STUDIES ON POLYNOMIAL RINGS THROUGH LOCALLY NILPOTENT DERIVATIONS

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Indian Statistical Institute
2019

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Thesis submitted to the Indian Statistical Institute in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy
June, 2019

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Acknowledgements

Indian Statistical Institute, Kolkata, has nurtured me to self-enhancement and is fortifying me always, with that power of knowledge that I am receiving continously through the tutelage and resources of the institute.

In my pursuit to study Commutative Algebra and Affine Algebraic Geometry, I certainly want to acknowledge the commendable supervision that I have received from my supervisor, Dr. Neena Gupta. She always welcomed my queries on the subject and encouraged me to explore the research topics with her valuable suggestions. She directed me to identify emerging mathematical problems, from which I tried to derive mathematical conclusions and possible solutions. Through intricate calculations and innovative ideas and methods that I adopted under her guidance, I have been able to conduct my research endeavour. Her deep observations on the subject motivated me to deal with the evolving dynamics of Affine Algebraic Geometry, specially locally nilpotent derivations. Indeed, while speculating on some challenging problems in Affine Algebraic Geometry, this theory's subsequent applications enabled me to excavate some new dimensions. The student-teacher collaboration did yield interesting mathematical results, which paved the way for this thesis and future research.

My journey to the submission of my thesis, had its beginning in the inspiration to learn Commutative Algebra from a stalwart like Prof. Amartya Kumar Dutta (our dear Sir) through his immensely interesting classes which I fortunately attended during my M. Math curriculum at ISI Kolkata. His lectures compelled my curious mind to visualize the concepts and expand the horizon of my knowledge. Prof. Amartya Kumar Dutta with his sea of experiential knowledge, emboldened me to scale newer heights of Commutative Algebra and Affine Algebraic Geometry, to which I have never arrived before. His meticulous attention to the mathematical details, not only taught me mathematical rigour but also did hone my skills to suitably and effectively project the resultant mathematical truth. Sir gave many engaging lectures on Affine Algebraic Geometry and locally nilpotent derivations. He introduced me to the myriad techniques of the subject and relevant research problems. His able guidance was always there in writing up the papers and his queries led us to new problems and solutions. In fact, the research on nice and quasi-nice derivations came out of his queries. The research courses conducted by him and Dr. Neena Gupta exposed me to the intricacies of excellent and notable research works, authored by themselves, which laid the foundation for my research.

I am particularly grateful to Prof. Mrinal K. Das for introducing me to projective modules and Euler class groups and related research problems. Indeed, the entire faculty of Stat-Math Unit of ISI, namely Prof. S.M. Srivastava, Prof. D. Goswami, Prof. R.P. Sarkar, Prof. B. Behera and also the visiting faculty all did enrich me with their enlightening classes during my days of M. Math.

Learning as a perpetual process flourishes through positive and constructive interactions and this platform was provided to me by ISI Kolkata. I had the opportunity to interact with eminent experts of Commutative Algebra, namely Prof. S.M. Bhatwadekar, Prof. A. Sathaye and Prof. T. Asanuma. I must acknowledge that the interactive sessions with each of them, who have devoted years in building a rich heritage of ground-breaking research, did give food to my mathematical thoughts.

Finally, I wish to express my deep gratitude to my parents and my wife for their love and support. This thesis would have been simply impossible without their encouragement. My heart-felt gratitude goes out to my pet cat (late) for his love and affection. I also thank all my friends at ISI, especially Mr. Animesh Lahiri, for their support and encouragement. I wish to express my gratitude to Dr. Sagnik Chakraborty for many fruitful and active discussions.

Notation

$\mathbb{N}$	:	Set of Natural Numbers.
$\mathbb{Z}$	:	Ring of Integers.
$\mathbb{Q}$	:	Field of Rational numbers.
$\mathbb{R}$	:	Field of Real numbers.
$\mathbb{C}$	:	Field of Complex numbers.
$k^{(n)}$	:	Field of rational functions in n variables over a field k.
DVR	:	Discrete Valuation Ring.
PID	:	Principal Ideal Domain.
UFD	:	Unique Factorization Domain.

For a commutative ring R, a prime ideal P of R, an R-algebra A and an R-module M, the following notation will be used:

R^*	:	Group of units of R.
$\mathrm{Pic}(R)$	:	Picard group of R.
$R^{[n]}$	:	Polynomial ring in n variables over R.
$\mathrm{Spec}(R)$	:	The set of all prime ideals of R.
$\mathrm{Max}(R)$	:	The set of all maximal ideals of R.
$\mathrm{ht}(P)$	:	Height of P.
$k(P)$	:	Residue field R_P/PR_P.
A_P	:	$S^{-1}A$ where $S = R\backslash P$; also identified with $A \otimes_R R_P$.
$\mathrm{Sym}_R(M)$	:	Symmetric algebra of M over R.
$\mathrm{Der}_k(B)$	:	Set of k-derivations of the k-algebra B.

For integral domains $R \subseteq A$,

$\mathrm{tr.deg}_R(A)$	:	Transcendence degree of the field of fractions of A over that of R.

If B is a subset of A, we shall use the notation $B \subseteq A$ when $B = A$ is a possibility and the notation $B \subsetneqq A$ for a proper subset when we want to emphasise that $B \neq A$.

Contents

Chapter 1

Introduction

The main aim of the thesis is to investigate the following problems :

(i) To find an algebraic characterization of the polynomial ring $k[X, Y, Z]$ over an algebraically closed field k of characteristic zero (in particular, an algebraic characterization of the affine three space).

(ii) To determine the structure of the kernel of a *nice derivation* on the polynomial ring $R[X, Y, Z]$ over a PID R containing $\mathbb{Q}$; in particular, the structure of the kernel of a *nice derivation* on $k[X_1, X_2, X_3, X_4]$ of *rank* 3, where k is a field of characteristic zero.

The first problem will be discussed in Chapter 3 under the heading "On algebraic characterization of the affine three space" while the second problem will be taken up in Chapter 4 entitled "On Nice and Quasi-Nice Derivations". Sections 1.2 and 1.3 of this chapter present an overview of the main results of Chapters 3 and 4, along with their contexts. In Chapter 2, we give the necessary definitions (Section 2.1) and state some well-known results on locally nilpotent derivations (Section 2.2) and on polynomial rings and projective modules (Section 2.3).

1.1 The concept of a locally nilpotent derivation

Let k be a field of characteristic zero, R a k-domain, and B an R-algebra. An R-derivation D on B is an R-linear map $D : B \to B$, which satisfies the Leibniz rule i.e., $D(ab) = aD(b) + bD(a)$ for all $a, b \in B$. In addition, if for each $a \in B$, there exists $n \in \mathbb{N}$ such that $D^n(a) = 0$, then D is said to be a *locally nilpotent derivation* on B. The set of all locally nilpotent R-derivations on B is denoted by $LND_R(B)$. When R is understood from the context (e.g. when $R = k$), we simply denote it by $LND(B)$. For any $D \in LND_R(B)$, the *kernel* of D is defined to be the subring $\{a \in B \mid D(a) = 0\}$. We denote the kernel of a locally nilpotent derivation D by $Ker\ D$.

Locally nilpotent derivations on affine domains over a field k are the ring theoretic version of $\mathbb{G}_a$-actions, i.e., actions of the algebraic group $(k, +)$ on affine varieties over k. More precisely, for an algebraically closed field k of characteristic zero and an affine k-domain B, there is a one to one correspondence between $LND(B)$ and the set of $\mathbb{G}_a$-actions on $\mathrm{Max}(B)$. The kernel of a locally nilpotent derivation on B corresponds to the ring of invariants of a $\mathbb{G}_a$-action on $\mathrm{Max}(B)$.

1.2 On algebraic characterization of the affine three space

A major theme in Affine Algebraic Geometry is the study of affine n-spaces (equivalently, polynomial rings) over a field. Investigations in the area often lead to the problem of determining whether a given affine domain is a polynomial ring. To show that an affine domain suspected to be a polynomial ring is indeed so, one approach could be to find a suitable set of coordinates (or variables). However, given an arbitrary polynomial, it is in general not at all easy to check whether it is a coordinate. In fact, many open problems in Affine Algebraic Geometry are closely related to the question of determining whether certain polynomials are coordinates. Moreover, the approach may not be applicable when the affine domain is abstractly defined as in the "Cancellation Problem" which asks whether for an affine domain A over a field k, $A^{[1]} = k^{[n+1]}$ necessarily implies $A = k^{[n]}$.

Another approach would be to find useful characterizations of the affine n-space, and then examine whether a given affine domain satisfies those characterizing conditions. This approach has often turned out to be fruitful. Consequently, the "Characterization Problem" is considered one of the most important problems on Affine Spaces, along with the "Cancellation Problem", "Embedding Problem", "Automorphism Problem", "Jacobian Problem" and other famous problems.

A simple example will illustrate the usefulness of having a good characterization of an affine space in the context of problems like the Cancellation Problem. If k is an algebraically closed field of characteristic zero, then an algebraic characterization of the polynomial ring $k^{[1]}$ is given by the fact that $k^{[1]}$ is the only one-dimensional UFD with trivial units. Now this immediately solves the Cancellation Problem for the affine line, i.e., that $A^{[1]} = k^{[2]}$ implies that $A = k^{[1]}$. A topological characterization of the affine line $\mathbb{A}^1_{\mathbb{C}}$ is given by the fact that the affine line $\mathbb{A}^1_{\mathbb{C}}$ is the only acyclic normal curve.

In his attempt to solve the Cancellation Problem, C.P. Ramanujam obtained a remarkable topological characterization of the affine plane $\mathbb{C}^2$ ([35]). Later an algebraic characterization of the polynomial ring $k^{[2]}$ was obtained by M. Miyanishi ([30]) for an algebraically closed field k of characteristic zero. This characterization involves the concept of "locally nilpotent derivation". Miyanishi's characterization theorem established that any two-dimensional affine domain B over an algebraically closed field k of characteristic zero which is a UFD, whose units are all in k^* and on which there exists a non-zero locally nilpotent derivation, must be $k^{[2]}$. This characterization was used by T. Fujita and Miyanishi-Sugie ([22], [33]) to solve the Cancellation Problem for the affine plane. Since then, there have been several attempts to give a characterization of $k^{[3]}$. Remarkable results were obtained by M. Miyanishi ([31]) and S. Kaliman ([25]). These results involved some topological invariants. In Chapter 3, we will use a variant of the Makar-Limanov invariant, to give new algebraic characterizations of $k^{[2]}$ and $k^{[3]}$. We recall the definition of the Makar-Limanov invariant and then define its variant which will be used in our characterization theorems.

The *Makar-Limanov invariant* of B, denoted by $ML(B)$, is defined to be

$$ML(B) := \bigcap_{D \in LND(B)} Ker\ D.$$

The Makar-Limanov invariant has been a powerful tool for solving some major problems in affine algebraic geometry like the Linearization Problem ([21, pp. 195–204]). L.G. Makar-Limanov used this invariant to show that the well-known Russell-Koras threefold, which was a candidate for a counterexample to the Linearization Problem, is not isomorphic to $\mathbb{C}^{[3]}$ ([28]). When k is an algebraically closed field of characteristic zero, the Makar-Limanov invariant gives the following characterization of $k^{[1]}$ ([8, Lemma 2.3]).

Theorem : Let k be a field of characteristic zero and B an affine k-domain with $\operatorname{tr.deg}_k B = 1$. Then $B = k^{[1]}$ if and only if $ML(B) = k$.

However, the triviality of the Makar-Limanov invariant alone does not characterize the affine 2-space (i.e., $\dim B = 2$ and $ML(B) = k \nRightarrow B = k^{[2]}$). There are two-dimensional affine domains B over any field k of characteristic zero called "Danielewski surfaces" (see Theorem 2.2.10) for which $ML(B) = k$ but which are not $k^{[2]}$.

In Section 3.1, we will show that under the additional condition that B has a locally nilpotent derivation D "with slice" (i.e., $1 \in \operatorname{Im}(D)$), the condition "$ML(B) = k$" does imply that $B = k^{[2]}$ when $\dim B = 2$.

We now define a variant of the Makar-Limanov invariant. This invariant is mentioned in the book *Algebraic Theory of Locally Nilpotent Derivations* by G. Freudenburg ([21, pg. 237]). Consider the subset $LND^*(B)$ of $LND(B)$ defined by

$$LND^*(B) = \{D \in LND(B) \mid Ds = 1 \text{ for some } s \in B\}.$$

Then we define

$$ML^*(B) := \bigcap_{D \in LND^*(B)} Ker\ D.$$

If $LND^*(B) = \emptyset$, we define $ML^*(B)$ to be B. Note that if $ML^*(B) = k$ then automatically $ML(B) = k$. Also note that $ML^*(k^{[n]}) = ML(k^{[n]}) = k$ for each $n \geqslant 1$.

Now we state our result on the characterization of the affine two space.

Theorem 3.1.8 *Let k be a field of characteristic zero and B a two-dimensional affine k-domain. Then the following are equivalent:*

(I) $B = k^{[2]}$.

(II) $ML^*(B) = k$.

(III) $ML(B) = k$ *and* $ML^*(B) \neq B$.

We will also show that if B is an affine UFD of dimension 3 over an algebraically closed field having a non-trivial locally nilpotent derivation D with a slice (i.e., $1 \in \operatorname{Im}(D)$), then $B = k^{[3]}$ iff $ML(B) = k$, i.e., we will prove

Theorem 3.2.6 *Let k be an algebraically closed field of characteristic zero and B an affine k-domain such that B is a UFD and $\dim B = 3$. Then the following are equivalent:*

(I) $B = k^{[3]}$.

(II) $ML^*(B) = k$.

(III) $ML(B) = k$ *and* $ML^*(B) \neq B$.

In Section 3.2, we will present a complete classification of three-dimensional affine factorial domains with $ML^*(B) = ML(B)$ (Proposition 3.2.3). In Section 3.3, we will also give examples of three-dimensional affine UFDs B for which $ML^*(B) = B$ but $ML(B) \subsetneqq ML^*(B)$. In the three examples (3.3.3, 3.3.4 and 3.3.5), $\operatorname{tr.deg}_k ML(B)$ will be two, one and zero respectively. We will also present an example (3.3.6) which will show that Theorem 3.2.5 does not extend to a four-dimensional affine regular UFD.

1.3 On Nice and Quasi-Nice Derivations

Let $B = R^{[n]}$. A locally nilpotent derivation D on B is said to be a *nice derivation* if $D^2(T_i) = 0$ for all $i \in \{1, \ldots, n\}$ for some coordinate system $(T_1, T_2, \ldots, T_n)$ of B. For any $D \in LND_R(B)$, the *rank* of D, denoted by *rank* D, is defined to be the least integer i for which there exists a coordinate system $(X_1, X_2, \ldots, X_n)$ of B satisfying $R[X_{i+1}, \ldots, X_n] \subseteq Ker\ D$.

We shall now discuss special cases of an important problem (Question 2 below) in Affine Algebraic Geometry. The problem is closely related to the celebrated Hilbert Fourteenth Problem which we recall below.

Question 1 (Hilbert's Fourteenth Problem):
Let k be a field of characteristic zero, L a subfield of $k(X_1, X_2, \ldots, X_n)(= k^{(n)})$ and $A := k[X_1, X_2, \ldots, X_n] \cap L$. Thus $k \subseteq A \subseteq k[X_1, X_2, \ldots, X_n]$ and L is the field of fractions of A. Is A finitely generated as a k-algebra?

It was shown by O. Zariski that the answer to Question 1 is affirmative if $\operatorname{tr.deg}_k L \leqslant 2$ ([40]). The first counterexample to Hilbert's Fourteenth Problem was given by M. Nagata ([34]) for $n = 32$ and $\operatorname{tr.deg}_k L = 4$ and later P. Roberts ([37]) gave a counterexample for the case $n = 7$ and $\operatorname{tr.deg}_k L = 6$. A'Campo-Neuen ([2]) and Deveney-Finston ([15]) showed that the example given by Roberts arises as the kernel of a locally nilpotent derivation on $k^{[7]}$. Later G. Freudenburg ([20]) and Daigle-Freudenburg ([12]) constructed locally nilpotent derivations on $k^{[n]}$ $(n = 6, n = 5)$ for which the kernels are not finitely generated as k-algebras. Soon S. Kuroda gave counterexamples to the Hilbert Fourteenth Problem for the cases $n = 4$ and $\operatorname{tr.deg}_k L = 3$ in [26] and $n = 3$ and $\operatorname{tr.deg}_k L = 3$ in [27]. In view of Zariski's theorem ([40]), Kuroda's examples are counterexamples to Hilbert's Fourteenth Problem for the lowest possible dimension. However, the examples of Kuroda cannot be viewed as kernels of some locally nilpotent derivations on $k^{[n]}$. This leads us to the following special case of the original Hilbert Fourteenth Problem.

Question 2′: Let k be a field of characteristic zero, $B = k^{[n]}$ and $D \in LND(B)$. Is $Ker\ D$ necessarily finitely generated as a k-algebra?

The answer is affirmative for $n \leqslant 3$ as shown by R. Rentschler ([36]) for $n = 2$ and by M. Miyanishi ([32]) for $n = 3$. In fact, Rentschler ([36]) and Miyanishi ([32]) proved that the kernels are polynomial rings. As mentioned earlier, for $n \geqslant 5$, there are counterexamples to Question $2'$ as shown by Daigle and Freudenburg ([20], [12]). Thus Question $2'$ reduces to

Question 2: Let k be a field of characteristic zero and $D \in LND(k^{[4]})$. Is $Ker\ D$ necessarily finitely generated?

Daigle-Freudenburg have constructed examples to show that given any integer $n \geqslant 3$, there exists a locally nilpotent derivation on $k^{[4]}$ of *rank* less or equal to 3 whose kernel cannot be generated by fewer than n elements ([14]). Question 2 is open. One therefore explores the following questions.

Question 3: Let k be a field of characteristic zero and $D \in LND(k^{[4]})$. Under what additional hypothesis is $Ker\ D$ finitely generated?

We mention two results addressing Question 3. First, Daigle-Freudenburg ([13]) showed that the kernel of any triangular k-derivation on $k^{[4]}$ is finitely generated over k. Later, Bhatwadekar-Daigle showed that the kernel is indeed finitely generated ([5, Theorem 1]) in the case when $rank\ D \leqslant 3$.

In view of the theorems of Rentschler and Miyanishi (that $Ker\ D$ in Question $2'$ is a polynomial ring for $n \leqslant 3$) and the theorem of Bhatwadekar-Daigle (that $Ker\ D$ in Question $2'$ is finitely generated for $n = 4$ when $rank\ D \leqslant 3$), one explores the following question.

Question 4: Let k be a field of characteristic zero and D be a locally nilpotent derivation on $k^{[4]}$ of *rank* at most 3. Under what additional hypothesis is $Ker\ D$ a polynomial ring?

Bhatwadekar-Gupta-Lokhande showed that if $Ker\ D$ is regular, then $Ker\ D$ is indeed $k^{[3]}$ ([7, Theorem 3.5]). One of the results of the thesis establishes the fact that $Ker\ D$ in Question 4 is a polynomial ring if D is a nice derivation (Corollary 4.1.8 below). This result will come as an outcome of our results on nice derivations on $R[X, Y, Z]$, where R is a PID containing $\mathbb{Q}$.

We now state our results on nice and the more general quasi-nice derivations on the polynomial ring $B = R[X, Y, Z]$, where R is an integral domain containing $\mathbb{Q}$. Let m be a positive integer $\leqslant 3$. We will call a locally nilpotent derivation D on B *quasi-nice or m-quasi* if $D^2(T_i) = 0$ for all $i \in \{1, \dots, m\}$ for some coordinate system (T_1, T_2, T_3) of B. Thus a quasi-nice derivation is a *nice derivation* if $m = 3$.

The case of the polynomial ring $k[X, Y, Z]$, where k is a field of characteristic zero was investigated by Z. Wang in [39]. He showed that *rank* D is less than 3 when $m = 2$ or 3 and that *rank* $D = 1$ when D is a nice derivation (i.e., when $m = 3$).

Now let R be a Noetherian domain containing $\mathbb{Q}$, say R is regular. It is natural to ask how far we can extend the results of Wang to $R[X, Y, Z](= R^{[3]})$. In particular, we consider the following question.

Question 5. *If D is a nice derivation on $R[X, Y, Z]$, then is rank $D = 1$, or, at least, is rank $D < 3$?*

In Section 4.1, we will give a complete description of the kernel of a nice derivation on $R[X, Y, Z]$ when R is a PID. We will show that the rank of D is indeed less than 3 and its kernel is a polynomial ring generated by two elements over R (Theorem 4.1.6). The precise result is the following:

Theorem 4.1.6. *Let R be a PID containing $\mathbb{Q}$ with field of fractions L and $B := R[X, Y, Z] = R^{[3]}$. Let $D(\neq 0) \in LND_R(B)$, and $A := Ker\ D$. Suppose that D is irreducible and $D^2X = D^2Y = D^2Z = 0$. Then there exists a coordinate system (U, V, W) of B related to (X, Y, Z) by a linear change such that the following hold:*

(i) *A contains a nonzero linear form of $\{X, Y, Z\}$.*

(ii) *rank $D \leqslant 2$. In particular, $A = R^{[2]}$.*

(iii) *$A = R[U, gV - fW]$, where $DV = f$, $DW = g$, and $f, g \in R[U]$ such that $gcd_{R[U]}(f, g) = 1$.*

(iv) *Either f and g are comaximal in B or they form a regular sequence in B. Moreover if they are comaximal (i.e., D is fixed-point free), then*

$B = A^{[1]}$ and rank $D = 1$; and if they form a regular sequence, then B is not A-flat and rank $D = 2$.

As a consequence of Theorem 4.1.6, we have the following result in response to Question 4.

Corollary 4.1.8. *Let k be a field of characteristic zero and let $k[X_1, X_2, X_3, X_4] = k^{[4]}$. Let $D \in LND(k[X_1, X_2, X_3, X_4])$, be such that D is irreducible and $DX_1 = 0$ and $D^2X_i = 0$ for $i = 2, 3, 4$. Then $Ker\ D = k^{[3]}$.*

In Section 4.1, we will also present a few examples and results which will explore how far Theorem 4.1.6 can be extended to more general rings in the context of Question 5. Example 4.1.3 will show that the kernel of a nice derivation on $R[X, Y, Z]$ need not be a polynomial ring, even when R is a Dedekind domain (cf. Theorem 4.1.6(ii)). However, Proposition 4.1.9 will show that the kernel is always generated by at most three elements when R is a Dedekind domain. We will construct a nice derivation D on $R = k^{[1]}$ with $rank\ D = 2$ (Example 4.1.5) showing that Wang's result over fields does not extend to PIDs. We will also construct a nice derivation D on $R = k^{[2]}$ with $rank\ D = 3$ (Example 4.1.10) showing that Theorem 4.1.6 does not extend to two-dimensional regular or factorial domains.

The following question on quasi-nice derivations arises in view of Wang's result that $rank\ D$ is less than 3 when $m = 2$.

Question 6. *If D is a locally nilpotent derivation on $R[X, Y, Z]$, such that D is irreducible and $D^2X = D^2Y = 0$, is then $rank\ D < 3$?*

In Section 4.2, we will investigate this question and present some partial results when R is a PID (Proposition 4.2.4) and a Dedekind domain (Proposition 4.2.6). Example 4.2.5 will show that Question 6 has a negative answer in general, even when R is a PID.

When k is a field of characteristic zero and D is a locally nilpotent derivation on $k[X, Y, Z]$ with $D^2(X) = 0$, D. Daigle had proved that the rank of D is less than 3 ([10, Theorems 5.1 and 5.2]). Moreover, he had shown that there exist $T_1, T_2, T_3 \in k[X, Y, Z]$ such that $k[X, Y, Z] = k[T_1, T_2, T_3]$ and $D^2(T_1) = D^2(T_2) = 0$. However, we shall construct a locally nilpotent derivation on $R[X, Y, Z]$ when $R = k^{[1]}$, such that $D^2(X) = 0$ and there does not exist any coordinate system (T_1, T_2, T_3) of $R[X, Y, Z]$ with $D^2(T_1) = D^2(T_2) = 0$ (Example 4.2.9).

Chapter 2

Preliminaries

Hence onwards by a "ring", we shall mean a "commutative ring with unity" and by an "algebra" a "commutative algebra". For a ring R, an R-algebra A and an integer n ($\geqslant 1$), we shall use the notation "$A = R^{[n]}$" to denote that A is isomorphic to a polynomial ring in n variables over R. Let $A = R[X_1, X_2, \dots, X_n](= R^{[n]})$ and $F \in A$. F is said to be a *coordinate in* A, if there exist $F_2, \dots, F_n \in A$ such that $A = R[F, F_2, \dots, F_n]$. A set of n polynomials $f_1, f_2, \dots, f_n$ in A are said to form a *coordinate system* if $A = R[f_1, f_2, \dots, f_n]$.

An integral domain B containing a field k will be called an "affine domain over k" if B is finitely generated as a k-algebra. For a ring A and a nonzerodivisor $f \in A$, we use the notation A_f to denote the localisation of A with respect to the multiplicatively closed set $\{1, f, f^2, \dots\}$. We denote the Krull dimension of a ring B by dim B. Capital letters like X, Y, Z, T, U, V will be used as indeterminates over respective ground rings; thus, $k[X, Y, Z] = k^{[3]}$, $R[U, V] = R^{[2]}$, etc.

A subring $A \subseteq B$ is defined to be *factorially closed* in B if, given nonzero $f, g \in B$, the condition $fg \in A \setminus \{0\}$ implies $f \in A$ and $g \in A$. When the ambient ring B is understood, we will simply say that A is factorially closed. A routine verification shows that a factorially closed subring of a UFD is a UFD. If A is a factorially closed subring of B, then A is algebraically closed in B; further if S is a multiplicatively closed set in A then $S^{-1}A$ is a factorially closed subring of $S^{-1}B$.

2.1 Definitions and some known results

Let k be a field of characteristic zero, R a k-domain, and B an R-domain.

Definition 2.1.1. (D1) An R-derivation D on B is an R-linear map $D : B \to B$, which satisfies the Leibniz rule i.e., $D(ab) = aD(b) + bD(a)$ for all $a, b \in B$. In addition, if for each $a \in B$, there exists $n \in \mathbb{N}$ such that $D^n(a) = 0$, then D is said to be a *locally nilpotent derivation* on B. The set of all locally nilpotent R-derivations on B is denoted by $LND_R(B)$. When R is understood from the context (e.g. when $R = k$), we simply denote it by $LND(B)$.

(D2) For any $D \in LND_R(B)$, the *kernel* of D is defined to be the subring

$$\{a \in B \mid D(a) = 0\}.$$

We denote the kernel of a locally nilpotent derivation D by $Ker\ D$.

(D3) A locally nilpotent derivation D is said to be *reducible* if there exists a non-unit $b \in B$ such that $DB \subseteq (b)B$; otherwise D is said to be *irreducible*. If B is a UFD and $\Delta \in LND_R(B)$, then there exists an irreducible $D \in LND_R(B)$ and $a \in B$ such that $\Delta = aD$ where D is unique up to multiplication by a unit ([21, Proposition 2.2]).

(D4) When $B := R^{[n]}$ and $D \in LND_R(B)$, the *rank* of D, denoted by *rank* D, is defined to be the least integer i for which there exists a coordinate system $(X_1, X_2, \ldots, X_n)$ of B satisfying $R[X_{i+1}, \ldots, X_n] \subseteq Ker\ D$.

(D5) An element $s \in B$ is called a *slice* if $Ds = 1$, and a *local slice* if $Ds \in Ker\ D$ and $Ds \neq 0$. Moreover, D is said to be *fixed-point free* if the B ideal $(DB) = B$.

(D6) Let B be a k-domain and D an element of $LND(B)$ with a local slice $r \in B$. The Dixmier map induced by r is defined to be the k-algebra homomorphism $\pi_r : B \to B_{Dr}$, given by

$$\pi_r(f) = \sum_{i \geqslant 0} \frac{(-1)^i}{i!} D^i f \frac{r^i}{(Dr)^i}.$$

(D7) Let B be a k-domain. The *Makar-Limanov invariant* of B, denoted by

$ML(B)$, is defined to be

$$ML(B) := \bigcap_{D \in LND(B)} Ker\ D.$$

Consider the subset $LND^*(B)$ of $LND(B)$ defined by

$$LND^*(B) = \{D \in LND(B) \mid Ds = 1 \text{ for some } s \in B\}.$$

Then we define

$$ML^*(B) := \bigcap_{D \in LND^*(B)} Ker\ D.$$

This invariant occurs in [21, p. 237]. If $LND^*(B) = \emptyset$, we define $ML^*(B)$ to be B. Note that if $ML^*(B) = k$ then automatically $ML(B) = k$. Also note that $ML^*(k^{[n]}) = ML(k^{[n]}) = k$ for each $n \geqslant 1$.

(D8) An affine k-domain B is defined to be *rigid* if it does not have any non-zero locally nilpotent derivation. Thus for a rigid ring B, $ML(B) = ML^*(B) = B$. B is defined to be *semi-rigid* if there exists a non-zero locally nilpotent derivation D on B such that $LND(B) = \{fD \mid f \in Ker\ D\}$. Thus for an affine k-domain B, with $LND(B) \neq \{0\}$, B is semi-rigid if and only if $ML(B) = Ker\ D$ for all non-zero $D \in LND(B)$.

(D8) We say two locally nilpotent derivations D_1 and $D_2 \in LND_R(B)$ are *distinct* if $Ker\ D_1 \neq Ker\ D_2$.

(D9) Let k be a field of characteristic zero, R a k-domain, $B := R^{[n]}$ and m be a positive integer $\leqslant n$. We will call a locally nilpotent derivation D on B *quasi-nice or m-quasi* if $D^2(T_i) = 0$ for all $i \in \{1, \dots, m\}$ for some coordinate system $(T_1, T_2, \dots, T_n)$ of B. Thus for any two positive integers r and m such that $1 \leqslant m < r \leqslant n$, it is easy to see that an r-quasi derivation is also an m-quasi derivation. We shall call an m-quasi derivation to be *strictly m-quasi* if it is not r-quasi for any positive integer $r > m$. When $m = n$, such a locally nilpotent derivation D on B is said to be a *nice derivation*.

Next we state a necessary and sufficient criterion, due to Nagata, for an integral domain to be a UFD ([29, Theorem 20.2]).

Lemma 2.1.2. *Let R be a Noetherian domain. If there exists a prime element x in R such that R_x is a UFD, then R is a UFD.*

We now quote a well-known result (for a reference, see the proof of [21, Lemma 2.8]).

Lemma 2.1.3. *Let k be an algebraically closed field of characteristic zero and C be an affine UFD over k of dimension one. Then $C = k[t, \frac{1}{p(t)}]$, where $k[t] = k^{[1]}$ and $p(t) \in k[t] \setminus \{0\}$. As a consequence, if $C^* = k^*$, then $C = k^{[1]}$.*

2.2 Some results on locally nilpotent derivations

The following lemma states some basic properties of locally nilpotent derivations on an affine domain ([21]).

Lemma 2.2.1. *Let k be a field of characteristic zero and B be an affine k-domain. Let $D \in LND(B)$ and $A := Ker\ D$. Then the following hold:*

(i) *A is a factorially closed subring of B.*

(ii) *For any multiplicatively closed subset S of $A \setminus \{0\}$, D extends to a locally nilpotent derivation on $S^{-1}B$ with kernel $S^{-1}A$ and $B \cap S^{-1}A = A$.*

(iii) *Moreover, if D is non-zero, then $\operatorname{tr.deg}_A B = 1$.*

As a consequence, $ML(B)$ and $ML^*(B)$ are factorially closed subrings of B and hence are algebraically closed in B.

The following important result is known as the Slice Theorem ([21, Corollary 1.22]).

Theorem 2.2.2. *Let k be a field of characteristic zero and B a k-domain. Suppose $D \in LND(B)$ admits a slice $s \in B$, and let $A = Ker\ D$. Then*

(a) *$B = A[s]$ and $D = \frac{\partial}{\partial s}$.*

(b) *$A = \pi_s(B)$ and $Ker\ \pi_s = sB$.*

(c) *If B is affine, then A is affine.*

The following theorem of Daigle and Freudenburg characterizes locally nilpotent derivations on $R^{[2]}$, where R is a UFD containing $\mathbb{Q}$ ([11, Theorem 2.4]).

Theorem 2.2.3. *Let R be a UFD containing $\mathbb{Q}$ with field of fractions K and let $B = R[X, Y](= R^{[2]})$. For an R-derivation $D \neq 0$ on B, the following are equivalent:*

(i) *D is locally nilpotent.*

(ii) $D = \alpha(\frac{\partial F}{\partial Y}\frac{\partial}{\partial X} - \frac{\partial F}{\partial X}\frac{\partial}{\partial Y})$*, for some* $F \in B$ *which is a variable of* $K[X,Y]$ *satisfying*
$gcd_B(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}) = 1$*, and for some* $\alpha \in R[F] \setminus \{0\}$.

Moreover, if the above conditions are satisfied, then $Ker\ D = R[F] = R^{[1]}$.

With the same notation as above, the following lemma gives interesting results when D satisfies some additional hypothesis ([39, Lemma 4.2]).

Lemma 2.2.4. *Let* R *be a UFD containing* $\mathbb{Q}$, $B = R[X,Y](= R^{[2]})$ *and* $D \in LND_R(B)$ *such that* D *is irreducible. Then the following hold:*

(i) *If* $D^2X = 0$*, then* $Ker\ D = R[bY + f(X)]$*, where* $b \in R$ *and* $f(X) \in R[X]$*. Moreover,* $DX \in R$ *and* $DY \in R[X]$.

(ii) *If* $D^2X = D^2Y = 0$*, then* $D = b\frac{\partial}{\partial X} - a\frac{\partial}{\partial Y}$ *for some* $a, b \in R$*. Moreover,* $Ker\ D = R[aX + bY]$.

(iii) *If* R *is a PID and* $D^2X = D^2Y = 0$*, then* D *has a slice.*

Over a Noetherian domain containing $\mathbb{Q}$, a necessary and sufficient condition for the kernel of a nonzero irreducible $D \in LND_R(R[X,Y])$ to be a polynomial ring is given by the following theorem ([6, Theorem 4.7]).

Theorem 2.2.5. *Let* R *be a Noetherian domain containing* $\mathbb{Q}$ *and let* D *be a non-zero irreducible locally nilpotent derivation on the polynomial ring* $R[X,Y]$*. Then the kernel* A *of* D *is a polynomial ring in one variable over* R *if and only if* DX *and* DY *either form a regular* $R[X,Y]$*-sequence or are comaximal in* $R[X,Y]$*. Moreover if* DX *and* DY *are comaximal in* $R[X,Y]$*, then* $R[X,Y]$ *is a polynomial ring in one variable over* A.

An important result on fixed-point free locally nilpotent derivations is the following ([21, Theorem 4.16]).

Theorem 2.2.6. *Let* R *be any* $\mathbb{Q}$*-algebra, and let* $B = R[X,Y] = R^{[2]}$*. Given* $D \in LND_R(R[X,Y])$*, the following conditions are equivalent:*

(1) *D is fixed-point free, i.e.,* $(DB) = B$*, where* (DB) *is the* B*-ideal generated by* DB.

(2) *There exists* $s \in B$ *with* $Ds = 1$.

In addition, when one of these conditions hold, $Ker\ D = R^{[1]}$.

The following result ensures that $LND^*(B) \neq \emptyset$ whenever B is a two-dimensional factorial affine domain over an algebraically closed field k of characteristic zero with $LND(B) \neq \emptyset$ ([21, Lemma 2.9]).

Lemma 2.2.7. *Let k be an algebraically closed field of characteristic zero and B an affine k-domain such that B is a UFD and $\dim B = 2$. Then every non-zero irreducible element of $LND(B)$ has a slice.*

We now state an important result for rigid domains by Crachiola and Makar-Limanov ([9, Theorem 3.1]).

Theorem 2.2.8. *Let k be a field of characteristic zero, C an affine k-domain and $C[T] = C^{[1]}$. Then the following hold:*

(i) $ML(C[T]) \subseteq ML(C)$.

(ii) *C is rigid if and only if $ML(C[T]) = C$.*

The following result gives a characterization of $k^{[1]}$ in terms of the Makar-Limanov invariant ([8, Lemma 2.3]).

Theorem 2.2.9. *Let k be a field of characteristic zero and A an affine k-domain with $\operatorname{tr.deg}_k A = 1$ such that k is algebraically closed in A. Then $A = k^{[1]}$ if it has a non-zero locally nilpotent derivation.*

We now recall a result proved by Makar-Limanov on Danielewski surfaces ([21, Theorem 9.1]).

Theorem 2.2.10. *Let k be a field of characteristic zero and $B := \frac{k[X,Y,Z]}{(X^nZ-p(Y))}$ where $n \in \mathbb{N}$ and $p(Y) \in k[Y]$. Let x be the image of X in B. Then the following hold:*

(i) *If $n = 1$ or if $\deg p(Y) = 1$, then $ML(B) = k$.*

(ii) *If $n \geqslant 2$ and $\deg p(Y) \geqslant 2$, then $ML(B) = k[x]$. Moreover, $Ker\ D = k[x]$ for every non-zero $D \in LND(B)$.*

2.3 Some results on polynomial rings and projective modules

We now state some well-known results on polynomial rings and projective modules which have been used to prove the results in this thesis.

For a ring containing $\mathbb{Q}$, the following cancellation theorem was proved by Hamann ([23, Theorem 2.8]).

Theorem 2.3.1. *Let R be a ring containing $\mathbb{Q}$ and A be an R-algebra such that $A^{[1]} = R^{[2]}$. Then $A = R^{[1]}$.*

The following is a well-known result of Abhyankar, Eakin and Heinzer ([1, Proposition 4.8]).

Theorem 2.3.2. *Let C be a UFD and let $X_1, \ldots, X_n$ be indeterminates over C. Suppose that A is an integral domain of transcendence degree one over C and that $C \subseteq A \subseteq C[X_1, \ldots, X_n]$. If A is a factorially closed subring of $C[X_1, \ldots, X_n]$, then $A = C^{[1]}$.*

The following local-global theorem was proved by Bass, Connell and Wright ([4]) and independently by Suslin ([38]).

Theorem 2.3.3. *Let R be a ring and A a finitely presented R-algebra. Suppose that for all maximal ideals $\mathfrak{m}$ of R, the $R_{\mathfrak{m}}$-algebra $A_{\mathfrak{m}}$ is isomorphic to the symmetric algebra of some $R_{\mathfrak{m}}$-module. Then $A \cong Sym_R(L)$ for some finitely presented R-module L.*

The following result is known as Serre's Splitting Theorem ([24, Theorem 7.1.8]).

Theorem 2.3.4. *Let A be a Noetherian ring of finite Krull dimension. Let P be a finitely generated projective A-module of rank greater than dimension of A. Then P has a unimodular element.*

Following is the famous Cancellation Theorem of Hyman Bass ([24, Theorem 7.1.11]).

Theorem 2.3.5. *Let R be a Noetherian ring of dimension d and P a finitely generated projective R-module of rank $> d$. Then P is "cancellative", i.e., $P \oplus Q \cong P^{'} \oplus Q$ for some finitely generated projective R-module Q implies that $P \cong P^{'}$.*

We now state a local-global result for a graded ring ([24, Theorem 4.3.11]).

Theorem 2.3.6. *Let $S = S_0 \oplus S_1 \oplus S_2 \oplus \dots$ be a graded ring and let M be a finitely presented S-module. Assume that for every maximal ideal $\mathfrak{m}$ of S_0, $M_{\mathfrak{m}}$ is extended from $(S_0)_{\mathfrak{m}}$. Then M is extended from S_0.*

For convenience, we state below an elementary result.

Lemma 2.3.7. *Let A and B be integral domains with $A \subseteq B$. If there exists f in A, such that $A_f = B_f$ and $fB \cap A = fA$, then $A = B$.*

Proof. Let $b \in B$. Suppose, if possible $b \notin A$. Now since $B_f = A_f$, we have $b \in A_f$. Hence there exist $a \in A$ and an integer $n > 0$ such that $b = a/f^n$. We may assume that n is the least possible. But then $a \in fB \cap A = fA$. Let $a = fa_1$ for some $a_1 \in A$. Then $b = a_1/f^{n-1}$, contradicting the minimality of n. □

Chapter 3

On algebraic characterization of the affine three space

3.1 A characterization of $k^{[2]}$

In this section we will describe an algebraic characterization of $k^{[2]}$ over a field k of characteristic zero (Theorem 3.1.8). We also investigate properties of a two-dimensional affine k-domain (say B) such that $ML(B) = ML^*(B)$. We first begin with a few general lemmas.

Lemma 3.1.1. *Let k be a field of characteristic zero, C an affine k-domain and $C[T] = C^{[1]}$. Then the following hold:*

(i) $ML^*(C[T]) \subseteq ML(C)$.

(ii) *C is rigid if and only if* $ML^*(C[T]) = C$.

Proof. (i) Given $D \in LND(C)$, we can extend D to $\tilde{D} \in LND^*(C[T])$ by $\tilde{D}T = 1$. Then:

$$ML^*(C[T]) \subseteq Ker\ \tilde{D} \cap Ker\ \frac{\partial}{\partial T} = Ker\ \tilde{D} \cap C = Ker\ D.$$

Thus, for any $D \in LND(C)$, we have $ML^*(C[T]) \subseteq Ker\ D$, and hence $ML^*(C[T]) \subseteq ML(C)$.

(ii) Now suppose $ML^*(C[T]) = C$. Then part (i) implies $C \subseteq ML(C)$, so C is rigid. Conversely, if C is rigid, then by Theorem 2.2.8 and part (i) we have:

$$C = ML(C[T]) \subseteq ML^*(C[T]) \subseteq ML(C) = C.$$

Hence $ML^*(C[T]) = C$. $\square$

Note that for an arbitrary affine k-domain B of dimension one, $ML(B) = ML^*(B)$. We have the following result on the equality of $ML(B)$ and $ML^*(B)$ for affine domains of dimension greater than one.

Lemma 3.1.2. *Let k be a field of characteristic zero and B be an affine k-domain of dimension $n \geqslant 2$. If $\operatorname{tr.deg}_k ML^*(B) = n-1$, then $ML(B) = ML^*(B)$ and B is a semi-rigid ring.*

Proof. Since $ML^*(B) \neq B$, by Theorem 2.2.2, there exists an $(n-1)$-dimensional subring C of B such that $B = C^{[1]}$ and $ML^*(B) \subseteq C$. Since $\operatorname{tr.deg}_k ML^*(B) = n-1 = \operatorname{tr.deg}_k C$ and both $ML^*(B)$ and C are algebraically closed in B, we have $ML^*(B) = C$. Hence, by Lemma 3.1.1, C is a rigid ring. Therefore, by Theorem 2.2.8, $ML(B) = C = ML^*(B)$.

Let $D(\neq 0) \in LND(B)$ and $A = Ker\ D$. Since $ML(B) \subseteq A$, $\operatorname{tr.deg}_k ML(B) = n-1 = \operatorname{tr.deg}_k A$ and both $ML(B)$ and A are algebraically closed in B, we have $A = ML(B)$, i.e., $Ker\ D = ML(B)$ for all $D(\neq 0) \in LND(B)$. Thus B is semi-rigid. $\square$

Lemma 3.1.3. *Let k be a field of characteristic zero and B an affine k-domain such that B is a semi-rigid ring. Then the following are equivalent:*

(I) $ML(B) = ML^*(B)$.

(II) *There exist a k-subalgebra C of B such that C is rigid and $B = C^{[1]}$.*

(III) *$ML(B)$ is rigid and $B = ML(B)^{[1]}$.*

Proof. (I) $\Rightarrow$ (II) Let D be a non-zero locally nilpotent derivation such that $LND(B) = \{fD \mid f \in Ker\ D\}$ and $C := Ker\ D$. Since $ML^*(B) = ML(B)$, D has a slice, say s. Thus by Theorem 2.2.2, $B = C[s] = C^{[1]}$. If $LND(C) \neq \{0\}$ and $d(\neq 0) \in LND(C)$, then d extends to $\tilde{d} \in LND(B)$ with $\tilde{d}(s) = 0$. But then $Ker\ \tilde{d} \neq C$, contradicting that B is semi-rigid. Thus $LND(C) = \{0\}$, i.e., C is rigid.

(II) $\Rightarrow$ (III) Trivial.

(III) $\Rightarrow$ (I) Follows from Lemma 3.1.1 (ii). $\square$

As a consequence we have the following sufficient condition for equality of the two invariants $ML(B)$ and $ML^*(B)$ for a two-dimensional affine domain B.

Lemma 3.1.4. *Let k be a field of characteristic zero and B a two-dimensional affine k-domain. Suppose that $ML^*(B) \neq B$. Then $ML^*(B) = ML(B)$.*

Proof. Clearly $\operatorname{tr.deg}_k ML^*(B) \leqslant 1$. Since $ML(B)$ and $ML^*(B)$ are algebraically closed subrings of B and $ML(B) \subseteq ML^*(B)$, it is enough to consider the case $\operatorname{tr.deg}_k ML^*(B) = 1$. The result now follows from Lemma 3.1.2. $\square$

Example 3.3.2 presents a two-dimensional affine domain B for which $ML(B) \subsetneqq ML^*(B) = B$. However the following consequence of Lemma 2.2.7 shows that such an example is not possible when B is a UFD.

Corollary 3.1.5. *Let k be an algebraically closed field of characteristic zero and B a two-dimensional affine k-domain such that B is a UFD. Then $ML(B) = ML^*(B)$.*

Proof. If $ML(B) = B$, then by definition, we have $ML^*(B) = B$. Now if $ML(B) \neq B$, then by Lemma 2.2.7, $LND^*(B) \neq \emptyset$, i.e., $ML^*(B) \neq B$ and hence $ML(B) = ML^*(B)$ by Lemma 3.1.4. $\square$

We have the following properties of a two-dimensional affine domain whenever the two invariants $ML(B)$ and $ML^*(B)$ are same.

Proposition 3.1.6. *Let k be a field of characteristic zero and B a two-dimensional affine k-domain. If $ML(B) = ML^*(B)$, then $ML(B)$ is rigid and B is a polynomial ring over $ML(B)$.*

Proof. Suppose $\operatorname{tr.deg}_k ML(B) = 2$. Then B is rigid and $ML(B) = B$.

Now suppose $\operatorname{tr.deg}_k ML(B) = 1$. Then by Lemma 3.1.2, B is semi-rigid and hence by Lemma 3.1.3 (III), $ML(B)$ is rigid and $B = ML(B)^{[1]}$.

Finally suppose $\operatorname{tr.deg}_k ML(B) = 0$. Then we have $ML(B) = ML^*(B) = L$, where L is the algebraic closure of k in B. Since $ML^*(B) \neq B$, by Theorem 2.2.2, there exists a one-dimensional subring C of B such that $B = C^{[1]}$. Now C is not rigid, otherwise by Theorem 2.2.8, $ML(B) = ML(C^{[1]}) = ML(C) = C$ contradicting that $ML(B) = L$. Hence, by Theorem 2.2.9, $C = L^{[1]}$. Thus $B = L^{[2]}$. $\square$

Remark 3.1.7. The proof of Proposition 3.1.6 shows that for a two-dimensional affine domain B over a field k of characteristic zero satisfying $ML(B) = ML^*(B)$ we have the following three cases:

(i) If $\operatorname{tr.deg}_k ML(B) = 2$, then B is rigid and $ML(B) = B$.

(ii) If $\operatorname{tr.deg}_k ML(B) = 1$, then B is semi-rigid and $B = C^{[1]}$ where $C = ML(B)$ is rigid.

(iii) If $\operatorname{tr.deg}_k ML(B) = 0$, then $B = L^{[2]}$ where $L = ML(B)$ is the algebraic closure of k in B.

As a consequence of Lemma 3.1.4 and Proposition 3.1.6, we have the following characterization of the affine 2-space.

Theorem 3.1.8. *Let k be a field of characteristic zero and B a two-dimensional affine k-domain. Then the following are equivalent:*

(I) $B = k^{[2]}$.

(II) $ML^*(B) = k$.

(III) $ML(B) = k$ *and* $ML^*(B) \neq B$.

Proof. Clearly (I) $\Rightarrow$ (II) $\Rightarrow$ (III). We now show that (III) $\Rightarrow$ (I). Since $ML^*(B) \neq B$, by Lemma 3.1.4 we have $ML^*(B) = ML(B) = k$. As $k(= ML(B))$ is algebraically closed in B, by Part (iii) of Proposition 3.1.6, we have $B = k^{[2]}$. $\square$

3.2 A characterization of $k^{[3]}$

In this section we will describe an algebraic characterization of $k^{[3]}$ over an algebraically closed field k of characteristic zero (Theorem 3.2.6). We also investigate properties of a three-dimensional affine k-domain (say B) over an algebraically closed field of characteristic zero for which $ML(B) = ML^*(B)$.

We first state a result for a polynomial ring in two variables over a one-dimensional affine UFD.

Lemma 3.2.1. *Let k be an algebraically closed field of characteristic zero, R a one-dimensional affine UFD and $B := R^{[2]}$. Then either $B = k^{[3]}$ or $ML(B) = R$.*

Proof. By Lemma 2.1.3, $R = k[t, \frac{1}{p(t)}]$ where $k[t] = k^{[1]}$ and $p(t) \in k[t] \setminus \{0\}$. Now, either $p(t) \in k$ or $p(t) \notin k$. If $p(t) \in k$, then $R = k^{[1]}$ and $B = k^{[3]}$. If $p(t) \notin k$, then $ML(B) = R$ since $p(t) \in ML(B)$ and $ML(B)$ is a factorially closed subring of B. $\square$

The next result shows that if a three-dimensional affine UFD B admits two non-zero distinct locally nilpotent derivations with slices, then there exists a k-subalgebra R of B, such that $B = R^{[2]}$. Example 3.3.7 shows that such a result does not extend to a four-dimensional affine UFD.

Lemma 3.2.2. *Let k be an algebraically closed field of characteristic zero and B an affine k-domain such that B is a UFD and dim $B = 3$. If B admits two non-zero distinct locally nilpotent derivations with slices, then there exists a k-subalgebra R of B, such that R is a UFD and $B = R^{[2]}$.*

Proof. Let D_1 and D_2 be two non-zero distinct locally nilpotent derivations on B with slices s_1, s_2. Let $Ker\ D_i = C_i$ for $i = 1, 2$. Then, by Theorem 2.2.2, $B = C_i[s_i] = {C_i}^{[1]}$ for each i. Now $ML(B) \subseteq ML^*(B) \subseteq C_1 \cap C_2 \subsetneqq C_i$. It follows that C_i is not rigid, otherwise by Theorem 2.2.8, $ML(B) = ML({C_i}^{[1]}) = ML(C_i) = C_i$. Since C_1 is a factorially closed subring of the UFD B, C_1 is a UFD. As C_1 is not rigid, by Lemma 2.2.7, C_1 has a locally nilpotent derivation with a slice and therefore by Theorem 2.2.2, $C_1 = R^{[1]}$ for some k-subalgebra R of C_1. Hence $B = R^{[2]}$. As R is a factorially closed subring of the UFD C_1, R is a UFD. □

The following result describes a classification of three-dimensional factorial affine domains B for which $ML(B) = ML^*(B)$.

Proposition 3.2.3. *Let k be an algebraically closed field of characteristic zero and B a three-dimensional affine UFD over k. If $ML(B) = ML^*(B)$, then $ML(B)$ is a rigid UFD and B is a polynomial ring over $ML(B)$.*

Proof. Suppose $\operatorname{tr.deg}_k ML(B) = 3$, then B is rigid and $ML(B) = B$. Now suppose $\operatorname{tr.deg}_k ML(B) = 2$. Then by Lemmas 3.1.2 and 3.1.3, $B = ML(B)^{[1]}$ and $ML(B)$ is rigid. Since B is a UFD, $ML(B)$ is also a UFD.

Now suppose $\operatorname{tr.deg}_k ML(B) \leqslant 1$. Then B admits two non-zero distinct locally nilpotent derivations on B with slices. Hence, by Lemma 3.2.2, there exists a one-dimensional k-subalgebra S of B such that $B = S^{[2]}$. Since B is a UFD, S is a UFD. If $\operatorname{tr.deg}_k ML(B) = 1$, then $B \neq k^{[3]}$ and hence $S \neq k^{[1]}$. Hence, by Theorem 2.2.9, S is rigid, and by Lemma 2.1.3, there exists $t \in B$ such that $S = k[t, \frac{1}{p(t)}]$, where $k[t] = k^{[1]}$ and $p(t) \in k[t] \setminus k$. In particular, $k^* \subsetneqq B^*$. Thus $ML(B) = S$ and $B = S^{[2]}$. Again if $\operatorname{tr.deg}_k ML(B) = 0$, then $ML(B) = k \neq S$. Hence by Lemma 3.2.1, $B = k^{[3]}$. □

Remark 3.2.4. The proof of Proposition 3.2.3 shows that for a three-dimensional affine UFD B over an algebraically closed field of characteristic zero satisfying $ML(B) = ML^*(B)$, we have the following four cases:

(i) If $\operatorname{tr.deg}_k ML(B) = 3$, then B is rigid.

(ii) If $\operatorname{tr.deg}_k ML(B) = 2$, then $B = C^{[1]}$, where C is rigid.

(iii) If $\operatorname{tr.deg}_k ML(B) = 1$, then $B = S^{[2]}$, where $S = k[t, \frac{1}{p(t)}]$ for some $p(t) \in k[t] \setminus k$.

(iv) If $\operatorname{tr.deg}_k ML(B) = 0$, then $B = k^{[3]}$.

The following result shows that for a three-dimensional factorial affine domain over an algebraically closed field, the equality of $ML(B)$ and $ML^*(B)$ holds whenever $ML^*(B) \neq B$.

Lemma 3.2.5. *Let k be an algebraically closed field of characteristic zero and B an affine k-domain such that B is a UFD and dim $B = 3$. If $ML^*(B) \neq B$, then $ML(B) = ML^*(B)$.*

Proof. Since $ML^*(B) \neq B$, we have $\operatorname{tr.deg}_k ML^*(B) \leq 2$. If $\operatorname{tr.deg}_k ML^*(B) = 2$, then the result follows from Lemma 3.1.2.

Now suppose $\operatorname{tr.deg}_k ML^*(B) = 1$. Then, by Lemma 3.2.2, there exists a k-subalgebra R of B such that R is a one-dimensional UFD and $B = R^{[2]}$. Thus $ML^*(B) \subseteq R$. As both $ML^*(B)$ and R are algebraically closed in B and have the same transcendence degree over k, we have $ML^*(B) = R$. As $ML^*(B) \neq k$, $B \neq k^{[3]}$ and hence $ML(B) = R$ by Lemma 3.2.1. Thus $ML(B) = ML^*(B)$.

If $\operatorname{tr.deg}_k ML^*(B) = 0$, then $ML^*(B) = k$ and hence $ML(B) = k = ML^*(B)$. □

We now state our main result.

Theorem 3.2.6. *Let k be an algebraically closed field of characteristic zero and B an affine k-domain such that B is a UFD and dim $B = 3$. Then the following are equivalent:*

(I) $B = k^{[3]}$.

(II) $ML^*(B) = k$.

(III) $ML(B) = k$ *and* $ML^*(B) \neq B$.

Proof. Clearly (I) $\Rightarrow$ (II) $\Rightarrow$ (III). We now show that (III) $\Rightarrow$ (I). Since $ML^*(B) \neq B$, by Lemma 3.2.5, $ML(B) = ML^*(B)$. Now by Part (iv) of Proposition 3.2.3, $B = k^{[3]}$. $\square$

Remark 3.2.7. (i) The hypothesis that $ML^*(B) \neq B$ is necessary in Lemma 3.2.5. We will show that for a three-dimensional affine UFD B, containing an algebraically closed field of characteristic zero, it may happen that $ML^*(B) = B$ but $\operatorname{tr.deg}_k ML(B)$ is zero (Example 3.3.5), one (Example 3.3.4), or two (Example 3.3.3) i.e. $ML^*(B) \neq ML(B)$.

(ii) Example 3.3.1 will show that both the hypotheses "k is an algebraically closed field" and "B is a UFD" are needed for the implication (III) $\implies$ (I) in Theorem 3.2.6.

Lemma 3.1.2 shows that there does not exist any three-dimensional affine k-domain B such that $ML(B) \subsetneqq ML^*(B)$ but $\operatorname{tr.deg}_k ML^*(B) = 2$. However we pose the following question.

Question 3.2.8. *Does there exist a three-dimensional affine k-domain B over a field k of characteristic zero such that $ML(B) = k$ but $\operatorname{tr.deg}_k ML^*(B) = 1$?*

Note that Theorem 3.2.6 shows that Question 3.2.8 has negative answer when k is an algebraically closed field and B is a UFD. If the answer to Question 3.2.8 is negative in general then the implication (III) $\implies$ (II) will hold in Theorem 3.2.6 even without the additional hypotheses that "k is an algebraically closed field" and "B is a UFD".

3.3 Some examples

In this section we shall present some examples to illustrate the hypotheses of the results stated earlier. The following example shows that both the hypotheses "k is algebraically closed" and "B is a UFD" are needed in Theorem 3.2.6.

Example 3.3.1. Let k be a field of characteristic zero, $R := \frac{k[X,Y,Z]}{(XY-Z^2-1)}$ and $B := R[T]$. Then the following hold:

(i) If k is an algebraically closed field, then B is not a UFD.

(ii) If $k = \mathbb{R}$, then B is a UFD.

(iii) $ML^*(B) = k$.

(iv) $B \neq k^{[3]}$.

Thus the conditions (II) and (III) of Theorem 3.2.6 hold but not (I).

Proof. Let x, y and z denote the images in B of X, Y and Z respectively.

(i) One can see that x is an irreducible element of B. Now if k is an algebraically closed field, then clearly x is not a prime element in B.

(ii) Suppose $k = \mathbb{R}$. Then x is a prime element in B and since $B[1/x](= k[x, 1/x]^{[2]})$ is a UFD, we have B is a UFD by Lemma 2.1.2.

(iii) Consider two locally nilpotent k-derivations on B, say D_1 and D_2 given by

$$D_1(x) = 0, \quad D_1(y) = 2z, \quad D_1(z) = x, \quad D_1(T) = 1 \quad \text{and}$$

$$D_2(x) = 2z, \quad D_2(y) = 0, \quad D_2(z) = y, \quad D_2(T) = 1.$$

Let $A_i = Ker\ D_i$ for $i = 1, 2$. Then by Theorem 2.2.2,

$$A_1 = k[x, y - 2zT + xT^2, z - xT] \quad \text{and}$$

$$A_2 = k[y, x - 2zT + yT^2, z - yT].$$

We now show that $A_1 \cap A_2 = k$. Consider A_1 as a subring of $A_1[\frac{1}{x}] = k[x, \frac{1}{x}, \frac{z}{x} - T]$ and A_2 as a subring of $A_2[\frac{1}{y}] = k[y, \frac{1}{y}, \frac{z}{y} - T]$. Let $\alpha \in A_1 \cap A_2$ and $n :=$ T-degree of α. Then there exist elements $a_i(x) \in k[x, \frac{1}{x}]$ and $b_j(y) \in k[y, \frac{1}{y}]$ for $i, j \in \{0, 1, \ldots, n\}$ such that

$$\alpha = \sum_{i=0}^{n} a_i(x)(\frac{z}{x} - T)^i = \sum_{j=0}^{n} b_j(x)(\frac{z}{y} - T)^j.$$

Comparing the coefficients of T^n from the two expressions, we have $(-1)^n a_n(x) = (-1)^n b_n(y) \in k[x, \frac{1}{x}] \cap k[y, \frac{1}{y}] = k$ (since x and y are algebraically independent over k). Again, comparing the coefficients of z^n from the two expressions we have $\frac{a_n(x)}{x^n} = \frac{b_n(y)}{y^n}$. Hence $n = 0$ and consequently $\alpha \in k$. Thus $ML^*(B) = k$.

(iv) Let $\bar{k}$ denote the algebraic closure of k. Then $B \bigotimes_k \bar{k}$ is not a UFD by (i). Hence $B \neq k^{[3]}$. $\square$

We now present examples of affine domains B for which $ML(B) \subsetneqq ML^*(B) = B$. We first present an example for $\dim B = 2$. By Corollary 3.1.5, such an example is not possible for two-dimensional factorial affine domains.

Example 3.3.2. Let k be an algebraically closed field of characteristic zero, $n \geqslant 1$ be an integer and $p(Y) \in k[Y]$ be such that $\deg p(Y) \geqslant 2$. Let $B = \frac{k[X,Y,Z]}{(X^nZ-p(Y))}$. Let x denote the image of X in B. B is not a UFD (since x is irreducible but not a prime in B). We have

(i) If $n = 1$, then $ML(B) = k$ by Theorem 2.2.10(i) but $ML^*(B) = B$ by Theorem 3.1.8 (since $B \neq k^{[2]}$).

(ii) If $n \geqslant 2$, then $ML(B) = k[x]$ and B is a semi-rigid ring by Theorem 2.2.10(ii) but $ML^*(B) = B$ by Theorem 2.2.2 (since $B \neq k^{[2]}$).

We now present examples of three-dimensional affine UFD B for which $ML^*(B) = B$ but $ML(B) \subsetneqq ML^*(B)$. In the three examples $\operatorname{tr.deg}_k ML(B)$ is two, one and zero respectively.

Example 3.3.3. Let $R := \frac{\mathbb{C}[X,Y,Z]}{(X^2+Y^3+Z^7)}$ and $B := \frac{R[U,V]}{(X^2U-Y^3V-1)}$. It has been proved by D.R. Finston and S. Maubach that B is a semi-rigid UFD of dimension 3 and $ML(B) = R$ ([19, Theorem 2]); in particular, $\operatorname{tr.deg}_k ML(B) = 2$. But $ML^*(B) = B$ by Theorem 2.2.2 (since $B \neq R^{[1]}$).

Example 3.3.4. Let $B := \frac{\mathbb{C}[X,Y,Z,T]}{(X+X^2Y+Z^2+T^3)}$. Let x denote the image of X in B. Since B_x is a UFD, by Lemma 2.1.2, B is a UFD. It has been proved by L.G. Makar-Limanov that $ML(B) = \mathbb{C}[x] = \mathbb{C}^{[1]}$ ([28, Lemma 8]); in particular $\operatorname{tr.deg}_k ML(B) = 1$. Since $B^* = \mathbb{C}^*$, we have $ML(B) \neq ML^*(B)$ by Proposition 3.2.3 (iii) and hence $ML^*(B) = B$ by Lemma 3.2.5.

Example 3.3.5. Let k be an algebraically closed field of characteristic zero and $B := \frac{k[X,Y,Z,T]}{(XY-ZT-1)}$. Let the images of X, Y, Z and T in B be denoted by x, y, z and t respectively. Since B_x is a UFD, by Lemma 2.1.2, B is a UFD. Moreover B is regular. Consider four non-zero locally nilpotent derivations D_1, D_2, D_3 and D_4 on B given by

(i) $D_1x = 0, \quad D_1y = z, \quad D_1z = 0, \quad D_1t = x.$

(ii) $D_2x = 0, \quad D_2y = t, \quad D_2z = x, \quad D_2t = 0.$

(iii) $D_3x = z, \quad D_3y = 0, \quad D_3z = 0, \quad D_3t = y.$

(iv) $D_4x = t, \quad D_4y = 0, \quad D_4z = y, \quad D_4t = 0.$

Let $Ker\ D_i = A_i$ for each $i = 1, 2, 3, 4$. Now $k[x, z] \subseteq A_1 \subseteq B$. Since both $k[x, z]$ and A_1 are algebraically closed in B and have the same transcendence degree over k, we have $A_1 = k[x, z]$. Similarly $A_2 = k[x, t]$, $A_3 = k[y, z]$, $A_4 = k[y, t]$ and $\bigcap_i A_i = k$. Thus $ML(B) = k$, i.e. $\operatorname{tr.deg}_k ML(B) = 0$. But $B \neq k^{[3]}$ (since the Whitehead group $K_1(B) \neq k^*$) and it follows from Theorem 3.2.6 that $ML^*(B) = B$.

We now present an example which shows that Theorem 3.2.6 does not extend to a four-dimensional affine regular UFD, i.e., a four-dimensional affine UFD $\widetilde{B}$ need not be $k^{[4]}$, even when $ML(\widetilde{B}) = ML^*(\widetilde{B}) = k$. We will follow the notation of Example 3.3.5.

Example 3.3.6. Let B be as in Example 3.3.5 and $\widetilde{B} := B[u] = B^{[1]}$. $\widetilde{B}$ is a regular UFD of dimension four. For each $i = 1, 2, 3, 4$, we extend the locally nilpotent derivation D_i on B to a locally nilpotent derivation $\widetilde{D_i}$ on $\widetilde{B}$, by defining $\widetilde{D_i}u = 1$. Let

$$\widetilde{D_5} = \frac{\partial}{\partial u} \text{ and } Ker\ \widetilde{D_i} = \widetilde{A_i}.$$

By Theorem 2.2.2, we have

$$\widetilde{A_1} = k[x, z, y - zu, t - xu],$$

$$\widetilde{A_2} = k[x, t, z - xu, y - tu],$$

$$\widetilde{A_3} = k[y, z, x - zu, t - yu],$$

$$\widetilde{A_4} = k[y, t, x - tu, z - yu] \text{ and}$$

$$\widetilde{A_5} = k[x, y, z, t].$$

Clearly $k[x, z + t - xu] \subseteq \widetilde{A_1} \cap \widetilde{A_2}$. Since $k[x, z + t - xu]$ and $\widetilde{A_1} \cap \widetilde{A_2}$ are algebraically closed in $B[u]$ and they have the same transcendence degree over k, we have

$$\widetilde{A_1} \cap \widetilde{A_2} = k[x, z + t - xu].$$

Similarly,

$$\widetilde{A_3} \cap \widetilde{A_4} = k[y, z + t - yu].$$

Again,

$$\widetilde{A_5} \cap k[x, z + t - xu] = k[x] \text{ and } \widetilde{A_5} \cap k[y, z + t - yu] = k[y].$$

Hence $\bigcap_i \widetilde{A_i} = k$. Thus $ML^*(\widetilde{B}) = ML(\widetilde{B}) = k$. But $\widetilde{B} \neq k^{[4]}$ (for instance, $K_1(\widetilde{B}) = K_1(B) \neq k^*$).

The following example shows that Lemma 3.2.2 need not be true for a four-dimensional affine UFD B, even if $ML(B) = ML^*(B)$.

Example 3.3.7. Let k be an algebraically closed field of characteristic zero and $R = k[X, Y, Z]/(X^2 + Y^3 + Z^7) = k[x, y, z]$, where x, y and z denote the images of X, Y and Z in R. Let $C = R[U, V]/(xU - yV - 1) = R[u, v]$, where u and v denote the images of U and V in C and $B = C[T] = C^{[1]}$. Then the following hold.

(i) B is a UFD of dimension 4.

(ii) $ML(B) = ML^*(B) = R$.

(iii) $B \neq R^{[2]}$.

(iv) $B \neq S^{[2]}$ for any k-subalgebra S of B.

Proof. (i) By Lemma 2.1.2, R and C are UFDs. Hence B is a UFD. Clearly $\dim B = 4$.

(ii) By [19, Lemma 2], $R \subseteq ML(B) \subseteq ML^*(B)$. Consider the R-linear derivations δ_1 and δ_2 on B as follows:

$$\delta_1(u) = y, \quad \delta_1(v) = x \quad \text{and} \quad \delta_1(T) = 1$$

and

$$\delta_2(u) = yT, \quad \delta_2(v) = xT \quad \text{and} \quad \delta_1(T) = 1.$$

Clearly they are locally nilpotent derivations with slices T. By Theorem 2.2.2, $A_1 := Ker\ \delta_1 = R[u - yT, v - xT]$ and $A_2 := Ker\ \delta_2 = R[2u - yT^2, 2v - xT^2]$. Then $A_{1x} = R_x[v - xT]$ and $A_{2x} = R_x[2v - xT^2]$ and the two rings A_1 and A_2 are clearly different. Therefore $A_1 \cap A_2 \subsetneqq A_2$. As $A_1 \cap A_2$ is a factorially closed subring of B containing R, we have $A_1 \cap A_2 = R$ by comparing the dimensions.

(iii) Since $(x, y)B = B$, it follows that $B \neq R^{[2]}$.

(iv) Suppose there exists a k-subalgebra S of B such that $B = S^{[2]}$. Then $R = ML(B) \subseteq S$. Since $\operatorname{tr.deg}_k R = \operatorname{tr.deg}_k S$ and both R and S are algebraically closed in B, it follows that $R = S$, contradicting (iii). Hence the result. $\square$

Remark 3.3.8. An important problem in Affine Algebraic Geometry asks whether, for the Russell-Koras threefold B defined in Example 3.3.4, $B^{[1]} = \mathbb{C}^{[4]}$. An affirmative answer will give a negative solution to the Zariski Cancellation Problem for the three-space in characteristic zero. It was shown by A. Dubouloz in [16] that $ML(B^{[1]}) = \mathbb{C}$ and A. Dubouloz and J. Fasel have shown that $X = Spec(B)$ is $\mathbb{A}^1$-contractible [17, Theorem 1.1]. It follows from Lemma 3.1.1 that $ML^*(B^{[2]}) \subseteq ML(B^{[1]}) = \mathbb{C}$. As a consequence, we have $ML^*(B^{[n]}) = \mathbb{C}$ for any $n \geqslant 2$. This leads to the question:

Question : Let B be as in Example 3.3.4. Is $ML^*(B^{[1]}) = \mathbb{C}$?

A negative answer to this problem will confirm that $B^{[1]} \neq \mathbb{C}^{[4]}$. However, it is possible that $B^{[n]} = \mathbb{C}^{[n+3]}$ for some $n \geqslant 2$.

Chapter 4

On Nice and Quasi-Nice Derivations

4.1 Nice Derivations

In this section, we shall explore generalisations of the following theorem of Z. Wang ([39, Proposition 4.6]).

Theorem 4.1.1. *Let k be a field of characteristic zero and $k[X,Y,Z] = k^{[3]}$. Suppose that $D(\neq 0) \in LND(k[X,Y,Z])$ satisfies $D^2X = D^2Y = D^2Z = 0$. Then the following hold:*

(i) *$Ker\ D$ contains a nonzero linear form of $\{X,Y,Z\}$.*

(ii) *$rank\ D = 1$.*

(iii) *If D is irreducible, then for some coordinate system $(X^{'}, Y^{'}, Z^{'})$ of $k[X,Y,Z]$ related to (X,Y,Z) by a linear change,*

$$D = f(X^{'})\frac{\partial}{\partial Y'} + g(X^{'})\frac{\partial}{\partial Z'}$$

where f, $g \in k[X^{'}]$ and $\gcd_{k[X']}(f,g) = 1$.

We first observe the following result.

Lemma 4.1.2. *Let R be a UFD containing $\mathbb{Q}$ and $D(\neq 0) \in LND_R(R[X,Y,Z])$ and $rank\ D < 3$. Then $Ker\ D = R^{[2]}$.*

Proof. Let $A := Ker\ D$. Since $rank\ D < 3$, there exists $X' \in R[X,Y,Z]$ such that $R[X,Y,Z] = R[X']^{[2]}$ and $X' \in A$. Then taking $C = R[X']$, it follows from Theorem 2.3.2 that $A(= Ker\ D) = R[X']^{[1]} = R^{[2]}$. $\square$

The following example shows that Lemma 4.1.2 does not extend to a Noetherian normal domain R which is not a UFD.

Example 4.1.3. Let $\mathbb{R}[a,b] = \mathbb{R}^{[2]}$ and $R := \frac{\mathbb{R}[a,b]}{(a^2+b^2-1)}$. Let $B := R[X,Y,Z] = R^{[3]}$ and D be an R-linear LND of B, such that

$$DX = a, \quad DY = b-1 \quad \text{and} \quad DZ = aY + (1-b)X.$$

Setting $u = aY + (1-b)X$, $v = (1+b)Y + aX$ and $w = 2Z + uY - vX$, we see that $Du = Dv = Dw = 0$ and $D^2X = D^2Y = D^2Z = 0$.

Let $A := Ker\ D$. Now $B_{(1+b)} = R_{(1+b)}[v,w,X]$ and $B_{(1-b)} = R_{(1-b)}[u,w,Y]$. Thus it follows that $A_{(1+b)} = R_{(1+b)}[v,w] = {R_{(1+b)}}^{[2]}$ and $A_{(1-b)} = R_{(1-b)}[u,w] = {R_{(1-b)}}^{[2]}$. Since $(1+b)$ and $(1-b)$ are comaximal elements of R, $A = R[u,v,w]$ and $A_m = {R_m}^{[2]}$ for every maximal ideal m of R.

Now $B = R[X,Y,Z] = R[X,Y,w]$ and $w \in A$; so $rank\ D < 3$. Setting $T = \frac{u}{a}$, we see that $A = R[aT, (1+b)T, w]$. By Theorems 2.3.3 and 2.3.4, $A = Sym_R(F \oplus P)$, where F is a free R-module of rank 1 and P is a rank 1 projective R-module given by the ideal $(a, 1+b)R$, which is not principal. Hence P is not stably free and so $A \neq R^{[2]}$ ([18, Lemma 1.3]).

Remark 4.1.4. In Proposition 4.1.9, we will see that over any Dedekind domain R, the kernel of a nice derivation on $R^{[3]}$ is generated by (at most) three elements.

The following example shows that part (ii) of Theorem 4.1.1 does not hold when K is replaced by a PID R.

Example 4.1.5. Let k be a field of characteristic zero, $R = k[t](= k^{[1]})$ and $B := R[X,Y,Z](= R^{[3]})$. Let $D \in LND_R(B)$ be such that

$$DX = 0, \quad DY = X - t \quad \text{and} \quad DZ = X + t.$$

Let $A = Ker\ D$ and $G := (X-t)Z - (X+t)Y$. We will show that

(i) $A = R[X,G]$.

(ii) $B \neq A^{[1]}$; in fact, B is not even A-flat.

(iii) *rank* $D = 2$.

Proof. (i) Let $C := R[X, G]$. We show that $C = A$. Clearly $C \subseteq A$. Set $f := X - t$. Then $B_f = R[X, G, Y]_f = {C_f}^{[1]}$. Hence, as both $C_f (\subseteq A_f)$ and A_f are factorially closed subrings of B_f and as $\operatorname{tr.deg}_{C_f} B_f = 1 = \operatorname{tr.deg}_A B$, we have $C_f = A_f$.

Now B/fB may be identified with $R[Y, Z] (= R^{[2]})$. Clearly $C/fC = R^{[1]}$ and the image of C/fC in B/fB is $R[tY] (= R^{[1]})$. Thus the natural map $C/fC \to B/fB$ is injective, i.e, $fB \cap C = fC$. Since A is factorially closed in B, we also have $fB \cap A = fA$ and hence $fA \cap C = fB \cap A \cap C = fB \cap C = fC$. Therefore as $C_f = A_f$, we have $C = A$ by Lemma 2.3.7.

(ii) $(X - t, X + t)B$ is a prime ideal of height 2 in B and $(X - t, X + t)B \cap A = (X, t, G)A$ is a prime ideal of height 3 in A, violating the going-down principle. Hence B is not A-flat and therefore $B \neq A^{[1]}$.

(iii) Since $DX = 0$, *rank* $D < 3$. If *rank* $D = 1$, then clearly $B = A^{[1]}$ contradicting (ii). Hence *rank* $D = 2$. □

We now prove an extension of Theorem 4.1.1 over a PID.

Theorem 4.1.6. *Let R be a PID containing $\mathbb{Q}$ with field of fractions L and $B := R[X, Y, Z] = R^{[3]}$. Let $D (\neq 0) \in LND_R(B)$ and $A := Ker\ D$. Suppose that D is irreducible and $D^2X = D^2Y = D^2Z = 0$. Then there exists a coordinate system (U, V, W) of B related to (X, Y, Z) by a linear change such that the following hold:*

(i) *A contains a nonzero linear form of $\{X, Y, Z\}$.*

(ii) *rank $D \leqslant 2$. In particular, $A = R^{[2]}$.*

(iii) *$A = R[U, gV - fW]$, where $DV = f$, $DW = g$, and $f, g \in R[U]$ such that $gcd_{R[U]}(f, g) = 1$.*

(iv) *Either f and g are comaximal in B or they form a regular sequence in B. Moreover if they are comaximal, (i.e., D is fixed-point free) then $B = A^{[1]}$ and rank $D = 1$; and if they form a regular sequence, then B is not A-flat and rank $D = 2$.*

Proof. (i) D extends to an LND of $L[X, Y, Z]$ which we denote by $\overline{D}$. By Theorem 4.1.1 there exists a coordinate system (U, V', W') of $L[X, Y, Z]$ related to (X, Y, Z) by a linear change and mutually coprime polynomials $p(U)$,

$q(U)$ in $L[U]$ for which

$$\overline{D} = p(U)\frac{\partial}{\partial V'} + q(U)\frac{\partial}{\partial W'}.$$

Multiplying by a suitable nonzero element of R, we can assume $U \in R[X, Y, Z]$. Clearly $A = Ker\ \overline{D} \cap R[X, Y, Z]$ and $U \in A$. Moreover without loss of generality we can assume that there exist $l, m, n \in R$ with $gcd_R(l, m, n) = 1$ such that $U = lX + mY + nZ$. As R is a PID, (l, m, n) is a unimodular row of R^3 and hence can be completed to an invertible matrix $M \in GL_3(R)$. Let $\begin{pmatrix} U \\ V \\ W \end{pmatrix} = M \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$.

Then $R[U, V, W] = R[X, Y, Z]$ and as $U \in A$, A contains a nonzero linear form in X, Y, Z.

(ii) Follows from (i) and Lemma 4.1.2.

(iii) $R[U]$ is a UFD and $B = R[U, V, W] = R[U]^{[2]}$. So D is a locally nilpotent $R[U]$-derivation on B. Now the proof follows from part (ii) of Lemma 2.2.4.

(iv) Since $B = R[U, V, W] = R[U]^{[2]}$, the first part follows from Theorem 2.2.5. Moreover, when f and g are comaximal in B, it also follows from Theorem 2.2.5 that $B = A^{[1]}$. Hence in this case $rank\ D = 1$.

If f and g form a regular sequence in B (and hence in A since A is factorially closed in B), $(f, g)B \cap A = (f, g, gV - fW)A$. But $(f, g, gV - fW)A$ is an ideal of height 3, while $(f, g)B$ is an ideal of height 2, violating the going-down principle. It follows that in this case B is not A-flat. In this case indeed $rank\ D = 2$, or else if $rank\ D = 1$, we would have $B = A^{[1]}$. □

The proof of Theorem 4.1.6 shows the following:

Corollary 4.1.7. *With the notation as above, the following are equivalent:*

(i) $B = A^{[1]}$.

(ii) $rank\ D = 1$.

(iii) B *is* A*-flat.*

Proof. (i)⇔(ii) and (ii)⇒(iii) are trivial. (iii)⇒(i) follows from Theorem 4.1.6(iv). □

As mentioned in the Introduction, Theorem 4.1.6 shows that the kernel of an irreducible nice derivation on $k^{[4]}$ of $rank \leqslant 3$ is $k^{[3]}$. More precisely, we have:

Corollary 4.1.8. *Let k be a field of characteristic zero and let $k[X_1, X_2, X_3, X_4] = k^{[4]}$. Let $D \in LND(k[X_1, X_2, X_3, X_4])$ be such that D is irreducible and $DX_1 = 0$ and $D^2X_i = 0$ for $i = 2, 3, 4$. Then $Ker\ D = k^{[3]}$.*

By a result of Bhatwadekar and Daigle ([5, Proposition 4.13]), we know that over a Dedekind domain R containing $\mathbb{Q}$, the kernel of any locally nilpotent R-derivation on $R^{[3]}$ is necessarily finitely generated. We now show that if D is a nice derivation, then the kernel is generated by at most three elements.

Proposition 4.1.9. *Let R be a Dedekind domain containing $\mathbb{Q}$ and $B := R[X, Y, Z](= R^{[3]})$. Let $D \in LND_R(B)$ such that D is irreducible and $D^2X = D^2Y = D^2Z = 0$. Let $A := Ker\ D$. Then the following hold:*

(i) *A is generated by at most 3 elements.*

(ii) *Moreover, if D is fixed-point free, then $rank\ D < 3$ and D has a slice. In particular, $rank\ D = 1$.*

Proof. (i) By Theorem 4.1.6, $A_p = {R_p}^{[2]}$ for each $p \in Spec(R)$. Hence by Theorem 2.3.3, $A \cong Sym_R(Q)$ for some rank 2 projective R-module Q. Since R is a Dedekind domain, by Theorem 2.3.4, $Q \cong Q_1 \oplus M$ where Q_1 is a rank 1 projective R-module and M is a free R-module of rank 1. Again, since R is a Dedekind domain Q_1 is generated by at most 2 elements. Hence A is generated by at most 3 elements.

(ii) Now assume D is fixed-point free. Let $DX = f_1$, $DY = f_2$ and $DZ = f_3$. Then, by Theorem 2.2.2, $B_{f_i} = {A_{f_i}}^{[1]}$ for each $i \in \{1, 2, 3\}$. Since $(f_1, f_2, f_3)B = B$, we have $B_{\tilde{p}} = {A_{\tilde{p}}}^{[1]}$, for each $\tilde{p} \in Spec(A)$. Hence, by Theorem 2.3.3, $B = Sym_A(P)$, where P is a projective A-module of rank 1. Now for each $p \in Spec(R)$, P_p is an A_p-module and as $A_p = {R_p}^{[2]}$, we have P_p is a free A_p-module since R_p is a discrete valuation ring and hence extended from R_p. Therefore, by Theorem 2.3.6, P is extended from R. Let $P = P_1 \otimes_R A$, where P_1 is a projective R-module of rank 1. Hence $B = Sym_A(P) = Sym_R(M \oplus Q_1 \oplus P_1)$, where M is a free R-module of rank 1. Since $B = R^{[3]}$, $M \oplus Q_1 \oplus P_1$ is a free R-module of rank 3 ([18, Lemma 1.3]). By Theorem 2.3.5, $Q_1 \oplus P_1$ is free of rank 2. Let $M = Rf$ and set $S := R[f]$. Then $B = R[f]^{[2]}$ and as $f \in A$, we have $rank\ D < 3$. Now $B = S^{[2]}$ and

$D \in LND_S(B)$ such that D is fixed-point free. Hence, by Theorem 2.2.6, D has a slice.

Let $B = R[f,g,h](= R^{[3]})$ and $s \in B$ be such that $Ds = 1$. Then by Theorem 2.2.2, $B(= S^{[2]}) = A[s](= A^{[1]})$. Hence by Theorem 2.3.1, $A = S^{[1]}$. Let $A = R[f,t]$. Then $B = R[f,g,h] = R[f,t,s]$ and $f,t \in A$. So *rank* $D = 1$. □

The following example shows that Theorem 4.1.6 does not extend to a higher-dimensional regular UFD, not even to $k^{[2]}$.

Example 4.1.10. Let k be a field of characteristic zero and $R = k[a,b] = k^{[2]}$. Let $B = R[X,Y,Z](= R^{[3]})$ and $D(\neq 0) \in LND_R(B)$ be such that

$$DX = b, \quad DY = -a \quad \text{and} \quad DZ = aX + bY.$$

Let $u = aX+bY$, $v = bZ-uX$, and $w = aZ+uY$. Then $Du = Dv = Dw = 0$, D is irreducible and $D^2X = D^2Y = D^2Z = 0$. Let $A = Ker\ D$. We show that

(i) $A = R[u,v,w]$.

(ii) $A = R[U,V,W]/(bW - aV - U^2)$, where $R[U,V,W] = R^{[3]}$ and hence $A \neq R^{[2]}$.

(iii) *rank* $D = 3$.

Proof. (i) Let $C := R[u,v,w]$. We show that $C = A$. Clearly $C \subseteq A$. Note that, $B_a = {C_a}^{[1]}$, so C_a is algebraically closed in B_a. But A is algebraically closed in B. So $A_a = C_a$. Similarly $A_b = C_b$. Since a,b is a regular sequence in C, $C_a \cap C_b = C$. Therefore $A \subseteq A_a \cap A_b = C_a \cap C_b = C$.

(ii) Let $\phi : R[U,V,W](= R^{[3]}) \twoheadrightarrow A$ be the R-algebra epimorphism such that $\phi(U) = u$, $\phi(V) = v$ and $\phi(W) = w$. Then $(bW - aV - U^2) \subseteq Ker\ \phi$ and $bW - aV - U^2$ is an irreducible polynomial of the UFD $R[U,V,W]$. Now $\operatorname{tr.deg}_R (R[U,V,W]/(bW - aV - U^2)) = 2 = \operatorname{tr.deg}_R A$. Hence $A \cong R[U,V,W]/(bW - aV - U^2)$. Let $F = bW - aV - U^2$. Now $(\frac{\partial F}{\partial U}, \frac{\partial F}{\partial V}, \frac{\partial F}{\partial W}, F)R[U,V,W] \neq R[U,V,W]$. So A is not a regular ring, in particular, $A \neq R^{[2]}$.

(iii) *rank* $D = 3$ by Lemma 4.1.2. □

4.2 Quasi-nice Derivations

In this section we discuss quasi-nice derivations. Let k be a field of characteristic zero, R a k-domain, $B := R^{[n]}$ and m be a positive integer $\leqslant n$. We shall call a quasi-nice R-derivation on B to be *m-quasi* if, for some coordinate system $(T_1, T_2, \ldots, T_n)$ of B, $D^2(T_i) = 0$ for all $i \in \{1, \ldots, m\}$. Thus for any two positive integers r and m such that $1 \leqslant m < r \leqslant n$, it is easy to see that an r-quasi derivation is also an m-quasi derivation. We shall call an m-quasi derivation to be *strictly m-quasi* if it is not r-quasi for any positive integer $r > m$.

Over a field k, Z. Wang ([39, Theorem 4.7 and Remark 5]) has proved the following result for 2-quasi derivations.

Theorem 4.2.1. *Let k be a field of characteristic zero and $k[X,Y,Z] = k^{[3]}$. Let $D(\neq 0) \in LND(k[X,Y,Z])$ be such that D is irreducible and $D^2X = D^2Y = 0$. Then one of the following holds:*

(I) *There exists a coordinate system (L_1, L_2, Z) of $k[X,Y,Z]$, where L_1 and L_2 are linear forms in X and Y such that*

  (i) $DL_1 = 0$.

  (ii) $DL_2 \in k[L_1]$.

  (iii) $DZ \in k[L_1, L_2] = k[X,Y]$.

  In this case, rank D can be either 1 or 2.

(II) *There exists a coordinate system (V, X, Y) of $k[X,Y,Z]$, such that $DV = 0$ and $DX, DY \in k[V]$. In particular, rank $D = 1$.*

Conversely, if $D \in Der(k[X,Y,Z])$ satisfies (I) or (II), then $D \in LND(k[X,Y,Z])$ and $D^2X = D^2Y = 0$.

The following two examples illustrate the cases *rank* $D = 1$ and *rank* $D = 2$ for part (I) of Theorem 4.2.1.

Example 4.2.2. Let $D \in LND(k[X,Y,Z])$ be such that $DX = DY = 0$ and $DZ = 1$. Then *rank* $D = 1$.

Example 4.2.3. Let $D \in LND(k[X,Y,Z])$ such that

$$DX = 0, DY = X, DZ = Y.$$

Setting $R = k[X]$, we see that $D \in LND_R(R[Y,Z])$ and D is irreducible. By Theorem 2.2.3, $D = \frac{\partial F}{\partial Z}\frac{\partial}{\partial Y} - \frac{\partial F}{\partial Y}\frac{\partial}{\partial Z}$ for some $F \in R[Y,Z] = K[X,Y,Z]$ such that $k(X)[Y,Z] = k(X)[F]^{[1]}$, $gcd_{R[Y,Z]}(\frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}) = 1$. Moreover $Ker\ D = R[F] = R^{[1]}$. Setting $F = XZ - \frac{Y^2}{2}$ we see $\frac{\partial F}{\partial Y} = -Y = -DZ$ and $\frac{\partial F}{\partial Z} = X = DY$.

Therefore, $Ker\ D = k[X,F]$. But F is not a coordinate in $k[X,Y,Z]$ since $(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z})k[X,Y,Z] \neq k[X,Y,Z]$. So $rank\ D = 2$.

We now address Question 2 of the Introduction, which gives a partial generalisation of Theorem 4.2.1.

Proposition 4.2.4. *Let R be a PID containing $\mathbb{Q}$ with field of fractions K. Let $D \in LND_R(R[X,Y,Z])$, where $R[X,Y,Z] = R^{[3]}$ such that D is irreducible and $D^2X = D^2Y = 0$. Let $\overline{D} \in LND(K[X,Y,Z])$ denote the extension of D to $K[X,Y,Z]$. Let $A := Ker\ D$. Suppose $\overline{D}$ satisfies condition (I) of Theorem 4.2.1. Then the following hold:*

(i) *$rank\ D < 3$.*

(ii) *There exists a coordinate system (L_1, L_2, Z) of $R^{[3]}$, such that L_1, L_2 are linear forms in X and Y, $DL_1 = 0$, $DL_2 \in R[L_1]$ and $DZ \in R[L_1, L_2] = R[X,Y]$. Moreover, $A = R[L_1, bZ + f(L_2)]$, where $b \in R[L_1]$ and $f(L_2) \in R[L_1, L_2]$.*

Proof. (i) Let $(\overline{L_1}, \overline{L_2}, Z)$ be a coordinate system of $K[X,Y,Z]$ such that $\overline{D}$ satisfies condition (I) of Theorem 4.2.1. Multiplying by a suitable nonzero constant from R, we can assume $\overline{L_1} \in R[X,Y]$. Let $\overline{L_1} = aX + bY$ where $a, b \in R$. Without loss of generality we can assume $gcd_R(a,b) = 1$. Since R is a PID, $(a,b,0)$ is a unimodular row in R^3 and hence can be completed to an invertible matrix (say N) in $GL_3(R)$. Thus $\overline{L_1}$ is a coordinate in $R[X,Y,Z]$. As $\overline{L_1} \in KerD = Ker\overline{D} \cap R[X,Y,Z]$, $rank\ D$ is at most 2 and hence $rank\ D \leqslant 2 < 3$.

(ii) Now set $L_1 = \overline{L_1}$. Since $gcd_R(a,b) = 1$, there exist $c, d \in R$ such that $ad - bc = 1$. Hence we can choose N as $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $N\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ Z \end{pmatrix}$.
Now the proof follows from part (i) of Lemma 2.2.4. $\square$

With the notation as above, if $\overline{D}$ satisfies condition (II) of Theorem 4.2.1, $rank\ D$ need not be 1. The following example shows that $rank\ D$ can even be 3.

Example 4.2.5. Let k be a field of characteristic zero, $R = k[t](= k^{[1]})$ with field of fractions L and $B := R[X, Y, Z](= R^{[3]})$. Let $D \in LND_R(B)$ be defined by

$$DX = t, \quad DY = tZ + X^2 \quad \text{and} \quad DZ = -2X.$$

Then D is irreducible and $D^2X = D^2Y = 0$. Let $\overline{D}$ denote the extension of D to $L[X, Y, Z]$. Let $F = -GX + tY$ where $G = tZ + X^2$. Then $F^2 - G^3 = tH$, where $H = tY^2 - 2tX^2Z^2 - 2tXYZ - 2X^3Y - X^4Z - t^2Z^3 \in R[X, Y, Z]$. Set $C := R[F, G, H]$. We show that

(i) $\overline{D}$ satisfies condition (II) of Theorem 4.2.1.

(ii) Then $C \cong R[U, V, W]/(U^2 - V^3 - tW)$, where $R[U, V, W] = R^{[3]}$ and hence $C \neq R^{[2]}$.

(iii) $Ker\ D = C$.

(iv) $rank\ D = 3$.

Proof. (i) $L[X, Y, Z] = L[X, Y, G]$, $DG = \overline{D}G = 0$ and $\overline{D}X, \overline{D}Y \in L[G]$. By Theorem 2.2.3, $Ker\ \overline{D} = L[F, G](= L^{[2]})$.

(ii) Consider the R-algebra epimorphism $\phi : R[U, V, W] \twoheadrightarrow R[F, G, H](= C)$, given by $\phi(U) = F$, $\phi(V) = G$ and $\phi(W) = H$. Clearly $(U^2 - V^3 - tW) \subseteq Ker\ \phi$. Since $U^2 - V^3 - tW$ is an irreducible polynomial in $R[U, V, W] = k[t, U, V, W] = k^{[4]}$ and $\operatorname{tr.deg}_k R[U, V, W]/(U^2 - V^3 - tW) = 3$, we have $Ker\ \phi = (U^2 - V^3 - tW)$ and hence $C \cong R[U, V, W]/(U^2 - V^3 - tW)$.

Set $f := U^2 - V^3 - tW$. Since $(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial U}, \frac{\partial f}{\partial V}, \frac{\partial f}{\partial W}, f) \subseteq (t, U, V, W)k[t, U, V, W]$, C is not regular and hence $C \neq R^{[2]}$.

(iii) Let $A := Ker\ D$. Then $A = L[F, G] \cap R[X, Y, Z]$. We note that since $Ker\ \overline{D}$ is factorially closed, $H \in Ker\ \overline{D}$ and hence $H \in A$.

Now $C_t = R_t[F, G, H] = R_t[F, G]$ and $R_t[X, Y, Z] = R_t[X, Y, G] = R_t[X, F, G]$. D extends to a locally nilpotent R_t-derivation (say $\tilde{D}$) on $R_t[X, Y, Z]$ and $\tilde{D}X \in {R_t}^*$. So by Theorem 2.2.2, $C_t = A_t$.

Clearly $C \subseteq A$. By Lemma 2.3.7, it is enough to show that the map $C/tC \to A/tA$ is injective. Since A is factorially closed in B, the natural map $A/tA \hookrightarrow B/tB$ is an inclusion. So we will be done if we show the composite map $\psi : C/tC \to B/tB$ is injective. For $g \in B$, let $\overline{g}$ denote the image of g in B/tB. In $\psi(C/tC)$, $\overline{G} = \overline{X}^2$, $\overline{F} = -\overline{X}^3$ and $\overline{H} = -2\overline{X}^3\overline{Y} - \overline{X}^4\overline{Z}$. Since $\overline{X}$ and $\overline{Z}$ are algebraically independent over k, $\operatorname{tr.deg}_k \psi(C/tC) = 2$. From (ii), it follows that C/tC is an integral domain and $\operatorname{tr.deg}_k C/tC =$

$\mathrm{tr.deg}_k\, \psi(C/tC) = 2$. Hence ψ is injective. So $C/tC \hookrightarrow A/tA$ and hence $C = A$ as desired.

(iv) By Lemma 4.1.2, $rank\ D = 3$. □

Over a Dedekind domain R, we have the following generalisation of Proposition 4.1.9 and Proposition 4.2.4.

Proposition 4.2.6. *Let R be a Dedekind domain containing $\mathbb{Q}$ with field of fractions K, and $B := R[X,Y,Z](= R^{[3]})$. Let $D \in LND_R(B)$ be irreducible and $D^2X = D^2Y = 0$ and $\overline{D}$ denote the extension of D to $K[X,Y,Z]$. Let $A := Ker\ D$. If $\overline{D}$ satisfies condition (I) of Theorem 4.2.1, then the following hold:*

(i) *A is generated by at most 3 elements.*

(ii) *Moreover, if D is fixed-point free, then $rank\ D < 3$ and D has a slice. In particular, $rank\ D = 1$.*

Proof. (i) By Proposition 4.2.4 and Lemma 4.1.2, $A_p = {R_p}^{[2]}$ for each $p \in Spec(R)$. Now the proof follows from the proof of part (i) of Proposition 4.1.9.

(ii) For each $p \in Spec(R)$, let D_p denote the extension of D to B_p. Then by Proposition 4.2.4 and Theorem 2.2.6, D_p has a slice. Thus $B_p = {A_p}^{[1]}$ for each $p \in Spec(R)$. Now the proof follows from the proof of part (ii) of Proposition 4.1.9. □

Remark 4.2.7. Example 4.1.10 shows that Proposition 4.2.4 does not extend to a higher-dimensional UFD, not even to $k^{[2]}$, where k is a field of characteristic zero. In fact, in that example, taking $L_1 = u$ and $L_2 = cX + dY$ for some $c, d \in k[a,b]$ such that $ad - bc \neq 0$, we find that $\overline{D}$ satisfies condition (I) of Theorem 4.2.1 and considering the coordinate system (u, X, Z), of $K[X,Y,Z]$ where K is the field of fractions of $k[a,b]$, we also see that $\overline{D}$ satisfies condition (II).

The following theorem of Daigle shows that over a field k of characteristic zero, there does not exist any strictly 1-quasi derivation on $k^{[3]}$ ([10, Theorem 5.1]).

Theorem 4.2.8. *Let k be a field of characteristic zero , $B = k^{[3]}$ and $D : B \to B$ be an irreducible locally nilpotent derivation. Assume that some variable Y*

of B satisfies $DY \neq 0$ and $D^2Y = 0$. Then there exist X, Z such that

$$B = k[X, Y, Z], \quad DX = 0, \quad DY \in k[X] \quad \textit{and} \quad DZ \in k[X, Y].$$

We now present an example of a strictly 1-quasi derivation on $R^{[3]}$ over a PID R containing $\mathbb{Q}$. Thus, Theorem 4.2.8 does not extend to a PID.

Example 4.2.9. Let k be a field of characteristic zero, $R = k[t](= k^{[1]})$ and $B = R[X, Y, Z](= R^{[3]})$. Let $D \in LND_R(B)$ be such that

$$DX = t, \quad DY = X \quad \text{and} \quad DZ = Y.$$

Then D is irreducible and $D^2X = 0$. Let $F := 2tY - X^2$, $G = 3t^2Z - 3tXY + X^3$ and $H = 8tY^3 + 9t^2Z^2 - 18tXYZ - 3X^2Y^2 + 6X^3Z$. Then $F^3 + G^2 = t^2H$. Set $C := R[F, G, H]$. We now show the following:

(i) Then $C \cong R[U, V, W]/(U^3 + V^2 - t^2W)$, where $R[U, V, W] = R^{[3]}$.

(ii) $Ker\ D = C$.

(iii) There does not exist any coordinate system (U_1, U_2, U_3) of B, such that $D^2(U_1) = D^2(U_2) = 0$.

Proof. (i) Consider the R-algebra epimorphism $\phi : R[U, V, W] \twoheadrightarrow R[F, G, H](= C)$, given by $\phi(U) = F$, $\phi(V) = G$ and $\phi(W) = H$. Clearly $(U^3+V^2-t^2W) \subseteq Ker\ \phi$. Since $U^3 + V^2 - t^2W$ is an irreducible polynomial in $R[U, V, W] = k[t, U, V, W] = k^{[4]}$, and $\operatorname{tr.deg}_k R[U, V, W]/(U^3+V^2-t^2W) = 3$ we have $Ker\ \phi = (U^3 + V^2 - t^2W)$ and hence $C \cong R[U, V, W]/(U^3 + V^2 - t^2W)$.

(ii) Let $A := Ker\ D$. Since A is factorially closed in B, $H \in A$. $C_t = R_t[F, G, H] = R_t[F, G]$. Also $R_t[X, Y, Z] = R_t[X, F, G]$. D extends to a locally nilpotent R_t-derivation (say $\tilde{D}$) on $R_t[X, Y, Z]$ and $\tilde{D}X \in {R_t}^*$. So by Theorem 2.2.2, $C_t = A_t$.

Clearly $C \subseteq A$. By Lemma 2.3.7, it is enough to show that the map $C/tC \to A/tA$ is injective. Since A is factorially closed in B, the natural map $A/tA \hookrightarrow B/tB$ is an inclusion. So we will be done if we show the composite map $\psi : C/tC \to B/tB$ is injective. For $g \in B$, let $\overline{g}$ denote the image of g in B/tB. In $\psi(C/tC)$, $\overline{F} = -\overline{X}^2$, $\overline{G} = \overline{X}^3$ and $\overline{H} = -3\overline{X^2Y^2} + 6\overline{X^3}\overline{Z}$. Since $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ are algebraically independent over k, $\operatorname{tr.deg}_k \psi(C/tC) = 2$. From (ii), it follows that C/tC is an integral domain and $\operatorname{tr.deg}_k C/tC =$

$\operatorname{tr.deg}_k \psi(C/tC) = 2$. Hence ψ is injective. So $C/tC \hookrightarrow A/tA$ and hence $C = A$ as desired.

(iii) For $f \in B$, let $\alpha :=$ coefficient of X in f, $\beta :=$ coefficient of Y in f and *deg* $f :=$ total degree of f. Then $\frac{\partial f}{\partial X}(0,0,0) = \alpha$ and $\frac{\partial f}{\partial Y}(0,0,0) = \beta$. If $f \in A$, let $f = p(F,G,H)$ for some $p \in R^{[3]}$. Then $\frac{\partial f}{\partial X} = \frac{\partial p}{\partial F}\frac{\partial F}{\partial X} + \frac{\partial p}{\partial G}\frac{\partial G}{\partial X} + \frac{\partial p}{\partial H}\frac{\partial H}{\partial X}$. We also have

$$\frac{\partial F}{\partial X} = -2X, \quad \frac{\partial G}{\partial X} = -3tY + 3X^2 \quad \text{and} \quad \frac{\partial H}{\partial X} = -18tYZ - 6XY^2 + 18X^2Z.$$

Thus $\alpha = 0$. Again, since

$$\frac{\partial F}{\partial Y} = 2t, \quad \frac{\partial G}{\partial Y} = -3tX, \quad \text{and} \quad \frac{\partial H}{\partial Y} = 24tY^2 - 18tXZ - 6X^2Y,$$

similarly we have $\beta = \lambda t$, for some $\lambda \in R$. Let U be a coordinate in B such that $D^2U = 0$. Since U is a coordinate, there exist $a, b, c \in R$, not all 0 and $V \in B$ with *deg* $V \geqslant 2$ and no linear term, such that $U = aX + bY + cZ + V$. Then $DU = at + bX + cY + (\frac{\partial V}{\partial X})t + (\frac{\partial V}{\partial Y})X + (\frac{\partial V}{\partial Z})Y$. Let $\gamma :=$ coefficient of X in $\frac{\partial V}{\partial X}$ and $\delta :=$ coefficient of Y in $\frac{\partial V}{\partial X}$. Then the coefficient of X in DU is $b + \gamma t$ and the coefficient of Y in DU is $c + \delta t$ (we can ignore terms from $(\frac{\partial V}{\partial Y})X$ and $(\frac{\partial V}{\partial Z})Y$ since neither of them has any linear term). Thus $b + \gamma t = 0$ and $c + \delta t = \lambda t$ for some $\lambda \in R$. Therefore, $b \in (t)R$ and $c \in (t)R$.

Let (U_1, U_2, U_3) be a coordinate system of B such that $D^2(U_1) = D^2(U_2) = 0$ and let $U_i = a_iX + b_iY + c_iZ + V_i$ where $a_i, b_i, c_i \in R$, not all 0 and $V_i \in B$ with *deg* $V_i \geqslant 2$ and no linear term, for $i = 1, 2$. Thus for each i, we have $b_i, c_i \in (t)R$. For $f \in B$, let $\overline{f}$ denote its image in $B/tB (= k^{[3]})$. Since $B/tB = k[\overline{U_1}, \overline{U_2}, \overline{U_3}]$, $\overline{U_1}, \overline{U_2}$ form a partial coordinate system in B/tB. But $\overline{U_i} = \overline{a_i}\overline{X} + \overline{V_i}$ and since $\overline{V_i}$ has no linear term, $\overline{a_i} \neq 0$, for each i. Then $\overline{a_2}\overline{U_1} - \overline{a_1}\overline{U_2}$ has no linear term, but it is a coordinate in B/tB. Hence we have a contradiction. $\square$

Let R be a Dedekind domain containing $\mathbb{Q}$ and $B := R[X,Y,Z] = R^{[3]}$. Let $D \in LND_R(B)$ such that D is irreducible and $D^2X = D^2Y = D^2Z = 0$ (i.e. D is nice). It has been proved in Proposition 4.1.9 that if D is fixed-point free, then *rank* $D = 1$. However, we do not know whether the result is true without the additional hypothesis that D is fixed-point free.

Example 4.2.10. Let $R := \mathbb{R}[a,b]/(a^2 + b^2 - 1)$ and $B := R[X,Y,Z]$. Set $u = aY + (1-b)X$, $v = (1+b)Y + aX$, $f = (1-b)uZ - (a+1)(u+1)Y$,

$g = (a+1)(u+1)X + auZ$ and $h = auZ - (1+b)(u+1)Y - vY + X$. Then we have the following relations:

(i) $au = (1-b)v$ (or $(1+b)u = av$).

(ii) $ag - (1+b)f = (a+1)(u+1)v$.

(iii) $u = (1-b)h - af$ (or $v = ah - (1+b)f$).

Let $D \in LND_R(B)$ such that

$$DX = -au, \quad DY = (1-b)u \quad \text{and} \quad DZ = (a+1)(u+1).$$

Set $A := Ker\ D$. Then $B_{(1+b)} = R_{(1+b)}[X, Z, v]$. Since $R_{(1+b)}$ is a PID, by Theorem 4.1.6, $A_{(1+b)} = R_{(1+b)}[v, g] = R_{(1+b)}[v, h]$. Similarly $A_{(1-b)} = R_{(1-b)}[v, f] = R_{(1-b)}[h, f]$. Since $(1+b)$ and $(1-b)$ are comaximal in R, $A = R[v, h, f] = R[h, f]$ (by (iii)).

Question: Is $rank\ D < 3$?

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