CHARACTERIZATION OF EIGENFUNCTIONS OF THE LAPLACE–BELTRAMI OPERATOR THROUGH RADIAL AVERAGES ON RANK ONE SYMMETRIC SPACES

MUNA NAIK

INDIAN STATISTICAL INSTITUTE, KOLKATA
2019

CHARACTERIZATION OF EIGENFUNCTIONS OF THE LAPLACE–BELTRAMI OPERATOR THROUGH RADIAL AVERAGES ON RANK ONE SYMMETRIC SPACES

MUNA NAIK

Thesis submitted to the Indian Statistical Institute
in partial fulfilment of the requirements
for the award of the degree of
Doctor of Philosophy.
2019

Indian Statistical Institute
203, B.T. Road, Kolkata, India.

DEDICATION

जय जय श्रीराधे जय जय श्रीराधे जय जय श्रीराधे

Jagadguru Shri Kripaluji Maharaj

गुरु गोविन्द दोउ खड़े काके लागूँ पाय ।
बलिहारी गुरुदेवकी गोविन्द दियो मिलाय ॥

Both Guru and God are standing before me,
To Whom Should I pay my respects first,
Glory to my Guru, Glory to my Guru
Who revealed God to me.

राधा-कृपा-कटाक्ष ते, पद-प्रसून-उपहार ।
प्रिया-प्रियहिँ-पद समर्पित, उर "कृपालु" उद्गार ॥
जगद्गुरु श्री कृपालु जी महाराज

Kripalu's heart welling up with emotions of love,
dedicates and offers up this gift of flowers
in the form of the padas or devotional songs
to the most beloved Radha Krishna
which have manifested by the merciful side glance of Shri Radha!
Jagadguru Shri Kripaluji Maharaj

Acknowledgements

I offer my deepest gratitude to my supervisor Rudra Sarkar without whom this thesis would not have possible. I was lucky enough to find a guide like him and it is only by his meticulous guidance and indomitable encouragement I was able to do my research work smoothly.

I am grateful to my teacher and collaborator Swagato K. Ray for his excellent lectures and several illuminating discussions throughout the years I was working for my PhD.

I would like to thank all faculty members of Stat–Math unit, in particular, Biswaranjan Behera, Kingsook Biswas, Shashi Mohan Srivastava, Mahuya Datta, Debashish Goswami and Paramita Das for various useful discussions and teaching.

My academic and nonacademic life is mostly influenced and moulded by teachers and I sincerely offer my veneration to them.

My sincere thanks to all my NISER teachers Binod Sahoo, Brundaban Sahu, K. L. Patra, Deepak Dalai, V. Muruganandam, N. Parhi, Anil Karn, Nabin Jana and Sanjoy Parui for my undergraduate training and to Purna Chandra Das who motivated me to pursue mathematics and would have been happier than me at this moment.

I am indebted to Prof. Swadheenananda Pattanayak, who is a great source of inspiration for many students and teachers of Odisha.

I do not have words to express gratitude towards my teacher Kedar Chandra Pattanayak and my family members.

I am extremely grateful to my group members, Mithun Bhowmick, Sayan Bagchi, Suparna Sen and Jayanta Sarkar for many effective discussions. I extend my sincere thanks to Moumita di, for her sisterly affection towards me and to all my friends, juniors and seniors especially Madhav da, Narayan da, Apurva da and Tapas da for being incredibly helpful.

I seek the blessings of Bhagwan Sri Ramakrishna, Sarada Maa, Swami Vivekananda and Maa Bhavatarini to increase selfless love and devotion at the lotus feet of my beloved spiritual master, Jagadguru Shree Kripaluji Maharaj and my *iṣṭa* Shri Radha Krishna. Finally I offer my obeisance to my Gurudev and Yugal Sarkar with the following verse:

असित गिरि समं स्यात् कज्जलं सिन्धु पात्रे सुरतरुवरशाखा लेखनी पत्रमुर्वी ।
लिखति यदि गृहीत्या शारदा सर्वकालं तदपि तब गुणानामीश पारं न याति ॥

Contents

Preface **vii**

List of symbols **ix**

0 Introduction **1**

0.1 1

0.2 3

0.3 12

1 Preliminaries **15**

1.1 Lorentz spaces 16

1.2 Symmetric space 16

1.3 Jacobi functions 21

1.4 Properties of elementary spherical function 23

1.5 Schwartz spaces and tempered distributions 25

1.6 Hardy-type spaces 27

1.7 Characterization of eigenfunctions and the Poisson transform 29

1.8 Kunze–Stein phenomenon and Herz's majorizing principle 31

2 Characterization of eigenfunctions from the equation $f * \mu = f$ **33**

2.1 Results from Spectral synthesis 34

2.2 Set up, statements and proofs of the results 35

2.2.1 Result leading to the two radius theorem 36

2.2.2 Characterization of eigenfunctions with some integrability conditions 40

2.2.3 A generalization of Furstenberg's result on harmonic function 42

3 Characterization of eigenfunctions via Roe–Strichartz type theorems **45**

3.1 Preparations . . . 45
3.1.1 Spectrum of the Laplacian . . . 46
3.1.2 Multiplier operator . . . 46
3.1.3 One radius theorem . . . 48
3.2 Characterization of L^p-tempered eigendistributions . . . 51
3.3 Characterization of L^2-tempered eigendistributions . . . 64
3.4 Comments on the sharpness of the results . . . 74

4 Mean value property in limit, a result of Plancherel–Pólya and Benyamini–Weit 75
4.1 Ball-MVP in limit for functions without growth restriction . . . 75
4.1.1 Statements of the results . . . 75
4.1.2 Convexity of distance . . . 76
4.1.3 Characterization of eigenfunctions by the mean value property 77
4.1.4 Proof of the main results . . . 78
4.2 Shell-MVP in limit for functions without growth restriction . . . 85
4.2.1 Statement of the main result . . . 85
4.2.2 Proof of the main result . . . 87
4.2.3 Not a mean value operator . . . 89
4.3 Sphere-MVP in limit for functions without growth restriction . . . 90
4.3.1 Proof of Theorem 4.3.1 . . . 91
4.4 Results for functions with growth conditions . . . 96
4.5 Examples and counter-examples . . . 100

5 Large and small time behaviour of heat propagation 107
5.1 Estimates of the heat kernel . . . 108
5.2 Large time behaviour of the heat propagation . . . 109
5.2.1 Counterexamples . . . 115
5.3 Asymptotic property of heat propagation and the characterization of eigenfunctions . . . 117
5.4 Small time behaviour of heat propagation . . . 124
5.4.1 Review of the result for Euclidean spaces . . . 125
5.4.2 Results on symmetric spaces . . . 126

Bibliography 136

PREFACE

Let X be a rank one Riemannian symmetric space of noncompact type and Δ be the Laplace–Beltrami operator of X. The space X can be identified with the quotient space G/K where G is a connected noncompact semisimple Lie group of real rank one with finite centre and K is a maximal compact subgroup of G. Thus G acts naturally on X by left translations. Through this identification, a function or measure on X is radial (i.e. depends only on the distance from eK), when it is invariant under the left-action of K. We consider right-convolution operators Θ on functions f on X defined by, $\Theta : f \mapsto f * \mu$ where μ is a radial (possibly complex) measure on X. These operators will be called multipliers. In particular Θ is a radial average when μ is a radial probability measure. Notable examples of radial averages are ball, sphere and annular averages. Another well known example is $f \mapsto f * h_t$, where h_t is the heat kernel on X. This will be called heat propagator and will be denoted by $e^{t\Delta}$. In this thesis we shall study the questions of the following genre. Below by eigenfunction we mean eigenfunction of Δ.

(i) Characterization of eigenfunctions from the equation $f * \mu = f$, which generalizes the classical question: Is a μ-harmonic function harmonic?

(ii) Fix a multiplier, in particular an averaging operator Θ. Suppose that $\{f_k\}_{k\in\mathbb{Z}}$ is a bi-infinite sequence of functions on X such that for all $k \in \mathbb{Z}$, $\Theta f_k = A f_{k+1}$ and $\|f_k\| < M$ for some constants $A \in \mathbb{C}$, $M > 0$ and a suitable norm $\|\cdot\|$. We try to infer that then f_0, hence every f_k, is an eigenfunction.

(iii) Let $B_t f$ be the ball (of radius t) average of f. Plancherel–Pólya (1931) and Benyamini–Weit (1989) proved that for continuous functions f, g on $\mathbb{R}^n$, if $B_t f \to g$ uniformly on compact sets as $t \to \infty$, then g is harmonic. We endeavour to generalize this result for eigenfunctions on X.

(iv) We explore the behaviour of heat propagator in X in large and small time to illustrate the differences with the corresponding results in $\mathbb{R}^n$. In particular we study the relation between the limiting behaviour of the ball-averages as radius tends to ∞ and that of the the heat propagator as time goes to ∞ and use this relation for the characterization of eigenfunctions.

List of symbols

Generalities

$\mathbb{N}$: the set of natural numbers

$\mathbb{Z}$: the set of integers

$\mathbb{Z}^+$: the set of nonnegative integers

$\mathbb{Z}^-$: the set of nonpositive integers

$\mathbb{R}$: the set of real numbers

$\mathbb{R}^+$: the set of positive real numbers

$\mathbb{R}^\times$: the set of nonzero real numbers

$\mathbb{C}$: the set of complex numbers

$\Re z$: the real part of $z \in \mathbb{C}$

$\Im z$: the imaginary part of $z \in \mathbb{C}$

$\overline{S}$: closure of a set S in a topological space

S° : interior of a set S in a topological space

∂S : boundary of a set S in a topological space

$|S|$: measure of a set S in a measure space

χ_E : indicator function of a set E

Supp T: support of a tempered distribution T

$p' = p/(p-1)$: conjugate exponent of p, p. 15

$L^p(S)$: L^p-Lebesgue space on S

$L^1_{loc}(S)$: space of locally integrable functions on S

$L^\infty_{loc}(S)$: space of locally bounded functions on S

$L^{p,q}(M)$: Lorentz space, p. 16

$\|f\|_{p,q}$: Lorentz norm of f, p. 16

$f_1 \asymp f_2$ for positive f_1, f_2: $C_1 f_1 \leq f_2 \leq C_2 f_1$ for $C_1 > 0, C_2 > 0$, p. 15

$f \lesssim g$: $|f| \leq C|g|$ for some constant C, p. 118

$\langle T, f \rangle$: action of a distribution T on f, p. 15

S^{n-1}: unit sphere in $\mathbb{R}^n$

$\mathrm{SO}(n)$: special orthogonal group

$\Delta_{\mathbb{R}^n}$: Laplacian on $\mathbb{R}^n$

Symmetric space

$X = G/K$: Riemmanian symmetric space, p. 16

$\mathfrak{g}$: Lie algebra of G

Δ : Laplace–Beltrami operator on X

$d(\cdot,\cdot)$: Riemmanian distance on X, p. 17

o : origin of X, p. 16

$|x| = d(o,x)$ for $x \in X$, p. 17

$x = k(x)\exp H(x)n(x)$: Iwasawa decomposition, p. 17

$\varphi_\lambda(x)$: elementary spherical function, p. 19

$\varphi_{\mu,k}(x) = \frac{\partial^k}{\partial\lambda^k}\varphi_\lambda(x)|_{\lambda=\mu}$, p. 20

$\phi_\lambda^{(\alpha,\beta)}(t)$: Jacobi function with parameter α and β, p. 21

$\mathrm{c}(\lambda)$: Harish-Chandra c-function, p. 19

$\gamma_p = 2/p - 1$ for $0 < p \leq \infty$ p. 23, 46

ρ : half sum of positive roots of $\mathfrak{g}$, p. 17

$S_p = \{z \in \mathbb{C} \mid |\Im z| \leq |\gamma_p|\rho\}$, p. 23, 46

$\Lambda(S_p)$: L^p-spectrum of Δ, p. 46

$\ell_x f$: left translation of a function f on $X = G/K$ by $x \in G$

Rf : radialization of a function f on X, p. 18

$\widehat{f}$: spherical Fourier transform of a function f on X, p. 20

$\widetilde{f}(\lambda,k)$: Helgason Fourier transform of a function f on X at $(\lambda,k) \in \mathbb{R}\times K/M$, p. 20

Function spaces

$C^\infty(X)$: space of infinitely differentiable (smooth) functions on X

$C^\infty(G//K)$: space of smooth K-biinvariant (radial) functions on X

$C_c^\infty(X)$: space of compactly supported smooth functions on X

$C_c^\infty(G//K)$: space of compactly supported radial smooth functions on X

$\mathcal{C}^p(X)$: L^p-Schwartz space on X, p. 25

$\mathcal{C}^p(G//K)$: set of radial functions in $\mathcal{C}^p(X)$, p. 25

$\mathcal{C}^p(X)'$: set of L^p-tempered distributions on X, p. 26

$\mathcal{H}_p^r(X)$: Hardy-type space on X, p. 27

$[f]_{p,r}$: Hardy-type norm, p. 27

Radial sets, averages

$B(x,r)$: geodesic ball of radius $r > 0$ with center at $x \in X$, p. 20

$\mathbb{A}_{r_1,r_2}(x)$: annulus centered at $x \in X$ with inner radius r_1 and outer radius r_2, p. 85

$M_t f$: spherical mean value operator, p. 20

$B_t f$: volume average of a function f, p. 21

$\psi_\lambda(r)$: spherical Fourier transform of $\chi_{B(o,r)}/|B(o,r)|$ at λ, p. 23

$\psi_{\mu,k}(r) = \frac{\partial^k}{\partial \lambda^k}\psi_\lambda(r)|_{\lambda=\mu}$, p. 23

V_r^λ : spherical Fourier transform at λ of $\chi_{B(o,r)}$, p. 21, 75

$m_r^\lambda = (V_r^\lambda)^{-1}\chi_{B(o,r)}$, p. 21, 76

$V_{r_1,r_2}^\lambda = V_{r_2}^\lambda - V_{r_1}^\lambda$, p. 85

$a_{r_1,r_2}^\lambda = (V_{r_1,r_2}^\lambda)^{-1}\,\chi_{\mathbb{A}_{r_1,r_2}(o)}$, p. 86

$\omega_t^\lambda(x) = \left(V_{\alpha t,\beta t}^\lambda\right)^{-1}\chi_{B(o,\beta t)}(x)$, p. 90

$\vartheta_t^\lambda(x) = \left(V_{\beta t}^\lambda\right)^{-1}\chi_{\mathbb{A}_{\alpha t,\beta t}(o)}(x)$, p. 99

$\varpi_t^\lambda(x) = \left(V_{\alpha t,\beta t}^\lambda\right)^{-1}\chi_{\mathbb{A}_{\alpha t,\beta t}(o)}(x)$, p. 99

Heat propagation

h_t : heat kernel, p. 46, 108

$e^{t\Delta}$: heat propagator given by right-convolution with h_t, p. 46, 108

$h_t^\lambda(x) = e^{t(\lambda^2+\rho^2)}h_t(x)$ for $\lambda \in \mathbb{C}$, p. 108

$\alpha_t^p = 2\gamma_p\rho t - r(t)$, $\beta_t^p = 2\gamma_p\rho t + r(t)$ for $r(t) > 0$, p. 118

$\mathbb{A}_t^p = \{k_1 a_s k_2 \mid s \in [\alpha_t^p, \beta_t^p],\ k_1, k_2 \in K\}$, p. 118

$h_{t,p} = \chi_{\mathbb{A}_t^p} h_t$, p. 118

$\bar{h}_{t,p} = \chi_{X\setminus\mathbb{A}_t^p} h_{t,p}$, p. 118

$\bar{h}_{t,p}^\lambda = e^{t(\lambda^2+\rho^2)}h_{t,p}$, p. 123

Chapter 0

Introduction

Let X be a rank one Riemannian symmetric space of noncompact type, which is equipped with a distance d and the Laplace–Beltrami operator Δ induced by its Riemannian structure. We fix a base point o of X, which we call origin. The prototypical example of such X is the hyperbolic spaces, in particular the upper half space. In this thesis we study some aspects of sphere, ball and other radial (with respect to the origin) averages of functions on X, leading to the characterization of eigenfunctions of Δ. The details of the problems we study are given in Section 0.2 below, after setting up the language in the next section.

0.1

The space X with the origin o can be realized as a quotient space $X = G/K$ where G is a noncompact connected semisimple Lie group with finite centre and of real rank one and K a maximal compact subgroup of G, so that o corresponds to the coset eK, where e is the identity element of G. Through this realization, functions on X, and radial functions on X are identified respectively with right K-invariant functions and K-biinvariant functions on G. Measures and distributions on X are also similarly identified with the corresponding objects on G. The group G acts on $X = G/K$ (and on the functions on X) naturally by left translation which we denote by $\ell_g, g \in G$. The Haar measure on G projects to a G-invariant Riemannian measure on X. Thus a radial average of a function f on X at a point $x \in X$, can be written as $f * \mu(x)$ where μ is a radial probability measure on X (identified as a K-biinvariant measure on G) and $*$ is the convolution of G. In particular the average of f on a sphere of radius t, centered at $x \in X$, denoted by $M_t f(x)$ is given by $M_t f(x) = f * \sigma_t(x)$ where σ_t is the normalized surface measure of the sphere of radius t. Similarly the average of f on a ball $B(x,r)$ of radius r, centered at $x \in X$ is

$$B_r f(x) = f * m_r(x)$$

where $m_r = \chi_{B(o,r)}/V_r$, V_r and $\chi_{B(o,r)}$ are the volume of $B(o,r)$ and its indicator function respectively. A function f on X is called harmonic if $\Delta f = 0$. The mean value theorem asserts that f is harmonic if and only if $M_t f(x) = f(x)$ (respectively $B_t f(x) = f(x)$) for all $t > 0$. To continue this discussion, we shall introduce some notation, without much elaboration. For $\lambda \in \mathbb{C}$, the elementary spherical function φ_λ is the unique radial eigenfunction of Δ with eigenvalue $-(\lambda^2 + \rho^2)$ satisfying $\varphi_\lambda(o) = 1$. Here ρ is the *half-sum of positive roots* (counted with their multiplicities), a positive number associated to the space X. We also have $\varphi_\lambda = \varphi_{-\lambda}$ and $\varphi_{i\rho} \equiv 1$. The generalized mean value property (MVP) states (see [41, 42]) that a continuous function f on X is an eigenfunction of Δ with eigenvalue $-(\lambda^2+\rho^2)$ for some $\lambda \in \mathbb{C}$, if and only if

$$f * \sigma_r = \varphi_\lambda(r) f \text{ for all } r > 0. \tag{0.1.1}$$

Above, $\varphi_\lambda(x)$ for $x \in X$ is interpreted as a function on distance of x from the origin o. Therefore such a function f also satisfies the ball mean value property:

$$f * \chi_{B(o,r)} = \left(\int_{B(o,r)} \varphi_\lambda(x)\, dx \right) f, \text{ for all } r > 0. \tag{0.1.2}$$

Taking $\lambda = i\rho$, we get back the standard MVP, characterizing harmonic functions. To put our study in perspective, let us recall some well known and relevant facts. These results illustrate the dichotomies between the space X and in particular the Euclidean spaces, which we shall experience as we shall go through the thesis. For simplicity we restrict first to harmonic functions. It follows from the Liouville theorem that bounded harmonic functions on $\mathbb{R}^n$ are constants. More generally if a harmonic function on $\mathbb{R}^n$ is of sublinear growth or nonnegative then it is constant and harmonic functions on $\mathbb{R}^n$ of polynomial growth of a fixed degree forms a finite dimensional vector space. These assertions are valid for any complete noncompact Riemannian manifold with nonnegative Ricci curvature and beyond. (See e.g. [19,21,45,84].) On the other hand the space X is of nonpositive Ricci curvatures and these results are not true for X. For instance, there exist nonconstant harmonic functions which are bounded or nonnegative. Indeed, the space of bounded (respectively nonnegative) harmonic functions in X is infinite dimensional. In general, for every $p > 2$, there is a wealth of L^p-eigenfunctions with complex eigenvalues in X, they are the Poisson transforms of suitable functions on the Poisson boundary of X. Two other distinguishing features of X in the context of the problems we shall deal with, are the exponential rate of volume growth of ball with radius and the dependence of L^p-spectrum of Δ on p.

0.2

Below we shall describe the problems we are concerned about, along with their motivations. The discussion on a particular chapter may be read as the preamble of that chapter. We hope this will help the readers to navigate through the thesis easily. Unless stated otherwise, from now on by eigenfunction we shall mean eigenfunction of Δ and by spectrum we mean the spectrum of Δ. Here and throughout this thesis, p' denotes the conjugate exponent of p, i.e. $p' = p/(p-1)$ and $\gamma_p = 2/p - 1$. In Chapter 1 we shall establish notation, terminologies and gather preliminary results which will be used in the thesis. We shall however assume the basics of analysis of Δ, as a detailed account on this is available in the literature (see e.g. [73–75]). Chapter 2 to Chapter 5 contain the results of this thesis, some representatives of which will be stated in these preambles.

Chapter 2

We take a radial (possibly complex) measure μ on X. For a function f on X, $f * \mu$ is a generalization of radial averages of f, whenever $f * \mu$ exists. If $f * \mu = f$, then f is called *μ-harmonic*, because it reduces to the standard mean value property when $\mu = \sigma_r$, the normalized surface measure of sphere of radius r. More generally, it follows from (0.1.1) that when f is an eigenfunction with eigenvalue $-(\lambda^2+\rho^2)$, then for a radial function (or measure) h on X, $f * h = \widehat{h}(\lambda) f$, whenever the convolution and the spherical Fourier transform $\widehat{h}(\lambda) = \int_X h(x)\varphi_\lambda(x)\,dx$ makes sense. Ball mean value property (0.1.2) is a particular case where $h = \chi_{B(o,r)}$. We may assume that $\widehat{h}(\lambda) = 1$, so that the equation simplifies to $f * h = f$. We consider the question if the converse is true, i.e., if $f * h = f$ for a radial function h for which $\widehat{h}(\lambda) = 1$ for a point $\lambda \in \mathbb{C}$, then is it true that f is an eigenfunction of Δ with eigenvalue $-(\lambda^2+\rho^2)$? There are a few obvious necessary conditions, which we need to consider for formulating such a result.

(1) The equation $f * h = f$ requires the existence of $f * h$, which is equivalent to the fact $\ell_g f \in L^1(X, h)$ for all most every $g \in G$ where $L^1(X, h)$ is the weighted L^1-space on X with weight h.

(2) The function f should be assumed to be in a suitable function space which accommodates eigenfunctions with the prescribed eigenvalue $-(\lambda^2 + \rho^2)$.

(3) To determine the eigenvalue $-(\lambda^2+\rho^2)$ uniquely from the equation $f*h = f$, we need to assume that $\widehat{h}(\lambda) = 1$ and $\widehat{h}(\nu) \neq 1$ whenever $\nu \neq \pm\lambda$ in the domain of definition of $\widehat{h}$.

A rather subtle point to note is that by the condition (3) above we are preventing eigenfunctions with eigenvalues other than $-(\lambda^2+\rho^2)$ to satisfy the equation $f*h = f$. We are expecting this to be sufficient to preclude all other functions which are not eigenfunctions to enter as a solution of $f * h = f$.

We now repeat the (abstract) formulation of the question. Take suitable f, h and fix an eigenvalue, so that the necessary conditions are satisfied. We ask what extra condition on h can ensure that $f * h = f$ implies that f is an eigenfunction with that specified eigenvalue? Instead of one h we can use several functions say $h_1, h_2, \dots$ in this formulation and adjust the necessary conditions accordingly, e.g. in the condition (3) above we can now assume that $\widehat{h_i}(\nu) \neq 1$ for at least one i.

Two prominent precursors to this study are Furstenberg's characterization of harmonic functions and Delsarte's two-radius theorem. Furstenberg proved in ([37, 38]) that if a bounded function f on X satisfies $f*\mu = f$ for an absolutely continuous probability measure μ then f is harmonic. (Furstenberg's proof is probabilistic. For another proof and a generalization see [81].) Suppose that μ above is given by the density h, i.e. $h \geq 0$ and $\int_X h = 1$. Then the domain of definition of $\widehat{h}$ is the Helgason–Johnson strip S_1, where

$$S_1 = \{\lambda \in \mathbb{C} \mid |\Im\lambda| \leq \rho\},$$

because, φ_λ are bounded if and only if $\lambda \in S_1$. Since $\varphi_{i\rho} \equiv 1$, we have $\widehat{h}(i\rho) = 1$. It also follows that $\widehat{h}(\nu) \neq 1$ whenever $\nu \neq \pm i\rho$ in S_1, because $|\varphi_\nu| < 1$ for those ν. Thus a function $f \in L^\infty(X)$ and h as above satisfy the necessary conditions given above. Noting that $(i\rho)^2 + \rho^2 = 0$, we arrive at the question answered affirmative by Furstenberg: does $f * \mu = f$ implies f is harmonic? But the question, which is paraphrased as: is a μ harmonic function harmonic, can be asked for other measures μ. Delsarte ([29, 30]) considered a characterization of harmonic functions through the (spherical) mean-value property. It was shown that if a continuous function on $\mathbb{R}^n$ satisfies the mean value property on spheres of two radii, then f is harmonic, unless the ratio of the radii belong to some finite set in $\mathbb{R}^+$. But there are nonharmonic functions which satisfy the mean value property with one radius. Once we notice that average of a function f over a sphere of radius r is $f * \sigma_r$ where σ_r is the normalized surface measure on the sphere of radius r around the origin, this falls in the genre of the questions we discussed above. It also indicates that sometimes we may have to use more than one measure to characterize harmonicity. Motivated by these results many authors considered various measures μ on X (and related spaces e.g. trees) and endeavored to find when a μ-harmonic function is harmonic. The paper by Ahern, Flores and Rudin [2] is of particular interest for us. They considered the Hermitian hyperbolic space and took the Lebesgue measure on $\mathbb{B}_n$, the unit ball in $\mathbb{C}^n$ (as the standard ball model of the space) as μ. Subsequently this result was explained and generalized by Koranyi [49] and Ben Natan, Weit [10], who intrigued our study. (See also [17, 58].) However, it appears that not much attention was paid in the literature for eigenfunctions other than the harmonic functions. Aim of this chapter is to consolidate and extend the ideas and methods from the results dealing with harmonic functions, after finding the proper set up to formulate the question for eigenfunctions. Our basic tool here can be described largely as spectral analysis and synthesis. But we recall that the Fourier transforms of L^p-functions extend analytically in a complex domain for $p \in [1, 2)$, which prevents us to use the standard method of determining the support of the Fourier transform of a dual object.

Chapter 3

In [67], Roe proved the following characterization of the sine function.

Theorem 0.2.1 (Roe). *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of functions on $\mathbb{R}$ such that $f_{k+1} = \frac{df_k}{dx}$ and $|f_k(x)| \leq C$ for all $k = 0, \pm1, \pm2, \cdots$ and $x \in \mathbb{R}$ for some $C > 0$. Then $f_0(x) = a\sin(x + b)$ where a and b are real constants.*

This theorem was generalized by Strichartz in [76] and Howard–Reese [44], where d/dx was replaced by the standard Laplacian $\Delta_{\mathbb{R}^n}$ of $\mathbb{R}^n$ and a characterization of bounded eigenfunctions of $\Delta_{\mathbb{R}^n}$ with eigenvalue -1 was obtained, although the proof works for other eigenvalues, for which there are bounded eigenfunctions.

Theorem 0.2.2 (Strichartz). *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of functions on $\mathbb{R}^n$ with $\Delta_{\mathbb{R}^n} f_k = \alpha f_{k+1}$ for some $\alpha > 0$, for all $k \in \mathbb{Z}$. If $\|f_k\|_{L^\infty(\mathbb{R}^n)} \leq C$ for all $k \in \mathbb{Z}$, for some $C > 0$, then $\Delta_{\mathbb{R}^n} f_0 = -\alpha f_0$.*

Among other things, it was demonstrated by a counter example in [76] that the result is not true for the hyperbolic 3-space. As observed in [52], such counter examples can be constructed in any Riemannian symmetric space of noncompact type. Indeed, the shape of the L^p-spectrum of Δ and the growth/decay of the elementary spherical functions are responsible for the failure of this result. Taking this into account, the story was further extended in [52], where they proved a version of Theorem 0.2.2 for eigenfunctions of Δ corresponding to nonzero real eigenvalues belonging to the interior of the L^1-spectrum, replacing L^∞-norm by suitable weak L^p-norm (denoted below by $\|\cdot\|_{p,\infty}$) in the formulation. A representative result in [52] is the following.

Theorem 0.2.3. *Let $\{f_k\}_{k\in\mathbb{Z}^+}$ be an infinite sequence of functions on X such that for some $p \in (1, 2)$, $\Delta f_k = -4\rho^2/pp' f_{k+1}$ for all $k \in \mathbb{Z}^+$. If $\|f_k\|_{p',\infty} \le C$ for some $C > 0$, for all $k \in \mathbb{Z}^+$, then $\Delta f_0 = -4\rho^2/pp' f_0$.*

If we take $f_k = f$ for all $k \in \mathbb{Z}^+$, where f is a weak $L^{p'}$-eigenfunction with eigenvalue $-4\rho^2/pp'$, then it is trivially true that $\|f_k\|_{p',\infty} \le M$ where $M = \|f\|_{p',\infty}$. Theorem 0.2.3 asserts that the apparent weak assumption of uniform-norm-boundedness of such a sequence leads to the strong conclusion that f_0 and hence all f_k are eigenfunctions. Perhaps, due to intrinsic difficulties, eigenfunctions with other (in particular complex) eigenvalues were not considered in [52].

We consider translation invariant continuous linear operators Θ on function spaces of X, which will be called *multipliers*. Indeed, they are radial (right) convolution operators $f \mapsto f * \mu$ with μ radial, which include suitable functions of the Laplacian. In the previous chapter we have considered them as generalization of radial averages. It appears to be natural to formulate the result above replacing Δ by such Θ, as we recall (and endeavour to extend) the heuristic principle: "an equation involving the Laplacian implies an analogous equation involving functions of the Laplacian" (see [74]). However, we realize that we cannot cast our net too wide to consider all such Θ and therefore content ourselves with some examples of Θ, e.g. ball and sphere averages, heat operators etc, which conforms with the concern of this thesis. Nevertheless, for the case $p = 2$, we strive to address the question in this generality and succeed partially, taking advantage of the one-dimensional L^2-spectrum. We may conjecture at this point that such an assertion for all multipliers should be true for other admissible p (i.e. for which there are eigenfunctions in $L^p(X)$). The typical results we prove here are the following:

Theorem 0.2.4. *Fix $t > 0$. For $1 \le p < 2$, let $\{f_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of measurable functions on X such that for all $k \in \mathbb{Z}$, $M_t f_k = A f_{k+1}$ for some constant $A \in \mathbb{C}$ and $\|f_k\|_{p',\infty} \le C$ for a constant $C > 0$.*

(a) *If $|A| = \varphi_{i\gamma_p\rho}(a_t)$, then f_0 is the Poisson transform at $-i\gamma_p\rho$ of a function $F \in L^{p'}(K/M)$, in particular, $\Delta f_0 = -\frac{4\rho^2}{pp'} f_0$.*

(b) *If* $|A| < \varphi_{i\gamma_p\rho}(a_t)$*, then* f_0 *may not be an eigenfunction.*

(c) *If* $|A| > \varphi_{i\gamma_p\rho}(a_t)$*, then* $f_0 = 0$*.*

Below by $\mathcal{C}^p(X)$ we denote the Harish-Chandra L^p-Schwartz space on X for $0 < p \leq 2$. Elements of the dual space of $\mathcal{C}^p(X)$ are called L^p-tempered distributions.

Theorem 0.2.5. *Let* $\Theta : \mathcal{C}^2(X) \to \mathcal{C}^2(X)$ *be a multiplier with real valued symbol* $m(\lambda) \in C^\infty(\mathbb{R})$*. Let* $\{f_k\}_{k\in\mathbb{Z}}$ *be a bi-infinite sequence of measurable functions such that* $\Theta f_k = A f_{k+1}$ *for all* $k \in \mathbb{Z}$*, for a nonzero constant* $A \in \mathbb{C}$ *and* $\|f_k\|_{2,\infty} \leq C$ *for a constant* $C > 0$*. Let* $m(\mathbb{R}) = \{m(\lambda) \mid \lambda \in \mathbb{R}\}$*. We have the following conclusions.*

(a) *If* $|A| \in m(\mathbb{R})$ *but* $-|A| \notin m(\mathbb{R})$*, then* $\Theta f_0 = |A| f_0$*.*

(b) *If* $-|A| \in m(\mathbb{R})$ *but* $|A| \notin m(\mathbb{R})$*, then* $\Theta f_0 = -|A| f_0$*.*

(c) *If both* $|A|, -|A| \in m(\mathbb{R})$*, then* f_0 *can be uniquely written as* $f_0 = f_+ + f_-$ *where* $f_+, f_- \in L^{2,\infty}(X)$ *satisfying* $\Theta f_+ = |A| f_+$ *and* $\Theta f_- = -|A| f_-$*.*

(d) *If neither* $|A|$ *nor* $-|A|$ *is in* $m(\mathbb{R})$*, then* $f_0 = 0$*.*

We note that apart from the spherical, ball mean value operators and heat propagator or simply a polynomial in Δ, Riesz and Bessel potentials, resolvent operator, heat operator in complex time z with $\Re z \geq 0$, right convolution by a radial $\mathcal{C}^p$-function for $0 < p \leq 2$ or by a radial L^q-functions with $1 \leq q < 2$ are some easily found examples of such multipliers acting on $\mathcal{C}^2(X)$.

These results cover eigenfunctions with eigenvalues in $(-\infty, -\rho^2]$, which are in some weak L^p-spaces. To enlarge the scope to all real eigenvalues, and to accommodate eigenfunctions without such growth condition (e.g. the powers of the Poisson kernel itself), we also formulate these results, using Hardy-type norms (see [13]) and L^p-tempered distributions instead of L^p or weak L^p as size estimates.

Finally, we shall also try to complement Theorem 0.2.3 by extending it for all complex eigenvalues, where Δ is replaced by a perturbation of it (see Theorem 3.2.8 and Corollaries 3.2.9, 3.2.10).

Chapter 4

Let $V_r = |B(o,r)|$, the volume of the ball of radius r and $m_r = V_r^{-1}\chi_{B(o,r)}$. We shall use these notation both for Euclidean spaces and symmetric spaces. Characterization of harmonic functions through asymptotic behaviour of ball or sphere averages of a function as the radius goes to *zero* is classical. For instance, we recall that [14, 15], f is harmonic on an open set D of $\mathbb{R}^n$ if and only if

$$\frac{n}{r^2}|f * \sigma_r(x) - f(x)| \to 0 \quad \text{as} \quad r \to 0+ \quad \text{for all } x \in D.$$

See e.g. Gray and Willmore [40] for more general results on Riemannian manifolds. However, the asymptotic behaviour of these averages as the radius tends to infinity seems to be less well known. We understand that the earliest result which considered this to characterize harmonic functions on $\mathbb{R}$ and $\mathbb{R}^2$ is by Plancherel and Pólya [59].

Theorem 0.2.6 (Plancherel and Pólya 1931)**.** *Suppose that for a function $f \in L^1_{loc}(\mathbb{R}^2)$,*

$$\lim_{r\to\infty} f * m_r(x) = \phi(x) \quad \textit{for all } x \in \mathbb{R}^2.$$

If there are $\psi \in L^1_{loc}(\mathbb{R}^2)^+$ and $r_0 \in L^\infty_{loc}(\mathbb{R}^2)^+$, such that for all $x \in \mathbb{R}^2$,

$$|f * m_r(x)| \leq \psi(x), \qquad \textit{for all } r \geq r_0(x),$$

then $\phi(x)$ is a harmonic function on $\mathbb{R}^2$.

In more recent years, Benyamini and Weit [12] obtained a version for $\mathbb{R}^n$.

Theorem 0.2.7 (Beniyamini and Weit 1989)**.** *If for a continuous function f on $\mathbb{R}^n$,*

$$\lim_{r\to\infty} f * m_r = f$$

uniformly on compact sets, then f is a harmonic function.

In fact, apart from Lebesgue measure, more general radial measures were considered on $\mathbb{R}^n$ and on the unit disc in [12]. It is plausible to formulate these for any metric-measure space, (where harmonic functions can be defined by spherical MVP, if there is no Laplacian), in particular for complete Riemannian manifolds. We set our goal to have such asymptotic version of the (generalized) mean value theorem as radius tends to infinity. As an analogue of Theorem 0.2.7 we offer,

Theorem 0.2.8. *Let f and g be two continuous functions on X. If for some $\lambda \in \mathbb{C}$,*

$$\left[\int_{B(o,r)} \varphi_\lambda(y)\, dy\right]^{-1} f * \chi_{B(o,r)}(x) \to g(x) \ \textit{as } r \to \infty$$

uniformly on compact sets, then $\Delta g = -(\lambda^2 + \rho^2) g$.

We shall also prove an exact analogue of Theorem 0.2.6 characterizing eigenfunctions on X. A takeaway from these results is that any condition on the growth of f is unnecessary. However, assumption on growth of f, may allow different (possibly easier) proof, and the conclusion can be sharpened, by getting more concrete realization of the limit function g as a Poisson transform of some L^p-function on the boundary. We pause briefly, to review the situation in the Euclidean spaces, for functions with growth conditions. To keep the discussion simple we restrict to the case of harmonic functions. We recall that if $f \in L^\infty(\mathbb{R}^n)$ is harmonic, then by MVP, $f(x) = f * m_r(x)$ for any point $x \in \mathbb{R}^n$. Let $A(r, R)$ be the annulus with outer radius R and inner radius r. We have, if f is harmonic, then as $r \to \infty$,

$$|f(0) - f(x)| = |f * m_r(0) - f * m_r(x)| \leq C\|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|} \to 0.$$

This shows that $f \in L^\infty(\mathbb{R}^n)$ is harmonic implies that f is constant. This proof works for functions with sublinear growth. The proof in fact shows that for an arbitrary (i.e. not necessarily harmonic) $f \in L^\infty(\mathbb{R}^n)$,

$$|f * m_r(0) - f * m_r(x)| \leq C\|f\|_\infty \frac{|A(r - |x|, r + |x|)|}{|B(o, r)|} \to 0 \text{ as } r \to \infty.$$

We conclude that if $f * m_r(x_0)$ oscillates at a point $x_0 \in \mathbb{R}^n$, then it oscillates 'the same way' at other points and if $f * m_r(x_0) \to C$, for a constant C then $f * m_r(x) \to C$ for any other point x as $r \to \infty$. So, $f * m_r(x) \to g(x)$, implies g is constant, hence harmonic. The crucial fact used in the proof is the polynomial growth of the ball. Indeed, growth of the ball is an important ingredient for similar results in other manifolds (e.g. see [21]). As mentioned above, in the symmetric spaces X, which we deal with here, geodesic balls grow exponentially which vindicates the failure of these results and prevents us to foresee the asymptotic behaviour of the ball averages of a function, even with appropriate growth assumption on it.

Let us come back to the space X. For convenience we shall use these notation:

$$V_r^\lambda = \int_{B(o,r)} \varphi_\lambda(x)\, dx, \qquad m_r^\lambda = \frac{1}{V_r^\lambda} \chi_{B(o,r)}, \text{ for any } \lambda \in \mathbb{C}.$$

A concern about the statement of Theorem 0.2.8 is that $V_r^\lambda = \int_{B(o,r)} \varphi_\lambda(y)\, dy$ in the denominator can possibly be zero for some $r > 0$. Fortunately, for $\lambda \notin \mathbb{R}^\times$, it can be shown that there exists $C_\lambda > 0$, such that for all $r > C_\lambda$, $V_r^\lambda \neq 0$. This resolves the issue for $\lambda \notin \mathbb{R}^\times$ as $r \to \infty$. If $\lambda \in \mathbb{R}^\times$, then V_r^λ can be zero for some r. Let D_0

be the set of discrete zeros of the analytic function $r \mapsto \int_{B(o,r)} \varphi_\lambda(y)\, dy = V_r^\lambda$. We interpret $r \to \infty$ as $r \to \infty$ through $\mathbb{R}^+ \setminus D_0$. But the situation is more involved in this case. We need to find a sequence $\{r_n\}$ of radii with $r_n \uparrow \infty$ and a $\delta > 0$ independent of n, such that for $r \in [r_n - \delta, r_n + \delta]$, $V_r^\lambda \neq 0$. We may stay away from these interpretational worries in this introduction, as they will be explained in detail in the chapter. But it may be worth pointing out that this adds difficulties to the proof and finding this sequence of intervals of a fixed minimum length where V_r^λ is nonzero is a crucial point of the proof. Another key-feature of X, we use, is the fact that distance function is geodesically convex in X. The proof is rather geometric than Fourier analytic and is influenced by [12].

From the result above for continuous or locally integrable functions, we can derive result for functions with growth conditions. A representative theorem is the following.

Theorem 0.2.9. *For $f, g \in L^p(X)$, $2 < p \leq \infty$ and $\lambda \in \mathbb{C}$ with $|\Im\lambda| < |\gamma_p \rho|$. if*

$$\|f * m_r^\lambda - g\|_p \to 0 \text{ as } r \to \infty,$$

then $\Delta g = -(\lambda^2 + \rho^2) g$.

For such a result for functions in Lebesgue or weak Lebesgue spaces, but with pointwise convergence replacing norm convergence, we shall wait till the next chapter.

We shall also consider theorems of this genre under spherical and annular averages, for functions with and without growth conditions. While the ball, spherical or annular averages are mean-value operators (i.e. eigenfunctions satisfy the MVP for them), we shall note that there are "averages" which are not mean-value operators, but still lead to the desired conclusion asymptotically. See e.g. Propositions 4.2.3 and 4.4.6.

Chapter 5

We shall explore some properties of the heat propagation on X, in large as well as in small time, again in the context of characterizing eigenfunctions and exhibit the differences with their Euclidean counter parts. We believe that the results here vindicate the well known fact that geometry of the space affects its heat kernel and the heat kernel illustrates many distinguishing features of the space.

Our first aim is to relate the large time behaviour of the heat operator acting on a function with the asymptotic property of its ball average. Using Wiener's Tauberain theorem on $\mathbb{R}$, Repnikov and Èĭdel'man ([65,66]) proved that for $f \in L^\infty(\mathbb{R}^n)$ and a fixed point $x_0 \in \mathbb{R}^n$, $f * m_r(x_0) \to L$ for a constant L as $r \to \infty$ if and only if $e^{t\Delta_{\mathbb{R}^n}} f(x_0) \to L$ as $t \to \infty$, where $\Delta_{\mathbb{R}^n}$ is the Laplacian on $\mathbb{R}^n$. This result was generalized by Li in [53] to complete n-dimensional Riemannian manifolds M with nonnegative Ricci curvature with the property that $|B(x_0, r)| \geq \theta r^n$ for all large r for some constant θ, which by Bishop–Gromov comparison theorem (see [56]) implies that ball around x_0 has polynomial volume growth. The proof uses the Euclidean result of Repnikov and Èĭdel'man mentioned above. We shall see that one side of the theorem fails for X, which, as mentioned above, is of nonpositive Ricci curvature and in which balls grow exponentially. Precisely there are functions $f \in L^\infty(X)$, such that $e^{t\Delta} f(x)$ converges for any $x \in X$ as $t \to \infty$, but $f * m_r(x)$ does not converge as $r \to \infty$. But the direct side of the assertion will be shown to be true for X, although Tauberain argument cannot be used here. We shall prove the following general statement. Here h_t is the heat kernel, the kernel of the operator $e^{t\Delta}$. We define $h_t^\lambda = e^{t(\lambda^2+\rho^2)} h_t$ which is the kernel of the operator $e^{(\Delta+\lambda^2+\rho^2)}$, in other words h_t^λ is the fundamental solution of the perturbed heat equation:

$$[\Delta + (\lambda^2 + \rho^2)]f = \frac{\partial}{\partial t} f.$$

The notation m_r^λ is as defined above in the discussion of Chapter 4.

Theorem 0.2.10. *Fix a $p > 2$. Let $\lambda = i(2/p-1)\rho$. Then for any weak L^p-function f and a point $x_0 \in X$,*

$$\lim_{r\to\infty} f * m_r^\lambda(x_0) = L \text{ implies } \lim_{t\to\infty} f * h_t^\lambda(x_0) = L,$$

where L is a constant. The converse is not true, i.e., there exists weak L^p-function f on X and point $x_0 \in X$, such that $f * h_t^\lambda(x_0)$ converges to a limit as $t \to \infty$ but $f * m_r^\lambda(x_0)$ does not as $r \to \infty$.

Through this relation and using that h_t^λ is a semigroup, we shall get a version of Theorem 0.2.9 under pointwise convergence. The upshot of this new argument is that it is free from the use of the geometric property of convexity of distance, albeit at the cost of the assumption on growth of the function f (so that $f * h_t^\lambda$ makes sense).

Our next aim is to illustrate that results involving heat propagation in small time also differs from the corresponding Euclidean results. Although these results are closely related to those in Chapter 2, we have included them here for the convenience of the presentation and to vindicate this point.

We start with the simple observation that if $\Delta f = 0$ for a suitable function f then $e^{t\Delta} f = f$ for any $t > 0$, whenever it makes sense. For the converse, we can appeal to the result of Furstenberg [37] (mentioned above in our discussion on Chapter 2) which states that if $f * \mu = f$ for any function $f \in L^\infty(X)$ and an absolutely continuous radial probability measure μ on X, then f is harmonic. The operator $e^{t\Delta}$ is given by convolution with the heat kernel h_t, i.e. for a suitable function f, $e^{t\Delta} f = f * h_t$. Since, for every $t > 0$, h_t is a nonnegative radial function and $\int_X h_t(x)\, dx = 1$, we can apply the result of Furstenberg, taking $d\mu = h_t(x)\, dx$. Thus if for a function $f \in L^\infty(X)$, $e^{t\Delta} f = f$ for some $t > 0$, then f is harmonic, i.e. $\Delta f = 0$.

As in other chapters, instead of only harmonic functions we consider eigenfunctions of Δ with nonzero eigenvalues. It is evident that if $\Delta f = cf$, for a function $f \in L^p(X)$ and a complex number c in the L^p-point spectrum of Δ, then $e^{t\Delta} f = e^{tc} f$ for any $t > 0$. Here we endeavour to explore the converse of this. The precise question is: if for f and c as above, $e^{t\Delta} f = e^{tc} f$ for some $t > 0$, is it necessarily true that $\Delta f = cf$? Indeed, while this converse holds true for Euclidean spaces, the situation is different for X as above. We shall show that the answer to such a question is affirmative only when t lies within a sharp range $0 < t < T$, where the *critical time* T depends on the proposed eigenvalue c and the integrability of the eigenfunction (or temperedness of the eigendistribution). In fact $f \in L^p(X)$ in the above discussion can be replaced by appropriately tempered distributions. As mentioned above, the tempered distributions being less restrictive, perhaps the ideal objects to consider in this setup. However as noted in the discussion on Chapter 2, the complex analytic extension of the L^p-Schwartz class functions on X, precludes the Euclidean technique of locating the support of the Fourier transform of a tempered distribution and adds to the difficulties.

0.3

We conclude with the following remarks. We recall that the Damek–Ricci (DR) spaces S are solvable Lie group $N \rtimes A$, where N is a nilpotent Lie group of Heisenberg type and A is isomorphic with $\mathbb{R}$. They are also known as harmonic NA groups. Through the Iwasawa decomposition $G = NAK$ of G, a rank one Riemannian symmetric space $X = G/K$ of noncompact type, can be realized as a DR space. Indeed, they are the most distinguished prototypes of the DR spaces, though they account for a very thin subcollection in the set of all DR spaces (see [5]). In general a DR space is not a symmetric space. The absence of semisimple group-action in a general DR space S offers many fresh challenges, as one tries to carry forward the results on X to them. One instance of the difficulties is that unlike on X, the decomposition of a function in K-types is unavailable in S, in particular the radial functions (and radialization) on S cannot be defined by group action. Keeping these in mind we have completely avoided such well-known techniques in our proofs. Most of the basic ingredients we used in the proofs are available for DR spaces. Thus the results in this thesis should be readily extendable to these spaces.

It is also plausible to ask questions similar to those considered in this thesis for higher rank symmetric spaces. We, however, note that some of the key results used in this thesis, e.g. Proposition 1.5.2 and Theorem 2.1.1 are specific to rank one symmetric spaces. They are used in Chapter 2 and Chapter 3. A starting point in Chapter 4 is the sphere or ball mean value property. We recall here that the mean value property in higher rank symmetric spaces is about averaging on the K-orbit of a point, which is not a geodesic sphere. In general, we feel that while some versions of the results in this thesis should be true in higher rank, in most of them, it may not be a straightforward extension.

Chapter 1

Preliminaries

The aim of this chapter is to establish the notation and to introduce some of the terminologies (especially those which are not so standard), that we assume in this thesis. *List of symbols* given in pages ix–xi covers most of the notation and we shall try to remind the readers at the places where it will be used. Therefore we omit repeating the basic notation, except a few important ones.

We shall follow the practice of using the letters C, C_1, C_2, C', c etc. for positive constants, whose value may change from one line to another. The constants may be suffixed to show their dependencies on important parameters. Everywhere in this thesis the symbol $f_1 \asymp f_2$ for two positive expressions f_1 and f_2 means that there are positive constants C_1, C_2 such that $C_1 f_1 \leq f_2 \leq C_2 f_1$. For any set E, we will denote its indicator function by χ_E. For a set S in a topological space $\overline{S}$ is its closure and S° is its interior and for a set S in a measure space $|S|$ is its measure. For $p \in [1, \infty]$, by L^p we denote the Lebesgue spaces. The space of locally integrable functions and locally bounded functions are denoted respectively by L^1_{loc} and L^∞_{loc}. The space of compactly supported C^∞ functions is denoted by C^∞_c. We shall not distinguish between two locally integrable functions which differ on a set of measure zero. For two functions f_1, f_2, the notation $\langle f_1, f_2 \rangle$ means $\int f_1 f_2$ if the integral makes sense. The expression $\langle f_1, f_2 \rangle$ may also mean that the distribution f_1 is acting on the function f_2. For $p \in (0,1) \cup (1, \infty)$, let $p' = p/(p-1)$, $p' = \infty$ if $p = 1$ and $p' = 1$ if $p = \infty$. Note that p' is negative for $0 < p < 1$. Support of a function f and the (distributional) support of a tempered distribution T are denoted by $\operatorname{Supp} f$ and $\operatorname{Supp} T$ respectively.

1.1 Lorentz spaces

Let (M, m) be a σ-finite measure space, $f : M \longrightarrow \mathbb{C}$ be a measurable function and $p \in [1, \infty)$, $q \in [1, \infty]$. We define

$$\|f\|_{p,q} = \begin{cases} \left(q \int_0^\infty [t d_f(t)^{1/p}]^q \, \frac{dt}{t}\right)^{1/q} & \text{if } q < \infty, \\ \\ \sup_{t>0} t d_f(t)^{1/p} & \text{if } q = \infty, \end{cases}$$

where for $\alpha > 0$, $d_f(\alpha) = m(\{x \mid |f(x)| > \alpha\})$ is the distribution function of f. Note that instead of the usual definition using the decreasing rearrangement $f^*(t) = \inf\{s \mid d_f(s) \leq t\}$ of f, we have used an alternative definition. For the equivalence see [63]. Let $L^{p,q}(M)$ be the set of all measurable functions $f : M \longrightarrow \mathbb{C}$ such that $\|f\|_{p,q} < \infty$. We note the following.

(i) The space $L^{p,\infty}(M)$ is known as the weak L^p-space.

(ii) $L^{p,p}(M) = L^p(M)$ and $\|\cdot\|_{p,p} = \|\cdot\|_p$ for $1 \leq p \leq \infty$.

(iii) For $1 < p, q < \infty$, the dual space of $L^{p,q}(M)$ is $L^{p',q'}(M)$ and the dual of $L^{p,1}(M)$ is $L^{p',\infty}(M)$.

(iv) If $q_1 \leq q_2 \leq \infty$ then $L^{p,q_1}(M) \subset L^{p,q_2}(M)$ and $\|f\|_{p,q_2} \leq \|f\|_{p,q_1}$.

The Lorentz "norm" $\|\cdot\|_{p,q}$ is actually a quasi-norm and $L^{p,q}(M)$ is a quasi Banach space. For $1 < p < \infty$, there is an equivalent norm which makes it a Banach space (see [72, Chapter V, Theorem 3.21, Theorem 3.22]). We shall slur over this difference and use the notation $\|\cdot\|_{p,q}$. For more details on Lorentz spaces we refer to [72].

1.2 Symmetric space

Notation and preliminaries required for symmetric spaces are standard and can be found for example in [41,42]. We recall that a rank one Riemannian symmetric space of noncompact type (which we denote by X throughout this thesis) can be realized as a quotient space G/K, where G is a connected noncompact semisimple Lie group with finite centre and of real rank one and K a maximal compact subgroup of G. Thus $o = \{eK\}$ is the origin of X. We shall frequently identify a function on X with a function on G which is invariant under the right K-action. The group G acts naturally on $X = G/K$ by left translations $\ell_g : xK \to gxK$ for $g \in G$. The Killing form on the Lie algebra $\mathfrak{g}$ of G induces a G-invariant Riemannian structure and a G-invariant measure on X. Let $d(\cdot,\cdot)$ be the distance and Δ be the Laplace–Beltrami operator on X, associated to this Riemannian structure on X. For $x \in X$, by $|x|$ we denote $d(o,x)$, the distance of x from the origin $o = eK$. Let $\mathfrak{k}$ be the Lie algebra of K, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then $\dim \mathfrak{a} = 1$ as G is of real rank one. We denote the real dual of $\mathfrak{a}$ by $\mathfrak{a}^*$. Let $\Sigma \subset \mathfrak{a}^*$ be the subset of nonzero roots of the pair $(\mathfrak{g}, \mathfrak{a})$. We recall that either $\Sigma = \{-\gamma, \gamma\}$ or $\{-2\gamma, -\gamma, \gamma, 2\gamma\}$ where γ is a positive root and the Weyl group W associated to Σ is $\{\mathrm{Id}, -\mathrm{Id}\}$ where Id is the identity operator. Let $m_\gamma = \dim \mathfrak{g}_\gamma$ and $m_{2\gamma} = \dim \mathfrak{g}_{2\gamma}$ where $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{2\gamma}$ are the root spaces corresponding to γ and 2γ. Then $\rho = \frac{1}{2}(m_\gamma + 2m_{2\gamma})\gamma$ denotes the half sum of positive roots. Let H_0 be the unique element in $\mathfrak{a}$ such that $\gamma(H_0) = 1/2$ and through this we identify $\mathfrak{a}$ with $\mathbb{R}$ as $t \mapsto tH_0$. Then $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \gamma(H) > 0\}$ is identified with the set of positive real numbers. We identify $\mathfrak{a}^*$ and its complexification $\mathfrak{a}^*_{\mathbb{C}}$ respectively with $\mathbb{R}$ and $\mathbb{C}$ by $t \mapsto t\gamma$, $t \in \mathbb{R}$ and $z \mapsto z\gamma$, $z \in \mathbb{C}$. By abuse of notation we will denote $\rho(H_0) = \frac{1}{4}(m_\gamma + 2m_{2\gamma})$ by ρ. Let $\mathfrak{n} = \mathfrak{g}_\gamma + \mathfrak{g}_{2\gamma}$, $N = \exp \mathfrak{n}$, $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}_+$ and $\overline{A^+} = \exp \overline{\mathfrak{a}_+}$. Then N is a nilpotent Lie group and A is a one dimensional vector subgroup identified with $\mathbb{R}$. Precisely A is parametrized by $a_s = \exp(sH_0)$. The Lebesgue measure on $\mathbb{R}$ induces a Haar measure on A by $da_s = ds$. Let M be the centralizer of A in K. The groups M and A normalizes N. We note that K/M is the Furstenberg boundary of X.

The group G has the Iwasawa decomposition $G = KAN$ and the polar decomposition $G = K\overline{A^+}K$. Through polar decomposition X is realized as $\overline{A^+} \times K$. Using the Iwasawa decomposition $G = KAN$, we write an element $x \in G$ uniquely as $k(x) \exp H(x) n(x)$ where $k(x) \in K, n(x) \in N$ and $H(x) \in \mathfrak{a}$. Let dk and dm respectively be the normalized Haar measures on K and M. let dg be the Haar measure on G uniquely given by the Reimannian measure on X and the measure dk on K so that

$$\int_G f(g)\, dg = \int_{G/K} \int_K f(gk)\, dk\, d(gK),$$

holds for all integrable functions f on G. Corresponding to Iwasawa and polar decompositions of G we have the following integral formulae for any integrable function f on G,

$$\int_G f(g)\, dg = \int_K \int_A \int_N f(ka_t n) e^{2\rho t}\, dn\, dt\, dk, \tag{1.2.1}$$

and

$$\int_G f(g)\, dg = \int_K \int_{\mathbb{R}^+} \int_K f(k_1 a_t k_2) J(t)\, dk_1\, dt\, dk_2, \tag{1.2.2}$$

where $J(t)$ is the Jacobian of polar decomposition given by

$$J(t) = \frac{2^n \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \left(\sinh \frac{t}{2}\right)^{m_\gamma + m_{2\gamma}} \left(\cosh \frac{t}{2}\right)^{m_{2\gamma}}$$

and $n = m_\gamma + m_{2\gamma} + 1$ is the dimension of the symmetric space. Since $\sinh t \asymp te^t/(1+t)$ for $t \geq 0$, it follows from (1.2.2) that

$$\begin{aligned} \int_G |f(g)| dg &\asymp C_1 \int_K \int_0^1 \int_K |f(k_1 a_t k_2)| t^{n-1}\, dk_1\, dt\, dk_2 \\ &+ C_2 \int_K \int_1^\infty \int_K |f(k_1 a_t k_2)| e^{2\rho t}\, dk_1\, dt\, dk_2 \end{aligned} \tag{1.2.3}$$

For an integrable function f on X, $\int_G f(g)\, dg = \int_X f(x)\, dx$ where in the left hand side f is considered as a right K-invariant function on G and dg is the Haar measure on G, while on the right hand side dx is the G-invariant measure on X. We shall slur over the difference between integrating over G and that on $X = G/K$ as we shall deal with functions on X.

A function f on X is called left K-invariant if $f(kx) = f(x)$ for all $k \in K$ and $x \in X$. Thus a left K-invariant function on X can be identified with a K-biinvariant function on G. Note that for a K-biinvariant function f, $f(x) = f(y)$ if $|x| = |y|$, i.e. they are radial. We shall use both the terms radial and K-biinvariant for such functions. For any function space $\mathcal{L}(X)$, by $\mathcal{L}(G//K)$ we denote the set of radial functions in $\mathcal{L}(X)$. A measure μ on X is radial or K-biinvariant if μ is invariant under left translation by elements on K, i.e. $\int_X f(x)\, d\mu(x) = \int_X f(kx)\, d\mu(x)$ for every $k \in K$. We note that for a radial function f, $f(x) = f(x^{-1})$, as $|x| = |x^{-1}|$. For a function $f \in L^1_{loc}(X)$ we define its *radialization* Rf by

$$Rf(x) := \int_K f(kx)\, dk.$$

Then Rf is a radial function and if f is radial then $Rf = f$. We also note that for $\phi, \psi \in C_c^\infty(X)$, we have (i) $\langle R\phi, \psi \rangle = \langle \phi, R\psi \rangle$ and (ii) $R(\Delta\phi) = \Delta(R\phi)$. From (i) it follows that $\int_X f(x)\, dx = \int_X Rf(x)\, dx$ and hence $\|Rf\|_1 \leq \|f\|_1$. We have also the trivial L^∞-boundedness of the operator R; $\|Rf\|_\infty \leq \|f\|_\infty$. Then a standard interpolation argument (see e.g. [72, p. 197]) yields that

$$\|Rf\|_{p,q} \leq \|f\|_{p,q} \text{ for } 1 < p < \infty, 1 \leq q \leq \infty.$$

For any $\lambda \in \mathbb{C}$, we define the elementary spherical function φ_λ by

$$\varphi_\lambda(x) = \int_K e^{-(i\lambda+\rho)H(x^{-1}k)}\, dk. \tag{1.2.4}$$

Then φ_λ is K-biinvariant, $\varphi_\lambda = \varphi_{-\lambda}$, $\varphi_\lambda(x) = \varphi_\lambda(x^{-1})$ and $\Delta\varphi_\lambda = -(\lambda^2+\rho^2)\varphi_\lambda$. It can be verified that $|\varphi_\lambda| \leq \varphi_{i\Im\lambda}$, $\varphi_\lambda(e) = 1$ for $\lambda \in \mathbb{C}$ and $\varphi_{i\rho} \equiv 1$. For $\lambda \in \mathbb{C}$ we denote

$$E_\lambda = \{u \in C^\infty(X) : \Delta u = -(\lambda^2+\rho^2)u\}.$$

It is well known that $u \in E_\lambda$ if and only if u satisfies the following functional equation (see [41, Proposition 2.4, p. 402]):

$$\int_K u(xky)\, dk = u(x)\varphi_\lambda(y), \text{ for all } x, y \in X. \tag{1.2.5}$$

Thus in particular

$$\int_K \varphi_\lambda(xky)\, dk = \varphi_\lambda(x)\varphi_\lambda(y), \ x, y \in X.$$

If $u \in E_\lambda$ is radial, then putting $x = e$ in (1.2.5) we get $u(y) = u(e)\varphi_\lambda(y)$. For $\Im\lambda < 0$ and $t > 0$, we have the following asymptotic estimate of φ_λ,

$$\lim_{t\to\infty} e^{-(i\lambda-\rho)t}\varphi_\lambda(a_t) = \mathrm{c}(\lambda) \tag{1.2.6}$$

where $\mathrm{c}(\lambda)$ is the Harish-Chandra c-function (see [41], Theorem 6.14, Ch IV). Since the c-function has neither zero nor pole in the region $\Im\lambda < 0$, from this we obtain the following. For any $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$, there is a $t_\lambda > 0$ such that

$$|\varphi_\lambda(a_t)| \asymp \varphi_{i\Im\lambda}(a_t) \asymp e^{(|\Im\lambda|-\rho)|t|} \text{ for } |t| > t_\lambda. \tag{1.2.7}$$

In particular for $0 < p < 2$ and for large $t > 0$,

$$|\varphi_{i\gamma_p\rho}(a_t)| \asymp e^{-\frac{2\rho t}{p'}}, \tag{1.2.8}$$

where $\gamma_p = 2/p - 1$. For $\lambda = 0$ we also have (see [3])

$$\varphi_0(a_t) \asymp (1+|t|)e^{-\rho|t|}. \tag{1.2.9}$$

For $\mu \in \mathbb{C}$ and $k \in \mathbb{N}$, let

$$\varphi_{\mu,k}(x) := \frac{\partial^k}{\partial \lambda^k} \varphi_\lambda(x)|_{\lambda=\mu}.$$

It satisfies the following estimate ([42, Ch 3, §1 Lemma 1.18 (iv)]):

$$\left| P\left(\frac{\partial}{\partial \lambda}\right) \varphi_\lambda(x) \right| \leq C(1+|x|)^r \varphi_{i\Im\lambda}(x),\ x \in X \tag{1.2.10}$$

for some constant $C > 0$, where P is a polynomial of degree r and $|x| = d(o,x)$. We also observe that $\varphi_{0,k} \equiv 0$ for odd $k \in \mathbb{N}$ as $\varphi_\lambda(x)$ is an even function in λ for any fixed $x \in X$ and hence $\varphi_{\lambda,k}(x)$ is an odd function in λ. For a measurable radial function f on X and $\lambda \in \mathbb{C}$, we define the spherical Fourier transform of f at λ by

$$\widehat{f}(\lambda) := \int_X f(x)\varphi_\lambda(x)\, dx, \tag{1.2.11}$$

whenever the integral makes sense. When the function f is radial we may simply refer $\widehat{f}$ as the Fourier transform of f. For a suitable function f on X, $\lambda \in \mathbb{C}$ and $k \in K/M$, we define the Helgason Fourier transform of f by

$$\widetilde{f}(\lambda, k) := \int_X f(x) e^{-(i\lambda+\rho)H(x^{-1}k)}\, dx. \tag{1.2.12}$$

If f is K-biinvariant then $\widetilde{f}(\lambda, k)$ is independent of $k \in K/M$ and reduces to its spherical Fourier transform $\widehat{f}(\lambda)$.

Let $B(x,t)$ be the geodesic ball of radius $t > 0$ centered at x in X and $|B(x,t)|$ be its volume. For $f \in L^1_{loc}(X)$, let $M_t f(x)$ denote the average of f over the geodesic sphere of radius t with center at xK in X. Precisely,

$$M_t f(x) = f * \sigma_t(x) = \int_K f(xka_t)\, dk,$$

where σ_t is the normalized surface measure of the geodesic sphere of radius t with center at the origin $o = eK$. The volume average of such a function f is denoted by $B_t f$ and is defined by,

$$B_t f(x) = \frac{1}{|B(o,t)|} f * \chi_{B(o,t)}(x) = \frac{1}{|B(o,t)|} \int_{B(x,t)} f(z)\, dz,$$

where $\chi_{B(o,t)}(x)$ is the indicator function of $B(o,t)$.

For a fixed $\lambda \in \mathbb{C}$ we define

$$V_r^\lambda = \int_{B(o,r)} \varphi_\lambda(x)\ dx = \int_0^r \varphi_\lambda(a_t) J(t)\ dt,$$

where $J(t)$ is the Jacobian of the polar decomposition defined in (1.2.2). Thus V_r^λ is the spherical Fourier transform at λ, of $\chi_{B(o,r)}$. Let

$$m_r^\lambda = (V_r^\lambda)^{-1} \chi_{B(o,r)}.$$

Note that, $V_r^{i\rho} = |B(o,r)|$ and $m_r^{i\rho} = (1/|B(o,r)|)\chi_{B(o,r)}$.

1.3 Jacobi functions

For $\alpha, \beta > -1/2$ and $t \geq 0$, let $\phi_\lambda^{(\alpha,\beta)}(t)$ denotes the Jacobi functions defined by

$$\phi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1\left(\frac{1}{2}(\alpha+\beta+1-i\lambda), \frac{1}{2}(\alpha+\beta+1+i\lambda); \alpha+1; -\sinh^2(t)\right)$$

where ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function. For a detailed account on Jacobi functions we refer to [48]. We recall that for rank one symmetric spaces, elementary spherical functions are Jacobi functions. They are related in the following way. Let $m = m_\gamma$, $k = m_{2\gamma}$ and $n = m + k + 1$. Then

$$\varphi_\lambda(a_t) = \phi_{2\lambda}^{(\alpha,\beta)}(t/2) \tag{1.3.1}$$

for $\alpha = (m+k-1)/2$ and $\beta = (k-1)/2$ (see [5], p. 650).

For $\Im\lambda < 0$ we have the following asymptotic estimate for Jacobi functions (see [48, 2.19, p. 8], cf. (1.2.6) – (1.2.8)).

$$\phi_\lambda^{(\alpha,\beta)}(t) = \mathrm{c}_{\alpha,\beta}(\lambda) e^{(i\lambda-\varrho)t}(1+o(1)) \text{ as } t \to \infty \tag{1.3.2}$$

where

$$\mathrm{c}_{\alpha,\beta}(\lambda) = \frac{2^{\varrho-i\lambda}\,\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{i\lambda+\varrho}{2})\Gamma(\frac{i\lambda+\alpha-\beta+1}{2})} \text{ and } \varrho = \alpha+\beta+1.$$

It follows that $\phi_\lambda^{(\alpha,\beta)} = \phi_{-\lambda}^{(\alpha,\beta)}$ for all $\lambda \in \mathbb{C}$ and $\mathrm{c}_{\alpha,\beta}(\lambda)$ has neither zero nor pole in the region $\Im\lambda < 0$. Hence from (1.3.2) and the fact that $\phi_\lambda^{(\alpha,\beta)} = \phi_{-\lambda}^{(\alpha,\beta)}$, we get for $\lambda \notin \mathbb{R}$,

$$|\phi_\lambda^{(\alpha,\beta)}(t)| \asymp e^{(|\Im\lambda|-\varrho)t} \text{ as } t \to \infty. \tag{1.3.3}$$

Following theorem (proved in [57, lemma 5.2(a)]) shows that V_r^λ, that is, the Fourier transform of $\chi_{B(o,r)}$ can be expressed in terms of the Jacobi functions.

Theorem 1.3.1. *Let* $\alpha' = \frac{m+k+1}{2}, \beta' = \frac{k+1}{2}$ *and* $n = m+k+1$, *for* m, k *as above. Then for* $\lambda \in \mathbb{C}$ *and* $r > 0$,

$$V_r^\lambda = \frac{2^n \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \sinh^n\left(\frac{r}{2}\right) \cosh^{k+1}\left(\frac{r}{2}\right) \phi_{2\lambda}^{(\alpha',\beta')}\left(\frac{r}{2}\right). \tag{1.3.4}$$

This theorem is proved in [57, lemma 5.2(a)]. However, we are including the proof here to remove a minor error involving the power of $\cosh(r/2)$ in the statement there.

Proof.

$$\begin{aligned}
V_r^\lambda &= \int_0^r \varphi_\lambda(a_t) J(t)\, dt \\
&= \frac{2^n \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^r \sinh^{m+k}\left(\frac{t}{2}\right) \cosh^k\left(\frac{t}{2}\right) \varphi_\lambda(a_t)\, dt \\
&= \frac{2^n \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^r A_t^{m+k,k}\, {}_2F_1\left(\frac{m}{4}+\frac{k}{2}-i\lambda, \frac{m}{4}+\frac{k}{2}+i\lambda; \frac{m+k+1}{2}; -\sinh^2\left(\frac{t}{2}\right)\right) dt
\end{aligned}$$

where

$$A_t^{m,k} := \sinh^m\left(\frac{t}{2}\right) \cosh^k\left(\frac{t}{2}\right).$$

For convenience we temporarily put

$$a = \frac{m}{4}+\frac{k}{2}-i\lambda,\ b = \frac{m}{4}+\frac{k}{2}+i\lambda,\ c = \frac{m+k+1}{2}$$

and substitute $z = -\sinh^2(t/2)$ in the integral above to get,

$$\begin{aligned}
&\int A_t^{m+k,k}\, {}_2F_1\left(\frac{m}{4}+\frac{k}{2}-i\lambda, \frac{m}{4}+\frac{k}{2}+i\lambda; \frac{m+k+1}{2}; -\sinh^2\left(\frac{t}{2}\right)\right) dt \\
= &\ (-1)^c \int z^{c-1}(1-z)^{a+b-c}\, {}_2F_1\big(a,b;c;z\big)\, dz \\
= &\ \frac{(-1)^c}{c} \int \frac{d}{dz}\Big(z^c(1-z)^{a+b+1-c}\, {}_2F_1\big(a+1,b+1;c+1;z\big)\Big)\, dz \\
= &\ \frac{(-1)^c}{c} z^c(1-z)^{a+b+1-c}\, {}_2F_1\big(a+1,b+1;c+1;z\big) + C \\
= &\ \frac{1}{c} A_t^{m+k+1,k+1}\, {}_2F_1\left(\frac{m}{4}+\frac{k}{2}+1-i\lambda, \frac{m}{4}+\frac{k}{2}+1+i\lambda; \frac{m+k+3}{2}; -\sinh^2\left(\frac{t}{2}\right)\right) + C.
\end{aligned}$$

Above we have used the following fact (see [1, 15.2.9]):

$$\frac{d}{dz}\Big(z^c(1-z)^{a+b+1-c}\,{}_2F_1\big(a+1,b+1;c+1;z\big)\Big) = cz^{c-1}(1-z)^{a+b-c}\,{}_2F_1\big(a,b;c;z\big).$$

Thus

$$\begin{aligned} V_r^\lambda &= c_n A_r^{m+k+1,k+1}\,{}_2F_1\left(\frac{m}{4}+\frac{k}{2}+1-i\lambda, \frac{m}{4}+\frac{k}{2}+1+i\lambda; \frac{m+k+3}{2}; -\sinh^2\left(\frac{r}{2}\right)\right) \\ &= c_n A_r^{m+k+1,k+1}\phi_{2\lambda}^{(\alpha',\beta')}\left(\frac{r}{2}\right) \end{aligned}$$

where $\alpha' = \frac{m+k+1}{2}, \beta' = \frac{k+1}{2}$ and $c_n = \frac{2^n \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$. □

Let $\psi_\lambda(r)$ denotes the spherical Fourier transform of $\frac{\chi_{B(o,r)}}{|B(o,r)|}$ at λ. Then from Theorem 1.3.4 we have

$$\psi_\lambda(r) = \frac{V_r^\lambda}{V_r^{i\rho}} = \frac{\phi_{2\lambda}^{(\alpha',\beta')}(\frac{r}{2})}{\phi_{2i\rho}^{(\alpha',\beta')}(\frac{r}{2})} \tag{1.3.5}$$

where $\alpha' = \frac{m+k+1}{2}, \beta' = \frac{k+1}{2}$. Using (1.3.3) and (1.3.5) for $\lambda \notin \mathbb{R}$ we obtain a $r_\lambda > 0$ depending on λ such that

$$\psi_\lambda(r) \asymp e^{(|\Im\lambda|-\rho)r} \text{ for } r \geq r_\lambda. \tag{1.3.6}$$

For $\mu \in \mathbb{C}$ and $k \in \mathbb{N}$, let

$$\psi_{\mu,k}(r) := \frac{\partial^k}{\partial\lambda^k}\psi_\lambda(r)|_{\lambda=\mu}.$$

1.4 Properties of elementary spherical function

In this section we shall gather some properties of φ_λ and ψ_λ which will be used throughout this thesis. We shall include proofs only when a suitable reference could not be found.

For $0 < p < \infty$, we define $\gamma_p := \frac{2}{p} - 1$ and $\gamma_\infty := -1$. Let

$$S_p := \{\,\lambda \in \mathbb{C} \mid |\Im\lambda| \leq |\gamma_p|\rho\,\},\ \partial S_p := \{\,\lambda \in \mathbb{C} \mid |\Im\lambda| = |\gamma_p|\rho\,\}.$$

Proposition 1.4.1. *The elementary spherical function φ_λ satisfies the following properties.*

(i) *For $1 \leq p < 2$, $\varphi_\lambda \in L^{p',\infty}(G//K)$ if and only if $\lambda \in S_p$.*

(ii) *For* $1 < p < 2$ *and* $1 \leq r \leq \infty$, $\varphi_\lambda \in L^{p',r}(G//K)$ *if and only if* $\lambda \in S_p^\circ$.

(iii) $\varphi_0 \notin L^{2,\infty}(G//K)$, $(1+|x|)^{-1}\varphi_0 \in L^{2,\infty}(G//K)$ *and* $\varphi_\lambda \in L^{2,\infty}(G//K)$ *for* $0 \neq \lambda \in \mathbb{R}$.

See [60, Proposition 2.1] and [50] for the proof of above proposition which uses the estimates (1.2.7) and (1.2.9).

Proposition 1.4.2. *Fix* $t > 0$. *Let* $0 < p \leq 2$ *and* $\lambda \in S_p$ *with* $\lambda \neq \pm i\gamma_p\rho$. *Then*

(a) $|\varphi_\lambda(a_t)| < \varphi_{i\gamma_p\rho}(a_t)$,

(b) $|\psi_\lambda(t)| < \psi_{i\gamma_p\rho}(t)$.

Proof. (a) Since $\varphi_\lambda = \varphi_{-\lambda}$, without loss of generality we assume that $\Im\lambda \geq 0$. Clearly $|\varphi_\lambda(a_t)| \leq \varphi_{i\Im\lambda}(a_t)$ for any $\lambda \in \mathbb{C}$. As $\lambda \mapsto \varphi_\lambda(a_t)$ is analytic, by maximum modulus principile, $|\varphi_\lambda(a_t)| < \varphi_{i\gamma_p\rho}(a_t)$ for $\lambda \in S_p^\circ$. Therefore it remains to prove the result for $\lambda = r + i\gamma_p\rho$, $r \in \mathbb{R}$. Seeking a contradiction, we assume that $|\varphi_\lambda(a_t)| = \varphi_{i\gamma_p\rho}(a_t)$. Then for some $b \in \mathbb{R}$,

$$\varphi_{i\gamma_p\rho}(a_t) = e^{-irb}\varphi_\lambda(a_t) = \int_K e^{-irb}e^{-(ir-\gamma_p\rho+\rho)H(a_t^{-1}k)}\,dk.$$

This implies that,

$$\int_K e^{(\gamma_p\rho-\rho)H(a_t^{-1}k)}[1 - e^{-ir(b+H(a_t^{-1}k))}]dk = 0$$

and taking the real part,

$$\int_K [1 - \cos r(H(a_t^{-1}k) + b)]e^{(\gamma_p\rho-\rho)H(a_t^{-1}k)}\,dk = 0.$$

Since $1 - \cos r(H(a_t^{-1}k) + b)$ is a nonnegative continuous function, we have $[1 - \cos r(H(a_t^{-1}k) + b)] = 0$ for all $k \in K$. But then $H(a_t^{-1}k) + b$ has to take values in a discrete set. As $k \mapsto H(a_t^{-1}k) + b$ is continuous and K is connected, it must be constant, which implies that $k \mapsto H(a_t^{-1}k)$ is constant which contradicts Kostant's convexity theorem ([41, Theorem 10.5, p. 476]). This completes the proof of (a) and (b) is an immediate consequence of (a). □

Proposition 1.4.3. *Fix* $t > 0$ *and* $0 < p \leq 2$. *Then as* $|\lambda| \to \infty$,

(a) $|\varphi_\lambda(a_t)| \to 0$ *uniformly on* S_p *and*

(b) $|\psi_\lambda(t)| \to 0$ *uniformly on* S_p.

Proof. For (a) see [34, Corollary 3] and (b) is an immediate consequence of (a). $\square$

Proposition 1.4.4. *Fix* $t > 0$. *Then we have the following conclusions:*

(a) $\varphi_{0,2}(a_t) \neq 0$ *and* $\varphi_{\lambda,1}(a_t) \neq 0$ *for nonzero* $\lambda \in i\mathbb{R}$,

(b) $\psi_{0,2}(t) \neq 0$ *and* $\psi_{\lambda,1}(t) \neq 0$ *for nonzero* $\lambda \in i\mathbb{R}$.

Proof. We shall prove only (a). Proof of (b) will be similar. It follows from (1.2.4) that for $\lambda \in i\mathbb{R}$, $\varphi_\lambda(a_t) > 0$ and $\varphi_{\lambda,2}(a_t) < 0$, so in particular $\varphi_{0,2}(a_t) \neq 0$. Since for a fixed t, $\lambda \mapsto \varphi_\lambda(a_t)$ is a non-constant entire function, by maximum modulus principle we have $\varphi_{iy_1}(a_t) < \varphi_{iy_2}(a_t)$, for $0 \leq y_1 < y_2$. That is, the function $f : y \mapsto \varphi_{iy}(a_t)$ is strictly increasing. As observed above, the second derivative in y of the function f is strictly positive. This implies that f has nonzero derivative at any $y > 0$, because otherwise it will have a local minimum. Therefore $\varphi_{\lambda,1}(a_t) \neq 0$ for any nonzero $\lambda \in i\mathbb{R}$. $\square$

1.5 Schwartz spaces and tempered distributions

For $0 < p \leq 2$, we define $\mathcal{C}^p(X)$ to be the space of all $u \in C^\infty(X)$ such that for any $D \in \mathcal{U}(\mathfrak{g})$ and integer $r \geq 0$, we have

$$\gamma_{r,D}(u) = \sup_{x \in G} (1 + |x|)^r \varphi_0(x)^{-\frac{2}{p}} |Du(x)| < \infty$$

where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. The seminorms $\gamma_{r,D}$ induce a Fréchet topology on $\mathcal{C}^p(X)$. Let $\mathcal{C}^p(G//K)$ be the set of radial functions in $\mathcal{C}^p(X)$. For $0 < p < 2$, let $\mathcal{C}^p(\widehat{G//K})$ be the set of even continuous functions on S_p which are holomorphic on the interior of S_p and satisfy

$$\nu_{l,m}(f) = \sup_{\lambda \in S_p} \left| \frac{d^l}{d\lambda^l} f(\lambda) \right| (1 + |\lambda|)^m < \infty$$

for all nonnegative integers l, m. For $p = 2$, $\mathcal{C}^2(\widehat{G//K})$ is the set of even Schwartz class functions on $\mathbb{R}$. We topologize $\mathcal{C}^p(\widehat{G//K})$ by the seminorms $\nu_{l,m}$. Then (see [4, 32, 39]) $f \mapsto \widehat{f}$ is a topological isomorphism from $\mathcal{C}^p(G//K)$ to $\mathcal{C}^p(\widehat{G//K})$. Let $\mathcal{C}^p(X)'$ denote the set of L^p-tempered distributions on X, i.e. the continuous linear functionals on $\mathcal{C}^p(X)$. When $f \in \mathcal{C}^p(X)'$ is a function, then the distribution is given by $\psi \mapsto \langle \psi, f \rangle = \int_X \psi(x) f(x)\, dx$, for $\psi \in \mathcal{C}^p(X)$. An L^p-tempered distribution T is called radial if

$$\langle T, \psi \rangle = \langle T, R(\psi) \rangle, \text{ for all } \psi \in \mathcal{C}^p(X).$$

In general the radial part $R(T)$ of an L^p-tempered distribution T is an L^p-tempered distribution defined by

$$\langle R(T), \psi \rangle = \langle T, R(\psi) \rangle, \text{ for all } \psi \in \mathcal{C}^p(X).$$

For future use we note that $\mathcal{C}^p(X) \subset L^{p,1}(X)$ for $1 \leq p \leq 2$ ([52, Lemma 6.1.1]). Therefore any $f \in L^{p',\infty}(X)$ defines an L^p-tempered distribution for $1 \leq p \leq 2$. Following proposition collects some other examples of L^p-tempered distributions.

Proposition 1.5.1.

(a) *Let $1 \leq p \leq 2$ and f be a measurable function on X such that $(1 + |x|)^r f \in L^{p',\infty}(X)$ for some $r \in \mathbb{Z}$. Then there exists a fixed seminorm ν of $\mathcal{C}^p(X)$ and a constant C independent of f satisfying*

$$|\langle f, \phi \rangle| \leq C \|(1 + |x|)^r f\|_{p',\infty} \nu(\phi)$$

for all $\phi \in \mathcal{C}^p(X)$. In particular f is an L^p-tempered distribution.

(b) *Let $0 < p \leq 2$ and $\lambda \in S_p$. Then for each fixed $k \in K$, the map $x \mapsto e^{-(i\lambda+\rho)H(x^{-1}k)}$ is an L^p-tempered distribution.*

(c) *Let $p \in (0, 2]$ and $\lambda \in S_p$. Then the map $x \mapsto \varphi_{\lambda,r}(x)$ is an L^p-tempered distribution for any $r \in \mathbb{Z}^+$.*

Proof. (a) The case $r = 0$ is proved in [52, Lemma 6.1.1]. Similar arguments can be used to prove other cases. For (b) see [52, Lemma 6.1.1].
(c) If $r = 0$, using (1.2.8) and (1.2.9) it is easy to see that the map $\psi \mapsto \int_X \psi(x) \varphi_\lambda(x)\, dx$ is a continuous linear functional on $\mathcal{C}^p(X)$. For other cases use (1.2.10). □

Following theorem, which is immediate from [10, Theorem 3.2] and isomorphism of $\mathcal{C}^p(G//K)$ with $\mathcal{C}^p(\widehat{G//K})$, will be used in the thesis.

Proposition 1.5.2. *Fix $0 < p < 2$ and $0 < q < p$. Let $\{g_0, g_1, g_2, \cdots, g_r\}$ be a finite collection of functions in $\mathcal{C}^q(G//K)$ with g_0 satisfying*

$$\limsup_{|t| \to \infty} e^{-\frac{\pi |t|}{2\gamma_q \rho}} \log |\widehat{g_0}(t)| = 0.$$

Let A be the set of common zeros of $\widehat{g_0}, \widehat{g_1}, \widehat{g_2}, \cdots, \widehat{g_r}$ in S_q and n_λ denote the minimal order of zero of the functions $\widehat{g_i}$ at the point λ. Assume that A is finite and $A \subset S_p$. Then the ideal generated by $\{g_0, g_1, g_2, \cdots, g_r\}$ is dense in the closed subspace V of $\mathcal{C}^p(G//K)$ of functions whose spherical Fourier transform vanish at all $\lambda \in A$ with order greater than or equal to n_λ.

1.6 Hardy-type spaces

We shall follow [13] for defining the Hardy-type norms for functions on X. Let $p \in (0, 2]$ be fixed. For a measurable function f on X let,

$$[f]_{p,r} = \sup_{t>0} \varphi_{i\gamma_p\rho}(a_t)^{-1} \left(\int_K |f(ka_t)|^r \, dk \right)^{1/r}, \ 1 \leq r < \infty$$

and

$$[f]_{p,\infty} = \sup_{x \in X} \varphi_{i\gamma_p\rho}(x)^{-1} |f(x)|.$$

Let $\mathcal{H}_p^r(X)$ and $\mathcal{H}_p^\infty(X)$ be the set of functions on X such that $[f]_{p,r} < \infty$, respectively, $[f]_{p,\infty} < \infty$. We shall refer these spaces as *Hardy-type spaces*. Let $L^1(G//K, \varphi_{i\gamma_p\rho})$ be the space of radial functions on X satisfying

$$\int_X |f(x)| \varphi_{i\gamma_p\rho}(x) \, dx < \infty.$$

These can be verified in a straightforward way using (1.2.8) and (1.2.9):

(i) $L^1(G//K, \varphi_{i\gamma_p\rho})$ is contained in the dual space of $\mathcal{H}_p^r(X)$,

(ii) $\varphi_\lambda \in \mathcal{H}_p^r(X)$ if and only if $\lambda \in S_p$,

(iii) for $h \in L^1(G//K, \varphi_{i\gamma_p\rho})$, $\widehat{h}(\lambda)$ extends analytically to S_p^o and is continuous on S_p.

Following are some other observations which we state in the form of a proposition.

Proposition 1.6.1. *Let $0 < p \leq 2$ and $r \in [1, \infty]$ be fixed. Then we have the following conclusions.*

(a) *For $1 \leq r < s \leq \infty$, $\mathcal{H}_p^s(X) \subset \mathcal{H}_p^r(X)$.*

(b) *For $0 < q < p \leq 2$, $\mathcal{H}_p^r(X) \subset \mathcal{H}_q^r(X)$.*

(c) *If $f \in \mathcal{H}_p^r(X)$ then $R(f) \in \mathcal{H}_p^r(X)$.*

(d) *For* $h \in L^1(G//K, \varphi_{i\gamma_p\rho})$ *and* $f \in \mathcal{H}_p^r(X)$, $f * h \in \mathcal{H}_p^r(X)$ *and*

$$[f * h]_{p,r} \leq [f]_{p,r} \int_X |h(x)| \varphi_{i\gamma_p\rho}(x)\ dx.$$

(e) *For* $0 < q \leq p \leq 2$, *let* $f \in \mathcal{H}_p^r(X)$. *Then there exists a fixed seminorm* γ *of* $\mathcal{C}^q(X)$ *and a constant* C *independent of* f *satisfying*

$$|\langle f, \phi \rangle| \leq C[f]_{p,r} \gamma(\phi)$$

for all $\phi \in \mathcal{C}^q(X)$. *Hence* $\mathcal{H}_p^r(X) \subset (\mathcal{C}^q(X))'$ *for* $0 < q \leq p \leq 2$.

(f) *For* $f \in \mathcal{H}_p^r(X)$ *and* $\phi \in \mathcal{C}^q(G//K)$ *with* $0 < q \leq p \leq 2$, $f * \phi \in C^\infty(X)$.

Proof. Assertions (a) and (b) are clear from the definition of $\mathcal{H}_p^r(X)$, (1.2.8) and (1.2.9). Assertion (c) is a consequence of Minkowski's integral inequality.

(d) Using Minkowski's integral inequality we have for $a \in \overline{A^+}$,

$$\begin{aligned} & \left(\int_K |f * h(ka)|^r\, dk \right)^{1/r} \\ \leq\ & \int_X |h(x)| \left(\int_K |f(kax)|^r\, dk \right)^{1/r} dx \\ \leq\ & \int_X |h(x)| \varphi_{i\gamma_p\rho}(ax) \left[\varphi_{i\gamma_p\rho}(ax)^{-1} \left(\int_K |f(kax)|^r\, dk \right)^{1/r} \right] dx \\ \leq\ & [f]_{p,r} \int_X |h(x)| \varphi_{i\gamma_p\rho}(ax)\, dx. \end{aligned}$$

Since $\int_K \varphi_\lambda(xky)\, dk = \varphi_\lambda(x)\varphi_\lambda(y)$, we get from above,

$$\begin{aligned} \left(\int_K |f * h(ka)|^r\, dk \right)^{1/r} &\leq [f]_{p,r} \int_X |h(x)| \varphi_{i\gamma_p\rho}(ax)\, dx \\ &\leq \varphi_{i\gamma_p\rho}(a)\, [f]_{p,r} \int_X |h(x)| \varphi_{i\gamma_p\rho}(x)\, dx. \end{aligned}$$

This proves (d).

For (e) let

$$\gamma(\phi) = \sup_{x \in X} |\phi(x)| \varphi_0^{-2/q}(x)(1 + |x|)^M$$

for some suitably large $M > 0$ be a seminorm of $\mathcal{C}^q(X)$. Then

$$\begin{aligned}\left|\int_X f(x)\phi(x)\,dx\right| &\leq \gamma(\phi)\int_X |f(x)|\varphi_0^{2/q}(x)(1+|x|)^{-M}\,dx \\ &\leq \gamma(\phi)\int_{\overline{A^+}} \varphi_0^{2/q}(a)(1+|a|)^{-M}\left(\int_K |f(ka)|^r\,dk\right)^{1/r} J(a)\,da \\ &\leq \gamma(\phi)[f]_{p,r}\int_{\overline{A^+}} \varphi_0^{2/q}(a)(1+|a|)^{-M}\varphi_{i\gamma_p\rho}(a)J(a)\,da.\end{aligned}$$

From (1.2.8) and (1.2.9), it follows that the integral in the last step converges for the given p, q. This proves the assertion.

(f) This is a consequence of the fact that translation commutes with convolution (see e.g. [68, Theorem 7.19]). □

1.7 Characterization of eigenfunctions and the Poisson transform

The following proposition is quoted from [52, Proposition 3.1.1].

Proposition 1.7.1. *Let u be a measurable function on X. Suppose that u satisfies one of these conditions:*

(i) $u \in L^{2,\infty}(X)$ *and* $\Delta u = -\rho^2 u$,

(ii) $u \in L^{2,q}(X)$ *for some* $1 \leq q < \infty$ *and* $\Delta u = -(\lambda^2+\rho^2)u$ *for a nonzero* $\lambda \in \mathbb{R}$,

(iii) $u \in L^{q',r}(X)$ *and* $\Delta u = -(\lambda^2+\rho^2)u$ *for some* $1 \leq p < q \leq 2$, $1 \leq r \leq \infty$ *and* $\lambda \in \partial S_p$,

(iv) $u \in L^{p',r}(X)$ *and* $\Delta u = -(\lambda^2+\rho^2)u$ *for some* $1 \leq p < 2$, $1 \leq r < \infty$ *and* $\lambda \in \partial S_p$.

Then $u = 0$.

For any $\lambda \in \mathbb{C}$ and $F \in L^1(K/M)$, we define the Poisson transform $\mathcal{P}_\lambda$ of F (see [41, p. 279]) by

$$\mathcal{P}_\lambda F(x) = \int_{K/M} F(k)e^{-(i\lambda+\rho)H(x^{-1}k)}\,dk \text{ for } x \in X.$$

Since for each fixed $\lambda \in \mathbb{C}$ and $k \in K/M$, the kernel: $x \mapsto e^{-(i\lambda+\rho)H(x^{-1}k)}$ is an eigenfunction of Δ with eigenvalue $-(\lambda^2 + \rho^2)$, it follows that,

$$\Delta \mathcal{P}_\lambda F = -(\lambda^2 + \rho^2)\mathcal{P}_\lambda F.$$

There are many results in the literature characterizing eigenfunctions of Δ which satisfy various size estimates as the Poisson transform of an appropriate object (e.g. a function in a Lebesgue class or a measure) on the boundary K/M of X. We shall mainly use two of them, which we state here.

Theorem 1.7.2 (Ben Saïd et. al.). *Let $\Im\lambda = -\gamma_p\rho$ for some $0 < p < 2$ (respectively $\lambda = 0$). Suppose that for a function $u \in C^\infty(X)$, $\Delta u = -(\lambda^2 + \rho^2)u$. Then $u = \mathcal{P}_\lambda f$ for some $f \in L^r(K/M)$ if and only if $u \in \mathcal{H}_p^r(X)$ (respectively $u \in \mathcal{H}_2^r(X)$) for $1 < r \leq \infty$. If $r = 1$ then $u = \mathcal{P}_\lambda \mu$ for some signed measure μ on K/M.*

The next one is an immediate corollary of the following theorem of Sjögren ([70, Theorem 6.1]).

Theorem 1.7.3 (Sjögren). *Let $u \in C^\infty(X)$ and $\Delta u = -(\lambda^2 + \rho^2)u$ for some $\lambda \in \mathbb{C}$ with $\Im\lambda < 0$ or $\lambda = 0$. For $1 < p \leq \infty$ and $\beta > 0$, the function $ka_t \mapsto \phi_{\Im\lambda}(a_t)^{-1}e^{-\beta t/p}u(ka_t)$ belongs to $L^{p,\infty}(X, m_\beta)$ if and only if $u = \mathcal{P}_\lambda f$ for some $f \in L^p(K/M)$. Here $dm_\beta(x) = dm_\beta(ka_t) = e^{(\beta-2\rho)t}J(t)\,dk\,dt$, $t > 0$, $k \in K/M$.*

Corollary 1.7.4. *Let $u \in C^\infty(X)$.*

(i) *Then $(1 + |x|)^{-1}u \in L^{2,\infty}(X)$ and $\Delta u = -\rho^2 u$ if and only if $u = \mathcal{P}_0 f$ for some $f \in L^2(K/M)$.*

(ii) *Let $1 \leq q \leq 2$ and $\lambda = \alpha - i\gamma_q\rho \neq 0$ for some $\alpha \in \mathbb{R}$. Then $u \in L^{q',\infty}(X)$ and $\Delta u = -(\lambda^2 + \rho^2)u$ if and only if $u = \mathcal{P}_\lambda f$ for some $f \in L^{q'}(K/M)$.*

Proof. For (i) take $\beta = 2\rho$ and $p = 2$ in Theorem 1.7.3 and use (1.2.9). For (ii) when $1 \leq q < 2$, take $\beta = 2\rho$ and $p = q'$ in Theorem 1.7.3 and use (1.2.7). For the case $q = 2$ (consequently $\lambda \in \mathbb{R}$, $\lambda \neq 0$) see [52, Theorem 4.3.5] and [50, Theorem 1.1]. □

Remark 1.7.5. The growth estimates on u in the hypothesis of Corollary 1.7.4 are justified by Proposition 1.7.1 and Proposition 1.4.1. We also note that to interpret the equation $\Delta u = -(\lambda^2 + \rho^2)u$, it is enough to assume that u (in Theorem 1.7.2, Theorem 1.7.3 and Corollary 1.7.4) is locally integrable because a locally integrable function is a distribution and hence u is infinitely differentiable by elliptic regularity theorem ([35, Corollary 6.34, p. 215]).

1.8 Kunze–Stein phenomenon and Herz's majorizing principle

We have the following result due to Herz [43] on convolution operators. (See also [23].)

Proposition 1.8.1 (Herz's majorizing principle)**.** *Let h be a radial function on X, and let $T_h : f \mapsto f * h$ be the corresponding right convolution operator on $L^p(X)$, $p \in [1, \infty]$. Then the operator norm of $T_h : L^p(X) \to L^p(X)$ obeys the following bound:*

$$\|T_h\|_{L^p \to L^p} \le \widehat{|h|}(i\gamma_p \rho)$$

where the equality holds if h is nonegative.

From Herz's majorizing principle we get the following results which we shall use in this thesis. See [51, Section 5] for more results.

Proposition 1.8.2 (Kunze–Stein phenomenon)**.** (a) *Let $f \in L^{p',\infty}(X)$ and h be a radial function on X such that $\widehat{|h|}(i\gamma_q \rho) < \infty$ for some $0 < q < p \le 2$. Then $f * h \in L^{p',\infty}(X)$ and*

$$\|f * h\|_{p',\infty} \le C\|f\|_{p',\infty}\widehat{|h|}(i\gamma_q \rho).$$

(b) *Let $f \in L^{p',\infty}(X)$ and $h \in L^q(G//K)$ for $1 \le q < p \le 2$. Then $f*h \in L^{p',\infty}(X)$ and*

$$\|f * h\|_{p',\infty} \le C\|f\|_{p',\infty}\|h\|_q.$$

(c) *Let $f \in L^{p',\infty}(X)$ and $h \in L^{p,1}(G//K)$ for $1 \le p < 2$. Then $f * h \in L^{p',\infty}(X)$ and*

$$\|f * h\|_{p',\infty} \le C\|f\|_{p',\infty}\|h\|_{p,1}.$$

Proof. We observe first that for functions f, g and h on X with h radial,

$$\langle f * h, g\rangle = \langle f, g * h\rangle \tag{1.8.1}$$

whenever both sides of the expression make sense.
(a) Choose p_1, p_2 with $q < p_1 < p < p_2 < 2$ if $p < 2$ and $q < p_1 < 2 < p_2 < q'$ if $p = 2$. Then $\varphi_{i\gamma_{p_1}\rho}(x) \le \varphi_{i\gamma_q\rho}(x)$ and $\varphi_{i\gamma_{p_2}\rho}(x) \le \varphi_{i\gamma_q\rho}(x)$ for all $x \in G$ and

consequently,

$$\widehat{|h|}(i\gamma_{p_1}\rho) \leq \widehat{|h|}(i\gamma_q\rho) < \infty \text{ and } \widehat{|h|}(i\gamma_{p_2}\rho) \leq \widehat{|h|}(i\gamma_q\rho) < \infty. \tag{1.8.2}$$

From Proposition 1.8.1 and a standard duality argument it follows that

$$\|f * h\|_{p_i'} \leq \widehat{|h|}(i\gamma_{p_i}\rho)\|f\|_{p_i'} \text{ for } i = 1, 2.$$

Using interpolation for restricted weak type operators ([72, Theorem 3.15, p. 197]), we get the assertion.
(b) We take a q_1 satisfying $q < q_1 < p$. From (a) it follows that

$$\|f * h\|_{p',\infty} \leq C\|f\|_{p',\infty}\widehat{|h|}(i\gamma_{q_1}\rho).$$

The assertion follows from above as

$$\widehat{|h|}(i\gamma_{q_1}\rho) = \int_X |h(x)|\varphi_{i\gamma_{q_1}\rho}(x)\, dx \leq \|h\|_q\|\varphi_{i\gamma_{q_1}\rho}\|_{q'},$$

and by Proposition 1.4.1(ii), $\varphi_{i\gamma_{q_1}\rho} \in L^{q'}(X)$.
(c) Note that for $1 \leq p < 2$, $L^{p,1}(G)$ is a Banach algebra (see [23]). From this, (1.8.1) and duality, the assertion follows.

□

Chapter 2

Characterization of eigenfunctions from the equation $f * \mu = f$

In this chapter we shall explore the following question. Suppose that a function f on X satisfies the equation $f * \mu = f$ where μ is a radial measure (or function) on X. When can we infer that f is an eigenfunction of Δ with a given eigenvalue $z \in \mathbb{C}$? We of course have to take f and μ from appropriate spaces of functions/measures, to make it meaningful. Here is a (non-exhaustive) list of some pairs of suitable spaces for f and h which conform to the necessary conditions (1) and (2), we mentioned in the introduction:

(a) $f \in C(X)$ and h is a compactly supported radial measure,

(b) $f \in C^\infty(X)$ and h is a compactly supported radial distribution,

(c) f is an L^p-tempered distribution and $h \in \mathcal{C}^q(G//K)$, $0 < q \leq p \leq 2$,

(d) $f \in L^{p',\infty}(X)$ and $h \in L^q(G//K)$ where $1 \leq q < p \leq 2$,

(e) $f \in L^{p',\infty}(X)$ and $h \in L^{p,1}(G//K)$, $1 \leq p < 2$,

(f) $f \in L^{p',\infty}(X)$ and h is a radial function on X such that $\widehat{|h|}(i\gamma_q\rho) < \infty$ for some $0 < q < p \leq 2$,

(g) $f \in \mathcal{H}_p^r(X), h \in L^1(G//K, \varphi_{i\gamma_q\rho})$ where $0 < q \leq p \leq 2$, $1 \leq r \leq \infty$,

(h) $f \in L^1_{2/p}(X)$ and $h \in L^\infty_{2/q}(G//K)$, $0 < q \leq p < 2$, where $L^1_{2/p}(X)$ is the set of measurable functions g on X such that

$$\int_X |g(x)|\varphi_0(x)^{2/p}\, dx < \infty$$

and $L^{\infty}_{2/q}(G//K)$ is the set of radial functions g on X such that

$$\sup_{x\in X} |g(x)| \varphi_0(x)^{-2/q} < \infty.$$

The last pair of spaces above was considered in [10]. We shall recall this list a few times in this chapter. We note that for each item in this, except perhaps (h), right convolution by h takes the space containing f to itself. For some cases, e.g. Lebesgue, Lorentz and Hardy-type spaces, the *Kunze–Stein phenomenon* (and Herz's principle) on X (see Proposition 1.6.1 and Proposition 1.8.2) can determine this condition. Although this is not a necessary criterion, the equation $f * h = f$ may appear as a suggestion for it.

We can distinguish the spaces above containing f in the following way. While in (a) and (b) f can be an eigenfunction with any eigenvalue in $\mathbb{C}$, the conditions in (c) to (h) restrict f to be an eigenfunction with eigenvalue in L^p-spectrum. We also note that apart from (c), (d), (e) and (f), other cases do not preclude unbounded functions, e.g. the oscillatory wave $x \mapsto e^{-(i\lambda+\rho)H(x^{-1}k)}$ for $k \in K$. However for our proofs, we divide these spaces in two classes, namely (i) the functions having some regularity, but no restriction on growth and (ii) measurable functions which satisfy certain growth conditions. Our working examples are (a) (and (b)) for (i) and (g) for (ii). A consequence of the first is a version of the two radius theorem for eigenfunctions.

We need some results from spectral synthesis, which we shall take up next.

2.1 Results from Spectral synthesis

In this section we collect some facts about mean periodic functions which is required for this chapter. See [57], [58] for more details. Let $C^{\infty}(G//K)$ and $C^{\infty}(G//K)'$ denote respectively the spaces of C^{∞}-radial functions and compactly supported radial distributions on X. Let $C(X)$ denote the space of continuous functions on X, $C(G//K)$ be its subspace of radial functions and $C(G//K)'$ be the space of compactly supported radial regular complex Borel measures. For $f \in C^{\infty}(G//K)$, we define

$$V(f) = \overline{\{f * T | T \in C^{\infty}(G//K)'\}}.$$

If a closed subspace V of $C^{\infty}(G//K)$ is invariant under convolution with elements of $C^{\infty}(G//K)'$, then we shall call V an invariant subspace. It is clear from the definition that $V(f)$ is a closed invariant subspace of $C^{\infty}(G//K)$. We call a function

$f \in C^\infty(G//K)$ *mean periodic* when $V(f)$ is a nonzero proper invariant subspace of $C^\infty(G//K)$. Henceforth we shall write $\partial_\lambda, \partial_\lambda^i$ to denote $\partial/\partial\lambda, \partial^i/\partial\lambda^i$ respectively. For $\mu \in \mathbb{C}$ and $k \in \mathbb{N}$, we recall that $\varphi_{\mu,k}(x) = \partial_\lambda^k \varphi_\lambda(x)|_{\lambda=\mu}$. We shall need the following results.

Theorem 2.1.1. *Let V be a nonzero proper closed invariant subspace of $C^\infty(G//K)$. Then*

$$V = \overline{\text{ span } \{\varphi_{\lambda,k} \,|\varphi_{\lambda,k} \in V, k \in \mathbb{N} \text{ and } \lambda \in \mathbb{C}\}}.$$

Hence any mean periodic function f is the limit of finite linear combinations of $\varphi_{\lambda,k} \in V(f)$ with $k \in \mathbb{N}, \lambda \in \mathbb{C}$.

Proof. See [57, Theorem 4.3]. □

Lemma 2.1.2. *Let $V \subset C^\infty(G//K)$ be as in Theorem 2.1.1. If $\varphi_{\lambda,k} \in V \setminus \{0\}$, then $\varphi_{\lambda,l} \in V$ for all $0 \leq l \leq k$.*

Proof. See [57, Lemma 4.2]. □

Lemma 2.1.3. *Both the sets $\{\varphi_{\lambda,k} \mid k = 0, 1, 2, \ldots\}$ for $\lambda \neq 0$ and $\{\varphi_{0,k} \mid k = 0, 2, 4 \ldots\}$ are linearly independent.*

Proof. We assume that $\lambda \neq 0$. Then $\lambda \in S_p$ for some $p \in (0, 2]$. Suppose that for some $N \geq 0$ there exists a nontrivial linear combination satisfying

$$\sum_{i=0}^{N} a_i \varphi_{\lambda,i} = 0 \text{ with } a_N \neq 0.$$

Choose $f \in \mathcal{C}^p(G//K)$ with $\partial_\lambda^i \widehat{f}(\lambda) = 0$ for $0 \leq i < N$ and $\partial_\lambda^N \widehat{f}(\lambda) \neq 0$. Then

$$0 = \left(\sum_{i=0}^{N} a_i \varphi_{\lambda,i} \right) * f = a_N \partial_\lambda^N \widehat{f}(\lambda),$$

which is a contradiction. Similar arguments can be used for the case $\lambda = 0$. □

2.2 Set up, statements and proofs of the results

The pair of function spaces collected in (a) to (h) can be divided in two different groups: functions without growth conditions and functions satisfying some integrability conditions. We shall prove one representative result from each group. Two radius theorem is a consequence of the result for the first group, i.e. for functions without growth conditions.

2.2.1 Result leading to the two radius theorem

We need the following result of complex analysis.

Lemma 2.2.1. *Let f be a nonconstant entire function such that $|f(z)| \leq Ce^{c|z|}$ for some constants $c, C > 0$ and*

$$\lim_{z \in \mathbb{R}, |z| \to \infty} f(z) = L \tag{2.2.1}$$

for some constant $L \in \mathbb{C}$. Then f has infinitely many zeros in the complex plane.

Proof. Let σ be the order of growth of f. Then it follows from the hypothesis that $\sigma \leq 1$. Assume that f has finitely many zeros and let $z_1, z_2, \cdots, z_n$ be the nonzero zeros of f. If $0 \leq \sigma < 1$, then by Hadamard's Factorization theorem ([71, Theorem 5.1, Chapter 5]),

$$f(z) = Az^m \prod_{k=1}^{n} (1 - \frac{z}{z_k}) = Q(z)$$

for some polynomial Q, $m \in \mathbb{Z}^+$ and constant A. If $\sigma = 1$, then again by Hadamard's Factorization theorem,

$$f(z) = e^{a_1 z + a_0} z^m \prod_{k=1}^{n} (1 - \frac{z}{z_k}) e^{\frac{z}{z_k}} = e^{\alpha z} P(z)$$

for some polynomial P, $m \in \mathbb{Z}^+$ and constant α. Thus for both the cases, $f(z) = e^{\alpha z} P(z)$. This contradicts (2.2.1) as f is nonconstant. □

We note that spherical Fourier transform of a radial compactly supported distribution T is entire and of exponential type, i.e.

$$|\widehat{T}(\lambda)| = |\langle T, \varphi_\lambda \rangle| \leq Ce^{c|\lambda|} \text{ for all } \lambda \in \mathbb{C} \tag{2.2.2}$$

for some positive constants C, c (see [31]). Thus it is clear from Lemma 2.2.1 that for any compactly supported distribution T, whose Fourier transform satisfies (2.2.1), there are infinitely many $\lambda \in \mathbb{C}$, such that $\widehat{T}(\lambda) = 1$. This rules out the possibility of characterization of eigenfunctions from an equation of the form $f * T = f$ for T as above. However if more than one such T satisfy the equation $f * T = f$, then they may serve the purpose. Spherical and ball averages are particular examples of such convolution by radial compactly supported distributions. Thus it follows that the (spherical or ball) mean value property for one single radius cannot characterize eigenfunctions of Δ. To illustrate, let us fix a $t > 0$. Let $f(z) = \varphi_z(a_t) - 1$. Owing to (2.2.2) and Proposition 1.4.3, it follows that f satisfies the hypothesis of Lemma 2.2.1. Hence we can find $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq \pm z_2$ and $\varphi_{z_1}(a_t) = 1 = \varphi_{z_2}(a_t)$. Clearly both $f = \varphi_{z_1}$ and $f = \varphi_{z_2}$ satisfy the equation $M_t f = \varphi_{z_1}(a_t) f$. This precludes the characterization of eigenfunctions with a particular eigenvalue from this equation. Argument is similar for the ball mean value property.

We shall take up the convolution equation $f * \mu = f$ for continuous functions f and compactly supported complex measures μ, which is the case (a) (and (b)) of the list. The two-radius theorem for eigenfunctions can be obtained as a consequence of this theorem.

Theorem 2.2.2. *Let f be a continuous function on X and $\mathcal{M}$ be a family of nonzero compactly supported radial regular Borel complex measures such that $f * \mu = 0$ for each $\mu \in \mathcal{M}$. Assume that $\lambda_0 \in \mathbb{C}$ is the only common zero of $\{\widehat{\mu} \mid \mu \in \mathcal{M}\}$. Further assume that:*

1. *if $\lambda_0 \neq 0$, then there exists $\mu \in \mathcal{M}$ with $\partial_\lambda \widehat{\mu}(\lambda_0) \neq 0$ and*
2. *if $\lambda_0 = 0$ then there exists $\mu \in \mathcal{M}$ with $\partial_\lambda^2 \widehat{\mu}(\lambda_0) \neq 0$.*

Then $\Delta f = -({\lambda_0}^2 + \rho^2) f$.

Proof. Without loss of generality we assume f to be smooth as f can be approximated uniformly on compact sets by functions of the form $f * h$, where h is a radial function in $C_c^\infty(X)$.

We shall first prove the assertion when f is a radial function.

$$\text{Let } \mathcal{V} = \{f \mid f * \mu = 0 \text{ for all } \mu \in \mathcal{M}\}.$$

Then $\mathcal{V}$ is a proper invariant subspace of $C^\infty(G//K)$. Hence by Theorem 2.1.1,

$$\mathcal{V} = \overline{\operatorname{span}\{\varphi_{\lambda,k} \mid \varphi_{\lambda,k} \in \mathcal{V}\}}.$$

We take a $\varphi_{\lambda,k} \neq 0$ from $\mathcal{V}$. Then for any $\mu \in \mathcal{M}$, $\varphi_{\lambda,k} * \mu = 0$ and

$$\varphi_{\lambda,k} * \mu = \partial_z^k (\varphi_z * \mu)|_{z=\lambda} = \partial_z^k (\widehat{\mu}(z)\varphi_z)_{z=\lambda} = \sum_{i=0}^{k} \binom{k}{i} \partial_\lambda^i \widehat{\mu}(\lambda) \varphi_{\lambda,k-i}.$$

From Lemma 2.1.3 we have the following conclusions.

(a) If $\lambda \neq 0$, then $\partial_\lambda^i \widehat{\mu}(\lambda) = 0$ for $0 \leq i \leq k$.

(b) If $\lambda = 0$, then $\partial_\lambda^i \widehat{\mu}(\lambda) = 0$ for all even i in $0 \leq i \leq k$. On the other hand as $\widehat{\mu}$ is an even function, $\partial_\lambda^i \widehat{\mu}(\lambda) = 0$ for all odd i in $0 \leq i \leq k$ as $\lambda = 0$.

Thus a nonzero $\varphi_{\lambda,k}$ satisfies $\varphi_{\lambda,k} * \mu = 0$ if and only if $\partial_\lambda^i \widehat{\mu}(\lambda) = 0$ for $0 \le i \le k$. But λ_0 is the only common zero of $\{\widehat{\mu} \mid \mu \in \mathcal{M}\}$. Therefore, $\mathcal{V} = \overline{\text{span}\{\varphi_{\lambda_0,k} \mid \varphi_{\lambda_0,k} \in \mathcal{V}\}}$. Additional conditions in the hypothesis along with Lemma 2.1.2 further assert that if $\lambda_0 \neq 0$, then $\varphi_{\lambda_0,k} \notin \mathcal{V}$ for $k \ge 1$ and if $\lambda_0 = 0$, then $\varphi_{0,k} \notin \mathcal{V}$ for $k \ge 2$. As odd derivatives of an even function are odd functions we also have $\varphi_{0,1} \equiv 0$. Thus in both the cases, $\mathcal{V} = \overline{\text{span}\{\varphi_{\lambda_0}\}}$. This implies $f = c\varphi_{\lambda_0}$ and in particular $\Delta f = -({\lambda_0}^2 + \rho^2) f$.

We recall that ℓ_x denotes the left translation by x and $R(\ell_x f)$ is the radial part of $\ell_x f$. To extend the result to the case when f is not radial, we first note that if for a continuous function g on X, $R(\ell_x g) = 0$ for all $x \in G$, then g is identically zero. Indeed $R(\ell_x g) = 0$ implies that $g * h(x) = \int_X R(\ell_x g)(y) h(y)\, dy = 0$, for any radial function h supported on a compact neighbourhood of o. Thus $g * h = 0$. As we can approximate g by functions $g * h$, for such h, uniformly on compact sets, we conclude that $g = 0$.

If $f * \mu = 0$, then clearly for all $x \in G$, $R(\ell_x f) * \mu = 0$ for $\mu \in \mathcal{M}$. Hence by the assertion proved above for radial functions, $\Delta R(\ell_x f) = -(\lambda_0^2 + \rho^2) R(\ell_x f)$. That is, for all $x \in G$,

$$R(\ell_x \Delta f) = R(\ell_x(-(\lambda_0^2 + \rho^2) f)).$$

Hence by the argument given above $\Delta f = -({\lambda_0}^2 + \rho^2) f$. $\square$

It is clear from the proof of the theorem above that if we take $f \in C^\infty(X)$, then $\mathcal{M}$ can be replaced by a set of compactly supported distributions. Theorem 2.2.2 along with Proposition 1.4.4 will establish the following two-radius theorem.

Theorem 2.2.3. *Let f be a continuous function on X and $\alpha \in \mathbb{C}$ be fixed. Suppose that for $t_1, t_2 > 0$,*

(i) *$M_{t_j} f = \varphi_\alpha(a_{t_j}) f$ (respectively, $B_{t_j} f = \psi_\alpha(t_j) f$) for $j = 1$ and 2,*

(ii) *the equations $\varphi_\lambda(a_{t_1}) = \varphi_\alpha(a_{t_1})$ and $\varphi_\lambda(a_{t_2}) = \varphi_\alpha(a_{t_2})$ (respectively, $\psi_\lambda(t_1) = \psi_\alpha(t_1)$ and $\psi_\lambda(t_2) = \psi_\alpha(t_2)$) have no common solution for $\lambda \in \mathbb{C} \setminus \{\pm\alpha\}$,*

(iii) *if $\alpha \notin i\mathbb{R}$, then either $\varphi_{\alpha,1}(a_{t_1}) \neq 0$ or $\varphi_{\alpha,1}(a_{t_2}) \neq 0$ (respectively, either $\psi_{\alpha,1}(t_1) \neq 0$ or $\psi_{\alpha,1}(t_2) \neq 0$).*

Then $\Delta f = -(\alpha^2 + \rho^2) f$.

Proof. Let δ_o denotes the Dirac mass at the origin. Then the hypothesis $M_{t_j} f = \varphi_\alpha(a_{t_j}) f$ can be written as $f * \mu_j = 0$, where $\mu_j = \sigma_{t_j} - \varphi_\alpha(a_{t_j})\delta_e$ for $j = 1, 2$. It follows from Proposition 1.4.4 and the conditions of the theorem, that μ_1 and μ_2 satisfy the hypothesis of Theorem 2.2.2, from which we assert that $\Delta f = -(\alpha^2+\rho^2)f$.

The version for ball average follows in a similar way, where we replace M_{t_j}, $\varphi_\alpha(a_{t_j})$ and σ_{t_j} by B_{t_j}, $\psi_\alpha(t_j)$ and $\chi_{B(o,t_j)}/|B(o,t_j)|$ respectively. $\square$

Remark 2.2.4. For a fixed $t_1 > 0$, almost every $t_2 > 0$ satisfy the hypothesis of Theorem 2.2.3. To see this we fix a $t_1 > 0$. As $\lambda \mapsto \varphi_\lambda(a_{t_1}) - \varphi_\alpha(a_{t_1})$ is analytic, the set $Z = \{\lambda \mid \varphi_\lambda(a_{t_1}) = \varphi_\alpha(a_{t_1}), \lambda \neq \pm\alpha\}$ is countable. For each fixed $\lambda \in Z$, we exclude those t_2 which satisfy: $\varphi_\lambda(a_{t_2}) = \varphi_\alpha(a_{t_2})$ and $\varphi_{\alpha,1}(a_{t_2}) = 0$. Since $\varphi_\lambda(a_t)$ is real analytic in t, we have to discard only countably many t_2 for each fixed $\lambda \in Z$. This settles the argument. For the ball mean value property the argument is similar.

Next result shows instead of taking two carefully chosen radii we can work with an arbitrary sequence of distinct radii for characterizing eigenfunctions with eigenvalues in $[-\rho^2, \infty)$.

Theorem 2.2.5. *Let $\alpha \in i\mathbb{R}$ and $\{t_j\}_{j\in\mathbb{N}}$ be a sequence of distinct positive real numbers. Suppose that for a function $f \in C(X)$, $M_{t_j} f = \varphi_\alpha(a_{t_j})f$ or $B_{t_j} f = \psi_\alpha(t_j)f$ for all $j \in \mathbb{N}$. Then $\Delta f = -(\alpha^2 + \rho^2)f$.*

Proof. We assume the first condition. In view of Theorem 2.2.3, it is sufficient to show that given any $\lambda \neq \pm\alpha$, there exists at least one $j \in \mathbb{N}$ such that $\varphi_\lambda(a_{t_j}) \neq \varphi_\alpha(a_{t_j})$.

Let $\lambda \neq \pm\alpha$. It follows from Proposition 1.4.2 that if $|\Im\lambda| \leq |\Im\alpha|$, then $|\varphi_\lambda(a_{t_j})| < \varphi_\alpha(a_{t_j})$ for any $t_j > 0$. Hence it is sufficient to consider λ such that $|\Im\lambda| > |\Im\alpha|$. We divide it in two cases.

Case 1: The sequence $\{t_j\}_{j\in\mathbb{N}}$ is unbounded. Since $\varphi_\lambda(a_t) \asymp e^{(|\Im\lambda|-\rho)t}$, for t_j sufficiently large $\varphi_\lambda(a_{t_j}) \neq \varphi_\alpha(a_{t_j})$.

Case 2: The sequence $\{t_j\}_{j\in\mathbb{N}}$ is bounded and $\varphi_\lambda(a_{t_j}) = \varphi_\alpha(a_{t_j})$ for all j. Passing to a subsequence if necessary, we assume that the sequence $\{t_j\}$ converges to a point s. But as the function $t \mapsto \varphi_\lambda(a_t) - \varphi_\alpha(a_t)$ is real analytic, that will imply that $\varphi_\lambda(a_t) = \varphi_\alpha(a_t)$ for all t, which is not possible unless either $\lambda = \alpha$ or $\lambda = -\alpha$. For the ball average the proof is similar. $\square$

We conclude this section with a counter example to show that Theorem 2.2.5 is not true for nonzero $\lambda \in \mathbb{R}$. To illustrate this consider $X = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. Then $\varphi_\lambda(a_t) = \sin(2\lambda t)/\lambda\sinh(2t)$ ([41, p. 433]). Take a nonzero $\lambda \in \mathbb{R}$ and define $t_n = n\pi/\lambda$. Then $\varphi_{2\lambda}(a_{t_n}) = \varphi_\lambda(a_{t_n}) = 0$. It is clear that $f = \varphi_{2\lambda}$ satisfies hypothesis of Theorem 2.2.5, but $\Delta f \neq -(\lambda^2 + \rho^2)f$.

2.2.2 Characterization of eigenfunctions with some integrability conditions

In the previous subsection we have used spectral synthesis crucially to solve the convolution equations $f * \mu = 0$ for compactly supported complex measures μ. Now we consider such equations when μ is no longer compactly supported. Following is a representative result for the cases (c) to (h).

Theorem 2.2.6. *Let $f \in \mathcal{H}_p^r(X)$ for some $0 < p < 2, 1 \leq r \leq \infty$ and let μ be an absolutely continuous radial measure on X with density h satisfying $\int_X |h(x)|\varphi_{i\gamma_q \rho}(x)\, dx < \infty$ for some $0 < q < p$. Suppose that $f * \mu = f$ and for a point $\beta \in S_p$,*

(i) $\widehat{\mu}(\pm\beta) = 1$ *and* $\widehat{\mu}'(\beta) \neq 0$,

(ii) $\widehat{\mu}(\pm\lambda) \neq 1$ *for any* $\lambda \in S_p \setminus \{\pm\beta\}$.

Then $\Delta f = -(\beta^2 + \rho^2) f$. Further, if $\beta \in \partial S_p$ and if we assume (changing β to $-\beta$ if necessary) that $\Im \beta < 0$, then we have these conclusions:

(a) $f = \mathcal{P}_\beta F$ *for some* $F \in L^r(K/M)$ *when* $1 < r \leq \infty$,

(b) $f = \mathcal{P}_\beta \nu$ *for some signed measure* ν *on* K/M *when* $r = 1$.

Proof. Let us first assume that f is a radial function. It is clear that $\widehat{h}$ extends analytically on $S_q^o \supset S_p$ and is continuous on S_q. By the hypothesis if $\lambda \in S_p$ and $\lambda \neq \pm\beta$, then $\widehat{h}(\lambda) \neq 1$. It can be verified that there exists a $\delta > 0$ such that $S_{p,\delta} \subset S_q^\circ$ and for any $\lambda \in S_{p,\delta} \setminus \{\pm\beta\}$, $\widehat{h}(\lambda) \neq 1$ where

$$S_{p,\delta} := \{\lambda \in \mathbb{C} \mid |\Im\lambda| \leq (\gamma_p + \delta)\rho\}.$$

Here is the argument. We start with a $\delta_1 > 0$ such that $S_{p,\delta_1} \subset S_q^o$. Since $\widehat{h}(\lambda) \to 0$ uniformly in S_{p,δ_1} as $|\lambda| \to \infty$ ([60, Proposition 4.5]), there exists $N > 0$ such that $|\widehat{h}(\lambda)| < \frac{1}{2}$ whenever $\lambda \in S_{p,\delta_1}$ and $|\Re\lambda| > N$. As the set $\{\lambda \mid \lambda \in S_{p,\delta_1},\ |\Re\lambda| \leq N\}$ is compact and $\widehat{h}(\lambda)$ is analytic, cardinality of $Z = \{\lambda \mid \lambda \in S_{p,\delta_1}, |\Re\lambda| \leq N,\ \widehat{h}(\lambda) = 1\}$ is finite. If $Z = \{\beta, -\beta\}$ then take $\delta = \delta_1$. Otherwise let $d = \min_{\lambda \in Z \setminus \{\beta\}} |\Im\lambda|$. Then $\gamma_p \rho < d < (\gamma_p + \delta_1)\rho$ and any $\delta < d/\rho - \gamma_p$ serves the purpose.

From analyticity and uniform boundedness of $\widehat{h}$ on $S_{p,\delta}$, mentioned above and by a standard argument using Cauchy's integral formula, it follows that $\widehat{h}e^{-(\lambda^2+\rho^2)}$ is in $\mathcal{C}^p(\widehat{G//K})$. Let W be the closed subspace of $\mathcal{C}^p(G//K)$ of functions whose Fourier transform vanishes at $\pm\beta$. Let $g \in \mathcal{C}^p(G//K)$ be such that $\widehat{g}(\lambda) = (\widehat{h}(\lambda) - 1)e^{-(\lambda^2+\rho^2)}$ for $\lambda \in S_p$. Then by Proposition 1.5.2, $\{g * u \mid u \in \mathcal{C}^p(G//K)\}$ is dense in W. We rewrite the hypothesis $f * h = f$ as $f * (h - \delta_o) = 0$, where δ_o is the Dirac measure at the origin. Then $f * g = 0$ and hence $\langle f, g * u\rangle = \langle f * g, u\rangle = 0$ for all $u \in \mathcal{C}^p(G//K)$. Using the continuity of $\phi \mapsto \langle f, \phi\rangle$ on $\mathcal{C}^p(G//K)$, we conclude that $\langle f, \phi\rangle = 0$ for all $\phi \in W$. But as φ_β annhilates W as an L^p-tempered distribution, we have $f = c_0\varphi_\beta$ ([68, Lemma 3.9]). In particular, $\Delta f = -(\beta^2 + \rho^2)f$. The argument used in Theorem 2.2.2 extends the result $\Delta f = -(\beta^2 + \rho^2)f$ for nonradial f. An application of Theorem 1.7.2 completes the proof of the assertion. □

Remark 2.2.7. As mentioned above, we can formulate and prove such a theorem using any of the pair of spaces in (c) to (h) from the list given at the beginning of this chapter. For instance, we may take up (h) (which is similar to what was considered in [10] for the characterization of harmonic functions). That is, we substitute $\mathcal{H}_p^r(X)$ by $L^1_{2/p}(X)$ and take the density h from $L^\infty_{2/q}(G//K)$ for some $0 < q < p < 2$, keeping the other conditions same, then the argument in the proof above, mutatis mutandis, leads the conclusion: $\Delta f = -(\beta^2 + \rho^2)f$. We shall not give the details, but add some preliminary results here, that are necessary for the proof. It is clear that $\mathcal{C}^p(G//K) \subset L^\infty_{2/p}(G//K)$ for $0 < p \leq 2$ and if $f \in L^1_{2/p}(G//K)$, then the map $T_f : \mathcal{C}^p(G//K) \to \mathbb{C}$ defined by $\psi \mapsto \int_X f(x)\psi(x)\, dx$ is a continuous linear functional on $\mathcal{C}^p(G//K)$. We also have the following properties.

Proposition 2.2.8. (a) *If $\lambda \in S_p^\circ$ for $0 < p < 2$, then $\varphi_\lambda \in L^1_{2/p}(G//K)$.*

(b) *If $h \in L^\infty_{2/p}(G//K)$ for $0 < p < 2$, then $\widehat{h}(\lambda)$ exists for each $\lambda \in S_p^\circ$ and is analytic on S_p°.*

(c) *$L^\infty_{2/p}(G//K) \subset L^q(G//K)$, for any $0 < p \leq 2$ and $q > p$.*

(d) *For $f \in L^1_{2/p}(X)$ with $0 < p \leq 2$ and $\psi_1, \psi_2 \in \mathcal{C}^p(G//K)$, $\langle f, \psi_1 * \psi_2\rangle = \langle f * \psi_1, \psi_2\rangle$.*

Proof. Parts (a) and (b) follow from the asymptotic estimates of φ_λ and φ_0 (see (1.2.7), (1.2.9)). For (c) we have from definition of $L^\infty_{2/p}(G//K)$ and estimate of φ_0,

$$\begin{aligned}\left(\int_X |h(y)|^q\, dy\right)^{1/q} &= \left(\int_X |h(y)|^q \varphi_0(y)^{-\frac{2q}{p}} \varphi_0(y)^{\frac{2q}{p}}\, dy\right)^{1/q} \\ &\leq \left[\sup_{x\in X} |h(x)|\varphi_0^{-2/p}\right] \left(\int_X \varphi_0(y)^{\frac{2q}{p}}\, dy\right)^{1/q} \\ &\leq C\left[\sup_{x\in X} |h(x)|\varphi_0^{-2/p}\right].\end{aligned}$$

Part (d) follows from Fubini's theorem along with the fact that $\mathcal{C}^p(G//K)$ is a convolution algebra. □

Instead of one measure we can use a (finite) family of measures for the formulation, which we state below for the sake of completion.

Theorem 2.2.9. *Let $0 < q < p < 2$ and $\mathcal{M}$ be a finite set of radial absolutely continuous complex radial measures on X of the form $d\mu(x) = h(x)\,dx$ with $h \in L^{\infty}_{2/q}(G//K)$ and $\widehat{h}(\beta) = 1$ for some $\beta \in S_p^{\circ}$. Suppose that the only points in S_p where $\widehat{\mu} = 1$ for all $\mu \in \mathcal{M}$ are $\pm\beta$ and for at least one $\mu \in \mathcal{M}$, $\widehat{\mu} - 1$ has a simple zero at $\lambda = \pm\beta$. Let $f \in L^1_{2/p}(G//K)$ be such that $f * \mu = f$ for all $\mu \in \mathcal{M}$, then $\Delta f = -(\beta^2 + \rho^2)f$.*

2.2.3 A generalization of Furstenberg's result on harmonic function

We come back to the Furstenberg's result ([37, 81]) that for any probability μ, all bounded μ-harmonic functions are harmonic. This result was reproved for unit disk in [11] using a Wiener's Tauberain theorem proved in the same paper. Our generalization for eigenfunctions is a adaptation of that proof using a more general Wiener's Tauberain theorem proved in [25]. Here is the statement of the result.

Theorem 2.2.10. *Fix a $0 < p < 2$. Let X be the quotient space $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ and let μ be a (essentially positive) non-atomic radial measure on X such that $\widehat{\mu}(i\gamma_p\rho) = L < \infty$ and $f \in \mathcal{H}_p^r(X)$ for some $1 \leq r \leq \infty$. If f satisfies $f * \mu = Lf$, then $\Delta f = -4\rho^2/pp'f$. Moreover,*

$$f = \mathcal{P}_{-i\gamma_p\rho}F \text{ for some } F \in L^r(K/M) \text{ if } 1 < r \leq \infty$$

and

$$f = \mathcal{P}_{-i\gamma_p\rho}\nu \text{ for some measure } \nu \text{ on } K/M \text{ when } r = 1.$$

If $p = 1$ and $L = 1$ then μ is a probability. If also $f \in L^{\infty}(X)$ then $f \in \mathcal{H}_1^r(X)$ for any $1 \leq r \leq \infty$ and we get back Furstenberg's result mentioned above. We need the following Wiener's Tauberian theorem which is a particular case of [25, Theorem 2.1]. Let us explain the notation used in it. Let us fix a $p \in (0, 2)$. Let $L^1(G//K, \varphi_{i\gamma_p\rho})$ be the set of radial measurable functions g on X satisfying $\int_X |g(x)|\varphi_{i\gamma_p\rho}(x)\,dx < \infty$. Then it is clear that for $g \in L^1(G//K, \varphi_{i\gamma_p\rho})$, $\widehat{g}$ extends as a continuous function on S_p which is analytic in the interior of S_p and

$$\lim_{|\xi| \to \infty} \widehat{g}(\xi + i\eta) = 0$$

uniformly in $\eta \in [-\gamma_p\rho, \gamma_p\rho]$ (see [60, Proposition 4.5]). Let $\lambda_0 = \alpha + i\gamma_p\rho$ be a point in ∂S_p for some $\alpha \in \mathbb{R}$. We define,

$$L^1_{\lambda_0}(G//K, \varphi_{i\gamma_p\rho}) = \{g \in L^1(G//K, \varphi_{i\gamma_p\rho}) \mid \widehat{g}(\lambda_0) = 0\}.$$

For a function $g \in L^1(G//K, \varphi_{i\gamma_p\rho})$, we also define

$$\delta_\infty(g) = -\limsup_{t\to+\infty} \exp\left(-\frac{\pi}{2\gamma_p\rho}t\right) \log|\widehat{g}(t)|$$

and

$$\delta_{\lambda_0}(g) = -\limsup_{t\to\gamma_p\rho^-}(\gamma_p\rho - t) \log|\widehat{g}(\Re\lambda_0 + it)|.$$

For a collection of functions $\mathcal{F}$ in $L^1(G//K, \varphi_{i\gamma_p\rho})$, let

$$\delta_\infty(\mathcal{F}) = \inf_{g\in\mathcal{F}} \delta_\infty(g) \text{ and } \delta_{\lambda_0}(\mathcal{F}) = \inf_{g\in\mathcal{F}} \delta_{\lambda_0}(g).$$

Here is the statement of the Wiener's Tauberian theorem.

Theorem 2.2.11 (Dahlner)**.** *Fix a $p \in (0, 2)$. Let $G = \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2)$. Let $\mathcal{F}$ be a family of functions in $L^1(G//K, \varphi_{i\gamma_p\rho})$ and $\lambda_0 = \alpha+i\gamma_p\rho$ for some $\alpha \in \mathbb{R}$. Let $I(\mathcal{F})$ be the smallest closed ideal in $L^1(G//K, \varphi_{i\gamma_p\rho})$ containing $\mathcal{F}$ and λ_0 be a point in ∂S_p. Then $I(\mathcal{F}) = L^1_{\lambda_0}(G//K, \varphi_{i\gamma_p\rho})$ if and only if the only common zero of $\{\widehat{g} \mid g \in \mathcal{F}\}$ is λ_0 and $\delta_\infty(\mathcal{F}) = \delta_{\lambda_0}(\mathcal{F}) = 0$.*

Proof of Theorem 2.2.10. Without loss of generality we assume that $L = 1$ and f is radial, for arguments used in Theorem 2.2.2 extend the result for nonradial f. Thus the hypothesis is $\widehat{\mu}(i\gamma_p\rho) = 1$ and $f*(\mu-\delta_o) = 0$. Clearly, $(\mu-\delta_o)*L^1(G//K, \varphi_{i\gamma_p\rho}) \subseteq L^1_{i\gamma_p\rho}(G//K, \varphi_{i\gamma_p\rho})$.

Step 1: Since for any $\lambda \in S_p \setminus \{\pm i\gamma_p\rho\}$, $|\varphi_\lambda(a_t)| < \varphi_{i\gamma_p\rho}(a_t)$ when $t > 0$ (Proposition 1.4.2), it follows that $\widehat{\mu}(\lambda) \neq 1$ for any such λ. Let $h_1 \in L^1(G//K)$ be defined by $\widehat{h_1}(\lambda) = e^{-(\lambda^2+\rho^2)}$. Then the function $g_1 = (\mu - \delta_o) * h_1 \in L^1_{i\gamma_p\rho}(G//K, \varphi_{i\gamma_p\rho})$ and the only zero of $\widehat{g_1}$ in S_p is λ_0.

Step 2: It is clear that there exists a function $g_2 \in (\mu - \delta_o) * L^1(G//K, \varphi_{i\gamma_p\rho})$ which satisfies $\delta_\infty(g_2) = 0$.

Step 3: Since $\mu \neq \delta_o$ we can find $A \subset X$ not containing o, such that $0 <$

$\int_A \varphi_{i\gamma_p\rho}(x)\, d\mu(x) < 1$. Let $\int_A \varphi_{i\gamma_p\rho}(x)\, d\mu(x) = \mu_p(A)$. Then for $0 < t < \gamma_p\rho$,

$$\begin{aligned} 1 - \widehat{\mu}(it) &= \int_X (\varphi_{i\gamma_p\rho}(x) - \varphi_{it}(x))\, d\mu(x) \\ &\geq \mu_p(A) \int_A (\varphi_{i\gamma_p\rho}(x) - \varphi_{it}(x))\mu_p(A)^{-1}\, d\mu(x). \end{aligned}$$

Since log is increasing and concave we have

$$\begin{aligned} &- (\gamma_p\rho - t) \log |\widehat{g_1}(it)| \\ &= -(\gamma_p\rho - t) \log \left[(1 - \widehat{\mu}(it))e^{-(\rho^2 - t^2)}\right] \\ &\leq -(\gamma_p\rho - t) \left[\log \mu_p(A) + \log \int_A (\varphi_{i\gamma_p\rho}(x) - \varphi_{it}(x))\mu_p(A)^{-1}\, d\mu(x) + \log e^{-(\rho^2 - t^2)}\right] \\ &\leq -(\gamma_p\rho - t)(\log \mu_p(A) + \int_A \log(\varphi_{i\gamma_p\rho}(x) - \varphi_{it}(x))\mu_p(A)^{-1}\, d\mu(x) + \log e^{-(\rho^2 - t^2)}). \end{aligned}$$

Since φ_{it} is a strictly increasing function for $t > 0$, we have,

$$\lim_{t \to \gamma_p\rho -} (\gamma_p\rho - t) \log(\varphi_{i\gamma_p\rho}(x) - \varphi_{it}(x)) = 0$$

From this using dominated convergence theorem, we conclude from above that $\delta_{i\gamma_p\rho}(g_1) = 0$.

Thus the hypothesis of Theorem 2.2.11 are satisfied. Therefore the smallest closed ideal containing $(\mu - \delta_o) * L^1(G//K, \varphi_{i\gamma_p\rho})$ is $L^1_{i\gamma_p\rho}(G//K, \varphi_{i\gamma_p\rho})$. Thus it follows from the hypothesis $f * h = 0$ for any $h \in L^1_{i\gamma_p\rho}(G//K, \varphi_{i\gamma_p\rho})$. As $\mathcal{C}^p(G//K) \subset L^1(G//K, \varphi_{i\gamma_p\rho})$, we get that $\langle f, h\rangle = f * h(e) = 0$ for all $h \in \mathcal{C}^p(G//K)$ with $\widehat{h}(i\gamma_p\rho) = 0$ vanish at $i\gamma_p\rho$. Since $\varphi_{i\gamma_p\rho}$ also annihilates all such h as an L^p-tempered distribution, therefore we get $f = C\varphi_{i\gamma_p\rho}$ (see [68, Lemma 3.9]) as L^p-tempered distribution. Hence the claim. □

Remark 2.2.12. Wiener's Tauberain theorem proved in [11] was generalized for all rank one symmetric spaces X of noncompact type in [9, 61]. It is expected that an adaptation of the proof of Theorem 2.2.11 will extend it to all such X. As the proof of Theorem 2.2.10 above does not use anything specific to $X = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$, we may conjecture that it will be true for all rank one symmetric spaces.

Chapter 3

Characterization of eigenfunctions via Roe–Strichartz type theorems

We recall briefly that generalizing a result of Roe [67], Strichartz [76] and Howard–Reese [44] proved that if a doubly infinite sequence $\{f_k\}$ of functions in $\mathbb{R}^n$ satisfying $\Delta_{\mathbb{R}^n} f_k = f_{k+1}$ for all $k \in \mathbb{Z}$, is uniformly bounded then $\Delta_{\mathbb{R}^n} f_0 = -f_0$. A counter example in [76] also shows that the result fails in hyperbolic 3-space. Similar counter examples can be constructed for any Riemannian symmetric space of noncompact type, as the failure can be explained by the spectral properties of the Laplace Beltrami operator Δ on X. However the situation was saved in [52] for X by choosing appropriate norm-boundedness replacing the uniform-boundedness of $\{f_k\}$. In this chapter, we aim to obtain versions of this result for translation invariant linear operators Θ, replacing Δ. While we achieve this goal when f_k are assumed to be L^2-tempered distributions, we have to restrict ourselves with particular examples, e.g. spherical and ball averages or heat operators, which is however coherent with the theme of this thesis. We shall also enlarge the scope of the theorem proved in [52], by including all complex eigenvalues. We shall begin with some definitions, preparatory discussions and results.

3.1 Preparations

Let us recollect the following notation from Chapter 1 as we shall use them frequently in this chapter. For a locally integrable function f, $M_t f$ and $B_t f$ respectively are sphere and ball averages of radius t of f. More explicitly we have

$$M_t f(x) = f * \sigma_t(x) \quad \text{and} \quad B_t f(x) = f * m_t(x),$$

where σ_t is the normalized surface measure of sphere of radius t and $m_t = |B(o,t)|^{-1}\chi_{B(o,t)}$. Spherical Fourier transform of σ_t and m_t at $\lambda \in \mathbb{C}$ are respectively, $\varphi_\lambda(a_t)$ and $\psi_\lambda(t)$. We recall that for $t > 0$, heat kernel h_t is defined as a radial function in $\mathcal{C}^p(X)$ for each $p \in (0,2]$, whose spherical Fourier transform is given by $\widehat{h_t}(\lambda) = e^{-t(\lambda^2+\rho^2)}$, $\lambda \in \mathbb{C}$. We define the heat propagator, denoted by $e^{t\Delta}$ as $f \mapsto f * h_t$ whenever the convolution makes sense.

3.1.1 Spectrum of the Laplacian

We recall that for $0 < p < \infty$, $\gamma_p = 2/p - 1$, $\gamma_\infty = -1$ and $S_p = \{\lambda \in \mathbb{C} \mid |\Im\lambda| \leq |\gamma_p|\rho\}$. Consider the map $\Lambda : \mathbb{C} \to \mathbb{C}$, given by $\Lambda(\lambda) = -(\lambda^2 + \rho^2)$. It follows that for $0 < p \leq \infty$, $\Lambda(S_p)$ is the closed region,

$$\left\{ z = x + iy \in \mathbb{C} \mid y^2 \leq -4\gamma_p^2\rho^2 \left(x + \frac{4\rho^2}{pp'} \right) \right\},$$

whose boundary $\partial\Lambda(S_p)$ is the parabola (see Figure 3.2 (a)):

$$y^2 = -4\gamma_p^2\rho^2 \left(x + \frac{4\rho^2}{pp'} \right). \tag{3.1.1}$$

For $p = 2$, the parabolic region degenerates to a ray $\Lambda(S_2) = \{x \in \mathbb{R} \mid x \leq -\rho^2\}$. We note that $\Lambda(S_r) \subsetneq \Lambda(S_q)$ for $0 < q < r \leq 2$. It is well known that the L^p-spectrum of the Laplace–Beltrami operator Δ is $\Lambda(S_p)$ for $1 \leq p \leq \infty$ ([78]). We shall call $\Lambda(S_p)$, the L^p-spectrum for any $0 < p \leq \infty$.

3.1.2 Multiplier operator

Fix $0 < p \leq 2$. Let m be an even C^∞-function defined on $\mathbb{R}$, which (if $p \neq 2$) extends analytically on S_p° and is continuous on ∂S_p. A continuous linear operator $\Theta : \mathcal{C}^p(X) \to \mathcal{C}^p(X)$ given by

$$\widetilde{\Theta f}(\lambda, k) = m(\lambda)\widetilde{f}(\lambda, k), \text{ for all } f \in \mathcal{C}^p(X),\, \lambda \in \mathbb{R},\, k \in K/M$$

will be called a multiplier on $\mathcal{C}^p(X)$ with symbol $m(\lambda)$. The operator Θ commutes with radialization operator R and left translations $\ell_x, x \in G$ and satisfies $\widetilde{\Theta f}(\lambda) = m(\lambda)\widetilde{f}(\lambda, k)$ for every $f \in \mathcal{C}^p(X), \lambda \in S_p$ and $k \in K/M$. See e.g. [22, 62] where multipliers on Schwartz spaces were considered. In our context some examples include the following:

(i) Let $g \in \mathcal{C}^q(G//K)$ for $0 < q \leq p \leq 2$. Then $\Theta_g : f \to f * g$ is a multiplier on $\mathcal{C}^p(X)$ with symbol $\widehat{g}(\lambda)$.

(ii) Let $g \in L^q(G//K)$ for $1 \leq q < p \leq 2$. Then $\Theta_g : f \to f * g$ is a multiplier on $\mathcal{C}^p(X)$ with symbol $\widehat{g}(\lambda)$.

(iii) For any polynomial P, $P(\Delta)$, spherical mean value operator M_t and volume mean value operator B_t are such multipliers on $\mathcal{C}^p(X)$ for $0 < p \leq 2$ with symbols $P(-(\lambda^2 + \rho^2))$, $\varphi_\lambda(a_t)$ and $\psi_\lambda(t)$ respectively.

(iv) The heat propagator $e^{t\Delta}$ given by convolution with heat kernel h_t is a multiplier having symbol $e^{-t(\lambda^2+\rho^2)}$. This is a particular case of (i) above as $h_t \in \mathcal{C}^p(G//K)$ for all $p \in (0, 2]$.

(v) Heat kernel h_z for complex time $z \in \mathbb{C}$ with $\Re z \geq 0$ defines multiplier $\Theta : \mathcal{C}^2(X) \to \mathcal{C}^2(X)$ given by $f \mapsto f * h_z$ and symbol $e^{-z(\lambda^2+\rho^2)}$.

For (i) it is enough to recall that $\mathcal{C}^p(G)$ is a convolution algebra and $\mathcal{C}^q(X) \subseteq \mathcal{C}^p(X)$ for $0 < q \leq p$. For (ii) we note that $\widehat{g}$ extends to an analytic function on $S_q^\circ \supset S_p$ and $\widehat{g} \to 0$ uniformly on S_{q_1} for any $q < q_1 < p$. Hence by Cauchy's integral formula, its derivatives are also uniformly bounded on S_{q_2} for any $q < q_1 < q_2 < p$. In (iii) for $P(\Delta)$ it follows from the definition of $\mathcal{C}^p(X)$. For M_t we can use its symbol $\varphi_\lambda(a_t)$, estimates (1.2.10) of its derivatives in λ and isomorphism of Schwartz spaces ([32]). For B_t whose symbol is $\psi_\lambda(t)$, the proof is similar. The following proposition gives a straightforward proof of this.

Proposition 3.1.1. *Let $0 < p \leq 2$ and $f \in \mathcal{C}^p(X)$. Then for any $t > 0$, $M_t f, B_t f \in \mathcal{C}^p(X)$.*

Proof. We note the following inequalities (see [39, Proposition 4.6.11 (iv)]):

$$(1 + |x|)/(1 + |xka_t|) \leq 1 + |t|, \quad (1 + |xka_t|)/(1 + |x|) \leq 1 + |t|.$$

Consider a seminorm $\gamma_{r,D}$ of $\mathcal{C}^p(X)$, for $r \in \mathbb{N}$ and $D \in \mathcal{U}(\mathfrak{g})$:

$$\gamma_{r,D}(f) := \sup_{x \in G} (1 + |x|)^r \varphi_0(x)^{-\frac{2}{p}} |Df(x)|.$$

Then using triangle inequality, the inequalities above and (1.2.9) we have,

$$\gamma_{r,D}(M_t f) \quad = \quad \int_K (1 + |x|)^r \varphi_0(x)^{-\frac{2}{p}} Df(xka_t)\, dk$$

$$
\begin{aligned}
&= \int_K \left(\frac{1+|x|}{1+|xka_t|}\right)^r \left(\frac{\varphi_0(x)}{\varphi_0(xka_t)}\right)^{-\frac{2}{p}} (1+|xka_t|)^r \varphi_0(xka_t)^{-\frac{2}{p}} Df(xka_t)\, dk \\
&\leq \gamma_{r,D}(f)(1+|t|)^r \int_K \left(\frac{\varphi_0(xka_t)}{\varphi_0(x)}\right)^{\frac{2}{p}} dk \\
&\leq C\gamma_{r,D}(f)(1+|t|)^r \int_K \left(\frac{(1+|xka_t|)e^{-\rho|xka_t|}}{(1+|x|)e^{-\rho|x|}}\right)^{\frac{2}{p}} dk \\
&\leq C\gamma_{r,D}(f)(1+|t|)^{\frac{2}{p}+r} \int_K e^{\frac{2\rho}{p}(|x|-|xka_t|)}\, dk \\
&\leq C\gamma_{r,D}(f)(1+|t|)^{\frac{2}{p}+r} e^{\frac{2\rho}{p}t} \\
&= C_t \gamma_{r,D}(f).
\end{aligned}
$$

This proves that $M_t f \in \mathcal{C}^p(X)$. Since, $B_t f(x) = \frac{1}{|B(0,t)|}\int_0^t (M_r f)(x) J(r)\, dr$, it is also clear that $B_t f \in \mathcal{C}^p(X)$. $\square$

The action of a multiplier Θ on $\mathcal{C}^p(X)$ extends naturally to the L^p-tempered distributions. For $T \in \mathcal{C}^p(X)'$, ΘT is an L^p-tempered distribution defined by

$$(\Theta T)(u) = T(\Theta u) \text{ for } u \in \mathcal{C}^p(X).$$

3.1.3 One radius theorem

The following result characterizes eigenfunctions of Δ through the generalized mean value theorem (see [42, p. 76, Prop 2.6; p. 414, Cor 2.3]).

Proposition 3.1.2. *Let f be a continuous function on X and $\lambda \in \mathbb{C}$. Then f satisfies $\Delta f = -(\lambda^2+\rho^2)f$ if and only if $M_t f(x) = f(x)\varphi_\lambda(a_t)$ for all $x \in X$ and all $t > 0$.*

An analogue of this result for ball-averages, which (in our notation) is obtained by substituting M_t by B_t and $\varphi_\lambda(a_t)$ by $\psi_\lambda(t)$ in the statement above, is also true. Recall that in Chapter 2, we have seen that a continuous function satisfying the generalized mean value property for a single radius is not necessarily an eigenfunction of Δ. However for functions with suitable growth conditions or for distributions with appropriate temperedness and for real eigenvalues in $[-\rho^2, \infty)$, it might be possible to characterize eigenfunctions/distributions through the mean value property using only one radius. We shall now state and prove such a *one radius theorem.* This theorem is structurally close to Theorem 2.2.10 in the previous chapter. We have however placed it here as it will be used for the main theorems of this chapter.

Theorem 3.1.3. *Let $t > 0$ be fixed and $p \in (0, 2]$. Let T be an L^p-tempered distribution on X such that $M_tT = \varphi_{i\gamma_p\rho}(a_t)T$ or $B_tT = \psi_{i\gamma_p\rho}(t)T$. Then T is an eigendistribution of Δ with eigenvalue $-\frac{4\rho^2}{pp'}$.*

Proof. Let δ_o be the Dirac mass at the origin $o = eK$. We recall that σ_t denotes the surface measure of the geodesic sphere of radius t. As $M_tT = T * \sigma_t$, the hypothesis $M_tT = \varphi_{i\gamma_p\rho}(a_t)T$ implies that $T * \nu = 0$ where $\nu = \sigma_t - \varphi_{i\gamma_p\rho}(a_t)\delta_o$.

First we shall assume that T is radial and $p \in (0, 2)$. Let $\psi_1(\lambda) = \widehat{\nu}(\lambda) = \varphi_\lambda(a_t) - \varphi_{i\gamma_p\rho}(a_t)$ for $\lambda \in \mathbb{C}$. We claim that it is possible to choose a $\delta > 0$ such that $\psi_1(\lambda)$ does not vanish on

$$S_{p,\delta} = \{\lambda \in \mathbb{C} \mid |\Im\lambda| \le (\gamma_p + \delta)\rho\}.$$

except when $\lambda = \pm i\gamma_p\rho$. From Proposition 1.4.2 it follows that if $\lambda \in S_p$ and $\lambda \neq \pm i\gamma_p\rho$, then $\varphi_\lambda(a_t) \neq \varphi_{i\gamma_p\rho}(a_t)$. Fix a $\delta_1 > 0$. By Proposition 1.4.3, there exists a $N > 0$ such that $|\varphi_\lambda(a_t)| < \frac{\varphi_{i\gamma_p\rho}(a_t)}{2}$ whenever $\lambda \in S_{p,\delta_1}$ and $|\Re\lambda| > N$. As the set $\{\lambda \mid \lambda \in S_{p,\delta_1},\ |\Re\lambda| \le N\}$ is compact and $\lambda \mapsto \varphi_\lambda(a_t)$ is analytic, the set $Z = \{\lambda \in S_{p,\delta_1} \mid \lambda \neq \pm i\gamma_p\rho, |\Re\lambda| \le N,\ \varphi_\lambda(a_t) = \varphi_{i\gamma_p\rho}(a_t)\}$ is finite. If Z is empty, take $\delta = \delta_1$, otherwise any $\delta < \frac{d}{\rho} - \gamma_p$ will serve our purpose if $d = \inf_{\lambda \in Z} |\Im\lambda|$ as it is evident that only zeros of ψ_1 in $S_{p,\delta}$ are $\pm i\gamma_p\rho$. As $\varphi_{i\gamma_p\rho,1}(a_t) \neq 0$ (Proposition 1.4.4), it follows that ψ_1 has simple roots at $\pm i\gamma_p\rho$.

Let $\psi_2(\lambda) = e^{-(\lambda^2+\rho^2)}\psi_1(\lambda)$ for $\lambda \in S_p$. Then $\psi_2 \in \mathcal{C}^p(\widehat{G//K})$. Let $g \in \mathcal{C}^p(G//K)$ such that $\widehat{g} = \psi_2$. Applying Proposition 1.5.2 we get $\{g * h \mid h \in \mathcal{C}^p(G//K)\}$ is dense in the space of all functions in $\mathcal{C}^p(G//K)$ whose Fourier transform vanishes at $\pm i\gamma_p\rho$. Since $\langle T, g * h\rangle = 0$, we have, $\langle T, \phi\rangle = 0$ for all $\phi \in \mathcal{C}^p(G//K)$ with $\widehat{\phi}(i\gamma_p\rho) = \widehat{\phi}(-i\gamma_p\rho) = 0$. But $\varphi_{i\gamma_p\rho}(x)$ is also a radial L^p-tempered distribution which annihilates all $\phi \in \mathcal{C}^p(G//K)$ whenever $\widehat{\phi}(i\gamma_p\rho) = 0$. Therefore, $T = \beta\varphi_{i\gamma_p\rho}$ (see [68, Lemma 3.9]) for some constant β. In particular, T is an eigendistribution of Δ with eigenvalue $-\frac{4\rho^2}{pp'}$.

Now we shall deal with the case $p = 2$ and T is radial. Applying Lemma 3.3.1, we will get that Supp $\widehat{T} \subset \{0\}$(where each $T_k = T$). Hence using Lemma 3.3.6 along with the fact that $\varphi_{0,k} \equiv 0$ for k odd, we get $T = \sum_{k=0}^{N} a_k\varphi_{0,2k}$ for some constants $a_0, a_1, \cdots, a_N$ with $a_N \neq 0$. We claim that $N = 0$. To establish the claim we note that,

$$M_t\varphi_{0,2k}(x) = \left.\frac{\partial^{2k}}{\partial^{2k}\lambda}\right|_{\lambda=0} (\varphi_\lambda * \sigma_t(x)) = \left.\frac{\partial^{2k}}{\partial^{2k}\lambda}\right|_{\lambda=0} (\varphi_\lambda(a_t)\varphi_\lambda(x))$$

$$= \sum_{i=0}^{2k} \binom{2k}{i} \varphi_{0,i}(a_t)\varphi_{0,2k-i}(x) = \sum_{i=0}^{k} \binom{2k}{2i} \varphi_{0,2i}(a_t)\varphi_{0,2(k-i)}(x).$$

If $N \geq 1$ and T satisfies $M_t T = \varphi_0(a_t)T$, comparing the coefficient of $\varphi_{0,2N-2}$ in both sides we get $N(2N-1)a_N\varphi_{0,2}(a_t) = 0$. But since $\varphi_{0,2}(a_t) \neq 0$ (see Proposition 1.4.4), we get $a_N = 0$ which is a contradiction. Hence $T = a_0\varphi_0$ and $\Delta T = -\rho^2 T$. This completes proof of the theorem for radial distributions.

Now we withdraw the assumption of radiality of T. This part of the argument is essentially same as that of Theorem 2.2.2. First we note that if for all $x \in G$, $R(\ell_x T_1) = 0$ for $T_1 \in \mathcal{C}^p(X)'$, then T_1 is zero as distribution. Indeed, we take a $h \in C_c^\infty(G//K)$ supported on a neighborhood of the origin $o = eK$. If $R(\ell_x T_1) = 0$ for all x, then $T_1 * h(x) = \langle R(\ell_x T_1), h\rangle = 0$. As we know that we can approximate T_1 by distributions of the form $T_1 * h$ in the topology of $\mathcal{C}^p(X)'$, we conclude that $T_1 = 0$. This proves the claim. The hypothesis $T * \sigma_t = \varphi_{i\gamma_p\rho}(a_t)T$ implies that

$$R(\ell_x T) * \sigma_t = R(\ell_x T * \sigma_t) = R(\ell_x(T * \sigma_t)) = \varphi_{i\gamma_p\rho}(a_t)R(\ell_x T).$$

Hence $R(\ell_x T)$ satisfies the hypothesis of the theorem. Therefore by the result proved above for radial distributions, $\Delta(R(\ell_x T)) = -\frac{4\rho^2}{pp'}R(\ell_x T)$ for all $x \in G$. This implies that $R(\ell_x(\Delta T)) = R(\ell_x(-\frac{4\rho^2}{pp'}T))$ for all $x \in G$. That is $R(\ell_x(\Delta T + \frac{4\rho^2}{pp'}T)) = 0$ for all $x \in G$. As noted above this implies that $\Delta T = -\frac{4\rho^2}{pp'}T$. Thus the proof of the theorem for M_t is complete. For B_t, replace $\varphi_\lambda(a_t), \varphi_{i\gamma_p\rho}(a_t)$ by $\psi_\lambda(t), \psi_{i\gamma_p\rho}(t)$ respectively. □

We have an analogue of the theorem above for the heat operator which we state here. It will be proved in Chapter 5, as part of a more general result.

Theorem 3.1.4. *Let $t > 0$ be fixed.*

(i) *Let T be an L^2-tempered distribution on X such that $e^{t\Delta}T = e^{-t(\lambda^2+\rho^2)}T$ for some $\lambda \in \mathbb{R}$. Then T is an eigendistribution of Δ with eigenvalue $-(\lambda^2 + \rho^2)$.*

(ii) *Let $p \in (0, 2)$ and T be an L^p-tempered distribution on X such that $e^{t\Delta}T = e^{-\frac{4\rho^2 t}{pp'}}T$. Then T is an eigendistribution of Δ with eigenvalue $-\frac{4\rho^2}{pp'}$.*

Proof. For (i) see Theorem 5.4.5 and for (ii) see Theorem 5.4.2. □

Remark 3.1.5. In the theorems above, if we substitute the distribution T by functions f with suitable decay, then clearly f can be characterized as a Poisson transform of an appropriate object on K/M. (See Theorem 1.7.2, Corollary 1.7.4, Proposition 1.5.1 and Proposition 1.6.1).

3.2 Characterization of L^p-tempered eigendistributions

In this section we consider possible analogue of the Roe–Strichartz theorem (see Theorem 0.2.3), for spherical (and ball) averages and the heat operator. We begin with a couple of preparatory lemmas. The first one is essentially proved in [44] (see also [52]) where the multiplier was the standard Laplacian of $\mathbb{R}^n$ acting on $L^\infty(\mathbb{R}^n)$. To make it applicable in a wider context we shall rewrite it with suitable modifications for a general multiplier $\Theta : \mathcal{C}^p(X) \to \mathcal{C}^p(X)$.

Lemma 3.2.1. *Fix $0 < p \leq 2$. Let $\Theta : \mathcal{C}^p(X) \to \mathcal{C}^p(X)$ be a multiplier and $\{T_k\}_{k\in\mathbb{Z}^+}$ be an infinite sequence of radial L^p-tempered distributions. Suppose that for all $k \in \mathbb{Z}^+$, $\Theta T_k = AT_{k+1}$ for a nonzero constant $A \in \mathbb{C}$ and $|\langle T_k, \psi\rangle| \leq M\gamma(\psi)$ for a fixed seminorm γ of $\mathcal{C}^p(X)$ and a constant $M > 0$. If $(\Theta - B)^{N+1}T_0 = 0$ for some $B \in \mathbb{C}$ with $|B| = |A|$ and $N \in \mathbb{N}$, then $\Theta T_0 = BT_0$.*

Proof. Since $(\Theta - B)^{N+1}T_0 = 0$, we have

$$\mathrm{Span}\{T_0, T_1, \cdots\} = \mathrm{Span}\{T_0, \Theta T_0, \cdots, \Theta^N T_0\} = \mathrm{Span}\{T_0, T_1, \cdots, T_N\}.$$

Suppose that $(\Theta - B)T_0 \neq 0$. Let k_0 be the largest positive integer such that $(\Theta - B)^{k_0}T_0 \neq 0$. Then $k_0 \leq N$.

Let $T = (\Theta - B)^{k_0-1}T_0$. Then $T \in$ Span $\{T_0, T_1, \cdots, T_N\}$. We assume that

$$T = a_0T_0 + \cdots + a_NT_N.$$

Then

$$\begin{aligned}(\Theta - B)^2T = (\Theta - B)^{k_0+1}T_0 = 0 \text{ and}\\ (\Theta - B)T = (\Theta - B)^{k_0}T_0 \neq 0.\end{aligned} \tag{3.2.1}$$

Using binomial expansion and (3.2.1) we get for any positive integer k,

$$\begin{aligned}\Theta^kT &= ((\Theta - B) + B)^kT\\ &= kB^{k-1}(\Theta - B)T + B^kT.\end{aligned}$$

This implies for any $\psi \in \mathcal{C}^p(G//K)$,

$$|\langle(\Theta - B)T, \psi\rangle| \leq \frac{1}{k}|A|^{1-k}|\langle\Theta^kT, \psi\rangle| + \frac{1}{k}|A||\langle T, \psi\rangle|. \tag{3.2.2}$$

Since,

$$\begin{aligned}|\langle \Theta^k T, \psi\rangle| &= |\langle \Theta^k (a_0T_0 + a_1T_1 + \cdots + a_NT_N), \psi\rangle| \\ &= |a_0A^k\langle T_k, \psi\rangle + \cdots + a_NA^k\langle T_{N+k}, \psi\rangle| \\ &\le |A|^k|a_0\langle T_k, \psi\rangle| + \cdots + |a_N\langle T_{N+k}, \psi\rangle| \\ &\le M|A|^k(|a_0| + \cdots + |a_N|)\gamma(\psi),\end{aligned}$$

it follows from (3.2.2) that,

$$|\langle (\Theta - B)T, \psi\rangle| \le M\frac{|A|}{k}(|a_0| + \cdots + |a_N|)\gamma(\psi) + \frac{|A|}{k}|\langle T, \psi\rangle|.$$

The right hand side of the inequality above goes to 0 as $k \to \infty$. Hence by (3.2.1), $(\Theta - B)^{k_0}T_0 = 0$ which contradicts the assumption on k_0. Therefore $N = 0$, i.e., $(\Theta - B)T_0 = 0$. □

Lemma 3.2.2. *Let $t > 0$ and $0 < p < 2$ be fixed.*

(i) *For $A \in \mathbb{C}$ with $|A| < \varphi_{i\gamma_p\rho}(a_t)$, there exists infinitely many $\lambda \in S_p$ with $|\varphi_\lambda(a_t)| = |A|$. Further if $A \neq 0$, then there are distinct $\lambda_1, \lambda_2 \in S_p$ such that $|\varphi_{\lambda_1}(a_t)| = |A| = |\varphi_{\lambda_2}(a_t)|$ and $\varphi_{\lambda_1}(a_t) \neq \varphi_{\lambda_2}(a_t)$.*

(ii) *Exact analogue of* (i) *holds true when $\varphi_{i\gamma_p\rho}(a_t), \varphi_\lambda(a_t)$ are replaced respectively by $\psi_{i\gamma_p\rho}(t), \psi_\lambda(t)$.*

Proof. We will only prove (i). Proof of (ii) will be similar. Let $A \neq 0$. Then as $|A| < \varphi_{i\gamma_p\rho}(a_t)$, there exists p_1 with $0 < p < p_1 \le 2$ with $\varphi_{i\gamma_{p_1}\rho}(a_t) \ge |A|$. Since $\varphi_\lambda(a_t) \to 0$ as $|\lambda| \to \infty$ in S_p (see Proposition 1.4.3), for each fixed q with $p < q \le p_1$, we easily obtain a $\lambda \in \mathbb{C}$ with $|\Im\lambda| = \gamma_q\rho$ and $|\varphi_\lambda(a_t)| = |A|$. Since cardinality of such λ is uncountable and zeros of analytic functions are isolated, one can choose λ_1 and λ_2 with $\varphi_{\lambda_1}(a_t) \neq \varphi_{\lambda_2}(a_t)$ and $|\varphi_{\lambda_1}(a_t)| = |\varphi_{\lambda_2}(a_t)| = |A|$. It follows from the explicit expression of Jacobi function that $\varphi_\lambda(a_t) = 0$ for infinitely many real λ (see e.g. [79, page 235, Proposition 2.2]). This takes care of the case $A = 0$. □

The first main result of this section is the following.

Theorem 3.2.3. *Fix $t > 0$. For $0 < p < 2$, let $\{T_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of L^p-tempered distributions on X satisfying for $k \in \mathbb{Z}$,*

(i) *$M_tT_k = AT_{k+1}$ for some $A \in \mathbb{C}$ and*

(ii) *$|\langle T_k, \psi\rangle| \le M\gamma(\psi)$ for all $\psi \in \mathcal{C}^p(X)$, for some fixed seminorm γ of $\mathcal{C}^p(X)$ and $M > 0$. Then we have the following conclusions.*

(a) *If* $|A| = \varphi_{i\gamma_p\rho}(a_t)$*, then* $\Delta T_0 = -\frac{4\rho^2}{pp'}T_0$*.*

(b) *If* $|A| > \varphi_{i\gamma_p\rho}(a_t)$*, then* $T_0 = 0$*.*

(c) *If* $|A| < \varphi_{i\gamma_p\rho}(a_t)$*, then* T_0 *may not be an eigendistribution of* Δ*. If* A *is also assumed to be nonzero, then* T_0 *may not be an eigendistribution of* M_t*.*

Proof. (a) Let us first assume that T_k are radial. Then from hypothesis it follows that $M_t^k T_{-k} = A^k T_0$ for all $k > 0$ and hence $A^k \widehat{T_0} = \varphi_\lambda(a_t)^k \widehat{T_{-k}}$. Therefore for $\varphi \in \mathcal{C}^p(\widehat{G//K})$ and $N \in \mathbb{N}$ we have,

$$\begin{aligned}
& |\langle(\varphi_\lambda(a_t) - \varphi_{i\gamma_p\rho}(a_t))^{N+1}\widehat{T_0}, \varphi\rangle| \\
= & |\langle\widehat{T_0}, (\varphi_\lambda(a_t) - \varphi_{i\gamma_p\rho}(a_t))^{N+1}\varphi\rangle| \\
= & |\langle\widehat{T_{-k}}, \left(\frac{\varphi_\lambda(a_t)}{A}\right)^k (\varphi_\lambda(a_t) - \varphi_{i\gamma_p\rho}(a_t))^{N+1}\varphi\rangle| \\
= & \left|\left\langle T_{-k}, \left(\left(\frac{\varphi_\lambda(a_t)}{A}\right)^k (\varphi_\lambda(a_t) - \varphi_{i\gamma_p\rho}(a_t))^{N+1}\varphi\right)^\vee\right\rangle\right| \\
\leq & C\gamma\left[\left(\left(\frac{\varphi_\lambda(a_t)}{A}\right)^k (\varphi_\lambda(a_t) - \varphi_{i\gamma_p\rho}(a_t))^{N+1}\varphi\right)^\vee\right] \\
\leq & C'\,\mu\left[\left(\frac{\varphi_\lambda(a_t)}{A}\right)^k (\varphi_\lambda(a_t) - \varphi_{i\gamma_p\rho}(a_t))^{N+1}\varphi\right], \qquad (3.2.3)
\end{aligned}$$

where for any $\phi \in \mathcal{C}^p(\widehat{G//K})$, $\phi^\vee \in \mathcal{C}^p(G//K)$ is its image under the Fourier inversion and the seminorm μ is given by

$$\mu(\phi) = \sup_{\lambda \in S_p}\left|\frac{d^\tau}{d\lambda^\tau}P(\lambda)\phi(\lambda)\right|,$$

for some even polynomial $P(\lambda)$ and derivative of even order τ. We shall first show that for $N = 17\tau + 7$, $(M_t - \varphi_{i\gamma_p\rho}(a_t))^{N+1}T_0 = 0$, equivalently, $\langle(\varphi_\lambda(a_t) - \varphi_{i\gamma_p\rho}(a_t))^{N+1}\widehat{T_0}, \varphi\rangle = 0$ for $\varphi \in \mathcal{C}^p(\widehat{G//K})$. In view of (3.2.3), it suffices to show that $\sup_{\lambda\in S_p} F^k(\lambda) \to 0$ as $k \to \infty$ where

$$F^k(\lambda) = \left|\frac{d^\tau}{d\lambda^\tau}P(\lambda)\left(\frac{\varphi_\lambda(a_t)}{A}\right)^k (\varphi_\lambda(a_t) - \varphi_{i\gamma_p\rho}(a_t))^{N+1}\varphi\right|.$$

Note that,

$$\frac{d^\tau}{d\lambda^\tau}\left(P(\lambda)\left(\frac{\varphi_\lambda(a_t)}{A}\right)^k(\varphi_\lambda(a_t)-\varphi_{i\gamma_p\rho}(a_t))^{N+1}\varphi\right) \tag{3.2.4}$$
$$=\sum_{i+j+l=\tau} C_{ijl}\frac{d^i}{d\lambda^i}\left(\frac{\varphi_\lambda(a_t)}{A}\right)^k\frac{d^j}{d\lambda^j}(\varphi_\lambda(a_t)-\varphi_{i\gamma_p\rho}(a_t))^{N+1}\frac{d^l}{d\lambda^l}(P(\lambda)\varphi)$$

Using the estimates of the derivatives of φ_λ (see (1.2.10)) along with the facts that $|\varphi_\lambda(a_t)/A| = |\varphi_\lambda(a_t)/\varphi_{i\gamma_p\rho}(a_t)| \leq 1$ and $\varphi \in \mathcal{C}^p(\widehat{G//K})$ we get for $\lambda \in S_p$,

$$F^k(\lambda) \leq C_1 k^\tau \left(\frac{\varphi_\lambda(a_t)}{\varphi_{i\gamma_p\rho}(a_t)}\right)^{k-\tau}\left|\frac{\varphi_\lambda(a_t)}{\varphi_{i\gamma_p\rho}(a_t)}-1\right|^{N+1-\tau} \tag{3.2.5}$$

for some constant C_1.

Since $\varphi_\lambda(a_t) \to 0$ uniformly in S_p as $|\lambda| \to \infty$ (Proposition 1.4.3(a)), we can find a compact connected neighborhood V of $i\gamma_p\rho$ in S_p such that if $\lambda \notin V$, then $|\varphi_\lambda(a_t)| < \varphi_{i\gamma_p\rho}(a_t)/2$. From (3.2.5), it is clear that $F^k \to 0$ as $k \to \infty$ uniformly on $S_p \setminus V$.

We need to show that

$$\sup_{\lambda\in V} k^\tau\left(\frac{\varphi_\lambda(a_t)}{\varphi_{i\gamma_p\rho}(a_t)}\right)^{k-\tau}\left|\frac{\varphi_\lambda(a_t)}{\varphi_{i\gamma_p\rho}(a_t)}-1\right|^{N+1-\tau} \to 0. \tag{3.2.6}$$

Let $\mathbb{D}$ be the open unit disk. Clearly, we can cover $\mathbb{D}\cup\{1\}$ by $\{\mathbb{D}_s\cup\{1\} \mid 0 < s < 1\}$ where $\mathbb{D}_s$ is a one-parameter family of open disks of radius s bounded by the circle $C_s : (x-(1-s))^2 + y^2 = s^2$ (see Figure 3.1(a)). It is clear that, if $s < s'$, then $\mathbb{D}_s \subset \mathbb{D}_{s'}$. Since V is compact and connected and since (see Proposition 1.4.2) $|\varphi_\lambda(a_t)/\varphi_{i\gamma_p\rho}(a_t)| < 1$ for $\lambda \in S_p, \lambda \neq \pm i\gamma_p\rho$, the image of V under the map $\lambda \mapsto \varphi_\lambda(a_t)/\varphi_{i\gamma_p\rho}(a_t)$, is a connected set contained inside the set $\mathbb{D}_s \cup \{1\}$ for some $0 < s < 1$. Without loss of generality we may assume $0 \in \mathbb{D}_s$. In view of (3.2.6), it suffices to show that $k^\tau z^{k-\tau}|z-1|^{N+1-\tau} \to 0$ uniformly as $k \to \infty$ on $\mathbb{D}_s \cup \{1\}$, which we shall take up now.

Let p_k and q_k be two points on C_s given by

$$p_k = (1-s) + se^{i\delta_k} \text{ and } q_k = (1-s) + se^{-i\delta_k},$$

for some $0 < \delta_k < \pi/2$ so that $s - s\cos\delta_k = k^{-1/4}$. Let V_k be the intersection of $\mathbb{D}_s \cup \{1\}$ with the minor circular segment of width $k^{-1/4}$ joining the points p_k, q_k

and 1 (as shown in Figure 3.1(b)). Precisely,

$$V_k = \{z \in \mathbb{D}_s \mid |1 - \Re(z)| < k^{-1/4}\} \cup \{1\}.$$

Note that,

$$|p_k|^2 = 1 - (2s - 2s^2)(1 - \cos(\delta_k)) = 1 - 2(1-s)k^{-1/4} \tag{3.2.7}$$

and

$$|p_k - 1|^2 = |1 - ((1-s) + se^{i\delta_k}))|^2 = 2sk^{-1/4}. \tag{3.2.8}$$

For $z \in \mathbb{D}_s \setminus V_k$, $|z| \leq |p_k|$ and hence for some constant C',

$$\begin{aligned} k^\tau |z|^{k-\tau} |z-1|^{N+1-\tau} &\leq C' k^\tau |p_k|^{k-\tau} \\ &= C' k^\tau (1 - 2(1-s)k^{-1/4})^{\frac{k-\tau}{2}}. \end{aligned} \tag{3.2.9}$$

For $z \in V_k$, $|z-1| \leq |p_k - 1|$ and therefore for some constants C' and C'',

$$\begin{aligned} k^\tau |z|^{k-\tau} |z-1|^{N+1-\tau} &\leq C' k^\tau |p_k - 1|^{N+1-\tau} \\ &= C' k^\tau (2sk^{-1/4})^{\frac{17\tau+7+1-\tau}{2}} \\ &\leq C'' k^{-(\tau+1)}. \end{aligned} \tag{3.2.10}$$

It is now immediate from (3.2.9) and (3.2.10) that as $k \to \infty$,

$$\sup_{\mathbb{D}_s \cup \{1\}} k^\tau |z|^{k-\tau} |z-1|^{N+1-\tau} \to 0.$$

Thus we have $(\varphi_\lambda(a_t) - \varphi_{i\gamma_p\rho}(a_t))^{N+1}\widehat{T_0} = 0$, hence $(M_t - \varphi_{i\gamma_p\rho}(a_t))^{N+1}T_0 = 0$. From this and Lemma 3.2.1, we get that $M_t T_0 = \varphi_{i\gamma_p\rho}(a_t)T_0$.

Now we shall extend the result to the case when T_ks are not necessarily radial. To make this part of the argument applicable in other situation in this chapter, we shall write M_t as Θ and consider it as a multiplier from $\mathcal{C}^p(X)$ to itself (see Subsection 3.1.2) for $0 < p \leq 2$ with symbol $m(\lambda) = \varphi_\lambda(a_t)$. Thus we have established above that $\Theta T_0 = m(i\gamma_p\rho)T_0$ when T_k are radial. Coming to the case of general T_k, we shall show that the condition in the hypothesis on the sequence $\{T_k\}$ implies that for any $y \in G$, the sequence $\{R\ell_y T_k\}$ of radial distributions also satisfies the hypothesis. Since Θ commutes with radialization and translations, it follows from the hypothesis that $\Theta T_k = AT_{k+1}$ that $\Theta R(\ell_y T_k) = AR(\ell_y T_{k+1})$. It remains to show that for

(a) (b)

Figure 3.1:

the seminorm γ of $\mathcal{C}^p(X)$ in the hypothesis of the theorem and $\psi_1 \in \mathcal{C}^p(G//K)$, $|\langle R(\ell_y T_k), \psi_1\rangle| \leq C_y M \gamma(\psi_1)$. Using (1.2.9) along with the fact $|xy| \leq |x| + |y|$ ([39, Proposition 4.6.11]) we have for any $\psi \in \mathcal{C}^p(X)$,

$$\begin{aligned}
\gamma(\ell_y \psi) &= \sup_{x \in X} |D\psi(y^{-1}x)| \varphi_0(x)^{-\frac{2}{p}} (1+|x|)^L \\
&= \sup_{x \in X} |D\psi(x)| \varphi_0(yx)^{-\frac{2}{p}} (1+|yx|)^L \\
&\asymp \sup_{x \in X} |D\psi(x)| e^{\frac{2\rho}{p}|yx|} (1+|yx|)^{L-\frac{2}{p}} \\
&\leq e^{\frac{2\rho}{p}|y|} (1+|y|)^{L-\frac{2}{p}} \sup_{x \in X} |D\psi(x)| e^{\frac{2\rho}{p}|x|} (1+|x|)^{L-\frac{2}{p}} \\
&\asymp e^{\frac{2\rho}{p}|y|} (1+|y|)^{L-\frac{2}{p}} \sup_{x \in X} |D\psi(x)| \varphi_0(x)^{-\frac{2}{p}} (1+|x|)^L \\
&= C_y \gamma(\psi),
\end{aligned}$$

where the constant C_y depends only on $y \in G$. Since $|\langle T_k, \psi\rangle| \leq M\gamma(\psi)$ for any $\psi \in \mathcal{C}^p(X)$, it follows that for $\psi_1 \in \mathcal{C}^p(G//K)$,

$$|\langle R(\ell_y T_k), \psi_1\rangle| = |\langle \ell_y T_k, \psi_1\rangle| = |\langle T_k, \ell_{y^{-1}} \psi_1\rangle| \leq M\gamma(\ell_{y^{-1}}\psi_1) \leq C_{y^{-1}} M \gamma(\psi_1).$$

Therefore from the result proved for radial distributions we conclude that

$$\Theta R(\ell_y(T_0)) = m(i\gamma_p \rho) R(\ell_y(T_0)) \text{ for all } y \in G.$$

Now appealing again to the fact that Θ commutes with translations and radialization we have $R(\ell_y(\Theta T_0)) = R(\ell_y(m(i\gamma_p\rho)T_0))$ for all $y \in G$. This implies

$\Theta T_0 = m(i\gamma_p\rho)T_0$ which was to be proved. In the last step we have used the fact that if for some $T \in \mathcal{C}^p(X)'$, $R(\ell_x T) = 0$ for all $x \in G$, then $T = 0$, which was noted in the proof of Theorem 3.1.3. Thus we have $M_t T_0 = \varphi_{i\gamma_p\rho}(a_t)T_0$. An application of Theorem 3.1.3 yields $\Delta T_0 = -\frac{4\rho^2}{pp'}T_0$.

(b) In view of (a), it suffices to prove the result in radial setup. For $\varphi \in \mathcal{C}^p(\widehat{G//K})$, we have

$$\begin{aligned}
|\langle \widehat{T_0}, \varphi\rangle| &= |\langle \widehat{T_{-k}}, \left(\frac{\varphi_\lambda(a_t)}{A}\right)^k \varphi\rangle| \\
&= \left|\left\langle T_{-k}, \left(\left(\frac{\varphi_\lambda(a_t)}{A}\right)^k \varphi\right)^\vee \right\rangle\right| \\
&\leq C\gamma\left[\left(\left(\frac{\varphi_\lambda(a_t)}{A}\right)^k \varphi\right)^\vee\right] \\
&\leq C'\,\mu\left[\left(\frac{\varphi_\lambda(a_t)}{A}\right)^k \varphi\right].
\end{aligned}$$

Since for $\lambda \in S_p$, $|\varphi_\lambda(a_t)| \leq \varphi_{i\gamma_p\rho}(a_t) < A$, it follows that

$$\mu\left[\left(\frac{\varphi_\lambda(a_t)}{A}\right)^k \varphi\right] \to 0 \text{ as } k \to \infty.$$

Hence $T_0 = 0$.

(c) As $|A| < \varphi_{i\gamma_p\rho}(a_t)$, by Lemma 3.2.2 there exists infinitely many λs in S_p with $|\varphi_\lambda(a_t)| = |A|$. Let λ_1 and λ_2 be two such distinct λs with $\lambda_1 \neq \pm\lambda_2$. Let $\varphi_{\lambda_1}(a_t) = |A|e^{i\theta_1}$ and $\varphi_{\lambda_2}(a_t) = |A|e^{i\theta_2}$. Let $T_k = e^{ik\theta_1}\varphi_{\lambda_1} + e^{ik\theta_2}\varphi_{\lambda_2}$ for $k \in \mathbb{Z}$. Then $M_t T_k = e^{ik\theta_1}M_t\varphi_{\lambda_1} + e^{ik\theta_2}M_t\varphi_{\lambda_2} = |A|T_{k+1}$. Clearly the sequence $\{T_k\}_{k\in\mathbb{Z}}$ satisfies hypothesis of the theorem (see Propositions 1.4.1 and 1.5.1) but T_0 is not an eigendistribution of Δ. If $A \neq 0$, in the above example choose λ_1 and λ_2 such that $\varphi_{\lambda_1}(a_t) \neq \varphi_{\lambda_2}(a_t)$ and $|\varphi_{\lambda_1}(a_t)| = |\varphi_{\lambda_2}(a_t)| = |A|$ (see Lemma 3.2.2(i)). □

Remark 3.2.4. In Theorem 3.2.3 if we substitute sphere average M_t by ball average B_t and $\varphi_{i\gamma_p\rho}(t)$ by $\psi_{i\gamma_p\rho}(t)$, we get the corresponding statement for ball average. To prove this we only need a step by step adaptation of the proof of Theorem 3.2.3. We omit this for brevity.

We now consider an analogous result for the heat operator.

Theorem 3.2.5. *Fix $t > 0$. For $0 < p < 2$, let $\{T_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of L^p-tempered distributions on X satisfying for all $k \in \mathbb{Z}$,*

(i) $e^{t\Delta}T_k = AT_{k+1}$ *for some $A \in \mathbb{C}$ and*

(ii) $|\langle T_k, \psi\rangle| \leq M\gamma(\psi)$ *for all $\psi \in \mathcal{C}^p(X)$, for some fixed seminorm γ of $\mathcal{C}^p(X)$ and $M > 0$. Then we have the following conclusions.*

(a) *If $|A| = e^{-\frac{4\rho^2 t}{pp'}}$, then $\Delta T_0 = -\frac{4\rho^2}{pp'}T_0$.*

(b) *If $|A| > e^{-\frac{4\rho^2 t}{pp'}}$, then $T_0 = 0$.*

(c) *If $0 \neq |A| < e^{-\frac{4\rho^2 t}{pp'}}$, then T_0 may not be an eigendistribution of $e^{t\Delta}$.*

(d) *If $A = 0$, then each $T_k = 0$ for all integer k.*

Proof. (a) Here again only a step by step adaptation of the argument given in the proof of Theorem 3.2.3 is required with the substitutions of $M_t, \varphi_\lambda(a_t), \varphi_{i\gamma_p\rho}(a_t)$ by $e^{t\Delta}, e^{-t(\lambda^2+\rho^2)}$ and $e^{-\frac{4t\rho^2}{pp'}}$ respectively to conclude that

$$e^{t\Delta}T_0 = e^{-\frac{4\rho^2 t}{pp'}}T_0.$$

Applying Theorem 3.1.4, we get the desired result.

(b) Use exactly the same argument as in (b) of Theorem 3.2.3 with suitable modification of appropriate symbols.

(c) As $|A| < e^{-\frac{4\rho^2 t}{pp'}}$, there exists p_1 with $0 < p < p_1 \leq 2$ with $e^{-\frac{4\rho^2 t}{p_1 p_1'}} \geq |A|$. Since $e^{-t(\lambda^2+\rho^2)} \to 0$ as $|\lambda| \to \infty$ for λ in S_p, for each fixed q with $p < q \leq p_1$ we have uncountably many $\lambda \in \mathbb{C}$ with $|\Im\lambda| = \gamma_q\rho$ and $|e^{-t(\lambda^2+\rho^2)}| = |A|$. Since the zeros of analytic functions are isolated one can always choose λ_1 and λ_2 with $e^{-t(\lambda_1^2+\rho^2)} \neq e^{-t(\lambda_2^2+\rho^2)}$ and $|e^{-t(\lambda_1^2+\rho^2)}| = |e^{-t(\lambda_2^2+\rho^2)}| = |A|$. Let $e^{-t(\lambda_1^2+\rho^2)} = |A|e^{i\theta_1}$ and $e^{-t(\lambda_2^2+\rho^2)} = |A|e^{i\theta_2}$. Let $T_k = e^{ik\theta_1}\varphi_{\lambda_1} + e^{ik\theta_2}\varphi_{\lambda_2}$ for $k \in \mathbb{Z}$. Then $e^{t\Delta}T_k = e^{ik\theta_1}e^{t\Delta}\varphi_{\lambda_1} + e^{ik\theta_2}e^{t\Delta}\varphi_{\lambda_2} = |A|T_{k+1}$. Clearly the sequence $\{T_k\}_{k\in\mathbb{Z}}$ satisfies hypothesis of the theorem but T_0 is not an eigendistribution of $e^{t\Delta}$.

(d) It suffices to prove that $T = 0$ whenever $e^{t\Delta}T = 0$ for a L^p tempered distribution T. Passing to $R(\ell_x T)$ for some $x \in G$, if necessary, without loss of generality we may assume that T is radial (see proof of Theorem 3.1.3). If $e^{t\Delta}T = 0$, we have $\langle e^{t\Delta}T, \phi\rangle = 0$ for all $\phi \in \mathcal{C}^p(G//K)$. That is $\langle T, \phi * h_t\rangle = 0$ for all $\phi \in \mathcal{C}^p(G//K)$. Since by Proposition 1.5.2, $\{h_t * \phi \mid \phi \in \mathcal{C}^p(G//K)\}$ is dense in in $\mathcal{C}^p(G//K)$, hence $\langle T, \phi\rangle = 0$ for all $\phi \in \mathcal{C}^p(G//K)$. Therefore $T = 0$. □

Restricting to measurable functions with appropriate growth as particular examples of L^p-tempered distributions, we have following consequences of the two theorems above.

Corollary 3.2.6. *Fix a $t > 0$. Let $\{f_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of measurable functions on X such that for all $k \in \mathbb{Z}$,*

$$M_t f_k = A f_{k+1} \quad (\textit{respectively } B_t f_k = A f_{k+1}, \quad e^{t\Delta} f_k = A f_{k+1}),$$

and

$$\|f_k\|_{p',\infty} \leq M \textit{ for a fixed } p \in [1,2),$$

for some constants $A \in \mathbb{C}$ and $M > 0$. If $|A| = \varphi_{i\gamma_p\rho}(a_t)$ (respectively $|A| = \psi_{i\gamma_p\rho}(t)$, $|A| = e^{-\frac{4\rho^2 t}{pp'}}$), then $\Delta f_0 = -\frac{4\rho^2}{pp'} f_0$ and $f_0 = \mathcal{P}_{-i\gamma_p\rho}F$ for some $F \in L^{p'}(K/M)$.

Corollary 3.2.7. *In the previous corollary if we substitute the condition "$\|f_k\|_{p',\infty} \leq M$ for $p \in [1,2)$" by "$[f_k]_{p,r} \leq M$ for a fixed $p \in (0,2)$ and $r \in [1,\infty]$", keeping the rest of the hypothesis same, then $\Delta f_0 = -\frac{4\rho^2}{pp'} f_0$. Moreover $f_0 = \mathcal{P}_{-i\gamma_p\rho}F$ for some $F \in L^r(K/M)$ if $r > 1$ and $f_0 = \mathcal{P}_{-i\gamma_p\rho}\mu$ for some signed measure μ on K/M if $r = 1$.*

The proofs of these corollaries are evident from the following steps.

(1) Functions f_k in $L^{p',\infty}(X)$ or in $\mathcal{H}_{p,r}(X)$ are L^p-tempered distributions. The uniform norm-boundedness condition i.e. $\|f_k\|_{p',\infty} \leq M$ or $[f_k]_{p,r} \leq M$, implies that

$$|\langle f_k, \psi\rangle| \leq C\gamma(\psi)$$

for a fixed seminorm γ of $\mathcal{C}^p(X)$, for all $\psi \in \mathcal{C}^p(X)$ and for some constant C (see Proposition 1.5.1(a) and Proposition 1.6.1(e)). Thus the hypotheses of Theorem 3.2.3 (along with Remark 3.2.4) and Theorem 3.2.5 are satisfied.

(2) Eigenfunctions of Δ satisfying suitable size estimates can be realized as the Poisson transforms of appropriate objects (functions, measures) on the boundary K/M. See Theorem 1.7.2 and Corollary 1.7.4.

Following the results of Strichartz [76], Howard–Reese [44] on $\mathbb{R}^n$ and of Kumar et. al. [52] on X, we have considered the problem of characterizing eigenfunctions with real eigenvalues. So far we have dealt with eigenfunctions having eigenvalues in $(-\rho^2, \infty)$ and in the next section we shall take up that for eigenvalues in $(-\infty, -\rho^2]$, which are in the L^2-spectrum. It is natural to try to extend the results for complex eigenvalues. It appears that there are intrinsic difficulties in addressing this question. We conclude this section with the following theorem which strives to formulate and prove such a result, characterizing eigenfunctions with arbitrary complex eigenvalues, although only for Δ as the multiplier. This will complement the results in [52].

We recall (see Section 3.1.1) that $\Lambda(S_p)$ is the L^p-spectrum of Δ and in particular $\Lambda(S_2) = (-\infty, -\rho^2]$. For $p \neq 2$, the boundary $\partial\Lambda(S_p)$ of $\Lambda(S_p)$ is a parabola given by (3.1.1). If z is a point in $\mathbb{C} \setminus \Lambda(S_2)$, then simple computation shows that z lies on $\partial\Lambda(S_p)$ for a unique $p \in (0, 2)$, which we shall denote by $p(z)$. If $z \in \Lambda(S_2)$ we define $p(z) = 2$. Precisely, for any $z \in \mathbb{C}$,

$$p(z) = \frac{2\sqrt{2}\rho}{\sqrt{\Re(z+\rho^2) + |z+\rho^2|} + \sqrt{2}\rho}.$$

For $p(z) \neq 2$, we define $N(z)$ to be the set of all points on the outward normal drawn to the parabola $\partial\Lambda(S_{p(z)})$ at the point z as shown in Figure 3.2 (a) (where $z = \Lambda(\alpha)$ and z_0 is a point on the normal). Using these notation we now state the theorem.

Theorem 3.2.8. *Let $\lambda_0 \in \mathbb{C} \setminus (-\infty, -\rho^2]$, $p = p(\lambda_0)$ and $z_0 \in N(\lambda_0)$. Let $\{T_k\}_{k\in\mathbb{Z}^+}$ be an infinite sequence of L^p-tempered distributions on X satisfying for all $k \in \mathbb{Z}^+$,*

(i) $(\Delta - z_0 I)T_k = AT_{k+1}$ *for some nonzero $A \in \mathbb{C}$ and*

(ii) $|\langle T_k, \psi\rangle| \leq M\gamma(\psi)$ *for all $\psi \in \mathcal{C}^p(X)$, for some fixed seminorm γ of $\mathcal{C}^p(X)$ and $M > 0$. Then we have the following conclusions.*

(a) *If $|A| = |\lambda_0 - z_0|$, then $\Delta T_0 = \lambda_0 T_0$.*

(b) *If $|A| < |\lambda_0 - z_0|$, then $T_0 = 0$.*

(c) *If $|A| > |\lambda_0 - z_0|$, then T_0 may not be an eigendistribution of Δ.*

We note that λ_0 is outside the L^2-spectrum and hence $p = p(\lambda_0) \neq 2$. The results involving L^2-spectrum will be considered in the next section. If $-\rho^2 < \lambda_0 < 0$ (hence $1 < p(\lambda_0) < 2$), then z_0 in the statement above can be taken to be the origin 0 in $\mathbb{C}$. In this case with $z_0 = 0$, the theorem above reduces essentially to Theorem 0.2.3 proved in [52].

Proof. It suffices to prove the theorem under the assumption that T_k are radial. The arguments at the end of Theorem 3.2.3 (a) extends the result from radial to the general case. Let $\lambda_0 = -(\alpha^2+\rho^2)$ with $\Im\alpha \geq 0$. Observe that $|\Lambda(\lambda)-z_0| \geq |\Lambda(\alpha)-z_0|$ for $\lambda \in S_p$ (see Figure 3.2(a)).

From hypothesis it follows that $(\Delta - z_0 I)^k T_0 = A^k T_k$ and hence

$$(-1)^k(\lambda^2 + \rho^2 + z_0)^k \widehat{T_0} = A^k \widehat{T_k},$$

where by $(\lambda^2+\rho^2+z_0)^k$ we mean the function $\lambda \mapsto (\lambda^2+\rho^2+z_0)^k$. Let $\phi \in \mathcal{C}^p(\widehat{G//K})$. Then

$$\begin{aligned} |\langle \widehat{T_0}, \phi \rangle| &= |\langle \widehat{T_k}, \left(\frac{A}{\lambda^2+\rho^2+z_0}\right)^k \phi \rangle| \\ &= \left| \left\langle T_k, \left(\left(\frac{A}{\lambda^2+\rho^2+z_0}\right)^k \phi \right)^\vee \right\rangle \right| \\ &\leq M\gamma \left[\left(\left(\frac{A}{\lambda^2+\rho^2+z_0}\right)^k \phi \right)^\vee \right] \\ &\leq M\,\mu \left[\left(\frac{A}{\lambda^2+\rho^2+z_0}\right)^k \phi \right], \end{aligned} \tag{3.2.11}$$

where for any $\phi \in \mathcal{C}^p(\widehat{G//K})$, $\phi^\vee \in \mathcal{C}^p(G//K)$ is its image under the Fourier inversion and the seminorm μ is given by $\mu(\phi) = \sup_{\lambda \in S_p} \left| \frac{d^\tau}{d\lambda^\tau} P(\lambda)\phi(\lambda) \right|$ for some even polynomial $P(\lambda)$ and derivative of even order τ. We note that in the definition of μ, it suffices to consider the supremum on $S_p^+ = \{\lambda \in S_p \mid \Im\lambda \geq 0\}$ as $\phi \in \mathcal{C}^p(\widehat{G//K})$ is an even function.

(a) Our aim is to show that for some $N \in \mathbb{N}$, $(\alpha^2 - \lambda^2)^{N+1}\widehat{T_0} = 0$, equivalently, $\langle (\alpha^2-\lambda^2)^{N+1}\widehat{T_0}, \phi \rangle = 0$ for all $\phi \in \mathcal{C}^p(\widehat{G//K})$. Substituting ϕ by $(\alpha^2-\lambda^2)^{N+1}\phi$ in (3.2.11) we get,

$$\begin{aligned} |\langle (\alpha^2-\lambda^2)^{N+1}\widehat{T_0}, \phi \rangle| &= |\langle \widehat{T_0}, (\alpha^2-\lambda^2)^{N+1}\phi \rangle| \\ &\leq M\,\mu \left[\left(\frac{A}{\lambda^2+\rho^2+z_0}\right)^k (\alpha^2-\lambda^2)^{N+1}\phi \right]. \end{aligned}$$

We note that from hypothesis $|A| = |\alpha^2 + \rho^2 + z_0|$. We fix $N = 6\tau + 1$, and write

$$F^k(\lambda) = \left| \frac{d^\tau}{d\lambda^\tau} P(\lambda) \left(\frac{A}{\lambda^2+\rho^2+z_0}\right)^k (\alpha^2-\lambda^2)^{N+1}\phi \right|.$$

We need to show that $\sup_{\lambda \in S_p^+} F^k(\lambda) \to 0$ as $k \to \infty$. We note that for constants

C_{lmn},

$$\frac{d^\tau}{d\lambda^\tau}\left(P(\lambda)\left(\frac{A}{\lambda^2+\rho^2+z_0}\right)^k(\alpha^2-\lambda^2)^{N+1}\phi\right)$$
$$= \sum_{\substack{l+m+n=\tau\\ l,m,n\in\mathbb{Z}^+}} C_{lmn}\frac{d^l}{d\lambda^l}\left(\frac{A}{\lambda^2+\rho^2+z_0}\right)^k\frac{d^m}{d\lambda^m}(\alpha^2-\lambda^2)^{N+1}\frac{d^n}{d\lambda^n}(P(\lambda)\phi).$$

Since $\phi \in \mathcal{C}^p(\widehat{G//K})$ and $\left|\frac{A}{\lambda^2+\rho^2+z_0}\right| = \left|\frac{\alpha^2+\rho^2+z_0}{\lambda^2+\rho^2+z_0}\right| \leq 1$ for $\lambda \in S_p^+$ we get for some constants C_1 and C_2,

$$F^k(\lambda) \leq C_1 k^\tau \left|\frac{\alpha^2+\rho^2+z_0}{\lambda^2+\rho^2+z_0}\right|^k \left|\alpha^2-\lambda^2\right|^{N+1-\tau} \tag{3.2.12}$$

and

$$F^k(\lambda) \leq C_2 k^\tau \left|\frac{\alpha^2+\rho^2+z_0}{\lambda^2+\rho^2+z_0}\right|^k. \tag{3.2.13}$$

Choose a compact connected neighborhood $\mathcal{U}$ of α in S_p^+ such that if $\lambda \notin \mathcal{U}$, then $\left|\frac{\alpha^2+\rho^2+z_0}{\lambda^2+\rho^2+z_0}\right| < \frac{1}{2}$. It follows from (3.2.13) that that $F^k(\lambda) \to 0$ uniformly in $S_p^+ \setminus \mathcal{U}$. Next we shall show that

$$\sup_{\lambda\in\mathcal{U}} k^\tau \left|\frac{\alpha^2+\rho^2+z_0}{\lambda^2+\rho^2+z_0}\right|^k \left|\alpha^2-\lambda^2\right|^{5\tau+2} \to 0 \text{ as } k \to \infty. \tag{3.2.14}$$

Since

$$\sup_{\lambda\in\mathcal{U}} k^\tau \left|\frac{\alpha^2+\rho^2+z_0}{\lambda^2+\rho^2+z_0}\right|^k \left|\alpha^2-\lambda^2\right|^{5\tau+2}$$
$$= \sup_{\lambda\in\mathcal{U}} k^\tau \left|\frac{\Lambda(\alpha)-z_0}{\Lambda(\lambda)-z_0}\right|^k \left|(\Lambda(\alpha)-z_0)-(\Lambda(\lambda)-z_0)\right|^{5\tau+2},$$

a careful examination of Figure 3.2(a) shows that this is equivalent to prove that

$$\sup_{z\in\Omega} k^\tau \left|\frac{\beta}{z}\right|^k |\beta - z|^{5\tau+2} \to 0,$$

where Ω is a compact region containing β, bounded by the parabolic arc and vertical line, lying on one side of tangent drawn at β opposite to the origin (shaded in Figure 3.2(b)). Again by applying a suitable rotation, we may assume that β lies on positive imaginary axis. We shall show that $\sup_H k^\tau \left|\frac{\beta}{z}\right|^k |\beta - z|^{5\tau+2} \to 0$ as $k \to \infty$ where $H = \{z \in \mathbb{C} \mid -\eta \leq \Re(z) \leq \eta,\ |\beta| \leq \Im(z) \leq \delta\}$ for any $\eta > 0$ and $\delta > |\beta|$ which suffices to establish our claim. Let $V_k = \{z \in H \mid |\Re(z-\beta)| < k^{-1/4}, |\Im(z-\beta)| < k^{-1/4}\}$ and $V_k^c = H \setminus V_k$. It follows that if $z \in V_k^c$, then $|z| \geq (|\beta|^2 + k^{-1/2})^{1/2}$. Hence for $z \in V_k^c$,

$$\left|\frac{\beta}{z}\right| \leq \frac{|\beta|}{(|\beta|^2 + k^{-1/2})^{1/2}} = (1 + \frac{c_1}{\sqrt{k}})^{-1/2},$$

where $c_1 = |\beta|^{-2}$. As H is compact, for some constant C_3,

$$\sup_{V_k^c} k^\tau \left|\frac{\beta}{z}\right|^k |\beta - z|^{5\tau+2} \leq C_3 k^\tau (1 + \frac{c_1}{\sqrt{k}})^{-k/2}. \tag{3.2.15}$$

If $\lambda \in V_k$, then $|\Re(z-\beta)| < k^{-1/4}$ and $|\Im(z-\beta)| < k^{-1/4}$. Therefore for some constant C_4,

$$\sup_{V_k} k^\tau \left|\frac{\beta}{z}\right|^k |\beta - z|^{5\tau+2} \leq C_4 k^\tau k^{-(5\tau+2)/4} = C_4 k^{-(\tau+2)/4}. \tag{3.2.16}$$

From (3.2.15) and (3.2.16), it is clear that $\sup_H k^\tau \left|\frac{\beta}{z}\right|^k |\beta - z|^{5\tau+2} \to 0$ as $k \to \infty$. Thus it is established that $F_k(\lambda) \to 0$ uniformly as $k \to \infty$.

If $\Theta = \Delta - z_0 I$, then Θ is a multiplier given by the symbol $m(\lambda) = -(\lambda^2 + \rho^2 + z_0)$. We have shown that $(\Theta - m(\alpha))^{N+1} T_0 = 0$. An application of Lemma 3.2.1 then gives $(\Theta - m(\alpha))T_0 = 0$, i.e. $\Delta T_0 = -(\alpha^2 + \rho^2)T_0 = \lambda_0 T_0$.

(b) By hypothesis $|A|/|\alpha^2 + \rho^2 + z_0| = |A|/|\lambda_0 - z_0| < 1$. Therefore for $\lambda \in S_p$,

$$\left|\frac{A}{\lambda^2 + \rho^2 + z_0}\right| = \left|\frac{A}{\alpha^2 + \rho^2 + z_0}\right| \left|\frac{\alpha^2 + \rho^2 + z_0}{\lambda^2 + \rho^2 + z_0}\right| \leq \left|\frac{A}{\alpha^2 + \rho^2 + z_0}\right| < 1. \tag{3.2.17}$$

Hence from (3.2.11), it follows that $|\langle \widehat{T_0}, \phi \rangle| = 0$ for all $\phi \in \mathcal{C}^p(\widehat{G//K})$. Thus $T_0 = 0$.

(c) As $|A| > |\lambda_0 - z_0|$, it is evident that there exist distinct $\lambda_1, \lambda_2 \in \Lambda(S_p)$ with $|\lambda_1 - z_0| = |A| = |\lambda_2 - z_0|$ (see Figure 3.2(a)). Suppose that $\lambda_1 = -(\alpha_1^2 + \rho^2)$, $\lambda_2 = -(\alpha_2^2 + \rho^2)$, for some $\alpha_1, \alpha_2 \in S_p$, $|\lambda_1 - z_0| = |A| e^{i\theta_1}$ and $|\lambda_2 - z_0| = |A| e^{i\theta_2}$ for $\theta_1, \theta_2 \in \mathbb{R}$. Define $T_k = e^{ik\theta_1}\varphi_{\alpha_1} + e^{ik\theta_2}\varphi_{\alpha_2}$ for $k \in \mathbb{Z}^+$. Then T_k are L^p-tempered distributions (see Propositions 1.4.1 and 1.5.1), they satisfy the hypothesis and

$$(\Delta - z_0)T_k = e^{ik\theta_1}(\Delta - z_0)\varphi_{\alpha_1} + e^{ik\theta_2}(\Delta - z_0)\varphi_{\alpha_2} = |A| T_{k+1}.$$

But T_0 is not an eigendistribution of Δ. □

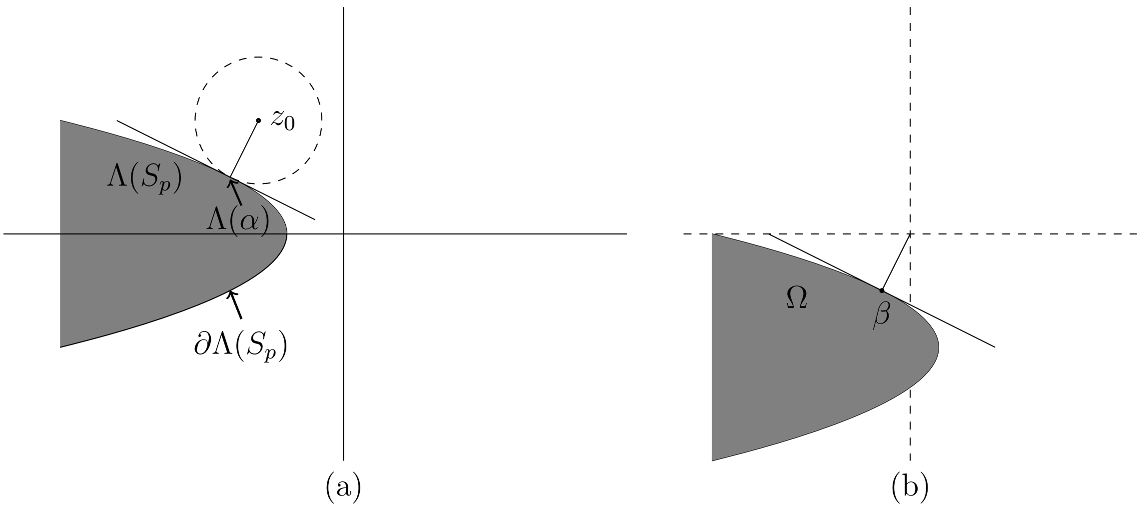

Figure 3.2:

Following results are immediate from the theorem above and Corollary 1.7.4, Theorem 1.7.2.

Corollary 3.2.9. *Let* $\lambda_0 \in \Lambda(S_1) \setminus (-\infty, -\rho^2]$, $p = p(\lambda_0)$ *and* $z_0 \in N(\lambda_0)$. *Let* $\{f_k\}_{k\in\mathbb{Z}^+}$ *be an infinite sequence of measurable functions on* X *such that* $||f_k||_{p',\infty} \leq M$ *for some constant* M. *If* $(\Delta - z_0 I)f_k = Af_{k+1}$ *for all* $k \in \mathbb{Z}^+$ *and for some constant* $A \in \mathbb{C}$ *with* $|A| = |\lambda_0 - z_0|$, *then* $\Delta f_0 = \lambda_0 f_0$ *and* $f_0 = \mathcal{P}_{\alpha - i\gamma_p\rho} F$ *for some* $F \in L^{p'}(K/M)$ *and* $\alpha \in \mathbb{R}$.

Corollary 3.2.10. *Let* $\lambda_0 \in \mathbb{C} \setminus (-\infty, -\rho^2]$, $p = p(\lambda_0)$ *and* $z_0 \in N(\lambda_0)$. *Let* $\{f_k\}_{k\in\mathbb{Z}^+}$ *be an infinite sequence of measurable functions on* X *such that* $[f_k]_{p,r} \leq M$ *for* $1 \leq r \leq \infty$ *and for some constant* M. *If* $(\Delta - z_0 I)f_k = Af_{k+1}$ *for all* $k \in \mathbb{Z}^+$ *and for some constant* $A \in \mathbb{C}$ *with* $|A| = |\lambda_0 - z_0|$, *then* $\Delta f_0 = \lambda_0 f_0$. *Moreover* $f_0 = \mathcal{P}_{\alpha - i\gamma_p\rho} F$ *for some* $\alpha \in \mathbb{R}$ *and* $F \in L^r(K/M)$ *if* $r > 1$ *and* $f_0 = \mathcal{P}_{\alpha - i\gamma_p\rho}\mu$ *for some* $\alpha \in \mathbb{R}$ *and signed measure* μ *on* K/M *if* $r = 1$.

3.3 Characterization of L^2-tempered eigendistributions

In the previous section we have dealt with polynomials in Δ, sphere and ball average and heat operator as multiplier from $\mathcal{C}^p(X)$ to itself for $0 < p < 2$. In this section we shall take up the case $p = 2$. Taking the advantage of the one-dimensionality of the L^2-spectrum, here we shall be able to deal with all the multipliers on $\mathcal{C}^2(X)$ together. We begin with a lemma.

Lemma 3.3.1. *Let Θ be a multiplier on $\mathcal{C}^2(X)$ with symbol $m(\lambda) \in C^\infty(\mathbb{R})$. Let $\{T_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of radial L^2-tempered distributions on X such that for all $k \in \mathbb{Z}$, $|\langle T_k, \psi\rangle| \leq M\gamma(\psi)$ for a fixed seminorm γ of $\mathcal{C}^2(X)$ and a constant $M > 0$. Let A be a complex constant. Then the following conclusions hold.*

(i) *If for all $k \in \mathbb{Z}^+$, $\Theta T_k = AT_{k+1}$, then* $\operatorname{Supp} \widehat{T_0} \subseteq \{\lambda \in \mathbb{R} \mid |m(\lambda)| \leq |A|\}$.

(ii) *If for all $k \in \mathbb{Z}^-$, $\Theta T_{k-1} = AT_k$, then* $\operatorname{Supp} \widehat{T_0} \subseteq \{\lambda \in \mathbb{R} \mid |m(\lambda)| \geq |A|\}$.

(iii) *If for all $k \in \mathbb{Z}$, $\Theta T_k = AT_{k+1}$, then* $\operatorname{Supp} \widehat{T_0} \subseteq \{\lambda \in \mathbb{R} \mid |m(\lambda)| = |A|\}$.

Proof. (i) Let $\Omega = \{\lambda \in \mathbb{R} \mid |m(\lambda)| \leq |A|\}$. We take a $\beta \notin \Omega$. Then there exists a positive constant $c < 1$ and a compact neighborhood V of β such that $|A| < c|m(\lambda)|$ for all $\lambda \in V$. Let $\phi \in \mathcal{C}^2(\widehat{G//K})$ be such that $\operatorname{Supp}\phi \subset V$. We claim that $\langle \widehat{T_0}, \phi\rangle = 0$.

Since ϕ is compactly supported and $m \in C^\infty(\mathbb{R})$, we have, $\phi(\lambda)A^k/m(\lambda)^k \in \mathcal{C}^2(\widehat{G//K})$. Suppose that for $\psi \in \mathcal{C}^2(G//K)$, $\widehat{\psi}(\lambda) = \phi(\lambda)A^k/m(\lambda)^k$.

The hypothesis implies that for all $k \in \mathbb{Z}^+$, $\Theta^k T_0 = A^k T_k$, equivalently,

$$m(\lambda)^k \widehat{T_0} = A^k \widehat{T_k}.$$

Therefore,

$$\begin{aligned} |\langle \widehat{T_0}, \phi\rangle| &= |\langle \widehat{T_k}, \left(\frac{A}{m(\lambda)}\right)^k \phi\rangle| = |\langle T_k, \psi\rangle| \\ &\leq M\gamma(\psi) \leq CM\mu_{n,\tau}\left[\left(\frac{A}{m(\lambda)}\right)^k \phi\right], \end{aligned}$$

where the seminorm $\mu_{n,\tau}$ is given by,

$$\mu_{n,\tau}\left[\left(\frac{A}{m(\lambda)}\right)^k \phi\right] = \sup_{\lambda\in V}(1+|\lambda|)^n \left|\frac{d^\tau}{d\lambda^\tau}\left(\frac{A}{m(\lambda)}\right)^k \phi(\lambda)\right|, \tag{3.3.1}$$

for $n, \tau \in \mathbb{Z}^+$. Since $m(\lambda), 1/m(\lambda) \in C^\infty(V)$ and $|\frac{A}{m(\lambda)}| < c$, it is clear that as $k \to +\infty$,

$$\sup_{\lambda\in V}(1+|\lambda|)^n \left|\frac{d^\tau}{d\lambda^\tau}\left(\frac{A}{m(\lambda)}\right)^k \phi(\lambda)\right| \to 0.$$

Thus $\beta \notin \operatorname{Supp}\widehat{T_0}$. Hence $\operatorname{Supp}\widehat{T_0} \subset \Omega$. This completes the proof of (i). A similar argument with negative integers and taking $k \to -\infty$ will prove (ii) and (iii) evidently follows from (i) and (ii). □

We state our first main result in this section.

Theorem 3.3.2. *Let $\Theta : \mathcal{C}^2(X) \to \mathcal{C}^2(X)$ be a multiplier with real valued symbol $m(\lambda) \in C^\infty(\mathbb{R})$. Let $\{T_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of elements of $\mathcal{C}^2(X)'$. Suppose that for all $k \in \mathbb{Z}$, $\Theta T_k = AT_{k+1}$ for a nonzero constant $A \in \mathbb{C}$ and $|\langle T_k, \psi\rangle| \leq M\gamma(\psi)$ for a fixed seminorm γ of $\mathcal{C}^2(X)$ and a constant $M > 0$. Let $m(\mathbb{R}) = \{m(\lambda) \mid \lambda \in \mathbb{R}\}$.*

(a) *If $|A| \in m(\mathbb{R})$ but $-|A| \notin m(\mathbb{R})$, then $\Theta T_0 = |A|T_0$.*

(b) *If $-|A| \in m(\mathbb{R})$ but $|A| \notin m(\mathbb{R})$, then $\Theta T_0 = -|A|T_0$.*

(c) *If both $|A|, -|A| \in m(\mathbb{R})$, then T_0 can be uniquely written as $T_0 = T_+ + T_-$ where $T_+, T_- \in \mathcal{C}^2(X)'$ satisfying $\Theta T_+ = |A|T_+$ and $\Theta T_- = -|A|T_-$.*

(d) *If neither $|A|$ nor $-|A|$ is in $m(\mathbb{R})$, then $T_0 = 0$.*

Proof. (a) First we shall prove the assertion with the assumption that the distributions T_k are radial. Let $|A| = m(\alpha)$ for some $\alpha \in \mathbb{R}$. In view of the Lemma 3.2.1 it suffices to show that for some $N \in \mathbb{N}$,

$$(\Theta - m(\alpha))^{N+1}T_0 = 0, \tag{3.3.2}$$

equivalently, $\langle (m(\lambda) - m(\alpha))^{N+1}\widehat{T_0}, \phi\rangle = 0$ for all $\phi \in C_c^\infty(\mathbb{R})$. Using that m is real valued and $-|A| \notin m(\mathbb{R})$ and Lemma 3.3.1(iii), we conclude that

$$\operatorname{Supp}\widehat{T_0} \subset \{\lambda \in \mathbb{R} \mid |m(\lambda)| = |A|\} = \{\lambda \in \mathbb{R} \mid m(\lambda) = m(\alpha)\}.$$

Let g be an even function in $C_c^\infty(\mathbb{R})$ such that $g \equiv 1$ on $[-1/2, 1/2]$ and support of g is contained in $(-1, 1)$. For $r > 0$, let g_r be defined by $g_r(\xi) = g(\xi/r)$.

Let $B = \max\{|\frac{d^k}{d\lambda^k}g(\lambda)| : \lambda \in [-1, 1], k \leq N\}$. Hence we have

$$\left|\frac{d^k}{d\lambda^k}g_r(\lambda)\right| \leq B/r^k \text{ for all } k \leq N. \tag{3.3.3}$$

Let $p(\lambda) = m(\lambda) - m(\alpha)$. For $\phi \in C_c^\infty(\mathbb{R})$, define

$$H_r(\lambda) = (m(\lambda) - m(\alpha))^{N+1}g_r(p(\lambda))\phi(\lambda).$$

Clearly $H_r \in \mathcal{C}^2(\widehat{G//K})$. Suppose that for $h_r \in \mathcal{C}^2(G//K)$, $\widehat{h_r} = H_r$. Since $H_r(\lambda) = (m(\lambda) - m(\alpha))^{N+1}\phi(\lambda)$ in a neighborhood of $\operatorname{Supp}\widehat{T_0}$, we have,

$$|\langle (m(\lambda) - m(\alpha))^{N+1}\widehat{T_0}, \phi\rangle| = |\langle \widehat{T_0}, H_r\rangle| = |\langle T_0, h_r\rangle| \leq M\gamma(h_r) \leq M\mu_{n,\tau}(H_r),$$

for some $n, \tau \in \mathbb{Z}^+$ where the seminorm $\mu_{n,\tau}$ is as given in (3.3.1). The proof of this step will be completed if we show that $\mu_{n,\tau}(H_r) \to 0$ as $r \to 0$ when H_r is defined using suitably large N.

Since g_r vanishes outside the set $\{\lambda \in \mathbb{R} \mid |p(\lambda)| \geq r\}$, it is enough to consider the supremum over the set $E = \{\lambda \in \mathbb{R} \mid |p(\lambda)| \leq r\} \cap \operatorname{Supp} \phi$. We note that for $\lambda \in E$, $|m(\lambda) - m(\alpha)| \leq r$. Since ϕ is compactly supported,

$$(1 + |\lambda|)^n |p^{(k)}(\lambda) m^{(l)}(\lambda) \phi^{(\tau-(k+l))}(\lambda)| \leq K_1 \text{ for } 0 \leq k, l \leq \tau. \tag{3.3.4}$$

for some constant K_1. Above the superscript (l) of a function denotes its l-th derivative. Using (3.3.3) and (3.3.4) we have for $N \geq \tau$,

$$\begin{aligned}
&\sup_{\lambda \in E} (1 + |\lambda|)^n \left| \frac{d^\tau}{d\lambda^\tau} \left[(m(\lambda) - m(\alpha))^{N+1} g_r(p(\lambda)) \phi(\lambda) \right] \right| \\
=& \sup_{\lambda \in E} (1 + |\lambda|)^n \left| \sum_{\substack{0 \leq k,l \leq \tau \\ k+l \leq \tau}} C_{k,l} g_r^{(k)}(p(\lambda)) p^{(k)}(\lambda) (m(\lambda) - m(\alpha))^{N+1-l} m^{(l)}(\lambda) \phi^{(\tau-(k+l))}(\lambda) \right| \\
\leq& \sup_{\lambda \in E} \sum_{\substack{0 \leq k,l \leq \tau \\ k+l \leq \tau}} C_{k,l} \left| (1 + |\lambda|)^n p^{(k)}(\lambda) m^{(l)}(\lambda) \phi^{(\tau-(k+l))}(\lambda) \right| \left| {g_r}^{(k)}(p(\lambda)) (m(\lambda) - m(\alpha))^{N+1-l} \right| \\
\leq& C \sum_{\substack{0 \leq k,l \leq \tau \\ k+l \leq \tau}} r^{-k} r^{N+1-l}
\end{aligned}$$

which goes to zero as $r \to 0$. Thus for $N \geq \tau$, $|\langle (m(\lambda) - m(\alpha))^{N+1} \widehat{T_0}, \phi \rangle| = 0$. By Lemma 3.2.1 this implies that $(\Theta - m(\alpha)) T_0 = 0$. The assertion is thus proved when T_k are radial. The assumption of radiality can be withdrawn as done in Theorem 3.2.3 (a).

(b) Since $-\Theta$ satisfies the hypothesis of part (a) with A replaced by $-A$, it follows from (a) that $-\Theta T_0 = |A| T_0$. Consequently $\Theta T_0 = -|A| T_0$

(c) Let $\Theta_0 = \Theta^2$. Then $\Theta_0 T_{2k} = A^2 T_{2k+2}$ for all $k \in \mathbb{Z}$. Hence the sequence $\{T_{2k}\}_{k \in \mathbb{Z}}$ satisfies the hypothesis of part (a), substituting Θ by Θ_0 and $|A|$ by $|A|^2$. Therefore $\Theta_0 T_0 = |A|^2 T_0$. Set $T_+ = \frac{|A|T_0 + \Theta T_0}{2|A|}$ and $T_- = \frac{|A|T_0 - \Theta T_0}{2|A|}$. Evidently $T_0 = T_+ + T_-$ and T_+ and T_- satisfy the required property. Uniqueness of this decomposition is clear because if $T_0 = S_+ + S_-$ with $\Theta S_+ = |A| S_+$ and $\Theta S_- = -|A| S_-$, then $\Theta T_0 = |A| S_+ - |A| S_-$. Therefore $S_+ = \frac{|A|T_0 + \Theta T_0}{2|A|}$ and $S_- = \frac{|A|T_0 - \Theta T_0}{2|A|}$.

(d) From Lemma 3.3.1 and proof of Theorem 3.2.3(a), it follows that distributional support of $R(\ell_x T_0)$ is empty for every $x \in G$. Hence $R(\ell_x T_0) = 0$ for every $x \in G$. Consequently $T_0 = 0$ by the argument given at the end of proof of Theorem 3.1.3. $\square$

Restricting to particular multipliers we have interesting corollaries of the theorem above. Here are a few representatives, written as a list for brevity.

Let $\{f_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of measurable functions on X such that for the multiplier Θ (which will be specified below), $\Theta f_k = A f_{k+1}$ for all $k \in \mathbb{Z}$ and for a constant $A \in \mathbb{C}$. We have these conclusions.

(i) $\Theta = \Delta$: Suppose that for all $k \in \mathbb{Z}$ either (a) $\|f_k\|_{2,\infty} \le M$ or (b) $[f_k]_{2,r} \le M$, where $r \in [1, \infty]$ for some $M > 0$; and $|A| > \rho^2$.

Then $\Delta f_0 = -|A| f_0$. Moreover under condition (a) $f = \mathcal{P}_\alpha F$ for some $F \in L^2(K/M)$ and for $\alpha \in \mathbb{R}$ satisfying $\alpha^2 + \rho^2 = |A|$.

(ii) $\Theta = e^{t\Delta}$ for a fixed $t > 0$: Suppose that for all $k \in \mathbb{Z}$ either (a) $\|f_k\|_{2,\infty} \le M$ or (b) $[f_k]_{2,r} \le M$ where $r \in [1, \infty]$, for some $M > 0$; and $|A| = e^{-t(\alpha^2+\rho^2)}$ for an $\alpha \in \mathbb{R}^\times$.

Then $\Delta f_0 = -(\alpha^2 + \rho^2) f_0$. Moreover under condition (a) $f = \mathcal{P}_\alpha F$ for some $F \in L^2(K/M)$.

(iii) $\Theta = M_t$ or B_t or $e^{t\Delta}$ for a fixed $t > 0$: Suppose that for all $k \in \mathbb{Z}$ either (a) $\|(1 + \|x\|)^{-1} f_k\|_{2,\infty} \le M$ or (b) $[f_k]_{2,r} \le M$ where $r \in [1, \infty]$ for some $M > 0$; and $|A| = \varphi_0(a_t)$ (respectively $|A| = \psi_0(t)$, $|A| = e^{-t\rho^2}$).

Then $\Delta f_0 = -(\alpha^2 + \rho^2) f_0$. Moreover under condition (a) $f = \mathcal{P}_0 F$ for some $F \in L^2(K/M)$ and under (b) $f_0 = \mathcal{P}_0 F$ for some $F \in L^r(K/M)$ if $r > 1$ and $f_0 = \mathcal{P}_0 \mu$ for some signed measure μ on K/M if $r = 1$.

(iv) $\Theta = M_t$ or B_t for a fixed $t > 0$: Suppose that for all $k \in \mathbb{Z}$ either (a) $\|f_k\|_{2,\infty} \le M$ or (b) $[f_k]_{2,r} \le M$ for $r \in [1, \infty]$ for some $M > 0$ for all $k \in \mathbb{Z}$; and $|A| < \varphi_0(a_t)$ (respectively $|A| < \psi_0(t)$).

Then either f_0 is an eigenfunction of M_t, respectively of B_t or f_0 is sum of two eigenfunctions of M_t, respectively of B_t with eigenvalues $|A|$ and -$|A|$.

The main argument of the proof of these assertions can be divided as the following:

A: Observe that parts (a) and (b) of Theorem 3.3.2 imply that when $m(\lambda)$ for $\lambda \in \mathbb{R}$, takes only positive or only negative values, then f_0 is an eigenfunction of the multiplier. We note that the symbol of Δ i.e. $-(\lambda^2 + \rho^2)$ takes only negative values, while the symbol of $e^{t\Delta}$ assumes only positive values on $\mathbb{R}$. On the other hand symbols of M_t and B_t which are respectively $\varphi_\lambda(a_t)$ and $\psi_\lambda(t)$ can have both positive and negative values.

B: The uniform norm-boundedness condition on functions f_k i.e. $\|f_k\|_{2,\infty} \leq M$ or $\|(1+|x|)^{-1}f_k\|_{2,\infty} \leq M$ or $[f_k]_{2,r} \leq M$ implies that each f_k is an L^2-tempered distribution and

$$|\langle f_k, \psi\rangle| \leq CM\gamma(\psi)$$

for some constant $C > 0$, for a fixed seminorm γ of $\mathcal{C}^2(X)$ and for all $\psi \in \mathcal{C}^2(X)$ (see Proposition 1.5.1(a) and Proposition 1.6.1(e)). Thus the hypothesis of Theorem 3.3.2 is satisfied.

C: We use the one radius theorem given in Section 3.1.3 which reduces an eigenfunction of a multiplier to an eigenfunction of Δ in some cases.

D: Eigenfunctions of Δ satisfying suitable condition on its growth can be realized as the Poisson transform of an appropriate object on K/M. See Theorem 1.7.2 and Corollary 1.7.4.

Remark 3.3.3. We emphasize that in (iv) above, both of the situations, i.e., f_0 is an eigenfunction of the multiplier or f_0 is a sum of two eigenfunctions of the multiplier are possible. Let us restrict to $\Theta = M_t$. It is easy to construct such examples for the particular symmetric space $X = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$, for which $\varphi_\lambda(a_t) = \sin(2\lambda t)/\lambda \sinh(2t)$ ([41, p. 433]). The function $\lambda \mapsto \varphi_\lambda(a_t)$ is an even oscillating function on $\mathbb{R}$ with decay. We can consider it as a function on $\mathbb{R}^+$. If $|A| < \varphi_0(a_t)$ is sufficiently close to $\varphi_0(a_t)$, then it is possible that $\varphi_\lambda(a_t) \neq -|A|$ for any $\lambda \in \mathbb{R}$. Hence for such an A in the hypothesis of (iv), f_0 will be an eigenfunction of M_t. On other hand if $|A|$ is small compared to $\varphi_0(a_t)$, then it is clear that there can be finitely many distinct $\lambda \in \mathbb{R}^+$, where $\varphi_\lambda(a_t)$ assumes the values $|A|$ and $-|A|$. Suppose that for $\lambda_1, \lambda_2 \in \mathbb{R}^+$ with $\lambda_1 \neq \lambda_2$, $\varphi_{\lambda_1}(a_t) = A$ and $\varphi_{\lambda_2}(a_t) = -A$. Let us take $f_k(x) = \varphi_{\lambda_1}(x) + (-1)^{|k|}\varphi_{\lambda_2}(x)$ for $k \in \mathbb{Z}$. Then $M_t f_k = Af_{k+1}$ and the sequence $\{f_k\}_{k\in\mathbb{Z}}$ satisfies the hypothesis of (iv) above, but f_0 is sum of two eigenfunctions of M_t with eigenvalues $|A|$ and $-|A|$.

For an arbitrary rank one symmetric space X, we can use the same argument, both for sphere and ball averages. Fixing a $t > 0$, we consider the map $\lambda \mapsto \varphi_\lambda(a_t)$ for M_t and the map $\lambda \mapsto \psi_\lambda(t)$ for B_t and momentarily call both of these functions h. Then h is an even function on $\mathbb{R}$, hence can be considered as a function on $\mathbb{R}^+$. Relating h to Jacobi functions (see (1.3.1), (1.3.5)) and using properties of Jacobi functions (see e.g. [79, Proposition 2.2, page 235]), we conclude that $\lambda \mapsto h(\lambda)$ is a damp oscillation on $\mathbb{R}^+$, decaying to 0 as $\lambda \to \infty$. Therefore again we can argue as above to conclude that we cannot get rid of any of the two possibilities. Lastly, we stress that the example of f_k given above shows that in this case we cannot arrive at the conclusion that f_0 is an eigenfunction of Δ.

So far in this section (in Theorem 3.3.2 and its consequences), we have dealt with multipliers whose symbols are real valued. Below we shall consider multipliers with complex valued symbols. Along with those we have already taken up in this section, the result we aim at will accommodate more multipliers, e.g. real sum of odd degree monomials in $i\Delta$, heat operator with complex time i.e. $e^{z\Delta}$ with $\Re z > 0$. We shall now state the result.

Theorem 3.3.4. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of measurable functions on X and $\Theta : \mathcal{C}^2(X) \to \mathcal{C}^2(X)$ be a multiplier with (possibly complex valued) symbol $m(\lambda) \in C^\infty(\mathbb{R})$. Suppose that for all $k \in \mathbb{Z}$, $\|f_k\|_{2,\infty} \leq M$ and $\Theta f_k = A f_{k+1}$ for constants $M > 0$, $A \in \mathbb{C}$. If $\{\lambda \in \mathbb{R} \mid |m(\lambda)| = |A|\}$ is finite, then f_0 can be uniquely written as $f_0 = g_1 + g_2 + \cdots + g_r$ for functions $g_i, i = 1, \cdots, r$ on X satisfying $\Theta g_i = A_i g_i$, where $A_i \in \mathbb{C}$ are distinct and $|A_i| = |A|$. Further if $m(\lambda) = c$, a constant for all $\lambda \in E$, then f_0 is an eigenfunction of Θ with eigenvalue c.*

We shall prove Theorem 3.3.4 through these intermediate steps, written as lemma and proposition.

Lemma 3.3.5. *Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct nonzero real numbers and $P_0, P_1, P_2, \cdots, P_k$ be polynomials. If $P_0(\partial_\lambda)\varphi_\lambda|_{\lambda=0} + P_1(\partial_\lambda)\varphi_\lambda|_{\lambda=\lambda_1} + \cdots + P_k(\partial_\lambda)\varphi_\lambda|_{\lambda=\lambda_k} \in L^{2,\infty}(X)$, then $P_0 = 0$ and $P_i, i = 1, \ldots, k$ are constant polynomials.*

Proof. Without loss of generality we assume that each $P_i \neq 0$ for $0 \leq i \leq k$. Let n_i be the degree of P_i and $N_i = \max_{j\neq i} n_j$. Let

$$f = P_0(\partial_\lambda)\varphi_\lambda|_{\lambda=0} + \cdots + P_k(\partial_\lambda)\varphi_\lambda|_{\lambda=\lambda_k}.$$

Choose a function $\psi \in \mathcal{C}^2(G//K)$ such that: $\partial_\lambda^j \widehat{\psi}|_{\lambda=\lambda_i} = 0$ for $1 \leq i \leq k$, $0 \leq j \leq N_0$; $\partial_\lambda^j \widehat{\psi}|_{\lambda=0} = 0$ for $0 \leq j < n_0$; and $(\partial_\lambda^{n_0}\widehat{\psi})(0) \neq 0$. Then,

$$\begin{aligned} f * \psi &= P_0(\partial_\lambda)(\varphi_\lambda * \psi)|_{\lambda=0} + P_1(\partial_\lambda)(\varphi_\lambda * \psi)|_{\lambda=\lambda_1} + \cdots + P_k(\partial_\lambda)(\varphi_\lambda * \psi)|_{\lambda=\lambda_k} \\ &= P_0(\partial_\lambda)(\widehat{\psi}(\lambda)\varphi_\lambda)_{\lambda=0} + P_1(\partial_\lambda)(\widehat{\psi}(\lambda)\varphi_\lambda)|_{\lambda=\lambda_1} + \cdots + P_k(\partial_\lambda)(\widehat{\psi}(\lambda)\varphi_\lambda)|_{\lambda=\lambda_k} \\ &= C\{\varphi_\lambda \partial_\lambda^{n_0}(\widehat{\psi}(\lambda)\}|_{\lambda=0} \\ &= C\varphi_0(\partial_\lambda^{n_0}\ \widehat{\psi})(0). \end{aligned}$$

Since by the hypothesis $f \in L^{2,\infty}(X)$ and $\psi \in \mathcal{C}^2(G//K)$, it follows that (see [69, Proposition 3.2 (iv)]) $f * \psi \in L^{2,\infty}(X)$. This leads to the conclusion that $\varphi_0 \in L^{2,\infty}(X)$ which is a contradiction (see Proposition 1.7.1). Hence $P_0 = 0$. Using similar arguments and the fact that $\partial_\lambda \varphi_\lambda|_{\lambda=\lambda_0} \notin L^{2,\infty}(X)$ for any $\lambda_0 > 0$ ([69, Lemma 4.5]), it can be shown that $P_i, 1 \leq i \leq k$ are constant polynomials. □

Lemma 3.3.6. *Let T be a radial L^2-tempered distribution on X such that distributional support of $\widehat{T}$ is $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$, then $T = P_1(\partial_\lambda)\varphi_\lambda|_{\lambda=\lambda_1} + P_2(\partial_\lambda)\varphi_\lambda|_{\lambda=\lambda_2} + \cdots + P_k(\partial_\lambda)\varphi_\lambda|_{\lambda=\lambda_k}$ for some polynomials $P_1, P_2, \cdots, P_k$. In particular if T is given by a measurable function $f \in L^{2,\infty}(G//K)$, then $f = c_1\varphi_{\lambda_1} + c_2\varphi_{\lambda_2} + \cdots + c_k\varphi_{\lambda_k}$ for some constants $c_1, c_2, \cdots, c_k$.*

Proof. As Supp $\widehat{T} = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$, it follows that (see [68, Theorem 6.25]), there exist polynomials P_i for $1 \leq i \leq k$ such that

$$\widehat{T} = \sum_{i=1}^{k} P_i(\partial_\lambda)\delta_{\lambda_i}$$

where δ_{λ_i} is the Dirac mass at λ_i. But if $S = P_i(\partial_\lambda)\varphi_\lambda|_{\lambda=\lambda_i}$, it is easy to see that $\widehat{S} = P_i(\partial_\lambda)\delta_{\lambda_i}$. By injectivity of the spherical Fourier transform of L^2-tempered distribution we conclude that

$$T = P_1(\partial_\lambda)\varphi_\lambda|_{\lambda=\lambda_1} + P_2(\partial_\lambda)\varphi_\lambda|_{\lambda=\lambda_2} + \cdots + P_k(\partial_\lambda)\varphi_\lambda|_{\lambda=\lambda_k}.$$

If T is given by a measurable function $f \in L^{2,\infty}(G//K)$, then owing to Lemma 3.3.5 we get $f = c_1\varphi_{\lambda_1} + c_2\varphi_{\lambda_2} + \cdots + c_k\varphi_{\lambda_k}$ for some constants $c_1, c_2, \cdots, c_k$. □

Remark 3.3.7. From the proof of the previous result, Propositions 1.7.1 and 1.4.1 it is clear that $0 \notin$ Supp $\widehat{f}$ if $f \in L^{2,\infty}(G//K)$ and Supp $\widehat{f}$ is finite.

Proposition 3.3.8. *Let f be a measurable function on X which can be written as a finite sum $f = f_1 + f_2 + \cdots + f_n$ where for some linear operator Θ, $\Theta f_i = \alpha_i f_i$ for $i = 1, \cdots, n$ with $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$ distinct. Then*

$$(\Theta - \alpha_1 I)(\Theta - \alpha_2 I)\cdots(\Theta - \alpha_n I)f = 0. \tag{3.3.5}$$

Conversely, if a measurable function f on X satisfies (3.3.5) *for distinct $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, then f can be uniquely written as a finite sum of eigenfunctions of Θ corresponding to the eigenvalues $\alpha_1, \alpha_2, \cdots, \alpha_n$.*

Proof. To prove the direct side we note that when $n = 1$, the result is trivially true. We shall use induction on n. We assume that the result is true for $n = m - 1$. Let $f = f_1 + f_2 + \cdots + f_m$ with $\Theta f_i = \alpha_i f_i, 1 \leq i \leq m$. Then $f - f_1 = \sum_{j=2}^{m} f_j$ is a sum of $m - 1$ eigenfunctions. By induction hypothesis

$$(\prod_{j=2}^{m}(\Theta - \alpha_j I))(f - f_1) = 0.$$

Applying $(\Theta - \alpha_1 I)$ to this equality, we get the result.

To prove the converse, we define polynomials

$$P_i(x) = \prod_{j \neq i} \frac{x - \alpha_j}{\alpha_i - \alpha_j}, \text{ for } 1 \leq i \leq n.$$

It is easy to see that $P_i(\alpha_j) = \delta_{ij}$ for $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta and $P_1 + P_2 + \cdots + P_n = 1$. Indeed, if $P = P_1 + P_2 + \cdots + P_n - 1$, then P is a polynomial of degree $n - 1$, with n roots (namely α_i, $1 \leq i \leq n$). Hence $P = 0$. Let $f_i = P_i(\Theta)f$ for $1 \leq i \leq n$. If f satisfies (3.3.5), then it is clear that $(\Theta - \alpha_i I)P_i(\Theta)f = 0$. Hence $\Theta(P_i(\Theta)f) = \alpha_i P_i(\Theta)f$. Therefore $\Theta f_i = \alpha_i f_i$ for $1 \leq i \leq n$. Since $P_1 + P_2 + \cdots + P_n = 1$, we have $f = f_1 + f_2 + \cdots + f_n$. Lastly, to prove the uniqueness of the representation $f = f_1 + f_2 + \cdots + f_n$, we note that $\Theta^j f = \alpha_1^j f_1 + \alpha_2^j f_2 + \cdots + \alpha_n^j f_n$ for $1 \leq j < n$, which can be written as the following matrix equation.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} f \\ \Theta f \\ \vdots \\ \Theta^{n-1} f \end{bmatrix}.$$

As α_i are distinct, the square matrix above is invertible. Thus f_i for $1 \leq i \leq n$ are uniquely determined in terms of $f, \Theta f, \cdots, \Theta^{n-1} f$. □

We shall now complete the proof of Theorem 3.3.4.

Proof of Theorem 3.3.4. If we assume that $f_k, k \in \mathbb{Z}$ are radial, then from Lemma 3.3.1(iii), we get that distributional support of $\widehat{f_0}$ is contained in the set $E = \{\lambda \in \mathbb{R} \mid |m(\lambda)| = |A|\}$. Let $E = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. By Lemma 3.3.6, we get that $f_0 = c_1\varphi_{\lambda_1} + c_2\varphi_{\lambda} + \cdots + c_n\varphi_{\lambda}$. Writing those φ_λ's together which have same eigenvalues (i.e. $m(\lambda)$) for Θ, we get that $f_0 = g_1 + g_2 + \cdots + g_r$ where $g_1, g_2, \cdots, g_r$ are the eigenfunctions for Θ corresponding to distinct eigenvalues say $A_1, A_2, \cdots, A_r$. Hence appealing to Proposition 3.3.8, we get that $(\Theta - A_1I)(\Theta - A_2I)\cdots(\Theta - A_rI)f_0 = 0$.

Rest of the argument is similar to one in the proof of Theorem 3.3.2. We include a quick sketch. If the sequence $\{f_k\}$ (whose elements f_k are not necessarily radial) satisfies the hypothesis, then for any $y \in G$, the sequence $\{R(\ell_y f_k)\}_{k\in\mathbb{Z}}$ is a sequence of radial functions which satisfies the hypothesis. Hence by the first part of the proof, $(\Theta - A_1I)(\Theta - A_2I)\cdots(\Theta - A_rI)R(\ell_y f_0) = 0$ for every $y \in G$. Therefore $R(\ell_y(\Theta - A_1I)(\Theta - A_2I)\cdots(\Theta - A_rI)f_0) = 0$ for every $y \in G$. Hence we get that $(\Theta - A_1I)(\Theta - A_2I)\cdots(\Theta - A_rI)f_0 = 0$. Applying Proposition 3.3.8, we have the desired conclusion. The last part of the assertion is immediate. □

The following are versions of the result listed as (i) after Theorem 3.3.2. The first one captures eigenfunctions with eigenvalues in $(-\infty, -\rho^2)$ and the second targets those with eigenvalue $-\rho^2$. While a bi-infinite sequence $\{f_k\}_{k\in\mathbb{Z}}$ is used in (i), here we shall use a sequence $\{f_k\}_{k\in\mathbb{Z}^+}$ of functions.

Theorem 3.3.9. *Let $z_0 = \alpha^2 + \rho^2 \pm i\beta$ for some $\alpha \in \mathbb{R}^\times$ and $\beta > 0$. Let $\{f_k\}_{k\in\mathbb{Z}^+}$ be a sequence of measurable functions on X such that for all $k \in \mathbb{Z}^+$, $\|f_k\|_{2,\infty} \leq M$ and $(\Delta + z_0I)f_k = Af_{k+1}$ for constants $M > 0$ and $A \in \mathbb{C}$ with $|A| = \beta$. Then $f_0 = \mathcal{P}_\alpha F$ for some $F \in L^2(K/M)$.*

Proof. As the previous result we may assume that f_k are radial. Let $\Theta = \Delta + z_0I$. Then Θ is a multiplier with symbol $\alpha^2 - \lambda^2 \pm i\beta$. From Lemma 3.3.1(i) it follows that $\operatorname{Supp}\widehat{f_0} \subseteq \{\alpha, -\alpha\}$. Hence by Lemma 3.3.6, $f_0 = c\varphi_\alpha$, in particular $\Delta f_0 = -(\alpha^2 + \rho^2)f_0$. Rest is an application of Corollary 1.7.4(ii). □

Theorem 3.3.10. *Suppose that for all $k \in \mathbb{Z}^+$, $\Delta f_k = Af_{k+1}$ for a constant $A \in \mathbb{C}$ with $|A| = \rho^2$. If some constant $M > 0$, either* (a) *$\|(1 + |x|)^{-1}f_k\|_{2,\infty} \leq M$, or* (b) *$[f_k]_{2,r} \leq M$ for $1 \leq r \leq \infty$, then $\Delta f_0 = -\rho^2 f_0$. We further conclude the following.*

(i) *If $\{f_k\}_{k\in\mathbb{Z}^+}$ satisfies* (a)*, then $f_0 = \mathcal{P}_0F$ for some $F \in L^2(K/M)$.*

(ii) *If $\{f_k\}_{k\in\mathbb{Z}^+}$ satisfies* (b) *with $r > 1$, then $f_0 = \mathcal{P}_0F$ for some $F \in L^r(K/M)$.*

(iii) *If $\{f_k\}_{k\in\mathbb{Z}^+}$ satisfies* (b) *with $r = 1$, then $f_0 = \mathcal{P}_0\nu$ for a signed measure ν on K/M.*

Proof. It is enough to prove the assertion assuming that f_k are radial. The result can be extended to the general case by argument used in Theorem 3.2.3 (a). Let $\Theta = \Delta$, which has the symbol $m(\lambda) = -(\lambda^2 + \rho^2)$. Since (a) and (b) in the hypothesis imply that f_k are L^2-tempered distributions (see Proposition 1.5.1), we have by Lemma 3.3.1(i) that distributional support of $\widehat{f_0}$ is contained in the set

$$\{\lambda \in \mathbb{R} \mid |m(\lambda)| \leq |A|\} = \{\lambda \in \mathbb{R} \mid |\lambda^2 + \rho^2| \leq \rho^2\} = \{0\} = \{\lambda \in \mathbb{R} \mid |m(\lambda)| = |A|\}.$$

From this it can be verified (see the proof of Theorem 3.3.2 (a) for the required line of argument) that, $\Delta f_0 = -\rho^2 f_0$. Realization of f_0 as the Poisson transform follows from Corollary 1.7.4(i) and Theorem 1.7.2. □

3.4 Comments on the sharpness of the results

We conclude this chapter pointing out that the growth estimates and the condition on the constant A, used in various results are not arbitrary. We shall take up Corollary 3.2.6 for the spherical mean value operator M_t only for this discussion. For other cases, the argument will be similar. We recall that a nonzero eigenfunction f_0 of Δ cannot be contained in $L^{p,r}(X)$ with any $p < 2$. Thus we have to take $f_k \in L^{q,r}(X)$ with $q \geq 2$. If we take $f_k \in L^{q',r}(X)$, with either $1 \leq p < q \leq 2$ and $1 \leq r \leq \infty$ or $q = p$ and $1 \leq r < \infty$, then any f_k cannot be an eigenfunction of Δ with the eigenvalue $-4\rho^2/pp'$, without being identically 0 (see Proposition 1.7.1).

The only possibility we are left with is $f_k \in L^{q',r}(X)$, with $1 \leq q < p \leq 2$ and $1 \leq r \leq \infty$. For this case, we can find $\lambda_1, \lambda_2 \in S_q$ with $\lambda_1 \neq \pm\lambda_2$ and $|\varphi_{\lambda_1}(a_t)| = |\varphi_{\lambda_2}(a_t)| = \varphi_{i\gamma_p\rho}(a_t)$ (see Lemma 3.2.2). Suppose that $\varphi_{\lambda_1}(a_t) = \varphi_{i\gamma_p\rho}(a_t)e^{i\theta_1}$ and $\varphi_{\lambda_2}(a_t) = \varphi_{i\gamma_p\rho}(a_t)e^{i\theta_2}$. We define

$$f_k = e^{ik\theta_1}\varphi_{\lambda_1} + e^{ik\theta_2}\varphi_{\lambda_2} \text{ for } k \in \mathbb{Z}.$$

Then

$$M_t f_k = e^{ik\theta_1} M_t \varphi_{\lambda_1} + e^{ik\theta_2} M_t \varphi_{\lambda_2} = \varphi_{i\gamma_p\rho}(a_t) f_{k+1}.$$

It can be verified that the sequence $\{f_k\}_{k\in\mathbb{Z}}$ satisfies the hypothesis of the theorem with $\|\cdot\|_{q',r}$-norm replacing $\|\cdot\|_{p',\infty}$-norm. However, f_0 is clearly not an eigenfunction of Δ. Thus the only suitable Lorentz norm in this case is the $\|\cdot\|_{p',\infty}$-norm.

It is also clear from Theorem 3.2.3 (b) and the example given to establish Theorem 3.2.3 (c), that taking an $A \in \mathbb{C}$ with $|A| \neq \varphi_{i\gamma_p\rho}(a_t)$ in the hypothesis of Corollary 3.2.6 is not meaningful.

Chapter 4

Mean value property in limit, a result of Plancherel–Pólya and Benyamini–Weit

In this chapter we set our task to obtain analogues of the theorems of Plancherel-Pólya and Beniyamini-Weit (Theorem 0.2.6 and Theorem 0.2.7) for ball, sphere and shell (i.e. annular region) averages of functions f on X which characterize eigenfunctions of Δ with arbitrary eigenvalues, instead of only harmonic functions. In these results, no restriction on the growth of f will be assumed. Using them we shall prove versions for functions in appropriate Lebesgue class, where pointwise convergence of the averages (as radius tend to infinity) will be replaced by norm convergence.

4.1 Ball-MVP in limit for functions without growth restriction

4.1.1 Statements of the results

We retain the notation used in the previous chapters. We recall, in particular that X is a rank one Riemannian symmetric space of noncompact type, $B(o,r)$ is the geodesic ball of radius $r > 0$ centered at the origin o in X, $|B(o,r)|$ is its volume and $\chi_{B(o,r)}$ is its indicator function. For a fixed $\lambda \in \mathbb{C}$, let

$$V_r^\lambda = \int_{B(o,r)} \varphi_\lambda(x)\, dx = \int_0^r \varphi_\lambda(a_t) J(t)\, dt,$$

where $J(t)$ denotes the Jacobian of the polar decomposition (see Section 1.2) and

$$m_r^\lambda = (V_r^\lambda)^{-1}\chi_{B(o,r)}.$$

For a function f, by abuse of language, $f * m_r^\lambda$ will be frequently referred to as the *ball-average*, when λ is fixed. Sphere and shell-average which will be taken up in sections 4.2 and 4.3 will have similar connotations.

Main result of this section is the following analogue of Theorem 0.2.6.

Theorem 4.1.1. *Suppose that for a function $f \in L^1_{loc}(X)$ and a $\lambda \in \mathbb{C}$,*

$$\lim_{r\to\infty} f * m_r^\lambda(x) \to g(x)$$

for some function g on X and for every $x \in X$. If there is a positive function $\psi \in L^1_{loc}(X)$ and a positive function $r_0 \in L^\infty_{loc}(X)$ such that $|f * m_r^\lambda(x)| \leq \psi(x)$ whenever $r \geq r_0(x)$, then $\Delta g = -(\lambda^2 + \rho^2)g$.

We also state a simpler version which is structurally similar to Theorem 0.2.7.

Theorem 4.1.2. *Fix a $\lambda \in \mathbb{C}$. Let f, g be two continuous functions on X. If for all $x \in X$, $f * m_r^\lambda(x) \to g(x)$ as $r \to \infty$ uniformly on compact sets, then $\Delta g = -(\lambda^2 + \rho^2)g$.*

It follows from (1.2.8), (1.2.9) that for large $r > 0$,

$$V_r^{i\gamma_p\rho} \asymp e^{\frac{2\rho r}{p}} \text{ for } 0 < p < 2, \text{ and } V_r^0 \asymp re^{\rho r}. \tag{4.1.1}$$

In fact, except for nonzero real λ, the quantity V_r^λ is nonzero for large r. For nonzero real λ, the situation is more delicate as V_r^λ can be zero for a countable discrete set of r. So we have to consider $r \to \infty$ avoiding these points. See Remark 4.1.8 below for more details which will justify the statements. We shall prove these results in Subsection 4.1.4 after gathering necessary ingredients.

4.1.2 Convexity of distance

We recall that the *distance function* is convex for hyperbolic spaces X. It is indeed a very general phenomenon ([16, p. 176, Prop 2.2, Chap II]). The result we shall use is the following.

Proposition 4.1.3. *If X is a $\mathbf{CAT}(0)$ space, then the distance from a point $x_0 \in X$, $x \mapsto d(x_0, x)$ is a convex function from $X \to \mathbb{R}$, i.e. given any geodesics $\gamma : [0, 1] \to$*

X, parameterized proportional to arc length, the following inequality holds for all $t \in [0,1]$:

$$d(x_0, \gamma(t)) \leq (1-t)d(x_0, \gamma(0)) + td(x_0, \gamma(1)).$$

Note that Riemannian manifolds of non-positive curvatures (hence in particular all rank one Riemannian symmetric spaces of noncompact type and Damek–Ricci spaces) are C**AT**(0) spaces. See also [8, p. 24, Chap 1, Prop 5.4].

4.1.3 Characterization of eigenfunctions by the mean value property

We recall that for a suitable function f on X, its mean value on the geodesic sphere of radius $t > 0$ is denoted by $M_t f(x)$ and is defined by

$$M_t f(x) = \int_K f(xka_t)\, dk.$$

We note that $M_t f$ is also a function on X and $M_t f(x) = f * \sigma_t(x)$ where σ_t denotes the normalized surface measure of the geodesic sphere of radius t. Note that we can also define

$$M_y f(x) = \int_{K/M} f(xky)\, dk \text{ for } y \in G.$$

Then it is clear that $M_y = M_t$ where $d(o, yK) = t$. Eigenfunctions of Δ can be characterized through the following generalized mean value theorem [41, p. 402, Prop. 2.4], [42, p. 76, Prop 2.6; p. 414, Cor 2.3].

Proposition 4.1.4. *Let f be a continuous function on X and $\lambda \in \mathbb{C}$. Then f satisfies $\Delta f = -(\lambda^2 + \rho^2) f$ if and only if $M_y f(x) = f(x)\varphi_\lambda(y)$ for all $x, y \in X$.*

It is indeed enough to assume this mean value property for almost all "radii" taken from a neighbourhood of the origin of $X = G/K$. To establish this, let us rewrite the main argument of the converse side. Precisely, we have the following proposition.

Proposition 4.1.5. *Let $f \in L^1_{loc}(X)$ and $\lambda \in \mathbb{C}$. Let f satisfy $M_y f(x) = f(x)\varphi_\lambda(y)$ for almost every $x \in X$ and for almost every $y \in N_o$ for some neighbourhood N_o of the origin o in X, then $\Delta f = -(\lambda^2 + \rho^2) f$.*

Proof. We take a ball $B(o, r)$ of radius r with center o inside N_o and a radial function $h \in C_c^\infty(X)$ with its support contained inside $B(o, r) \subset N_o$ which satisfies

$\int_{B(o,r)} \varphi_\lambda(z)h(z)\,dz = 1$. Then for almost every $x \in X$,

$$\begin{aligned} f * h(x) &= \int_G f(xz)h(z)\,dz \\ &= \int_0^r \int_K f(xka_t)h(a_t)J(t)\,dk\,dt \\ &= f(x)\int_0^r \varphi_\lambda(a_t)h(a_t)J(t)\,dt \\ &= f(x). \end{aligned}$$

Therefore $M_y(f*h)(x) = (f*h)(x)\varphi_\lambda(y)$. We can thus assume that f is smooth. Consequently, for any fixed x, $y \mapsto M_y f(x)$ is a smooth function and

$$M_y f(x) = \int_K f(xky)\,dk = f(x)\varphi_\lambda(y) \text{ for all } x \in X, \text{ and for all } y \in N_o. \quad (4.1.2)$$

We define $F_x(y) = M_y f(x)$. Then clearly F_x is a function on X and

$$\Delta_y F_x(y) = f(x)\Delta_y \varphi_\lambda(y) = -(\lambda^2+\rho^2)f(x)\varphi_\lambda(y) \quad (4.1.3)$$

for all $x \in X$ and $y \in N_o$. Here we write Δ_y for Δ to emphasize that Δ is acting on F_x which is a function in y variable. Hence in particular $\Delta_y F_x(o) = -(\lambda^2+\rho^2)f(x)$. On the other hand from (4.1.2) we have

$$\Delta_y F_x(y) = \Delta_y \int_K f(xky)\,dk = \int_K \Delta_y f(xky)dk = \int_K (\Delta f)(xky)dk$$

by translation invariance of Δ and hence

$$\Delta_y F_x(o) = \int_K (\Delta f)(xk)dk = \Delta f(x). \quad (4.1.4)$$

From (4.1.3) and (4.1.4) we get

$$\Delta f(x) = -(\lambda^2+\rho^2)f(x).$$

□

4.1.4 Proof of the main results

We shall first prove Theorem 4.1.2. We shall isolate a few steps of the proof in the following lemmas.

Lemma 4.1.6. *Let $x_0 \in X$ and $r > d(x_0, o)$ be fixed. Suppose that for some $s_0 > 0$, $d(x_0, a_{s_0}) = r$. Then for any positive s, $d(x_0, a_s) > r$ if and only if $s > s_0$.*

Proof. We note that by triangle inequality,

$$s_0 = d(a_{s_0}, o) \geq d(a_{s_0}, x_0) - d(x_0, o) = r - d(x_0, o), \tag{4.1.5}$$

and for any s_2 satisfying $0 < s_2 < r - d(x_0, o)$,

$$d(a_{s_2}, x_0) \leq d(a_{s_2}, o) + d(o, x_0) = s_2 + d(o, x_0) < r. \tag{4.1.6}$$

To prove the converse side of the assertion, let us take a $s_1 > s_0$. Then for s_2 as above we have by (4.1.5),

$$s_2 < r - d(x_0, o) \leq s_0 < s_1.$$

Thus we have $s_2 < s_0 < s_1$, and by (4.1.6) $d(a_{s_2}, x_0) < r$. We assume that $d(a_{s_1}, x_0) \leq r$ and take $\gamma(t) = a_{(1-t)s_2 + ts_1}, t \in [0, 1]$. Then there exists a $t_0 \in [0, 1]$ such that $(1 - t_0)s_2 + t_0 s_1 = s_0$. Applying Proposition 4.1.3 we get

$$d(x_0, a_{s_0}) = d(x_0, \gamma(t_0)) \leq (1 - t_0)d(x_0, a_{s_2}) + t_0 d(x_0, a_{s_1}) < (1 - t_0)r + t_0 r = r,$$

which contradicts the hypothesis.

We shall now prove the forward side of the assertion. We have, $d(x_0, a_s) > r$ which implies that $d(x_0, o) + d(o, a_s) \geq d(x_0, a_s) > r$, i.e. $s > r - d(x_0, o)$. Hence for s_2 as above,

$$s_2 < r - d(x_0, o) < s.$$

Therefore if we assume that $s < s_0$, then we have $s_2 < s < s_0$, $d(a_{s_2}, x_0) < r$ by (4.1.6) and $d(a_{s_0}, x_0) = r$ by the hypothesis. Applying Proposition 4.1.3 as before we conclude that $d(a_s, x_0) < r$ which contradicts the hypothesis. Therefore $s \geq s_0$. But $d(a_{s_0}, x_0) = r$ by hypothesis. Hence $s > s_0$. This completes the proof of the forward side and hence the lemma. □

Lemma 4.1.7. *Fix a $\lambda \in \mathbb{C}$. Then there exists a sequence $\{r_n\}_{n \in \mathbb{N}}$ of positive real numbers with $r_n \uparrow \infty$ and a $\delta > 0$ such that for any $r, s \in [r_n - \delta, r_n + \delta]$, $|V_r^\lambda|/|V_s^\lambda| \leq C$ for some constant C independent of n, r and s.*

Proof. We shall deal with three separate sets of λ which will exhaust $\mathbb{C}$.

Case 1: We take complex λ such that $\lambda \notin \mathbb{R}$. Since $\varphi_\lambda = \varphi_{-\lambda}$, we have $V_r^\lambda = V_r^{-\lambda}$. Hence without loss of generality we may assume that $\Im\lambda < 0$. As for large t,

$\sinh t \asymp \cosh t \asymp e^t$ and $\phi_\lambda^{(\alpha,\beta)}(a_t) \asymp e^{(|\Im\lambda|-\varrho)t}$ (see (1.3.3)), from (1.3.4) we can find $r_0 > 0$ and constant C' such that $|V_r^\lambda| \asymp e^{C'r}$ whenever $r > r_0$. Hence for any $\delta > 0$ and any $R > 0$ with $R-\delta > r_0$, for any $r, s \in [R-\delta, R+\delta]$ we get $\frac{|V_r^\lambda|}{|V_s^\lambda|} \asymp e^{2C'\delta} < C$.

Case 2: We take $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. It is enough to consider $\lambda > 0$ as $\varphi_\lambda = \varphi_{-\lambda}$. We shall use the notation of Subsection 1.3. From (1.3.4) we have

$$V_r^\lambda = C \sinh^{\frac{m+k}{2}}\left(\frac{r}{2}\right) \cosh^{\frac{k}{2}}\left(\frac{r}{2}\right) u_\lambda(r),$$

where

$$\begin{aligned} u_\lambda(r) &= \sinh^{\frac{m+k+2}{2}}\left(\frac{r}{2}\right) \cosh^{\frac{k+2}{2}}\left(\frac{r}{2}\right) \phi_{2\lambda}^{(\alpha',\beta')}\left(\frac{r}{2}\right) \\ &= \sinh^{\alpha'+\frac{1}{2}}\left(\frac{r}{2}\right) \cosh^{\beta'+\frac{1}{2}}\left(\frac{r}{2}\right) \phi_{2\lambda}^{(\alpha',\beta')}\left(\frac{r}{2}\right). \end{aligned} \tag{4.1.7}$$

It follows that (see [83, (6.12)-(6.15)]),

$$u_\lambda(r) = C_\lambda(\cos(\lambda r + \theta_\lambda) + \epsilon_\lambda^*(r)), \tag{4.1.8}$$

where $C_\lambda > 0$, $\theta_\lambda \in \mathbb{R}$ and $\epsilon_\lambda^*(r) = O(e^{-r})$. We find a $t_0 > 0$ such that for all $r > t_0$, $|\epsilon_\lambda^*(r)| < 1/4$. For $n \in \mathbb{N}$, we define $r_n = (2n\pi - \theta_\lambda)/\lambda$ and take $\delta = \pi/3\lambda$. For $r \in [r_n - \delta, r_n + \delta]$, $\lambda r + \theta_\lambda \in [2n\pi - \pi/3, 2n\pi + \pi/3]$, consequently $\cos(\lambda r + \theta_\lambda) \geq 1/2$.

Hence for large $n \in \mathbb{N}$ so that $r_n - \delta > t_0$ and for $r \in [r_n - \delta, r_n + \delta]$, from (4.1.8) we have

$$0 < C_\lambda/4 \leq u_\lambda(r) \leq 2C_\lambda.$$

Therefore for $r, s \in [r_n - \delta, r_n + \delta]$, V_r^λ and V_s^λ are positive and

$$\frac{|V_r^\lambda|}{|V_s^\lambda|} = \frac{V_r^\lambda}{V_s^\lambda} \asymp \frac{e^{C'r} u_\lambda(r)}{e^{C's} u_\lambda(s)} \leq C e^{2C'\delta}$$

for some constants C, C' independent of n.

Case 3: We take $\lambda = 0$. We shall use the estimate $\varphi_0(a_t) \asymp (1+t)e^{-\rho t}$ and the estimate of the Jacobian of the polar decomposition that for $s > 1$, $J(s) \asymp e^{2\rho s}$.

Let $V_1 = \int_0^1 \varphi_0(a_s) J(s)\, ds$. Then

$$V_r^0 = \int_0^r \varphi_0(a_s) J(s)\, ds$$

$$
\begin{aligned}
&= V_1 + \int_1^r \varphi_0(a_s) J(s)\, ds \\
&\asymp V_1 + \int_1^r s\, e^{\rho s}\, ds \\
&\asymp re^{\rho r} \quad \text{for some } r > r_0. \qquad (4.1.9)
\end{aligned}
$$

We choose any $\delta > 0$. For sufficiently large $R > 0$ and $r, s \in [R-\delta, R+\delta]$,

$$\frac{V_r^0}{V_s^0} \asymp \frac{re^{\rho r}}{se^{\rho s}} \leq \frac{(R+\delta)e^{\rho(R+\delta)}}{(R-\delta)e^{\rho(R-\delta)}}.$$

Hence

$$\limsup_R \frac{V_r^0}{V_s^0} \leq C_\delta$$

for some constant C_δ. It is now easy to get the desired sequence.

Thus for each $\lambda \in \mathbb{C}$, there is a sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive real numbers with $r_n \uparrow \infty$ and there exists a $\delta > 0$ such that for any $r, s \in [r_n - \delta, r_n + \delta]$, $|V_r^\lambda|/|V_s^\lambda| \leq C_\delta$ for some constant C_δ which depends on δ and is independent of n. □

Remark 4.1.8. It is clear from Case 1 and Case 3 of the lemma above, that for a complex λ which is not a nonzero real number, there exists a constant $C_\lambda > 0$, such that for all $r > C_\lambda$, the quantity V_r^λ is nonzero. Therefore for these λ all the statements of theorems and lemmas above, involving $(V_r^\lambda)^{-1}$, do not lead to any confusion. However if λ is a nonzero real number, it can be easily deduced from Case 2 of Lemma 4.1.7 that V_r^λ is zero for countably many radii r. Hence for the nonzero real λ, in all the statements $r \to \infty$ is interpreted as $r \to \infty$ through $\mathbb{R}^+ \setminus D_0$ where D_0 is the set of these discrete zeros.

Lemma 4.1.9. *Fix a $\lambda \in \mathbb{C}$. Let f be a radial function in $L^1_{loc}(X)$ such that it satisfies $\lim_{r\to\infty} f * m_r^\lambda(o) = L$ for some constant L. Then there exists a neighbourhood N_o of the origin and a sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive real numbers with $r_n \uparrow \infty$ such that $\lim_{n\to\infty} f * m_{r_n}^\lambda(x) = L\varphi_\lambda(x)$ for any $x \in N_o$.*

Proof. As λ is fixed, in this proof we shall write m_r for m_r^λ, V_r for V_r^λ etc. for convenience. Since m_r is radial, hence $m_r(ky) = m_r(y)$ and $m_r(y^{-1}) = m_r(y)$, we have

$$\varphi_\lambda * m_r(x) = \int_X \varphi_\lambda(xy) m_r(y)\, dy = \varphi_\lambda(x) \int_X \varphi_\lambda(y) m_r(y)\, dy = \varphi_\lambda(x).$$

In particular $\varphi_\lambda * m_r(o) = 1$. Take $g(x) = f(x) - L\varphi_\lambda(x)$. Then

$$g * m_r(o) = f * m_r(o) - L\varphi_\lambda * m_r(o) = f * m_r(o) - L.$$

Therefore from the hypothesis we have $g * m_r(o) \to 0$ as $r \to \infty$. It suffices to prove that $g * m_r(x) \to 0$, because

$$g * m_r(x) = f * m_r(x) - L\varphi_\lambda * m_r(x) = f * m_r(x) - L\varphi_\lambda(x).$$

Thus our modified statement to prove is the following: Let f be a radial locally integrable function on X. If $\lim_{r\to\infty} f * m_r(o) = 0$, then there exists a neighbourhood N_o of the origin and positive sequence $\{r_n\}_{n\in\mathbb{N}}$ with $r_n \uparrow \infty$ such that $\lim_{n\to\infty} f * m_{r_n}(x) = 0$ for any $x \in N_o$. Let $s, t > 0$. We have by triangle inequality,

$$d(o, a_{-s}ka_t) \leq d(o, a_{-s}) + d(a_{-s}, a_{-s}ka_t) = d(o, a_{-s}) + d(o, ka_t) = s + t$$

and

$$\begin{aligned} s = d(o, a_{-s}) &\leq d(o, a_{-s}ka_t) + d(a_{-s}ka_t, a_{-s}) \\ &= d(o, a_{-s}ka_t) + d(o, ka_t) = d(o, a_{-s}ka_t) + t. \end{aligned}$$

Hence for $s > 0, t > 0$,

$$s - t \leq d(o, a_{-s}ka_t) \leq s + t, \text{ for all } k \in K.$$

Therefore for $t < r$, if $s < r - t$ then $d(o, a_{-s}ka_t) < r$ for all $k \in K$ and if $s > r + t$ then $d(o, a_{-s}ka_t) > r$ for all $k \in K$.

For a fixed $k \in K$ and $t > 0$, define a continuous function $\alpha = \alpha_k$ in s by

$$\alpha(s) = d(ka_t, a_s).$$

Then

$$\alpha(s) < r \text{ if } s < r - t \text{ and } \alpha(s) > r \text{ if } s > r + t.$$

So there exists $s_0 = s_0(k) \in [r - t, r + t]$ such that $\alpha(s_0) = r$. By Lemma 4.1.6, $s > s_0$ if and only if $\alpha(s) > r$.

Therefore, we have for a $r > 0$ and $0 < t < r$,

$$f * m_r(a_t) = \frac{1}{V_r} \int_X f(x)\chi_{B_r}(x^{-1}a_t)\, dx \tag{4.1.10}$$

$$= \frac{1}{V_r}\int_K \int_{\mathfrak{a}_+} f(a_s)\chi_{B_r}(a_{-s}ka_t)J(s)\,ds\,dk$$
$$= \frac{1}{V_r}\int_K \int_0^{s_0} f(a_s)J(s)\,ds\,dk$$
$$= \int_K \frac{V_{s_0}}{V_r}\frac{1}{V_{s_0}}\int_0^{s_0} f(a_s)J(s)\,ds\,dk$$

where s_0 depends on k.

By Lemma 4.1.7, we have a sequence $r_n \uparrow \infty$ and a fixed $\delta > 0$ such that for any $s, s' \in [r_n - \delta, r_n + \delta]$, $\frac{|V_s|}{|V_{s'}|} \leq C$ for some constant C independent of n. We shall show that for this sequence $\{r_n\}_{n\in\mathbb{N}}$ and for $0 < t < \delta$, $f * m_{r_n}(a_t) \to 0$ as $n \to \infty$.

In (4.1.10) we take $r = r_n$ and $0 < t < \delta$. Then $r_n, s_0 \in (r_n - t, r_n + t) \subset (r_n - \delta, r_n + \delta)$ and $\frac{|V_{s_0}|}{|V_{r_n}|} \leq C$. Hence

$$|f * m_{r_n}(a_t)| \leq C\int_K \left|\frac{1}{V_{s_0}}\int_0^{s_0} f(a_s)J(s)\,ds\right| dk.$$

By hypothesis for any given $\epsilon > 0$ there is a M such that for $u > M$,

$$\left|\frac{1}{V_u}\int_0^{u} f(a_s)J(s)\,ds\right| < \epsilon.$$

Thus if we take n so large that $r_n > M + t$, then $s_0 = s_0(k) > r_n - t > M$ and

$$|f * m_{r_n}(a_t)| < C\epsilon.$$

This proves the assertion for $N_o = B(o, \delta)$. □

We shall now complete the proof of Theorem 4.1.2.

Proof of Theorem 4.1.2. Since λ is fixed, as above we shall drop the superscript λ and write m_r for m_r^λ in this proof. We recall that the left translation of a function f by $x \in G$ is denoted by $\ell_x f$.

The proof is now based on these observations.

(i) As m_r is radial, $R(f * m_r) = R(f) * m_r$.

(ii) Since for any fixed $x \in X$, the set Kx is compact in X, from hypothesis we conclude that $f * m_r(y) \to g(y)$ uniformly for $y \in Kx$. As

$$R(f * m_r)(x) = \int_K f * m_r(kx)\ dk,$$

we have, $R(f * m_r)(x) \to R(g)(x)$. Together with (i) above this gives $R(f) * m_r(x) \to R(g)(x)$.

(iii) We note that $\ell_y f * m_r(x) = \ell_y(f * m_r)(x)$ for any $y \in G, x \in X$. Therefore if $f * m_r \to g$ uniformly on compact subsets of X, then for any fixed $y \in G$,

$$\ell_y f * m_r = \ell_y(f * m_r) \to \ell_y g$$

uniformly on compact subsets of X.

(iv) By (ii) and (iii) $R(\ell_y f) * m_r(x) \to R(\ell_y g)(x)$ for all $x \in X$ and any $y \in G$.

(v) For any locally integrable function F on G/K, $RF(o) = F(o)$.

From (iv), we have for any $y \in G$,

$$R(\ell_y f) * m_r(o) \to R(\ell_y g)(o) = \ell_y g(o) = g(y).$$

Since $R(\ell_y f)$ is K-biinvariant, from this and Lemma 4.1.9, we have a neighbourhood N_o of the origin and positive sequence $\{r_n\}_{n\in\mathbb{N}}$ with $r_n \uparrow \infty$ such that

$$R(\ell_y f) * m_{r_n}(x) \to g(y)\varphi_\lambda(x) \text{ for any } x \in N_o.$$

Together with (iv) this implies that for all $x \in N_o$ and all $y \in G$,

$$M_x g(y) = R(\ell_y g)(x) = g(y)\varphi_\lambda(x).$$

An application of Proposition 4.1.5 now completes the proof. □

Proof of Theorem 4.1.1. Our target is to reach the step (iv) of the previous proof. The result follows from that step and Lemma 4.1.9. By the hypothesis for all $x \in X, y \in G$ and $k \in K$,

$$\ell_y(f * m_r^\lambda)(kx) \to \ell_y g(kx) \text{ as } r \to \infty.$$

We need to show that as $r \to \infty$,

$$\int_K \ell_y(f * m_r^\lambda)(kx)dk \to \int_K \ell_y g(kx)dk,$$

which is same as (iv) above.

Since r_0 is locally bounded, for almost every fixed $x \in X$ and $y \in G$, there exists a constant $C_{x,y} > 0$ which depends on x, y, such that $r_0(ykx) \leq C_{x,y}$ for all $k \in K$. By the hypothesis

$$|f * m_r^\lambda(x)| \leq \psi(x)$$

if $r > C_{x,e}$. Therefore $|g(x)| \leq \psi(x)$ for almost every $x \in X$. Hence $g \in L^1_{loc}(X)$. Similarly by the hypothesis for almost every fixed $x \in X, y \in G$, $k \in K$,

$$|\ell_y(f * m_r^\lambda)(kx)| \leq \ell_y\psi(kx)$$

whenever $r > C_{x,y}$. We also note that, $k \mapsto \ell_y\psi(kx)$ is an integrable function on K for almost every fixed $x \in X$, $y \in G$. Therefore by dominated convergence theorem,

$$\int_K \ell_y(f * m_r^\lambda)(kx)dk \to \int_K \ell_y g(kx)dk,$$

which was our target. □

4.2 Shell-MVP in limit for functions without growth restriction

Aim of this section is to prove an analogue of Theorem 4.1.2 replacing ball-averages by shell-averages. For technical reasons and for keeping the exposition simple, we shall only consider eigenvalues in $(-\infty, -\rho^2]$, vis-à-vis, the spectral parameter λ in $i\mathbb{R}$.

4.2.1 Statement of the main result

For $0 < r_1 < r_2$, let $\mathbb{A}_{r_1,r_2}(x)$ denote the annulus or shell centered at x with inner radius r_1 and outer radius r_2. For $\lambda \in i\mathbb{R}$ we define

$$\begin{aligned} V^\lambda_{r_1,r_2} &:= \int_{\mathbb{A}_{r_1,r_2}(x)} \varphi_\lambda(x)\ dx \\ &= \int_{r_1}^{r_2} \varphi_\lambda(a_t)\ J(t)\ dt \\ &= V^\lambda_{r_2} - V^\lambda_{r_1}. \end{aligned}$$

We recall from (1.3.3) and (1.3.4) that for each $\lambda \notin \mathbb{R}$ there exists an $r_\lambda > 0$ depending on λ, such that for all $r > r_\lambda$,

$$V_r^\lambda \asymp e^{(|\Im\lambda|+\rho)r}. \tag{4.2.1}$$

From (4.1.1) we also have

$$V_r^0 \asymp re^{\rho r} \text{ for all sufficiently large } r. \tag{4.2.2}$$

For $0 < r_1 < r_2$ and $\lambda \in i\mathbb{R}$, let

$$a_{r_1,r_2}^\lambda := (V_{r_1,r_2}^\lambda)^{-1}\, \chi_{\mathbb{A}_{r_1,r_2}(o)}.$$

We fix $d > 0, \delta > 0$ and consider radii r_1, r_2 which satisfy

$$d < r_2 - r_1 < d + \delta.$$

For a continuous function f, we say

$$f * a_{r_1,r_2}^\lambda(x) \to g(x)$$

uniformly on compact sets as $r_1 \to \infty$ with $d < r_2 - r_1 < d + \delta$, to mean that for a compact set $\mathcal{K}$ of X, given an $\epsilon > 0$, there exists $M > 0$, such that for all $r_1 > M$ and all $r_2 > 0$ satisfying $d < r_2 - r_1 < d + \delta$,

$$|f * a_{r_1,r_2}^\lambda(x) - g(x)| < \epsilon$$

for all $x \in \mathcal{K}$. When $\mathcal{K} = \{x\}$, we simply write $\lim_{r_1\to\infty} f * a_{r_1,r_2}^\lambda(x) = g(x)$ with $d < r_2 - r_1 < d + \delta$. With this notation we offer the main result of this section.

Theorem 4.2.1. *Fix a $\lambda \in i\mathbb{R}$, $d > 0$ and $\delta > 0$. Let f be a continuous function on X such that*

$$f * a_{r_1,r_2}^\lambda(x) \to g(x)$$

uniformly on compact sets as $r_1 \to \infty$ with $d < r_2 - r_1 < d + \delta$. Then $\Delta g = -(\lambda^2 + \rho^2)g$.

4.2.2 Proof of the main result

Following lemma is an intermediate step. Below λ, d, δ and $a^{\lambda}_{r_1,r_2}$ are as in Theorem 4.2.1. For convenience, we shall write $r_1 \to \infty$ to mean $r_1 \to \infty$ with $d < r_2 - r_1 < d + \delta$.

Lemma 4.2.2. *Let $\mu_j = a^{\lambda}_{j,j+d+\frac{\delta}{2}}$ for $j \in \mathbb{N}$. Let f be a radial continuous function on X such that it satisfies $\lim_{r_1\to\infty} f * a^{\lambda}_{r_1,r_2}(o) = L$. Then there exists a neighbourhood N_o of the origin such that $\lim_{j\to\infty} f * \mu_j(x) = L\varphi_\lambda(x)$ for any $x \in N_o$.*

Proof. As λ is fixed we shall write a_{r_1,r_2} for $a^{\lambda}_{r_1,r_2}$ and V_{r_1,r_2} for $V^{\lambda}_{r_1,r_2}$, unless it is required to mention λ. It is also easy to see that $\varphi_\lambda * a_{r_1,r_2}(x) = \varphi_\lambda(x)$ and hence in particular $\varphi_\lambda * a_{r_1,r_2}(o) = 1$.

Take $g(x) = f(x) - L\varphi_\lambda(x)$. Then

$$g * a_{r_1,r_2}(o) = f * a_{r_1,r_2}(o) - L\varphi_\lambda * a_{r_1,r_2}(o) = f * a_{r_1,r_2}(o) - L.$$

Therefore the modified hypothesis is $g * a_{r_1,r_2}(o) \to 0$. Using this hypothesis we shall show that $g * \mu_j(x) \to 0$ as $j \to \infty$. Indeed this is enough, because,

$$g * \mu_j(x) = f * \mu_j(x) - L\varphi_\lambda * \mu_j(x) = f * \mu_j(x) - L\varphi_\lambda(x).$$

Thus our modified statement to prove is the following:

Let f be a radial continuous function on X. If $\lim_{r_1\to\infty} f * a_{r_1,r_2}(o) = 0$, then there exists a neighbourhood N_o of the origin such that $\lim_{j\to\infty} f * \mu_j(x) = 0$ for any $x \in N_o$.

As f is radial, it follows from polar decomposition that

$$\begin{aligned} f * a_{r_1,r_2}(o) &= \frac{1}{V_{r_1,r_2}} \int_K \int_{r_1}^{r_2} f(ka_s) J(s)\, ds\, dk \\ &= \frac{1}{V_{r_1,r_2}} \int_K \int_{r_1}^{r_2} f(a_s) J(s)\, ds\, dk \\ &= \frac{1}{V_{r_1,r_2}} \int_{r_1}^{r_2} f(a_s) J(s)\, ds. \end{aligned}$$

Hence from hypothesis we have

$$\lim_{r_1\to\infty} \frac{1}{V_{r_1,r_2}} \int_{r_1}^{r_2} f(a_s) J(s)\ ds = 0 \text{ whenever } d < r_2 - r_1 < d + \delta. \qquad (4.2.3)$$

Fix $x \in X$ with $|x| < \frac{\delta}{4}$. For $t \geq 0$ and $k \in K$, we claim that

$$t - |x| \leq |a_{-t}kx| \leq t + |x|, \tag{4.2.4}$$

as

$$|a_{-t}kx| = d(0, a_{-t}kx) \leq d(0, a_{-t}k) + d(a_{-t}k, a_{-t}kx) = t + |x|$$

and

$$t = d(0, a_{-t}k) \leq d(0, a_{-t}kx) + d(a_{-t}kx, a_{-t}k) = |a_{-t}kx| + |x|.$$

From (4.2.4) it follows that

$$|a_{-t}kx| > r_2 \text{ if } t > r_2 + |x| \text{ and } |a_{-t}kx| < r_2 \text{ if } t < r_2 - |x|.$$

Hence by continuity and by Lemma 4.1.6 for fixed $k \in K$, we can find a unique $t_k \in (r_2 - |x|, r_2 + |x|)$ with $|a_{-t_k}kx| = r_2$ and $|a_{-t}kx| < r_2$ if and only if $t < t_k$. Similarly for fixed $k \in K$, we can find a unique $s_k \in (r_1 - |x|, r_1 + |x|)$ with $|a_{-s_k}kx| = r_1$ and $|a_{-t}kx| < r_1$ if and only if $t < s_k$. Clearly $s_k < t_k$ and

$$r_2 - r_1 - 2|x| < t_k - s_k < r_2 - r_1 + 2|x|. \tag{4.2.5}$$

Therefore

$$\begin{aligned} |f * a_{r_1,r_2}(x)| &= \left| \frac{1}{V_{r_1,r_2}} \int_S f(y) \chi_{\mathbb{A}_{r_1,r_2}}(y^{-1}x) dy \right| \\ &= \left| \frac{1}{V_{r_1,r_2}} \int_K \int_{\mathbb{R}^+} f(ka_t) \chi_{\mathbb{A}_{r_1,r_2}}(a_{-t}kx) J(t)\ dt\ dk \right| \\ &= \left| \frac{1}{V_{r_1,r_2}} \int_K \int_{s_k}^{t_k} f(a_t) J(t)\ dt\ dk \right| \\ &= \left| \int_K \frac{V_{s_k,t_k}}{V_{r_1,r_2}} \frac{1}{V_{s_k,t_k}} \int_{s_k}^{t_k} f(a_t) J(t)\ dt\ dk \right| \\ &\leq \int_K \frac{V_{r_1-|x|,r_2+|x|}}{V_{r_1,r_2}} \left| \frac{1}{V_{s_k,t_k}} \int_{s_k}^{t_k} f(a_t) J(t)\ dt \right| dk. \end{aligned} \tag{4.2.6}$$

We recall that there exists $r_\lambda > 0$ and $C_\lambda \in \mathbb{R}$ such that $V_r^\lambda \asymp e^{C_\lambda r}$ for $0 \neq \lambda \in i\mathbb{R}$ and $V_r^0 \asymp re^{C_0 r}$ for all $r > r_\lambda$ (see (4.1.1)). Hence there exists $r_0 > 0$ such that $V_{r_1-|x|,r_2+|x|}/V_{r_1,r_2} \leq C_x$ for some constant C_x whenever $r_1 > r_0$. Hence from (4.2.6) we get for $r_1 > r_0$,

$$|f * a_{r_1,r_2}(x)| = C_x \int_K \left| \frac{1}{V_{s_k,t_k}} \int_{s_k}^{t_k} f(a_t) J(t)\ dt \right| dk \tag{4.2.7}$$

If $r_1 = j$ and $r_2 = j + d + \frac{\delta}{2}$ in (4.2.6), then from (4.2.5) we get

$$d < t_k - s_k < d + \delta. \tag{4.2.8}$$

From (4.2.3), (4.2.7) and (4.2.8), it follows that $\lim_{j\to\infty} f * \mu_j(x) \to 0$ as $j \to \infty$. □

Completion of proof of Theorem 4.2.1. Let $\{h_i\}_{i\in\mathbb{N}}$ be a sequence of continuous functions converging uniformly to h over compact sets. Then we have the following observations.

(a) For any fixed $x \in G$, $\ell_x h_i \to \ell_x h$ as $i \to \infty$ uniformly over compact sets.

(b) $R(h_i) \to R(h)$ pointwise as $i \to \infty$.

Fix a point $x \in G$. By the hypothesis and observations (a), (b) we have

$$\ell_x(f * a_{r_1,r_2}) \to \ell_x g$$

uniformly on compact sets as $r_1 \to \infty$ and

$$R(\ell_x(f * a_{r_1,r_2})) \to R(\ell_x g),$$

pointwise as $r_1 \to \infty$. Since $R(\ell_x f) * a_{r_1,r_2} = R(\ell_x(f * a_{r_1,r_2}))$, we have

$$R(\ell_x f) * a_{r_1,r_2} \to R(\ell_x g),$$

pointwise as $r_1 \to \infty$. In particular $R(\ell_x f) * a_{r_1,r_2}(o) \to R(\ell_x g)(o)$, where o is the origin of X. By Lemma 4.2.2 (and using its notation), there exists a neighbourhood N_o of o, such that for all $y \in N_o$,

$$\lim_{j\to\infty} R(\ell_x f) * \mu_j(y) = R(\ell_x g)(o)\varphi_\lambda(y).$$

Hence $R(\ell_x g)(y) = R(\ell_x g)(o)\varphi_\lambda(y)$ for all $y \in N_o$. But as $M_y g(x) = R(\ell_x g)(y)$, we have $M_y g(x) = g(x)\varphi_\lambda(y)$ for all $y \in N_o$. Proposition 4.1.5 now asserts that $\Delta g = -(\lambda^2 + \rho^2)g$. □

4.2.3 Not a mean value operator

We conclude this section with an example of a right convolution operator which is *not a mean value operator*, yet shares the property with ball and shell-averages in the limit. Fix $0 < \alpha < \beta$ and a complex number λ which is not a nonzero real number. We define for every $t > 0$,

$$\omega_t^\lambda(x) = (V_{\alpha t,\beta t}^\lambda)^{-1} \chi_{B(o,\beta t)}(x).$$

Then right-convolution by ω_t^λ is not a mean value operator, e.g. in general for a harmonic function f, $f * \omega_t^{i\rho} \neq f$. However we shall see that it has the property in the limit.

Proposition 4.2.3. *Let $0 < \alpha < \beta$ and $\lambda \notin \mathbb{R} \setminus \{0\}$ be fixed. If for a continuous function f on X such that $\lim_{t\to\infty} f * \omega_t^\lambda(x) \to g(x)$ uniformly on compact sets, then $\Delta g = -(\lambda^2 + \rho^2)g$.*

Proof. From (4.2.1) and (4.2.2) it follows that

$$\lim_{t\to\infty} \frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda} = 0. \tag{4.2.9}$$

We also note that

$$\begin{aligned} \frac{1}{V_{\beta t}^\lambda} \int_{B(x,\beta t)} f(y)\, dy &= \frac{V_{\alpha t,\beta t}^\lambda}{V_{\beta t}^\lambda} \frac{1}{V_{\alpha t,\beta t}^\lambda} \int_{B(x,\beta t)} f(y)\, dy \\ &= \left(1 - \frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda}\right) \frac{1}{V_{\alpha t,\beta t}^\lambda} \int_{B(x,\beta t)} f(y)\ dy. \end{aligned} \tag{4.2.10}$$

Therefore $\lim_{r\to\infty} f * m_r^\lambda(x) \to g(x)$ uniformly on compact sets. Applying Theorem 4.1.2 we get the desired result. □

Remark 4.2.4. From (4.2.9) and (4.2.10) it follows that for a locally integrable function f on X and $x \in X$, $\lim_{t\to\infty} f * \omega_t^\lambda(x) \to L$ if and only if $\lim_{r\to\infty} f * m_r^\lambda(x) \to L$, where ω_t^λ is as defined in the proposition above.

4.3 Sphere-MVP in limit for functions without growth restriction

In this section we shall prove an analogue of Theorem 4.1.2 for sphere averages. We shall see that the result can actually be proved for an arbitrary normalized measure on K/M, in particular for surface measure on sphere. Let X be a Riemannian symmetric space of non-compact type of rank one and σ_t be the normalized surface measure of geodesic sphere of radius t with center at the origin $o = eK$. We recall that the average of a function f on a sphere of radius $t > 0$ is

$$M_t f(x) = f * \sigma_t(x) = \int_K f(xka_t)\, dk = \int_K f(xky)\, dk = M_y f(x),$$

where σ_t is the normalized surface measure on the sphere of radius t and $y \in X$ satisfies $|y| = d(y, o) = t$. The statement we aim to prove is the following.

Theorem 4.3.1. *Fix a $\lambda \in \mathbb{C}$. Let μ be a finite (normalized) Borel measure on K/M. If for two functions $f, g \in C(X)$,*

$$\frac{1}{\varphi_\lambda(y)} \int_K f(xky)\, d\mu(k) \to g(x)$$

uniformly on compact sets of X as $|y| \to \infty$, then $\Delta g = -(\lambda^2 + \rho^2) g$.

See [80] for the result on $\mathbb{R}^n$ characterizing harmonic function.

Remark 4.3.2. We recall that $\varphi_\lambda(y)$ is positive for $\lambda \in i\mathbb{R}$ and for $\lambda \notin \mathbb{R}$, $\varphi_\lambda(y) \neq 0$ when $|y|$ is sufficiently large (see (1.2.7)). However for nonzero real λ, $\varphi_\lambda(y) = 0$ for a set of measure zero of y. Hence for those λ, the statement of Theorem 4.3.1, is interpreted as $|y| \to \infty$ avoiding a discrete set of positive real numbers. See statement of Theorem 4.1.1 for similar situation and Remark 4.1.8 for the interpretation.

If μ is the normalized surface measure σ on the unit sphere and $\sigma_t^\lambda = \varphi_\lambda(a_t)^{-1}\sigma_t$, where σ_t is as defined above, then the result in Theorem 4.3.1 reduces to an analogue of Theorem 4.1.2 which reads as follows.

Theorem 4.3.3. *If for a $\lambda \in \mathbb{C}$ and for two functions $f, g \in C(X)$,*

$$f * \sigma_r^\lambda \to g$$

uniformly on compact sets of X as $r \to \infty$, then $\Delta g = -(\lambda^2 + \rho^2) g$.

4.3.1 Proof of Theorem 4.3.1

We shall slur over the difference between a compact subset $\mathcal{K}$ of G and its projection $\pi(\mathcal{K})$ on X. As in other places we shall not distinguish between integration on K/M and that on K. The following functional equation will be used frequently:

$$\int_K \varphi_\lambda(xky)\, dk = \varphi_\lambda(x)\varphi_\lambda(y). \tag{4.3.1}$$

First we shall show that it is enough to prove the result for the surface measure σ. Since y converges to ∞ through any direction, we can replace y by $k_1 y$ for any $k_1 \in K$, to get from the hypothesis that as $|y| \to \infty$,

$$\frac{1}{\varphi_\lambda(y)} \int_K \int_K f(xkk_1y)\, d\mu(k)\, dk_1 \to g(x).$$

Since $\int_K d\mu(k) = 1$, the left side reduces to

$$\frac{1}{\varphi_\lambda(y)} \int_K \int_K f(xkk_1y)\, dk_1\, d\mu(k) = \frac{1}{\varphi_\lambda(y)} \int_K f(xk_1y)\, dk_1.$$

Therefore our modified aim is the following which is Theorem 4.3.3 stated in a different way:

Theorem 4.3.4. *Fix a $\lambda \in \mathbb{C}$. If for two functions $f, g \in C(X)$,*

$$\frac{1}{\varphi_\lambda(y)} \int_K f(xky)\, dk \to g(x)$$

uniformly on compact sets of X as $|y| \to \infty$, then $\Delta g = -(\lambda^2 + \rho^2) g$.

Rest of the subsection is devoted to the proof of this theorem.

Proof of 4.3.4. Observe that the theorem will be proved if we can show

$$M_z(g) = \varphi_\lambda(z) g$$

for all $z \in G$ (or for $z \in G$ with $|z| < \epsilon$ for some $\epsilon > 0$) (see Proposition 4.1.5). We shall divide the proof in two cases.

Case 1: We take $\lambda \in \mathbb{C} \setminus \mathbb{R}^\times$. Fix $z \in G$. We shall verify that

$$\frac{1}{\varphi_\lambda(y)} M_z(M_y f) \to M_z g \tag{4.3.2}$$

uniformly on compact sets of X as $|y| \to \infty$.

Let us fix a compact subset $\mathcal{K}$ of G. Then $\widetilde{\mathcal{K}} = \mathcal{K}Kz$ is also a compact subset of G. By the hypothesis for a given $\epsilon > 0$ we have a $r_o > 0$ such that if $|y| > r_o$, then

$$\left| \frac{1}{\varphi_\lambda(y)} \int_K f(wky)\ dk - g(w) \right| < \epsilon \text{ for all } w \in \widetilde{\mathcal{K}}.$$

Take $x \in \mathcal{K}$. Then $xk_1z \in \widetilde{\mathcal{K}}$, for $k_1 \in K$. Hence for $|y| > r_o$ we have

$$\begin{aligned}
&\left|\frac{1}{\varphi_\lambda(y)} M_z(M_y f)(x) - M_z g(x)\right| \\
=&\left|\frac{1}{\varphi_\lambda(y)} \int_K \int_K f(xk_1zk_2y)\, dk_2\, dk_1 - \int_K g(xk_1z)\ dk_1\right| \\
=&\left|\int_K \left[\frac{1}{\varphi_\lambda(y)} \int_K f(xk_1zk_2y)\, dk_2 - g(xk_1z)\right]\ dk_1\right| \\
\leq& \int_K \left|\frac{1}{\varphi_\lambda(y)} \int_K f(xk_1zk_2y)\, dk_2 - g(xk_1z)\right|\ dk_1 \\
<&\epsilon.
\end{aligned}$$

This completes the verification that

$$\frac{1}{\varphi_\lambda(y)} M_z(M_y f) \to M_z g$$

uniformly on the compact set $\mathcal{K}$ as $|y| \to \infty$, which is asserted above.

We fix a compact subset $\mathcal{K}$ of G. From the hypothesis we know that there exists $r_o > 0$ such that if $|w| > r_o$, then

$$\left|\frac{1}{\varphi_\lambda(w)} \int_K f(xkw)dk - g(x)\right| < \frac{\epsilon}{\varphi_{i\Im\lambda}(z)} \text{ for all } x \in \mathcal{K}.$$

Take $|y| > r_o + |z|$. Then

$$\begin{aligned}
&\left|\frac{1}{\varphi_\lambda(y)} M_z(M_y f)(x) - \varphi_\lambda(z) g(x)\right| \\
=&\left|\frac{1}{\varphi_\lambda(y)} \int_K \int_K f(xk_1zk_2y)\, dk_2\, dk_1 - \varphi_\lambda(z) g(x)\right| \\
=&\left|\frac{1}{\varphi_\lambda(y)} \int_K \int_K f(xk_1zk_2y)\, dk_1\, dk_2 - \varphi_\lambda(z) g(x)\right| \\
=&\left|\frac{1}{\varphi_\lambda(y)} \int_K \int_K f(xk_1zk_2y)\, dk_1\, dk_2 - \int_K \frac{\varphi_\lambda(zk_2y)}{\varphi_\lambda(y)} g(x)\ dk_2\right| \\
=&\left|\int_K \frac{\varphi_\lambda(zk_2y)}{\varphi_\lambda(y)} \left[\frac{1}{\varphi_\lambda(zk_2y)} \int_K f(xk_1zk_2y)\, dk_1 - g(x)\right] dk_2\right| \\
\leq& \int_K \frac{|\varphi_\lambda(zk_2y)|}{|\varphi_\lambda(y)|} \left|\frac{1}{\varphi_\lambda(zk_2y)} \int_K f(xk_1zk_2y)\, dk_1 - g(x)\right| dk_2. \qquad (4.3.3)
\end{aligned}$$

Since $|y| > r_o + |z|$, by triangle inequality we have $|zk_2y| \geq |y| - |z| > r_o$. Hence

$$\left| \frac{1}{\varphi_\lambda(zk_2y)} \int_K f(xk_1zk_2y)\, dk_1 - g(x) \right| < \frac{\epsilon}{\varphi_{i\Im\lambda}(z)}.$$

Thus

$$\left| \frac{1}{\varphi_\lambda(y)} M_z(M_yf)(x) - \varphi_\lambda(z)g(x) \right| < \frac{\epsilon}{\varphi_{i\Im\lambda}(z)} \int_K \frac{|\varphi_\lambda(zk_2y)|}{|\varphi_\lambda(y)|} dk_2. \tag{4.3.4}$$

If $\lambda \in i\mathbb{R}$, then $\varphi_\lambda(y)$ is positive. For such λ using $\int_K \varphi_\lambda(zk_2y)\, dk_2 = \varphi_\lambda(z)\varphi_\lambda(y)$, we get from (4.3.4) that

$$\left| \frac{1}{\varphi_\lambda(y)} M_z(M_yf)(x) - \varphi_\lambda(z)g(x) \right| < \epsilon$$

for λ belonging to imaginary axis and for all $|y| > r_o + |z|$.

Otherwise i.e. for $\lambda \notin \mathbb{R} \cup i\mathbb{R}$ from (4.3.4) we get

$$\left| \frac{1}{\varphi_\lambda(y)} M_z(M_yf)(x) - \varphi_\lambda(z)g(x) \right| < \frac{\epsilon}{\varphi_{i\Im\lambda}(z)} \int_K \frac{\varphi_{i\Im\lambda}(zk_2y)}{|\varphi_\lambda(y)|}\, dk_2. \tag{4.3.5}$$

Since $C'\varphi_{i\Im\lambda}(y) \leq |\varphi_\lambda(y)| \leq C''\varphi_{i\Im\lambda}(y)$ for some constants C' and C'' and sufficient large $|y|$ (see (1.2.7)), from (4.3.5) it follows that

$$\left| \frac{1}{\varphi_\lambda(y)} M_z(M_yf)(x) - \varphi_\lambda(z)g(x) \right| \quad < \quad \frac{\epsilon}{C'\varphi_{i\Im\lambda}(z)} \int_K \frac{\varphi_{i\Im\lambda}(zk_2y)}{\varphi_{i\Im\lambda}(y)}\, dk_2.$$

Hence

$$\left| \frac{1}{\varphi_\lambda(y)} M_z(M_yf)(x) - \varphi_\lambda(z)g(x) \right| < C\epsilon$$

for sufficiently large $|y|$ and for some constant C. Hence

$$\frac{1}{\varphi_\lambda(y)} M_z(M_yf) \to \varphi_\lambda(z)g$$

uniformly on compact sets of X as $|y| \to \infty$ which combined with (4.3.2) gives $M_z(g) = \varphi_\lambda(z)g$ for all $z \in G$. Thus $\Delta g = -(\lambda^2 + \rho^2)g$.

Case 2: We now take $\lambda \in \mathbb{R}^\times$. We shall find a $\delta > 0$ and a sequence $\{y_n\}$ of elements in G with the following properties:

(a) $\varphi_\lambda(y_n)$ is positive and $\varphi_\lambda(y) > 0$ whenever $|y_n| - \delta \leq |y| \leq |y_n| + \delta$,

(b) $\varphi_\lambda(z)$ is positive for $0 \leq |z| \leq \delta$,

(c) $|y_n| \to \infty$ as $n \to \infty$.

We recall that for $\lambda \in \mathbb{R}^\times$ and $t \geq 1$ the Harish-Chandra series for φ_λ implies

$$\varphi_\lambda(a_t) = e^{-\rho t}[\mathrm{c}(\lambda)e^{i\lambda t} + c(-\lambda)e^{-i\lambda t} + E(\lambda, t)]$$

where $|E(\lambda, t)| \leq Ce^{-2t}$ (see [46, (3.11)]). Let $\mathrm{c}(\lambda) = a(\lambda) + ib(\lambda)$. Using $\mathrm{c}(-\lambda) = \overline{\mathrm{c}(\lambda)}$, we get

$$\begin{aligned} \varphi_\lambda(a_t) &= e^{-\rho t}[\Re(\mathrm{c}(\lambda)e^{i\lambda t}) + E(\lambda, t)] \\ &= e^{-\rho t}[a(\lambda)\cos(\lambda t) - b(\lambda)\sin(\lambda t) + E(\lambda, t)] \\ &= e^{-\rho t}[C_\lambda \cos(\lambda t + \theta_\lambda) + E(\lambda, t)] \end{aligned}$$

for some constant $C_\lambda > 0$. Thus the zeros of $\varphi_\lambda(a_t)$ are the zeros of

$$u(t) = C_\lambda \cos(\lambda t + \theta_\lambda) + E(\lambda, t).$$

We find a $t_0 > 0$ such that $|E(\lambda, t)| < \frac{C_\lambda}{2}$ for $t > t_0$. Let $R_n = \frac{2n\pi - \theta_\lambda}{\lambda}$ and $\delta_1 = \frac{\pi}{3\lambda}$. If $R_n - \delta_1 \leq s \leq R_n + \delta_1$, then

$$2n\pi - \frac{\pi}{3} \leq \lambda s + \theta_\lambda \leq 2n\pi + \frac{\pi}{3} \text{ and } \frac{1}{2} \leq \cos(\lambda s + \theta_\lambda) \leq 1.$$

Hence if $R_n - \delta_1 > t_0$, then $\varphi_\lambda(a_s) > 0$ whenever $R_n - \delta_1 \leq s \leq R_n + \delta_1$. Therefore $\varphi_\lambda(y) > 0$ for $R_n - \delta_1 \leq |y| \leq R_n + \delta_1$. Further as φ_λ is real valued and continuous on $\mathbb{R}$ and $\varphi_\lambda(e) = 1$, there exists $\delta_2 > 0$ such that $\varphi_\lambda(z)$ is positive for $0 \leq |z| \leq \delta_2$. Choose $\delta = \min\{\delta_1, \delta_2\}$ and $y_n = a_{R_n}$. It is clear that we get the desired sequence $\{y_n\}_{n \in \mathbb{N}}$ possibly after re-indexing suitably.

Through steps similar to that of Case 1, we get for $z \in G$ with $|z| < \delta$,

$$\frac{1}{\varphi_\lambda(y_n)} M_z(M_{y_n} f) \to M_z g, \text{ as } n \to \infty, \tag{4.3.6}$$

uniformly on compact sets of X. Rest of the proof is also similar to Case 1. We shall only include a sketch for brevity.

We fix a compact subset $\mathcal{K}$ of G and $z \in G$ with $|z| < \delta$. From the hypothesis we know that there exists $r_o > 0$ such that if $|w| > r_o$, then

$$\left| \frac{1}{\varphi_\lambda(w)} \int_K f(xkw)\, dk - g(x) \right| < \frac{\epsilon}{\varphi_0(z)} \text{ for all } x \in \mathcal{K}.$$

Choose N_0 such that $|y_n| > r_o + \delta$ for all $n \geq N_0$. Then by triangle inequality we have

$$|y_n| - \delta \leq |zk_2y_n| \leq |y_n| + \delta.$$

Therefore both $\varphi_\lambda(zk_2y_n)$ and $\varphi_\lambda(y_n)$ are positive. For $n \geq N_0$ we have,

$$\begin{aligned}
\left|\frac{1}{\varphi_\lambda(y_n)}M_z(M_{y_n}f)(x) - \varphi_\lambda(z)g(x)\right| &< \frac{\epsilon}{\varphi_0(z)}\int_K \frac{|\varphi_\lambda(zk_2y_n)|}{|\varphi_\lambda(y_n)|}\,dk_2 \\
&< \frac{\epsilon}{\varphi_0(z)}\int_K \frac{\varphi_\lambda(zk_2y_n)}{\varphi_\lambda(y_n)}\,dk_2 \\
&< \frac{\epsilon}{\varphi_0(z)}\varphi_\lambda(z) \\
&\leq \epsilon.
\end{aligned}$$

Hence

$$\frac{1}{\varphi_\lambda(y_n)}M_z(M_{y_n}f) \to \varphi_\lambda(z)g \text{ as } n \to \infty$$

uniformly on compact sets of X. This and (4.3.6) imply that $M_zg = \varphi_\lambda(z)g$ for all $z \in G$ with $|z| < \delta$, hence $\Delta g = -(\lambda^2 + \rho^2)g$ (see Proposition 4.1.5). □

Remark 4.3.5. Note that in the first part of the argument in Case 2 in the proof above we have used a result about φ_λ available in the literature. Alternatively one can use properties of Jacobi function, as done in Lemma 4.1.7.

4.4 Results for functions with growth conditions

In this section we shall consider functions in some integrability classes and endeavour to obtain results analogous to those of the previous section. Indeed, we shall use Theorem 4.1.2 to prove the corresponding results for such functions, where pointwise convergence of the averages will be replaced by the convergence in these integrability classes. We shall mainly work with ball averages, although similar results can be obtained for other two averages, considered in the last section. The growth condition enables us to get a more concrete realization of the limit function g (see Theorem 4.4.3 below) as the Poisson transform of an L^p function on the boundary K/M of X for an appropriate p.

We start with L^p and weak L^p-functions. We recall that no eigenfunction of Δ can reside in $L^p(X)$ with $p \leq 2$ (see Proposition 1.7.1). Thus the range of p to consider is $2 < p \leq \infty$. We also recall that for a fixed $2 < p \leq \infty$, if $|\Im\lambda| \geq |\gamma_p\rho|$ then $-(\lambda^2 + \rho^2)$ is not in the L^p-point spectrum (see e.g. [5, Corollary 4.18]). These restrictions justify the formulation of the theorem we state below.

Theorem 4.4.1. *If for two functions $f, g \in L^{p,\infty}(X)$, $2 < p \leq \infty$, and $\lambda \in S_p$,*

$$\|f * m_r^\lambda - g\|_{p,\infty} \to 0 \text{ as } r \to \infty,$$

then g is a C^∞-function satisfying $\Delta g = -(\lambda^2 + \rho^2)g$. Moreover if $\Im\lambda \leq 0$, then $g = \mathcal{P}_\lambda F$ for some $F \in L^p(K/M)$, otherwise $g = \mathcal{P}_{-\lambda} F$ for some $F \in L^p(K/M)$.

Proof. We note that f, g are $L^{p'}$-tempered distributions (see Proposition 1.5.1(a)), hence in the statement of the theorem Δg is interpreted in the sense of an $L^{p'}$-tempered distribution.

We take $h \in \mathcal{C}^{p'}(X)$. Noting that $f * h = f * R(h), g * h = g * R(h)$ and that $\|R(h)\|_{p',1} \leq \|h\|_{p',1}$ we have for all $x \in X$,

$$\begin{aligned} |f * h * m_r^\lambda(x) - g * h(x)| &= |f * R(h) * m_r^\lambda(x) - g * R(h)(x)| \\ &= |f * m_r^\lambda * R(h)(x) - g * R(h)(x)| \\ &\leq \|f * m_r^\lambda - g\|_{p,\infty} \|h\|_{p',1} \to 0 \end{aligned}$$

as $r \to \infty$. Thus $\|f * h * m_r^\lambda - g * h\|_\infty \to 0$ as $r \to \infty$. Since $f * h$ and $g * h$ are continuous functions, by Theorem 4.1.2 we have $\Delta(g * h) = -(\lambda^2 + \rho^2) g * h$. It can be verified that $g * h \in C^\infty(X) \cap L^{p,\infty}(X)$ (see Proposition 1.8.2) and hence in particular an $L^{p'}$-tempered distribution. We note that $\Delta(g * h) = (\Delta g) * h$. Indeed writing $\psi = R(h)$ for convenience we have, for any $\phi \in \mathcal{C}^{p'}(X)$,

$$\langle \Delta(g*h), \phi\rangle = \langle g*h, \Delta\phi\rangle = \langle g, \Delta\phi*\psi\rangle = \langle g, \Delta(\phi*\psi)\rangle = \langle \Delta g, \phi*\psi\rangle = \langle \Delta g*\psi, \phi\rangle.$$

Thus as a distribution $\Delta g * h = \Delta(g * h)$ and hence $(\Delta g) * h = [-(\lambda^2 + \rho^2) g] * h$. We note that $\Delta g * h(x) = \Delta g(\ell_x h)$, where $\ell_x h$ is the left translation of h by x. Putting $x = eK$, we have $\Delta g * h(eK) = \Delta g(h)$. Therefore from above (taking $x = eK$) for all $h \in \mathcal{C}^{p'}(X)$,

$$\langle \Delta g, h\rangle = \langle -(\lambda^2 + \rho^2) g, h\rangle.$$

That is, as an $L^{p'}$-tempered distribution, $\Delta g = -(\lambda^2 + \rho^2) g$. Since $\Delta - (\lambda^2 + \rho^2) I$ is hypoelliptic this implies that $g \in C^\infty(X)$. (see [35, Corollary 6.34, p. 215]). Final part of the theorem follows from Corollary 1.7.4.

□

Remark 4.4.2. Instead of weak L^p-functions we can take L^p-functions for $2 < p \leq$

∞, but in that case the spectral parameter λ has to be in S_p° (see Subsection 3.1.1). Our next result is an analogue of Theorem 4.4.1 using Hardy-type norm.

Theorem 4.4.3. *Let $p \in (0, 2]$ and $f, g \in \mathcal{H}_p^r(X)$ for some $r \in [1, \infty]$. If for a $\lambda \in S_p$, $f * m_r^\lambda \to g$ in $\mathcal{H}_p^r(X)$ as $r \to \infty$, then $\Delta g = -(\lambda^2 + \rho^2)g$. In particular if $|\Im\lambda| = \gamma_p\rho$ and $\Im\lambda < 0$ or $\lambda = 0$,*

(i) *$r > 1$, then $g = \mathcal{P}_\lambda F$ for some $F \in L^r(K/M)$,*

(ii) *$r = 1$, then $g = \mathcal{P}_\lambda\mu$ for a signed measure μ on K/M.*

Proof. Take a function $h \in \mathcal{C}^p(X)$. Then using Propsition 1.6.1(d) it follows that

$$|f * h * m_r^\lambda(x) - g * h(x)| \leq [f * m_r^\lambda - g]_{p,r} \int_X |h(x)|\varphi_{i\gamma_p\rho}(x)\, dx \to 0, \text{ as } r \to \infty.$$

Following the steps of Theorem 4.4.1 we can prove $\Delta g * h = -(\lambda^2 + \rho^2) g * h$ and finally $\Delta g = -(\lambda^2 + \rho^2)g$. Assertion (i) and (ii) are immediate consequences of this and Theorem 1.7.2. □

Remark 4.4.4. One can formulate an annulus versions of Theorem 4.4.1 and Theorem 4.4.3, replacing m_r^λ by a_{r_1,r_2}^λ (defined for Theorem 4.2.1). A step by step adaptation of the proofs and application of Theorem 4.2.1 will prove the assertions.

The following theorem, (in which pointwise convergence replaces norm convergence) will be proved in Chapter 5, although we shall use it to derive some results in this section.

Theorem 4.4.5. *Fix a $p \in [1, 2)$ and let $\lambda = i\gamma_p\rho$. Suppose that for $f \in L^{p',\infty}(X)$ and a measurable function g on X,*

$$\lim_{r\to\infty} f * m_r^\lambda(x) = g(x), \quad \textit{for almost every } x \in X.$$

Then $\Delta g = -(\lambda^2 + \rho^2)g$.

(See Theorem 5.2.8 and its proof in the next chapter.)

In Subsection 4.2.3 we have seen that there are right convolution operators T_t which are not mean value operators (e.g. for a harmonic function f, $T_t f \neq f$), but they act as mean value operator in the limit. We shall consider below two examples of the averages. The first one falls in this category.

Fix $0 < \alpha < \beta$ and $\lambda \in \mathbb{C}$ which is not a nonzero real number. For $t > 0$, we define:

$$\vartheta_t^\lambda(x) = \left(V_{\beta t}^\lambda\right)^{-1} \chi_{\mathbb{A}_{\alpha t, \beta t}(o)}(x),$$

and

$$\varpi_t^\lambda(x) = \left(V_{\alpha t, \beta t}^\lambda\right)^{-1} \chi_{\mathbb{A}_{\alpha t, \beta t}(o)}(x),$$

where $o = eK$ is the origin. It is clear that the right-convolution operator defined by ϑ_t^λ is not a mean value operator. In the following two propositions we will investigate their asymptotic behaviour.

Proposition 4.4.6. *Fix a $p \in [1, 2)$. Let ϑ_t^λ be as defined above. If $f, g \in L^{p',\infty}(X)$ satisfy one of these conditions:*

(a) $\lim_{t\to\infty} f * \vartheta_t^\lambda(x) = g(x)$ *for almost every $x \in X$ where $\lambda = i\gamma_p\rho$,*

(b) $\|f * \vartheta_t^\lambda - g\|_{p',\infty} \to 0$ *as $t \to \infty$, where $\lambda \in \partial S_p$,*

then $\Delta g = -(\lambda^2 + \rho^2)g$.

Proof. It is clear from (4.2.1) and (4.2.2) that $\lim_{t\to\infty} V_{\alpha t}^\lambda / V_{\beta t}^\lambda = 0$. Further we note that for any $f \in L^1_{loc}(X)$ and any $\lambda \in \mathbb{C}$ which is not a nonzero real number,

$$\begin{aligned}
f * m_{\beta t}^\lambda(x) =& \frac{1}{V_{\beta t}^\lambda} \int_{B(x,\beta t)} f(y)\, dy \\
=& \frac{1}{V_{\beta t}^\lambda} \left(\int_{B(x,\alpha t)} f(y)\, dy + \int_{\mathbb{A}_{\alpha t,\beta t}(x)} f(y)\, dy \right) \\
=& \frac{1}{V_{\beta t}^\lambda} \int_{\mathbb{A}_{\alpha t,\beta t}(x)} f(y)\, dy + \left(\frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda} \right) \frac{1}{V_{\alpha t}^\lambda} \int_{B(x,\alpha t)} f(y)\, dy \\
=& f * \vartheta_t^\lambda(x) + \left(\frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda} \right) f * m_{\alpha t}^\lambda(x).
\end{aligned} \tag{4.4.1}$$

For $f \in L^{p',\infty}$ and $\lambda \in \partial S_p$, by Kunze–Stein phenomenon ([51, Theorem 5.6(e)]) and (4.2.1) we have

$$\|f * m_{\alpha t}^\lambda\|_{p',\infty} \leq C \|f\|_{p',\infty} \|m_{\alpha t}^\lambda\|_{p,1} \leq C \|f\|_{p',\infty} \frac{(V_{\alpha t}^{i\rho})^{\frac{1}{p}}}{|V_{\alpha t}^\lambda|} \leq C' \tag{4.4.2}$$

for some constant C', since $L^{p,1}$-norm of $\chi_{B(x,\alpha t)}$ is $(V_{\alpha t}^{i\rho})^{1/p}$. It is clear that (4.4.1), (4.4.2), hypotheses (a) and (b) and applications of Theorem 4.4.5 and Theorem 4.4.1 prove the assertion. ☐

Proposition 4.4.7. *Fix a $p \in [1, 2)$. Let ϖ_t^λ be as defined above. If $f, g \in L^{p',\infty}(X)$ satisfy one of these conditions:*

(a) $\lim_{t\to\infty} f * \varpi_t^\lambda(x) = g(x)$ *for almost every* $x \in X$ *where* $\lambda = i\gamma_p\rho$,

(b) $\|f * \varpi_t^\lambda - g\|_{p',\infty} \to 0$ *as* $t \to \infty$ *where* $\lambda \in \partial S_p$,

then $\Delta g = -(\lambda^2 + \rho^2)g$.

Proof. As we have noted in the proof of Theorem 4.4.6, $\lim_{t\to\infty} V_{\alpha t}^\lambda / V_{\beta t}^\lambda = 0$. For any $f \in L^1_{loc}(X)$ and any $\lambda \in \mathbb{C}$ which is not a nonzero real number we also have,

$$\begin{aligned}
f * m_{\beta t}^\lambda(x) =& \frac{1}{V_{\beta t}^\lambda} \int_{B(x,\beta t)} f(y)\, dy \\
=& \frac{1}{V_{\beta t}^\lambda} \left(\int_{B(x,\alpha t)} f(y)\, dy + \int_{\mathbb{A}_{\alpha t,\beta t}(x)} f(y)\, dy \right) \\
=& \left(\frac{V_{\alpha t,\beta t}^\lambda}{V_{\beta t}^\lambda} \right) \frac{1}{V_{\alpha t,\beta t}^\lambda} \int_{\mathbb{A}_{\alpha t,\beta t}(x)} f(y)\, dy + \left(\frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda} \right) \frac{1}{V_{\alpha t}^\lambda} \int_{B(x,\alpha t)} f(y)\, dy \\
=& \left(1 - \frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda} \right) \frac{1}{V_{\alpha t,\beta t}^\lambda} \int_{\mathbb{A}_{\alpha t,\beta t}(x)} f(y)\, dy + \left(\frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda} \right) \frac{1}{V_{\alpha t}^\lambda} \int_{B(x,\alpha t)} f(y)\, dy \\
=& \left(1 - \frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda} \right) f * \varpi_t^\lambda(x) + \left(\frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda} \right) f * m_{\alpha t}^\lambda(x). \qquad (4.4.3)
\end{aligned}$$

Using (4.4.3) and applying Theorem 4.4.5 on the hypothesis (a), we get the assertion (a). From (4.4.3) it follows that

$$\|f * m_{\beta t}^\lambda - g\|_{p',\infty} = \left(1 - \frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda} \right) \|f * \varpi_t^\lambda - g\|_{p',\infty} + \left(\frac{V_{\alpha t}^\lambda}{V_{\beta t}^\lambda} \right) \|f * m_{\alpha t}^\lambda + g\|_{p',\infty} \quad (4.4.4)$$

If $\lambda \in \partial S_p$ from (4.4.4), (4.4.2) and Theorem 4.4.1 assertion (b) follows. □

Remark 4.4.8. It may be of some interest to note that it follows from (4.4.1) and (4.4.3) that for $f \in L^1_{loc}(X)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}^\times$, and some $x \in X$, if $f * m_r^\lambda(x) \to L$ as $r \to \infty$ then $f * \vartheta_t^\lambda(x) \to L$ and $f * \varpi_t^\lambda(x) \to L$ as $t \to \infty$.

4.5 Examples and counter-examples

In this concluding section of the chapter we shall:

(1) present a simple illustration of asymptotic behaviour of ball averages $f * m_r^\lambda$ of some continuous functions f and

(2) construct a counter example to show that the condition $r \to \infty$ in the hypothesis of the results obtained, cannot be replaced by "r approaches to ∞ through a sequence".

We begin with a lemma on Jacobi functions (see Section 1.3).

Lemma 4.5.1. *Let $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \mu$. Let $\phi_\lambda(t) = \phi_\lambda^{(\alpha',\beta')}(t)$ and $\phi_\mu(t) = \phi_\mu^{(\alpha',\beta')}(t)$ where $\alpha' = \frac{m+k+1}{2}$ and $\beta' = \frac{k+1}{2}$. Then we have the following conclusions.*

(a) *If $|\Im\lambda| > |\Im\mu|$, then $\frac{\phi_\mu(t)}{\phi_\lambda(t)} \to 0$ as $t \to \infty$.*

(b) *If $|\Im\lambda| < |\Im\mu|$, then $|\frac{\phi_\mu(t)}{\phi_\lambda(t)}|$ diverges to ∞ as $t \to \infty$.*

(c) *If $|\Im\lambda| = |\Im\mu|$ and $\lambda \neq 0, \mu \neq 0$, then $\frac{\phi_\mu(t)}{\phi_\lambda(t)}$ oscillates as $t \to \infty$.*

(d) *If $|\Im\lambda| = |\Im\mu|, \lambda = 0$ and $\mu \neq 0$, then $\frac{\phi_\mu(t)}{\phi_\lambda(t)} \to 0$ as $t \to \infty$.*

(e) *If $|\Im\lambda| = |\Im\mu|, \lambda \neq 0$ and $\mu = 0$, then $|\frac{\phi_\mu(t)}{\phi_\lambda(t)}|$ diverges to ∞ as $t \to \infty$.*

Proof. Without loss of generality we shall assume that $\Im\lambda, \Im\mu \leq 0$. If $\Im\lambda < 0$ from (1.3.2) and (1.3.3), we have

$$\lim_{t\to\infty} e^{(-i\lambda+\varrho)t}\phi_\lambda(t) = \mathrm{c}_{\alpha',\beta'}(\lambda) \text{ and } |\phi_\lambda(t)| \asymp e^{-(\Im\lambda+\varrho)t} \text{ as } t \to \infty. \tag{4.5.1}$$

From (1.3.4) we get

$$\phi_{2\lambda}\left(\frac{r}{2}\right) = \frac{\Gamma(\frac{n}{2}+1)V_r^\lambda}{2^n \pi^{\frac{n}{2}} \sinh^{m+k+1}(\frac{r}{2}) \cosh^{k+1}(\frac{r}{2})}.$$

As for large r, $V_r^0 \asymp re^{\rho r}$ (see Case 3 of Lemma 4.1.7) and $\sinh(r) \asymp \cosh(r) \asymp e^r$, hence

$$|\phi_0(r)| \asymp \frac{2re^{2(\frac{m}{4}+\frac{k}{2})r}}{e^{(m+2k+2)r}} \asymp re^{-\varrho r}. \tag{4.5.2}$$

Similarly for $0 \neq \lambda \in \mathbb{R}$, using (4.1.7) and (4.1.8) we get

$$|\phi_\lambda(r)| \leq Ce^{-\varrho r} \tag{4.5.3}$$

for some constant C and for sufficiently large r.

Using (4.5.1), (4.5.2) and (4.5.3), it is clear that if $\Im\mu > \Im\lambda$, then

$$\lim_{t\to\infty} \frac{|\phi_\mu(t)|}{|\phi_\lambda(t)|} = 0$$

and if $\Im\mu < \Im\lambda$, then

$$\lim_{t\to\infty} \frac{|\phi_\mu(t)|}{|\phi_\lambda(t)|} = \infty.$$

This proves (a) and (b).

For (c) we have the following two cases.

Case (i) Let $\Im\mu = \Im\lambda < 0$. Then,

$$\lim_{t\to\infty}\frac{\phi_\mu(t)}{\phi_\lambda(t)} = \lim_{t\to\infty}\frac{e^{(-i\mu+\varrho)(t)}\phi_\mu(t)}{e^{(-i\lambda+\varrho)t}\phi_\lambda(t)}\frac{e^{(-i\lambda+\varrho)t}}{e^{(-i\mu+\varrho)t}} = \frac{\mathrm{c}_{\alpha',\beta'}(\mu)}{\mathrm{c}_{\alpha',\beta'}(\lambda)}\lim_{t\to\infty} e^{-i(\lambda-\mu)t}. \quad (4.5.4)$$

Since $\lambda - \mu \in \mathbb{R}$, $\lim_{t\to\infty}\phi_\mu(t)/\phi_\lambda(t)$ is oscillatory.

Case (ii) Let $\Im\mu = \Im\lambda = 0$. Using (4.1.7) and (4.1.8) we get

$$\lim_{t\to\infty}\frac{\phi_\mu(t)}{\phi_\lambda(t)} = \lim_{t\to\infty}\frac{u_{\mu/2}(2t)}{u_{\lambda/2}(2t)} = \frac{C_{\mu/2}}{C_{\lambda/2}}\lim_{t\to\infty}\frac{\cos(\mu t+\theta_{\mu/2})+\epsilon^*_{\mu/2}(2t)}{\cos(\lambda t+\theta_{\lambda/2})+\epsilon^*_{\lambda/2}(2t)}. \quad (4.5.5)$$

We shall show that

$$\lim_{t\to\infty}\frac{\cos(\mu t+\theta_{\mu/2})+\epsilon^*_{\mu/2}(2t)}{\cos(\lambda t+\theta_{\lambda/2})+\epsilon^*_{\lambda/2}(2t)}$$

is oscillatory, dividing it in two subcases.

We assume first that $\xi = \mu/\lambda$ is irrational. Then by Kronecker's approximation theorem $\{2n\pi\xi \pmod{2\pi} \mid n\in\mathbb{N}\}$ is dense in $[0,2\pi]$. We construct a sequence $\{t_n\}_{n\in\mathbb{N}}$ where

$$t_n = \frac{2n\pi}{\lambda} - \frac{\theta_{\lambda/2}}{\lambda},\ n\in\mathbb{N}.$$

Then

$$\lim_{n\to\infty}\cos(\lambda t_n+\theta_{\lambda/2})+\epsilon^*_{\lambda/2}(2t_n) = 1.$$

Since $\mu t_n + \theta_{\mu/2} = 2n\pi\xi - \theta_{\lambda/2}\xi + \theta_{\mu/2}$, it follows that

$$\{\mu t_n + \theta_{\mu/2} \pmod{2\pi} \mid n\in\mathbb{N}\}$$

is also dense in $[0,2\pi]$. Therefore for any $L\in[-1,1]$, there is a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ of $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\cos(\mu t_{n_k}+\theta_{\mu/2}) = L.$$

Hence,

$$\lim_{k\to\infty}\frac{\cos(\mu t_{n_k}+\theta_{\mu/2})+\epsilon^*_{\mu/2}(2t_{n_k})}{\cos(\lambda t_{n_k}+\theta_{\lambda/2})+\epsilon^*_{\lambda/2}(2t_{n_k})} = L.$$

If μ/λ is rational we assume that $\mu/\lambda = m/n$, for $m, n \in \mathbb{Z}$. We choose a $\xi \in \mathbb{R}$ such that $\cos(\lambda\xi + \theta_{\lambda/2}) \neq 0$ and construct a sequence $\{t_k\}_{k\in\mathbb{N}}$ with $t_k = \xi + \frac{2n\pi k}{\lambda}, k \in \mathbb{N}$. Then,

$$\lim_{k\to\infty} \frac{\cos(\mu t_k + \theta_{\mu/2}) + \epsilon^*_{\mu/2}(2t_k)}{\cos(\lambda t_k + \theta_{\lambda/2}) + \epsilon^*_{\lambda/2}(2t_k)} = \frac{\cos(\mu\xi + \theta_{\mu/2})}{\cos(\lambda\xi + \theta_{\lambda/2})},$$

which is oscillatory as $\xi \in \mathbb{R}$ is arbitrary and $\mu \neq \lambda$. Since

$$\lim_{t\to\infty} \frac{\cos(\mu t + \theta_{\mu/2}) + \epsilon^*_{\mu/2}(2t)}{\cos(\lambda t + \theta_{\lambda/2}) + \epsilon^*_{\lambda/2}(2t)}$$

is oscillatory, we have the assertion for this case. This completes the proof of (c). Using (4.5.3) and (4.5.2), (d) and (e) also follows easily. □

An immediate consequence of this lemma is the following.

Proposition 4.5.2. *Let $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \mu$ and $f = \varphi_\lambda + \varphi_\mu$. Then we have the following conclusions.*

(a) *If $|\Im\lambda| > |\Im\mu|$, then $f * m_r^\lambda(x) \to \varphi_\lambda(x)$ for all $x \in X$ as $r \to \infty$.*

(b) *If $|\Im\lambda| < |\Im\mu|$, then $f * m_r^\lambda(x)$ diverges for all $x \in X$ as $r \to \infty$.*

(c) *If $|\Im\lambda| = |\Im\mu|$ and $\lambda \neq 0, \mu \neq 0$, then $f * m_r^\lambda(x)$ oscillates for all $x \in X$ as $r \to \infty$.*

(d) *If $|\Im\lambda| = |\Im\mu|, \lambda = 0$ and $\mu \neq 0$, then $f * m_r^\lambda(x) \to \varphi_\lambda(x)$ for all $x \in X$ as $r \to \infty$.*

(e) *If $|\Im\lambda| = |\Im\mu|, \lambda \neq 0$ and $\mu = 0$, then $f * m_r^\lambda(x)$ diverges for all $x \in X$ as $r \to \infty$.*

Proof. Without loss of generality we shall assume that $\Im\lambda, \Im\mu \leq 0$. It is easy to see that $f * m_r^\lambda(x) = \varphi_\lambda(x) + \frac{V_r^\mu}{V_r^\lambda}\varphi_\mu(x)$ for all $x \in X$. From (1.3.4) we get

$$\frac{V_r^\mu}{V_r^\lambda} = \frac{\phi_{2\mu}^{(\alpha',\beta')}(\frac{r}{2})}{\phi_{2\lambda}^{(\alpha',\beta')}(\frac{r}{2})}.$$

Hence in view of Lemma 4.5.1, the proposition follows. □

On $\mathbb{R}^n$ it was shown in [12] that there exists a non-harmonic continuous function f and a sequence $r_n \uparrow \infty$ such that $f * m_{r_n} \to f$ uniformly on compact sets. We shall conclude this section with a counter example of the same genre. Precisely we shall show that given a $\alpha = -(\lambda^2 + \rho^2) \in \mathbb{C}$ there exist continuous functions f, g with g not an eigenfunction with eigenvalue α and a sequence $r_n \uparrow \infty$ such that $f * m_{r_n}^\lambda(x) \to g(x)$ uniformly on compact sets. That is Theorem 4.4.1 is not true if the radius r approaches ∞ via an arbitrary sequence. Here is the example.

We fix a $\lambda \in \mathbb{C}$ with $\Im\lambda < 0$ and then take $\mu \in \mathbb{C}, \mu \neq \lambda$, but $\Im\mu = \Im\lambda$. Let $f(x) = \varphi_\mu(x)$. Since $\lambda - \mu \in \mathbb{R}$, we have the sequence $r_n = 2n\pi/(\lambda - \mu)$ of positive real numbers diverging to ∞ and $e^{-i(\lambda-\mu)r_n} = 1$. We recall that

$$f * m_r^\lambda(x) = \frac{V_r^\mu}{V_r^\lambda}\varphi_\mu(x) = \frac{\phi_{2\mu}^{(\alpha',\beta')}(\frac{r}{2})}{\phi_{2\lambda}^{(\alpha',\beta')}(\frac{r}{2})}\varphi_\mu(x).$$

Hence by (4.5.4) we have $\lim_{n\to\infty} f * m_{r_n}^\lambda(x) = \frac{\mathbf{c}_{\alpha',\beta'}(2\mu)}{\mathbf{c}_{\alpha',\beta'}(2\lambda)}\varphi_\mu(x)$. If $g = \frac{\mathbf{c}_{\alpha',\beta'}(2\mu)}{\mathbf{c}_{\alpha',\beta'}(2\lambda)}\varphi_\mu$, then $\Delta g \neq -(\lambda^2 + \rho^2)g$. In a similar fashion, it is possible to construct a counterexample for the case $0 \neq \lambda \in \mathbb{R}$. Precisely, take $f(x) = \varphi_\mu(x)$ with $\lambda \neq \mu \in \mathbb{R}^\times$. As $\mu \neq \lambda$, owing to Lemma 4.5.1(c), we can obtain a real number $L \neq 0$ and sequence $\{r_n\}_{n\in\mathbb{N}}$ with $r_n \uparrow \infty$ and

$$\lim_{n\to\infty} \frac{\phi_{2\mu}^{(\alpha',\beta')}(\frac{r_n}{2})}{\phi_{2\lambda}^{(\alpha',\beta')}(\frac{r_n}{2})} = L.$$

Hence we get

$$\lim_{n\to\infty} f * m_{r_n}^\lambda(x) = L\varphi_\mu(x),$$

and the limit is not an eigenfunction of Δ with the prescribed eigenvalue.

Here we have dealt only with ball-averages. Similar results and counter examples can be obtained for other two averages. For instance, as $\varphi_\lambda(a_t)$ is a Jacobi function, arguments as above will lead to the following.

Proposition 4.5.3. *Let $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \mu$ and $f = \varphi_\lambda + \varphi_\mu$. Then we have the following conclusions.*

(a) *If $|\Im\lambda| > |\Im\mu|$, then $f * \sigma_r^\lambda(x) \to \varphi_\lambda(x)$ for all $x \in X$ as $r \to \infty$.*

(b) *If $|\Im\lambda| < |\Im\mu|$, then $f * \sigma_r^\lambda(x)$ diverges for all $x \in X$ as $r \to \infty$.*

(c) *If $|\Im\lambda| = |\Im\mu|$ and $\lambda \neq 0, \mu \neq 0$, then $f * \sigma_r^\lambda(x)$ oscillates for all $x \in X$ as $r \to \infty$.*

(d) *If $|\Im\lambda| = |\Im\mu|, \lambda = 0$ and $\mu \neq 0$, then $f * \sigma_r^\lambda(x) \to \varphi_\lambda(x)$ for all $x \in X$ as $r \to \infty$.*

(e) *If* $|\Im\lambda| = |\Im\mu|, \lambda \neq 0$ *and* $\mu = 0$, *then* $f * \sigma_r^\lambda(x)$ *diverges for all* $x \in X$ *as* $r \to \infty$.

We can also show that it is not possible to take a sequence in Theorem 4.3.3. As an example take $f = \varphi_\mu$ and r_n as above.

Chapter 5

Large and small time behaviour of heat propagation and the characterization of eigenfunctions

Aim of this chapter is to illustrate some distinguishing features of the heat propagation on X and relate them to the characterization of eigenfunctions. Repnikov and Èĭdel'man [65, 66] proved the following theorem.

Theorem 5.0.1 (Repnikov and Èĭdel'man)**.** *For a function $f \in L^\infty(\mathbb{R}^n)$ and a fixed point $x_0 \in \mathbb{R}^n$, $f * m_r(x_0) \to L$ for a constant L as $r \to \infty$ if and only if $e^{t\Delta_{\mathbb{R}^n}} f(x_0) \to L$ as $t \to \infty$.*

This result was generalized by Li [53], to complete n-dimensional Riemannian manifolds M with nonnegative Ricci curvature with the property that $|B(x_0, r)| \geq \theta r^n$ for all large r for some constant θ, where the Euclidean result stated above was used. We shall see that one side of the theorem fails for X, precisely, there are functions $f \in L^\infty(X)$, such that $e^{t\Delta} f(x)$ converges for any $x \in X$ as $t \to \infty$, but $f * m_r(x)$ does not converge as $r \to \infty$, however, the forward side of the assertion is true for X. In fact we shall obtain a generalized version of the forward side of this result for X (see Theorem 5.2.1 below). We shall use this to pass from ball averages to heat kernel averages. Using the heat semigroup we shall get a result of the genre of Chapter 4, i.e. a mean value property in limit, for functions in Lebesgue or weak Lebesgue classes, but with pointwise convergence replacing norm convergence. This argument is free from the use of the geometric property of convexity of distance.

Using another non-Euclidean feature of the heat propagation in X, we shall reinforce our observation in Chapter 4, that there are certain right convolution operators for which eigenfunctions do not satisfy the mean value property, but asymptotically they can also characterize eigenfunctions.

While the results described above are about large time behaviour of heat propagation, we shall show that in small time also heat propagation in X has distinctive behaviour, in the context of the characterization of eigenfunctions.

5.1 Estimates of the heat kernel

We recall that the heat operator $e^{t\Delta}$ on X is the same as the convolution operator $\phi \mapsto \phi * h_t$ where h_t is the heat-kernel, i.e. the fundamental solution of the heat equation

$$\Delta f = \frac{\partial}{\partial t} f.$$

Precisely, for $t > 0$, h_t is defined as a radial function in the Harish-Chandra L^p-Schwartz space $\mathcal{C}^p(X), 0 < p \leq 2$, whose spherical Fourier transform is given by (see (3.1) [6], [5, Section 5]),

$$\widehat{h_t}(\lambda) = e^{-t(\lambda^2+\rho^2)},\ \lambda \in \mathfrak{a}^* = \mathbb{R}.$$

We need the following estimates of the heat kernel and its derivative (see [5, Theorem 5.9, Corollaries 5.49, 5.55]).

$$h_t(a_r) \asymp t^{-\frac{3}{2}}(1+r)\left(1+\frac{1+r}{t}\right)^{(n-3)/2} e^{-\rho^2 t}e^{-\rho r}e^{-r^2/4t},\ \text{for } t>0, r\geq 0. \quad (5.1.1)$$

$$-\frac{d}{dr}h_t(a_r) \asymp t^{-\frac{3}{2}}r\left(1+\frac{1+r}{t}\right)^{(n-1)/2} e^{-\rho^2 t}e^{-\rho r}e^{-r^2/4t},\ \text{for } t>0, r\geq 0. \quad (5.1.2)$$

We note that $|\frac{\partial}{\partial r}h_t(a_r)| = -\frac{\partial}{\partial r}h_t(a_r)$ ([5, p. 669]) and hence $-\frac{\partial}{\partial r}h_t(a_r)$ is nonnegative.

For $\lambda \in \mathbb{C}$ and $t > 0$ we define,

$$h_t^\lambda = e^{t(\lambda^2+\rho^2)}h_t.$$

Thus h_t^λ is the fundamental solution of the *perturbed heat equation*

$$[\Delta + (\lambda^2+\rho^2)]f = \frac{\partial}{\partial t}f.$$

From above it is clear that $h_t^\lambda = h_t^{-\lambda}$ and $e^{t(\Delta - c)}f = f * h_t^\lambda$ where $c = -(\lambda^2+\rho^2)$.

5.2 Large time behaviour of the heat propagation

We offer the following generalization of the Euclidean result (Theorem 5.0.1), connecting the large time behaviour of (perturbed) heat operator with asymptotic behaviour of the ball average. The notation h_t^λ is defined in the previous section and m_r^λ is as given in Section 1.2, precisely,

$$m_r^\lambda = (V_r^\lambda)^{-1}\chi_{B(o,r)}, \quad V_r^\lambda = \int_0^r \varphi_\lambda(t)J(t)\,dt.$$

Theorem 5.2.1. *Fix a $p \in [1,2)$. Let $\lambda = i\gamma_p\rho$. Suppose that for $f \in L^{p',\infty}(X)$ and for a fixed $x_0 \in X$,*

$$\lim_{r\to\infty} f * m_r^\lambda(x_0) = L.$$

Then

$$\lim_{t\to\infty} f * h_t^\lambda(x_0) = L.$$

Unlike the proof of the Euclidean result cited above, Wiener's Tauberian theorem cannot be used to prove this result. We shall show in Subsection 5.2.1 that the converse of this result is not true, i.e. there exists weak $L^{p'}$-function f on X and point $x_0 \in X$, such that $f * h_t^\lambda(x_0)$ converges to a limit as $t \to \infty$ but $f * m_r^\lambda(x_0)$ does not converge as $r \to \infty$. As a technical tool, we require to estimate $L^{p,1}$-norm of h_t, which we shall do next. For estimate on L^p-norm of h_t see [6, 24].

Lemma 5.2.2. *For $t > 1$ and $p \in [1,\infty)$,*

$$\|h_t\|_{p,1} \le Ce^{-\frac{4\rho^2}{pp'}t}$$

for some constant C. Further if $p \in [1,2)$ we have,

$$\|h_t\|_{p,1} \asymp e^{-\frac{4\rho^2}{pp'}t}.$$

Proof. We write $h_t(r)$ for $h_t(a_r)$ with $r > 0$ and view h_t as a function on $(0,\infty)$. Since $h_t(r)$ is a strictly decreasing function in r (as $\frac{d}{dr}h_t(a_r)$ is negative), using the explicit expression of the G-invariant measure in polar coordinate (see Section 1.2) we find the distribution function of h_t for $\alpha \in (0,\infty)$ as,

$$d_{h_t}(\alpha) = |\{r \in (0,\infty) \mid h_t(r) > \alpha\}| \le \int_0^{h_t^{-1}(\alpha)} e^{2\rho r}\,dr = \frac{e^{2\rho h_t^{-1}(\alpha)} - 1}{2\rho}.$$

Let $M = h_t(0) = \max_{s\geq 0} h_t(s)$. Therefore

$$\|h_t\|_{p,1} \leq C \int_0^M \left(\frac{e^{2\rho h_t^{-1}(\alpha)} - 1}{2\rho} \right)^{\frac{1}{p}} d\alpha = C \int_0^\infty (e^{2\rho s} - 1)^{\frac{1}{p}} |h_t'(s)| \, ds$$

where we have used the substitution $\alpha = h_t(s)$. Using (5.1.2) and dominating s and $t + 1 + s$ by $2t + s$, we have

$$\begin{aligned} \|h_t\|_{p,1} &\leq C \int_0^\infty e^{\frac{2\rho s}{p}} t^{-\frac{3}{2}} s \left(1 + \frac{1+s}{t} \right)^{\frac{n-1}{2}} e^{-\rho^2 t} e^{-\rho s} e^{-\frac{s^2}{4t}} \, ds \\ &\leq C t^{-\frac{n}{2}-1} e^{-\rho^2 t} \int_0^\infty (2t+s)^{\frac{n+1}{2}} e^{\gamma_p \rho s} e^{-\frac{s^2}{4t}} \, ds. \end{aligned}$$

By the substitution $s = 2tu$ and collecting powers of t we get,

$$\begin{aligned} \|h_t\|_{p,1} &\leq C t^{\frac{1}{2}} e^{-\rho^2 t} \int_0^\infty (1+u)^{\frac{n+1}{2}} e^{-t(u^2 - 2u\gamma_p \rho)} \, du \\ &= C t^{\frac{1}{2}} e^{-\rho^2 t (1-\gamma_p^2)} \int_0^\infty (1+u)^{\frac{n+1}{2}} e^{-t(u-\gamma_p \rho)^2} \, du \\ &= C t^{\frac{1}{2}} e^{-\frac{4\rho^2 t}{pp'}} \left[\int_0^{1+\gamma_p \rho} (1+u)^{\frac{n+1}{2}} e^{-t(u-\gamma_p \rho)^2} \, du + \int_{1+\gamma_p \rho}^\infty (1+u)^{\frac{n+1}{2}} e^{-t(u-\gamma_p \rho)^2} \, du \right]. \end{aligned}$$

For $u \in [1 + \gamma_p \rho, \infty)$ we have $u - \gamma_p \rho \asymp u$. Using the elementary estimate $e^{-tu^2} \leq \frac{(n+1)!}{(tu^2)^{n+1}}$ it follows that

$$\int_{1+\gamma_p \rho}^\infty (1+u)^{\frac{n+1}{2}} e^{-t(u-\gamma_p \rho)^2} \, du \leq \frac{C_1}{t^{n+1}}.$$

By the substitution $u = s + \gamma_p \rho$ we also get

$$\int_0^{1+\gamma_p \rho} (1+u)^{\frac{n+1}{2}} e^{-t(u-\gamma_p \rho)^2} \, du \leq C_2 \int_{-\gamma_p \rho}^1 e^{-ts^2} \, ds \leq C_2 \int_0^\infty e^{-ts^2} \, ds = \frac{C_2}{\sqrt{t}}.$$

Hence

$$\|h_t\|_{p,1} \leq C \sqrt{t} e^{-\frac{4\rho^2 t}{pp'}} \left[\frac{C_1}{t^{n+1}} + \frac{C_2}{\sqrt{t}} \right] \leq C_3 e^{-\frac{4\rho^2 t}{pp'}}.$$

This completes the proof of first assertion. Since $\widehat{h_t}(i\gamma_p \rho) = e^{-4\rho^2 t/pp'}$, we have

$$e^{-4\rho^2 t/pp'} \leq \|h_t\|_{p,1} \|\varphi_{i\gamma_p \rho}\|_{p',\infty}, \text{ for } 1 \leq p < 2.$$

Owing to Proposition 1.4.1 we get the second assertion. $\square$

As a step towards the proof of Theorem 5.2.1 we shall first prove a lemma.

Lemma 5.2.3. *Let $f \in L^{q',\infty}(X)$ and $\lambda = i\gamma_p\rho$ for some $p, q \in [1,\infty]$. Then for any $x \in X$ and for all $r > 0$,*

$$\int_{B(o,r)} f(xy)h_t^\lambda(y)\,dy = -\int_0^r V_s^\lambda \frac{d}{ds}(h_t^\lambda(a_s))f * m_s^\lambda(x)\,ds + V_r^\lambda(f * m_r^\lambda)(x)h_t^\lambda(a_r).$$

Consequently for any $x \in X$,

$$f * h_t^\lambda(x) = \lim_{r\to\infty}\int_{B(o,r)} f(xy)h_t^\lambda(y)\,dy = -\int_0^\infty V_s^\lambda \frac{d}{ds}(h_t^\lambda(a_s))f * m_s^\lambda(x)\,ds.$$

We need to use the following result on integration by parts (see e.g. [27, p. 163]) to prove Lemma 5.2.3.

Proposition 5.2.4. *Let $\phi, \psi \in L^1(a,b)$ for $a, b \in \mathbb{R}$ with $a < b$. For each $x \in (a,b)$, let*

$$\Phi(x) = \int_a^x \phi(t)\,dt \text{ and } \Psi(x) = \int_a^x \psi(t)\,dt.$$

Then $\Phi\psi, \phi\Psi \in L^1(a,b)$ and for each $x \in (a,b)$,

$$\Phi(x)\Psi(x) - \Phi(a)\Psi(a) = \int_a^x (\Phi\psi + \phi\Psi)(t)\,dt.$$

Proof of Lemma 5.2.3. Let $x \in X$ be fixed. We recall that $\sigma_u, u > 0$ is the normalized surface measure on the sphere around the origin of radius u and $M_s f = f * \sigma_s$. We take

$$\phi(u) = f * \sigma_u(x)J(u) \text{ and } \psi(u) = \frac{d}{du}h_t^\lambda(a_u), \text{ for } u > 0.$$

Then,

$$\|\phi\|_{L^1(0,r)} \leq \int_0^r \int_K |f(xka_u)|\,dk\,J(u)\,du = \int_{B(x,r)} |f(y)|\,dy \leq \|f\|_{q',\infty}|B(x,r)|^{1/q},$$

where $B(x,r)$ is the ball in X of radius r around x. Above we have used that $\|\chi_E\|_{q,1} = |E|^{1/q}$. From (5.1.2) it is clear that $\psi \in L^1(0,r)$. We have

$$\Phi(s) = \int_0^s \phi(u)\,du = \int_0^s f * \sigma_u(x)J(u)\,du = \int_0^s \int_K f(xka_u)\,dk\,J(u)\,du.$$

Thus $\Phi(s) = V_s^{i\rho}(f * m_s^{i\rho})(x) = V_s^\lambda(f * m_s^\lambda)(x)$, where V_s^λ, m_s^λ are as defined in Section 1.2 and Subsection 4.1.1. It is clear that $\Psi(s) = h_t^\lambda(a_s) - h_t^\lambda(o)$. Applying

Proposition 5.2.4 with these ϕ and ψ and noting that $\Phi(0) = 0 = \Psi(0)$, we get,

$$\int_0^r f * \sigma_s(x) J(s)(h_t^\lambda(a_s) - h_t^\lambda(o))\, ds + \int_0^r V_s^\lambda(f * m_s^\lambda)(x) \frac{d}{ds}(h_t^\lambda(a_s))\, ds$$
$$= V_r^\lambda(f * m_r^\lambda)(x)(h_t^\lambda(a_r) - h_t^\lambda(o)).$$

This implies

$$\begin{aligned}
&\int_0^r f * \sigma_s(x) J(s) h_t^\lambda(a_s)\, ds \\
&= \int_0^r f * \sigma_s(x) J(s) h_t^\lambda(o)\, ds - \int_0^r V_s^\lambda(f * m_s^\lambda)(x) \frac{d}{ds}(h_t^\lambda(a_s))\, ds \\
&+ V_r^\lambda(f * m_r^\lambda)(x) h_t^\lambda(a_r) - V_r^\lambda(f * m_r^\lambda)(x) h_t^\lambda(o).
\end{aligned}$$

Since $\int_0^r f * \sigma_s(x) J(s)\, ds = V_r^\lambda(f * m_r^\lambda)(x)$ we have,

$$\int_0^r f*\sigma_s(x) J(s) h_t^\lambda(a_s)\, ds = -\int_0^r V_s^\lambda(f*m_s^\lambda)(x) \frac{d}{ds}(h_t^\lambda(a_s))\, ds + V_r^\lambda(f*m_r^\lambda)(x) h_t^\lambda(a_r).$$

Finally we note that,

$$\int_{B(o,r)} f(xy) h_t^\lambda(y)\, dy = \int_K \int_0^r f(xka_s) h_t^\lambda(a_s) J(s)\, dk\, ds = \int_0^r f * \sigma_s(x) h_t^\lambda(a_s) J(s)\, ds$$

and this proves the first part of the assertion. For the second part we note that

$$|f * m_r^{i\rho}(x)| \le \int_X |f(xy)| m_r^{i\rho}(y)\, dy \le \|f\|_{q',\infty} \|m_r^{i\rho}\|_{q,1} \le C \|f\|_{q',\infty} \frac{(V_r^{i\rho})^{1/q}}{V_r^{i\rho}}$$

for some constant C. It follows from (4.1.1) and (5.1.1) that $(V_r^{i\rho})^{1/q} h_t(a_r) \to 0$ as $r \to \infty$. Hence

$$\lim_{r\to\infty} V_r^\lambda(f * m_r^\lambda)(x) h_t^\lambda(a_r) = e^{t(\lambda^2+\rho^2)} \lim_{r\to\infty} V_r^{i\rho}(f * m_r^{i\rho})(x) h_t(a_r) = 0. \qquad \square$$

We are now ready to complete the proof of the theorem.

Proof of Theorem 5.2.1. Since

$$\frac{d}{ds} V_s^\lambda = J(s)\varphi_\lambda(a_s) \text{ and } \int_X h_t^\lambda(x)\varphi_\lambda(x)\, dx = \int_0^\infty h_t^\lambda(a_s)\varphi_\lambda(a_s) J(s)\, ds = 1,$$

we have,

$$\int_0^\infty V_s^\lambda \left(-\frac{d}{ds} h_t^\lambda(a_s)\right) ds = \int_0^\infty h_t^\lambda(a_s)\varphi_\lambda(a_s)J(s)\, ds = 1.$$

We fix $x_0 \in X$ and an $\epsilon > 0$. Hence by Lemma 5.2.3,

$$|f * h_t^\lambda(x_0) - L| = \left|\int_0^\infty V_s^\lambda \left(-\frac{d}{ds} h_t^\lambda(a_s)\right) (f * m_s^\lambda(x_0) - L)\, ds\right|.$$

Then by hypothesis there exists $r_0 > 0$ such that for $r > r_0$, $|f * m_r^\lambda(x_0) - L| < \epsilon$. Therefore

$$|f * h_t^\lambda(x_0) - L| \leq C \int_0^{r_0} V_s^\lambda \left(-\frac{d}{ds} h_t^\lambda(a_s)\right) ds + \epsilon \int_{r_0}^\infty V_s^\lambda \left(-\frac{d}{ds} h_t^\lambda(a_s)\right) ds.$$

We have

$$\int_{r_0}^\infty V_s^\lambda \left(-\frac{d}{ds} h_t^\lambda(a_s)\right) ds \leq \int_0^\infty V_s^\lambda \left(-\frac{d}{ds} h_t^\lambda(a_s)\right) ds = \int_0^\infty \varphi_\lambda(a_s) h_t^\lambda(a_s) J(s)\, ds = 1.$$

Since $V_s^\lambda \leq V_{r_0}^\lambda$ for $0 < s < r_0$, it follows from the estimate (5.1.2) that

$$\int_0^{r_0} V_s^\lambda \left(-\frac{d}{ds} h_t^\lambda(a_s)\right) ds \leq \epsilon$$

when t is suitably large. Therefore from above we conclude that

$$|f * h_t^\lambda(x_0) - L| \leq 2\epsilon$$

for appropriately large $t > 0$. ☐

Using that h_t^λ is a semigroup, we have the following results.

Proposition 5.2.5. *Let $\lambda = i\gamma_p\rho$ for a fixed $p \in [1, 2)$. Suppose that for $f \in L^{p',\infty}(X)$ and a measurable function g on X,*

$$\lim_{t\to\infty} f * h_t^\lambda(x) = g(x), \quad \text{for almost every } x \in X.$$

Then $\Delta g = -(\lambda^2 + \rho^2)g$.

Proof. Since $\|h_t^\lambda\|_{p,1}$ is uniformly bounded for $t > 1$ (see Lemma 5.2.2) we note that,

$$|f * h_t^\lambda(x)| \leq \int_X |f(xy^{-1})| h_t^\lambda(y)\, dy \leq \|f\|_{p',\infty} \|h_t^\lambda\|_{p,1} \leq C\|f\|_{p',\infty}.$$

for a fixed constant C. Hence $|g(x)| \leq C\|f\|_{p',\infty}$ for almost every $x \in X$. Thus $g \in L^\infty(X)$ and the convolution $g * h_s^\lambda$ makes sense. Therefore,

$$|(f * h_t^\lambda - g)(xy^{-1})|h_s^\lambda(y) \leq C\|f\|_{p',\infty} h_s^\lambda(y), \text{ for all } t > 0.$$

Applying dominated convergence theorem, we get as $t \to \infty$, $f*h_t^\lambda*h_s^\lambda(x) \to g*h_s^\lambda(x)$ for almost all $x \in X$. On the other hand for almost every x,

$$f * h_t^\lambda * h_s^\lambda(x) = f * h_{t+s}^\lambda(x) \to g(x) \text{ as } t \to \infty.$$

Therefore for any $s > 0$, $g * h_s^\lambda(x) = g(x)$ for almost every $x \in X$ and equivalently, $g * h_s(x) = e^{-s(\lambda^2+\rho^2)} g(x)$. Using these relations and the fact that $g * h_t$ is a solution of the heat equation, we have,

$$\begin{aligned}
\Delta g = \Delta(g * h_t^\lambda) &= e^{t(\lambda^2+\rho^2)}\Delta(g * h_t) \\
&= e^{t(\lambda^2+\rho^2)}\partial_t(g * h_t) \\
&= e^{t(\lambda^2+\rho^2)}\partial_t(e^{-t(\lambda^2+\rho^2)}g) \\
&= -(\lambda^2 + \rho^2)g.
\end{aligned}$$

□

Here is a version of the proposition above under norm-convergence.

Proposition 5.2.6. *Let $\lambda = i\gamma_p\rho$ for a fixed $p \in [1, 2)$. Let $f \in L^{p',\infty}(X)$ and g be a measurable function on X such that $\lim_{t\to\infty} \|f*h_t^\lambda - g\|_{p',\infty} = 0$. Then $\Delta g = -(\lambda^2+\rho^2)g$ and $g = \mathcal{P}_{-\lambda}F$ for some $F \in L^{p'}(K/M)$.*

Proof. Using Kunze–Stein phenomenon (Proposition 1.8.2(c)) and uniform boundedness in t of $\|h_t^\lambda\|_{p,1}$ (Lemma 5.2.2), we get

$$\|f * h_t^\lambda\|_{p',\infty} \leq C'\|f\|_{p',\infty}\|h_t^\lambda\|_{p,1} = C\|f\|_{p',\infty}$$

for all large $t > 0$. So $\|g\|_{p',\infty} \leq C\|f\|_{p',\infty}$. Also

$$\|(f * h_t^\lambda - g) * h_s^\lambda\|_{p',\infty} \leq C\|f * h_t^\lambda - g\|_{p',\infty} \to 0 \text{ as } t \to \infty.$$

Thus we get $f * h_{t+s}^\lambda \to g * h_s^\lambda$ in $L^{p',\infty}(X)$. Therefore, $g(x) = g * h_s^\lambda(x)$ for almost every $x \in X$ and for every $s > 0$. In the last part of the proof of Proposition 5.2.5 it is shown that this implies $\Delta g = -(\lambda^2 + \rho^2)g$. Last assertion is immediate from Corollary 1.7.4. □

Instead of weak L^p-norm we can use convergence in Hardy-type norm (see Section 1.6). This enables us to capture eigenfunctions with eigenvalues in $[-\rho^2, \infty)$.

Proposition 5.2.7. *Fix a $p \in (0,2]$ and $r \geq 1$. Let $\lambda = i\gamma_p\rho$. Suppose that for functions $f \in \mathcal{H}_p^r(X)$ and a measurable function g on X,*

$$\lim_{t\to\infty} [f * h_t^\lambda - g]_{p,r} = 0.$$

Then $\Delta g = -(\lambda^2 + \rho^2)g$. In particular $g = \mathcal{P}_{-\lambda}F$ for some $F \in L^r(K/M)$ if $r > 1$ and $g = \mathcal{P}_{-\lambda}\mu$ for a signed measure μ on K/M if $r = 1$.

Proof. From Proposition 1.6.1 (d) we have,

$$[f * h_t^\lambda]_{p,r} \leq [f]_{p,r} \int_X h_t^\lambda(x)\varphi_{i\gamma_p\rho}(x)\, dx = [f]_{p,r}$$

for all $t > 0$. Hence $[g]_{p,r} \leq [f]_{p,r}$ and

$$[(f * h_t^\lambda - g) * h_s^\lambda]_{p,r} \leq [f * h_t^\lambda - g]_{p,r} \int_X h_s^\lambda(x)\varphi_{i\gamma_p\rho}(x)\, dx = [f * h_t^\lambda - g]_{p,r} \to 0$$

as $t \to \infty$. Therefore $f * h_t^\lambda * h_s^\lambda = f * h_{t+s}^\lambda \to g * h_s^\lambda$ in $\mathcal{H}_p^r(X)$ as $t \to \infty$. Thus $g(x) = g * h_s^\lambda(x)$ for almost every $x \in X$ and $s > 0$. Applying exactly same argument as in last part of Proposition 5.2.5 we get $\Delta g = -(\lambda^2 + \rho^2)g$. Last assertion follows from Theorem 1.7.2. □

Finally we use Theorem 5.2.1 to prove an analogue of Theorem 4.1.1, for functions with integrability condition and under pointwise convergence on X.

Theorem 5.2.8. *Fix a $p \in [1,2)$ and let $\lambda = i\gamma_p\rho$. Suppose that for $f \in L^{p',\infty}(X)$ and a measurable function g on X,*

$$\lim_{r\to\infty} f * m_r^\lambda(x) = g(x), \quad \textit{for almost every } x \in X.$$

Then $\Delta g = -(\lambda^2 + \rho^2)g$.

Proof. From Theorem 5.2.1 it follows that $\lim_{t\to\infty} f * h_t^\lambda(x) = g(x)$ for almost every $x \in X$. The assertion now follows from Proposition 5.2.5. □

5.2.1 Counterexamples

(a) We shall show that Theorem 5.2.1 is not true for $\lambda = \alpha + i\gamma_p\rho$ with $\alpha \neq 0$. Fix $p \in (0,2)$. Let $\lambda = \alpha + i\gamma_p\rho$ for some nonzero real number α. Choose q with

$p < q \leq 2$ sufficiently close to p such that $\alpha^2 > (\gamma_p^2 - \gamma_q^2)\rho^2$. Take $f = \varphi_{\alpha+i\gamma_p\rho} + \varphi_{i\gamma_q\rho}$. If $p \in [1,2)$ then $f \in L^{p',\infty}(X)$. It follows from Proposition 4.5.2 (a) that $f * m_r^\lambda(x) \to \varphi_\lambda(x)$ as $r \to \infty$. On the other hand,

$$\begin{aligned} f * h_t^\lambda(x) &= e^{t[(\alpha+i\gamma_p\rho)^2+\rho^2]}\big(\widehat{h_t}(\alpha+i\gamma_p\rho)\varphi_{\alpha+i\gamma_p\rho}(x)+\widehat{h_t}(i\gamma_q\rho)\varphi_{i\gamma_q\rho}(x)\big) \\ &= \varphi_{\alpha+i\gamma_p\rho}(x) + e^{t[(\alpha+i\gamma_p\rho)^2+\rho^2]}e^{-t[(i\gamma_q\rho)^2+\rho^2]}\varphi_{i\gamma_q\rho}(x) \\ &= e^{t(\alpha^2-(\gamma_p^2-\gamma_q^2)\rho^2)}e^{2ti\alpha\gamma_p\rho}\varphi_{i\gamma_q\rho}(x) + \varphi_{\alpha+i\gamma_p\rho}(x), \end{aligned}$$

which diverges as $t \to +\infty$.

(b) As mentioned above, unlike in $\mathbb{R}^n$, converse of Theorem 5.2.1 is not true for X. We take for instance $f = \varphi_{\alpha+i\gamma_p\rho} + \varphi_{i\gamma_p\rho}$ for some nonzero real number α and $0 < p < 2$. Then,

$$\begin{aligned} f * h_t^{i\gamma_p\rho}(x) &= e^{t[(i\gamma_p\rho)^2+\rho^2]}\left(\widehat{h_t}(\alpha+i\gamma_p\rho)\varphi_{\alpha+i\gamma_p\rho}(x) + \widehat{h_t}(i\gamma_p\rho)\varphi_{i\gamma_p\rho}(x)\right) \\ &= e^{t[(i\gamma_p\rho)^2+\rho^2]}e^{-t[(\alpha+i\gamma_p\rho)^2+\rho^2]}\varphi_{\alpha+i\gamma_p\rho}(x) + \varphi_{i\gamma_p\rho}(x) \\ &= e^{-t\alpha^2}e^{-2ti\alpha\gamma_p\rho}\varphi_{\alpha+i\gamma_p\rho}(x) + \varphi_{i\gamma_p\rho}(x). \end{aligned}$$

Therefore as $t \to +\infty$, $f * h_t^{i\gamma_p\rho}(x) \to \varphi_{i\gamma_p\rho}(x)$. But on the other hand, it follows from Proposition 4.5.2 (c) that $f * m_r^{i\gamma_p\rho}(x)$ oscillates as $r \to \infty$. Note that when $p \in [1,2)$ then f defined above is in $L^{p',\infty}(X)$. We can somewhat strengthen this (counter) example by producing a nonnegative function f, having the same property. We fix again a $p \in (0,2)$. Let $\varphi_{\alpha+i\gamma_p\rho}(x) = u(x) + iv(x)$, where u, v are real-valued functions on X. Then,

$$\begin{aligned} u * h_t^{i\gamma_p\rho}(x) + iv * h_t^{i\gamma_p\rho}(x) &= \varphi_{\alpha+i\gamma_p\rho} * h_t^{i\gamma_p\rho}(x) \\ &= e^{t[(i\gamma_p\rho)^2+\rho^2]}\widehat{h_t}(\alpha+i\gamma_p\rho)\varphi_{\alpha+i\gamma_p\rho}(x) \\ &= e^{t[(i\gamma_p\rho)^2+\rho^2]}e^{-t[(\alpha+i\gamma_p\rho)^2+\rho^2]}\varphi_{\alpha+i\gamma_p\rho}(x) \\ &= e^{-t\alpha^2}e^{-2ti\alpha\gamma_p\rho}\varphi_{\alpha+i\gamma_p\rho}(x), \end{aligned}$$

which tends to 0 as $t \to \infty$. Therefore as $t \to \infty$, both $u * h_t^{i\gamma_p\rho}(x) \to 0$ and $v * h_t^{i\gamma_p\rho}(x) \to 0$. But since (see Proposition 4.5.2), $\varphi_{\alpha+i\gamma_p\rho} * m_r^{i\gamma_p\rho}(x)$ oscillates as $r \to \infty$, we conclude that either $u * m_r^{i\gamma_p\rho}(x)$ or $v * m_r^{i\gamma_p\rho}(x)$ do not converge as $r \to \infty$. If $u * m_r^{i\gamma_p\rho}(x)$ does not converge take $f = 2\varphi_{i\gamma_p\rho} - u$, otherwise take $f = 2\varphi_{i\gamma_p\rho} - v$. Then it is easy to see that $f > 0$ (see Proposition 1.4.2) and $f * h_t^{i\gamma_p\rho}(x) \to 2\varphi_{i\gamma_p\rho}(x)$ as $t \to \infty$ but $f * m_r^{i\gamma_p\rho}(x)$ does not converges as $r \to \infty$.

Remark 5.2.9. In passing we observe the following immediate consequences of

Lemma 5.2.3, although they are not connected with our concern in this thesis.

1. Let Mf be the central Hardy–Littlewood maximal function of f, i.e.

$$Mf(x) = \sup_{r>0} |f| * m_r(x).$$

From the second assertion of Lemma 5.2.3, it follows (for $\lambda = i\rho$) that

$$\begin{aligned}|f * h_t(x)| \leq & -\int_0^\infty V_s \frac{d}{ds}(h_t(a_s)) f * m_s(x)\, ds \\ \leq & Mf(x) \int_0^\infty h_t(a_s) J(s)\, ds = Mf(x),\end{aligned}$$

where $V_s = V_s^{i\rho}$ and $m_s = m_s^{i\rho}$. Thus the heat maximal function is bounded by the central Hardy-Littlewood maximal function. From this and the mapping properties of Mf ([20], [77], [5]), we get the mapping properties of the heat maximal operator. In [5] the mapping properties of heat maximal operator was obtained directly, without comparing it with Mf.

2. A nonempty subset $\Gamma \subset X$ is said to be a non-analytic set if the only real analytic function defined on an open set containing Γ which vanishes on Γ is the zero function. If closure of Γ has positive measure then clearly Γ is such a set. But there are interesting "thin" non-analytic sets Γ. See [55] for various examples. It can be shown that for a function $f \in L^{p,\infty}(X)$ with $1 \leq p \leq \infty$ and such a set Γ, if $f * \chi_{B(r,o)}(x) = 0$ for all $x \in \Gamma$ and for all $r > 0$, then $f \equiv 0$. In other words, the linear span of elements of the set $\{\chi_{B(x,r)} \mid r > 0, x \in \Gamma\}$ is dense in $L^{p,1}(X)$, hence in $L^p(X)$. Indeed by the second assertion of Lemma 5.2.3, $f * \chi_{B(r,o)}(x) = 0$ for all $x \in \Gamma$ and all $r > 0$ implies that $f * h_t$ vanishes on Γ. Since $f * h_t$ is a solution of the heat equation, it follows from analytic regularity theorem of parabolic equation (see [36, p. 324]) that $f * h_t(x)$ is analytic in x. This reproves the result (for the particular case of rank one symmetric spaces) obtained in [54, 55].

5.3 Asymptotic property of heat propagation and the characterization of eigenfunctions

In the previous section, we have proved that for $p \in [1, 2)$, $\lambda = i\gamma_p\rho$, $f \in L^{p',\infty}(X)$ and a measurable function g on X, if $f * h_t^\lambda(x) \to g(x)$ for almost every $x \in X$, then $\Delta g = -(\lambda^2 + \rho^2)g$. We shall revisit the result in the light of finite propagation speed of heat diffusion, as observed by Davies [26, Corollary 5.7.3] for real hyperbolic spaces, and was generalized by Anker and Setti [7] for a class of manifolds which includes all Riemannian symmetric spaces of noncompact type (see also [18]). It was shown that on these spaces, heat concentrates at a finite speed to an annulus moving to infinity. Restricting to rank one symmetric spaces the statement reads:

Theorem 5.3.1 (Davies, Anker–Setti)**.** *Let $r(t)$ be a positive function with $r(t)/t^{1/2} \to \infty$ as $t \to \infty$ and*

$$\mathbb{A}_t^1 = \{k_1 a_s k_2 \mid 2\rho t - r(t) < |s| < 2\rho t + r(t),\, k_1, k_2 \in K\}.$$

Then

$$\int_{\mathbb{A}_t^1} h_t(x)\, dx \to 1, \ \text{ as } t \to \infty.$$

As pointed out in [7,26] this behaviour sharply differs from what happens in $\mathbb{R}^n$. For $1 \leq p \leq 2$, an L^p-version of this result is given in [6, Theorem 4.1.2]. Our aim in this section is to obtain a suitable version of this result, and use it to modify the results characterizing eigenfunctions (obtained in the previous section) through this asymptotic behaviour of the heat propagation.

Below by $f \lesssim g$ we mean that $|f| \leq C|g|$ for some constant C. For a $p \in (0, 2)$, let

$$\alpha_t^p = 2\gamma_p \rho t - r(t) \ \text{ and } \ \beta_t^p = 2\gamma_p \rho t + r(t)$$

where $0 < r(t) < 2\gamma_p \rho t$ for all $t > 0$ and $r(t)/\sqrt{t} \to \infty$ as $t \to \infty$. We define the annulus,

$$\mathbb{A}_t^p = \{k_1 a_s k_2 \mid s \in [\alpha_t^p, \beta_t^p],\ k_1, k_2 \in K\}.$$

Let

$$h_{t,p} = \chi_{\mathbb{A}_t^p} h_t \quad \text{and} \quad \bar{h}_{t,p} = \chi_{X \setminus \mathbb{A}_t^p} h_t.$$

In these notation, Theorem 5.3.1 asserts that $\|\bar{h}_{t,1}\|_1 \to 0$. Next two propositions are generalizations of Theorem 5.3.1 in two different set up.

Proposition 5.3.2. *For any fixed $p \in [1, 2)$,*

$$e^{\frac{4\rho^2 t}{pp'}} \|\bar{h}_{t,p}\|_{p,1} \to 0$$

as $t \to \infty$.

Proof. Since p is fixed we shall drop the superscript p and write α_t, β_t and $\mathbb{A}_t$ for α_t^p, β_t^p and $\mathbb{A}_t^p$ respectively. As $\bar{h}_{t,p}$ is a nonnegative radial function, it can be viewed as a function on $(0,\infty)$. For $\alpha > 0$, let $d_{\bar{h}_{t,p}}(\alpha)$ denotes the distribution function of $\bar{h}_{t,p}$, i. e.

$$d_{\bar{h}_{t,p}}(\alpha) = \left|\{r \in (0,\infty) \mid \bar{h}_{t,p}(r) > \alpha\}\right|.$$

From (5.1.2) it is clear that h_t is a radial decreasing function. As for $r \in [0, \alpha_t)$, $\bar{h}_{t,p}(r) = h_t(r)$, we have for $\alpha \in [h_t(\alpha_t), h_t(0)]$,

$$d_{\bar{h}_{t,p}}(\alpha) \leq \int_0^{h_t^{-1}(\alpha)} e^{2\rho r}\, dr \lesssim e^{2\rho h_t^{-1}(\alpha)}. \tag{5.3.1}$$

For $\alpha \in [h_t(\beta_t), h_t(\alpha_t)]$,

$$d_{\bar{h}_{t,p}}(\alpha) \;\leq\; \int_0^{\alpha_t} e^{2\rho r}\, dr \lesssim e^{2\rho\alpha_t} \tag{5.3.2}$$

and for $\alpha \in [0, h_t(\beta_t)]$,

$$\begin{aligned} d_{\bar{h}_{t,p}}(\alpha) &\leq \int_0^{\alpha_t} e^{2\rho r}\, dr + \int_{\beta_t}^{h_t^{-1}(\alpha)} e^{2\rho r}\, dr \\ &\lesssim e^{2\rho\alpha_t} + (e^{2\rho h_t^{-1}(\alpha)} - e^{2\rho\beta_t}) \\ &\lesssim e^{2\rho h_t^{-1}(\alpha)} \text{ (as } \beta_t > \alpha_t). \end{aligned} \tag{5.3.3}$$

Let $M = h_t(0) = \max_{s\geq 0} h_t(s)$. Then

$$\begin{aligned} \|\bar{h}_{t,p}\|_{p,1} &= \int_0^M d_{\bar{h}_{t,p}}(\alpha)^{\frac{1}{p}}\, d\alpha \\ &= \int_0^{h_t(\beta_t)} d_{\bar{h}_{t,p}}(\alpha)^{\frac{1}{p}}\, d\alpha + \int_{h_t(\beta_t)}^{h_t(\alpha_t)} d_{\bar{h}_{t,p}}(\alpha)^{\frac{1}{p}}\, d\alpha + \int_{h_t(\alpha_t)}^{M} d_{\bar{h}_{t,p}}(\alpha)^{\frac{1}{p}}\, d\alpha. \end{aligned} \tag{5.3.4}$$

Let us denote the first, second and third integral above respectively by I_1, I_2 and I_3. Then for $t > 1$ sufficiently large so that $r(t) > 4\sqrt{t}$,

$$\begin{aligned} e^{\frac{4\rho^2 t}{pp'}} I_1 = e^{\frac{4\rho^2 t}{pp'}} &\int_0^{h_t(\beta_t)} d_{\bar{h}_{t,p}}(\alpha)^{\frac{1}{p}}\, d\alpha \\ \lesssim e^{\rho^2 t(1-\gamma_p^2)} &\int_0^{h_t(\beta_t)} e^{\frac{2\rho}{p} h_t^{-1}(\alpha)}\, d\alpha \\ \lesssim e^{\rho^2 t(1-\gamma_p^2)} &\int_{\beta_t}^{\infty} e^{\frac{2\rho s}{p}} h_t'(s)\, ds, \end{aligned}$$

where we have used the substitution $\alpha = h_t(s)$. Using (5.1.2) and $\sqrt{1+u} \asymp 1 + \sqrt{u}$

for $u > 0$ we get,

$$\begin{aligned}
e^{\frac{4\rho^2 t}{pp'}} I_1 &\lesssim t^{-\frac{3}{2}} \int_{\beta_t}^{\infty} s\left(1+\frac{1+s}{t}\right)^{\frac{n-1}{2}} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\ ds \\
&\lesssim t^{-\frac{3}{2}} \int_{\beta_t}^{\infty} s\left(1+\frac{s}{t}\right)^{\frac{n-1}{2}} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\ ds \\
&\lesssim t^{-\frac{3}{2}} \int_{\beta_t}^{\infty} s\left(1+\sqrt{\frac{s}{t}}\right)^{n-1} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\ ds.
\end{aligned}$$

Expanding the second term of the integrand binomially, substituting $(s - 2t\gamma_p\rho)$ by $2\sqrt{t}s$ we get,

$$\begin{aligned}
e^{\frac{4\rho^2 t}{pp'}} I_1 &\lesssim \sum_{k=0}^{n-1} t^{-\frac{k+3}{2}} \int_{\beta_t}^{\infty} s^{\frac{k}{2}+1} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\ ds \\
&\lesssim \sum_{k=0}^{n-1} t^{-\frac{k+3}{2}} \int_{\frac{r(t)}{2\sqrt{t}}}^{\infty} (2\sqrt{t}s + 2t\gamma_p\rho)^{\frac{k}{2}+1} e^{-s^2} 2\sqrt{t}\ ds \\
&\lesssim \sum_{k=0}^{n-1} \int_{\frac{r(t)}{2\sqrt{t}}}^{\infty} (s+1)^{\frac{k}{2}+1} e^{-s^2}\ ds.
\end{aligned}$$

For I_3 we proceed through steps as above to conclude first that for t large as above,

$$e^{\frac{4\rho^2 t}{pp'}} I_3 \lesssim\ t^{-\frac{3}{2}} \int_{0}^{\alpha_t} s\left(1+\frac{1+s}{t}\right)^{\frac{n-1}{2}} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\ ds.$$

Hence,

$$\begin{aligned}
e^{\frac{4\rho^2 t}{pp'}} I_3 &\lesssim t^{-\frac{1}{2}} \int_{0}^{\alpha_t} \frac{s}{t}\left(1+\frac{1+s}{t}\right)^{\frac{n-1}{2}} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\ ds \\
&\lesssim t^{-\frac{1}{2}} \int_{0}^{\alpha_t} \left(1+\frac{1+s}{t}\right)^{\frac{n+1}{2}} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\ ds \\
&\lesssim t^{-\frac{1}{2}} \int_{0}^{\alpha_t} \left(1+\frac{1+\alpha_t}{t}\right)^{\frac{n+1}{2}} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\ ds \\
&\lesssim t^{-\frac{1}{2}} \int_{0}^{\alpha_t} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\ ds.
\end{aligned}$$

In the last step we have used that for large t, $1 + (1 + \alpha_t)/t \lesssim 1$. Substituting $(s - 2t\gamma_p\rho)$ by $2\sqrt{t}s$, we have from above

$$e^{\frac{4\rho^2 t}{pp'}} I_3 \lesssim t^{-\frac{1}{2}} \int_{-\sqrt{t}\gamma_p\rho}^{-\frac{r(t)}{2\sqrt{t}}} e^{-s^2} 2\sqrt{t}\ ds$$

$$\lesssim \int_{-\sqrt{t}\gamma_p\rho}^{-\frac{r(t)}{2\sqrt{t}}} e^{-s^2}\, ds.$$

Similarly for t large as above,

$$\begin{aligned} e^{\frac{4\rho^2 t}{pp'}} I_2 =& e^{\frac{4\rho^2 t}{pp'}} \int_{h_t(\beta_t)}^{h_t(\alpha_t)} d_{\bar{h}_{t,p}}(\alpha)^{\frac{1}{p}}\, d\alpha \\ \lesssim& e^{\rho^2 t(1-\gamma_p^2)} \int_{h_t(\beta_t)}^{h_t(\alpha_t)} e^{\frac{2\rho}{p}\alpha_t}\, d\alpha \\ \lesssim& e^{\rho^2 t(1-\gamma_p^2)} e^{\frac{2\rho}{p}\alpha_t} h_t(\alpha_t). \end{aligned}$$

Using (5.1.1) we have,

$$\begin{aligned} e^{\frac{4\rho^2 t}{pp'}} I_2 &\lesssim t^{-\frac{1}{2}} \left(\frac{1+\alpha_t}{t}\right) \left(1 + \frac{1+\alpha_t}{t}\right)^{\frac{n-3}{2}} e^{-\frac{(\alpha_t - 2t\gamma_p\rho)^2}{4t}} \\ &\lesssim t^{-\frac{1}{2}} \left(1 + \frac{1+\alpha_t}{t}\right)^{\frac{n-1}{2}} \lesssim t^{-\frac{1}{2}}. \end{aligned}$$

As $\lim_{t\to\infty} r(t)/\sqrt{t} = \infty$, from the estimates of I_1, I_2, I_3 obtained above, the assertion follows. □

Below is another generalization of Theorem 5.3.1.

Proposition 5.3.3. *For $p \in (0,2)$,*

$$e^{\frac{4\rho^2 t}{pp'}} \int_{\mathbb{A}_t^p} h_t(x)\varphi_{i\gamma_p\rho}(x)\, dx \to 1, \ \text{as } t \to \infty.$$

Proof. For convenience we shall drop the superscript p and write α_t, β_t and $\mathbb{A}_t$ respectively for α_t^p, β_t^p and $\mathbb{A}_t^p$. Since

$$e^{\frac{4\rho^2 t}{pp'}} \int_X h_t(x)\varphi_{i\gamma_p\rho}(x)\, dx = 1, \tag{5.3.5}$$

we need to show that

$$e^{\frac{4\rho^2 t}{pp'}} \int_{X\setminus\mathbb{A}_t} h_t(x)\varphi_{i\gamma_p\rho}(x)\, dx \to 0, \ \text{as } t \to \infty. \tag{5.3.6}$$

For $p \in [1,2)$, this follows from the proposition above, as,

$$e^{\frac{4\rho^2 t}{pp'}} \int_{X\setminus\mathbb{A}_t} h_t(x)\varphi_{i\gamma_p\rho}(x)\, dx \leq e^{\frac{4\rho^2 t}{pp'}} \|\bar{h}_{t,p}\|_{p,1} \|\varphi_{i\gamma_p\rho}\|_{p',\infty}.$$

However, in the proof below we consider all $p \in (0,2)$. We assume that $t > 1$ and such that $r(t) > 4\sqrt{t}$. Using (1.2.2), (1.2.8) and (5.1.1) we get,

$$
\begin{aligned}
I_1 &= \tfrac{4\rho^2 t}{pp'} \int_0^{\alpha_t} h_t(a_s)\varphi_{i\gamma_p\rho}(a_s)J(s)\,ds \\
&\lesssim t^{-\frac{3}{2}} \int_0^{\alpha_t} (1+s)\left(1+\frac{1+s}{t}\right)^{(n-3)/2} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds \\
&\lesssim t^{-\frac{1}{2}} \int_0^{\alpha_t} \frac{1+s}{t}\left(1+\frac{1+s}{t}\right)^{(n-3)/2} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds \\
&\lesssim t^{-\frac{1}{2}} \int_0^{\alpha_t} \left(1+\frac{1+s}{t}\right)^{(n-1)/2} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds \\
&\lesssim t^{-\frac{1}{2}} \int_0^{\alpha_t} \left(1+\frac{1+\alpha_t}{t}\right)^{(n-1)/2} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds.
\end{aligned}
$$

We substitute $(s - 2t\gamma_p\rho)$ by $2\sqrt{t}s$ and note that $1 + (1+\alpha_t)/t \lesssim 1$ to get from above,

$$
I_1 \lesssim t^{-\frac{1}{2}} \int_0^{\alpha_t} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds \lesssim \int_{-\sqrt{t}\gamma_p\rho}^{-\frac{r(t)}{2\sqrt{t}}} e^{-s^2}\,ds.
$$

Similarly we get,

$$
\begin{aligned}
I_2 &= \tfrac{4\rho^2 t}{pp'} \int_{\beta_t}^{\infty} h_t(a_s)\varphi_{i\gamma_p\rho}(a_s)J(s)\,ds \\
&\lesssim t^{-\frac{3}{2}} \int_{\beta_t}^{\infty} (1+s)\left(1+\frac{1+s}{t}\right)^{(n-3)/2} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds \\
&\lesssim t^{-\frac{1}{2}} \int_{\beta_t}^{\infty} \frac{1+s}{t}\left(1+\frac{1+s}{t}\right)^{(n-3)/2} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds \\
&\lesssim t^{-\frac{1}{2}} \int_{\beta_t}^{\infty} \left(1+\frac{1+s}{t}\right)^{(n-1)/2} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds.
\end{aligned}
$$

Since for $u > 0$, $\sqrt{1+u} \asymp 1 + \sqrt{u}$, from this using binomial expansion we have,

$$
\begin{aligned}
I_2 &\lesssim t^{-\frac{1}{2}} \int_{\beta_t}^{\infty} \left(1+\sqrt{\frac{s}{t}}\right)^{n-1} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds \\
&\lesssim \sum_{k=0}^{n-1} \binom{n-1}{k} t^{-\frac{k+1}{2}} \int_{\beta_t}^{\infty} s^{\frac{k}{2}} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds \\
&\lesssim \sum_{k=0}^{n-1} t^{-\frac{k+1}{2}} \int_{\beta_t}^{\infty} s^{\frac{k}{2}} e^{-\frac{(s-2t\gamma_p\rho)^2}{4t}}\,ds.
\end{aligned}
$$

Substituting $s - 2t\gamma_p\rho$ by $2\sqrt{t}s$ we get,

$$\begin{aligned} I_2 &\lesssim \sum_{k=0}^{n-1} t^{-\frac{k+1}{2}} \int_{\frac{r(t)}{2\sqrt{t}}}^{\infty} (2\sqrt{t}s + 2t\gamma_p\rho)^{\frac{k}{2}} e^{-s^2} 2\sqrt{t}\, ds \\ &\lesssim \sum_{k=0}^{n-1} \int_{\frac{r(t)}{2\sqrt{t}}}^{\infty} (s+1)^{\frac{k}{2}} e^{-s^2}\, ds. \end{aligned}$$

As $\lim_{t\to\infty} r(t)/\sqrt{t} = \infty$, from the estimates of I_1, I_2 obtained above, it follows that $e^{\frac{4\rho^2 t}{pp'}} \int_{X\setminus \mathbb{A}_t} h_t(a_s)\varphi_{i\gamma_p\rho}(a_s)J(s)\, ds \to 1$ as $t \to \infty$. □

As immediate consequences of the two propositions above, we get the following modifications of Propositions 5.2.5, 5.2.6 and 5.2.7. Below for any $\lambda \in \mathbb{C}$,

$$h_{t,p}^{\lambda} := e^{t(\lambda^2+\rho^2)} h_{t,p}.$$

Theorem 5.3.4. *Fix a $p \in [1, 2)$. Let $\lambda = i\gamma_p\rho$. Suppose that for $f \in L^{p',\infty}(X)$ and a measurable function g on X*

$$\lim_{t\to\infty} f * h_{t,p}^{\lambda}(x) = g(x)$$

for almost every $x \in X$. Then $\Delta g = -(\lambda^2 + \rho^2)g$.

Proof. By Holder's inequality we have,

$$\left| \int_{X\setminus \mathbb{A}_t^p} f(xy^{-1}) h_t^{\lambda}(y)\, dy \right| = e^{\frac{4\rho^2 t}{pp'}} |f * \bar{h}_{t,p}(x)| \leq \|f\|_{p',\infty} e^{\frac{4\rho^2 t}{pp'}} \|\bar{h}_{t,p}\|_{p,1}.$$

Therefore by Proposition 5.3.2, the left side of the inequality above goes to 0 as $t \to \infty$. Thus the hypothesis implies that, $\lim_{t\to\infty} f * h_t^{\lambda}(x) = g(x)$ for almost every $x \in X$. Owing to Proposition 5.2.5 the result follows. □

Theorem 5.3.5. *Let p, λ, f and g be as in the previous theorem. If*

$$\lim_{t\to\infty} \|f * h_{t,p}^{\lambda} - g\|_{p',\infty} = 0,$$

then $\Delta g = -(\lambda^2 + \rho^2)g$ and $g = \mathcal{P}_{-\lambda}F$ for some $F \in L^{p'}(K/M)$.

Proof. We have,

$$|f * h_t^{\lambda}(x) - g(x)| \leq \left| \int_{X\setminus \mathbb{A}_t^p} f(xy^{-1}) h_t^{\lambda}(y)\, dy \right| + \left| \int_{\mathbb{A}_t^p} f(xy^{-1}) h_t^{\lambda}(y)\, dy - g(x) \right|$$

$$= e^{\frac{4\rho^2 t}{pp'}} |f * \bar{h}_{t,p}(x)| + |f * h^\lambda_{t,p}(x) - g(x)|.$$

From this, using Kunze–Stein phenomenon (Proposition 1.8.2(c)) we get,

$$\|f * h^\lambda_t - g\|_{p',\infty} \leq e^{\frac{4\rho^2 t}{pp'}} \|f\|_{p',\infty} \|\bar{h}_{t,p}\|_{p,1} + \|f * h^\lambda_{t,p} - g\|_{p',\infty}.$$

From the hypothesis, Proposition 5.3.2 and Proposition 5.2.6 the result follows. □

Theorem 5.3.6. *Let p, λ and g be as in the previous theorem and $f \in \mathcal{H}^r_p(X)$. If*

$$\lim_{t\to\infty} [f * h^\lambda_{t,p} - g]_{p,r} = 0,$$

then $\Delta g = -(\lambda^2 + \rho^2) g$. In particular $g = \mathcal{P}_{-\lambda} F$ for some $F \in L^r(K/M)$ if $r > 1$ and $g = \mathcal{P}_{-\lambda}\mu$ for a signed measure μ on K/M if $r = 1$.

Proof. Following steps as in the previous theorem, we have $[f * h^\lambda_t - g]_{p,r} \leq e^{\frac{4\rho^2 t}{pp'}} [f * \bar{h}_{t,p}]_{p,r} + [f * h^\lambda_{t,p} - g]_{p,r}$. From Proposition 1.6.1(d) we have

$$e^{\frac{4\rho^2 t}{pp'}} [f * \bar{h}_{t,p}]_{p,r} \leq [f]_{p,r} \left[e^{\frac{4\rho^2 t}{pp'}} \int_{X\setminus A_t} h_t(x) \varphi_{i\gamma_p \rho}(x)\, dx \right],$$

which goes to zero as $t \to \infty$ owing to (5.3.6). From hypothesis it also follows that $[f * h^\lambda_{t,p} - g]_{p,r} \to 0$ as $t \to \infty$. Consequently $[f * h^\lambda_t - g]_{p,r} \to 0$ as $t \to \infty$. From Proposition 5.2.7 we get the desired result. □

Remark 5.3.7. We note that unlike $\{h^\lambda_t\}$, $\{h^\lambda_{t,p}\}$ is not a semigroup. Right-convolution by $h^\lambda_{t,p}$ is not a mean value operator. Thus the three theorems above are somewhat parallel to Proposition 4.2.3 and Proposition 4.4.6 in the previous chapter.

5.4 Small time behaviour of heat propagation

We consider again the heat propagator $e^{t\Delta}$ on X, which, we recall, is given by the right-convolution by h_t. Purpose of this section is to illustrate that in the results characterizing eigenfunctions through heat propagation in small time also manifests dichotomies with the Euclidean spaces. If f is a suitable eigenfunction with eigenvalue $-(\lambda^2+\rho^2)$ for some $\lambda \in \mathbb{C}$, it is straightforward to see that $f * h^\lambda_t = f$ for any $t > 0$. We shall explore here when the converse of this result is true. Note that if $\lambda = i\rho$, then $h^\lambda_t = h_t$ which is a probability and a classical result of

Furstenberg [37] asserts that the converse is true for this case when $f \in L^\infty(X)$. We shall consider other $\lambda \in \mathbb{C}$, and appropriate function (or distribution) spaces on X which can accommodate eigenfunctions with eigenvalue $-(\lambda^2 + \rho^2)$. A reader will observe that the main result in this section is closely related to the result in Chapter 2. We have chosen to isolate it here to emphasize this difference.

We begin with the corresponding result on the Euclidean spaces.

5.4.1 Review of the result for Euclidean spaces

It follows from the mean value property that there is no nonzero harmonic function belonging to $L^p(R^n)$ for $1 \leq p < \infty$. More generally, there are no nonzero eigenfunction of $\Delta_{\mathbb{R}^n}$ belonging to any $L^p(R^n)$ except for negative eigenvalues and if $\Delta_{\mathbb{R}^n} f = -cf$ for some $c > 0$ and $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \frac{2n}{n-1}$, then f is necessarily zero. Keeping these in view, we formulate the following theorem. Temporarily for this subsection only, by $\widehat{f}$ we denote the Euclidean Fourier transform of a function f on $\mathbb{R}^n$.

Theorem 5.4.1. *Let f be a nonzero measurable function.*

(i) *If $f \in L^\infty(\mathbb{R}^n)$ and satisfies $e^{t\Delta_{\mathbb{R}^n}} f = f$ for some $t > 0$, then f is harmonic.*

(ii) *If $f \in L^p(\mathbb{R}^n)$ with $p > \frac{2n}{n-1}$ and satisfies $e^{t\Delta_{\mathbb{R}^n}} f = e^{-tc} f$ for some $c > 0$ and $t > 0$, then $\Delta_{\mathbb{R}^n} f = -cf$.*

As we could not locate a reference, we shall include its proof. We shall use these notation. For $\lambda, \mu \in \mathbb{C}$, $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, let

$$\phi_\lambda(x) = \int_{S^{n-1}} e^{i\lambda x.w} \, dw, \quad \phi_{\mu,k}(x) = \frac{\partial^k}{\partial \lambda^k} \phi_\lambda(x)|_{\lambda=\mu}.$$

Proof. Suppose that $c = \alpha^2$ with $\alpha \geq 0$. Let p_t, p_t^c be the Gaussian, respectively the shifted Gaussian defined through their Fourier transforms: $\widehat{p_t}(\xi) = e^{-t|\xi|^2}$ and $\widehat{p_t^c}(\xi) = e^{-t(|\xi|^2 - c)}$. Then the hypotheses in (i) and (ii) can be rewritten as $f * p_t^c = f$ where $c = 0$ for (i). Let $g(x) = \exp(-|x|^2)$ and $\mathcal{K} = g * \mu$ where $\mu = p_t^c - \delta_0$ and δ_0 is the dirac at 0. We note that $\mathcal{K} \in L^1(\mathbb{R}^n)$, $\widehat{\mathcal{K}}(\xi) = 0$ if and only if $|\xi| = \alpha$, $g * f \in L^\infty(\mathbb{R}^n)$ and the hypotheses further boils down to $f * \mu = 0$. This implies that $\mathcal{K} * g * f = 0$. By [68, Theorem 9.3], support of the Fourier transform of the tempered distribution $g * f$ is contained inside the set $\{\xi \in \mathbb{R}^n \mid |\xi| = \alpha\}$. Consequently, same is true for the support of $\widehat{f}$. We shall now deal with (i) and (ii) of the statements separately.

(i) As $c=0$ we have supp $(\widehat{f})\subset\{0\}$. Therefore $\widehat{f}=\sum_{\beta} c_\beta D^\beta\delta_0$ (see [68, Theorem 6.25]). From this we get $f(x)=\sum_{\beta} c_\beta x^\beta$. But as $f\in L^\infty(\mathbb{R}^n)$, it is a constant function, in particular harmonic.

(ii) We shall first prove the result under the assumption that f is radial. Here $c>0$, hence $\alpha\neq 0$ and the tempered distribution $\widehat{f}$ is supported on the sphere $|\xi|=\alpha$. Therefore, $f=\sum_{k=0}^N a_k\phi_{\alpha,k}$ for some nonzero constants $a_0,a_1,\cdots,a_N$. (see [64, Lemma 2.2]). We claim that $N=0$. To establish the claim we note that,

$$\begin{aligned} e^{t\Delta_{\mathbb{R}^n}}\phi_{\alpha,k}(x) = \frac{\partial^k}{\partial^k\lambda}\,|_{\lambda=\alpha}\,(\phi_\lambda * p_t(x)) &= \frac{\partial^k}{\partial^k\lambda}\Big|_{\lambda=\alpha}\left(e^{-t\lambda^2}\phi_\lambda(x)\right)\\ &=\sum_{i=0}^{k}\binom{k}{i}\frac{\partial^i}{\partial^i\lambda}(e^{-t\lambda^2})\Big|_{\lambda=\alpha}\phi_{\alpha,k-i}(x). \end{aligned}$$

If $N\geq 1$, comparing the coefficient of $\phi_{\alpha,N-1}$ in both sides of identity $e^{t\Delta_{\mathbb{R}^n}}f=e^{-t\alpha^2}f$ we get $a_N=0$ which is a contradiction. Hence $f=a_0\phi_\alpha$ and $\Delta_{\mathbb{R}^n}f=-cf$. This completes the proof for radial functions.

Now we withdraw the assumption that f is radial. Seeking to meet a contradiction, we suppose that $\Delta_{\mathbb{R}^n}f\neq -cf$. Then by Godement's mean value theorem ([41, p. 409]), there exists a point $x_0\in\mathbb{R}^n$ such that $\int_{\mathrm{SO}(n)} f(x_0+Ay)\,dA\neq\phi_\alpha(y)f(x_0)$, where dA is the Haar measure on SO(n). Let $F(y)=\int_{\mathrm{SO}(n)} f(x_0+Ay)\,dA$. Then F is a radial function and the hypothesis $f*\mu=0$ implies that $F*\mu=0$. Applying the result for radial functions proved above, we have $\Delta_{\mathbb{R}^n}F=-cF$. But this implies that

$$\int_{\mathrm{SO}(n)} f(x_0+Ay)\,dA = F(y) = \int_{\mathrm{SO}(n)} F(Ay)\,dA = \phi_\alpha(y)F(0)=\phi_\alpha(y)f(x_0)$$

which is a contradiction. Thus it is proved that $\Delta_{\mathbb{R}^n}f=-cf$. □

See [82] for the characterization of harmonic functions on weighted spaces in $\mathbb{R}^n$, which is close in spirit to the results of this section.

5.4.2 Results on symmetric spaces

For $p\in(0,2)$, recall from Subsection 3.1.1 that $\Lambda(S_p)$ is given by

$$\Lambda(S_p)=\{x+iy\in\mathbb{C}\mid x\leq -4\rho^2/pp',|y|\leq 2\gamma_p\rho(-4\rho^2/pp'-x)^{1/2}\},\qquad(5.4.1)$$

which is a closed region in $\mathbb{C}$ enclosed by the parabola : $y^2 = -4\gamma_p^2\rho^2(x + \frac{4\rho^2}{pp'})$. For $p = 2$ the region $\Lambda(S_2)$ degenerates to the ray $(-\infty, -\rho^2]$. For a fixed $p \in (0, 2]$ and $c \in \Lambda(S_p)$, we define the critical time T_c^p by

$$T_c^p = \begin{cases} \pi \left((-\frac{4\rho^2}{pp'} - \Re c)^{1/2}\gamma_p\rho + \frac{|\Im c|}{2} \right)^{-1} \text{if } p \neq 2 \text{ and } c \neq -4\rho^2/pp' \\ \infty \text{ otherwise.} \end{cases} \tag{5.4.2}$$

The definition makes sense since for $c \in \Lambda(S_p)$, $\frac{4\rho^2}{pp'} + \Re c \leq 0$. Whenever p is fixed, we shall write T_c for T_c^p.

First we shall deal with L^p-tempered distributions with $p \in (0, 2)$ and withhold the case $p = 2$ until the end of this subsection.

Theorem 5.4.2. *Following conclusions hold.*

(a) *Fix a $p \in (0, 2)$. Let γ be an L^p-tempered distribution on X which satisfies $e^{t\Delta}\gamma = e^{tc}\gamma$ for some $c \in \Lambda(S_p)$ and $0 < t < T_c$. Then γ is an eigendistribution of Δ with eigenvalue c.*

(b) *Suppose that $p \in [1, 2)$ and for a function $f \in L^{p',\infty}(X)$, $e^{t\Delta}f = e^{tc}f$, for some $c \in \Lambda(S_p)$ and $0 < t < T_c$. Then $f = \mathcal{P}_\alpha F$ for some $F \in L^{p'}(K/M)$ and $\alpha \in \mathbb{C}$ satisfying $\alpha^2 + \rho^2 = -c$ and $\Im\alpha \leq 0$.*

(c) *Suppose that $p \in (0, 2)$, $1 \leq r \leq \infty$ and for a function $f \in \mathcal{H}_p^r(X)$, $e^{t\Delta}f = e^{tc}f$, for some $c \in \Lambda(S_p)$ and $0 < t < T_c$. Then $f = \mathcal{P}_\alpha F$ for some $F \in L^r(K/M)$ and $\alpha \in \mathbb{C}$ satisfying $\alpha^2 + \rho^2 = -c$ and $\Im\alpha \leq 0$ if $r \neq 1$, and $f = \mathcal{P}_\alpha\mu$ for some signed measure μ on X and α as above if $r = 1$.*

(d) *Fix a $p \in (0, 2)$. Then for any $T \geq T_c$, there exist a nonzero L^p-tempered distribution γ on X such that $e^{t\Delta}\gamma \neq e^{tc}\gamma$ for all $t < T$, but $e^{T\Delta}\gamma = e^{Tc}\gamma$. In particular γ is not an eigendistribution of Δ.*

Proof. (a) We suppose that $c = c_1 + ic_2$, $c_1, c_2 \in \mathbb{R}$ and $c = -(\alpha^2 + \rho^2)$ for $\alpha \in S_p$. First we shall assume that γ is a radial L^p-tempered distribution. For $\lambda \in \mathbb{C}$, we define a function $\psi_1(\lambda) = e^{-(\lambda^2+\rho^2+c)t} - 1$, which is the spherical Fourier transform of $h_t^\alpha - \delta_o$ at λ, where δ_o is the dirac at the origin o. For any $\delta > 0$ we define an augmented p-strip $S_{p,\delta}$ by

$$S_{p,\delta} = \{\lambda \mid |\Im\lambda| \leq (1 + \delta)\gamma_p\rho\}.$$

We claim that there exists a $\delta > 0$ such that if $\lambda \in S_{p,\delta}$ is a zero of ψ_1 then $\lambda^2 + \rho^2 = -c$. Indeed, $\psi_1(\lambda) = 0$ implies $t(\lambda^2 + \rho^2 + c) = i2n\pi$ for some $n \in \mathbb{Z}$. We write $\lambda = x + iy\gamma_p\rho \in S_{p,\delta}$ for $y \in \mathbb{R}$ with $|y| \leq 1 + \delta$, to get

$$\begin{aligned} i2n\pi =& t((x + iy\gamma_p\rho)^2 + \rho^2 + c) \\ =& t(x^2 - y^2\gamma_p^2\rho^2 + 2ixy\gamma_p\rho + \rho^2 + c) \\ =& t(x^2 + (1 - y^2\gamma_p^2)\rho^2 + c_1) + i(2txy\gamma_p\rho + tc_2). \end{aligned}$$

Equating the real and imaginary parts, we have

$$x^2 = -c_1 - (1 - y^2\gamma_p^2)\rho^2, \quad 2txy\gamma_p\rho + tc_2 = 2n\pi.$$

When $(-\frac{4\rho^2}{pp'} - \Re c)^{1/2}\gamma_p\rho + \frac{|\Im c|}{2} \neq 0$ then from (5.4.2) and the hypothesis $t < T_c$ we have

$$\left(-\frac{4\rho^2}{pp'} - c_1\right)^{1/2} \gamma_p\rho < \frac{\pi}{t} - \frac{|c_2|}{2}.$$

If $(-\frac{4\rho^2}{pp'} - \Re c)^{1/2}\,\gamma_p\rho + \frac{|\Im c|}{2} = 0$, then trivially

$$\left(-\frac{4\rho^2}{pp'} - c_1\right)^{1/2} \gamma_p\rho < \frac{\pi}{t} - \frac{|c_2|}{2}.$$

Thus in both the cases

$$\left(-\frac{4\rho^2}{pp'} - c_1\right)^{1/2} \gamma_p\rho = (-c_1 - (1 - \gamma_p^2)\rho^2)^{1/2}\gamma_p\rho < \frac{\pi}{t} - \frac{|c_2|}{2}.$$

We choose $\delta > 0$ sufficiently small so that

$$(-c_1 - (1 - y^2\gamma_p^2)\rho^2)^{1/2}\ |y|\gamma_p\rho < \frac{\pi}{t} - \frac{|c_2|}{2}$$

whenever $1 \leq |y| \leq 1 + \delta$. For $|y| \leq 1$,

$$(-c_1 - (1 - y^2\gamma_p^2)\rho^2)^{1/2}\ |y|\gamma_p\rho \leq (-c_1 - (1 - \gamma_p^2)\rho^2)^{1/2}\,\gamma_p\rho \ < \ \frac{\pi}{t} - \frac{|c_2|}{2}.$$

Hence

$$\begin{aligned} 2|n|\pi &= |2txy\gamma_p\rho + tc_2| \\ &\leq 2t|x||y|\gamma_p\rho + t|c_2| \\ &= 2t\left(-c_1 - (1 - y^2\gamma_p^2)\rho^2\right)^{1/2} |y|\gamma_p\rho + t|c_2| \end{aligned}$$

$$< 2t\left(\frac{\pi}{t} - \frac{|c_2|}{2}\right) + t|c_2| = 2\pi.$$

This is possible only when $n = 0$. Therefore $\lambda^2 + \rho^2 = -c$ and $\lambda = \pm\alpha$. This establishes the claim about the zero of the function ψ_1.

Let $\psi_2(\lambda) = e^{-(\lambda^2+\rho^2)}\psi_1(\lambda)$. Then $\psi_2 \in \mathcal{C}^p(\widehat{G//K})$. Let $g \in \mathcal{C}^p(G//K)$ be the inverse image of ψ_2 under the spherical Fourier transform. We note that the hypothesis, $e^{t\Delta}\gamma = e^{tc}\gamma$ equivalently, $\gamma * h_t^\alpha = \gamma$ implies that $\gamma * (h_t^\alpha - \delta_o) = 0$. Hence $\gamma * g = 0$ as a L^p-tempered distribution. We also observe that (i) if $\alpha \neq 0$, then ψ_2 has a simple root at α and (ii) if $\alpha = 0$ then ψ_2 has a zero of order two at α. We shall deal with these two cases separately.

If $\alpha \neq 0$, then by Proposition 1.5.2, $\{g * \xi \mid \xi \in \mathcal{C}^p(G//K)\}$ is dense in the space of all functions in $\mathcal{C}^p(G//K)$ whose Fourier transform vanish at α. Since γ is a radial L^p-tempered distribution and

$$\langle \gamma, g * \xi \rangle = \langle \gamma * g, \xi \rangle = 0, \quad \text{for all } \xi \in \mathcal{C}^p(G//K),$$

we have, $\langle \gamma, \phi \rangle = 0$ for all $\phi \in \mathcal{C}^p(G//K)$ whenever $\widehat{\phi}(\alpha) = 0$. But φ_α is also a radial L^p-tempered distribution which annihilates all $\phi \in \mathcal{C}^p(G//K)$ whenever $\widehat{\phi}(\alpha) = 0$. Therefore by [68, Lemma 3.9], $\gamma = C\varphi_\alpha$ for some constant C. Thus γ is an eigendistribution of Δ with eigenvalue c.

We take now $\alpha = 0$. As $\alpha = 0$ is a zero of order two of ψ_2, by Proposition 1.5.2, $\{g * \xi \mid \xi \in \mathcal{C}^p(G//K)\}$ is dense in the space of all functions in $\mathcal{C}^p(G//K)$ whose Fourier transform and its first derivative vanish at zero. Since $\langle \gamma, g * \xi \rangle = 0$ we have, $\langle \gamma, \phi \rangle = 0$ for all $\phi \in \mathcal{C}^p(G//K)$ whenever $\widehat{\phi}(0) = 0$ and $\frac{\partial \widehat{\phi}(\lambda)}{\partial \lambda}|_{\lambda=0} = 0$. But as $\widehat{\phi}$ is an even function, $\frac{\partial \widehat{\phi}(\lambda)}{\partial \lambda}|_{\lambda=0} = 0$. Hence $\langle \gamma, \phi \rangle = 0$ for all $\phi \in \mathcal{C}^p(G//K)$ whenever $\widehat{\phi}(0) = 0$. Therefore $\gamma = C\varphi_0$ for constants C (see [68, Lemma 3.9]). Thus γ is an eigenfunction of Δ with eigenvalue c. This completes the proof of (a) under the assumption that γ is radial.

Now we remove the assumption that γ is radial. We claim that for an L^p-tempered distribution γ, if $R(\ell_x\gamma) = 0$ for all $x \in G$, then $\gamma = 0$. Indeed, $R(\ell_x\gamma) = 0$ for all $x \in G$ implies that for any $\psi \in \mathcal{C}^p(G//K)$, $\gamma * \psi = 0$ as a distribution. We take $\psi = \psi_\epsilon$, where $\{\psi_\epsilon\}$ is a C_c^∞-approximate identity to conclude that $\gamma = 0$.

Since Δ commutes with radialization R and translation ℓ_x, from the hypothesis $e^{t\Delta}\gamma = e^{tc}\gamma$, it follows that $e^{t\Delta}R(\ell_x\gamma) = e^{tc}R(\ell_x\gamma)$ for all $x \in G$. Applying the result for radial function we have $\Delta R(\ell_x\gamma) = cR(\ell_x\gamma)$ for all $x \in G$. Using again the commutativity of Δ with R and ℓ_x, we conclude that $R(\ell_x(\Delta\gamma - c\gamma)) = 0$ for all $x \in G$. This implies that $\Delta\gamma = c\gamma$ by the claim established above. This completes part (a).

(b) and (c) By Proposition 1.5.1 and Proposition 1.6.1, f is an L^p-tempered distribution, hence by part (a) an eigendistribution with eigenvalue c. Assertions now follow from Corollary 1.7.4 and Theorem 1.7.2.

(d) As in (a) we take $c = c_1 + ic_2$ for $c_1, c_2 \in \mathbb{R}$ and $c = -(\alpha^2 + \rho^2)$ for $\alpha \in S_p$. We claim that given any $T \geq T_c$ we can find a $\beta \in S_p$ such that either $T(\beta^2 + \rho^2 + c) = 2\pi i$ or $T(\beta^2 + \rho^2 + c) = -2\pi i$. To see this let us first assume that $c_2 \geq 0$. Since $c \in \Lambda(S_p)$ and $T \geq T_c$ we have by (5.4.1) and (5.4.2)

$$-2\gamma_p\rho\left(-\frac{4\rho^2}{pp'} - c_1\right)^{1/2} \leq c_2 - \frac{2\pi}{T} < c_2 \leq 2\gamma_p\rho\left(-\frac{4\rho^2}{pp'} - c_1\right)^{1/2}.$$

Therefore there exists $\beta \in S_p$ (see (5.4.1)) such that $-(\beta^2 + \rho^2) = c_1 + i(c_2 - \frac{2\pi}{T})$. This implies

$$T(\beta^2 + \rho^2 + c) = 2i\pi.$$

Similarly, if $c_2 < 0$, then the conditions $c \in \Lambda(S_p)$ and $T \geq T_c$ implies that

$$-2\gamma_p\rho\left(-\frac{4\rho^2}{pp'} - c_1\right)^{1/2} \leq c_2 < c_2 + \frac{2\pi}{T} \leq 2\gamma_p\rho\left(-\frac{4\rho^2}{pp'} - c_1\right)^{1/2}$$

and hence again by (5.4.1) we can find a $\beta \in S_p$ such that $-(\beta^2 + \rho^2) = c_1 + i(c_2 + \frac{2\pi}{T})$. That is $T(\beta^2 + \rho^2 + c) = -2i\pi$.

Let $f = \varphi_\alpha + \varphi_\beta$. Then f is an L^p-tempered distribution which is clearly not an eigenfunction of Δ, but

$$f * h_T^c = \varphi_\alpha + e^{-T(\beta^2+\rho^2+c)}\varphi_\beta = \varphi_\alpha + \varphi_\beta = f.$$

That is $e^{T\Delta}f = e^{Tc}f$. It is also clear that if $t < T$, then $f * h_t^\alpha \neq f$ as $t|\beta^2 + \rho^2 + c| < 2|\pi|$ for such t. Thus $\gamma = f$ is the L^p-tempered distribution, required to find for part (b). $\square$

A consequence of Theorem 5.4.2 is the following.

Corollary 5.4.3. *Let γ be an L^p-tempered distribution on X for some $p \in (0, 2)$ and $c = -(\alpha^2 + \rho^2) \in \Lambda(S_p)$ for some $\alpha \in \mathbb{C}$. Suppose that for two distinct positive numbers t_1, t_2 with $|t_1 - t_2| < T_c$, $\gamma * h_{t_1}^\alpha = \gamma * h_{t_2}^\alpha$. Then f is an eigendistribution of Δ with eigenvalue c. For two arbitrary positive numbers t_1, t_2 with $|t_1 - t_2| \geq T_c$,* *there exist a nonzero L^p-tempered distribution γ which is not an eigendistribution of Δ but satisfies $\gamma * h^\alpha_{t_1} = \gamma * h^\alpha_{t_2}$.*

Proof. First we shall show that for an L^p-tempered distribution γ_1, $\gamma_1 * h^\alpha_t = 0$ for some fixed $t > 0$ implies that $\gamma_1 = 0$. Passing to $R(\ell_x\gamma_1)$ for some $x \in G$, if necessary, without loss of generality we may assume that each γ_1 is radial (see proof of Theorem 5.4.2). If $\gamma_1 * h^\alpha_t = 0$, we have $\langle \gamma_1 * h^\alpha_t, \phi \rangle = 0$ for all $\phi \in \mathcal{C}^p(G//K)$. That is $\langle \gamma_1, \phi * h^\alpha_t \rangle = 0$ for all $\phi \in \mathcal{C}^p(G//K)$. Since by Proposition 1.5.2, $\{h^\alpha_t * \phi \mid \phi \in \mathcal{C}^p(G//K)\}$ is dense in $\mathcal{C}^p(G//K)$, we have $\langle \gamma_1, \phi \rangle = 0$ for all $\phi \in \mathcal{C}^p(G//K)$. That is the distribution $\gamma_1 = 0$.

Without loss of generality we assume that $t_1 > t_2$ and let $t = t_1 - t_2$. If $|t_1 - t_2| < T_c$ and $\gamma * h^\alpha_{t_1} = \gamma * h^\alpha_{t_2}$, then it follows that $\gamma_2 * h^\alpha_t = \gamma_2$ for $0 < t < T_c$ where $\gamma_2 = \gamma * h^\alpha_{t_2}$. Hence from Theorem 5.4.2(a) it follows that $\Delta(\gamma * h^\alpha_{t_2}) = c(\gamma * h^\alpha_{t_2})$, which in turn implies that $(\Delta\gamma - c\gamma) * h^\alpha_{t_2} = 0$. From the argument given in the preceding paragraph, first part of the assertion follows. For the second part if $t > T_c$ using (b) of Theorem 5.4.2, we can obtain a nonzero γ satisfying $\gamma * h^\alpha_t = \gamma$ but γ is not an eigendistribution of Δ. It is also easy to see that γ satisfies $\gamma * h^\alpha_{t_1} = \gamma * h^\alpha_{t_2}$. □

Thus if we miss the critical time, we need two observations in a small interval of time. The corollary we state and prove below finds another way to determine whether f is an eigendistribution, taking observations in the intermediate time. This is a reminiscent of the *two radius theorem* for the spherical mean operator ([28], [33]).

Corollary 5.4.4. *We fix a $p \in (0, 2)$ and a point $c = -(\alpha^2 + \rho^2) \in \Lambda(S_p)$, for some $\alpha \in \mathbb{C}$. Suppose that for an L^p-tempered distribution γ on X, $\gamma * h^\alpha_s = \gamma = \gamma * h^\alpha_t$ for some $s, t > 0$ with t/s irrational, then γ is an eigendistribution of Δ with eigenvalue c.*

Though the proof is a simple consequence of Corollary 5.4.3 and Kronecker's approximation theorem, we need to check the necessity of the hypothesis. We note that if $s < T_c$ or $t < T_c$ or $|s - t| < T_c$ then $\gamma * h^\alpha_t = \gamma = \gamma * h^\alpha_s$ implies $\Delta\gamma = c\gamma$ by Theorem 5.4.2 and Corollary 5.4.3 and thus in these cases it is irrelevant whether s/t is irrational or not. Therefore we have to check the necessity only for the complementary cases. We shall see below that for a fixed $p \in (0, 2)$, given $s > 0, t > 0$ with $s \neq t$ and s/t rational, there exist a point $c \in \Lambda(S_p)$ such that $s \geq T_c$, $t \geq T_c$, $|s - t| \geq T_c$ and an L^p-tempered distribution γ which is not an eigendistribution of Δ but satisfies $\gamma * h^\alpha_s = \gamma = \gamma * h^\alpha_t$ where $c = -(\alpha^2 + \rho^2)$.

Suppose that $s/t = m/n$ where m, n are relatively prime positive integers. Let $c_2 = 2m\pi/s$. We choose $c_1 < 0$ sufficiently negative such that $c_1 + ic_2 \in \Lambda(S_p)$. It can be verified in a straightforward way from (5.4.1) that $s = 2m\pi/c_2 \geq T_{c_1}, t = sn/m = 2n\pi/c_2 \geq T_{c_1}$ and $|s-t| = 2|n-m|\pi/c_2 \geq T_{c_1}$. We find $\alpha, \beta \in S_p$ such that $-(\alpha^2+\rho^2) = c_1$ and $-(\beta^2+\rho^2) = c_1+ic_2$. This is possible as both $c_1, c_1+ic_2 \in \Lambda(S_p)$. We take $c = c_1$. Since $-(\alpha^2 + \rho^2) = c_1 = c$ we have

$$\varphi_\alpha * h_t^\alpha = \varphi_\alpha = \varphi_\alpha * h_s^\alpha.$$

We also have

$$\varphi_\beta * h_s^\alpha = e^{s(c_1+ic_2-c_1)}\varphi_\beta = e^{isc_2}\varphi_\beta = e^{i2m\pi}\varphi_\beta = \varphi_\beta$$

and using $t = (n/m)s$,

$$\varphi_\beta * h_t^\alpha = e^{t(\Delta-c_1)}\varphi_\beta = e^{itc_2}\varphi_\beta = e^{i(n/m)sc_2}\varphi_\beta = e^{i2n\pi}\varphi_\beta = \varphi_\beta.$$

It is established that $\gamma = \varphi_\alpha + \varphi_\beta$ then $\gamma * h_s^\alpha = \gamma = \gamma * h_t^\alpha$ but γ is not an eigendistribution of Δ.

Lastly, we note that in the hypothesis of Corollary 5.4.4, the condition $\gamma * h_s^\alpha = \gamma = \gamma * h_t^\alpha$ cannot be substituted by $\gamma * h_s^\alpha = \gamma * h_t^\alpha$. Because by Corollary 5.4.3 when $|s-t| \geq T_c$, there exists an L^p-tempered distribution γ which satisfies $\gamma * h_s^\alpha = \gamma * h_t^\alpha$ but γ is not an eigendistribution of Δ. For the sake of completion, let us now prove the corollary.

Proof. As discussed above, it is enough to prove the assertion with the assumption that $t > T_c, s > T_c$. Let $\xi = t/s$. Then $\xi > 0$ is an irrational number. We take $0 < \epsilon < T_c$ which implies $0 < \epsilon/s < 1$. By Kronecker's approximation theorem, there exist $n \in \mathbb{N}$, such that $0 < n\xi - m < \epsilon/s$, where $m \in \mathbb{N}$ is the integer part of $n\xi > 0$. Hence $0 < nt - ms < \epsilon < T_c$. Since h_t^α is a semigroup, it follows from the hypothesis that $f * h_{nt}^\alpha = f * h_{ms}^\alpha$. The assertion follows from these and Corollary 5.4.3. □

So far we have not considered L^2-tempered distributions, which we shall do now. First we recall some relevant information.

(i) The space $\mathcal{C}^2(\widehat{G//K})$ is isomorphic to the space of even Schwartz class functions on $\mathbb{R}$. In particular, for a function $\psi \in \mathcal{C}^2(G//K)$, $\widehat{\psi}$ has no complex analytic extension, in general.

(ii) The L^2-spectrum $\Lambda(S_2) = (-\infty, -\rho^2]$.

(iii) For $p = 2$ and any $c \in \Lambda(S_2)$, $T_c = \infty$ (see (5.4.2)).

(iv) Weak L^2-functions and functions in $\mathcal{H}_2^r(X)$ are L^2-tempered distributions (Proposition 1.5.1 (a), Proposition 1.6.1 (d)).

(v) If $\Delta u = -\rho^2 u$ for some $u \in L^{2,\infty}(X)$, then $u = 0$ (see Proposition 1.7.1).

(vi) $(1 + |x|)^{-1}\varphi_0 \in L^{2,\infty}(X)$ and $\Delta\varphi_0 = -\rho^2\varphi_0$ (see Proposition 1.4.1).

As, (i), (ii) and (iii) suggest that (for $p = 2$) the situation is close to Euclidean, in particular the proof of Theorem 5.4.2, which deals with analytic functions on complex domain will not work in this situation.

Theorem 5.4.5. *Let γ be an L^2-tempered distribution on X and let $t > 0$ be fixed. Let f be a measurable function on X. Let $t > 0$ be fixed.*

(a) *If $e^{t\Delta}\gamma = e^{tc}\gamma$ for some $c = -(\alpha^2 + \rho^2) \leq -\rho^2$, then $\Delta\gamma = c\gamma$.*

(b) *If γ in (a) is a function $f \in L^{2,\infty}(X)$ and $c < -\rho^2$, then $f = \mathcal{P}_\alpha F$ for some $F \in L^2(K/M)$.*

(c) *If γ in (a) is a function f satisfying $(1 + |x|)^{-1} f \in L^{2,\infty}(X)$ and $c = -\rho^2$, then $f = \mathcal{P}_0 F$ for some $F \in L^2(K/M)$.*

(d) *If γ in (a) is a function $f \in \mathcal{H}_2^r(X)$ for $1 < r \leq \infty$ and $c = -\rho^2$ then $f = \mathcal{P}_0 F$ for some $F \in L^r(K/M)$. If $r = 1$ and other conditions are same then $f = \mathcal{P}_0\mu$ for some signed measure μ on K/M.*

Proof. As we have seen in the proof of Theorem 5.4.2, it is enough to prove (a) under the assumption that γ is radial. The hypothesis can be rewritten as $\gamma * h_t^\alpha = \gamma$. This implies that $\gamma * h_{nt}^\alpha = \gamma$, hence

$$\widehat{\gamma} = e^{nt(\alpha^2 - \lambda^2)}\widehat{\gamma} \text{ for any } n \in \mathbb{N}.$$

We take $\phi, \psi \in \mathcal{C}^2(\widehat{G//K})$. As ϕ, ψ are even functions on $\mathbb{R}$, we consider them as functions on $[0, \infty)$. Let us first assume $\alpha > 0$. Suppose that ϕ is supported on $[\alpha + \epsilon, \infty)$, and ψ is supported on $[0, \alpha - \epsilon]$ for some $0 < \epsilon < \alpha$. For $n \in \mathbb{N}$, define

$$\psi_{nt}(\lambda) = \psi(\lambda)e^{nt(\lambda^2 - \alpha^2)} \in \mathcal{C}^2(\widehat{G//K}).$$

Then for any $n \in \mathbb{N}$,

$$\langle \widehat{\gamma}, \phi \rangle = \langle \widehat{\gamma}, e^{nt(\alpha^2-\lambda^2)}\phi \rangle \text{ and } \langle \widehat{\gamma}, \psi \rangle = \langle \widehat{\gamma} e^{nt(\alpha^2-\lambda^2)}, e^{-nt(\alpha^2-\lambda^2)}\psi \rangle = \langle \widehat{\gamma}, \psi_{nt} \rangle.$$

We note that $e^{nt(\alpha^2-\lambda^2)}\phi \to 0$ and $\psi_{nt} \to 0$ as $n \to \infty$ in the topology of $\mathcal{C}^2(\widehat{G//K})$. Therefore $\langle \widehat{\gamma}, \phi \rangle = 0$ and $\langle \widehat{\gamma}, \psi \rangle = 0$. This establishes that $\widehat{\gamma}$ is supported on $\{\pm\alpha\}$.

If $\alpha = 0$, we argue the same way, except that we need to consider only ϕ and not ψ above to conclude that $\widehat{\gamma}$ is supported on $\{0\}$.

From this and since γ is radial, we conclude that (see [68, Theorem 6.25]), there exists a polynomial P such that

$$\widehat{\gamma} = P(\partial_\lambda)\delta_\lambda|_{\lambda=\alpha}.$$

Using injectivity of the spherical Fourier transform we get,

$$\gamma = \sum_{l=0}^{N} a_l \varphi_{\alpha,l},$$

for some $a_1, a_2, \ldots, a_N \in \mathbb{C}$, $a_N \neq 0$, and $N \in 2\mathbb{Z}$ for the case $\alpha = 0$ (as odd derivatives in λ of φ_λ is identically 0). Rest of the argument is as the proof of Theorem 5.4.1: we assume that $N \geq 1$, use the hypothesis $e^{t\Delta}\gamma = e^{tc}\gamma$, i.e. $\gamma * h_t = e^{ct}\gamma$, equate the coefficients of $\varphi_{\alpha,N-1}$ if $\alpha \neq 0$ and coefficients of $\varphi_{\alpha,N-2}$ if $\alpha = 0$ in the both sides, and conclude that $a_N = 0$. Thus $\gamma = C\varphi_\alpha$, in particular $\Delta\gamma = c\gamma$. This completes the proof for radial γ and hence of the assertion (a).

Assertions (b), (c), and (d) follow from (a), applying Propositions 1.5.1, 1.6.1, Corollary 1.7.4 and Theorem 1.7.2. □

See [47] for other characterization of eigenfunctions with real eigenvalues as the Poisson transform of L^2-function on K/M, arising from Strichartz conjecture ([75]). This can be used instead of weak L^2-norm to obtain analogous result.

Remark 5.4.6. Following remarks are in order.

(1) If we take $c = 0$, then $c \in \Lambda(S_1)$ and $c \notin \Lambda(S_p)$ for any other $1 < p \leq 2$. In this case Theorem 5.4.2 gives back the classical result: $f * h_t = f$ for $f \in L^\infty(X)$ implies that f is harmonic, which we have discussed at the beginning.

(2) We have used the results for the characterization of L^p, weak L^p-eigenfunctions or eigenfunctions in the Hardy-type spaces as the Poisson transform (of Lebesgue functions or measure on K/M). As mentioned above, unlike the Lebesgue spaces

and Lorentz spaces for $1 \leq p \leq \infty$, the Hardy-type spaces can accommodate eigenfunctions with arbitrary complex eigenvalues. There are other size estimates in the literature, through which such characterization is possible. See for instance subsection 4.1 in [51] for a brief survey.

Bibliography

[1] Abramowitz, M.; Stegun, I. A. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* National Bureau of Standards Applied Mathematics Series, 55, Washington, D.C. 1964.

[2] Ahern, P.; Flores, M.; W. Rudin. *An invariant volume-mean-value property.* J. Funct. Anal. 111 (1993), 380–397.

[3] Anker, J-P. *La forme exacte de l'estimation fondamentale de Harish-Chandra.* C. R. Acad. Sci. Paris. Sér. I Math. 305 (1987), no. 9, 371–374

[4] Anker, J-P. *The Spherical Fourier Transform of Rapidly Decreasing Functions. A Simple Proof of a Characterization due to Harish-Chandra, Helgason, Trombi, and Varadarajan.* Journal of Functional Analysis 96, 331–349(1991).

[5] Anker, J-P.; Damek, E.; Yacoub, C. *Spherical analysis on harmonic AN groups.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 23 (1996), no. 4, 643–679 (1997).

[6] Anker, J-P.; Ji, L. *Heat kernel and Green function estimates on noncompact symmetric spaces.* Geom. Funct. Anal. **9** (1999), no. 6, 1035–1091.

[7] Anker, J-P.; Setti, A. G. *Asymptotic finite propagation speed for heat diffusion on certain Riemannian manifolds.* J. Funct. Anal. 103 (1992), no. 1, 50–61.

[8] Ballmann, W. *Lectures on Spaces of nonpositive curvature.* DMV Seminar, 25. Birkhäuser Verlag, Basel, 1995.

[9] Ben Natan, Y. *Wiener's Tauberian Theorem on G/K and Harmonic Functions on G/K.* Thesis, Technion, Israel-Institute of Technology.

[10] Ben Natan, Y.; Weit, Y. *Integrable harmonic functions on symmetric spaces of rank one.* Journal of Functional Analysis 160 (1998), 141–149.

[11] Ben Natan, Y.; Benyamini, Y; Hedenmalm, H.; Weit, Y. *Wiener's Tauberian theorem for spherical functions on the automorphism group of the unit disk.* Ark. Mat. 34 (1996), no. 2, 199–224.

[12] Benyamini Y.; Weit, Y. *Functions satisfying the mean value property in the limit*, J. Analyse Math. 52 (1989), 167–198.

[13] Ben Saïd, S.; Oshima, T.; Shimeno, N. *Fatou's theorems and Hardy-type spaces for eigenfunctions of the invariant differential operators on symmetric spaces.* Int. Math. Res. Not. 2003, no. 16, 915–931.

[14] Blaschke, W. *Ein Mittelwertsatz und eine kennzeichnende Eigenschaft des logaritmischen potentials*, Ber. Ver. Sächs. Akad. Wiss. Leipzig 68 (1916), 3–7.

[15] Blaschke, W. *Mittelwertsatz der Potentialtheorie*, Jahresber. Deutsch. Math. Verein. 27 (1918), 157–160.

[16] Bridson, M. R.; Haefliger, A. *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999.

[17] Casadio Tarabusi, E; Cohen, J. M.; Korányi, A.; Picardello, M. A. *Converse mean value theorems on trees and symmetric spaces.* J. Lie Theory 8 (1998), no. 2, 229–254.

[18] Cheeger, J.; Gromov, M.; Taylor, M. *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds.* J. Differential Geom. 17 (1982), no. 1, 15–53.

[19] Cheng, S.Y.; Yau, S.T. *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. 28 (1975), 333–354.

[20] Clerc, J. L.; Stein, E. M. *L^p-multipliers for non-compact symmetric spaces* , Proc. Nat. Acad. Sci. USA 71 (1974), 3911–3912.

[21] Colding, T. H.; Minicozzi, W. P. *Harmonic functions with polynomial growth.* J. Differential Geom. 46 (1997), no. 1, 1–77.

[22] Corwin, L. *Tempered distributions on Heisenberg groups whose convolution with Schwartz class functions is Schwartz class.* J. Funct. Anal. 44 (1981), no. 3, 328–347.

[23] Cowling, M. *Herz's "principe de majoration" and the Kunze-Stein phenomenon.* 73–88, CMS Conf. Proc., 21, Amer. Math. Soc., Providence, RI, 1997.

[24] Cowling, M.; Giulini, S.; Meda, S. *$L^p - L^q$ estimates for functions of the Laplace–Beltrami operator on noncompact symmetric spaces.* I. Duke Math. J. 72 (1993), no. 1, 109–150.

[25] Dahlner, A. *A Wiener Tauberian theorem for weighted convolution algebras of zonal functions on the automorphism group of the unit disc.* 67–102, Contemp. Math., 404, Amer. Math. Soc., Providence, RI, 2006.

[26] Davies, E. B. *Heat kernels and spectral theory.* Cambridge University Press, Cambridge, 1989.

[27] de Barra, G. *Measure theory and integration.* Horwood Publishing Limited, Chichester, 2003.

[28] Delsarte, J. *Lectures on topics in mean periodic functions and the two-radius theorem.* Tata Institute of Fundamental Research, Bombay, 1961.

[29] Delsarte, J. *Note sur une propriété nouvelle des fonctions harmoniques.* C. R. Acad. Sci. Paris. A–B, 246 (1958), 1358–1360.

[30] Delsarte, J.; Lions, J. L. *Moyennes généralisées.* Comment. Math. Helv. 33 (1959), 59–69.

[31] Eguchi, M.; Hashizume, M; Okamoto, K. *The Paley-Wiener Theorem for Distributions on Symmetric Spaces.* Hiroshima Math. J. 3(1973), 109–120.

[32] Eguchi, M.; Hashizume, M; Okamoto, K. *Asymptotic Expansions of Eisenstein Integrals and Fourier Transform on Symmetric Spaces.* J. Funct. Anal. 34, 167–216 (1979).

[33] Flatto, Leopold. *The converse of Gauss's mean value theorem for harmonic functions.* Journal of differential equations 1, 483–490(1965).

[34] Flensted-Jensen, M.; Razozin, David L. *Spherical functions are fourier transform of L_1-functions.* Ann. Sci. cole Norm. Sup. (4) 6 (1973), 457–458 (1974).

[35] Folland, G. B. *Introduction to partial differential equations.* 2nd edition. Princeton University Press, Princeton, NJ, 1995.

[36] Friedman, A. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964.

[37] Furstenberg, H. *A Poisson formula for semi-simple Lie groups*. Ann. of Math (2) 77 (1963), 335–386.

[38] Furstenberg, H. *Boundaries of Riemannian symmetric spaces*. Symmetric spaces. 359–377. Pure and Appl. Math., Vol. 8, Dekker, New York, 1972.

[39] Gangolli, R.; Varadarajan, V. S. *Harmonic analysis of spherical functions on real reductive groups*. Springer-Verlag, Berlin, 1988.

[40] Gray, A.; Willmore, T. J. *Mean-value theorems for Riemannian manifolds*. Proc. Roy. Soc. Edinburgh Sect. A 92 (1982), no. 3-4, 343–364

[41] Helgason, S. *Groups and geometric analysis. Integral geometry, invariant differential operators and spherical functions*. Academic Press, Inc., Orlando, FL, 1984.

[42] Helgason, S. *Geometric analysis on symmetric spaces*. Math. Surveys and Monographs, 39. AMS, Providence, RI, 2008.

[43] Herz, C. *Sur le phénomène de Kunze-Stein*. C. R. Acad. Sci. Paris Sér. A **271** (1970) 491–493.

[44] Howard, R.; Reese, M. *Characterization of eigenfunctions by boundedness conditions*. Canad. Math. Bull. 35 (1992), no. 2, 204–213.

[45] Hua, B. *Generalized Liouville theorem in nonnegatively curved Alexandrov spaces*. Chin. Ann. Math. Ser. B 30 (2009), no. 2, 111–128.

[46] Ionescu, A. D. *On the Poisson transform on symmetric spaces of real rank one*. J. Funct. Anal. 174 (2000), no. 2, 513–523.

[47] Kaizuka, K. *A characterization of the L^2-range of the Poisson transform related to Strichartz conjecture on symmetric spaces of noncompact type*. Adv. Math. 303 (2016), 464–501

[48] Koornwinder, T. H. *Jacobi function and analysis on noncompact semisimple Lie groups* in: Special functions: group theoretical aspects and applications, (eds) Richard Askey et. al. (1984), Math. Appl. (Dordrecht: Reidel) pp. 1–85.

[49] Korányi, A. *On a mean value property for hyperbolic spaces*, Contemp. Math. 191 (1995), 107–116.

[50] Kumar, P. *Fourier restriction theorem and characterization of weak L^2 eigenfunctions of the Laplace–Beltrami operator*. J. Funct. Anal. 266 (2014), no. 9, 5584–5597.

[51] Kumar, P.; Ray, S. K.; Sarkar, R. P. *The role of restriction theorems in harmonic analysis on harmonic NA groups*. J. Funct. Anal. 258 (2010), no. 7, 2453–2482.

[52] Kumar, P.; Ray, S. K.; Sarkar, R. P. *Characterization of almost L^p-eigenfunctions of the Laplace–Beltrami operator*. Transactions of the American mathematical society. 366 (2014), no. 6, 3191–3225.

[53] Li, P. *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature*. Ann. of Math. (2) 124 (1986), no. 1, 1–21.

[54] Narayanan, E. K.; Sitaram, A. *Some questions on integral geometry on noncompact symmetric spaces of higher rank*. Monatsh. Math. 170 (2013), no. 2, 195–203.

[55] Pati, V.; Sitaram, A. *Some questions on integral geometry on Riemannian manifolds*. Sankhya Ser. A 62 (2000), no. 3, 419–424.

[56] Petersen, P. *Riemannian geometry* Springer, New York, 2006.

[57] Peyerimhoff, N.; Samiou, E. *Spherical spectral synthesis and two-radius theorems on Damek-Ricci spaces*. Ark. Mat. 48 (2010), no. 1, 131–147.

[58] Peyerimhoff, Norbert; Samiou, E. *Integral Geometric Properties of Noncompact Harmonic Spaces*. J Geom Anal (2015) 25, 122–148.

[59] Plancherel, M.; Pólya, G. *Sur les valeurs moyennes des fonctions réelles définies pour toutes les valeurs de la variable*. Comment. Math. Helv. 3 (1931), no. 1, 114–121.

[60] Pusti, Sanjoy; Ray, Swagato K.; Sarkar, Rudra P. *Wiener-Tauberian type theorems for radial sections of homogeneous vector bundles on certain rank one Riemannian symmetric spaces of noncompact type*. Math. Z. 269 (2011), no. 1-2, 555–586.

[61] Pusti, S.; Samanta, A. *Wiener Tauberian theorem for rank one semisimple Lie groups and for hypergeometric transforms.* Math. Nachr. 290 (2017), no. 13, 2024–2051.

[62] Ramsey, T.; Weit, Y. *Multipliers on Schwartz spaces of some Lie groups.* Colloq. Math. 51 (1987), 295–301.

[63] Ray, S. K.; Sarkar, R. P. *Fourier and Radon transform on harmonic NA groups.* Trans. Amer. Math. Soc. 361 (2009), no. 8, 4269–4297.

[64] Rawat, R.; Sitaram, A. *The injectivity of the Pompieu transform and L^p-analogues of the Weiner-Tauberian Theorem.* Isr. J. Math. 91 (1995), 307–316.

[65] Repnikov, V. D.; Èĭdel'man, S. D. *Necessary and sufficient conditions for establishing a solution to the Cauchy problem.* Soviet Math. Dokl. 7 (1966), 388–391.

[66] Repnikov, V. D.; Èĭdel'man, S. D. *A new proof of the theorem on the stabilization of the solution of the Cauchy problem for the heat equation.* Mat. Sb. (NS) 73 (115) 1967, 155–159.

[67] Roe, J. *A characterization of the sine function.* Math. Proc. Cambridge Philos. Soc. 87 (1980), no. 1, 69–73.

[68] Rudin, W. *Functional analysis.* McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.

[69] Sarkar, R. P. *Eigenfunction of the Laplacian as a degenerate case of a function with its Fourier transform supported in an annulus.* Publ. Res. Inst. Math. Sci. 54 (2018), no. 2, 351–378.

[70] Sjögren, P. *Characterizations of Poisson integrals on symmetric spaces.* Math. Scand. 49 (1981), 229–249.

[71] Stein; Elias M, Shakarchi, Rami. *Complex analysis* . Princeton University Press, 2006.

[72] Stein, E. M.; Weiss, G. *Introduction to Fourier analysis on Euclidean spaces.* Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971.

[73] Strichartz, R. S. *Analysis of the Laplacian on the complete Riemannian manifold*. J. Funct. Anal. 52 (1983), no. 1, 48–79.

[74] Strichartz, R. S. *Mean value properties of the Laplacian via spectral theory*. Trans. Amer. Math. Soc. 284 (1984), no. 1, 219–228.

[75] Strichartz, R. S. *Harmonic analysis as spectral theory of Laplacians*. J. Funct. Anal. 87 (1989), no. 1, 51–148.

[76] Strichartz, R. S. *Characterization of eigenfunctions of the Laplacian by boundedness conditions*. Trans. Amer. Math. Soc. 338 (1993), no. 2, 971–979.

[77] Stromberg, J. O. *Weak type L^1-estimates for maximal functions on non-Compact symmetric spaces*, Annals of Mathematics, Second Series, Vol. 114, No. 1, 1981, pp. 115–126.

[78] Taylor, M. E. *L^p-estimates on functions of the Laplace operator*. Duke Math. J. 58 (1989), no. 3, 773–793.

[79] Volchkov, Valery. V.; Volchkov, Vitaly. V. *Offbeat Integral Geometry on Symmetric Spaces*, Birkhauser, 2013.

[80] Weit, Y. *On a generalized asymptotic mean value property*. Aequationes Math. 41 (1991), no. 2-3, 242–247.

[81] Weit, Y. *On Furstenberg's characterization of harmonic functions on symmetric spaces*. Isr. J. Math. 114 (1999) 265–269.

[82] Weit, Y. *On the characterization of harmonic functions on weighted spaces and the heat equation*. Indian J. Math. 55 (2013), suppl., 159–165.

[83] Wong, R; Wang Q. Q. *On the asymptotics of the Jacobi function and its zeros*. SIAM J. MATH. ANAL, 23 (1992), no. 6, pp 1637–1649.

[84] Yau, S. T. *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975) 201–228.