mathfrak{X})$ of coherent

$\mathscr{O}_{\mathfrak{X}}$-modules. Let $\mathcal{K}(\mathfrak{X})$ be the homotopy category of $\mathcal{A}(\mathfrak{X})$ complexes and $\mathbf{D}(\mathfrak{X})$ be the corresponding derived category.

For any full subcategory $\mathcal{A}_*(\mathfrak{X})$ of $\mathcal{A}(\mathfrak{X})$ we will denote by $\mathbf{D}_*(\mathfrak{X})$ the full subcategory of $\mathbf{D}(\mathfrak{X})$ of those complexes whose homology sheaves lie in $\mathcal{A}_*(\mathfrak{X})$. We will denote by $\mathbf{D}_*^+(\mathfrak{X})$ (resp. $\mathbf{D}_*^-(\mathfrak{X})$) the full subcategory of $\mathbf{D}_*(\mathfrak{X})$ whose objects are those complexes $\mathcal{F} \in \mathbf{D}_*(\mathfrak{X})$ such that $\mathcal{H}^m(\mathcal{F})$ vanishes for $m \ll 0$ (resp. $m \gg 0$).

2.0.1 Sign Convention

We will use the following sign convention. Let $\mathcal{A}$ be an abelian category and $\mathcal{K}(\mathcal{A})$ be the homotopy category of $\mathcal{A}$. Then $\mathcal{K}(\mathcal{A})$ is a triangulated category. For $A^\bullet, B^\bullet$ complexes in $\mathcal{K}(\mathcal{A})$ we have the following isomorphism

$$\theta_{i,j} : A^\bullet[i] \otimes B^\bullet[j] \xrightarrow{\sim} (A^\bullet \otimes B^\bullet)[i+j] \tag{2.1}$$

satisfying for every pair $p, q \in \mathbb{Z}$

$$\theta_{i,j}|(A^{p+i} \otimes B^{q+j}) = \text{multiplication by}(-1)^{pj}. \tag{2.2}$$

Similarly, for fixed $A^\bullet$ we have a family of isomorphisms

$$\theta(B^\bullet)\colon A^\bullet \otimes B^\bullet \xrightarrow{\sim} B^\bullet \otimes A^\bullet$$

defined locally by

$$\theta(B^\bullet)(a \otimes b) \to (-1)^{pq}(b \otimes a) \text{ for } a \in A^p \text{ and } b \in B^q. \tag{2.3}$$

For objects A, B, C in $\mathcal{A}$, the following diagram of isomorphism commutes and both the compositions $\theta_{m+n,l} \circ (\theta_{m,n} \otimes 1)$ and $\theta_{m,n+l} \circ (1 \otimes \theta_{n,l})$ have the sign $(-1)^{mn+lm+ln}$.

$$\begin{tikzcd}
A[m] \otimes B[n] \otimes C[l] \arrow[r, "\theta_{m,n}\otimes 1", "(-1)^{nm}"'] \arrow[d, "1\otimes\theta_{n,l}"', "(-1)^{ln}"] & (A \otimes B)[m+n] \otimes C[l] \arrow[d, "(-1)^{l(m+n)}"', "\theta_{m+n,l}"] \\
A[m] \otimes (B \otimes C)[n+l] \arrow[r, "(-1)^{(n+l)m}", "\theta_{m,n+l}"'] & (A \otimes B \otimes C)[m+n+l]
\end{tikzcd} \tag{2.4}$$

2.4.2 The torsion functor

Let $(X, \mathscr{O}_X)$ be a ringed space. For any $\mathscr{O}_X$-ideal $\mathscr{I}$ and any $\mathcal{M} \in \mathcal{A}(X)$, set

$$\varGamma_{\mathscr{I}}\mathcal{M} := \varinjlim_{n} \mathcal{H}om_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathcal{M}). \tag{2.5}$$

For $\mathcal{M}, \mathcal{N} \in \mathcal{A}(X)$, we have the following isomorphism from [**AJL2**, page 20]

$$\mathbf{R}\varGamma_{\mathscr{I}}(\mathcal{M} \otimes \mathcal{N}) \simeq \mathbf{R}\varGamma_{\mathscr{I}}\mathcal{M} \otimes \mathcal{N}. \tag{2.6}$$

For a formal scheme $\mathfrak{X}$ with ideal of definition $\mathscr{I}$, we set

$$\varGamma'_{\mathfrak{X}} := \varGamma_{\mathscr{I}}. \tag{2.7}$$

This definition is independent of the choice of the defining ideal $\mathscr{I}$. We call $M \in A(\mathfrak{X})$ a torsion $\mathscr{O}_{\mathfrak{X}}$-module if $\varGamma'_{\mathfrak{X}}M = M$. Let $\mathcal{A}_{\mathrm{t}}(\mathfrak{X}) := \mathcal{A}_{\mathscr{I}}(\mathfrak{X})$ be the thick subcategory of $\mathcal{A}(\mathfrak{X})$ whose objects are all the torsion $\mathscr{O}_{\mathfrak{X}}$-modules; and set $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) := \mathcal{A}_{\mathrm{qc}}(\mathfrak{X}) \cap \mathcal{A}_{\mathrm{t}}(\mathfrak{X})$. Let $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ denote the full subcategory of $\mathbf{D}(\mathfrak{X})$ consisting of complexes of $\mathscr{O}_{\mathfrak{X}}$-modules whose cohomologies lie in $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$.

There is a natural inclusion functor $j^t_{\mathfrak{X}} : \mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) \hookrightarrow \mathcal{A}(\mathfrak{X})$, and whenever $\mathfrak{X}$ is a separated noetherian formal scheme this $j^t_{\mathfrak{X}}$, by [**AJL**, prop 5.3.1], induces an equivalence of categories, $\mathbf{D}^+(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})) \xrightarrow{\approx} \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$.

Chapter 3

The pseudofunctors $(-)_t^{\times}$ and $(-)_t^{\mathrm{s}}$

In this section we will define the functor $(-)_t^{\times}$ on the category of pseudoproper morphisms of noetherian formal schemes and the functor $(-)_t^{\mathrm{s}}$ on the category of smooth morphisms. We will go on to show that these are in fact pseudofunctors on their respective subcategories, which will establish the conditions [A] and [B] of the input data mentioned in [**Nk**], required for pasting $(-)_t^{\times}$ and $(-)_t^{\mathrm{s}}$ over the category of composites of smooth and pseudoproper morphisms.

3.0.1 Right-adjoint to the derived direct image functor for pseudoproper maps

For a pseudofinite type map $f : \mathfrak{X} \to \mathfrak{Y}$ of noetherian formal schemes, the functor $\mathbf{R}f_*$ takes $\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$ to $\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$ and has a right $\triangle$-adjoint $f_t^{\times} : \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y}) \to \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$, i.e., there exists a morphism of $\triangle$-functors $\tau_f : \mathbf{R}f_*f_t^{\times} \to \mathbf{1}$, called the trace map, such that for all $\mathcal{G} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$ and $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$, the composed map (in the derived category of abelian groups),

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})}(\mathcal{G}, f_t^{\times}\mathcal{F}) &\longrightarrow \mathrm{Hom}_{\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})}(\mathbf{R}f_*\mathcal{G}, \mathbf{R}f_*f_t^{\times}\mathcal{F}) \\ &\xrightarrow{\tau_f} \mathrm{Hom}_{\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})}(\mathbf{R}f_*\mathcal{G}, \mathcal{F}) \qquad (3.1)\end{aligned}$$

is an isomorphism. By construction, the pair $(f_t^{\times}, \tau_f)$ is unique upto a unique isomorphism. For the construction of $f_t^{\times}$ and the proof of the above statements we refer the reader to [**AJL**, section 4,5].

Furthermore, it holds that the assignment $f \longmapsto f^{\times}_t$ (via the trace maps τ_f), forms a pseudofunctor, that is, given pseudofinite-type maps $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}$ of noetherian formal schemes, there is a comparison isomorphism of functors

$$c^{\times}_{f,g} : f^{\times}_t g^{\times}_t \xrightarrow{\sim} (gf)^{\times}_t \tag{3.2}$$

such that for pseudofinite-type maps of noetherian formal schemes, $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z} \xrightarrow{h} \mathfrak{W}$ we have the following commutative diagram of isomorphism of functors.

$$\begin{tikzcd}
f^{\times}_t g^{\times}_t h^{\times}_t \arrow[r, "f^{\times}_t(c^{\times}_{g,h})"] \arrow[d, "c^{\times}_{f,g}"'] & f^{\times}_t (hg)^{\times}_t \arrow[d, "c^{\times}_{f,hg}"] \\
(gf)^{\times}_t h^{\times}_t \arrow[r, "c^{\times}_{gf,h}"] & (hgf)^{\times}_t
\end{tikzcd} \tag{3.3}$$

The comparison maps $c^{\times}_{-,-}$ are dependent on the choice of the trace maps τ_{-} though we usually suppress reference to them. The commutativity of the diagram above follows from the pseudofunctoriality of $\mathbf{R}(-)_*$ and the right adjointness of $(-)^{\times}_t$ above.

3.3.2 Projection formula and flat-base-change isomorphism

For a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of noetherian formal schemes, and for $\mathcal{E} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$ and $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$, the functorial *projection map*

$$\mathbf{R}f_*\mathcal{E} \otimes \mathcal{F} \to \mathbf{R}f_*(\mathcal{E} \otimes \mathbf{L}f^*\mathcal{F}) \tag{3.4}$$

defined as the adjoint to the natural composition

$$\mathbf{L}f^*(\mathbf{R}f_*\mathcal{E} \otimes \mathcal{F}) \to \mathbf{L}f^*\mathbf{R}f_*\mathcal{E} \otimes \mathbf{L}f^*\mathcal{F} \to \mathcal{E} \otimes \mathbf{L}f^*\mathcal{F} \tag{3.5}$$

is an isomorphism. While, we do not necessarily need $\mathcal{E} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$ and $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$ for the projection map to be an isomorphism, it is true for more general complexes (see [**AJL**, page 30]), but for us restriction to $\mathbf{D}^+_{\mathrm{qct}}(-)$ suffices. The details can be found in [**L**, pages 123-125], with appropriate modifications for the formal case, see [**AJL**, page 30].

The projection isomorphism (3.4) leads to an adjoint version for $(-)^{\times}_t$ as follows.

Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a pseudoproper map for noetherian formal schemes, $\mathcal{F}$ a complex in $\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$ and $\mathfrak{L}$ a bounded complex of locally free $\mathscr{O}_{\mathfrak{Y}}$-modules, i.e. a *perfect complex.* Then, consider the map,

$$pr^{\times} : f^{\times}_t \mathcal{F} \otimes_{\mathfrak{X}} f^{*}\mathfrak{L} \longrightarrow f^{\times}_t(\mathcal{F} \otimes_{\mathfrak{Y}} \mathfrak{L}) \tag{3.6}$$

defined as the adjoint of the following composition,

$$\mathbf{R}f_*(f^{\times}_t \mathcal{F} \otimes_{\mathfrak{X}} f^{*}\mathcal{L}) \simeq \mathbf{R}f_* f^{\times}_t \mathcal{F} \otimes_{\mathfrak{Y}} \mathcal{L} \to \mathcal{F} \otimes_{\mathfrak{Y}} \mathcal{L}. \tag{3.7}$$

The map $pr^{\times}$ is an isomorphism; let $\mathscr{E} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$ and consider the map

$$\alpha_f : \mathrm{Hom}_{\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})}(\mathscr{E}\ ,\ f^{\times}_t \mathcal{F} \otimes_{\mathfrak{X}} f^{*}\mathfrak{L}) \to \mathrm{Hom}_{\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})}(\mathscr{E}\ ,\ f^{\times}_t(\mathcal{F} \otimes_{\mathfrak{Y}} \mathfrak{L})) \tag{3.8}$$

defined via the commutatitivity of the following diagram.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\mathscr{E}, f^{\times}_t \mathcal{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} f^{*}\mathfrak{L}) \arrow[r, "\alpha_f"] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\mathscr{E}, f^{\times}_t(\mathcal{F} \otimes_{\mathscr{O}_{\mathfrak{Y}}} \mathfrak{L}) \\
\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\mathscr{E} \otimes_{\mathscr{O}_{\mathfrak{X}}} (f^{*}\mathfrak{L})^{\vee}, f^{\times}_t \mathcal{F}) \arrow[d, "\wr"] & \mathrm{Hom}_{\mathbf{D}(\mathfrak{Y})}(\mathbf{R}f_*\mathscr{E}, \mathcal{F} \otimes_{\mathscr{O}_{\mathfrak{Y}}} \mathfrak{L}) \arrow[u, "\wr"] \\
\mathrm{Hom}_{\mathbf{D}(\mathfrak{Y})}(\mathbf{R}f_*(\mathscr{E} \otimes_{\mathscr{O}_{\mathfrak{X}}} (f^{*}\mathfrak{L})^{\vee}), \mathcal{F}) \arrow[r, "(3.4)"] & \mathrm{Hom}_{\mathbf{D}(\mathfrak{Y})}(\mathbf{R}f_*\mathscr{E} \otimes_{\mathscr{O}_{\mathfrak{Y}}} \mathfrak{L}^{\vee}, \mathcal{F}) \arrow[u, "\wr"]
\end{tikzcd} \tag{3.9}$$

The vertical arrows in both the columns are isomorphisms on account of the adjointness between Rf_* and $f^{\times}_t$ and the fact the $\mathfrak{L}$ is a bounded complex of locally free modules. The bottom row isomorphism is given by the projection isomorphism. Since all the maps are isomorphisms, hence α_f is an isomorphism. Since $\mathscr{E}$ was an arbitrary complex in $\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$, $pr^{\times}$ is an isomorphism.

Along with the right adjointness of $(-)^{\times}_t$ and pseudofunctoriality, we have a *flat-base-change isomorphism* for $(-)^{\times}_t$, that is, given the fibered square of noetherian formal schemes

$$\begin{tikzcd}
\mathfrak{X}' \arrow[r, "v"] \arrow[d, "g"'] & \mathfrak{X} \arrow[d, "f"] \\
\mathfrak{Y}' \arrow[r, "u"] & \mathfrak{Y}
\end{tikzcd}$$

where u, v are flat and f, g pseudo-proper, we have a base change isomorphism

$$\beta : \mathbf{R}\varGamma'_{\mathfrak{X}'} v^* f^{\times}_t(\mathcal{F}) \xrightarrow{\sim} g^{\times}_t \mathbf{R}\varGamma'_{\mathfrak{Y}'} u^*(\mathcal{F}) \tag{3.10}$$

where $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$. For the proof of flat-base-change see [**AJL**, Theorem 7.4].

3.10.3 Differential forms and smoothness over formal schemes

Recall that a continuous homomorphism $\phi : A \to B$ of noetherian adic rings is said to be *formally smooth* if for every discrete topological A-algebra C and every nilpotent ideal I of C, any continuous A-homomorphism $B \to C/I$ factors as $B \xrightarrow{\nu} C \twoheadrightarrow C/I$, where ν is a continuous A-homomorphism.

Recall that a morphism of formal schemes $f : \mathfrak{X} \to \mathfrak{Y}$ is said to be formally smooth if for any morphism $Z \to \mathfrak{Y}$ where $Z = \mathrm{Spec}(C)$ is an affine scheme, and for any closed subscheme $Z_0 \subset Z$ defined by a nilpotent ideal in C, every $\mathfrak{Y}$-morphism $Z_0 \to \mathfrak{X}$ extends to a $\mathfrak{Y}$-morphism $Z \to \mathfrak{X}$.

We call a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ *smooth* if it is essentially of pseudofinite type and formally smooth.

For a brief introduction of basic properties of *smooth* maps for formal schemes we refer the reader to [**LNS**, section 2].

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an essentially pseudofinite type map of formal schemes with defining ideals $\mathscr{I}, \mathscr{J}$ respectively. Consider the induced map of ordinary schemes

$$f_n : X_n := (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}^{n+1}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{J}^{n+1}) =: Y_n.$$

Let $j_n : X_n \hookrightarrow \mathfrak{X}$ be the canonical closed immersion. For $m \geq 0$ let $\Omega^m_{X_n/Y_n}$ be the m-th exterior power of the sheaf of relative differentials for f_n, and set $\widehat{\Omega}^m_f = \widehat{\Omega}^m_{\mathfrak{X}/\mathfrak{Y}} = \varprojlim_n j_{n*}\Omega^m_{X_n/Y_n}$. Here $\widehat{\Omega}^m_f$ is independent of the choice of $\mathscr{I}, \mathscr{J}$. Moreover, it is a coherent $\mathscr{O}_{\mathfrak{X}}$-module.

If f is smooth then f is flat and $\widehat{\Omega}^m_{\mathfrak{X}/\mathfrak{Y}}$ is locally free of finite rank. This rank is constant on the connected components of $\mathfrak{X}$, and if the rank is same on all connected components, say n, then f is said to be of pure relative dimension n. In that case we define the *canonical invertible module* as $\widehat{\omega}_f := \bigwedge^n \widehat{\Omega}_f$. For morphisms of formal schemes which do not have a pure relative dimension, we adopt the convention that the relative dimension n is a locally constant integer-valued function on $\mathfrak{X}$. Henceforth, unless stated otherwise, by relative dimension of a smooth morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of noetherian formal schemes we shall mean this locally constant integer-valued function on $\mathfrak{X}$. Likewise, if $\mathcal{F}$ is a complex on $\mathfrak{X}$ and n a locally constant integer-valued function on $\mathfrak{X}$, then $\mathcal{F}[n]$ has the obvious meaning, namely, that for any connected component U of $\mathfrak{X}$, $\mathcal{F}[n]_{|U} := \mathcal{F}_{|U}[n_{|U}]$.

We shall state a few properties and results associated with $\widehat{\omega}$ here.

Property 1. *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ and $g : \mathfrak{Y} \longrightarrow \mathfrak{Z}$ be smooth maps of relative dimensions d and e respectively. Then gf is smooth of relative dimension $d+e$ and there is a canonical isomorphism*

$$\phi : f^*\widehat{\omega}_g \otimes_{\mathscr{O}_{\mathfrak{X}}} \widehat{\omega}_f \xrightarrow{\sim} \widehat{\omega}_{gf}.$$

In particular, for positive integers m, n we have the following isomorphism.

$$\phi : f^*\widehat{\omega}_g[m] \otimes_{\mathscr{O}_{\mathfrak{X}}} \widehat{\omega}_f[n] \xrightarrow{\sim} \widehat{\omega}_{gf}[m+n]. \tag{3.11}$$

Property 2. *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Z}$ and $g : \mathfrak{Y} \longrightarrow \mathfrak{Z}$ be formal scheme maps with f essentially of pseudo-finite type, so that the projection $q : \mathfrak{W} := \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \longrightarrow \mathfrak{Y}$ is also of pseudo-finite type.*

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} =: \mathfrak{W} \arrow[r, "q"] \arrow[d, "p"] & \mathfrak{Y} \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Z} \end{tikzcd}$$

Then for the projection $p : \mathfrak{W} \longrightarrow \mathfrak{X}$ there is a natural isomorphism

$$\varphi : p^*\widehat{\Omega^1}_{\mathfrak{X}/\mathfrak{Z}} \xrightarrow{\sim} \widehat{\Omega^1}_{\mathfrak{W}/\mathfrak{Y}}. \tag{3.12}$$

Property 3. *With the hypotheses as in Property 2., assume further that f is smooth of constant relative dimension d. Then q is also of smooth of relative dimension d. Furthermore there exists a natural isomorphism*

$$\psi : p^*\widehat{\omega}_f[d] \xrightarrow{\sim} \widehat{\omega}_q[d]. \tag{3.13}$$

For the proofs for the above statements we refer the reader to [**LNS**, Proposition 2.6.5] and [**LNS**, Proposition 2.6.6].

3.13.4 Pseudofunctors for smooth maps.

Let $u : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a smooth map of noetherian formal schemes of relative dimension n. Then we define a functor, $u^{\mathrm{s}} : \mathbf{D}(\mathfrak{Y}) \longrightarrow \mathbf{D}(\mathfrak{X})$ as

$$u^{\mathrm{s}}(\mathcal{F}) := \mathbf{L}u^*(\mathcal{F}) \otimes_{\mathfrak{X}} \widehat{\omega}_u[n]. \tag{3.14}$$

Here, since u is smooth, and hence flat, so $u^* = \mathbf{L}u^*$ and $\underset{=}{\otimes} = \otimes$ as $\widehat{\omega}_u$ is a locally free sheaf. The functor $u_t^{\mathrm{s}} : \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y}) \to \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$ is defined as

$$u_t^{\mathrm{s}} := \mathbf{R}\varGamma'_{\mathfrak{X}} u^{\mathrm{s}} = \mathbf{R}\varGamma'_{\mathfrak{X}}(u^* \otimes_{\mathfrak{X}} \widehat{\omega}_u[n]) \simeq \mathbf{R}\varGamma'_{\mathfrak{X}} u^* \otimes_{\mathfrak{X}} \widehat{\omega}_u[n]. \tag{3.15}$$

Proposition 3.1. *The functor $(-)^s$ defined in (3.14) results in a pseudofunctor over the category of smooth maps, while $(-)_t^{\mathrm{s}}$ defined in (3.15) results in a $\mathbf{D}^+_{\mathrm{qct}}$-valued pseudofunctor over the category of smooth maps.*

Thus, given any composition of smooth morphisms $\mathfrak{X} \xrightarrow{u} \mathfrak{Y} \xrightarrow{v} \mathfrak{Z}$ we have an isomorphism of functors $c^s_{u,v} : u^s v^s \xrightarrow{\sim} (vu)^s$ such that for any triple composition of smooth morphisms $\mathfrak{X} \xrightarrow{u} \mathfrak{Y} \xrightarrow{v} \mathfrak{Z} \xrightarrow{w} \mathfrak{W}$ of relative codimensions m, n, l respectively, the following diagram of isomorphisms commutes, and an analogous statement hold for $(-)_t^{\mathrm{s}}$.

$$\begin{tikzcd} u^{\mathrm{s}}v^{\mathrm{s}}w^{\mathrm{s}} \arrow[r, "u^{\mathrm{s}}(c^{\mathrm{s}}_{v,w})"] \arrow[d, "(c^{\mathrm{s}}_{u,v})w^{\mathrm{s}}"'] & u^{\mathrm{s}}(vw)^{\mathrm{s}} \arrow[d, "c^{\mathrm{s}}_{u,wv}"] \\ (vu)^{\mathrm{s}}w^{\mathrm{s}} \arrow[r, "c^{\mathrm{s}}_{vu,w}"] & (wvu)^{\mathrm{s}} \end{tikzcd} \tag{3.16}$$

To see this, we define the comparison isomorphism $c^{\mathrm{s}}_{u,v} : u^{\mathrm{s}}v^{\mathrm{s}} \xrightarrow{\sim} (vu)^{\mathrm{s}}$ to be the composition of following isomorphisms

$$\begin{aligned} u^{\mathrm{s}}v^{\mathrm{s}} &= u^*(v^* \otimes \widehat{\omega}_v[n]) \otimes \widehat{\omega}_u[m] \\ &\xrightarrow{\sim} u^*v^* \otimes (u^*\widehat{\omega}_v[n] \otimes \widehat{\omega}_u[m]) \\ &\xrightarrow{c^*_{u,v}} (vu)^* \otimes (u^*\widehat{\omega}_v[n] \otimes \widehat{\omega}_u[m]) \\ &\xrightarrow{\sim} (vu)^* \otimes (u^*\widehat{\omega}_v \otimes \widehat{\omega}_u)[m+n] \\ &\xrightarrow{(3.11)} (vu)^* \otimes \widehat{\omega}_{vu}[m+n] \\ &= (vu)^{\mathrm{s}} \end{aligned} \tag{3.17}$$

and proving the commutativity of diagram of isomorphisms (3.16) above reduces to proving the commutativity of the following diagram.

$$\begin{tikzcd} u^*v^*\widehat{\omega}_w[l] \otimes u^*\widehat{\omega}_v[n] \otimes \widehat{\omega}_u[m] \arrow[r, "\sim"] \arrow[d, "\wr"'] & u^*\widehat{\omega}_{wv}[n+l] \otimes \widehat{\omega}_u[m] \arrow[d, "\wr"] \\ (vu)^*\widehat{\omega}_w[l] \otimes \widehat{\omega}_{vu}[m+n] \arrow[r, "\sim"] & \widehat{\omega}_{wvu}[m+n+l] \end{tikzcd} \quad . \tag{3.18}$$

The commutativity of (3.18) follows from the standard properties of differential forms and the commutativity of the diagram (2.4).

We will now define the comparison isomorphism $u^{\mathrm{s}}_t v^{\mathrm{s}}_t \xrightarrow{\sim} (v \circ u)^{\mathrm{s}}_t$. First, observe that the following composition is an isomorphism.

$$\begin{aligned} \mathbf{R}\varGamma'_{\mathfrak{X}} u^{\mathrm{s}} \mathbf{R}\varGamma'_{\mathfrak{Y}} v^{\mathrm{s}} &\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}}(u^* \mathbf{R}\varGamma'_{\mathfrak{Y}} v^{\mathrm{s}} \otimes \widehat{\omega}^{\bullet}_u) \\ &\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}} u^* \mathbf{R}\varGamma'_{\mathfrak{Y}} v^{\mathrm{s}} \otimes \widehat{\omega}^{\bullet}_u \\ &\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}} u^* v^{\mathrm{s}} \otimes \widehat{\omega}^{\bullet}_u \\ &\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}}(u^* v^{\mathrm{s}} \otimes \widehat{\omega}^{\bullet}_u) \\ &\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}} u^{\mathrm{s}} v^{\mathrm{s}} \end{aligned} \tag{3.19}$$

The third isomorphism follows from [**AJL**, Proposition 5.2.8(c)], see also (8.6) in section 9. Now the isomorphism (3.19), together with the isomorphism (3.17), give the required comparison isomorphism $\mathbf{R}\varGamma'_{\mathfrak{X}}(v \circ u)^{\mathrm{s}} \to \mathbf{R}\varGamma'_{\mathfrak{X}} u^{\mathrm{s}} v^{\mathrm{s}}$. Moreover, these comparison isomorphisms also satisfy the associativity rule of (3.16), whose verification reduces easily to that of (3.18).

Chapter 4

Smooth Base Change Isomorphism

In this section we will define and prove a *smooth-base-change isomorphism* for a fibered diagram involving a smooth map and pseudoproper map. For its proof we shall use the *flat-base-change isomorphism* mentioned in theorem (3.10). We will then show that the smooth-base-change isomorphism is horizontally and vertically transitive. This isomorphism along with its transitivities would provide us with the condition [C] of the input data in [**Nk**] needed for gluing $(-)_t^{\times}$ on pseudoproper maps with $(-)_t^{\mathrm{s}}$ on smooth maps.

Consider a fibered square of noetherian formal schemes

$$\begin{tikzcd} \mathscr{V} \arrow[r, "v"] \arrow[d, "g"] & \mathfrak{X} \arrow[d, "f"] \\ \mathscr{U} \arrow[r, "u"] & \mathfrak{Y} \end{tikzcd} \tag{4.1}$$

where f, (and hence) g are *pseudo-proper* and u, (and hence) v are smooth. Then the smooth base-change isomorphism is the map

$$\beta^{\mathrm{s}}(\mathcal{F}) : v_t^{\mathrm{s}} f_t^{\times} \mathcal{F} \longrightarrow g_t^{\times} u_t^{\mathrm{s}} \mathcal{F}, \qquad \mathcal{F} \in \mathbf{D}^{+}_{\mathrm{qct}}(\mathfrak{Y}) \tag{4.2}$$

defined via the commutativity of the following diagram

$$\begin{tikzcd}
\mathbf{R}\Gamma'_{\mathscr{V}} v^{\mathrm{s}} f_t^{\times}\mathcal{F} \arrow[rr, dashed, "\beta^{\mathrm{s}}(\mathcal{F})"] \arrow[d, equal] & & g_t^{\times}\mathbf{R}\Gamma'_{\mathscr{U}} u^{\mathrm{s}}\mathcal{F} \arrow[d, equal] \\
\mathbf{R}\Gamma'_{\mathscr{V}}(v^*(f_t^{\times}\mathcal{F})\otimes\widehat{\omega}_v[n]) \arrow[d, "\wr"] & & g_t^{\times}\mathbf{R}\Gamma'_{\mathscr{U}}(u^*\mathcal{F}\otimes\widehat{\omega}_u[n]) \\
\mathbf{R}\Gamma'_{\mathscr{V}} v^*(f_t^{\times}\mathcal{F})\otimes\widehat{\omega}_v[n] \arrow[d, "\wr", "(3.13)"'] & & g_t^{\times}(\mathbf{R}\Gamma'_{\mathscr{U}} u^*\mathcal{F}\otimes\widehat{\omega}_u[n]) \arrow[u, "\wr"'] \\
\mathbf{R}\Gamma'_{\mathscr{V}} v^* f_t^{\times}\mathcal{F}\otimes g^*\widehat{\omega}_u[n] \arrow[rr, "(3.10)"] & & g_t^{\times}\mathbf{R}\Gamma'_{\mathscr{U}} u^*\mathcal{F}\otimes g^*(\widehat{\omega}_u[n]) \arrow[u, "\wr"', "(3.6)"]
\end{tikzcd} \tag{4.3}$$

Now we'll prove that this β^{s} satisfies horizontal and vertical transitivity vis-á-vis extending (4.1) horizontally and vertically.

For a map $f:\mathfrak{X}\to\mathfrak{Y}$ set $f_t^* := \mathbf{R}\Gamma'_{\mathfrak{X}} f^*$.

Proposition 4.1. *Given the following fibered diagram of formal schemes,*

$$\begin{tikzcd}
\mathfrak{X}'' \arrow[r, "v'"] \arrow[d, "f''"'] \arrow[dr, phantom, "\sigma_1"] & \mathfrak{X}' \arrow[r, "u'"] \arrow[d, "f'"] \arrow[dr, phantom, "\sigma_2"] & \mathfrak{X} \arrow[d, "f"] \\
\mathfrak{Y}'' \arrow[r, "v"] & \mathfrak{Y}' \arrow[r, "u"] & \mathfrak{Y}
\end{tikzcd} \tag{4.4}$$

where f, f', f'' are pseudo-proper and u, u', v, v' are smooth morphisms of formal schemes, we have following commutative diagram of isomorphisms for $\mathcal{F}\in\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$.

$$\begin{tikzcd}
v_t'^{\mathrm{s}} u_t'^{\mathrm{s}} f_t^{\times}\mathcal{F} \arrow[r] \arrow[d] & v_t'^{\mathrm{s}} f_t'^{\times} u_t^{\mathrm{s}}\mathcal{F} \arrow[r] & f_t''^{\times} v_t^{\mathrm{s}} u_t^{\mathrm{s}}\mathcal{F} \arrow[d] \\
(u'v')_t^{\mathrm{s}} f_t^{\times}\mathcal{F} \arrow[rr] & & f_t''^{\times}(uv)_t^{\mathrm{s}}\mathcal{F}
\end{tikzcd} \tag{4.5}$$

Proof. We expand the diagram (4.5) to diagram (4.6) below. We use $\widehat{\omega}_-^{\bullet}$ in place of $\widehat{\omega}_-[-]$ for convenience. We verify that all the subdiagrams of (4.6) commute.

- The commutativity of the squares 3 and 6 is obvious from the definitions of the labelled maps and the sign convention in (2.2).
- The square 2 commutes functorially.
- The squares 9 and 10 commute via the definition of definition of $c^s_{(u',v')}$ and $c^s_{(u,v)}$ respectively, see (3.17).

- The squares (1) and (11) commute via the definition of *smooth-base-change*, see (4.2).

- And finally the square (8) commutes via the transitivity of *flat-base-change* proved in [**AJL**, section 7.5.1].

Via the natural isomorphism $\widehat{\omega}^{\bullet}_{v'} \simeq f''^{*}\widehat{\omega}^{\bullet}_{v}$, checking the commutativity of (5) reduces to checking that of (4.7). Using the adjointness $f''_{*} \dashv f''^{\times}_{t}$, the commutativity of the septagon in (4.7) reduces to proving the commutativity of the border of the diagram (4.8) where $A = u_t^{*}\mathcal{F}$ and $B = \widehat{\omega}_u$. The unlabelled subdiagrams of (4.8) commute functorially. The diagrams labelled ♣ commute by the definitions of the maps involved and we notice that the commutativity of the subdiagram ♠ is the commutativity of the projection isomorphism with the base change isomorphism, whose proof has been detailed in [**L**, Proposition 3.7.3]. □

$$
\begin{tikzcd}[column sep=small]
v_t'^{\mathrm{s}} u_t'^{\mathrm{s}} f_t^{\times} \mathcal{F} \arrow[rr, "v_t'^{\mathrm{s}} \circ \beta^{\mathrm{s}}_{\sigma_1}"] \arrow[dr, equal, "\text{def.}"] \arrow[ddddddd, dashed, "(c^{\mathrm{s}}_{(u',v')}) \circ f_t^{\times}"'] & & v_t'^{\mathrm{s}} f_t'^{\times} u_t^{\mathrm{s}} \mathcal{F} \arrow[rr, "\beta^{\mathrm{s}}_{\sigma_2} \circ u_t^{\mathrm{s}}"] \arrow[d, equal, "\text{def.}"] & & f_t''^{\times} v_t^{\mathrm{s}} u_t^{\mathrm{s}} \mathcal{F} \arrow[dl, equal, "\text{def.}"'] \arrow[ddddddd, dashed, "f_t''^{\times} \circ (c^{\mathrm{s}}_{(u,v)})"] \\
& v_t'^{*}(u_t'^{*} f_t^{\times} \mathcal{F} \otimes \widehat{\omega}^{\bullet}_{u'}) \otimes \widehat{\omega}^{\bullet}_{v'} \arrow[r, "v_t'^{\mathrm{s}} \circ \beta^{\mathrm{s}}_{\sigma_1}"] \arrow[d, "(3.13)"] \arrow[dr, phantom, "1"] & v_t'^{*} f_t'^{\times}(u_t^{*} \mathcal{F} \otimes \widehat{\omega}^{\bullet}_{u}) \otimes \widehat{\omega}^{\bullet}_{v'} \arrow[r, "\beta^{\mathrm{s}}_{\sigma_2} \circ u_t^{\mathrm{s}}"] \arrow[d, "(3.6)"] \arrow[ddr, phantom, "5"] & f_t''^{\times} v_t^{*}(u_t^{*} \mathcal{F} \otimes \widehat{\omega}^{\bullet}_{u}) \otimes \widehat{\omega}^{\bullet}_{v} \arrow[d] & \\
& v_t'^{*}(u_t'^{*} f_t^{\times} \mathcal{F} \otimes f'^{*} \widehat{\omega}^{\bullet}_{u}) \otimes \widehat{\omega}^{\bullet}_{v'} \arrow[r, "\beta'_{\sigma_1} \otimes 1"] \arrow[d] \arrow[dr, phantom, "2"] & v_t'^{*}(f_t'^{\times} u_t^{*} \mathcal{F} \otimes f'^{*} \widehat{\omega}^{\bullet}_{u}) \otimes \widehat{\omega}^{\bullet}_{v'} \arrow[d] & f_t''^{\times}(v_t^{*} u_t^{*} \mathcal{F} \otimes v^{*} \widehat{\omega}^{\bullet}_{u} \otimes \widehat{\omega}^{\bullet}_{v}) \arrow[d, "(3.6)"] & \\
\arrow[r, phantom, "9"] & v_t'^{*} u_t'^{*} f_t^{\times} \mathcal{F} \otimes v'^{*} f'^{*} \widehat{\omega}^{\bullet}_{u} \otimes \widehat{\omega}^{\bullet}_{v'} \arrow[r, "\beta'_{\sigma_1} \otimes 1"] \arrow[dd, "(3.13)"] \arrow[ddr, phantom, "3"] & v_t'^{*} f_t'^{\times} u_t^{*} \mathcal{F} \otimes v'^{*} f'^{*} \widehat{\omega}^{\bullet}_{u} \otimes \widehat{\omega}^{\bullet}_{v'} \arrow[r, "\beta'_{\sigma_2}"] \arrow[d, "(3.13)"] \arrow[ddr, phantom, "6"] & f_t''^{\times} v_t^{*} u_t^{*} \mathcal{F} \otimes f''^{*}(v^{*} \widehat{\omega}^{\bullet}_{u} \otimes \widehat{\omega}^{\bullet}_{v}) \arrow[dd, "{(3.11),(3.13)}"] \arrow[r, phantom, "10"] & \\
& & v_t'^{*} f_t'^{\times} u_t^{*} \mathcal{F} \otimes v'^{*} \widehat{\omega}^{\bullet}_{u'} \otimes \widehat{\omega}^{\bullet}_{v'} \arrow[d, "(3.11)"] & & \\
& v_t'^{*} u_t'^{*} f_t^{\times} \mathcal{F} \otimes \widehat{\omega}^{\bullet}_{u'v'} \arrow[r, "\beta'_{\sigma_1} \otimes 1"] \arrow[d] \arrow[drr, phantom, "8"] & v_t'^{*} f_t'^{\times} u_t^{*} \mathcal{F} \otimes \widehat{\omega}^{\bullet}_{u'v'} \arrow[r, "\beta'_{\sigma_2} \otimes 1"] & f_t''^{\times} v_t^{*} u_t^{*} \mathcal{F} \otimes \widehat{\omega}^{\bullet}_{u'v'} \arrow[d] & \\
& (u'v')^{*}_t f_t^{\times} \mathcal{F} \otimes \widehat{\omega}^{\bullet}_{u'v'} \arrow[rr, "\beta'_{\sigma} \otimes 1"] \arrow[dl, equal, "\text{def.}"'] \arrow[drr, phantom, "11"] & & f_t''^{\times} (uv)^{*}_t \mathcal{F} \otimes \widehat{\omega}^{\bullet}_{u'v'} \arrow[dr, "(3.6) \circ (3.13)"] & \\
(u'v')^{\mathrm{s}}_t f_t^{\times} \mathcal{F} \arrow[rrrr, "\beta^{\mathrm{s}}_{\sigma}"'] & & & & f_t''^{\times} (uv)^{\mathrm{s}}_t \mathcal{F}
\end{tikzcd}
\tag{4.6}
$$

$$
\begin{tikzcd}
v_t'^{*} f_t'^{\times}(u_t^{*}\mathcal{F} \otimes \widehat{\omega}_u^{\bullet}) \otimes f''^{*}\widehat{\omega}_v^{\bullet} \arrow[r] \arrow[d] & f_t''^{\times} v_t^{*}(u_t^{*}\mathcal{F} \otimes \widehat{\omega}_u^{\bullet}) \otimes f''^{*}\widehat{\omega}_v^{\bullet} \arrow[r] \arrow[dr] & f_t''^{\times}(v_t^{*}(u_t^{*}\mathcal{F} \otimes \widehat{\omega}_u^{\bullet}) \otimes \widehat{\omega}_v^{\bullet}) \arrow[d] \\
v_t'^{*}(f_t'^{\times} u_t^{*}\mathcal{F} \otimes f'^{*}\widehat{\omega}_u^{\bullet}) \otimes f''^{*}\widehat{\omega}_v^{\bullet} \arrow[d] & & f_t''^{\times}(v_t^{*}u_t^{*}\mathcal{F} \otimes v^{*}\widehat{\omega}_u^{\bullet}) \otimes f''^{*}\widehat{\omega}_v^{\bullet} \\
v_t'^{*} f_t'^{\times} u_t^{*}\mathcal{F} \otimes v'^{*} f'^{*}\widehat{\omega}_u^{\bullet} \otimes f''^{*}\widehat{\omega}_v^{\bullet} \arrow[r] & f_t''^{\times} v_t^{*}u_t^{*}\mathcal{F} \otimes v^{*} f'^{*}\widehat{\omega}_u^{\bullet} \otimes f''^{*}\widehat{\omega}_v^{\bullet} \arrow[r] & f_t''^{\times} v_t^{*}u_t^{*}\mathcal{F} \otimes f''^{*} v^{*}\widehat{\omega}_u^{\bullet} \otimes f''^{*}\widehat{\omega}_v^{\bullet} \arrow[u]
\end{tikzcd}
\tag{4.7}
$$

$$
\begin{tikzcd}
f''_* v'^*_t f'^{\times}_t (A \otimes B) \arrow[r] \arrow[d] & v'^*_t f'_* f'^{\times}_t (A \otimes B) \arrow[r, "\text{trace}"] \arrow[d, "\text{def.}"] \arrow[dr, phantom, "\clubsuit"] & v'^*_t (A \otimes B) \arrow[ddr] & \\
f''_* v'^*_t (f'^{\times}_t A \otimes f'^* B) \arrow[r] \arrow[d] & v'^*_t f'_* (f'^{\times}_t A \otimes f'^* B) \arrow[r] & v'^*_t (f'_* f'^{\times}_t A \otimes B) \arrow[u, "\text{trace}"'] \arrow[d] & \\
f''_* (v'^*_t f'^{\times}_t A \otimes v'^* f'^* B) \arrow[dd] & f''_* v'^*_t f'^{\times}_t A \otimes v^* B \arrow[r] \arrow[dr] \arrow[dd, "\text{proj.}"] & v^*_t f'_* f'^{\times}_t A \otimes v^* B \arrow[r] \arrow[d, phantom, "\clubsuit"] & v^*_t A \otimes v^* B \\
\arrow[r, phantom, "\spadesuit"] & & f''_* f''^{\times}_t v^*_t A \otimes v^* B \arrow[ur, "\text{trace}"] \arrow[dr, phantom, "\clubsuit"] & \\
f''_* (v'^*_t f'^{\times}_t A \otimes f''^* v'^* B) \arrow[r, equal] & f''_* (v'^*_t f'^{\times}_t A \otimes f''^* v'^* B) \arrow[r] & f''_* (f''^{\times}_t v^*_t A \otimes f''^* v^* B) \arrow[u] \arrow[r, equal] & f''_* (f''^{\times}_t v^*_t A \otimes f''^* v^* B) \arrow[uu]
\end{tikzcd}
\tag{4.8}
$$

Proposition 4.2. *Given the following fibered diagram*

$$\begin{tikzcd}
\mathfrak{X}'_1 \arrow[r, "u''"] \arrow[d, "g'"'] \arrow[dr, phantom, "\sigma_1"] & \mathfrak{X}_1 \arrow[d, "g"] \\
\mathfrak{X}' \arrow[r, "u'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\sigma_2"] & \mathfrak{X} \arrow[d, "f"] \\
\mathfrak{Y}' \arrow[r, "u"] & \mathfrak{Y}
\end{tikzcd}$$

where f, f', g, g' are pseudo-proper and u, u', u'' are smooth maps, the following diagram of isomorphisms commutes.

$$\begin{tikzcd}
u''^{\mathrm{s}}_t g_t^{\times} f_t^{\times} \arrow[r] \arrow[d] & g'^{\times}_t u'^{\mathrm{s}}_t f_t^{\times} \arrow[r] & g'^{\times}_t f'^{\times}_t u^{\mathrm{s}}_t \arrow[d] \\
u''^{\mathrm{s}}_t (gf)_t^{\times} \arrow[rr] & & (g'f')_t^{\times} u^{\mathrm{s}}_t
\end{tikzcd} \tag{4.9}$$

Proof. We expand the diagram (4.9) to the diagram (4.10) below. It suffices to prove that all the subdiagrams commute.

- The commutativity of squares 2 and 4 is straightforward.
- The square 7 commutes via the definition of $c^{\times}_{f,g}$.
- The diagram 8 is the transitivity of the adjoint projection map $pr^{\times}$ defined (3.6) and its commutativity follows from the Exercise 4.7.3.4 (d) in [**L**, page 197].
- The squares 1, 6 and 3 commute via the definition of smooth-base-change.
- And finally the square 5 commutes via the transitivity of *flat-base-change* proved in [**AJL**, Section 7].

$\square$

$$
\begin{tikzcd}
u''^{\mathrm{s}}_t g^\times_t f^\times_t \mathcal{F} \arrow[rr, "\beta^{\mathrm{s}}_{\sigma_1}\circ f^\times_t"] \arrow[dr, equal, "\text{def.}"] \arrow[dddddd, dashed, "u''^{\mathrm{s}}_t\circ(c^\times_{(f,g)})"'] & & g'^\times_t u'^s_t f^\times_t \mathcal{F} \arrow[rr, "g'^\times_t\circ\beta^{\mathrm{s}}_{\sigma_2}"] \arrow[d, equal, "\text{def.}"] & & g'^\times_t f'^\times_t u^{\mathrm{s}}_t \mathcal{F} \arrow[dl, equal, "\text{def.}"'] \arrow[dddddd, dashed, "(c^\times_{(f',g')})\circ u^{\mathrm{s}}_t"] \\
& u''^*_t g^\times_t f^\times_t \mathcal{F} \otimes \widehat{\omega}^\bullet_{u''} \arrow[r, "\beta^{\mathrm{s}}_{\sigma_1}"] \arrow[d, equal] \arrow[dr, phantom, "1"] & g'^\times_t (u'^*_t f^\times_t \mathcal{F} \otimes \widehat{\omega}^\bullet_{u'}) \arrow[r, "\beta^{\mathrm{s}}_{\sigma_2}"] \arrow[d, "(3.6)"] \arrow[dr, phantom, "3"] & g'^\times_t f'^\times_t (u^*_t \mathcal{F} \otimes \widehat{\omega}^\bullet_u) \arrow[d, "(3.6)"] & \\
& u''^*_t g^\times_t f^\times_t \mathcal{F} \otimes \widehat{\omega}^\bullet_{u''} \arrow[r, "\beta'_{\sigma_1}"] \arrow[d, equal] \arrow[dr, phantom, "2"] & g'^\times_t u'^*_t f^\times_t \mathcal{F} \otimes g'^* \widehat{\omega}^\bullet_{u'} \arrow[r, "\beta'_{\sigma_2}"] \arrow[d, "(3.13)"] \arrow[dr, phantom, "4"] & g'^\times_t f'^\times_t u^*_t \mathcal{F} \otimes g'^* f'^* (\widehat{\omega}^\bullet_u) \arrow[d, "(3.13)"] & \\
\arrow[r, phantom, "7"] & u''^*_t g^\times_t f^\times_t \mathcal{F} \otimes \widehat{\omega}^\bullet_{u''} \arrow[r, "\beta'_{\sigma_1}\otimes 1"] \arrow[d, "u''^*\circ c^\times_{(f,g)}\otimes 1"'] & g'^\times_t u'^*_t f^\times_t \mathcal{F} \otimes \widehat{\omega}^\bullet_{u''} \arrow[r, "\beta'_{\sigma_2}\otimes 1"] \arrow[d, phantom, "5"] & g'^\times_t f'^\times_t u^*_t \mathcal{F} \otimes \widehat{\omega}^\bullet_{u''} \arrow[d, "c^\times_{(f',g')}\circ u^*\otimes 1"] \arrow[r, phantom, "8"] & \\
& u''^*_t (fg)^\times_t \mathcal{F} \otimes \widehat{\omega}^\bullet_{u''} \arrow[rr, "\beta'_\sigma\otimes 1"] \arrow[ddl, equal, "\text{def.}"] \arrow[drr, phantom, "6"] & & (f'g')^\times_t u^*_t \mathcal{F} \otimes \widehat{\omega}^\bullet_{u''} \arrow[d, "(3.13)"] & \\
& & & (f'g')^\times_t u^*_t \mathcal{F} \otimes g'^* f'^* (\widehat{\omega}^\bullet_u) \arrow[dr, "(3.6)"] & \\
u''^{\mathrm{s}}_t (fg)^\times_t \mathcal{F} \arrow[rrrr, "\beta^{\mathrm{s}}_\sigma"] & & & & (f'g')^\times_t u^{\mathrm{s}}_t \mathcal{F}
\end{tikzcd}
\tag{4.10}
$$

Chapter 5

Fundamental Local Isomorphism

The final requirement of input conditions for pasting two pseudofunctors in [**Nk**] involves factorization of identity maps (see input condition [D]). More precisely, for any object $\mathfrak{X}$ and for any composition $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{u} \mathfrak{X}$ factoring the identity map on $\mathfrak{X}$ such that u is smooth, there is a *fundamental isomorphism* $\phi_{f,u} : f_t^{\times} u_t^{\mathrm{s}} \xrightarrow{\;\sim\;} \mathbf{1}_{\mathfrak{X}}$, which is compatible with the comparison isomorphisms of $(-)_t^{\mathrm{s}}$ and $(-)_t^{\times}$ and the base change isomorphisms. To establish this we need to study the composition $f_t^{\times} u_t^{\mathrm{s}}$ locally first. In this section, we give a concrete description of the functor $f_t^{\times}$, when f is a local complete intersection. We denote this concrete version of $f_t^{\times}$ by $f^{\natural}$. After constructing $f^{\natural}$ we will establish a smooth-base-change isomorphism for fibered squares involving smooth morphisms and local complete intersection morphisms and prove that this is compatible with the smooth-base-change isomorphism β^{s} defined in (4.2).

We begin by briefly recalling a more concrete description of the functor $f_t^{\times}$, when f is a closed immersion.

5.0.1 Twisted Inverse Image for Closed Immersions

Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a closed immersion of noetherian formal schemes. The functor $f_* : \mathcal{A}(\mathfrak{X}) \longrightarrow \mathcal{A}(\mathfrak{Y})$ is exact, so $\mathbf{R}f_* = f_*$. Let $\mathscr{I}$ be the kernel of the surjective map $\mathscr{O}_{\mathfrak{Y}} \twoheadrightarrow f_*\mathscr{O}_{\mathfrak{X}}$ and let $\overline{\mathfrak{Y}}$ be the ringed space $(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{I})$, so that f factors naturally as $\mathfrak{X} \xrightarrow{\overline{f}} \overline{\mathfrak{Y}} \xrightarrow{i} \mathfrak{Y}$, and the map $\overline{f}$ is flat.

The functor $\mathcal{H}_{\mathscr{I}} : \mathcal{A}(\mathfrak{Y}) \to \mathcal{A}(\overline{\mathfrak{Y}})$ defined by $\mathcal{H}_{\mathscr{I}}(\mathcal{G}) := \mathcal{H}om(\mathscr{O}_{\mathfrak{Y}}/\mathscr{I}, \mathcal{G})$ has an exact left adjoint, namely $i_* : \mathcal{A}(\overline{\mathfrak{Y}}) \longrightarrow \mathcal{A}(\mathfrak{Y})$, so $\mathcal{H}_{\mathscr{I}}$ preserves K-injectivity and $\mathbf{R}\mathcal{H}_{\mathscr{I}}$ is the right-adjoint to $i_* : \mathbf{D}(\overline{\mathfrak{Y}}) \longrightarrow \mathbf{D}(\mathfrak{Y})$. Hence the functor $f^\flat : \mathbf{D}(\mathfrak{Y}) \longrightarrow \mathbf{D}(\mathfrak{X})$ defined by,

$$f^\flat(\mathcal{G}) := \overline{f}^* \mathbf{R}\mathcal{H}_{\mathscr{I}}(\mathcal{G}) = \overline{f}^* \mathbf{R}\underline{\mathcal{H}om}^\bullet(\mathscr{O}_{\mathfrak{Y}}/\mathscr{I}, \mathcal{G}) \simeq \overline{f}^* \mathbf{R}\underline{\mathcal{H}om}^\bullet(f_*\mathscr{O}_{\mathfrak{X}}, \mathcal{G}) \tag{5.1}$$

is the right-adjoint to $f_* = i_*\overline{f_*}$ and $f^\flat$ sends $\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$ to $\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$. Via the trace map

$$\mathbf{R}f_*f^\flat\mathcal{G} \simeq \mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(f_*\mathscr{O}_{\mathfrak{X}}, \mathcal{G}) \to \mathcal{G} \tag{5.2}$$

we therefore obtain a canonical map

$$f^\flat \to f_t^\times \tag{5.3}$$

which is an isomorphism.

A closed immersion $f : \mathfrak{X} \to \mathfrak{Y}$ of noetherian formal schemes with ideal sheaf $\mathscr{I}$ is called a *local complete intersection* of codimension n, if for any $x \in \mathfrak{X}$, there exists an affine open neighbourhood $U \subset \mathfrak{Y}$ such that, over U the ideal $\mathscr{I}_{\mathfrak{X}}(U) \subset \mathscr{O}_{\mathfrak{Y}}(U)$ is generated by a regular sequence of length n. For a general local complete intersection map, the codimension n is a locally constant integer-valued function on $\mathfrak{X}$ and as in subsection 3.10.3, in future references, by the codimension of a local complete intersection map we shall mean a locally constant integer-valued function on $\mathfrak{X}$.

Let $\mathfrak{X} \to \mathfrak{Y}$ be a closed immersion of noetherian formal schemes which is locally a complete intersection of codimension r. Let $\mathscr{I}$ be the ideal sheaf corresponding to this closed immersion. We define the *normal sheaf* as $(\mathscr{I}/\mathscr{I}^2)^\vee := \underline{\mathcal{H}om}_{\mathfrak{X}}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_{\mathfrak{X}})$, where $(\mathscr{I}/\mathscr{I}^2)$ is naturally an $\mathscr{O}_{\mathfrak{X}}$-module. This is a locally free sheaf of rank r. We define $\mathcal{N}_f := \bigwedge^r(\mathscr{I}/\mathscr{I}^2)^\vee$ as the *normal bundle* on $\mathfrak{X}$.

Proposition 5.1. *(Fundamental Local Isomorphism). Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a closed immersion of noetherian formal schemes, where $\mathfrak{X}$ is locally a complete intersection in $\mathfrak{Y}$ of codimension n with ideal sheaf $\mathscr{I}$. Then there is a natural functorial isomorphism,*

$$\phi : \overline{f}^* \boldsymbol{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(f_*\mathscr{O}_{\mathfrak{X}}, \mathscr{O}_{\mathfrak{Y}}) \xrightarrow{\sim} \bigwedge^n(\mathscr{I}/\mathscr{I}^2)^\vee[-n]. \tag{5.4}$$

Proof. We shall use the proof for the scheme case as given in [**H2**]. We shall proceed with the proof after setting up the $\mathcal{K}oszul$ resolution for local complete intersection of formal schemes.

Around any point $x \in \mathfrak{X}$ and $y = f(x) \in \mathfrak{Y}$, we may find affine open subschemes $\mathscr{V} = \mathrm{Spf}(A) \subset \mathfrak{Y}$, $\mathscr{U} = \mathrm{Spf}(B) \subset \mathfrak{X}$ containing y and x respectively, such that $f^{-1}(\mathscr{V}) = \mathscr{U}$. Let $U = \mathrm{Spec}(B)$ and $V = \mathrm{Spec}(A)$. Then

$$\begin{array}{ccc} \mathscr{U} & \xrightarrow{\kappa_{\mathscr{U}}} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_0} \\ \mathscr{V} & \xrightarrow{\kappa_{\mathscr{V}}} & V \end{array} \tag{5.5}$$

is a fibered square and $f_0 : U \to V$ is a local complete intersection of ordinary schemes and $\kappa_{\mathscr{U}}, \kappa_{\mathscr{V}}$ are flat completion morphisms. The induced map of rings $\tilde{f}_0 : A \longrightarrow B$, by virtue of f being a closed immersion is a surjective map. In view of the hypothesis we may further assume by shrinking $\mathscr{U}$ and $\mathscr{V}$, if necessary, that $I = \ker(\tilde{f}_0)$ is generated by a regular sequence of length n, i.e., $B \simeq A/I$, and $I = (\underline{\alpha})$, where $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a regular sequence in A.

The regular sequence $(\underline{\alpha})$ gives rise to a Koszul resolution of A/I over A and hence over $\mathscr{O}_V$ a Koszul resolution of $\mathscr{O}_V/\mathcal{I}$, where $\mathcal{I}$ is the ideal sheaf of U in V. Then $\kappa^*_{\mathscr{V}}(\mathcal{I}) = \mathscr{I}_{\mathscr{V}}$, is the ideal sheaf of $\mathscr{U}$ in $\mathscr{V}$, which is also the kernel sheaf of the natural map $\mathscr{O}_{\mathscr{V}} \to f_*\mathscr{O}_{\mathscr{U}}$. We define $\mathcal{K}_p(\underline{\alpha}) := \bigwedge^p \mathscr{O}_V^n$. Let $\{e_i\}$ be the basis of $\mathscr{O}_V^p$. We define the differential map, $d_p : \mathcal{K}_p(\underline{\alpha}) \longrightarrow \mathcal{K}_{p-1}(\underline{\alpha})$ via,

$$d_p(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}) = \Sigma(-1)^j \alpha_j e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_p}. \tag{5.6}$$

For an $\mathscr{O}_V$-module $\mathcal{F}$, let $\mathcal{K}^\bullet(\underline{\alpha}; \mathcal{F}) := \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_V}(\mathcal{K}_\bullet(\underline{\alpha}), \mathcal{F})$. The augumentation map

$$\epsilon_0 : \mathcal{K}_0(\underline{\alpha}) = \mathscr{O}_V \to \mathscr{O}_V/\mathcal{I} \tag{5.7}$$

is the obvious canonical map, inducing a map of complexes $\mathcal{K}_\bullet(\underline{\alpha}) \xrightarrow{\epsilon} \mathscr{O}_V/\mathcal{I}$, which is moreover a quasi-isomorphism. Hence,

$$\kappa^*_{\mathscr{V}}\mathcal{K}_\bullet(\underline{\alpha}) \to \mathscr{O}_{\mathscr{V}}/\mathscr{I} \tag{5.8}$$

is a resolution of $\mathscr{O}_{\mathscr{V}}/\mathscr{I}$ and we have the following isomorphisms.

$$\begin{aligned}\overline{f}^*\kappa_{\mathscr{V}}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_V}(f_{0*}\mathscr{O}_U,\mathscr{O}_V) &\xrightarrow{\sim} \overline{f}^*\kappa_{\mathscr{V}}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathscr{V}}}((\mathscr{O}_V/\mathcal{I}),\mathscr{O}_V)\\ &\xrightarrow{\sim} \overline{f}^*\kappa_{\mathscr{V}}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathscr{V}}}(\mathcal{K}_\bullet(\underline{\alpha}),\mathscr{O}_V)\\ &\xrightarrow{\sim} \overline{f}^*\kappa_{\mathscr{V}}^*\mathcal{H}(\mathcal{K}^\bullet(\underline{\alpha}))[-n]\\ &\xrightarrow{\sim} \overline{f}^*\kappa_{\mathscr{V}}^*(\mathscr{O}_V/\mathcal{I})[-n]\\ &\xrightarrow{\sim} \bigwedge^n(\mathscr{I}/\mathscr{I}^2)^\vee[-n]. \end{aligned} \tag{5.9}$$

Here the second, fourth and fifth isomorphisms are dependent on the choice of the generators of $\mathcal{I}$, but the composition

$$\overline{f}^*\mathbf{R}^n\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathscr{V}}}(f_*\mathscr{O}_{\mathscr{U}},\mathscr{O}_{\mathscr{V}}) \xrightarrow{\sim} \overline{f}^*\kappa_{\mathscr{V}}^*\mathbf{R}^n\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_V}(f_{0*}\mathscr{O}_U,\mathscr{O}_V) \xrightarrow{\sim} \bigwedge^n(\mathscr{I}/\mathscr{I}^2)^\vee \tag{5.10}$$

is independent of the choice of generators. These canonical isomorphism glue together to give the following desired global isomorphism of $\mathscr{O}_{\mathfrak{Y}}$-modules.

$$f^\flat\mathscr{O}_{\mathfrak{Y}} = \overline{f}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(f_*\mathscr{O}_{\mathfrak{X}},\mathscr{O}_{\mathfrak{Y}}) \to \bigwedge^n(\mathscr{I}/\mathscr{I}^2)^\vee[-n] = \mathcal{N}_f[-n] \tag{5.11}$$

□

For $\mathcal{G}^\bullet \in \mathbf{D}(\mathfrak{Y})$, define the functor $f^\natural : \mathbf{D}(\mathfrak{Y}) \to \mathbf{D}(\mathfrak{X})$ as follows:

$$f^\natural(\mathcal{G}) := \mathbf{L}f^*(\mathcal{G}) \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathcal{N}_f[-n]. \tag{5.12}$$

Now since $f_*\mathscr{O}_{\mathfrak{X}}$ has finite tor dimension, so for any $\mathcal{G} \in \mathbf{D}(\mathfrak{Y})$, we have

$$\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(f_*\mathscr{O}_{\mathfrak{X}},\mathcal{G}) \xrightarrow{\ \sim\ } \mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(f_*\mathscr{O}_{\mathfrak{X}},\mathscr{O}_{\mathfrak{Y}}) \otimes_{\mathfrak{Y}} \mathcal{G} \xrightarrow{\ \sim\ } \mathcal{G} \otimes_{\mathfrak{Y}} \mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(f_*\mathscr{O}_{\mathfrak{X}},\mathscr{O}_{\mathfrak{Y}})$$

and hence the following isomorphism (see sign convention in (2.2) and (2.3)).

$$f^\flat\mathcal{G} = \overline{f}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(f_*\mathscr{O}_{\mathfrak{X}},\mathcal{G}) \xrightarrow{\ \sim\ } \mathbf{L}f^*\mathcal{G} \otimes_{\mathfrak{X}} f^\flat\mathscr{O}_{\mathfrak{Y}} \xrightarrow{\ \sim\ } \mathbf{L}f^*\mathcal{G} \otimes_{\mathfrak{X}} \mathcal{N}_f[-n] = f^\natural\mathcal{G} \tag{5.13}$$

Using (5.3) and (5.13) we have the following isomorphism.

$$\eta : f_t^\times(\mathcal{G}^\bullet) \xrightarrow{\ \sim\ } f^\flat(\mathcal{G}^\bullet) \xrightarrow{\ \sim\ } f^\natural(\mathcal{G}^\bullet) \tag{5.14}$$

For local complete intersections, $(-)^\natural$ serves as a concrete version of $(-)^\times_t$. Moreover, for a local complete intersection $w : \mathfrak{X}' \to \mathfrak{Y}'$ of codimension n, we define the trace morphism $\tau^\natural_w : w_*\mathcal{N}_w[-n] \to \mathscr{O}_{\mathfrak{Y}'}$ as the following composition

$$w_*\mathcal{N}_w[-n] \xrightarrow{(5.14)} w_*\bar{w}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}'}}(w_*\mathscr{O}_{\mathfrak{X}'}, \mathscr{O}_{\mathfrak{Y}'}) \xrightarrow{\sim} \mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}'}}(w_*\mathscr{O}_{\mathfrak{X}'}, \mathscr{O}_{\mathfrak{Y}'}) \to \mathscr{O}_{\mathfrak{Y}'}. \tag{5.15}$$

Consider the morphisms $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}$ where f and g are local complete intersections of codimensions m and n respectively. Then $(g \circ f)$ is a local complete intersection of codimension $m+n$. Moreover, for $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Z})$ we have a following commutative diagram.

$$\begin{tikzcd} f^\times_t g^\times_t \mathcal{F} \arrow[r] \arrow[d] & (gf)^\times_t \mathcal{F} \arrow[d] \\ f^\flat g^\flat \mathcal{F} \arrow[r] \arrow[d] & (gf)^\flat \mathcal{F} \arrow[d] \\ f^\natural g^\natural \mathcal{F} \arrow[r] & (gf)^\natural \mathcal{F} \end{tikzcd} \tag{5.16}$$

In the case of ordinary scheme, the commutativity of the diagram on top follows from the description of the isomorphism $f^\times_t \xrightarrow{\sim} f^\flat$ and [H, III Proposition 6.6(1)] and the commutativity of bottom diagram follows from [Co, Theorem 2.5.1]. The proof of commutativity in the case of formal schemes are similar and hence have been omitted.

For the fibered product of a smooth map and a local complete intersection, there is a concrete version of the smooth-base-change isomorphism involving $(-)^\natural$. Consider the following fibered diagram.

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "v"] \arrow[d, "q"'] & \mathfrak{Y}' \arrow[d, "p"] \\ \mathfrak{X} \arrow[r, "u"] & \mathfrak{Y} \end{tikzcd} \tag{5.17}$$

where u, v are smooth maps of relative dimension d and p, q are local complete intersections of codimension r. For $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$, we define the base-change map $\beta^\natural \colon v^{\mathrm{s}}_t p^\natural \mathcal{F} \to$

$q^{\natural}u_t^{\mathrm{s}}\mathcal{F}$, using the following composition.

$$
\begin{aligned}
v_t^{\mathrm{s}}p^{\natural}\mathcal{F} &=\!=\!= v_t^*(\mathbf{L}p^*\mathcal{F}\otimes\mathcal{N}_p[-r])\otimes\widehat{\omega}_v[d]\\
&\xrightarrow{\ \sim\ } v_t^*\mathbf{L}p^*\mathcal{F}\otimes v^*\mathcal{N}_p[-r]\otimes\widehat{\omega}_v[d]\\
&\xrightarrow{\ \sim\ } \mathbf{L}q^*u_t^*\mathcal{F}\otimes\widehat{\omega}_v[d]\otimes v^*\mathcal{N}_p[-r]\\
&\xrightarrow{\ \sim\ } \mathbf{L}q^*u_t^*\mathcal{F}\otimes\mathbf{L}q^*\widehat{\omega}_u[d]\otimes\mathcal{N}_q[-r]\\
&\xrightarrow{\ \sim\ } q^*(u_t^*\mathcal{F}\otimes\widehat{\omega}_u[d])\otimes\mathcal{N}_q[-r]\\
&\xrightarrow{\ \sim\ } q^{\natural}u_t^{\mathrm{s}}\mathcal{F}
\end{aligned}
\tag{5.18}
$$

The third isomorphism is induced by the canonical isomorphism $A\otimes B\simeq B\otimes A$. In particular, the sign convention in (2.3) applies. Since all the maps involved are isomorphisms, so is $\beta^{\natural}$.

Also, for the fibered diagram (5.17) we have a base-change isomorphism $\beta^{*,\flat}\colon v_t^*p^{\flat}\mathcal{F}\xrightarrow{\ \sim\ } q^{\flat}u_t^*\mathcal{F}$ which induces the base-change isomorphism $\beta^{\mathrm{s},\flat}\colon v_t^{\mathrm{s}}p^{\flat}\mathcal{F}\xrightarrow{\ \sim\ } q^{\flat}u_t^{\mathrm{s}}\mathcal{F}$. The isomorphism β is defined via the following composition.

$$
\begin{aligned}
v_t^*p^{\flat}\mathcal{F} &\xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathfrak{X}'}v^*\bar{p}^*\mathbf{R}\underline{\mathcal{H}om}^{\bullet}_{\mathscr{O}_{\mathfrak{Y}}}(p_*\mathscr{O}_{\mathfrak{Y}'},\mathcal{F})\\
&\xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathfrak{X}'}\bar{q}^*u^*\mathbf{R}\underline{\mathcal{H}om}^{\bullet}_{\mathscr{O}_{\mathfrak{Y}}}(p_*\mathscr{O}_{\mathfrak{Y}'},\mathcal{F})\\
&\xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathfrak{X}'}\bar{q}^*\mathbf{R}\underline{\mathcal{H}om}^{\bullet}_{\mathscr{O}_{\mathfrak{X}}}(u^*p_*\mathscr{O}_{\mathfrak{Y}'},u^*\mathcal{F})\\
&\xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathfrak{X}'}\bar{q}^*\mathbf{R}\underline{\mathcal{H}om}^{\bullet}_{\mathscr{O}_{\mathfrak{X}}}(q_*v^*\mathscr{O}_{\mathfrak{Y}'},u^*\mathcal{F})\\
&\xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathfrak{X}'}\bar{q}^*\mathbf{R}\underline{\mathcal{H}om}^{\bullet}_{\mathscr{O}_{\mathfrak{X}}}(q_*\mathscr{O}_{\mathfrak{X}'},u^*\mathcal{F})\\
&\xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathfrak{X}'}q^{\flat}u^*\mathcal{F}\\
&\xrightarrow{\ \sim\ } q^{\flat}\mathbf{R}\varGamma'_{\mathfrak{X}}u^*\mathcal{F}\\
&\xrightarrow{\ \sim\ } q^{\flat}u_t^*\mathcal{F}
\end{aligned}
\tag{5.19}
$$

The isomorphism $\beta^\flat$ is defined as follows.

$$
\begin{aligned}
v_t^{\mathrm{s}} p^\flat \mathcal{F} &\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}'} v^* \bar{p}^* \mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(p_* \mathscr{O}_{\mathfrak{Y}'}, \mathcal{F}) \otimes \widehat{\omega}_v^\bullet \\
&\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}'} \bar{q}^* u^* \mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(p_* \mathscr{O}_{\mathfrak{Y}'}, \mathcal{F}) \otimes q^* \widehat{\omega}_u^\bullet \\
&\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}'} \bar{q}^* \mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{X}}}(u^* p_* \mathscr{O}_{\mathfrak{Y}'}, u^* \mathcal{F}) \otimes q^* \widehat{\omega}_u^\bullet \\
&\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}'} \bar{q}^* \mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{X}}}(q_* v^* \mathscr{O}_{\mathfrak{Y}'}, u^* \mathcal{F}) \otimes q^* \widehat{\omega}_u^\bullet \\
&\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}'} \bar{q}^* \mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{X}}}(q_* \mathscr{O}_{\mathfrak{X}'}, u^* \mathcal{F}) \otimes q^* \widehat{\omega}_u^\bullet \\
&\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}'} \bar{q}^* (\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{X}}}(q_* \mathscr{O}_{\mathfrak{X}'}, u^* \mathcal{F}) \otimes \widehat{\omega}_u^\bullet) \\
&\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}'} \bar{q}^* \mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{X}}}(q_* \mathscr{O}_{\mathfrak{X}'}, u^* \mathcal{F} \otimes \widehat{\omega}_u^\bullet) \\
&\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}'} q^\flat (u^* \mathcal{F} \otimes \widehat{\omega}_u^\bullet) \\
&\xrightarrow{\sim} q^\flat \mathbf{R}\varGamma'_{\mathfrak{X}} (u^* \mathcal{F} \otimes \widehat{\omega}_u^\bullet) \\
&\xrightarrow{\sim} q^\flat u_t^{\mathrm{s}} \mathcal{F}
\end{aligned}
$$

We will now show that via the isomorphism η defined in (5.14), $\beta^\natural$ is compatible with the smooth-base-change isomorphism β^{s} for $(-)_t^\times$.

Proposition 5.2. *For the fibered square of (5.17) and for $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$, the following diagram of isomorphisms commutes.*

$$
\begin{tikzcd}
v_t^{\mathrm{s}} p^\natural \mathcal{F} \arrow[r, "\beta^\natural"] \arrow[d, "\eta"'] & q^\natural u_t^{\mathrm{s}} \mathcal{F} \arrow[d, "\eta"] \\
v_t^{\mathrm{s}} p_t^\times \mathcal{F} \arrow[r, "\beta"] & q_t^\times u_t^{\mathrm{s}} \mathcal{F}
\end{tikzcd}
\tag{5.20}
$$

Proof. Consider the following diagram

$$
\begin{tikzcd}
v_t^{\mathrm{s}} p^\natural \mathcal{F} \arrow[r, "\beta^\natural"] \arrow[d, "\eta"'] \arrow[dr, phantom, "1"] & q^\natural u_t^{\mathrm{s}} \mathcal{F} \arrow[d, "\eta"] \\
v_t^{\mathrm{s}} p^\flat \mathcal{F} \arrow[r, "\beta^{\mathrm{s},\flat}"] \arrow[d] \arrow[dr, phantom, "2"] & q^\flat u_t^{\mathrm{s}} \mathcal{F} \arrow[d] \\
v_t^{\mathrm{s}} p_t^\times \mathcal{F} \arrow[r, "\beta"] & q_t^\times u_t^{\mathrm{s}} \mathcal{F}
\end{tikzcd}
\tag{5.21}
$$

We first the reduce the diagram (2) into the following diagram.

$$\begin{tikzcd} v_t^{\mathrm{s}} p^\flat \mathcal{F} \arrow[r] \arrow[d] & q^\flat(u_t^* \mathcal{F} \otimes \widehat{\omega}_u^\bullet) \arrow[r, dashed, "pr^\flat"] \arrow[d] & q^\flat u_t^* \mathcal{F} \otimes q^* \widehat{\omega}_u^\bullet \arrow[d] \\ v_t^* p_t^\times \mathcal{F} \otimes \widehat{\omega}_v^\bullet \arrow[r] & q_t^\times(u_t^* \mathcal{F} \otimes \widehat{\omega}_u^\bullet) \arrow[r] & q_t^\times u_t^* \mathcal{F} \otimes q^* \widehat{\omega}_u^\bullet \end{tikzcd} \tag{5.22}$$

The adjoint projection map $pr^\flat$ in the diagram (5.22) is obtained via the base-change isomorphism $\beta^\flat$ (equivalently it can also be obtained via the isomorphism (5.3) and the adjointness of $(-)_t^\times$) making the diagram (5.22) commute. Now using the isomorphism $\widehat{\omega}_v^\bullet \simeq q^* \widehat{\omega}_u^\bullet$, the commutativity of the subdiagram (2) reduces to the commutativity of the following diagram.

$$\begin{tikzcd} v_t^* p^\flat \mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "3"] & q^\flat u_t^* \mathcal{F} \arrow[d] \\ v_t^* p_t^\times \mathcal{F} \arrow[r] & q_t^\times u_t^* \mathcal{F} \end{tikzcd} \tag{5.23}$$

To show that the diagram (3) commutes, we use the adjointness of q_* to $q_t^\times$ to reduce to proving the commutativity of the diagram (5.24).

In the diagram (5.24) below, the commutativity of subdiagram $\spadesuit_1$ follows from the definition of the base change isomorphism $u^* p_* \simeq q_* v^*$ and the commutativity of subdiagram $\spadesuit_2$ follows from the application of trace map to the isomorphism (5.3). The remaining diagrams commute for trivial reasons.

$$
\begin{tikzcd}
\mathbf{R}\varGamma'_{\mathfrak{X}} q_* v^* \bar{p}^* \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(p_*\mathscr{O}_{\mathfrak{X}}, \mathcal{F}) \arrow[rrr, "\text{def.}"] \arrow[dr, "\sim"] \arrow[dd, "\wr"] \arrow[dr, phantom, shift right=6ex, "\spadesuit_1"] & & & q_* v_t^* p_t^{\times} \mathcal{F} \arrow[d] \\
& \mathbf{R}\varGamma'_{\mathfrak{X}} u^* p_* \bar{p}^* \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(p_*\mathscr{O}_{\mathfrak{X}}, \mathcal{F}) \arrow[rr, "(5.3)"] \arrow[d, "\wr"] & \spadesuit_2 & \mathbf{R}\varGamma'_{\mathfrak{X}} u_* p_* p_t^{\times} \mathcal{F} \arrow[d, "\tau_p"] \\
\mathbf{R}\varGamma'_{\mathfrak{X}} q_* \bar{q}^* u^* \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(p_*\mathscr{O}_{\mathfrak{X}}, \mathcal{F}) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathbf{R}\varGamma'_{\mathfrak{X}} u^* \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(p_*\mathscr{O}_{\mathfrak{X}}, \mathcal{F}) \arrow[r] \arrow[d, "\wr"] & \mathbf{R}\varGamma'_{\mathfrak{X}} u^* \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(\mathscr{O}_{\mathfrak{Y}}, \mathcal{F}) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathbf{R}\varGamma'_{\mathfrak{X}} u^* \mathcal{F} \arrow[dd, equal] \\
\mathbf{R}\varGamma'_{\mathfrak{X}} q_* \bar{q}^* \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{X}}}(u^* p_*\mathscr{O}_{\mathfrak{X}}, u^*\mathcal{F}) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathbf{R}\varGamma'_{\mathfrak{X}} \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}'}}(u^* p_*\mathscr{O}_{\mathfrak{X}}, u^*\mathcal{F}) \arrow[r] \arrow[d, "\wr"] & \mathbf{R}\varGamma'_{\mathfrak{X}} \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}'}}(u^*\mathscr{O}_{\mathfrak{Y}}, u^*\mathcal{F}) \arrow[d, "\wr"] & \\
\mathbf{R}\varGamma'_{\mathfrak{X}} q_* \bar{q}^* \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}'}}(q_*\mathscr{O}_{\mathfrak{X}'}, u^*\mathcal{F}) \arrow[r, "\sim"] & \mathbf{R}\varGamma'_{\mathfrak{X}} \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}'}}(q_*\mathscr{O}_{\mathfrak{X}'}, u^*\mathcal{F}) \arrow[r, "\sim"] & \mathbf{R}\varGamma'_{\mathfrak{X}} \underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}'}}(\mathscr{O}_{\mathfrak{Y}'}, u^*\mathcal{F}) \arrow[r] & u_t^* \mathcal{F}
\end{tikzcd}
\tag{5.24}
$$

We now expand the diagram (1) in (5.21) using the definitions of the $(-)^{\mathrm{s}}_t$, $(-)^{\flat}$ and $(-)^{\natural}$.

$$\begin{tikzcd}
v_t^*(\mathbf{L}p^*\mathcal{F}\otimes\mathcal{N}_p^\bullet)\otimes\widehat{\omega}_v^\bullet \arrow[rr] \arrow[d] & & \mathbf{L}q^*(u_t^*\mathcal{F}\otimes\widehat{\omega}_u^\bullet)\otimes\mathcal{N}_q^\bullet \arrow[dd] \\
v_t^*\mathbf{L}p^*\mathcal{F}\otimes v^*\bar{p}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(p_*\mathscr{O}_{\mathfrak{X}},\mathscr{O}_{\mathfrak{Y}})\otimes\widehat{\omega}_v^\bullet \arrow[d] & & \\
v_t^*\bar{p}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(p_*\mathscr{O}_{\mathfrak{X}},\mathcal{F})\otimes\widehat{\omega}_v^\bullet \arrow[rr] & & \bar{q}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}'}}(q_*\mathscr{O}_{\mathfrak{X}'},u^*\mathcal{F}\otimes\widehat{\omega}_u^\bullet)
\end{tikzcd} \tag{5.25}$$

Since $p_*\mathscr{O}_{\mathfrak{X}}$ and $q_*\mathscr{O}_{\mathfrak{X}'}$ have finite-tor dimension, we can rewrite the above diagram as follows.

$$\begin{tikzcd}
v_t^*\mathbf{L}p^*\mathcal{F}\otimes v^*\mathcal{N}_p^\bullet\otimes\widehat{\omega}_v^\bullet \arrow[r] \arrow[d] & \mathbf{L}q^*u_t^*\mathcal{F}\otimes\mathcal{N}_q^\bullet\otimes q^*\widehat{\omega}_u^\bullet \arrow[d] \\
v_t^*\mathbf{L}p^*\mathcal{F}\otimes v^*\bar{p}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(p_*\mathscr{O}_{\mathfrak{X}},\mathscr{O}_{\mathfrak{Y}})\otimes\widehat{\omega}_v^\bullet \arrow[r] & \mathbf{L}q^*u_t^*\mathcal{F}\otimes\bar{q}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}'}}(q_*\mathscr{O}_{\mathfrak{X}'},\mathscr{O}_{\mathfrak{Y}'})\otimes q^*\widehat{\omega}_u^\bullet
\end{tikzcd} \tag{5.26}$$

Now the commutativity of (5.26) follows from the commutativity of the diagram (5.27) which follows from (5.19) (using $\mathcal{F}=\mathscr{O}_{\mathfrak{Y}}$) and the isomorphism (5.14).

$$\begin{tikzcd}
v_t^*\mathcal{N}_p^\bullet \arrow[r] \arrow[d] & \mathcal{N}_q^\bullet \arrow[d] \\
v_t^*\bar{p}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}}}(p_*\mathscr{O}_{\mathfrak{X}},\mathscr{O}_{\mathfrak{Y}}) \arrow[r] & \bar{q}^*\mathbf{R}\underline{\mathcal{H}om}^\bullet_{\mathscr{O}_{\mathfrak{Y}'}}(q_*\mathscr{O}_{\mathfrak{X}'},\mathscr{O}_{\mathfrak{Y}'})
\end{tikzcd} \tag{5.27}$$

$\square$

Chapter 6

Tor-independent Base Change Isomorphism

In this section we give a base-change isomorphism for fibered squares of pseudoproper maps and local complete intersections.

A fibered square

$$\begin{tikzcd} \mathscr{V} \arrow[r, "v"] \arrow[d, "g"'] & \mathfrak{X} \arrow[d, "f"] \\ \mathscr{U} \arrow[r, "u"] & \mathfrak{Y} \end{tikzcd} \tag{6.1}$$

where f is essentially of pseudofinite type and u is a local complete intersection of codimension n will be called a *tor-independent* square if for any $y = f(x)$, any affine open neighbourhood U of y such that the ideal of $u^{-1}U$ in $\mathscr{U}$ is given by a regular sequence $(t) = (t_1, \dots, t_n)$ and any affine open neighbourhood V of x, the natural image of (t) over V is also regular. Thus v is also a local complete intersection of codimension n.

Proposition 6.1. *Suppose that in the tor-independent square of (6.1) f, g are pseudo-proper maps. Then the following statements are true.*

(a) The morphism $\theta : \mathbf{L}u^*\boldsymbol{R}f_*\mathcal{F} \to \boldsymbol{R}g_*\mathbf{L}v^*\mathcal{F}$ obtained via the following composition

$$\mathbf{L}u^*\boldsymbol{R}f_*\mathcal{F} \to \mathbf{L}u^*\boldsymbol{R}f_*v_*\mathbf{L}v^*\mathcal{F} \xrightarrow{\sim} \mathbf{L}u^*u_*\boldsymbol{R}g_*\mathbf{L}v^*\mathcal{F} \to \boldsymbol{R}g_*\mathbf{L}v^*\mathcal{F}$$

is an isomorphism for $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$.

(b) The base change map $\beta : \mathbf{L}v^* f_t^{\times} \mathcal{F} \to g_t^{\times} \mathbf{L}u^* \mathcal{F}$ *obtained via the following natural composition*

$$\mathbf{L}v^* f_t^{\times} \mathcal{F} \to g_t^{\times} \boldsymbol{R}g_* \mathbf{L}v^* f_t^{\times} \mathcal{F} \xrightarrow{\theta} g_t^{\times} \mathbf{L}u^* \boldsymbol{R}f_* f_t^{\times} \mathcal{F} \xrightarrow{\tau_f} g_t^{\times} \mathbf{L}u^* \mathcal{F}$$

is an isomorphism for $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$*.*

For the proof of the statement *(a)* we use the proof of [**AJL**, Prop 7.2.1(b)] where u, v are assumed to be flat. We proceed using the following alterations. The only ingredients in the proof of [**AJL**, Prop 7.2.1(b)] which do not work for lci maps are [**AJL**, Lemmas 7.2.1, 7.2.2].

The analog of [**AJL**, Lemma 7.2.1] in our case holds, for if $\mathfrak{X}$ and $\mathfrak{Y}$ are ordinary schemes and u, v are lci then it implies that $\mathscr{U}$ and $\mathscr{V}$ are ordinary schemes as well, whereupon we employ Lipman's proof of base change for independent squares for ordinary schemes, see [**L**, Theorem 4.4.1].

We now prove the analog of [**AJL**, Lemma 7.2.2] in our case as follows.

Proposition 6.2. *Let* $\mathscr{I}$ *be the ideal of definition of a noetherian formal scheme* $\mathfrak{Y}$*, let* Y_n *be the ordinary scheme* $(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{I}^n)$ *and let* $i_n : Y_n \hookrightarrow \mathfrak{Y}$ *be the canonical closed immersion. Also, let* $u : \mathscr{U} \to \mathfrak{Y}$ *be a local complete intersection of codimension* r*. Assume further that the following fibered square is tor-independent (see (6.1)).*

$$\begin{tikzcd} \mathscr{U}' = Y_n \times \mathscr{U} \arrow[r, "p_n"] \arrow[d, "u_n"'] & \mathscr{U} \arrow[d, "u"] \\ Y_n \arrow[r, "i_n"] & \mathfrak{Y} \end{tikzcd}$$

Then the map $\alpha_{\mathcal{F}} : \mathbf{L}u^* i_{n*} \mathcal{F} \to p_{n*} \mathbf{L}u_n^* \mathcal{F}$ *adjoint to the following natural composition,*

$$i_{n*}\mathcal{F} \xrightarrow{\tau} i_{n*} u_{n*} \mathbf{L}u_n^* \mathcal{F} \to u_* p_{n*} \mathbf{L}u_n^* \mathcal{F} \tag{6.2}$$

is an isomorphism for $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(Y_n)$*.*

Proof. Since u is a closed immersion, to show that $\alpha_{\mathcal{F}}$ is an isomorphism, it is enough to show

$$u_* \alpha_{\mathcal{F}} : u_* \mathbf{L}u^* i_{n*} \mathcal{F} \to u_* p_{n*} \mathbf{L}u_n^* \mathcal{F} \tag{6.3}$$

is an isomorphism. It is enough to check that $u_*\alpha$ is an isomorphism on affine patches. Let $V \subset \mathfrak{Y}$ be an affine open subset such that i_n is given by a regular sequence (t) of length r. Let $\mathcal{K} = \mathcal{K}_\bullet(t)$ be the Koszul resolution of $\mathscr{O}_V/\mathscr{I}(V)$. Using tor-independence of the square (6.2), we have a Koszul resolution induced by the natural image of the sequence (t) over $\mathscr{O}_{Y_n}$. We will denote it by $\mathcal{K}_n = \mathcal{K}_\bullet(t)$. For convenience, we consider $\mathfrak{U}', \mathfrak{U}, Y_n$ as ringed spaces on $\mathfrak{Y}$, extended by zero outside their support. Thus now $u_* = u_{n*} = p_{n*} = i_{n*} = identity$. Similarly we consider a complex $\mathcal{F}$ to be a complex on $\mathfrak{Y}$, extended by zero outside its support.

Thus if $\mathcal{M}$ is a complex of $\mathscr{O}_{Y_n}$-modules and $\mathcal{N}$ a complex of $\mathscr{O}_{\mathfrak{X}}$-modules, there are natural maps

$$\begin{aligned}
&\mathcal{M} \to \mathcal{M} \otimes^{\mathbf{L}}_{Y_n} \mathscr{O}_{\mathfrak{U}'} \simeq \mathbf{L}u_n^*\mathcal{M},\\
&\mathbf{L}u^*\mathcal{N} \simeq \mathcal{N} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}} \to \mathcal{N} \otimes_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}} \simeq \mathcal{N}.
\end{aligned}$$

It follows that $\alpha_{\mathcal{F}}$ is the composite $\mathbf{L}u^*\mathcal{F} \to \mathbf{L}u^*\mathbf{L}u_n^*\mathcal{F} \to \mathbf{L}u_n^*\mathcal{F}$. Moreover there are natural isomorphisms.

$$\begin{aligned}
&\mathbf{L}u^*\mathcal{F} \simeq \mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}},\\
&\mathbf{L}u_n^*\mathcal{F} \simeq \mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{Y_n}} \mathscr{O}_{\mathfrak{U}'} \simeq \mathcal{F} \otimes_{\mathscr{O}_{Y_n}} \mathcal{K}_n \simeq \mathcal{F} \otimes_{\mathscr{O}_{\mathfrak{Y}}} \mathcal{K} \simeq \mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}},\\
&\mathbf{L}u^*\mathbf{L}u_n^*\mathcal{F} \simeq \mathbf{L}u^*(\mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}}) \simeq (\mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}}.
\end{aligned}$$

In the following commutative diagram the bottom row composes to the identity map, hence $\alpha_{\mathcal{F}}$, which is the composite of the top row, is an isomorphism.

$$\begin{tikzcd}
\mathbf{L}u^*\mathcal{F} \arrow[r] \arrow[d, "\wr"] & \mathbf{L}u^*\mathbf{L}u_n^*\mathcal{F} \arrow[r] \arrow[d, "\wr"] & \mathbf{L}u_n^*\mathcal{F} \arrow[d, "\wr"] \\
\mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}} \arrow[r, "\beta"] & (\mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}} \arrow[r, "\gamma"] & \mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}}
\end{tikzcd} \tag{6.4}$$

In the bottom row, β is induced by applying $-\otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}}$ to $\mathcal{F} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}}$, while γ is induced by the canonical map $\mathscr{O}_{\mathscr{U}} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_{\mathscr{U}} \to \mathscr{O}_{\mathscr{U}}$. $\square$

Plugging the proposition 6.2 in the proof of [**AJL**, Theorem 7.2] as a replacement for [**AJL**, Lemma 7.2.2], we now have the proof of proposition 6.1(a).

We will now prove 6.1(b). In the independent square (6.1), f, g are pseudoproper maps and u, v are closed immersions which are locally complete intersections maps. Let $h = fv = ug$ be the composition. Now, since closed immersions are pseudoproper maps, we have a comparison isomorphism.

$$c^{\times} : h_t^{\times}\mathcal{F} \xrightarrow{\sim} v_t^{\times} f_t^{\times}\mathcal{F} \xrightarrow{\sim} g_t^{\times} u_t^{\times}\mathcal{F} \tag{6.5}$$

Moreover, for local complete intersection maps, by (5.14) we have the isomorphisms $u_t^{\times}\mathcal{G} \simeq u^{\natural}\mathcal{G}$ and $v_t^{\times}\mathcal{F} \simeq v^{\natural}\mathcal{F}$ for $\mathcal{F}, \mathcal{G}$ in $\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$ and $\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$ respectively. Now, using β, and the additional information that u, v are local complete intersection, we consider the map β' defined via following composition.

$$\begin{aligned} v^{\natural} f_t^{\times}\mathcal{F} &= v_t^{*} f_t^{\times}\mathcal{F} \otimes \mathcal{N}_v[-n] \\ &\xrightarrow{\beta_t} g_t^{\times} u_t^{*}\mathcal{F} \otimes \mathcal{N}_v[-n] \\ &\xrightarrow{\sim} g_t^{\times} u_t^{*}\mathcal{F} \otimes g^{*}\mathcal{N}_u[-n] \\ &\xrightarrow{\sim} g_t^{\times}(u_t^{*}\mathcal{F} \otimes \mathcal{N}_u[-n]) \\ &\xrightarrow{\sim} g_t^{\times} u^{\natural}\mathcal{F}. \end{aligned} \tag{6.6}$$

Hence, β is an isomorphism if and only if β' is an isomorphism, and to show that the map β' is an isomorphism, it is enough to show that for $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$, the following diagram commutes.

$$\begin{tikzcd} v^{\natural} f_t^{\times}\mathcal{F} \arrow[r, "\beta'_t"] \arrow[d, "\eta"'] & g_t^{\times} u^{\natural}\mathcal{F} \arrow[d, "\eta"] \\ v_t^{\times} f_t^{\times}\mathcal{F} \arrow[r, "c^{\times}"] & g_t^{\times} u_t^{\times}\mathcal{F} \end{tikzcd} \tag{6.7}$$

Using the definition of β', we rewrite the diagram as follows.

$$\begin{tikzcd} v^{\natural} f_t^{\times}\mathcal{F} \arrow[rr] \arrow[d, equal] & & g_t^{\times} u^{\natural}\mathcal{F} \arrow[d, equal] \\ \mathbf{L}v_t^{*} f_t^{\times}\mathcal{F} \otimes \mathcal{N}_v[-n] \arrow[r] \arrow[d] & g_t^{\times}\mathbf{L}u_t^{*}\mathcal{F} \otimes g^{*}\mathcal{N}_u[-n] \arrow[r] & g_t^{\times}(\mathbf{L}u_t^{*}\mathcal{F} \otimes \mathcal{N}_u[-n]) \arrow[d] \\ v_t^{\times} f_t^{\times}\mathcal{F} \arrow[rr] & & g_t^{\times} u_t^{\times}\mathcal{F} \end{tikzcd} \tag{6.8}$$

Applying $\mathbf{R}h_*(\xrightarrow{\sim} \mathbf{R}f_* v_* \simeq u_* \mathbf{R}g_*)$ to the lower part of the diagram (6.8), via adjointness of $\mathbf{R}h_*$ with $h_t^{\times}$, it suffices to consider the diagram (6.10) below where $\mathcal{N}_v^{\bullet} =$

$\mathcal{N}_v[-n], \mathcal{N}_u^\bullet = \mathcal{N}_u[-n]$ and $h_* = \mathbf{R}h_*$. The unlabelled diagrams commute for obvious reasons. We will now show that the subdiagram $\spadesuit_1$ in (6.10) commutes. We first record that the following diagram is commutative.

$$\begin{tikzcd} v_*\mathcal{N}_v^\bullet \arrow[r, "\sim"] \arrow[d, "\tau_v"'] & f^*u_*\mathcal{N}_u^\bullet \arrow[d, "\tau_u"] \\ \mathscr{O}_{\mathfrak{X}} \arrow[r, "\sim"] & f^*\mathscr{O}_{\mathfrak{Y}} \end{tikzcd} \tag{6.9}$$

To prove the commutativity of $\spadesuit_1$ we expand it into the diagram (6.11) where the commutativity of the sub-diagram $\clubsuit_1$ in (6.11) is shown by expanding it into the diagram (6.12) with $\mathcal{M} = f_t^\times \mathcal{F}$ and $\mathcal{N} = \mathcal{N}_v^\bullet$.

The commutativity of the unlabelled diagrams of (6.12) is straight-forward to check. The diagram $\blacklozenge$ commutes via the adjointness of $f^* \dashv f_*$ and [**L**, Proposition 3.7.3] and the subdiagram 1 commutes via the definition of the adjoint base change isomorphism.

$$
\begin{tikzcd}
h_*(v^*f_t^\times\mathcal{F}\otimes\mathcal{N}_v^\bullet) \arrow[r, equal] \arrow[d, "\wr"] & h_*(v^*f_t^\times\mathcal{F}\otimes g^*\mathcal{N}_u^\bullet) \arrow[r, "\beta"] \arrow[d, "\wr"] & h_*(g_t^\times u^*\mathcal{F}\otimes g^*\mathcal{N}_u^\bullet) \arrow[r] \arrow[d, "\wr"] & h_*g_t^\times(u^*\mathcal{F}\otimes\mathcal{N}_u^\bullet) \arrow[d, "\wr"] \\
f_*v_*(v^*f_t^\times\mathcal{F}\otimes\mathcal{N}_v^\bullet) \arrow[r] \arrow[d, "\mathrm{proj}_v"] & u_*g_*(v^*f_t^\times\mathcal{F}\otimes g^*\mathcal{N}_u^\bullet) \arrow[r, "\beta"] \arrow[d, "\mathrm{proj}_g"] & u_*g_*(g_t^\times u^*\mathcal{F}\otimes g^*\mathcal{N}_u^\bullet) \arrow[r] \arrow[d, "\mathrm{proj}_g"] & u_*g_*g_t^\times(u^*\mathcal{F}\otimes\mathcal{N}_u^\bullet) \arrow[dd, "\tau_g"] \\
f_*(f_t^\times\mathcal{F}\otimes v_*\mathcal{N}_v^\bullet) \arrow[d, "\tau_v"] & u_*(g_*v^*f_t^\times\mathcal{F}\otimes\mathcal{N}_u^\bullet) \arrow[r, "\beta"] \arrow[d] & u_*(g_*g_t^\times u^*\mathcal{F}\otimes\mathcal{N}_u^\bullet) \arrow[d, "\tau_g"] & \\
f_*(f_t^\times\mathcal{F}\otimes\mathscr{O}_{\mathfrak{X}}) \arrow[r, phantom, "\spadesuit_1"] \arrow[dd] & u_*(u^*f_*f_t^\times\mathcal{F}\otimes\mathcal{N}_u^\bullet) \arrow[r, "\tau_f"] \arrow[d, "\mathrm{proj}_u"] & u_*(u^*\mathcal{F}\otimes\mathcal{N}_u^\bullet) \arrow[r, equal] \arrow[d, "\mathrm{proj}_u"] & u_*(u^*\mathcal{F}\otimes\mathcal{N}_u^\bullet) \arrow[d, "\mathrm{proj}_u"] \\
 & f_*f_t^\times\mathcal{F}\otimes u_*\mathcal{N}_u^\bullet \arrow[r, "\tau_f"] \arrow[d, "\tau_u"] & \mathcal{F}\otimes u_*\mathcal{N}_u^\bullet \arrow[r, equal] \arrow[d, "\tau_u"] & \mathcal{F}\otimes u_*\mathcal{N}_u^\bullet \arrow[d, "\tau_u"] \\
f_*f_t^\times\mathcal{F} \arrow[d] & f_*f_t^\times\mathcal{F}\otimes\mathscr{O}_{\mathfrak{Y}} \arrow[d, "\tau_f"] & \mathcal{F}\otimes\mathscr{O}_{\mathfrak{Y}} \arrow[d] & \mathcal{F}\otimes\mathscr{O}_{\mathfrak{Y}} \arrow[d] \\
\mathcal{F} & \mathcal{F} \arrow[l] \arrow[r] & \mathcal{F} \arrow[r, equal] & \mathcal{F}\otimes\mathscr{O}_{\mathfrak{Y}}
\end{tikzcd}
\tag{6.10}
$$

$$
\begin{tikzcd}
f_*v_*(v^*f_t^{\times}\mathcal{F}\otimes\mathcal{N}_v^{\bullet}) \arrow[rrrr, "c_*"] \arrow[d, "\mathrm{proj}_v"'] & & {} \arrow[d, phantom, "\clubsuit_1"] & & u_*g_*(v^*f_t^{\times}\mathcal{F}\otimes g^*\mathcal{N}_u^{\bullet}) \arrow[d, "\mathrm{proj}_g"] \\
f_*(f_t^{\times}\mathcal{F}\otimes v_*\mathcal{N}_v^{\bullet}) \arrow[r, "\sim"] \arrow[d, "\tau_v"'] \arrow[dr, phantom, "(6.9)"] & f_*(f_t^{\times}\mathcal{F}\otimes f^*u_*\mathcal{N}_u^{\bullet}) \arrow[d, "\tau_u"] \arrow[drr, "\mathrm{proj}_f"] & {} & & u_*(g_*v^*f_t^{\times}\mathcal{F}\otimes\mathcal{N}_u^{\bullet}) \arrow[d, "\text{adj. bc}"] \\
f_*(f_t^{\times}\mathcal{F}\otimes\mathscr{O}_{\mathfrak{X}}) \arrow[r, "\sim"] \arrow[d] & f_*(f_t^{\times}\mathcal{F}\otimes f^*\mathscr{O}_{\mathfrak{Y}}) \arrow[d, "\mathrm{proj}_f"] & & f_*f_t^{\times}\mathcal{F}\otimes u_*\mathcal{N}_u^{\bullet} \arrow[d, "\tau_f"] \arrow[dll, "\tau_u"] & u_*(u^*f_*f_t^{\times}\mathcal{F}\otimes\mathcal{N}_u^{\bullet}) \arrow[l, "\mathrm{proj}_u"] \arrow[d, "\tau_f"] \\
f_*f_t^{\times}\mathcal{F} \arrow[r] \arrow[d] & f_*f_t^{\times}\mathcal{F}\otimes\mathscr{O}_{\mathfrak{Y}} \arrow[dr, "\tau_f"] & & \mathcal{F}\otimes u_*\mathcal{N}_u^{\bullet} \arrow[dl, "\tau_u"] & u_*(u^*\mathcal{F}\otimes\mathcal{N}_u^{\bullet}) \arrow[l, "\mathrm{proj}_u"] \\
\mathcal{F} \arrow[rr] & & \mathcal{F}\otimes\mathscr{O}_{\mathfrak{Y}} & &
\end{tikzcd}
\tag{6.11}
$$

$$
\begin{tikzcd}
f_*v_*(v^*\mathcal{M}\otimes g^*\mathcal{N}) \arrow[rr, "\sim"] & & h_*(v^*\mathcal{M}\otimes g^*\mathcal{N}) & & u_*g_*(v^*\mathcal{M}\otimes g^*\mathcal{N}) \arrow[ll, "\sim"'] \\
& f_*v_*(v^*f_t^*f_*\mathcal{M}\otimes g^*\mathcal{N}) \arrow[ul, "\tau_f"] \arrow[r, "\overset{1}{\sim}", "c_*"'] \arrow[d, "\wr"] & f_*v_*(g_t^*u^*f_*\mathcal{M}\otimes g^*\mathcal{N}) \arrow[r, "\sim"] \arrow[d, "\wr"] & f_*v_*(g_t^*g_*v^*\mathcal{M}\otimes g^*\mathcal{N}) \arrow[ulll, "\tau_g"'] \arrow[r, "\sim", "c_*"'] \arrow[d, "\wr"] & u_*g_*(g_t^*g_*v^*\mathcal{M}\otimes g^*\mathcal{N}) \arrow[u, "\tau_g"] \\
f_*(\mathcal{M}\otimes v_*g^*\mathcal{N}) \arrow[uu, "\wr", "\mathrm{proj}_v"'] & f_*(f_t^*f_*\mathcal{M}\otimes v_*g^*\mathcal{N}) \arrow[l, "\tau_f"] \arrow[dd, "\text{adj. bc}", "\wr"'] \arrow[r, phantom, "\blacklozenge"] & f_*v_*g_t^*(u^*f_*\otimes\mathcal{N}) \arrow[r, "\sim", "\text{adj. bc}"'] & f_*v_*g_t^*(g_*v^*\mathcal{M}\otimes\mathcal{N}) \arrow[r, "\sim", "c_*"'] & u_*g_*g_t^*(g_*v^*\mathcal{M}\otimes\mathcal{N}) \arrow[u, "\wr"] \\
& & f_*f_t^*u_*(u^*f_*\mathcal{M}\otimes\mathcal{N}) \arrow[u, "\text{adj. bc}"] \arrow[r, "\sim", "\text{adj. bc}"'] & f_*f_t^*u_*(g_*v^*\mathcal{M}\otimes\mathcal{N}) \arrow[u, "\text{adj. bc}"] & u_*(g_*v^*\mathcal{M}\otimes\mathcal{N}) \arrow[l, "\tau_f"] \arrow[u, "\tau_g"] \\
f_*(\mathcal{M}\otimes f^*u_*\mathcal{N}) \arrow[uu, "\wr", "\text{adj. bc}"'] & f_*(f_t^*f_*\mathcal{M}\otimes f^*u_*\mathcal{N}) \arrow[l, "\tau_f"] & f_*f_t^*(f_*\mathcal{M}\otimes u_*\mathcal{N}) \arrow[l, "\sim"] \arrow[u, "\wr", "\mathrm{proj}_u"'] & f_*\mathcal{M}\otimes u_*\mathcal{N} \arrow[l, "\tau_f"] \arrow[r, "\sim", "\mathrm{proj}_u"'] & u_*(u^*f_*\mathcal{M}\otimes\mathcal{N}) \arrow[u, "\wr", "\text{adj. bc}"'] \arrow[ull, "\tau_f"]
\end{tikzcd}
\tag{6.12}
$$

Chapter 7

Identity Factorization

After defining $(-)_t^\times$ for local complete intersections and obtaining a fundamental local isomorphism in chapter 5, we are now in position to establish the input condition [D] of [**Nk**, Subsection 2.1, Page 7], to glue the pseudofunctors $(-)_t^s$ and $(-)_t^\times$. We will use the Proposition 7.1 below and the isomorphism (5.14) to give the isomorphism (7.6). We will then show this isomorphism is compatible for extension of base by smooth or pseudoproper morphisms. Finally, we will show that this isomorphism is compatible with compositions and smooth-base-change isomorphism.

Proposition 7.1. *Consider a sequence $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{u} \mathfrak{X}$ of separated pseudofinite-type maps of noetherian formal schemes, such that u is smooth of relative dimension n, and $uf = 1_{\mathfrak{X}}$. Then f is a local complete intersection of codimension n.*

Proof. Consider a point x in $\mathfrak{X}$. Let $f(x) = y$ so that $u(y) = uf(x) = x$. Let $A = \mathscr{O}_{\mathfrak{X},x}$ and $B = \mathscr{O}_{\mathfrak{Y},y}$, so we have induced maps of local rings,

$$A \to B \twoheadrightarrow A$$

and hence induced maps

$$\widehat{A} \to \widehat{B} \to \widehat{A} \tag{7.1}$$

where the completions are along the respective maximal ideals. Let $k = A/m_A$. Then since the first map from the induced sequence $k \to \widehat{B}/m_A\widehat{B} \to k$ is smooth, we know that there is an isomorphism

$$\rho : k[\![T]\!] \xrightarrow{\sim} \widehat{B}/m_A\widehat{B} \tag{7.2}$$

for suitable variables $T = (t_1, t_2, \ldots, t_n)$. Consider the following diagram where r is a lift of the map ρ.

$$\begin{tikzcd} & I \arrow[d] & \\ & \widehat{A}[\![T]\!] \arrow[d, "r"] & \\ \widehat{A} \arrow[ur] \arrow[r] & \widehat{B} \arrow[r] & \widehat{A} \end{tikzcd} \tag{7.3}$$

Here I is the kernel of the map r. By complete Nakayama [**Mat**, Theorem 8.4], r is surjective. Going modulo m_A in (7.3) and using the isomorphism (7.2) and the flatness of $\widehat{B}$ over $\widehat{A}$, we conclude that $I/m_AI = 0$. By Nakayama, we conclude that $I = 0$. Thus r is an isomorphism. Now if $J = \ker(B \to A)$ and $\widehat{J}$ is its completion along the maximal ideal m_B, then $\widehat{J}$ is generated by the regular sequence $(t_1, t_2, \ldots, t_n)$ of length n. Since, $J/m_BJ \simeq \widehat{J}/m_B\widehat{J}$, thus, J is generated by a sequence of length n which is regular by faithful flatness. $\square$

Consider the sequence, $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{u} \mathfrak{X}$, where u is smooth separated map such that $uf = 1$ hence f is locally a complete intersection. For such a factorization of identity map, we want to define a canonical isomorphism $\phi_{f,u} \colon f_t^{\times} u_t^{\mathrm{s}} \xrightarrow{\sim} 1_{\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})}$. Moreover, we want this isomorphism to behave well with smooth or pseudo-proper base change on $\mathfrak{X}$. We begin with an important special case.

Proposition 7.2. *If $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is a smooth separated map of relative dimension n, and $i : \mathfrak{Y} \to \mathfrak{X}$ is a section of f given by an ideal $\mathscr{I} \subset \mathscr{O}_{\mathfrak{X}}$, then we have a canonical isomorphism*

$$\zeta'_{i,f} : \mathcal{N}_{\mathfrak{Y}/\mathfrak{X}}[-n] \otimes i^*\widehat{\omega}_{\mathfrak{X}/\mathfrak{Y}}[n] \simeq \mathscr{O}_{\mathfrak{Y}} \tag{7.4}$$

where $\widehat{\omega}_{\mathfrak{X}/\mathfrak{Y}} = \bigwedge^n \Omega^1_{\mathfrak{X}/\mathfrak{Y}}$ and $\mathcal{N}_{\mathfrak{Y}/\mathfrak{X}} = \bigwedge^n (i^*(\mathscr{I}/\mathscr{I}^2))^\vee = \mathcal{N}_f$.

Here is a description of the isomorphism (7.4) using local coordinates on $\mathfrak{X}$. Since $f : \mathfrak{X} \to \mathfrak{Y}$ is smooth of relative dimension n, for any $x \in \mathfrak{X}$ we can choose an open neighbourhood $\mathscr{U} \subset \mathfrak{X}$ containing x and pick n ordered sections, say, $x_1, x_2, \ldots, x_n$ of $\mathscr{O}_{\mathfrak{X}}$ over $\mathscr{U}$ which will be the local coordinates relative to f. Then $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ is a local generator of $\widehat{\omega}_{\mathfrak{X}/\mathfrak{Y}}$ over the open set $\mathscr{U}$. Now $i : \mathfrak{Y} \to \mathfrak{X}$ is a local complete intersection of codimension n, hence we can, by Proposition 7.1, ensure that $x_1, x_2, \ldots, x_n$ chosen above are also the local generators of the ideal sheaf $\mathscr{I}$ over the open set $\mathscr{U}$. Then $\overset{\vee}{x_1} \wedge \overset{\vee}{x_2} \wedge \cdots \wedge \overset{\vee}{x_n}$ generate $\mathcal{N}_{\mathfrak{Y}/\mathfrak{X}} = \bigwedge^n (i^*(\mathscr{I}_{\mathfrak{Y}}/\mathscr{I}^2_{\mathfrak{Y}}))^{\vee}$ over the open set $\mathscr{U}$. The morphism ζ' is now locally defined by the following map of generators.

$$\overset{\vee}{x_1} \wedge \overset{\vee}{x_2} \wedge \cdots \wedge \overset{\vee}{x_n} \otimes i^*(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n) \longmapsto 1 \tag{7.5}$$

This map is independent of the choice of generators $x_1, x_2, \ldots, x_n$. It then follows that this local definition globalizes. For a detailed proof of the Proposition 7.2 in the case of ordinary schemes, see [**Co**, Section 2.7].

We will now define $\phi_{f,u}$ for the composition of maps, $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{u} \mathfrak{X}$ described above. For any $\mathcal{G}^{\bullet} \in \mathbf{D}^{+}_{\mathrm{qct}}(\mathfrak{X})$ we look at the following isomorphism $\zeta'_{f,u}$ obtained via the composition of the following natural isomorphisms.

$$\begin{aligned} \zeta'_{f,u} : \mathcal{G}^{\bullet} &\xrightarrow{\ \sim\ } \mathcal{G}^{\bullet} \otimes_{\mathfrak{Y}} \mathscr{O}_{\mathfrak{Y}} \\ &\xrightarrow{(7.4)} \mathcal{G}^{\bullet} \otimes_{\mathfrak{Y}} (\mathcal{N}_f[-n] \otimes f^*\widehat{\omega}_u[n]) \\ &\xrightarrow{\ \sim\ } \mathbf{L}f^*u_t^*\mathcal{G}^{\bullet} \otimes f^*\widehat{\omega}_u[n] \otimes \mathcal{N}_f[-n] \\ &\xrightarrow{\ \sim\ } \mathbf{L}f^*(u_t^*\mathcal{G}^{\bullet} \otimes \widehat{\omega}_u[n]) \otimes \mathcal{N}_f[-n] \\ &\overset{\text{def.}}{=\!=} f^{\natural}u_t^{\mathrm{s}}\mathcal{G}^{\bullet} \end{aligned} \tag{7.6}$$

Using $\zeta'_{f,u}$ we define $\phi^{\natural}_{f,u} : f^{\natural}u_t^{\mathrm{s}} \xrightarrow{\ \sim\ } \mathbf{1}_{\mathfrak{X}}$ to be the inverse of the $\zeta'_{f,u}$. We define the abstract fundamental isomorphism $\phi^{\times} : f_t^{\times}u_t^{\mathrm{s}} \to \mathbf{1}_{\mathfrak{X}}$ via the following composition of isomorphisms, see (5.14).

$$f_t^{\times}u_t^{\mathrm{s}}\mathcal{F} \xrightarrow{\ \sim\ } f^{\natural}u_t^{\mathrm{s}}\mathcal{F} \xrightarrow{\phi^{\natural}} \mathcal{F} \tag{7.7}$$

Now we will check that, $\phi_{f,u}$ is compatible with the base change isomorphism (4.2) and the comparison isomorphism (3.2) defined earlier.

Proposition 7.3. *Consider the following fibered diagram of separated noetherian formal schemes*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "f'"] \arrow[d, "g"'] & \mathfrak{Y}' \arrow[r, "u'"] \arrow[d, "h"] & \mathfrak{X}' \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \arrow[r, "u"] & \mathfrak{X} \end{tikzcd} \tag{7.8}$$

where u and u' are smooth morphisms, $uf = 1_{\mathfrak{X}}$, $u'f' = 1_{\mathfrak{X}'}$ so that by Proposition 7.1, *f, f' are local complete intersection maps.*

(a) If g (hence h) is a pseudoproper morphism then the following diagram of isomorphism commutes for $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$.

$$\begin{tikzcd} f_t^{\times}u_t'^{\mathrm{s}}g_t^{\times}\mathcal{F} \arrow[r, "\beta"] \arrow[d, "\phi"'] & f_t'^{\times}h_t^{\times}u_t^{\mathrm{s}}\mathcal{F} \arrow[r, "c^{\times}"] & g_t^{\times}f_t^{\times}u_t^{\mathrm{s}}\mathcal{F} \arrow[d, "\phi"] \\ 1_{\mathbf{D}(\mathfrak{X}')}g_t^{\times}\mathcal{F} \arrow[r, equal] & g_t^{\times}\mathcal{F} \arrow[r, equal] & g_t^{\times}\mathcal{F} \end{tikzcd} \tag{7.9}$$

(b) If g (hence h) is a smooth morphism, then the following diagram of isomorphisms commutes for $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$.

$$\begin{tikzcd} f_t'^{\times}u_t'^{\mathrm{s}}g_t^{\mathrm{s}}\mathcal{F} \arrow[r, "c^{\mathrm{s}}"] \arrow[d, "\phi"'] & f_t'^{\times}h_t^{\mathrm{s}}u_t^{\mathrm{s}}\mathcal{F} \arrow[r, "\beta^{\natural}"] & g_t^{\mathrm{s}}f_t^{\times}u_t^{\mathrm{s}}\mathcal{F} \arrow[d, "\phi"] \\ 1_{\mathbf{D}(\mathfrak{X}')}g_t^{\mathrm{s}}\mathcal{F} \arrow[r, equal] & g_t^{\mathrm{s}}\mathcal{F} \arrow[r, equal] & g_t^{\mathrm{s}}\mathcal{F} \end{tikzcd} \tag{7.10}$$

Proof. For part (a) we use the isomorphism (5.14) in the diagram (7.9) and obtain the following diagram.

$$\begin{tikzcd} f_t^{\times}u_t'^{\mathrm{s}}g_t^{\times}\mathcal{F} \arrow[r, "\beta_{u,g}"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\clubsuit"] & f_t'^{\times}h_t^{\times}u_t^{\mathrm{s}}\mathcal{F} \arrow[r, "c^{\times}"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\spadesuit"] & g_t^{\times}f_t^{\times}u_t^{\mathrm{s}}\mathcal{F} \arrow[d, "\eta"] \\ f'^{\natural}u_t'^{\mathrm{s}}g_t^{\times}\mathcal{F} \arrow[r, "\beta^{\mathrm{s}}"] \arrow[d, "\phi"'] & f'^{\natural}h_t^{\times}u_t^{\mathrm{s}}\mathcal{F} \arrow[r, "\beta^{\natural}"] \arrow[d, phantom, "1"] & g_t^{\times}f^{\natural}u_t^{\mathrm{s}}\mathcal{F} \arrow[d, "\phi"] \\ 1_{\mathbf{D}(\mathfrak{X}')}g_t^{\times}\mathcal{F} \arrow[r, equal] & g_t^{\times}\mathcal{F} \arrow[r, equal] & g_t^{\times}\mathcal{F} \end{tikzcd} \tag{7.11}$$

The diagram ♣ commutes via (5.20) and the diagram ♠ commutes via (6.7). We will now prove that the diagram 1 is commutative. We expand it below. As before we use $f^* = \mathbf{L}f^*$, $f'^* = \mathbf{L}f'^*$, $\widehat{\omega}^{\bullet} = \widehat{\omega}[n]$ and $\mathcal{N}^{\bullet} = \mathcal{N}[-n]$, where n is the relative dimension of

g.

$$\begin{tikzcd}[column sep=small]
f'^{\natural}u'^{\mathrm{s}}_t g^{\times}_t\mathcal{F} \arrow[r, "f'^{\natural}\beta^{\mathrm{s},g}"] \arrow[d, equal] & f'^{\natural}h^{\times}_t u^{\mathrm{s}}_t\mathcal{F} \arrow[r, "\beta'_{f,h}(u^{\mathrm{s}}_t)"] \arrow[d, equal] & g^{\times}_t f^{\natural}u^{\mathrm{s}}_t\mathcal{F} \arrow[d, equal] \\
f'^*(u'^*_t g^{\times}_t\mathcal{F}\otimes\widehat{\omega}^{\bullet}_{u'})\otimes\mathcal{N}^{\bullet}_{f'} \arrow[r, "f'^{\natural}\beta^{\mathrm{s}}_{u,g}"] \arrow[dd, "\wr"] & f'^*h^{\times}_t(u^*_t\mathcal{F}\otimes\widehat{\omega}^{\bullet}_u)\otimes\mathcal{N}^{\bullet}_{f'} \arrow[r, "\beta'_{f,h}(u^{\mathrm{s}}_t)"] \arrow[d, "\wr"] & g^{\times}_t(f^*(u^*_t\mathcal{F}\otimes\widehat{\omega}^{\bullet}_u)\otimes\mathcal{N}^{\bullet}_f) \arrow[d, "\wr"] \\
 & f'^*h^{\times}_t u^*_t\mathcal{F}\otimes f'^*h^*\widehat{\omega}^{\bullet}_u\otimes\mathcal{N}^{\bullet}_{f'} \arrow[d, "\wr"] & g^{\times}_t(f^*u^*_t\mathcal{F}\otimes f^*\widehat{\omega}^{\bullet}_u\otimes\mathcal{N}^{\bullet}_f) \arrow[d, "pr^{\times}"', "\wr"] \\
f'^*u'^*_t g^{\times}_t\mathcal{F}\otimes f'^*\widehat{\omega}^{\bullet}_{u'}\otimes\mathcal{N}^{\bullet}_{f'} \arrow[r] \arrow[d, "\zeta'_{f',u'}"] & f'^*h^{\times}_t u^*_t\mathcal{F}\otimes f'^*\widehat{\omega}^{\bullet}_{u'}\otimes\mathcal{N}^{\bullet}_{f'} \arrow[r] \arrow[d, "\zeta'_{f',u'}"'] & g^{\times}_t f^*u^*_t\mathcal{F}\otimes g^*(f^*\widehat{\omega}^{\bullet}_u\otimes\mathcal{N}^{\bullet}_f) \arrow[d, "\zeta'_{f,u}"] \\
f'^*u'^*_t g^{\times}_t\mathcal{F}\otimes\mathscr{O}_{\mathfrak{X}'} \arrow[r, "f'^*\beta_{u,g}"] \arrow[d, "\wr"] & f'^*h^{\times}_t u^*_t\mathcal{F}\otimes\mathscr{O}_{\mathfrak{X}'} \arrow[r, "\beta_{f,h}(u^{\mathrm{s}}_t)"] \arrow[d, "\wr"] & g^{\times}_t f^*u^*_t\mathcal{F}\otimes g^*\mathscr{O}_{\mathfrak{X}} \arrow[d, "\wr"] \\
f'^*u'^*_t g^{\times}_t\mathcal{F} \arrow[r, "f'^*\beta_{u,g}"] \arrow[d, "u'f'=1"] & f'^*h^{\times}_t u^*_t\mathcal{F} \arrow[r, "\beta_{f,h}(u^{\mathrm{s}}_t)"] \arrow[d, phantom, "2"] & g^{\times}_t f^*u^*_t\mathcal{F} \arrow[d, "uf=1"] \\
1_{\mathfrak{X}'}g^{\times}_t\mathcal{F} \arrow[r, equal] & g^{\times}_t\mathcal{F} \arrow[r, equal] & g^{\times}_t 1_{\mathfrak{X}}\mathcal{F}
\end{tikzcd} \tag{7.12}$$

In the above diagram the commutativity of all the unlabelled subdiagrams is straightforward to check. We only need to show the commutativity of diagram 2. By adjointness of g_* to $g^{\times}_t$, we reduce to proving the commutativity of the following diagram.

$$\begin{tikzcd}
 & g_*f'^*u'^*_t g^{\times}_t\mathcal{F} \arrow[r, "\beta_{u,g}"] \arrow[d, "\beta_{f,g}"] \arrow[dl] \arrow[dr, phantom, "\clubsuit_1"] & g_*f'^*h^{\times}_t u^*_t\mathcal{F} \arrow[d, "\beta_{f,g}"] \\
 & f^*h_*u'^*_t g^{\times}_t\mathcal{F} \arrow[r, "\beta_{u,g}"] \arrow[d, "\beta_{u,g}"] & f^*h_*h^{\times}_t u^*_t\mathcal{F} \arrow[dd, "\tau_h"] \\
g_*g^{\times}_t\mathcal{F} \arrow[drr] & f^*u^*_t g_*g^{\times}_t\mathcal{F} \arrow[l] \arrow[d, "\tau_g"] \arrow[r, phantom, "\clubsuit_2"] & {} \\
 & f^*u^*_t\mathcal{F} \arrow[r, equal] \arrow[d, "uf=1"] & f^*u^*_t\mathcal{F} \arrow[d] \\
 & \mathcal{F} \arrow[r, equal] & \mathcal{F}
\end{tikzcd} \tag{7.13}$$

In the diagram (7.13) the subdiagram $\clubsuit_1$ commutes functorially and the diagram $\clubsuit_2$ commutes via the definitions of smooth-base-change isomorphisms.

For part (b), using the Proposition 7.1 we expand the diagram (7.10) as follows.

$$\begin{tikzcd}
{f'}_t^{\times} u_t^{\mathrm{s}} g_t^{\mathrm{s}} \mathcal{F} \arrow[r, "c^{\mathrm{s}}"] \arrow[d] & {f'}_t^{\times} {g'}_t^{\mathrm{s}} u_t^{\mathrm{s}} \mathcal{F} \arrow[r, "\beta_{f,g'}"] \arrow[d] & g_t^{\mathrm{s}} f_t^{\times} u_t^{\mathrm{s}} \mathcal{F} \arrow[d] \\
f'^{\natural} u_t^{\mathrm{s}} g_t^{\mathrm{s}} \mathcal{F} \arrow[r, "c^{\mathrm{s}}"] \arrow[d] & f'^{\natural} {g'}_t^{\mathrm{s}} u_t^{\mathrm{s}} \mathcal{F} \arrow[r, "\beta^{\natural}_{f,g'}"] & g_t^{\mathrm{s}} f^{\natural} u_t^{\mathrm{s}} \mathcal{F} \arrow[d] \\
1_{\mathfrak{X}'} g_t^{\mathrm{s}} \mathcal{F} \arrow[r, equal] & g_t^{\mathrm{s}} \mathcal{F} \arrow[r, equal] & g_t^{\mathrm{s}} 1_{\mathfrak{X}} \mathcal{F}
\end{tikzcd} \tag{7.14}$$

Here again the top left diagram commutes functorially and the top right diagram commutes by Proposition 5.2 and expanding the bottom diagram using the definitions of the functors involved we obtain the diagram (7.15).

The commutativity of (7.15) is straight-forward from the definition of the labelled maps and the comparison maps for the pseudofunctor $(-)_t^*$. $\square$

$$
\begin{tikzcd}
f'^{\natural}u_t'^{\mathrm{s}}g_t^{\mathrm{s}}\mathcal{F} \arrow[r, "c^{\mathrm{s}}"] \arrow[d, equal] & f'^{\natural}g_t'^{\mathrm{s}}u_t^{\mathrm{s}}\mathcal{F} \arrow[r, "\beta^{\natural}"] \arrow[d, equal] & g_t^{\mathrm{s}}f^{\natural}u_t^{\mathrm{s}}\mathcal{F} \arrow[d, equal] \\
f'^*(u_t'^*(g_t^*\mathcal{F}\otimes\widehat{\omega}_g^\bullet)\otimes\widehat{\omega}_{u'}^\bullet)\otimes\mathcal{N}_{f'}^\bullet \arrow[r, "c^s"] \arrow[d, "\wr"] & f'^*(g_t'^*(u_t^*\mathcal{F}\otimes\widehat{\omega}_u^\bullet)\otimes\widehat{\omega}_{g'}^\bullet)\otimes\mathcal{N}_{f'}^\bullet \arrow[r, "\beta^{\natural}"] \arrow[d, "\wr"] & g_t^*(f^*(u_t^*\mathcal{F}\otimes\widehat{\omega}_u^\bullet)\otimes\mathcal{N}_f^\bullet)\otimes\widehat{\omega}_g^\bullet \arrow[d, "\wr"] \\
f'^*u_t'^*g_t^*\mathcal{F}\otimes f'^*u^*\widehat{\omega}_g^\bullet\otimes f'^*\widehat{\omega}_{u'}^\bullet\otimes\mathcal{N}_{f'}^\bullet \arrow[r, "c^*"] \arrow[dd, "\zeta'_{f',u'}"'] & f'^*g_t'^*u_t^*\mathcal{F}\otimes f'^*g'^*\widehat{\omega}_u^\bullet\otimes f'^*\widehat{\omega}_{g'}^\bullet\otimes\mathcal{N}_{f'}^\bullet \arrow[r, "c^*"] \arrow[d, "\wr"] & g_t^*f^*u_t^*\mathcal{F}\otimes g^*(f^*\widehat{\omega}_u^\bullet\otimes\mathcal{N}_f^\bullet)\otimes\widehat{\omega}_g^\bullet \arrow[dd, "\zeta'_{f,u}"] \\
 & f'^*g_t'^*u_t^*\mathcal{F}\otimes f'^*\widehat{\omega}_{u'}^\bullet\otimes\mathcal{N}_{f'}^\bullet\otimes\widehat{\omega}_g^\bullet \arrow[d, "\zeta'_{f',u'}"] & \\
f'^*u_t'^*g_t^*\mathcal{F}\otimes\mathscr{O}_{\mathfrak{X}'}\otimes\widehat{\omega}_g^\bullet \arrow[r, "c^*"] \arrow[d, "\wr"] & f'^*g_t'^*u_t^*\mathcal{F}\otimes\mathscr{O}_{\mathfrak{X}'}\otimes\widehat{\omega}_g^\bullet \arrow[r, "c^*"] \arrow[d, "\wr"] & g_t^*f^*u_t^*\mathcal{F}\otimes g^*\mathscr{O}_{\mathfrak{X}}\otimes\widehat{\omega}_g^\bullet \arrow[d, "\wr"] \\
f'^*u_t'^*g_t^*\mathcal{F}\otimes\widehat{\omega}_g^\bullet \arrow[r, "c^*"] \arrow[d, "u'f'=1"'] & f'^*g_t'^*u_t^*\mathcal{F}\otimes\widehat{\omega}_g^\bullet \arrow[r, "c^*"] & g_t^*f^*u_t^*\mathcal{F}\otimes\widehat{\omega}_g^\bullet \arrow[d, "uf=1"] \\
1_{\mathfrak{X}'}g_t^*\mathcal{F}\otimes\widehat{\omega}_g^\bullet \arrow[d] & & g_t^*1_{\mathfrak{X}}\mathcal{F}\otimes\widehat{\omega}_g^\bullet \arrow[d] \\
g_t^*\mathcal{F}\otimes\widehat{\omega}_g^\bullet \arrow[r, equal] & g_t^{\mathrm{s}}\mathcal{F} \arrow[r, equal] & g_t^*\mathcal{F}\otimes\widehat{\omega}_g^\bullet
\end{tikzcd}
\tag{7.15}
$$

Proposition 7.4. *For the following diagram of separated noetherian formal schemes,*

$$\begin{tikzcd}
\mathfrak{X} \arrow[d, "f"] & & \\
\mathfrak{Y} \arrow[r, "v"] \arrow[d, "g"] & \mathfrak{X} \arrow[d, "h"] & \\
\mathfrak{Z} \arrow[r, "u"] & \mathfrak{W} \arrow[r, "w"] & \mathfrak{X}
\end{tikzcd} \tag{7.16}$$

where the middle square is fibered, f, g, h are pseudoproper and u, v, w are smooth maps, such that $uf = 1 = wh$ (hence, $(gf)(wu) = 1$), the following diagram of isomorphisms commutes for $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})$.

$$\begin{tikzcd}
f_t^{\times} g_t^{\times} u_t^{\mathrm{s}} w_t^{\mathrm{s}} \mathcal{F} \arrow[r] \arrow[d] & (gf)_t^{\times} (wu)_t^{\mathrm{s}} \mathcal{F} \arrow[d] \\
f_t^{\times} v_t^{\mathrm{s}} h_t^{\times} w_t^{\mathrm{s}} \mathcal{F} \arrow[r] & \mathcal{F}
\end{tikzcd} \tag{7.17}$$

Proof. Here again we will use Proposition 7.1 and the isomorphism η of (5.14) to obtain the following diagram.

$$\begin{tikzcd}
f_t^{\times} g_t^{\times} u_t^{\mathrm{s}} w_t^{\mathrm{s}} \mathcal{F} \arrow[rrrr, "c^{\times}(c^{\mathrm{s}})"] \arrow[dr, "\eta"] \arrow[ddd, "f_t^{\times}(\beta) w_t^{\mathrm{s}}"'] & & & & (gf)_t^{\times} (wu)_t^{\mathrm{s}} \mathcal{F} \arrow[dl, "\eta"'] \arrow[ddd, dashed, "\phi^{\times}_{vf,wh}"] \\
& f^{\natural} g^{\natural} u_t^{\mathrm{s}} w_t^{\mathrm{s}} \mathcal{F} \arrow[rr, "c^{\natural}(c^{\mathrm{s}})"] \arrow[d, "f^{\natural}(\beta) w_t^{\mathrm{s}}"'] \arrow[drr, phantom, "\clubsuit"] & & (gf)^{\natural} (wu)_t^{\mathrm{s}} \mathcal{F} \arrow[d, "\phi_{vf,wh}"] & \\
& f^{\natural} v_t^{\mathrm{s}} h^{\natural} w_t^{\mathrm{s}} \mathcal{F} \arrow[rr, "\phi_{f,v}(\phi_{h,w})"] & & 1_{\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})} \mathcal{F} \arrow[dr, equal] & \\
f_t^{\times} v_t^{\mathrm{s}} h_t^{\times} w_t^{\mathrm{s}} \mathcal{F} \arrow[ur, "\eta"] \arrow[rrrr, dashed, "\phi^{\times}_{f,v}(\phi^{\times}_{h,w})"] & & & & 1_{\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X})} \mathcal{F}
\end{tikzcd} \tag{7.18}$$

The commutativity of the trapezium on the left follows from the commutativity of the diagram (5.20). The commutativity of the trapezium on the top follows from the commutativity of the diagram (5.16). The trapezium in the bottom commutes via the definition of $\phi^{\times}_{f,v}$ and $\phi^{\times}_{h,w}$. The trapezium on the right commutes via the definition of $\phi^{\times}_{vf,wh}$. We will now prove the commutativity of the diagram $\clubsuit$.

Using the definition of the functors involved in the middle square of the diagram (7.18) we get the diagram (7.19).

$$
\begin{tikzcd}
f^*(g^*(u_t^*(w_t^*\mathcal{F} \otimes \widehat{\omega}_w^\bullet) \otimes \widehat{\omega}_u^\bullet) \otimes \mathcal{N}_g^\bullet) \otimes \mathcal{N}_f^\bullet \arrow[r] \arrow[d] & f^*g^*((wu)_t^*\mathcal{F} \otimes u^*\widehat{\omega}_w^\bullet \otimes \widehat{\omega}_w^\bullet) \otimes f^*\mathcal{N}_g^\bullet \otimes \mathcal{N}_f^\bullet \arrow[r] & (gf)^*((wu)_t^*\mathcal{F} \otimes \widehat{\omega}_{wu}^\bullet) \otimes \mathcal{N}_{gf}^\bullet \arrow[d] \\
f^*g^*u_t^*w_t^*\mathcal{F} \otimes f^*g^*(u^*\widehat{\omega}_w^\bullet \otimes \widehat{\omega}_u^\bullet) \otimes f^*\mathcal{N}_g^\bullet \otimes \mathcal{N}_f^\bullet \arrow[rr] \arrow[d] & & (gf)^*(wu)_t^*\mathcal{F} \otimes (gf)^*\widehat{\omega}_{wu}^\bullet \otimes \mathcal{N}_{gf}^\bullet \arrow[d] \\
f^*g^*u_t^*w_t^*\mathcal{F} \otimes f^*g^*u^*\widehat{\omega}_w^\bullet \otimes f^*g^*\widehat{\omega}_u^\bullet \otimes f^*\mathcal{N}_g^\bullet \otimes \mathcal{N}_f^\bullet \arrow[d] & & (gf)^*(wu)_t^*\mathcal{F} \otimes ((gf)^*\widehat{\omega}_{wu}^\bullet \otimes \mathcal{N}_{gf}^\bullet) \arrow[ddd] \\
f^*v_t^*h^*w_t^*\mathcal{F} \otimes f^*v^*h^*\widehat{\omega}_w^\bullet \otimes f^*\widehat{\omega}_v^\bullet \otimes f^*v^*\mathcal{N}_h^\bullet \otimes \mathcal{N}_f^\bullet \arrow[d] & & \\
(vf)_t^*(wh)_t^*\mathcal{F} \otimes (vf)^*h^*\widehat{\omega}_w^\bullet \otimes (vf)^*\mathcal{N}_h^\bullet \otimes f^*\widehat{\omega}_v^\bullet \otimes \mathcal{N}_f^\bullet \arrow[d] & & \\
\mathcal{F} \otimes h^*\widehat{\omega}_w^\bullet \otimes \mathcal{N}_h^\bullet \otimes f^*\widehat{\omega}_v^\bullet \otimes \mathcal{N}_f^\bullet \arrow[d] & & \mathcal{F} \otimes (gf)^*\widehat{\omega}_{wu}^\bullet \otimes \mathcal{N}_{gf}^\bullet \arrow[d] \\
\mathcal{F} \otimes \mathscr{O}_{\mathfrak{X}} \otimes \mathscr{O}_{\mathfrak{X}} \arrow[rr] & & \mathcal{F} \otimes \mathscr{O}_{\mathfrak{X}}
\end{tikzcd}
\tag{7.19}
$$

The commutativity of diagram (7.19) reduces to showing commutativity of the following diagram of differentials.

$$
\begin{tikzcd}
f^*g^*(u^*\widehat{\omega}^\bullet_w \otimes \widehat{\omega}^\bullet_u)\otimes f^*\mathcal{N}^\bullet_g\otimes\mathcal{N}^\bullet_f \arrow[r] \arrow[d, "\wr"] & f^*g^*\widehat{\omega}^\bullet_{wu}\otimes\mathcal{N}^\bullet_{gf} \arrow[r, "c^*"] & (gf)^*\widehat{\omega}^\bullet_{wu}\otimes\mathcal{N}^\bullet_{gf} \arrow[ddddddd, "\zeta'_{gf,wu}"] \\
f^*g^*u^*\widehat{\omega}^\bullet_w \otimes f^*\widehat{\omega}^\bullet_v \otimes f^*v^*\mathcal{N}^\bullet_h\otimes \mathcal{N}^\bullet_f \arrow[d, "c^*"', "\wr"] & & \\
f^*v^*h^*\widehat{\omega}^\bullet_w \otimes f^*v^*\mathcal{N}^\bullet_h \otimes f^*\widehat{\omega}^\bullet_v\otimes\mathcal{N}^\bullet_f \arrow[d, "\wr"] & \clubsuit & \\
f^*v^*(h^*\widehat{\omega}^\bullet_w\otimes \mathcal{N}^\bullet_h)\otimes f^*\widehat{\omega}^\bullet_v\otimes\mathcal{N}^\bullet_f \arrow[d, "vf=1"', "\wr"] & & \\
h^*\widehat{\omega}^\bullet_w\otimes\mathcal{N}^\bullet_h\otimes f^*\widehat{\omega}^\bullet_v\otimes\mathcal{N}^\bullet_f \arrow[d, "\zeta'_{h,w}"', "\wr"] & & \\
\mathscr{O}_{\mathfrak{X}}\otimes f^*\widehat{\omega}^\bullet_v\otimes\mathcal{N}^\bullet_f \arrow[d, "\wr"] & & \\
f^*\widehat{\omega}^\bullet_v\otimes\mathcal{N}^\bullet_f \arrow[d, "\zeta'_{f,v}"', "\wr"] & & \\
\mathscr{O}_{\mathfrak{X}} \arrow[rr, equal] & & \mathscr{O}_{\mathfrak{X}}
\end{tikzcd}
\tag{7.20}
$$

The diagram (7.20) is a statement on sheaves and its commutativity can be checked over local rings. Let $x \in \mathfrak{X}$ and let $y = v^{-1}(x) = f(x), z = (wu)^{-1}(x) = (gf)(x)$ and $t = w^{-1}(x) = h(x)$ be its pre-images in $\mathfrak{Y}, \mathfrak{Z}$ and $\mathfrak{W}$ respectively.

We can choose $t_1, \ldots, t_m$ to be the ordered sections in $\mathscr{O}_{\mathfrak{W},t}$ as the local coordinates of w around t and $z_1, \ldots, z_n$ as local coordinates of u around z in $\mathscr{O}_{\mathfrak{Z},z}$. Moreover, we can choose $\{t_i\}$ in such a way that these are also the local generators of ideal sheaves $\mathscr{I}_h$, where $\mathscr{I}_h$ is the ideal sheaf for the local complete intersection h. We first observe the following.

1. The sections $\{v^*t_i\}$ are also local generators of $\mathscr{I}_g$, the ideal sheaf of the local complete immersion g.

2. The sections $\{g^*z_j\}$ are local coordinates of v around y in $\mathscr{O}_{\mathfrak{Y}}$ and can be chosen in such a way that they are also the local generators of the ideal sheaf $\mathscr{I}_f$, the

ideal sheaf of the local complete intersection f. We will abuse of notation slightly here by calling $\{z_j\}$ as the local coordinates of v around y in $\mathfrak{Y}$.

3. By abuse of notation, $\{t_1, \ldots, t_m, z_1, \ldots, z_n\}$ can be chosen as the local coordinates of (wu) around the point z such that they are also the local generators of $\mathscr{I}_{gf}$, the ideal sheaf for the local complete intersection (gf).

In light of observations (1)-(3) above, we have the following.

$$\begin{aligned}
dt_1 \wedge \cdots \wedge dt_m &\text{ generates } \widehat{\omega}^{\bullet}_w \\
dz_1 \wedge \cdots \wedge dz_n &\text{ generates } \widehat{\omega}^{\bullet}_u \\
dz_1 \wedge \cdots \wedge dz_n &\text{ generates } \widehat{\omega}^{\bullet}_v \\
dz_1 \wedge \cdots \wedge dz_n \wedge dt_1 \wedge \cdots \wedge dt_m &\text{ generates } \widehat{\omega}^{\bullet}_{wu} \\
v^*(\overset{\vee}{z_1} \wedge \cdots \wedge \overset{\vee}{z_n}) &\text{ generates } \mathcal{N}^{\bullet}_f \\
u^*(\overset{\vee}{t_1} \wedge \cdots \wedge \overset{\vee}{t_m}) &\text{ generates } \mathcal{N}^{\bullet}_g \\
\overset{\vee}{t_1} \wedge \cdots \wedge \overset{\vee}{t_m} &\text{ generates } \mathcal{N}^{\bullet}_h \\
\overset{\vee}{z_1} \wedge \cdots \wedge \overset{\vee}{z_n} \wedge \overset{\vee}{t_1} \wedge \cdots \wedge \overset{\vee}{t_m} &\text{ generates } \mathcal{N}^{\bullet}_{gf}
\end{aligned}$$

The commutativity of the diagram (7.20) now follows from the local description of the maps $\zeta'_{h,w}, \zeta'_{f,v}$ and $\zeta'_{gf,wu}$ given below.

$$\begin{aligned}
\zeta'_{h,w}(dt_1 \wedge \cdots \wedge dt_m \otimes \overset{\vee}{t_1} \wedge \cdots \wedge \overset{\vee}{t_m}) &= 1 \\
\zeta'_{f,v}(dz_1 \wedge \cdots \wedge dz_n \otimes \overset{\vee}{z_1} \wedge \cdots \wedge \overset{\vee}{z_n}) &= 1 \\
\zeta'_{gf,wu}(dz_1 \wedge \cdots \wedge dz_n \wedge dt_1 \wedge \cdots \wedge dt_m \otimes \overset{\vee}{z_1} \wedge \cdots \wedge \overset{\vee}{z_n} \wedge \overset{\vee}{t_1} \wedge \cdots \wedge \overset{\vee}{t_m}) &= (-1)^{mn} 1
\end{aligned} \tag{7.21}$$

The $(-1)^{mn}$ in (7.21) is on account of the sign change encountered in isomorphism (second vertical map on the left)

$$f^*g^*u^*\widehat{\omega}^{\bullet}_w \otimes f^*\widehat{\omega}^{\bullet}_v \otimes f^*v^*\mathcal{N}^{\bullet}_h \otimes \mathcal{N}^{\bullet}_f \xrightarrow{\sim} f^*v^*h^*\widehat{\omega}^{\bullet}_w \otimes f^*v^*\mathcal{N}^{\bullet}_h \otimes f^*\widehat{\omega}^{\bullet}_v \otimes \mathcal{N}^{\bullet}_f$$

in the diagram (7.20), since using the local description, the isomorphism

$$f^*v^*\mathcal{N}^{\bullet}_h \otimes f^*\widehat{\omega}^{\bullet}_v \xrightarrow{\sim} (-1)^{mn} f^*\widehat{\omega}^{\bullet}_v \otimes f^*v^*\mathcal{N}^{\bullet}_h$$

has the sign $(-1)^{mn}$ via the properties of wedge product. $\square$

Chapter 8

The Output

In section 3 we defined the functor $(-)^{\mathrm{s}}_t$ and $(-)^{\times}_t$ for smooth and pseudoproper morphisms of noetherian formal schemes respectively and proved that these are in fact a $\mathbf{D}^+_{\mathrm{qct}}$-valued pseudofunctors on the respective subcategories, see Proposition 3.1 and diagram (3.3). We then gave a base change map for smooth-proper squares of noetherian formal schemes and proved that this map is an isomorphism and is transitive for vertical and horizontal extensions, see Propositions 4.1 and 4.2. We proved the fundamental local isomorphism in the formal case for maps factoring identity in and showed that this isomorphism is compatible with extensions by smooth and pseudoproper morphisms, see Proposition 7.3. Thus we have established all the input conditions stated in [**Nk**, Section 2.1] required to glue two pseudofunctors, see [**Nk**, Theorem 7.1.3]. And as a result we have the following output.

Theorem 8.1. *Let $\mathcal{C}$ be the category of noetherian formal schemes whose morphisms are composites of separated smooth morphisms and pseudoproper morphisms. Then on $\mathcal{C}$, we have a $\mathbf{D}^+_{\mathrm{qct}}$-valued pseudofunctor $(-)^!$, with the following properties.*

(a) If $f : \mathfrak{X} \to \mathfrak{Y}$ is a separated smooth morphism of relative dimension n, then

$$f^!(\mathcal{G}) = \boldsymbol{R}\varGamma'_{\mathfrak{X}}(f^*(\mathcal{G}) \otimes \widehat{\omega}_f[n]).$$

(b) If $f : \mathfrak{X} \to \mathfrak{Y}$ is a pseudoproper morphism, then $f^! \simeq f^{\times}_t$, where $f^{\times}_t$ is the $\triangle$-functorial right adjoint of $\boldsymbol{R}f_*$.

In [**Nk**], $f^!$ was defined for the composites of *etale* and *pseudoproper* maps of noetherian formal schemes, so the definition of $f^!$ which we have obtained, extends the previous definition of $f^!$ to the composites of *smooth* and *pseudoproper* maps.

Moreover, glueing the pseudofunctors for smooth and pseudoproper also generalizes the two classical approaches to constuct $f^!$ mentioned in the introduction.

8.0 Extension to Non-torsion version

We will now introduce the derived category $\tilde{\mathbf{D}}(-)$ and its associated derived subcategories $\tilde{\mathbf{D}}_{\mathrm{qc}}(-)$ and $\tilde{\mathbf{D}}^+_{\mathrm{qc}}(-)$ and state a *non-torsion* version of Theorem 8.1. The category $\tilde{\mathbf{D}}_{\mathrm{qc}}(-)$, contains $\mathbf{D}_{\mathrm{qc}}(-)$ and hence $\mathbf{D}_{\mathrm{qct}}(-)$ as a full subcategory and the functor $\mathbf{R}\varGamma'_{(-)}$ maps $\tilde{\mathbf{D}}_{\mathrm{qc}}(-)$ inside $\mathbf{D}_{\mathrm{qct}}(-)$. In fact $\mathbf{R}\varGamma'_{(-)} : \tilde{\mathbf{D}}_{\mathrm{qc}} \to \mathbf{D}_{\mathrm{qct}}(-)$ has a $\vartriangle$-functorial right adjoint Λ. By using $f^!$ together with these functors we can define a functor $(-)^{\tilde{!}}$, which would have $\tilde{\mathbf{D}}_{\mathrm{qc}}(-)$ as its source category. As explained below, $(-)^{\tilde{!}}$ turns out to be a pre-pseudofunctor (see [**NS**]) instead of a pseudofunctor on $\tilde{\mathbf{D}}_{\mathrm{qc}}(-)$. However, its restriction to $\mathbf{D}_{\mathrm{c}}(-)$ is a pseudofunctor.

For a locally noetherian formal scheme $\mathfrak{X}$, we define

$$\tilde{\mathbf{D}}_{\mathrm{qc}}(\mathfrak{X}) := \mathbf{R}\varGamma'^{-1}_{\mathfrak{X}}(\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})) \tag{8.1}$$

to be the full subcategory of $\mathbf{D}(\mathfrak{X})$ whose objects are complexes $\mathcal{F}$, such that $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$. Also, $\tilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathfrak{X}) :\ = \tilde{\mathbf{D}}_{\mathrm{qc}}(\mathfrak{X}) \cap \mathbf{D}^+(\mathfrak{X})$.

It is immediate from the definition that $\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X}) \subset \mathbf{D}^+_{\mathrm{qc}}(\mathfrak{X}) \subset \tilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathfrak{X})$. We refer the reader to [**AJL**, section 5] for a detailed treatment of the torsion functor $\varGamma'_{\mathfrak{X}}$ and its properties. We recall a few results involving the torsion functor here. The functor $\mathbf{R}\varGamma'_{\mathfrak{X}} : \mathbf{D}(\mathfrak{X}) \to \mathbf{D}(\mathfrak{X})$ has a $\vartriangle$-functorial right adjoint given by

$$\Lambda_{\mathfrak{X}} := \mathbf{R}\mathrm{Hom}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}, -). \tag{8.2}$$

Also, we have natural morphisms of functors,

$$\mathbf{R}\varGamma'_{\mathfrak{X}} \to 1 \to \Lambda_{\mathfrak{X}}. \tag{8.3}$$

Via the above morphisms, the functors $\mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}}$ and $\Lambda_{\mathfrak{X}}$ are idempotent, and in fact we have the following isomorphisms.

$$\begin{aligned}\mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}}\mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}} \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}} \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}}\Lambda_{\mathfrak{X}} \\ \Lambda_{\mathfrak{X}}\mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}} \xrightarrow{\sim} \Lambda_{\mathfrak{X}} \xrightarrow{\sim} \Lambda_{\mathfrak{X}}\Lambda_{\mathfrak{X}}\end{aligned} \tag{8.4}$$

Nayak and Sastry in [**NS**], construct a duality pre-pseudofunctor $(-)^\sharp$ over $\tilde{\mathbf{D}}^+_{\mathrm{qc}}(-)$. Roughly, a *pre-pseudofunctor* $(-)^\sharp$ satisfies all but one condition of being a pseudofunctor, namely, it is no longer required that $(-)^\sharp$ is isomorphic to the identity functor for identity morphisms and instead one only gets a morphism to the identity functor. More precisely, for every $\mathfrak{X}$ there exists a natural map $1^\sharp_{\mathfrak{X}} \to 1$, so that for any morphism $f : \mathfrak{X} \to \mathfrak{Y}$, the canonical maps $f^\sharp 1^\sharp_{\mathfrak{Y}} \to f^\sharp$ and $1^\sharp_{\mathfrak{X}} f^\sharp \to f^\sharp$ are the comparison isomorphisms.

Let us now construct an analogue of the pseudofunctor $(-)^!$ defined in Theorem 8.1 to obtain a $\tilde{\mathbf{D}}^+_{\mathrm{qc}}(-)$-valued *pre-pseudofunctor* $(-)^{\tilde{!}}$.

For a finite-type separated morphism $f : \mathfrak{X} \to \mathfrak{Y}$, consider the diagram below.

$$\begin{tikzcd} \mathbf{R}f_* : \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{X}) \arrow[r] \arrow[d, hook] & \mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y}) \arrow[d, hook] \\ \mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}} : \tilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathfrak{X}) \arrow[r] \arrow[ur, dashed] & \tilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathfrak{Y}) \end{tikzcd} \tag{8.5}$$

The dotted arrow indicates that the actual target where the image of $\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}}$ lies is the full subcategory $\mathbf{D}^+_{\mathrm{qct}}(\mathfrak{Y})$ of $\tilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathfrak{Y})$. We need to make a small comment here regarding [**NS**, equation (1.1.1)], which says that for any map $f : \mathfrak{X} \to \mathfrak{Y}$, the following are isomorphisms.

$$\mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}}\mathbf{L}f^*\mathbf{R}\mathit{\Gamma}'_{\mathfrak{Y}} \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}}\mathbf{L}f^* \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}}\mathbf{L}f^*\Lambda_{\mathfrak{Y}} \tag{8.6}$$

$$\Lambda_{\mathfrak{X}}\mathbf{L}f^*\mathbf{R}\mathit{\Gamma}'_{\mathfrak{Y}} \xrightarrow{\sim} \Lambda_{\mathfrak{X}}\mathbf{L}f^* \xrightarrow{\sim} \Lambda_{\mathfrak{X}}\mathbf{L}f^*\Lambda_{\mathfrak{Y}} \tag{8.7}$$

We would like to add that for a smooth map $u : \mathfrak{X} \to \mathfrak{Y}$ of noetherian formal schemes, in light of (10.2), the following are isomorphisms too.

$$\mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}} u^{\mathrm{s}}_t \mathbf{R}\mathit{\Gamma}'_{\mathfrak{Y}} \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}} u^{\mathrm{s}}_t \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}} u^{\mathrm{s}}_t \Lambda_{\mathfrak{Y}}$$
$$\Lambda_{\mathfrak{X}} u^{\mathrm{s}}_t \mathbf{R}\mathit{\Gamma}'_{\mathfrak{Y}} \xrightarrow{\sim} \Lambda_{\mathfrak{X}} u^{\mathrm{s}}_t \xrightarrow{\sim} \Lambda_{\mathfrak{X}} u^{\mathrm{s}}_t \Lambda_{\mathfrak{Y}} \tag{8.8}$$

For a map $f\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathcal{C}$, we define $f^{\tilde{!}} = \Lambda_{\mathfrak{X}} f^{!} \mathbf{R}\mathit{\Gamma}'_{\mathfrak{Y}} : \tilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathfrak{Y}) \to \tilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathfrak{X})$. It follows easily from the definiton that $(-)^{\tilde{!}}$ is a $\tilde{\mathbf{D}}_{\mathrm{qc}}(-)$-valued pre-pseudofunctor on $\mathcal{C}$. Moreover, if f is pseudoproper, the $f^{\tilde{!}}$ is right adjoint to $\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathfrak{X}}$ while if f is smooth of relative dimension n, then $f^{\tilde{!}} = \Lambda_{\mathfrak{X}}(f^* \otimes \omega_f[n])$.

The restriction of $(-)^{\tilde{!}}$ to $\mathbf{D}^+_{\mathrm{c}}(-)$ works out to be a pseudofunctor because $\Lambda_{|\mathbf{D}^+_{\mathrm{c}}(-)}$ is isomorphic to the identity functor. For any smooth map $f\colon \mathfrak{X} \to \mathfrak{Y}$ of relative dimension n and any $\mathcal{F} \in \mathbf{D}^+_{\mathrm{c}}(\mathfrak{Y})$, we have

$$\Lambda_{\mathfrak{X}} f^* \mathcal{F} \xrightarrow{\sim} f^* \mathcal{F}$$

which induces the isomorphism

$$\Lambda_{\mathfrak{X}} f^{\mathrm{s}} \mathcal{F} \xrightarrow{\sim} f^{\mathrm{s}} \mathcal{F} \simeq f^* \mathcal{F} \otimes \widehat{\omega}_f[n].$$

Theorem 8.2. *On the category $\mathcal{C}$ of noetherian formal schemes whose morphisms are separated and essentially of pseudofinite-type and furthermore are composites of smooth and pseudoproper morphisms, there is a $\tilde{\mathbf{D}}^+_{\mathrm{qc}}(-)$-valued pre-pseudofunctor $f^{\tilde{!}}$ with the following properties.*

(a) If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a pseudoproper morphism, then $f^{\tilde{!}} = \Lambda_{\mathfrak{X}} f_t^{\times} \boldsymbol{R}\mathit{\Gamma}'_{\mathfrak{Y}}$ and is right adjoint to $\boldsymbol{R}f_*\boldsymbol{R}\mathit{\Gamma}'_{\mathfrak{X}}$.

(b) If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a separated smooth morphism, then

$$f^{\tilde{!}} \simeq \Lambda_{\mathfrak{X}} f^{\mathrm{s}}_t \boldsymbol{R}\mathit{\Gamma}'_{\mathfrak{Y}} \simeq \Lambda_{\mathfrak{X}}(f^* \otimes \omega_f[n]).$$

(c) Finally, for any $f\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathcal{C}$, $f^{\tilde{!}}(\mathbf{D}^+_{\mathrm{c}}(\mathfrak{Y})) \subset \mathbf{D}^+_{\mathrm{c}}(\mathfrak{X})$, and thus $f^{\tilde{!}}|_{\mathbf{D}^+_{\mathrm{c}}}$ is a pseudofunctor such that if f is smooth of relative dimension n, then $f^{\tilde{!}} \simeq f^* \otimes \omega_f[n]$.

Part II

Reconstruction of Formal Schemes using their Derived Categories

Chapter 9

Introduction and Preliminaries

Balmer in [**B1**] defines the spectrum $\mathrm{Spc}(\mathcal{K})$ of a *tensor triangulated category* $\mathcal{K}$ and shows that it admits a structure of a locally ringed space. He shows that if $\mathcal{K} = \mathbf{D}^{\mathrm{perf}}(X)$, where X is a topologically noetherian scheme, and $\mathbf{D}^{\mathrm{perf}}(X)$ is the full subcategory of perfect complexes in the derived category of $\mathscr{O}_X$-modules $\mathbf{D}(X)$, then $\mathrm{Spc}(\mathcal{K}) \simeq X$. The underlying space and the topology on $\mathrm{Spc}(\mathcal{K})$ is obtained using Thomason's classification of $\otimes$-thick subcategories of the triangulated category $\mathcal{K}$, see [**T1**], which goes back to earlier work of Hopkins and Neeman when X is affine, see [**Ho**], [**N1**]. In the global case, Thomason also uses the fact that for any closed subset $Y \subset X$, of a noetherian scheme, there exists a perfect complex over X, say $\mathcal{F}$, such that $\mathrm{Supph}(\mathcal{F}) = Y$. However, we not know of such results for perfect complexes over formal schemes.

Alonso-Tarrio, Jeremias-Lopez and Souto-Salorio in [**AJS**], for a noetherian formal scheme $\mathfrak{X}$, give a correspondence between $\otimes$-compatible localizing subcategories of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ the derived category of complexes with quasi-coherent and torsion homology (see section 3), and the specialization closed subsets of $\mathfrak{X}$. We use this classification to modify the definition of spectrum of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, define a topology on it and show that $\mathrm{Spc}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ can be equipped with a ringed structure in a canonical way. Moreover, together with this ringed structure, we have an isomorphism of formal schemes $\mathrm{Spc}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})) \simeq \mathfrak{X}$. Moreover, we obtain a faithful embedding of the category $\mathbb{F}$ of separated noetherian formal schemes into the category $\mathcal{T}$ of tensor triangulated categories with unit, see Theorems 12.1 and 12.2.

9.0 Spectrum for Tensor Triangulated Categories

In this section we will briefly recap a few definitions and properties from [**B1**]. The reader is referred to [**B1**] for an introduction to the notion of spectrum of a tensor triangulated category.

Let $\mathcal{K}$ be a triangulated category. We say that a subcategory $\mathcal{K}' \subset \mathcal{K}$ is a *full triangulated subcategory* if it is closed under translations, isomorphisms and cones of the morphisms in $\mathcal{K}'$. A full triangulated subcategory $\mathcal{K}'$ of $\mathcal{K}$ is called a *thick* subcategory if the condition $A \oplus B \in \mathcal{K}'$, where A, $B \in \mathcal{K}$, implies that either $A \in \mathcal{K}'$ or $B \in \mathcal{K}'$.

A *tensor triangulated category* is a triangulated category $\mathcal{K}$ with a covariant functor $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ which is exact in each variable. A morphism of $\otimes$-triangulated categories is a triangulated functor which commutes with $\otimes$ up to isomorphism. *A priori*, there is no assumption on the associativity or commutativity of this $\otimes$, however, since we will be considering only full subcategories of derived categories of $\mathscr{O}_{\mathfrak{X}}$-modules, where $\mathfrak{X}$ is a noetherian formal scheme, the $\otimes$ (derived tensor product) will be associative and commutative upto isomorphism and have a unit element.

A *thick* subcategory $\mathcal{K}'$ of a tensor triangulated category $(\mathcal{K}, \otimes)$ is called $\otimes$-*thick* if $P \in \mathcal{K}'$ implies that $P \otimes Q \in \mathcal{K}'$ and $Q \otimes P \in \mathcal{K}'$ for all $Q \in \mathcal{K}$.

Let $C \subset \mathcal{K}$ be a collection of objects in a tensor triangulated category $\mathcal{K}$. We will denote by $\langle C \rangle$ the smallest $\otimes$-thick subcategory of $\mathcal{K}$ containing C.

Definition 9.1. A $\otimes$-thick subcategory of a tensor triangulated subcategory $A \subset (\mathcal{K}, \otimes)$ is called *atomic* if whenever $A \subset \langle D \rangle$ for a collection of objects $D \subset \mathcal{K}$, then there exists an element $d \in D$ such that $A \subset \langle d \rangle$.

Remark 9.1. It is easy to see that definition 9.1 is equivalent to: whenever there is a collection of $\otimes$-thick subcategories $\{B_i \subset \mathcal{K} | i \in I\}$ such that $A \subset \langle \bigcup_i B_i \rangle$, then there exists an $i \in I$ such that $A \subset B_i$. Moreover, an *atomic* $\otimes$-thick subcategory is by definition *principal* (generated by a single element).

Definition 9.2. Let $(\mathcal{K}, \otimes)$ be a tensor triangulated category. We define the *spectrum* of $(\mathcal{K}, \otimes)$ (denoted by $\mathrm{Spc}(\mathcal{K})$) to be the collection of all non-zero atomic subcategories of $(\mathcal{K}, \otimes)$.

$\mathrm{Spc}(\mathcal{K})$ is a topological space with the basic open sets given by

$$U(a) := \{B \in \mathrm{Spc}(\mathcal{K}) | B \not\subset \langle a \rangle\}. \tag{9.1}$$

We will now state some conditions on the morphisms in $\mathcal{K}$ for which the construction of $\mathrm{Spc}(-)$ is functorial, see [**B1**].

Definition 9.3. A morphism $\varphi : (\mathcal{K}, \otimes) \to (\mathcal{L}, \otimes)$ of $\otimes$-triangulated categories is called *dense* if $\langle \varphi(\mathcal{K}) \rangle = \mathcal{L}$. The morphism φ is called *geometric* if the following condition is satisfied: For any collection of $\otimes$-thick subcategories $\{C_i | i \in I\}$ in $\mathcal{K}$ one has

$$\langle \varphi(\bigcap C_i) \rangle = \bigcap \langle \varphi(C_i) \rangle. \tag{9.2}$$

The proofs of the following two propositions can be found in [**B1**]. Proposition 9.1 corresponds to the [**B1**, Proposition and Definition 4.1] and Proposition 9.2 corresponds to [**B1**, Proposition 4.11].

Proposition 9.1. *Let $\varphi : \mathcal{K} \to \mathcal{L}$ be a morphism of $\otimes$-triangulated categories. Assume that φ is geometric and dense. Let $C \in \mathrm{Spc}(\mathcal{L})$. Define*

$$\Phi(C) := \bigcap_{H \subset \mathcal{K}\ \otimes\text{-thick}\ s.t.\ C \subset \langle \varphi(H) \rangle} H. \tag{9.3}$$

The following conditions hold.

1. *$\Phi(C)$ is a non-zero atomic $\otimes$-thick subcategory of $\mathcal{K}$, hence Φ defines a map of topological spaces $\mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$, which is moreover continuous.*

2. *$C \subset \langle \varphi(\Phi(C)) \rangle$.*

Proposition 9.2. *The association $\varphi \mapsto \Phi$ of Proposition 9.1 induces a contravariant functor $\mathrm{Spc}(-)$ from the category of $\otimes$-triangulated categories with geometric and dense morphisms to the category of topological spaces.*

9.3.1 Triangular presheaves

Let $(\mathcal{K}, \otimes)$ be a $\otimes$-triangulated category. Let $V \subset \mathrm{Spc}(\mathcal{K})$ be an open subset and $V^c = \mathrm{Spc}(\mathcal{K}) \setminus V$ be its complement. Set $J(V^c) := \langle \bigcup_{A \in V^c} A \rangle$. Define

$$\mathcal{K}(V) := \mathcal{K}/J(V^c). \tag{9.4}$$

Thus, $\mathcal{K}(V)$ is the Verdier quotient of the triangulated category $\mathcal{K}$ by the full subcategory $J(V^c)$.

Here, we differ slightly from the definition of $\mathcal{K}(V)$ given in [**B1**]. Balmer defines $\mathcal{K}(V)$ as the idempotent completion of the above mentioned localization and when $\mathcal{K} = \mathbf{D}^{\mathrm{perf}}(X)$ where X is a topologically noetherian scheme, the idempotent completion $\widetilde{\mathcal{K}/J(V^c)}$ is equivalent to the category $\mathbf{D}^{\mathrm{perf}}(V)$, see [**B1**, Theorem 7.8]. However we do not know of any such "lifting" result in the category $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ where $\mathfrak{X}$ is a noetherian formal scheme, so we shall work with only the quotient. Later, we will show that the association $V \mapsto \mathcal{K}(V)$ is sufficient to recover the structure sheaf $\mathscr{O}_{\mathfrak{X}}$, which will be sufficient to reconstruct the formal scheme $\mathfrak{X}$.

Chapter 10

Localizing subcategories of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$

In [**AJS**], Alonso Tarrio, Jeremias Lopez and Souto Salorio classify $\otimes$-*compatible localizing subcategories* of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ for a noetherian formal scheme $\mathfrak{X}$. Over an ordinary noetherian scheme such subcategories are generated by perfect complexes and hence their classification can be viewed as a generalization of Thomason's classification of $\otimes$-*thick* subcategories of $\mathbf{D}^{\mathrm{perf}}(X)$. We will use the results from [**AJS**] to identify the points of the spectrum of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and construct a triangular presheaf on the topological space $\mathrm{Spc}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$. We begin by recalling the definition of the derived torsion functor $\mathbf{R}\varGamma'_{\mathfrak{X}}$.

10.0.1 Torsion functors

Let $(X, \mathscr{O}_X)$ be a ringed space. For any $\mathscr{O}_X$-ideal $\mathscr{I}$ and any $\mathcal{M} \in \mathcal{A}(X)$, set

$$\varGamma_{\mathscr{I}}\mathcal{M} := \varinjlim_{n} \mathcal{H}om_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathcal{M}). \tag{10.1}$$

For $\mathcal{M}, \mathcal{N} \in \mathcal{A}(X)$, there is an isomorphism

$$\mathbf{R}\varGamma_{\mathscr{I}}(\mathcal{M} \otimes \mathcal{N}) \simeq \mathbf{R}\varGamma_{\mathscr{I}}\mathcal{M} \otimes \mathcal{N}. \tag{10.2}$$

For a formal scheme $\mathfrak{X}$ with ideal of definition $\mathscr{I}$, we set

$$\varGamma'_{\mathfrak{X}} := \varGamma_{\mathscr{I}}. \tag{10.3}$$

Moreover if $Z \subset \mathfrak{X}$ is a closed subset given by ideal sheaf $\mathcal{I}$,

$$\varGamma'_Z := \varGamma_{\mathcal{I}}.$$

This definition is independent of the choice of the defining ideal $\mathscr{I}$. We call $M \in A(\mathfrak{X})$ a torsion $\mathscr{O}_{\mathfrak{X}}$-module if $\varGamma'_{\mathfrak{X}}M = M$. Let $\mathcal{A}_{\mathrm{t}}(\mathfrak{X})$ be the thick subcategory of $\mathcal{A}(\mathfrak{X})$ whose objects are all the torsion $\mathscr{O}_{\mathfrak{X}}$-modules; and set $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) := \mathcal{A}_{\mathrm{qc}}(\mathfrak{X}) \cap \mathcal{A}_{\mathrm{t}}(\mathfrak{X})$. Let $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ denote the full subcategory of $\mathbf{D}(\mathfrak{X})$ consisting of complexes of $\mathscr{O}_{\mathfrak{X}}$-modules whose cohomologies lie in $\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})$.

There is a natural inclusion functor $j^t_{\mathfrak{X}} : \mathcal{A}_{\mathrm{qct}}(\mathfrak{X}) \hookrightarrow \mathcal{A}(\mathfrak{X})$, and whenever $\mathfrak{X}$ is a separated noetherian formal scheme, this $j^t_{\mathfrak{X}}$, by [**AJL**, prop 5.3.1], induces an equivalence of categories, $\mathbf{D}(\mathcal{A}_{\mathrm{qct}}(\mathfrak{X})) \xrightarrow{\approx} \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$.

For a locally noetherian formal scheme $\mathfrak{X}$, we define

$$\tilde{\mathbf{D}}_{\mathrm{qc}}(\mathfrak{X}) := \mathbf{R}\varGamma_{\mathfrak{X}}'^{-1}(\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})) \tag{10.4}$$

to be the full subcategory of $\mathbf{D}(\mathfrak{X})$ whose objects are complexes $\mathcal{F}$, such that $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$.

It is immediate from the definition that $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}) \subset \mathbf{D}_{\mathrm{qc}}(\mathfrak{X}) \subset \tilde{\mathbf{D}}_{\mathrm{qc}}(\mathfrak{X})$. We refer the reader to [**AJL**, section 5] for a detailed treatment of the torsion functor $\varGamma'_{\mathfrak{X}}$ and its properties. We recall a few results involving the torsion functor here. The functor $\mathbf{R}\varGamma'_{\mathfrak{X}} : \mathbf{D}(\mathfrak{X}) \to \mathbf{D}(\mathfrak{X})$ has a $\triangle$-functorial right adjoint given by

$$\Lambda_{\mathfrak{X}} := \mathbf{R}\mathrm{Hom}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}, -) \tag{10.5}$$

Also, we have the natural morphisms of functors,

$$\mathbf{R}\varGamma'_{\mathfrak{X}} \to 1 \to \Lambda_{\mathfrak{X}}. \tag{10.6}$$

Via above morphisms the functors $\mathbf{R}\varGamma'_{\mathfrak{X}}$ and $\Lambda_{\mathfrak{X}}$ are idempotent, and in fact we have the following isomorphisms.

$$\begin{aligned} \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{R}\varGamma'_{\mathfrak{X}} \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}} \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathfrak{X}}\Lambda_{\mathfrak{X}} \\ \Lambda_{\mathfrak{X}}\mathbf{R}\varGamma'_{\mathfrak{X}} \xrightarrow{\sim} \Lambda_{\mathfrak{X}} \xrightarrow{\sim} \Lambda_{\mathfrak{X}}\Lambda_{\mathfrak{X}} \end{aligned} \tag{10.7}$$

For a closed immersion $Z \hookrightarrow \mathfrak{X}$, we have the functor $\mathbf{R}\varGamma_Z : \mathbf{D}_{\mathrm{qc}}(\mathfrak{X}) \to \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, which is the right derived functor of sheaf of sections supported along Z. Along with $\mathbf{R}\varGamma_Z$, we also have the derived torsion functor $\mathbf{R}\varGamma'_Z$ on $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ (see Chapter 2). These two functors are naturally isomorphic over $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, see [**AJS**, Section 4].

10.7.2 Localizing subcategories

We will now recall a few results on localizing subcategories of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, see [**AJS**].

The category $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ has a natural tensor bi-functor $\underset{=}{\otimes}^L$ which makes it into a $\otimes$-triangulated category. A triangulated subcategory $\mathcal{L} \subset \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ is called *localizing* if it is closed under taking coproducts. A localizing subcategory $\mathcal{L} \subset \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ is called *rigid* if it is $\otimes$-thick.

In addition, the tensor multiplication in $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ is commutative and associative upto isomorphism and the object $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}$ is the identity for this tensor multiplication in $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. That is, for $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, there are the following natural isomorphisms.

$$\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{F} \simeq \mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \simeq \mathbf{R}\varGamma'_{\mathfrak{X}}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}}) \xrightarrow{\sim} \mathcal{F} \tag{10.8}$$

We will start with the set-up given in section 4 and 5 of [**AJS**] and briefly recap a few results leading up to their classification theorem.

Let $\mathcal{K}$ be a tensor triangulated category and $C \subset \mathcal{K}$ be a collection of objects. We will denote by $\langle C \rangle$ the smallest $\otimes$-thick subcategory of $\mathcal{K}$ that contains C.

Let $\mathfrak{X}$ be a noetherian formal scheme and let $\mathcal{I}$ be its ideal of definition. Let $x \in \mathfrak{X}$ and denote by $i_x : \mathfrak{X}_x \hookrightarrow \mathfrak{X}$ the canonical inclusion map where $\mathfrak{X}_x = \mathrm{Spf}(\widehat{\mathcal{O}_{\mathfrak{X},x}})$. Let $\kappa(x)$ denote the residue field of the local ring $\widehat{\mathcal{O}_{\mathfrak{X},x}}$, and let $\mathcal{K}_x$ denote the quasi-coherent torsion sheaf associated to $\widehat{\mathcal{O}_{\mathfrak{X},x}}$-module $\kappa(x)$. Define $\mathcal{K}(x) := \mathbf{R}i_{x*}\mathcal{K}_x = i_{x*}\mathcal{K}(x)$. Note that $\mathcal{K}(x) = \mathbf{R}\varGamma_{\overline{\{x\}}}\mathcal{K}(x) \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$.

For $x \in \mathfrak{X}$, denote by $\mathcal{L}_x := \langle \mathcal{K}(x) \rangle$, the smallest $\otimes$-thick localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ generated by $\mathcal{K}(x)$. For any subset $Z \subset \mathfrak{X}$, define $\mathcal{L}_Z$ to be smallest localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ that contains $\{\mathcal{K}(x) | x \in Z\}$. If $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and $x \in \mathfrak{X}$, then $\mathbf{R}\varGamma_{\overline{\{x\}}}(\mathbf{R}i_{x*}i_x^*\mathcal{F}) \in \mathcal{L}_x$ for $i_x \colon \mathfrak{X}_x \hookrightarrow \mathfrak{X}$, see [**AJS**, Lemma 4.1].

We will state below a few properties, the proofs of which can be found in [**AJS**]

Property 1. *The smallest localizing subcategory $\mathcal{L}$ of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ that contains $\mathcal{K}(x)$ for every $x \in \mathfrak{X}$ is the whole of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$.*

Property 2. *If $x, y \in \mathfrak{X}$ and $x \neq y$, then $\mathcal{K}(x) \otimes^{\mathbf{L}} \mathcal{K}(y) = 0$.*

Property 3. *For every subset $Z \subset \mathfrak{X}$, the localizing subcategory $\mathcal{L}_Z$ is rigid.*

Property 4. *If Z and Y are subsets of $\mathfrak{X}$ such that $Z \cap Y = \emptyset$, then $\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G} = 0$ for every $\mathcal{F} \in \mathcal{L}_Z$ and $\mathcal{G} \in \mathcal{L}_Y$.*

Property 5. *For $x \in \mathfrak{X}$ and $\mathcal{F} \in \mathcal{L}_x$, we have $\mathcal{F} = 0 \Leftrightarrow \mathcal{F} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x) = 0$.*

Property 6. *For a noetherian formal scheme $\mathfrak{X}$ there is a bijection between the class of all rigid localizing subcategories of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ denoted by $\mathbf{Loc}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and the set of all subsets of $\mathfrak{X}$. Thus, if $\mathbf{P}(\mathfrak{X})$ denotes the power set of $\mathfrak{X}$, then there are following inverse bijections ϕ, ψ*

$$\psi : \mathbf{Loc}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}) \leftrightarrows \mathbf{P}(\mathfrak{X}) : \phi, \tag{10.9}$$

where on one hand, $\phi(Z) := \mathcal{L}_Z$ for any subset $Z \subset \mathfrak{X}$ and on the other hand, for any $\otimes$-thick localizing subcategory $\mathcal{L} \subset \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$,

$$\psi(\mathcal{L}) := \{x \in \mathfrak{X} |\ \text{there exists } \mathcal{G} \in \mathcal{L} \text{ such that } \mathcal{K}(x) \otimes^{\mathbf{L}}_{\mathfrak{X}} \mathcal{G} \neq 0\}.$$

10.9.3 Tensor-compatible localizations

Associated to every localizing subcategory $\mathcal{L}$ of a tensor triangulated category $\mathcal{K}$ there are endofunctors ℓ, the Bousfield localization functor and γ, the colocalization functor. For $\mathcal{F} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, there is a canonical distinguished triangle

$$\gamma\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \ell\mathcal{F} \xrightarrow{+} \tag{10.10}$$

such that $\gamma\mathcal{F} \in \mathcal{L}$ and $\ell\mathcal{F} \in \mathcal{L}^{\perp}$. The endofunctors ℓ and γ are idempotent and for $\mathcal{F}, \mathcal{G} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ there are following canonical isomorphisms

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\gamma\mathcal{F}, \gamma\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\gamma\mathcal{F}, \mathcal{G}) \tag{10.11}$$

$$\mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\ell\mathcal{F}, \ell\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(\mathfrak{X})}(\mathcal{F}, \ell\mathcal{G}) \tag{10.12}$$

If $\mathcal{L}$ is rigid, then the above isomorphisms also hold for sheafified inner-*Homs*. Moreover, there are natural morphisms

$$t : \mathcal{F} \otimes^{\mathbf{L}} \gamma\mathcal{G} \longrightarrow \gamma(\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}) \tag{10.13}$$

$$p : \mathcal{F} \otimes^{\mathbf{L}} \ell\mathcal{G} \longrightarrow \ell(\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}) \tag{10.14}$$

such that the diagram

$$\begin{tikzcd}
\mathcal{F} \otimes^{\mathbf{L}} \gamma\mathcal{G} \arrow[r] \arrow[d, "t"] & \mathcal{F} \otimes^{\mathbf{L}} \mathcal{G} \arrow[r] \arrow[d, equal] & \mathcal{F} \otimes^{\mathbf{L}} \ell\mathcal{G} \arrow[r, "+"] \arrow[d, "p"] & {} \\
\gamma(\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}) \arrow[r] & \mathcal{F} \otimes^{\mathbf{L}} \mathcal{G} \arrow[r] & \ell(\mathcal{F} \otimes^{\mathbf{L}} \mathcal{G}) \arrow[r, "+"] & {}
\end{tikzcd} \tag{10.15}$$

is a morphism of distinguished triangles.

The localizing subcategory $\mathcal{L}$ (or the localizing functor ℓ) is called $\otimes$-*compatible* if the canonical morphism t of (10.13) (or equivalently p of (10.14)) is an isomorphism.

Let $\mathfrak{X}$ be a noetherian formal scheme, then for $\mathcal{K} = \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, and for any specialization-closed subset $Z \subset \mathfrak{X}$, the functor $\mathbf{R}\varGamma_Z$ together with the natural map $\mathbf{R}\varGamma_Z \to 1$, satisfies the formal properties of a localization functor for the localizing pair $(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}), \mathcal{L}_Z)$. Moreover, since over $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ the functors $\mathbf{R}\varGamma'_Z$ and $\mathbf{R}\varGamma_Z$ are isomorphic and for $\mathcal{F}, \mathcal{G} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ we have the natural isomorphism $\mathbf{R}\varGamma_Z(\mathcal{F} \otimes \mathcal{G}) \simeq \mathcal{F} \otimes \mathbf{R}\varGamma_Z\mathcal{G}$, the morphism t defined in (10.13) is an isomorphism. Hence $\mathcal{L}_Z$ is a $\otimes$-compatible localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$.

Theorem (5.3) in [**AJS**] states that the image of a $\otimes$-compatible localizing subcategory under the bijection ψ mentioned in the remark (10.9) is a *specialization-closed* subset of $\mathfrak{X}$. And conversely, if $Z \subset \mathfrak{X}$ is a *specialization-closed* subset, then $\phi(Z)$ is a $\otimes$-compatible localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. We summarize this as follows.

There is a bijection between the class of $\otimes$-compatible localizing subcategories of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ denoted by $\mathbf{Loc}^{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ and the set of subsets stable for specialization of $\mathfrak{X}$. That is, ϕ and ψ of (10.9) induce inverse bijections:

$$\psi : \mathbf{Loc}^{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})) \leftrightarrows \{Z \subset \mathfrak{X} | Z \text{ is specialization closed}\} : \phi. \tag{10.16}$$

Chapter 11

The topological space $\mathrm{Spc}(\mathbf{D}_{\mathrm{qct}}(-))$

We will now define the spectrum of tensor triangulated category $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ for a noetherian formal scheme $\mathfrak{X}$, where $\mathfrak{X}$ is either separated or of finite Krull dimension. Owing to the fact that the cohomological support of an object in $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ is a *specialization-closed* subset and not necessarily a closed subset of $\mathfrak{X}$, our definition of spectrum of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ will be different from Balmer's definition. In order to avoid confusion we will use a different notation $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ to denote the spectrum. We will then try to study the underlying topological space of $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$. For this construction we require our tensor triangulated category to have a unit object, which serves as the multiplicative identity for the tensor product. For $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, the tensor multiplication is the derived tensor product $\otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}}$ and the unit element is $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}$. We also need to make a few additional assumptions on the tensor triangulated category which we will state below and show that $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ satisfies these conditions.

Proposition 11.1. *Let $\mathfrak{X}$ be a noetherian formal scheme and let $\mathcal{K} = \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. The intersection of all rigid localizing subcategories of a tensor triangulated category $(\mathcal{K}, \otimes)$ is (0).*

The proof follows from [**AJS**, Theorem 4.2, Cor. 4.9].

Proposition 11.2. *If $\{\mathcal{L}_i | i \in I\}$ is a collection of $\otimes$-compatible rigid localizing subcategories of a tensor triangulated category $(\mathcal{K}, \otimes)$, then $\cap_i \mathcal{L}_i$ is a $\otimes$-compatible rigid localizing subactegory of $(\mathcal{K}, \otimes)$.*

It is easy to see that $\cap_i \mathcal{L}_i$ is rigid and localizing for any tensor triangulated category $\mathcal{K}$. And for $\mathcal{K} = \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, tensor compatibility of $\cap_i \mathcal{L}_i$ can be obtained using [**AJS**, Theorem 5.3].

Henceforth, for any collection of objects D in tensor triangulated category $\mathcal{K}$, we will denote by $\langle D \rangle_{\otimes}$ the smallest $\otimes$-compatible rigid localizing subcategory of $\mathcal{K}$ containing D.

Definition 11.1. Let $(\mathcal{K}, \otimes)$ be a tensor triangulated category with the following properties.

1. $\mathcal{K}$ is closed under coproducts.
2. Intersection of $\otimes$-compatible rigid localizing subcategories of $\mathcal{K}$ is also $\otimes$-compatible rigid localizing subcategory of $\mathcal{K}$.
3. $\mathcal{K}$ is molecular, that is, every $\otimes$-compatible rigid localizing subcategory of $\mathcal{K}$ is generated by the $\otimes$-compatible rigid localizing atomic subcategories contained in it.

Then we define the spectrum of $\mathcal{K}$, denoted by $\mathrm{Spc}_{\otimes}(\mathcal{K})$, to be collection of all $\otimes$-compatible rigid atomic localizing subcategories of $\mathcal{K}$.

We define the topology on $\mathrm{Spc}_{\otimes}(\mathcal{K})$ by defining

$$\mathcal{B} = \{U(\mathcal{L}) | \mathcal{L} \in \mathrm{Spc}_{\otimes}(\mathcal{K})\} \tag{11.1}$$

as the collection of sub-basic open subsets where, for any $\mathcal{L} \in \mathrm{Spc}_{\otimes}(\mathcal{K})$,

$$U(\mathcal{L}) := \{\mathcal{L}' \in \mathrm{Spc}_{\otimes}(\mathcal{K}) | \mathcal{L}' \not\subset \mathcal{L}\}. \tag{11.2}$$

Equivalently, we can use the collection $\mathcal{B}' = \{F(\mathcal{L}) | \mathcal{L} \in \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))\}$, where

$$F(\mathcal{L}) := \{\mathcal{L}' \in \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})) | \mathcal{L}' \subset \mathcal{L}\}, \tag{11.3}$$

to generate the closed subsets of the topology on $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$, that is, any closed subset in $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ is an arbitrary intersection of finite union of elements of $\mathcal{B}'$.

Lemma 11.3. *Let $\mathfrak{X}$ be a noetherian formal scheme. Let $\{Y_i | i \in I\}$ be a collection of specialization-closed subsets of $\mathfrak{X}$. Then their union $Y = \cup_i Y_i$ is also specialization closed and*

$$\mathcal{L}_Y = \langle \bigcup_i \mathcal{L}_{Y_i} \rangle_{\otimes}. \tag{11.4}$$

Proof. The first assertion is obvious, as taking any point $y \in Y$ amounts to $y \in Y_i$ for some i, and since $Y_i' s$ are specialization-closed, $\overline{\{y\}} \subset Y_i \subset Y$. The subcategory $\langle \bigcup_i \mathcal{L}_{Y_i} \rangle_{\otimes}$ is a $\otimes$-compatible rigid localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and by (10.16), we know that any $\otimes$-compatible rigid localizing subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ is of the form $\mathcal{L}_Z$ for some specialization-closed subset $Z \subset \mathfrak{X}$. Let $\mathcal{L}_Z = \langle \bigcup_i \mathcal{L}_{Y_i} \rangle_{\otimes}$. Since the definition of $\mathcal{L}_Z$ makes the (10.16) correspondence inclusion preserving, hence Z must be smallest specialization-closed subset containing all the $Y_i' s$, whence $Z = \cup_i Y_i$. $\square$

Definition 11.2. Let $\mathcal{K}, \mathcal{K}'$ be tensor triangulated categories satisfying conditions (1)-(3) of (11.1) and let $\varphi : \mathcal{K}' \to \mathcal{K}$ be a morphism of tensor triangulated categories. Then we say

1. φ is geometric if for any collection $\{\mathcal{L}_i\}$ of $\otimes$-compatible rigid localizing subcategories of $\mathcal{K}'$
$$\langle \varphi(\bigcap \mathcal{L}_i) \rangle_{\otimes} = \bigcap \langle \varphi(\mathcal{L}_i) \rangle_{\otimes}. \tag{11.5}$$
2. φ is dense if $\langle \varphi(\mathcal{K}') \rangle_{\otimes} = \mathcal{K}$.

Proposition 11.4. *Let $\mathfrak{X}$ be a noetherian formal scheme and let $Y \subset \mathfrak{X}$ be a specialization-closed subset. Then $\mathcal{L}_Y$ is a non-zero $\otimes$-compatible atomic subcategory of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ if and only if Y is non-empty, closed and irreducible.*

The proof of the above proposition is similar to that of [**B1**, Proposition 7.2].

Thus,

$$\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})) := \{\mathcal{L}_{\overline{\{x\}}} | x \in \mathfrak{X}\}. \tag{11.6}$$

Now that we have have identified the points in $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$, let us identify the topology of $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$.

Recall that the a topology on $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ by describing the subbasis of open subsets. For any $\mathcal{L} \in \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$, we have

$$U(\mathcal{L}) := \{\mathcal{L}' \in \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})) | \mathcal{L}' \not\subset \mathcal{L}\}, \tag{11.7}$$

which using Proposition 11.4 can be written as

$$U(\mathcal{L}) = \{\mathcal{L}_{\overline{\{x\}}} \in \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})) | \mathcal{L}_{\overline{\{x\}}} \not\subset \mathcal{L}\}.$$

We use the collection $\mathcal{B} = \{U(\mathcal{L}) | \mathcal{L} \in \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))\}$ as a sub-basis of open subsets to give a topology on $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$.

Proposition 11.5. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of noetherian formal schemes. Let $Z \subset \mathfrak{Y}$ be a specialization-closed subset. Then $f^{-1}Z$ is a specialization-closed subset of $\mathfrak{X}$ and*

$$\langle \boldsymbol{R}\varGamma'_{\mathfrak{X}} \mathbf{L} f^*(\mathcal{L}_Z) \rangle_{\otimes} = \mathcal{L}_{f^{-1}Z}. \tag{11.8}$$

Proof. Let $x \in f^{-1}Z$. Since Z is specialization-closed and f is continuous, we have $f(\overline{\{x\}}) \subset \overline{\{f(x)\}} \subset Z$. Thus, $\overline{\{x\}} \subset f^{-1}Z$.

Since $\mathcal{L}_Z$ is generated by $\mathcal{K}(y)$ for $y \in Z$, proving (11.8) reduces to proving the following for every $y \in Z$.

$$\langle \mathbf{R}\varGamma'_{\mathfrak{X}} \mathbf{L} f^* \mathcal{L}_{\overline{\{y\}}} \rangle_{\otimes} = \mathcal{L}_{f^{-1}\overline{\{y\}}} \tag{11.9}$$

The inclusion $\langle \mathbf{R}\varGamma'_{\mathfrak{X}} \mathbf{L} f^* \mathcal{L}_{\overline{\{y\}}} \rangle_{\otimes} \subset \mathcal{L}_{f^{-1}\overline{\{y\}}}$ follows from [**AJS**, Theorem 5.6], since $\mathbf{R}\varGamma'_{\mathfrak{X}} \mathbf{L} f^* \mathcal{K}(y)$ is supported inside $f^{-1}\overline{\{y\}}$.

We will now prove the inclusion $\langle \mathbf{R}\varGamma'_{\mathfrak{X}} \mathbf{L} f^* \mathcal{L}_{\overline{\{y\}}} \rangle_{\otimes} \supset \mathcal{L}_{f^{-1}\overline{\{y\}}}$. Let $x \in f^{-1}\overline{\{y\}}$ be any point. We will show that $\mathcal{K}(x)$ is a summand of $\mathbf{R}\varGamma'_{\mathfrak{X}} \mathbf{L} f^* \mathcal{K}(y) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x)$. First observe that

$$\mathbf{R}\varGamma'_{\mathfrak{X}} \mathbf{L} f^* \mathcal{K}(y) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x) \simeq \mathbf{L} f^* \mathcal{K}(y) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathbf{R}\varGamma'_{\mathfrak{X}} \mathcal{K}(x) \simeq \mathbf{L} f^* \mathcal{K}(y) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x). \tag{11.10}$$

Thus

$$\mathcal{H}^i(\mathbf{R}\varGamma'_{\mathfrak{X}} \mathbf{L} f^* \mathcal{K}(y) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x)) = 0 \text{ for } i > 0 \text{ and}$$
$$\mathcal{H}^0(\mathbf{R}\varGamma'_{\mathfrak{X}} \mathbf{L} f^* \mathcal{K}(y) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x)) \simeq \mathcal{H}^0(\mathbf{L} f^* \mathcal{K}(y) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x) \simeq f^* \mathcal{K}(y) \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x) \simeq \mathcal{K}(x).$$

Now consider the commutative diagram

$$\begin{tikzcd}
\mathcal{K}(x) \arrow[r, "\sim"] \arrow[ddrr, equal] & \mathscr{O}_{\mathfrak{X}} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x) \arrow[r, "\sim"] & \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*\mathscr{O}_{\mathfrak{Y}} \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x) \arrow[d, "\beta"] \\
 & & \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*\mathcal{K}(y) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x) \arrow[d, "\alpha"] \\
 & & \mathcal{K}(x)
\end{tikzcd} \tag{11.11}$$

where β is induced by the canonical map $\mathscr{O}_{\mathfrak{Y}} \to \mathcal{K}(y)$ while α is the canonical truncation map in degree 0. Thus $\mathcal{K}(x)$ is a summand of the complex $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*\mathcal{K}(y) \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}} \mathcal{K}(x)$. Since $\langle \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*\mathcal{K}(y)\rangle_{\otimes}$ is closed under summands and tensor, hence $\langle \mathcal{K}(x)\rangle_{\otimes} \subset \langle \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*\mathcal{K}(y)\rangle_{\otimes}$. Since, $x \in f^{-1}\overline{\{y\}}$ was arbitrarily chosen, thus $\langle \mathcal{K}(x)\rangle_{\otimes} \subset \langle \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*\mathcal{K}(y)\rangle_{\otimes}$ for all $x \in f^{-1}\overline{\{y\}}$, hence $\mathcal{L}_{f^{-1}\overline{\{y\}}} \subset \langle \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*\mathcal{L}_{\overline{\{y\}}}\rangle_{\otimes}$. □

Proposition 11.6. *Let $\mathfrak{X}$ be a noetherian formal scheme. Then there is a homeomorphism of topological spaces as follows:*

$$\begin{aligned} E\colon \mathfrak{X} &\longrightarrow \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})) \\ x &\longmapsto \mathcal{L}_{\overline{\{x\}}}. \end{aligned} \tag{11.12}$$

Proof. By Proposition 11.4, $\mathcal{L}_{\overline{\{x\}}} \in \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$, and hence we obtain E as defined above. By (10.16), E is bijective. Let $Y \subset \mathfrak{X}$ be any closed set. Then $E(Y) = \{\mathcal{L}_{\overline{\{y\}}} | y \in Y\}$. If Y is irreducible say $Y = \overline{\{\eta\}}$, then

$$E(Y) = \{\mathcal{L}_{\overline{\{y\}}} | y \in \overline{\{\eta\}}\} = \{\mathcal{L}_{\overline{\{y\}}} | \mathcal{L}_{\overline{\{y\}}} \subset \mathcal{L}_{\overline{\{\eta\}}}\} = F(\mathcal{L}_{\overline{\{\eta\}}}), \tag{11.13}$$

and hence $E(Y)$ is a closed set. In general, since $\mathfrak{X}$ is noetherian, Y is a finite union of irreducible closed subsets, hence $E(Y)$ is closed. Hence E is a closed map of topological spaces. Moreover,

$$E^{-1}(F(\{\mathcal{L}_{\overline{\{y\}}}\})) = \{y' | \mathcal{L}_{\overline{\{y'\}}} \in \mathcal{L}_{\overline{\{y\}}}\} = \{y' | y' \in \overline{\{y\}}\} = \overline{\{y\}}. \tag{11.14}$$

Thus E^{-1} is a closed map of topological spaces and hence E is a homeomorphism. □

For a map $f\colon \mathfrak{X} \to \mathfrak{Y}$ of noetherian formal schemes, consider the functor $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^* : \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}) \to \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. The functor $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*$ induces a map of topological spaces

$\Phi_f := \mathrm{Spc}_{\otimes}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*)\colon \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{Y})) \to \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ described as follows. For a $\mathcal{L} \in \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y})$, define

$$\Phi_f(\mathcal{L}) := \bigcap_{\substack{H \subset \mathcal{K}\ \otimes\text{-compatible rigid loc.} \\ \text{s.t. } \mathcal{L} \subset \langle \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*(H)\rangle_{\otimes}}} H. \tag{11.15}$$

The proof of the fact that $\Phi_f(\mathcal{L}) \in \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ is similar to the given in [**B1**, Proposition 4.6]

Proposition 11.7. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of noetherian formal schemes. Then the morphism $\boldsymbol{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^* : \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}) \to \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ between the tensor triangulated categories is geometric, dense and continuous (see Definition 11.2). Moreover, the following diagram of maps of topological spaces is commutative.*

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "f"] \arrow[d, "E"', "\wr"] & \mathfrak{Y} \arrow[d, "E"', "\wr"] \\
\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})) \arrow[r, "\Phi_f"] & \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}))
\end{tikzcd} \tag{11.16}$$

Proof. Let $\{C_i | i \in I\}$ be a collection of $\otimes$-compatible rigid localizing triangulated subcategories of $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}))$. In view of the classification (10.16), there exist specialization-closed subsets Z_i of $\mathfrak{Y}$ corresponding to C_i, that is, $C_i = \mathcal{L}_{Z_i}$. Let $Z = \cap Z_i$. Thus we have

$$\langle \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*(\bigcap_i C_i)\rangle_{\otimes} = \langle \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*(\bigcap_i \mathcal{L}_{Z_i})\rangle_{\otimes} = \langle \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*\mathcal{L}_Z\rangle_{\otimes} = \mathcal{L}_{f^{-1}Z} \tag{11.17}$$

$$= \bigcap_i \mathcal{L}_{f^{-1}(Z_i)} = \bigcap_i \langle \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*(\mathcal{L}_{Z_i})\rangle_{\otimes} = \bigcap_i \langle \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*(C_i)\rangle_{\otimes}, \tag{11.18}$$

where the second equality in the first line and the first equality in the second line follows from the fact that for a collection $\{W_i\}$ of specialization-closed subsets of $\mathfrak{Y}$, $\bigcap_i \mathcal{L}_{W_i} = \mathcal{L}_{\cap W_i}$. The last equality in the first line and the second equality in the second line follows from Proposition (11.5). Since, $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*\mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}) \xrightarrow{\sim} \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, hence φ is dense (see Definition (11.2). The proof of continuity of $\Phi_f(-)$ is similar to the one given in [**B1**, Proposition 4.8].

For the proof of commutativity of the diagram (11.16) we will use a notation similar to the one given in [**B1**, Theorem 7.7] for simplicity. Let $x \in \mathfrak{X}$. We will use the following notation:

$$\mathcal{K} := \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}),\ \mathcal{K}' := \mathbf{D}_{\mathrm{qct}}(\mathfrak{X}),\ \varphi = \mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^* : \mathcal{K} \to \mathcal{K}'$$
$$\Phi_f := \mathrm{Spc}(\varphi) : \mathrm{Spc}(\mathcal{K}') \to \mathrm{Spc}(\mathcal{K}).$$

By definition we have

$$\Phi_f(E(x)) := \bigcap_{\substack{H \subset \mathcal{K}\ \otimes\text{-compatible rigid loc.} \\ \text{s.t. } E(x) \subset \langle \varphi(H) \rangle_{\otimes}}} H. \tag{11.19}$$

On the right hand side we are taking the intersection of those $\otimes$-compatible rigid localizing subcategories of $\mathcal{K}$ which contain the $\otimes$-compatible rigid localizing subcategory $E(x)$ of $\mathcal{K}$. Now using the classification given in (10.16), we may assume that $\Phi_f(E(x)) = \mathcal{L}_W$, that $H = \mathcal{L}_Z$, where Z and W are specialization-closed subsets of $\mathfrak{Y}$. Thus

$$\mathcal{L}_W = \bigcap_{\mathcal{L}_Z \subset \mathcal{K}\ \text{ s.t. } \mathcal{L}_{\overline{\{x\}}} \subset \mathcal{L}_{f^{-1}Z}} \mathcal{L}_Z \tag{11.20}$$

or

$$W = \bigcap_{Z \subset \mathfrak{Y}\ \text{sp. closed s.t. } x \in f^{-1}Z} Z. \tag{11.21}$$

This means that W is the smallest specialization-closed subset of $\mathfrak{Y}$ which contains $f(x)$. Thus $\Phi_f(E(x)) = \mathcal{L}_{\overline{\{f(x)\}}} = E(f(x))$. $\square$

Chapter 12

Reconstruction of the structure sheaf $\mathscr{O}_{\mathfrak{X}}$

We saw in the previous section that a map $f\colon \mathfrak{X} \to \mathfrak{Y}$ of noetherian formal schemes induces a continuous map $\Phi_f : \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})) \to \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}))$ of topological spaces via the functor $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathbf{L}f^*\colon \mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}) \to \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. Now consider a morphism $\varphi\colon (\mathcal{K}', \otimes) \to (\mathcal{K}, \otimes)$ of tensor triangulated categories. This morphism induces a (continuous) map Φ of topological spaces defined as follows. For any $\mathcal{L} \in \mathrm{Spc}_{\otimes}(\mathcal{K})$,

$$\Phi(\mathcal{L}) := \bigcap_{\substack{H \subset \mathcal{K}\ \otimes\text{-compatible rigid loc.} \\ \text{s.t. } \mathcal{L} \subset \langle \varphi(H) \rangle_{\otimes}}} H. \tag{12.1}$$

The difference between the map Φ and the map Φ_f defined in (11.15) for $\mathcal{K} = \mathbf{D}_{\mathrm{qct}}(-)$, is that the latter is induced by an actual morphism of formal schemes whereas the former is induced by a morphism of tensor triangulated categories and does not assume the existence of any morphism of underlying formal schemes.

We will now proceed to define a sheaf of rings on the space $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ but before that we will define a presheaf of tensor triangulated categories on it.

Let $\mathcal{K}$ be a tensor triangulated category and let $\mathrm{Spc}_{\otimes}(\mathcal{K})$ be its spectrum. Let $J(U^c) := \langle \bigcup_{\mathcal{L} \in U^c} \mathcal{L} \rangle_{\otimes}$ and let $\mathscr{K}(U) = \mathcal{K}/J(U^c)$ be the Verdier quotient of $\mathcal{K}$ by the $\otimes$-compatible rigid localizing subcategory $J(U^c)$. The association

$$U \rightsquigarrow \mathscr{K}(U) \tag{12.2}$$

forms a presheaf of tensor triangulated categories on $\mathrm{Spc}_{\otimes}(\mathcal{K})$ and the pair $(\mathrm{Spc}_{\otimes}(\mathcal{K}), \mathscr{K})$ becomes a *triangulated presheaf* in the sense of [**B1**, Definition 5.6]. Moreover, if $\varphi\colon \mathcal{K} \to \mathcal{K}'$ is a morphism of tensor triangulated categories such that $\mathcal{K}$ and $\mathcal{K}'$ satisfy properties 1 and 2 of the Definition 11.1 and that $\otimes$-compatible rigid localizing sucategories of $\mathcal{K}'$ are generated by the $\otimes$-compatible rigid localizing atomic subcategories contained in them, then the following holds.

1. $\varphi(J(U^c)) \subset J(\Phi^{-1}(U^c))$
2. *There is a morphism of presheaves of tensor triangulated categories on* $\mathrm{Spc}_{\otimes}(\mathcal{K})$

$$\mathfrak{F}\colon \mathscr{K} \to \Phi_* \mathscr{K}'$$

where the presheaf $\Phi_* \mathscr{K}$ *is defined on* $\mathrm{Spc}_{\otimes}(\mathcal{K})$ *as follows. For any open subset* $U \subset \mathrm{Spc}_{\otimes}(\mathcal{K})$,

$$\Phi_* \mathscr{K}'(U) = \mathscr{K}(\Phi^{-1}(U)).$$

The proofs of the above statements are the same as that of the corresponding statements in [**B1**, Section 5].

Let $\mathfrak{X}$ be a noetherian formal scheme and let $U \subset \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ be an open subset and let $V = E^{-1}U$ be the open subset in $\mathfrak{X}$, where E is the homeomorphism defined in (11.12). It is easy to see that for the tensor triangulated category $\mathcal{K} = \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, the localizing subcategory $J(U^c) := \langle \bigcup_{\mathcal{L} \in U^c} \mathcal{L} \rangle_{\otimes}$, is the subcategory $\mathcal{L}_{U^c}$ and hence the Verdier quotient $\mathscr{K}(U)$ is given by

$$\mathscr{K}(U) = \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})/\mathcal{L}_{U^c}.$$

12.2.1 Presheaf of rings

Let $\mathfrak{X}$ be a noetherian formal scheme and let $\mathscr{K}$ be the presheaf of triangulated categories on $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ defined above.

In the rest of the this section we shall drop the usage of the homeomorphism map E and use the same symbol to denote a subset of $\mathfrak{X}$ and its homeomorphic image in $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ or vice-versa.

Let $V \subset \mathfrak{X}$ be an open subset, let $Z = \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})) \setminus V$ be its complement in $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ and let $\mathscr{K}(V)$ be as defined above. Since $\mathcal{L}_Z$ is a rigid $\otimes$-compatible localizing subcategory it is easy to verify that the tensor structure over $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ induces, in a natural way, a tensor multiplication over $\mathscr{K}(V)$ with same multiplicative identity namely $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}$. Recall that $\mathscr{K}(V) = \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))/\mathcal{L}_Z$ is the category whose objects are the same as the objects of $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ and morphisms are obtained by inverting those $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$-morphisms whose *cones* lies in $\mathcal{L}_Z$. From (10.8) it follows immediately that $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}$ is the multiplicative identity in $\mathscr{K}(V)$ for all open subsets V of $\mathfrak{X}$. Consider the set of endomorphisms of the unit object $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}$ in the category $\mathscr{K}(V)$, namely,

$$\mathrm{End}_{\mathscr{K}(V)}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) = \mathrm{Hom}_{\mathscr{K}(V)}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}, \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}). \tag{12.3}$$

Moreover, if $V = \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$, that is $\mathscr{K}(V) = \mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$, then

$$\begin{aligned}\mathrm{End}_{\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) &= \mathrm{Hom}_{\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}, \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})}(\mathscr{O}_{\mathfrak{X}}, \Lambda_{\mathfrak{X}}\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})}(\mathscr{O}_{\mathfrak{X}}, \mathscr{O}_{\mathfrak{X}}) \\ &\xrightarrow{\sim} \varGamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}).\end{aligned} \tag{12.4}$$

Here, the first isomorphism follows from the adjointness of $\mathbf{R}\varGamma'_{\mathfrak{X}}$ and $\Lambda_{\mathfrak{X}}$ and the second isomorphism which is induced by the co-unit map $\Lambda_{\mathfrak{X}}\mathbf{R}\varGamma'_{\mathfrak{X}} \to 1$, holds by Greenlees-May duality (see [**AJL**, 6.2.1]).

Proposition 12.1. *Let $V \subset \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ be an open subset, let Z be its complement and let $\mathscr{K}(V) = \mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))/\mathcal{L}_Z$ be the corresponding localized triangulated category. Then there is a natural isomorphism of rings*

$$\mathrm{End}_{\mathscr{K}(V)}(\boldsymbol{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) \simeq \mathscr{O}_{\mathfrak{X}}(V). \tag{12.5}$$

Proof. We will first construct a map in either direction. An endomorphism of $\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}$ in $\mathscr{K}(V)$ is given by a fraction in the derived category over $\mathfrak{X}$, say,

$$\begin{tikzcd} \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \arrow[dr, "a"'] & & \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \arrow[dl, "h"] \\ & C & \end{tikzcd} \tag{12.6}$$

where the cone $c(h)$ of h lies in $\mathcal{L}_Z$. To this diagram we apply the functor j^*, where $j : V \hookrightarrow \mathfrak{X}$ is the inclusion map, to obtain the following diagram.

$$\begin{tikzcd}
\mathbf{R}\Gamma'_V\mathscr{O}_V \simeq j^*\mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \arrow[dr, "j^*a"'] & & j^*\mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \simeq \mathbf{R}\Gamma'_V\mathscr{O}_V \arrow[dl, "j^*h"] \\
& j^*C &
\end{tikzcd} \tag{12.7}$$

We know that $j^*(c(h)) = 0$, hence the map j^*h is an isomorphism, and hence (12.7) gives a morphism $(j^*h)^{-1}j^*a\colon j^*\mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \to j^*\mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}$, which using (12.4), gives an element of $\Gamma(V, \mathscr{O}_{\mathfrak{X}}) = \mathscr{O}_{\mathfrak{X}}(V)$. It is easy to see that this element is independent of the choice of the fraction (12.6) representing the endomorphism. Thus we get a map $\mathrm{End}_{\mathscr{K}(V)}(\mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) \to \mathscr{O}_{\mathfrak{X}}(V)$ which we denote by α.

Given an element $s \in \mathscr{O}_{\mathfrak{X}}(V)$, it defines a map $s\colon \mathscr{O}_{\mathfrak{X}}(V) \to \mathscr{O}_{\mathfrak{X}}(V)$. The canonical map $t : \mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \to \mathbf{R}\Gamma'_{\mathfrak{X}}j_*j^*\mathscr{O}_{\mathfrak{X}}$ is an isomorphism over V, hence its cone $c(t)$ lies in $\mathcal{L}_Z$. Thus t induces an isomorphism in $\mathscr{K}(V)$. Also, s induces a natural map $j_*j^*\mathscr{O}_{\mathfrak{X}} \to j_*j^*\mathscr{O}_{\mathfrak{X}}$, which gives the following fraction.

$$\begin{tikzcd}
\mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \arrow[dr, "t"] & & & \\
& \mathbf{R}\Gamma'_{\mathfrak{X}}j_*j^*\mathscr{O}_{\mathfrak{X}} \arrow[dr, "\mathbf{R}\Gamma'_{\mathfrak{X}}j_*s"] & & \mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \arrow[dl, "t"'] \\
& & \mathbf{R}\Gamma'_{\mathfrak{X}}j_*j^*\mathscr{O}_{\mathfrak{X}} &
\end{tikzcd} \tag{12.8}$$

Thus we get an element of $\mathrm{End}_{\mathscr{K}(V)}(\mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}})$ and we will denote the resulting map $\mathscr{O}_{\mathfrak{X}}(V) \to \mathrm{End}_{\mathscr{K}(V)}(\mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}})$ by β.

We will now show that $\alpha \circ \beta \simeq \mathbf{1}$. For $s \in \mathscr{O}_{\mathfrak{X}}(V)$, construct a diagram as in (12.8) above. Applying j^* to this diagram and using $j^*\mathbf{R}\Gamma'_{\mathfrak{X}}j_*s \simeq \mathbf{R}\Gamma'_{\mathfrak{X}}s$ and that j^*t is an isomorphism and finally that j^*t commutes with $\mathbf{R}\Gamma'_{\mathfrak{X}}s$, we see that $(\alpha \circ \beta)(s) = s$. It follows that α is surjective and it remains to see that α is injective. Let $g \in \mathrm{End}_{\mathscr{K}(V)}(\mathbf{R}\Gamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}})$ be such that $\alpha(g) = 0$ in $\mathscr{O}_{\mathfrak{X}}(V)$. Assume that the map g is given by the following fraction, where the cone of h lies in $\mathcal{L}_Z$.

$$\begin{tikzcd} \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \arrow[dr, "u"] & & \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \arrow[dl, "h"'] \\ & C & \end{tikzcd} \tag{12.9}$$

Since $\alpha(g) = 0$, and $j^*(h)$ is an isomorphism, therefore $j^*u : j^*\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \to j^*C$ is the zero map, thus the bottom fraction in the diagram (12.10) below represents the zero map. Then, the commutativity of the diagram (12.10) and the fact that the cones of all the vertical arrows lie in $\mathcal{L}_Z$ implies that the fraction on the top is isomorphic to the fraction in the bottom in $\mathscr{K}(V)$. Thus the fraction g is zero.

$$\begin{tikzcd} \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \arrow[d] \arrow[drr, "u"] & & & & \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}} \arrow[d] \arrow[dll, "h"'] \\ \mathbf{R}\varGamma'_{\mathfrak{X}}j_*j^*\mathscr{O}_{\mathfrak{X}} \arrow[drr, "j_*j^*u"] & & C \arrow[d] & & \mathbf{R}\varGamma'_{\mathfrak{X}}j_*j^*\mathscr{O}_{\mathfrak{X}} \arrow[dll, "j_*j^*h"'] \\ & & \mathbf{R}\varGamma'_{\mathfrak{X}}j_*j^*C & & \end{tikzcd} \tag{12.10}$$

□

Now let $V_1 \subset V_2$ be two open subsets of $\mathfrak{X}$ (and hence of $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$) and let Z_1, Z_2 be their respective complements. Then $Z_2 \subset Z_1$ and hence $\mathcal{L}_{Z_2} \subset \mathcal{L}_{Z_1}$. Now from universal property of localization, it follows that there exists a unique functor $q_{12} : \mathscr{K}(V_2) \to \mathscr{K}(V_1)$and q_1 factors upto isomorphism as $q_{12} \circ q_2$.

$$\begin{tikzcd} \mathscr{K} \arrow[rr, "q_1"] \arrow[dr, "q_2"'] & & \mathscr{K}(V_1) \\ & \mathscr{K}(V_2) \arrow[ur, "q_{12}"'] & \end{tikzcd} \tag{12.11}$$

Thus for every inclusion of open subsets $V_1 \subset V_2$, we have obtained a functor $q_{12} : \mathscr{K}(V_2) \to \mathscr{K}(V_1)$. The factorization $q_1 \simeq q_{12} \circ q_2$ is an isomorphism of functors and not an equality. For the following inclusion of open subsets $V_1 \subset V_2 \subset V_3$ in $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$, we can use the universal property and obtain an isomorphism of functors.

$$q_{13} \simeq q_{12} \circ q_{23} \tag{12.12}$$

For the inclusion $j_{12} : V_1 \hookrightarrow V_2$, the restriction map $q_{12}\colon \mathscr{K}(V_2) \to \mathscr{K}(V_1)$ induces a map of endomorphism rings of identity,

$$\mathrm{res}_{12}\colon \mathrm{Hom}_{\mathscr{K}(V_2)}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}, \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) \to \mathrm{Hom}_{\mathscr{K}(V_1)}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}, \mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) \tag{12.13}$$

such that the following is a commutative diagram of rings.

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(V_2)}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) \arrow[r, "\alpha"] \arrow[d, "\mathrm{res}_{12}"'] & \mathscr{O}_{\mathfrak{X}}(V_2) \arrow[d, "j_{12}^*"] \\
\mathrm{End}_{\mathscr{K}(V_1)}(\mathbf{R}\varGamma'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) \arrow[r, "\alpha"] & \mathscr{O}_{\mathfrak{X}}(V_1)
\end{tikzcd} \tag{12.14}$$

The isomorphism (12.5) in Proposition 12.1 and the restriction maps discussed in the preceding paragraph give a locally ringed structure on $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ which is isomorphic to $\mathscr{O}_{\mathfrak{X}}$. Thus the topology on $\mathrm{Spc}_{\otimes}(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}))$ and the isomorphism (12.5) give the following proposition.

Proposition 12.2. *Let $\mathfrak{X}$ be a noetherian formal scheme, then the spectrum of the category $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$ is a locally ringed space isomorphic to $\mathscr{O}_{\mathfrak{X}}$.*

So far we have reconstructed the locally ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ as a spectrum of the tensor triangulated category $\mathbf{D}_{\mathrm{qct}}(\mathfrak{X})$. Next we claim that there is only one equivalence class of coherent ideals $\mathscr{I} \subset \mathscr{O}_{\mathfrak{X}}$ for which the $\mathscr{I}$*-adic* topology makes $\mathfrak{X}$ into a formal scheme. Recall that two coherent ideals $\mathscr{I}_1, \mathscr{I}_2 \subset \mathscr{O}_{\mathfrak{X}}$ are called equivalent if there exist positive integers m and n such that $\mathscr{I}_1^n \subset \mathscr{I}_2$ and $\mathscr{I}_2^m \subset \mathscr{I}_1$.

Now let $\mathscr{I}_1$ and $\mathscr{I}_2$ be two coherent ideals in $\mathscr{O}_{\mathfrak{X}}$ which make $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ into two formal schemes denoted by $\mathfrak{X}_1 = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}, \mathscr{I}_1 \subset \mathscr{O}_{\mathfrak{X}})$ and $\mathfrak{X}_2 = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}, \mathscr{I}_2 \subset \mathscr{O}_{\mathfrak{X}})$ respectively. We can and will further assume that $\mathscr{I}_1$ and $\mathscr{I}_2$ are radical ideals and hence the largest ideals of definition of the respective formal schemes. Since, $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_2)$ is an ordinary scheme with the underlying space $\mathfrak{X}$ and $\mathscr{I}_1$ is the largest defining ideal of $\mathfrak{X}_1$, hence, $\mathscr{I}_1^m \subset \mathscr{I}_2$ for some integer m. By symmetry of the above argument, we also have $\mathscr{I}_2^n \subset \mathscr{I}_1$ for some integer n. Thus $\mathscr{I}_1$ and $\mathscr{I}_2$ are equivalent ideals.

We now have following generalizations of Corollary (8.6) and Theorem (9.7) of [**B1**] respectively.

Theorem 12.1. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two noetherian formal schemes which are either separated or of finite Krull dimension. Assume that there is an equivalence of $\otimes$-triangulated* *categories with identity*

$$(\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}), \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}}, \boldsymbol{R}\mathit{\Gamma}'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}}) \simeq (\mathbf{D}_{\mathrm{qct}}(\mathfrak{Y}), \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{Y}}}, \boldsymbol{R}\mathit{\Gamma}'_{\mathfrak{Y}}\mathscr{O}_{\mathfrak{Y}}).$$

Then there exists an isomorphism of noetherian formal schemes $\mathfrak{X} \simeq \mathfrak{Y}$, inducing the above equivalence.

Theorem 12.2. *Consider the functor $\mathcal{D}\colon \mathbb{F} \to \mathcal{T}$ from the category $\mathbb{F}$ of noetherian formal schemes which are either separated or of finite Krull dimension to the category $\mathcal{T}$ of tensor triangulated categories with unit, given by $\mathfrak{X} \mapsto (\mathbf{D}_{\mathrm{qct}}(\mathfrak{X}), \otimes^{\mathbf{L}}_{\mathscr{O}_{\mathfrak{X}}}, \boldsymbol{R}\mathit{\Gamma}'_{\mathfrak{X}}\mathscr{O}_{\mathfrak{X}})$. Then this functor is faithful and takes isomorphisms to isomorphisms. Moreover, over the subcategory $\mathcal{T}' \subset \mathcal{T}$ comprising tensor triangulated categories satisfying Properties 1, 2 and 3 of Definition 11.1, there exists a functor from $\mathcal{T}$ into ringed spaces such that its pre-composition with the functor $\mathcal{D}$ yields the natural inclusion of $\mathbb{F}$ into the ringed spaces.*

Bibliography

[**AJL**] L. Alonso Tarrio, A. Jeremias Lopez and J. Lipman *Duality and flat base change on formal schemes* 1999: Contemporary Mathematics, 244, American Mathematical Soc., Providence, RI pp. 3-90.

[**AJL2**] L. Alonso Tarrio, A. Jeremias Lopez, Marta Perez *Local structure theorems for smooth maps of formal schemes* 2009: Journal of Pure and Applied Algebra pp. 1373-1398.

[**AJS**] L.A. Tarrio, A.J. Lopez and M.J.S. Salorio *Bousfiled localization on formal schemes* 2004: Journal of Algebra, Vol. 278, pp. 585-610.

[**AK**] A. Altman and S. Kleinman *Introduction to Grothendieck Duality Theory* 1970: Lecture Notes in Mathematics.

[**B1**] P. Balmer *Presheaves of triangulated categories and reconstruction of schemes* 2002: Mathematische Annalen, Vol. 324, no.3, 557-580.

[**Co**] B. Conrad *Grothendieck Duality and base change* 2000: Lecture Notes in Mathematics.

[**H**] R. Hartshorne *Residues and Duality* 1966: Lecture Notes in Mathematics.

[**Ho**] M. J. Hopkins *Global methods in homotopy theory* 1987: Homotopy Theory (conference at Durham 1985), London Math. Soc. Lecture Notes 117 (1987) 73-96.

[**L**] J. Lipman and M. Hashimoto *Foundations of Grothendieck Duality for Diagrams of Schemes* 2009: Lecture Notes in Mathematics.

[**LNS**] J. Lipman, P. Sastry, S. Nayak *Pseudofunctorial Behaviour of Cousin Complexes* 1999: Contemporary Math., pp. 3-133.

[**Mat**] Hideyuki Matsumura *Commutative Ring Theory* 1986: Cambridge University Press.

[**N**] A. Neeman *Grothendieck Duality theorem via Bousfield's techniques and Brown representability* 1996: Journal of American Mathematical Scociety, Vol. 9, pp. 205-236.

[**N1**] A. Neeman *The chromatic tower of* $D(R)$ 1992: Topology(31) no.3, pp. 205-236.

[**Nk**] S. Nayak *Pasting Pseudofunctors* 2005: Contemporary Math. 375, Amer. Math. Soc., Providence, RI, pp. 195-271.

[**NS**] P. Sastry, S. Nayak *Explicit Verdier* 2017: Preprint.

[**T1**] R. W. Thomason*The classification of triangulated subcategories* 1997: Composito Mathematica, Vol. 105, pp. 1-27.

[**TT**] R. W. Thomason and T. Trobaugh *Higher algebraic K-theory of schemes and derived categories* 1990: The Grothendieck Festschrift. Progress in Mathematics, vol 88. Birkhuser, Boston, MA, pp. 247-435.

[**V**] J.L. Verdier *Base change for twisted inverse image of coherent sheaves* 1969: Algebraic Geometry, Oxford Univ. press, pp. 393-408.

[**Y**] A. Yekutieli *Rigid Dualizing Complexes via Differential Graded Algebras (Survey), in Triangulated Categories,* 2010: London Math. Soc, Lecture Note Series 375, pp. 452-463.

[**YJ**] A. Yekutieli and J.J. Zhang *Rigid Dualizing Complexes over Commutative Rings,* 2009: Algebras and Representation Theory 12, Number 1, pp. 19-52.

List of Publications

1. Suresh Nayak and Saurabh Singh:
   Duality Pseudofunctor over the Composites of Smooth and Pseudoproper Morphisms of Noetherian Formal Schemes.
   Preprint, 2018. (To be submitted).

2. Suresh Nayak and Saurabh Singh:
   Reconstruction of Formal Schemes using their Derived Categories.
   Preprint, 2018. (To be submitted).