On free-type rigid C*-tensor categories and their annular representations

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To every other person in my life...

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Contents

Notations

$\mathbb{N}$	The set of natural numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{R}$	The set of real numbers
$M_n(\mathbb{C})$	The set of $n \times n$ complex matrices
$\mathcal{L}(\mathcal{H})$	Set of bounded operators on the Hilbert space $\mathcal{H}$
S^1	The unit circle

Chapter 0

Introduction

Rigid C*-tensor categories provide a unifying language for a variety of phenomena encoding "quantum symmetries". For example, they appear as the representation categories of Woronowicz' compact quantum groups, and as "gauge symmetries" in the algebraic quantum field theory of Haag and Kastler. Perhaps most prominently, they arise as categories of finite index bimodules over operator algebras, taking center stage in Jones' theory of subfactors. The construction and classification of these categories is a very active area of research. Much of the work in this area has been focused on *unitary fusion categories*, which are rigid C*-tensor categories with finitely many isomorphism classes of simple objects with simple tensor unit. Categories with infinitely many isomorphism classes of simple objects are called *infinite depth*, and the primary examples come from either discrete groups, representation categories of compact quantum groups, or general categorical constructions such as the free product.

Planar algebras as an invariant for finite index subfactors was introduced by Vaughan Jones. This invariant can be axiomatized by Ocneanu's paragroups in the finite depth case [Ocn88], and in general by Popa's λ-lattices [Pop95] or Jones' subfactor planar algebras [Jon99]. The planar algebra approach to standard invariants has become important in classification programs for subfactors [JMS14, BJ00] and has provided useful tools for constructions of exotic fusion categories [BPMS12]. These invariants corresponding to specific types of subfactors have been calculated by several authors in different contexts. In [Jon01], the notion of *annular modules over a planar algebra* or *annular representations* was introduced to construct subfactors with principal graphs E_6 and E_8. These representations in the case of Temperley-Lieb planar algebra (TL) were worked out in the same paper. In [Gho06], the author showed that the category of the representations of group planar algebra (i.e., the planar algebra associated to the fixed point subfactor arising from an outer action of a finite group) is equivalent to the category of representations of a non-trivial quotient of the quantum double of the corresponding group. It was observed that the quotient can be avoided if one restricts the isotopies to the ones which fixes boundaries of annulus at all times (called *affine isotopies*, introduced in [JR06]). With this isotopy in place, the representations are called *affine representations* or *modules*. Further in [JR06], affine annular category of a planar algebra $\mathcal{A}P$ was introduced and the same for Temperley-Lieb planar algebras was studied. The structure of affine annular category of a planar algebra was further studied in [DGG14a] and they introduced the braided monoidal category of Hilbert representations of the affine annular category of a planar alegebra, denoted by $Rep(\mathcal{A}P)$ and showed that in the case of finite depth planar algebras, $Rep(\mathcal{A}P) \overset{\otimes}{\cong} Z(\text{proj}(P))$, where $Z(\text{proj}(P))$ denotes the unitary Drinfeld center of the projection category, $\text{proj}(P)$, of the planar algebra P.

The unitary representation theory of the annular Temperley-Lieb planar algebras has played an important role in the construction and classification of small index subfactors [Jon01, JMS14]. Annular categories also play a role in the realm of $2+1$ TQFT's [Wal06]. Ocneanu's tube algebra was also been shown to be closely related to the Drinfeld center of a fusion categories, with equivalence classes of irreducible representations of the tube algebra being in $1-1$ correspondence with simple objects in the center [Izu99, Müe03]. Analyzing the tube structure has been useful for computing the S and T matrices of $Z(\mathcal{C})$ [Izu99].

From a different direction, representation theory for rigid C*-tensor categories was introduced in a remarkable paper of Popa-Vaes [PV15] in which they introduce the concept of cp-multipliers for $\mathcal{C}$ which are a class of functions in $l^\infty(\text{Irr}(\mathcal{C}))$. These cp-multipliers give positive linear functionals on the fusion algebra $\mathbb{C}[\text{Irr}(\mathcal{C})]$ after a certain normalization. A $*$-representation is said to be *admissible* if every vector state is a normalization of some cp-multiplier. The class of admissible representations of the fusion algebra provides a good notion for the representation theory for $\mathcal{C}$, generalizing unitary representations of a discrete group G if $\mathcal{C}$ is equivalent to $\mathbf{Vec}(G)$. One can take the universal C*-completion of the fusion algebra, denoted by $C_u^*(\mathcal{C})$, with respect to these admissible representations. In this context, they define approximation and rigidity properties, generalizing the definitions from the world of discrete groups. They show that if $\mathcal{C}$ is equivalent to the category of M-M-bimodules in the standard invariant of a finite index inclusion $N \subseteq M$ of II_1 factors, then the definitions of approximation and rigidity properties given via cp-multipliers [PV15] are equivalent to the definitions given via the symmetric enveloping algebra for the subfactor $N \subseteq M$ given by Popa [Pop94b, Pop99].

In the process of unification of these approaches, Ghosh-Jones [GJ16] introduced *annular algebras* $\mathcal{A}\Lambda$ with weight set $\Lambda \subseteq [\mathrm{Obj}(\mathcal{C})]$ for any rigid C*-tensor category $\mathcal{C}$. Choosing $\Lambda := \mathrm{Irr}(\mathcal{C})$ yields the tube algebra of Ocneanu, $\mathcal{A}$, whereas if we choose Λ on a planar algebra description gives the affine annular category $\mathcal{A}P$ of Jones. The category of non-degenate $*$-representations of the annular algebra $\mathcal{A}\Lambda$, $Rep(\mathcal{A}\Lambda)$, is called the *annular representation category of* $\mathcal{C}$ *with weight* Λ. It was shown that all sufficiently large (full) weight sets, annular algebras are isomorphic after tensoring with the $*$-algebra of matrix units with countable index set, hence have equivalent representation categories, unifying the two perspectives and providing a means of translating results from planar algebras to the tube algebra in a direct way. This representation category also provides a representation-theoretic characterization of the category $Z(\text{Ind-}\mathcal{C})$, introduced by Neshveyev and Yamashita [NY16] to provide a categorical understanding of analytic properties (see [PSV18]).

For each $X \in \Lambda$, there is a corner of the annular algebra, denoted by $\mathcal{A}\Lambda_{X,X}$, which is a unital $*$-algebra. One can also take universal C*-completions for these corner algebras (denoted by $C^*_u(\mathcal{A}\Lambda_{X,X})$) with respect to the representations which extend to the whole of annular algebra (such representations are called *weight* X *admissible representations*). The corner algebra corresponding to the unit object $\mathbb{1}$, $\mathcal{A}\Lambda_{\mathbb{1},\mathbb{1}}$, turns out to be canonically $*$-isomorphic to the fusion algebra of the category. Thus weight $\mathbb{1}$ admissible representations are precisely the admissible representations in the sense of Popa-Vaes [PV15] and $C^*_u(\mathcal{C}) \cong C^*_u(\mathcal{A}\Lambda_{\mathbb{1},\mathbb{1}})$. Hence the admissible representation theory of Popa and Vaes is the restriction of ordinary representation theory of the tube algebra. This allowed them to define analytical properties like amenability, Haagerup property and property (T) in terms of weight X representations.

Unlike rigid C*-tensor categories themselves, whose underlying categorical structure is trivial due to semi-simplicity, the representation category of the tube algebra is a large W*-category, and is complicated to describe. Thus an important problem is to find concrete descriptions of these large representation categories in terms of representation categories of more familiar C*-algebras such as group C*-algebras. Basic philosophy in Mathematics is to describe new objects in terms of the objects already well-known. There are many procedures for producing new rigid C*-tensor categories from old ones, such as Deligne tensor product, equivariantization, G-graded extensions, etc. A natural question is, if we understand the annular structure of our starting categories, can we describe the annular representation category of the one we have produced?

The first part of this thesis deals with the description of annular representations of one such constructed category, namely, free product of two rigid C*-tensor categories, in terms of the annular representations of the individual categories.

Into a different direction, consider a finite index subfactor $N \subseteq M$ and suppose N and M are both hyperfinte. Then we can choose an isomorphism $\varphi : N \to M$ and consider $\mathcal{H}_\varphi := {}_N L^2(M)_{\varphi(N)}$ as an N-N bimodule, with left action given by the inclusion as usual, but the right action uses the isomorphism φ and the right action by M. The bi-category constructed from alternating powers of $\mathcal{H}_\varphi$ and $\overline{\mathcal{H}}_\varphi$ recovers the standard invariant of the subfactor (see [Gho11, Pop96]). However, the whole tensor category generated by $\mathcal{H}_\varphi$ and $\overline{\mathcal{H}}_\varphi$ contains more information than just the subfactor standard invariant This information is captured by an oriented planar algebra $\mathcal{P}_+$, whose alternating part is isomorphic to the subfactor planar algebra $\mathcal{P}_{N\subseteq M}$ associated to the original inclusion $N \subseteq M$. We call such a planar algebra an *oriented extension* of $\mathcal{P}_{N\subseteq M}$. In the second part of this thesis, we introduce a universal such extension called the *free oriented extension* of a subfactor planar algebra, and try to study several of its properties.

Now we try to give an idea of the contents of each of the chapters.

The Chapter 1 discusses the concepts and results needed in the later chapters of the thesis. For the sake of completeness, in Section 1.1.1, we begin with brief discussion of how one can have the tower of basic construction and hence the *standard invariant* in terms of relative commutants for a given finite index subfactor $N \subseteq M$. In Section 1.1.2, we move on to discussing bifinite bimodules over II_1 factors and describe the relative tensor product bifinite bimodules. We quickly give the intertwiner picture of the standard invariant in Section 1.1.3 and define planar algebras from the scratch in the next subsection (Section 1.1.4). We then state, in Section 1.1.5, a landmark result of Jones giving the relation between subfactor planar algebras and extremal subfactors thus giving the planar algebraic version of the standard invariant. The next subsection, Section 1.1.6, is devoted to defining the oriented planar algebras which are generalizations of shaded planar algebras. The category theoretic background required for the later parts of the thesis is provided in detail in Section 1.2. Definitions and statements of various results are presented leading to rigid semi-simple C*-tensor categories and C*-2-categories. Examples are given at appropriate points to make the things clear to the reader. The Section 1.3 of the chapter is intended to give a picture of interconnections between subfactor planar algebras and singly generated C*-2-categories, and oriented planar algebras and singly generated C*-tensor categories. Construction of free product of two rigid semi-simple C*-tensor categories is presented in the Section 1.4. Free product has already appeared in work of Bisch-Jones [BJ97] as free composition of subfactors and in the work of Wang

[Wan95] in the context of compact quantum groups. To fit our requirements, we see the free product as *unitary idempotent completion* of the category of non-crossing partitions of words with letters from objects of the underlying categories. We end the chapter with Section 1.5 which gives a review of annular representation theory for rigid semi-simple C*-tensor categories introduced by Ghosh-Jones [GJ16]. We see that the notion of "admissible representations" of Popa-Vaes [PV15] coincides with "weight $\mathbb{1}$ admissible representations" of Ghosh-Jones [GJ16], thus facilitating the translation of approximation and rigidity properties into the language of annular representations.

Chapter 2 is about annular representations of free product categories which is essentially the whole of [GJR18a]. In the first section, we define a new weight set Λ which has a distinguished subset $\mathbf{W}$ of words with specific properties. We see that though Λ is not full, the annular representation category with this weight set Λ is unitarily equivalent as a *-linear category to the representation category of the tube algebra of the free product category, thus enabling us to work with the much smaller weight set Λ. By dividing the section into subsections corresponding to the length of words in Λ, we analyze the corresponding centralizer algebras. Suppose $\mathcal{B}$ is an arbitrary rigid C*-tensor category, and $\Gamma \subseteq [\mathrm{Obj}(\mathcal{B})]$ be an arbitrary weight set containing $\mathbb{1}$ which is "essentially full". Then we have $\mathcal{J}\Gamma_0 := \mathcal{A}\Gamma \cdot \mathcal{A}\Gamma_{\mathbb{1},\mathbb{1}} \cdot \mathcal{A}\Gamma$, an ideal in $\mathcal{A}\Gamma$ generated by $\mathcal{A}\Gamma_{\mathbb{1},\mathbb{1}}$. Let $Rep_0(\mathcal{A}\Gamma)$ be the category of *admissible representations of the fusion algebra* with respect to Γ which is equivalent to $Rep(C^*_u(\mathcal{B}))$ and $Rep_+(\mathcal{A}\Gamma) := Rep(\mathcal{A}\Gamma/\mathcal{J}\Gamma_0)$ be the representations of $\mathcal{A}\Gamma$ which contain $\mathcal{J}\Gamma_0$ in their kernel. We then make a crucial observation that $Rep(\mathcal{A}\Gamma) \cong Rep_0(\mathcal{A}\Gamma) \oplus Rep_+(\mathcal{A}\Gamma)$. Whenever $\Gamma = \mathrm{Irr}(\mathcal{C})$, we write $\mathcal{J}\mathcal{C}_0$ for $\mathcal{J}\Gamma_0$. Using this along with some more analysis of the annular algebra we finally end this section by proving that

$$Rep(\mathcal{A}\,(\mathcal{C} * \mathcal{D})) \cong Rep(C^*_u(\mathcal{C}) * C^*_u(\mathcal{D})) \oplus Rep_+(\mathcal{A}\mathcal{C}) \oplus Rep_+(\mathcal{A}\mathcal{D}) \oplus Rep(\mathbb{Z})^{\oplus \mathbf{W}_0},$$

where $\mathbf{W}_0$- set of cyclic equivalence classes of words in $\mathbf{W}$.

In the next section we present an example of the category of G-graded Hilbert spaces, $\mathbf{Hilb}_{f.d.}(G)$, and show how this matches another known result. By using the results of this chapter and [DGG14a, NY18, JR06] we then explain how one can have a description of the representations of Fuss-Catalan categories.

Chapter 3 consists of [GJR18b] and deals with oriented extensions of subfactor planar algebras. We begin the first section by setting up a few notations. We denote by $\mathcal{P}_{sh}$ (resp., $\mathcal{P}_{or}$) the category of all subfactor planar algebras (resp., oriented factor planar algebras). There is a natural shading functor $\mathcal{S} : \mathcal{P}_{or} \to \mathcal{P}_{sh}$ which is a forgetful functor sending every oriented factor planar algebra to its shaded part which is a subfactor planar algebra. We then define what we mean by an oriented extension of a subfactor planar algebra and construct a canonical extension which we call the free oriented extension. Further we show that it enjoys a universal property, namely, for any oriented extension there exists a canonical sub planar algebra isomorphic to the free oriented extension. The free oriented extension can also be viewed as a functor from $\mathcal{P}_{sh}$ to $\mathcal{P}_{or}$ and we show that, by virtue of the universal property, it is a left adjoint to the shading functor $\mathcal{S}$. It was suggested by V.F.R. Jones that the free oriented extension should be related to free products of categories. We show that this is indeed true. Namely, if Q is any oriented extension, then the free oriented extension is realized inside the free product of the projection category of Q with the category of $\mathbb{Z}$-graded finite dimensional Hilbert spaces. In the last section, we use this result, combined with a result of Vaes, to show that the free oriented extension of any hyperfinite II_1 subfactor planar algebra is realized in the category of bimodules of the hyperfinite II_1 factor.

Of the two component articles [GJR18a, GJR18b] which are included in this thesis, the first one, [GJR18a] has been accepted for publishing in the Journal of Non-Commutative Geometry and the second one, [GJR18b] has been published in the International Journal of Mathematics.

Chapter 1

Preliminaries

This chapter essentially gives the background required for understanding the next chapters. Since many of the results are directly taken from some or the other source (mentioned accordingly), we omit proofs for most of them.

1.1 Subfactors, bimodules, planar algebras

1.1.1 Subfactors and the tower of basic construction

An infinite dimensional von Neumann algebra with trivial center and a faithful tracial state is said to be a II_1 **factor**. It can be proved that such a trace is unique. By a **subfactor** we mean a unital inclusion of II_1 factors.

Given a subfactor $N \subseteq M$ and the faithful tracial state tr on M, uniqueness enforces the restriction of tr to N to be faithful on N. Let $\mathcal{H} := L^2(M)$, the Hilbert space underlying the GNS representation of M associated with tr and Ω be the distinguished cyclic vector. Then the subspace $\mathcal{H}_1 := [N\Omega]$ of $\mathcal{H}$ can be identified with $L^2(N)$.

Let e_1 denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_1$. It is also true that $e_1(M\Omega) \subseteq N\Omega$; hence e_1 induces(by restriction) a map $E : M \to N$. The map E is called the tr-preserving **conditional expectation** of M onto N. It satisfies the following properties:

(i) $e_1 x e_1 = E(x) e_1 \ \forall x \in M$. Thus E defines a Banach space projection of M onto N.

(ii) E is $N-N$-bilinear, i.e., $E(n_1 m n_2) = n_1 E(m) n_2$.

(iii) E is tr-preserving, i.e., $\mathrm{tr} \circ E = \mathrm{tr}$.

The map $x\Omega \mapsto x^*\Omega$ is a conjugate-linear isometry from $M\Omega \subseteq L^2(M)$ onto itself. We denote its extension to $L^2(M)$ by J and call it the **modular conjugation operator** for M. J is an anti-unitary involution, i.e., $J = J^* = J^{-1}$ (where J^* is defined by the equation $\langle J\xi, \eta\rangle = \overline{\langle \xi, J^*\eta\rangle}$). It is easy to see that $Je_1 = e_1J$.

The space $\mathcal{H}$ can be seen as an M-M-bimodule with actions $x.[a\Omega] := [xa\Omega]$ and $[a\Omega].y := [Jy^*Ja\Omega]$. If we denote $\mathcal{H}_\infty := \mathcal{H} \otimes \ell^2$, then one of the striking results is the following (See [JS97]):

Theorem 1.1.1. *Any separable M-module $\mathcal{K}$ is isomorphic to a submodule of $\mathcal{H}_\infty$. Thus we have a projection $p \in M_\infty(M)$ such that $\mathcal{K} \cong \mathcal{H}_\infty p$.* □

The **M-module dimension** of $\mathcal{K}$ is defined to be $\dim_M \mathcal{K} := \text{Tr}\ p$, where Tr the trace on $M_\infty(M) = \mathcal{B}(l^2) \otimes M$ defined by $\text{Tr} := \text{Tr}_{\mathcal{B}(l^2)} \otimes \text{tr}$.

The **index** of the subfactor $N \subseteq M$, $[M : N]$, is defined as $\dim_N L^2(M)$. V. Jones proved in [Jon83] that $[M : N] \in \left\{4cos^2\frac{\pi}{n} : n = 3, 4, \cdots\right\} \cup [4, \infty]$ for any inclusion $N \subseteq M$ of II_1 factors and each of these values is realized.

Given such a unital inclusion $N \subseteq M$, we construct the von Neumann algebra $M_1 = \langle M, e_1\rangle = (M \cup e_1)''$ and consequently to the tower $N \subseteq M \subseteq M_1$. It is a fact that M_1 is a II_1 factor if and only if $[M : N] < \infty$, in which case, $[M_1 : M] = [M : N]$. Repeating the construction with $M \subseteq M_1$ and projection e_2, we get a new tower $M \subseteq M_1 \subseteq M_2$ with $M_2 = \langle M_1, e_2\rangle$. Continuing this process, we have a tower called **tower of basic construction**:

$$N \subseteq M \subseteq M_1 \subseteq \cdots\cdots$$

with projections $e_i \in M_i$, $M_i = \langle M, e_i\rangle$, and $[M_{i+1} : M_i] = [M : N]$ for $i = 1, 2, \ldots$. It is a consequence of properties of the index that, when $[M : N] < \infty$, $N' \cap M$ is finite-dimensional. One can then have, by taking relative commutants, a grid of finite dimensional C*-algebras called the **standard invariant of** $N \subseteq M$:

$$\begin{array}{ccccccc} \mathbb{C} = N' \cap N \subseteq N' \cap M \subseteq & N' \cap M_1 \subseteq & N' \cap M_2 \subseteq & \cdots\cdots \\ \cup & \cup & \cup & \\ \mathbb{C} = M' \cap M \subseteq & M' \cap M_1 \subseteq & M' \cap M_2 \subseteq & \cdots\cdots \end{array}$$

It is called an "invariant" because it turns our that it is in fact an invariant for finite index subfactors but not a complete invariant in the most general case. It is a complete invariant on the class of *amenable* subfactors (see [Pop94a]).

Note that, for a finite index subfactor $N \subseteq M$, $N' \cap M$ is contained in both N' and M. We have two traces coming from each of them (when seen inside $\mathcal{L}(L^2(M))$). We say that the subfactor $N \subseteq M$ is *extremal* if both these traces coincide.

Next, we will see the grid in terms of intertwiner spaces of bimodules $L^2(M_n)$. Even before that we will have an overview of tensor products of bifinite bimodules over II_1 factors.

1.1.2 Tensor product of bifinite bimodules over II_1 factors

Suppose M and P are II_1 factors and ${}_M\mathcal{H}$ (resp. $\mathcal{K}_P$) be a left M-module (resp., right P-module) such that dim ${}_M\mathcal{H} < \infty$ (resp., dim $\mathcal{K}_P < \infty$), equivalently, $M' := {}_M\mathcal{L}(\mathcal{H})$ (resp., $P' := \mathcal{L}_P(\mathcal{K})$) is a II_1-factor.

We say a vector $\xi \in \mathcal{H}$ (resp. $\eta \in \mathcal{K}$) is **left-bounded** (resp., **right-bounded**) if there exists a constant $K > 0$ such that $\|m \cdot \xi\|^2 \leq K \; tr_M(m^* m) \; \forall m \in M$ (resp., $\|\eta \cdot p\|^2 \leq K \; tr_P(p^* p) \; \forall p \in P$). The set of left-bounded vectors (resp., right-bounded vectors) is denoted by $({}_M\mathcal{H})^0$ (resp., $(\mathcal{K}_P)^0$). The set $({}_M\mathcal{H})^0$ (resp., $(\mathcal{K}_P)^0$) form a dense subspace of $\mathcal{H}$ (resp., $\mathcal{K}$) and is closed under the action of M' (resp., P').

Using the Radon-Nikodym derivative with respect to the faithful trace, one can obtain the M-valued (resp., P-valued) inner product ${}_M\langle \cdot, \cdot \rangle : ({}_M\mathcal{H})^0 \times ({}_M\mathcal{H})^0 \to M$ (resp., $\langle \cdot, \cdot \rangle_P : (\mathcal{K}_P)^0 \times (\mathcal{K}_P)^0 \to P$) defined by the equation $tr_M \left(m \left({}_M\langle \xi, \xi' \rangle\right)\right) = \langle m \cdot \xi, \xi' \rangle$ (resp., $tr_P \left(p \left(\langle \eta, \eta' \rangle_P\right)\right) = \langle \eta \cdot p, \eta' \rangle$). This inner product satisfies the following properties (see [JS97]):

(1) ${}_M\langle \xi, \xi' \rangle \geq 0$ (resp., $\langle \eta, \eta' \rangle_P \geq 0$)

(2) ${}_M\langle \xi, \xi' \rangle = \left({}_M\langle \xi', \xi \rangle\right)^*$ (resp., $\langle \eta, \eta' \rangle_P = \left(\langle \eta', \eta \rangle_P\right)^*$)

(3) ${}_M\langle \xi, m \cdot \xi' \rangle = m \cdot \left({}_M\langle \xi, \xi' \rangle\right)$ and ${}_M\langle m \cdot \xi, \xi' \rangle = \left({}_M\langle \xi, \xi' \rangle\right) \cdot m^*$ (resp., $\langle \eta, \eta' \cdot p \rangle_P = \left(\langle \eta, \eta' \rangle_P\right) \cdot p$ and $\langle \eta \cdot p, \eta' \rangle_P = p^* \cdot \left(\langle \eta, \eta' \rangle_P\right)$)

(4) ${}_M\langle \xi, m' \cdot \xi' \rangle = {}_M\langle m'^* \cdot \xi, \xi' \rangle$ (resp., $\langle \eta, p' \cdot \eta' \rangle_P = \langle p'^* \cdot \eta, \eta' \rangle_P$)

for every $\xi, \xi' \in ({}_M\mathcal{H})^0$, $m \in M$ and $m' \in M'$ (resp., $\eta, \eta' \in (\mathcal{K}_P)^0$, $p \in P$ and $p' \in P'$).

Further, there exists a finite subset $\{\xi_i\}_i$ (resp., $\{\eta_j\}_j$) of $({}_M\mathcal{H})^0$ (resp., $(\mathcal{K}_P)^0$) satisfying $id_{({}_M\mathcal{H})^0} = \sum\limits_i {}_M\langle \xi_i, \cdot \rangle \xi_i$ (resp. $id_{(\mathcal{K}_P)^0} = \sum\limits_j \eta_j \langle \eta_j, \cdot \rangle_P$). Such a set is called *basis* for the module and it satisfies the following properties which are routine to verify:

Proposition 1.1.2.

(1) $tr_{M'}(m') = [\dim {}_M\mathcal{H}]^{-1} \sum\limits_i \langle \xi_i, m' \cdot \xi_i \rangle$ *(resp.,* $tr_{P'}(p') = [\dim \mathcal{K}_P]^{-1} \sum\limits_j \langle \eta_j, \eta_j \cdot p' \rangle$*) for every* $m' \in M'$ *(resp.,* $p' \in P'$*) which implies* $({}_M\mathcal{H})^0 = ({}_{M'}\mathcal{H})^0$ *(resp.,* $(\mathcal{K}_P)^0 = ({}_{P'}\mathcal{K})$*)*

(2) $\sum\limits_i {}_{M'}\langle \xi_i, \xi_i \rangle = (\dim {}_M\mathcal{H}) \, 1_{M'}$ *(resp.,*$\sum\limits_j {}_{P'}\langle \eta_j, \eta_j \rangle = (\dim \mathcal{K}_P) \, 1_{P'}$*), where* ${}_{M'}\langle \cdot, \cdot \rangle$ *(resp.,* $\langle \xi, m' \cdot \xi' \rangle_{P'}$*) is the* M' *(resp.* P'*)-valued inner product by viewing* $\mathcal{H}$ *as a left* $M' = {}_M\mathcal{L}(\mathcal{H})$*-module (resp.,* $\mathcal{K}$ *as a right* $P' = \mathcal{L}_P(\mathcal{K})$*-module).* □

The dual Hilbert space or the contragredient $\bar{\mathcal{H}} := \{\bar{\xi} : \xi \in \mathcal{H}\}$ (resp., $\bar{\mathcal{K}} := \{\bar{\eta} : \eta \in \mathcal{K}\}$) can be equpped with a right M-module (resp., left P-module) structure given by $\bar{\xi} \cdot m := \overline{m^*\xi}$ for $m \in M$ and $\xi \in \mathcal{H}$ (resp., $p \cdot \bar{\eta} := \overline{\eta \cdot p^*}$ for $p \in P$ and $\eta \in \mathcal{K}$). Then the following are easy to see:

(1) $\left(\bar{\mathcal{H}}_M\right)^0 = \overline{\left({}_M\mathcal{H}\right)^0}$ (resp., $\left({}_P\bar{\mathcal{K}}\right)^0 = \overline{\left(\mathcal{K}_P\right)^0}$)

(2) $\langle \bar{\xi}, \bar{\xi'} \rangle_M = {}_M\langle \xi, \xi' \rangle$ for $\xi, \xi' \in \left({}_M\mathcal{H}\right)^0$ (resp., ${}_P\langle \bar{\eta}, \bar{\eta'} \rangle = \langle \eta, \eta' \rangle_P$ for $\eta, \eta' \in \left(\mathcal{K}_P\right)^0$)

(3) $\{\bar{\xi_i}\}_i$ (resp., $\{\bar{\eta_j}\}_j$) is a basis for $\bar{\mathcal{H}}_M$ (resp., ${}_P\bar{\mathcal{K}}$)

(4) $\dim\, {}_M\mathcal{H} = \dim\, \bar{\mathcal{H}}_M$ (resp., $\dim\, \mathcal{K}_P = \dim\, {}_P\bar{\mathcal{K}}$).

Suppose that M, P are II_1 factors and $\mathcal{H}$ is a M-P-bimodule. If $\dim\, \mathcal{H}_P < \infty$, then the index of subfactor $M \subseteq P'$ turns out to be $[P' : M] = \dim\, {}_M\mathcal{H} \cdot \dim\, \mathcal{H}_P$. We define $\mathrm{index}({}_M\mathcal{H}_P) := \dim\, {}_M\mathcal{H} \cdot \dim\, \mathcal{H}_P$. The bimodule ${}_M\mathcal{H}_P$ is said to be *bifinite* if $\mathrm{index}({}_M\mathcal{H}_P) < \infty$. If ${}_M\mathcal{L}_{\mathcal{P}}(\mathcal{H})$ is one-dimensional, then ${}_M\mathcal{H}_P$ is an irreducible M-P-bimodule. The bifinite bimodule ${}_M\mathcal{H}_P$ is called *extremal* if the canonical traces of the II_1-factors ${}_M\mathcal{L}(\mathcal{H})$ and $\mathcal{L}_{\mathcal{P}}(\mathcal{H})$ coincide on the intertwiner space ${}_M\mathcal{L}_P(\mathcal{H})$ (which has finite complex dimension due to the finiteness of the index). It is true that, in the case of bifinite bimodules, a vector is left-bounded if and only if it is right-bounded. Hence we will talk simply of **bounded vectors** and denote the collection of bounded vectors in bifinite bimodule $\mathcal{H}$ by $\mathcal{H}^0$.

Remark 1.1.3. If we take $\mathcal{H} = L^2(M)$ for a finite index subfactor $N \subseteq M$, then saying that the subfactor $N \subseteq M$ is extremal is equivalent to saying that the bimodule ${}_N L^2(M)_M$ is extremal.

The following result from [JS97] gives the universal property possessed by the tensor product of bifinite bimodules over a II_1 factor.

Theorem 1.1.4. *Let M, P and Q be II_1 factors, and suppose that $\mathcal{H}$(resp., $\mathcal{K}$) is a bifinite M-P-bimodule (resp., P-Q-bimodule). Then there exists a bifinite M-Q-bimodule, denoted by $\mathcal{H} \underset{P}{\otimes} \mathcal{K}$, which is determined by the following universal property:*

There exists a surjective linear map, ι, from the algebraic tensor product $\mathcal{H}^0 \underset{P}{\otimes} \mathcal{K}^0$ onto $\left(\mathcal{H} \underset{P}{\otimes} \mathcal{K}\right)^0$, satisfying:

(a) $\iota(\xi \cdot p \otimes \eta) = \iota(\xi \otimes p \cdot \eta)$;

(b) $\iota(m \cdot \xi \otimes \eta \cdot q) = m \cdot \iota(\xi \otimes \eta) \cdot q$

(c) $_M\langle \iota(\xi\otimes\eta), \iota(\xi'\otimes\eta')\rangle = {}_M\langle \xi, \xi' \cdot {}_P\langle \eta, \eta'\rangle\rangle$ *and* $\langle \iota(\xi\otimes\eta), \iota(\xi'\otimes\eta')\rangle_Q = \langle \eta, \langle \xi, \xi'\rangle_P \cdot \eta'\rangle_Q$

for every $\xi, \xi' \in \mathcal{H}^0, \eta, \eta' \in \mathcal{K}^0, m \in M, p \in P$ *and* $q \in Q$.

Moreover, $\mathcal{H} \underset{P}{\otimes} \mathcal{K}$ *is unique up to unique unitary isomorphism. That is, if* $(\mathcal{V}, \iota')$ *is another pair which satisfies the properties (a)-(c) stated above, then there is a unique unitary isomorphism, say,* $\psi : \mathcal{H} \underset{P}{\otimes} \mathcal{K} \to \mathcal{V}$ *such that* $\psi \circ \iota = \iota'$. □

By taking a hint from Theorem 1.1.4(c), a candidate for $\mathcal{H}\underset{P}{\otimes}\mathcal{K}$ would be the completion of the quotient of $\mathcal{H}^0 \underset{\text{alg}}{\otimes} \mathcal{K}^0$ by *span* $\{\xi \cdot p \otimes \eta - \xi \otimes p \cdot \eta\}$ with respect to the inner product $\langle \xi \underset{P}{\otimes} \eta, \xi' \underset{P}{\otimes} \eta'\rangle = \langle \xi, \xi' \cdot {}_P\langle \eta, \eta'\rangle\rangle = \langle \eta, \langle \xi, \xi'\rangle_P \cdot \eta'\rangle$. Further, if $\{\xi_i\}_i$ and $\{\eta_j\}_j$ are the basis for $_M\mathcal{H}$ and $_P\mathcal{K}$ (resp., $\mathcal{H}_P$ and $\mathcal{K}_Q$), then $\{\xi_i \underset{P}{\otimes} \eta_j\}_{i,j}$ forms a basis for $_M(\mathcal{H} \underset{P}{\otimes} \mathcal{K})$ (resp., $(\mathcal{H} \underset{P}{\otimes} \mathcal{K})_Q$). Also, the left dimension, the right dimension and the index of the bifinite bimodules are multiplicative with respect to this tensor product. The map $\mathcal{L}_P(\mathcal{H}) \ni x \mapsto x \underset{P}{\otimes} id_{\mathcal{K}} \in \mathcal{L}_Q(\mathcal{H} \underset{B}{\otimes} \mathcal{K})$ (resp., $_P\mathcal{L}(\mathcal{K}) \ni y \mapsto id_{\mathcal{H}} \underset{P}{\otimes} y \in {}_M\mathcal{L}(\mathcal{H} \underset{B}{\otimes} \mathcal{K})$) is an inclusion of unital $*$-algebras.

Let $\mathcal{H}$ be an M-P-bimodule with dim $_M\mathcal{H} < \infty$ (resp., dim $\mathcal{H}_P < \infty$) and $\{\xi_i\}_i$ (resp., $\{\eta_j\}_j$) be a basis for $_M\mathcal{H}$ (resp., $\mathcal{H}_P$). Then, it is easy to see that the bounded vector $\sum\limits_j \eta_j \underset{P}{\otimes} \overline{\eta_j}$ (resp., $\sum\limits_i \overline{\xi_i} \underset{M}{\otimes} \xi_i$) is independent of the basis and M-M-central, that is, $m \cdot (\sum\limits_j \eta_j \underset{P}{\otimes} \overline{\eta_j}) = (\sum\limits_j \eta_j \underset{P}{\otimes} \overline{\eta_j}) \cdot m$ for all $m \in M$ (resp., P-P-central).

1.1.3 Intertwiners and relative commutants

Given a finite index II_1 subfactor $N \subseteq M$, with the convention $M_{-1} = N$ and $M_0 = M$, we have the following result from [JS97] which gives the intertwiner picture of the standard invariant of $N \subseteq M$.

Proposition 1.1.5. *For each* $n \geq 0$

(i) $({}_{M_n}L^2(M_n)_N) \underset{N}{\otimes} ({}_N L^2(M)_M) \cong ({}_{M_n}L^2(M_{n+1})_M)$ *as* M_n*-*M*-bimodules*

(ii) there exists an isomorphism of commuting squares of finite-dimensional C^**-algebras, as follows:*

$$\begin{pmatrix} N' \cap M_{2n} \subseteq & N' \cap M_{2n+1} \\ \cup & \cup \\ M' \cap M_{2n} \subseteq & M' \cap M_{2n+1} \end{pmatrix} \cong \begin{pmatrix} {}_N\mathcal{L}_M(L^2(M_n)) \subseteq & {}_N\mathcal{L}_N(L^2(M_n)) \\ \cup & \cup \\ {}_M\mathcal{L}_M(L^2(M_n)) \subseteq & {}_M\mathcal{L}_N(L^2(M_n)) \end{pmatrix}$$

□

Remark 1.1.6. The minimal projections in the relative commutants (on the left) correspond to isotypic components of the corresponding bimodule (on the right) under the isomorphism of Proposition 1.1.5 (ii).

Remark 1.1.7. An important observation can be made at this juncture keeping Proposition 1.1.5 (i) in mind. Set $\alpha = {}_N L^2(M)_M$. With $\bar{\alpha}$ denoting the conjugate M-N-bimodule we have:

$$N' \cap M_{2n} \cong {}_N\mathcal{L}_M\left((\alpha\bar{\alpha})^n\,\alpha\right)$$

$$N' \cap M_{2n+1} \cong {}_N\mathcal{L}_N\left((\alpha\bar{\alpha})^{n+1}\right)$$

$$M' \cap M_{2n} \cong {}_M\mathcal{L}_M\left((\bar{\alpha}\alpha)^n\right)$$

$$M' \cap M_{2n+1} \cong {}_M\mathcal{L}_N\left(\bar{\alpha}\,(\alpha\bar{\alpha})^n\right)$$

with the symbol $\otimes$ omitted whenever there is no point of confusion and tensor being taken over N or M accordingly.

If we have the data of submodules of tensor powers of α and $\bar{\alpha}$, then we have the data of the whole standard invariant of $N \subseteq M$! Thus the point of interest now turns to the N-M, N-N, M-M, and M-N-bimodules which appear as submodules of tensor powers of $\alpha = {}_N L^2(M)_M$ and $\bar{\alpha} = {}_M\overline{L^2(M)}_N$. More about this will be discussed in terms of categories but before that we will now move onto *planar algebras* which is yet another way of looking at the standard invariant, this time, in terms of pictures!

1.1.4 (Shaded) Planar algebras

We now briefly recall basic machinery required to understand the widely used pictorial invariant called as *planar algebras*. We then see how it is related to the standard invariant of a finite index subfactor described in Section 1.1.1. We will be more interested in what are called the *subfactor* planar algebras which form a special class of *shaded* planar algebras. For a more detailed exposition of planar algebras, we refer the reader to [Jon99]. The definition presented here is an amalgamation of many such from [Jon99, Gho11, DGG14b], for instance.

For $k, k_1, k_2, \ldots k_n \in \mathbb{N} \cup \{0\}$ and $\varepsilon, \varepsilon_1, \ldots, \varepsilon_n \in \{+,-\}$, a **planar tangle diagram of type** $((\boldsymbol{\varepsilon_1}, \boldsymbol{k_1}), (\boldsymbol{\varepsilon_2}, \boldsymbol{k_2}), \ldots, (\boldsymbol{\varepsilon_n}, \boldsymbol{k_n}); (\boldsymbol{\varepsilon}, \boldsymbol{k}))$ consists of:

- an external disc D_0 and finitely many (possibly none) internal discs $D_1, D_2, \ldots, D_n$ in the interior of D_0, each of which is homorphic to the unit disc.
- each of these discs have even number (possibly none) of marked points, say, $2k_i$, on its boundaries dividing the boundary into several segments.

- each disc has a distinguished boundary segment marked by ε_i, from which the marked points are numbered clockwise.
- non-intersecting paths (referred to as *strings*) in $D_0 \setminus \left[\bigcup_{i=1}^{n} \operatorname{int}(D_i)\right]$, such that each of the strings have no end points (i.e., loops) or has end points at two distinct marked points and the strings exhaust all marked points.

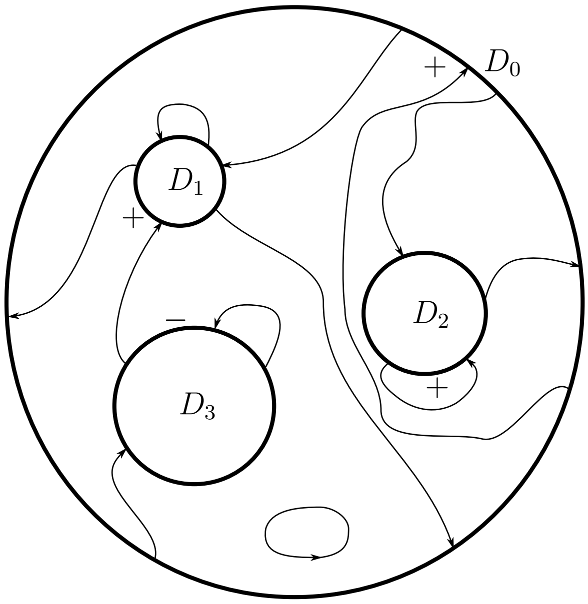


Figure 1.1: Planar tangle diagram of type $((+,3),(+,2),(-,2);(+,4))$

The sign ε_i on each disc indicates *shading* or *orientation* of the region bounded by strings and boundary segments. The convention followed is that + (resp. −) indicates the anti-clockwise (resp. clockwise) orientation. These orientations of the strings induce orientations on the strings as shown in Figure 1.1. It is to be observed that, on each disc, the orientation of string arising out of two consecutive marked points is always opposite to each other.

Two planar tangle diagrams T_1, T_2 are said to be *planar isotopic* if there exists a continuous map $\varphi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi_0 = \mathrm{id}_{\mathbb{R}^2}$, φ_t is a homeomorphism for all $t \in [0,1]$ and $\varphi_1(T_1) = T_2$ preserving the marked points and the orientations of the regions (thus strings). The planar isotopy class of a planar tangle diagram of type $((\varepsilon_1,k_1),(\varepsilon_2,k_2),\ldots,(\varepsilon_n,k_n);(\varepsilon,k))$ is called a *planar tangle of type* $((\varepsilon_1,k_1),(\varepsilon_2,k_2),\ldots,(\varepsilon_n,k_n);(\varepsilon,k))$.

If there are $2k$ marked points on a disc D (external or internal) and the distinguished boudary segment is denoted by sign ε, then we say that D has *color* εk. Let

Col := $\{\varepsilon k : (\varepsilon, k) \in \{+,-\} \times (\mathbb{N} \cup \{0\})\}$. If a tangle T has internal discs D_i with colors $\varepsilon_i k_i, i = 1, 2, \ldots, n$ respectively and external disc with color εk, then we write it as $T : (\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_k k_n) \to \varepsilon k$. For example, the tangle in Figure 1.1 is given by $T : (+3, +2, -2) \to +4$. If the tangle T has no internal discs, then we write it as $T : \emptyset \to \varepsilon k$. The set of all tangles with εk as the color of external disc is denoted by $\mathscr{T}_{\varepsilon k}$.

The composition of two tangles $T : (\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_n k_n) \to \varepsilon k$ and $S : (\delta_1 l_1, \delta_2 l_2, \ldots, \delta_m l_m) \to \varepsilon_i k_i$ (resp. $S : \emptyset \to \varepsilon_i k_i$), denoted by $(T \underset{i}{\circ} S) : (\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_{i-1} k_{i-1}, \delta_1 l_1, \delta_2 l_2, \ldots, \delta_m l_m, \varepsilon_{i+1} k_{i+1}, \ldots, \varepsilon_n k_n) \to \varepsilon k$ (resp. $(T \underset{i}{\circ} S) : (\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_{i-1} k_{i-1}, \varepsilon_{i+1} k_{i+1}, \ldots, \varepsilon_n k_n \to \varepsilon k)$, is obtained by gluing the external boundary of S with the boundary of the ith internal disc of T preserving the marked points on either of them with the help of isotopy, and then erasing the common boundary. There will be re-numbering of the discs which is done as follows.

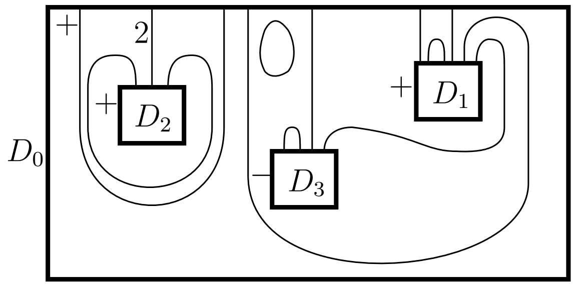
Let D_j^T denote the j^{th} internal disc of T. If $S : \emptyset \to \varepsilon_i k_i$, then $T \underset{i}{\circ} S$ has $n-1$ many internal discs with

$$D_j^{T \underset{i}{\circ} S} = \begin{cases} D_j^T \text{ for } j = 1, 2, \ldots, i-1, \\ D_{j+1}^T \text{ for } j = i, i+1, \ldots, n-1. \end{cases}$$

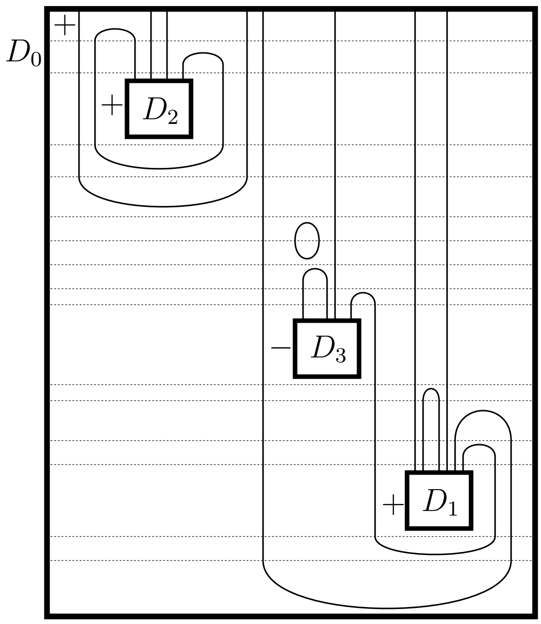
Suppose S has internal discs and D_j^S denote the j^{th} internal disc of S for $j = 1, 2, \ldots, m$. Then $T \underset{i}{\circ} S$ has $n+m-1$ many internal discs with

$$D_j^{T \underset{i}{\circ} S} = \begin{cases} D_j^T \text{ for } j = 1, 2, \ldots, i-1, \\ D_{j-i+1}^S \text{ for } j = i, i+1, \ldots, i+m-1, \\ D_{j-m+1}^T \text{ for } j = i+m, \ldots, n+m-1. \end{cases}$$

While drawing tangles it is convenient to replace discs by rectangles with sides parallel to the coordinate axes in $\mathbb{R}^2$ and all the marked points on the top where strings end transversally. We always mark the distinct boundary segment on the left of the rectangle. As the orientation on the strings is induced by the sign ε, we often suppress the directions of the strings while drawing tangles. Also, whenever there is no scope of ambiguity, we often replace n parallel strands by a single strand with the number n written adjacent to it. For instance, the tangle given in the Figure 1.1, with these conventions, will look like:



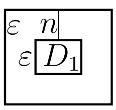
In the isotopy class of tangle in this form, the tangle which can be sliced into horizontal strips each of which contains exactly one local maxima or local minima or an internal rectangle, is often called a *standard form* of the given tangle. A standard form of tangle in Figure 1.1 is as follows:

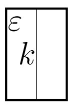


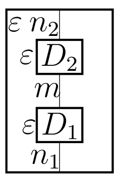
Remark 1.1.8. Note that, one standard form representative of a tangle can be obtained from another applying finitely many moves of three types, namely, *sliding move, rotation move* and *wiggling move* (See [Jon99, Gho11]).

Remark 1.1.9. In its standard form, any tangle is a product of tangles each of which contain exactly one local maxima or local minima or an internal rectangle.

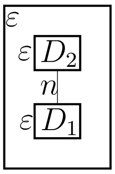
Let $\varepsilon \in \{+,-\}$, $n, m, n_1, n_2 \in \mathbb{N} \cup \{0\}$ be such that $n = 2k, m + n_1 = 2k_1$ and $m+n_2 = 2k_2$ for some $k, k_1, k_2 \in \mathbb{N}\cup\{0\}$. Thus, we have $n_1+n_2 = 2l$ with $l = k_1+k_2-m$. With the above conventions, we give a list of some important tangles (not essentially in the standard form) which are used to describe several structures such as multiplication, inclusion, involution etc.

- *Identity tangle:* $I_{\varepsilon k} :=$  $: \varepsilon k \to \varepsilon k$.

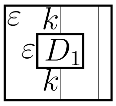
- *Unit tangle:* $1_{\varepsilon k} :=$  $: \emptyset \to \varepsilon k$.

- *Multiplication tangle:* $M^{\varepsilon l}_{\varepsilon k_1, \varepsilon k_2} :=$  $: (\varepsilon k_1, \varepsilon k_2) \to \varepsilon l$. When $n = n_1 = n_2$

(hence $l = k_1 = k = 2$), we denote the multiplication tangle simply by $M_{\varepsilon l}$.

- *Inner product tangle:* $\mathrm{IP}_{\varepsilon k} :=$ 

$: (\varepsilon k, \varepsilon k) \to \emptyset$.

- *Right inclusion tangle:* $\mathrm{RI}_{\varepsilon k} :=$ 

$: \varepsilon k \to \varepsilon(k+1)$.

Definition 1.1.10. A **planar algebra** P consists of collection of complex vector spaces $\{P_{\varepsilon k}\}_{(\varepsilon,k)\in \mathbf{Col}}$ and for every tangle $T : (\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_n k_n) \to \varepsilon k$ (resp, $T : \emptyset \to \varepsilon k$), there exists a multi-linear map $P_T : P_{\varepsilon_1 k_1} \times \cdots \times P_{\varepsilon_n k_n} \to P_{\varepsilon k}$ (resp, an element $P_T \in P_{\varepsilon k}$) (which will be referred as action of T) such that the action

- preserves composition of tangles, that is, for tangles $T : (\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_n k_n) \to \varepsilon k$ and $S : (\delta_1 l_1, \delta_2 l_2, \ldots, \delta_m l_m) \to \varepsilon_i k_i$, we have

$$P_{T \underset{i}{\circ} S} = P_T \circ (\mathrm{id}_{P_{\varepsilon_1 k_1}} \times \cdots \times \mathrm{id}_{P_{\varepsilon_{i-1} k_{i-1}}} \times P_S \times \mathrm{id}_{P_{\varepsilon_{i+1} k_{i+1}}} \times \cdots \times \mathrm{id}_{P_{\varepsilon_n k_n}})$$

- preserves identity, that is, $P_{I_{\varepsilon k}} = \mathrm{id}_{P_{\varepsilon k}}$,

- intertwines the action of permutation on the numbering of internal discs in a tangle with that of the inputs in the multi-linear map, that is, if $T : (\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_n k_n) \to \varepsilon k$, $x_j \in P_{\varepsilon_j k_j}$ for $1 \le j \le n$ and $\sigma \in S_n$, we have $P_{\sigma(T)}(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) = P_T(x_1, \ldots, x_n)$ where the tangle $\sigma(T) : (\varepsilon_{\sigma^{-1}(1)} k_{\sigma^{-1}(1)}, \ldots, \varepsilon_{\sigma^{-1}(n)} k_{\sigma^{-1}(n)}) \to \varepsilon k$ is obtained by renaming the j-th internal disc D_j in T as $D_{\sigma(j)}$ in $\sigma(T)$ for $1 \le j \le n$.

Note that a planar algebra has a unital filtered algebra structure with multiplication, unit and inclusion respectively given by the actions of the multiplication, unit and right inclusion tangles described earlier.

Definition 1.1.11. A planar algebra P is said to be a **$*$-planar algebra**, if there exists conjugate linear involutions $\{*_{\varepsilon k} : P_{\varepsilon k} \to P_{\varepsilon k}\}$ such that $[P_T(x_1, \ldots, x_n)]^* = P_{T^*}(x_1^*, \ldots, x_n^*)$, for each tangle $T : (\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_n k_n) \to \varepsilon k$ (resp. $T : \emptyset \to \varepsilon k$) and $x_i \in P_{\varepsilon_i k_i}$, $1 \le i \le n$, where the adjoint T^* of a tangle T is obtained by reflecting it about a horizontal line keeping the orientations of the regions and distinguished boundary components intact.

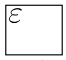
Definition 1.1.12. A planar algebra P is said to be

1. **connected**, if $\dim(P_{\pm 0}) = 1$.

2. **finite dimensional**, if $\dim(P_{\varepsilon k}) < \infty$ for every .

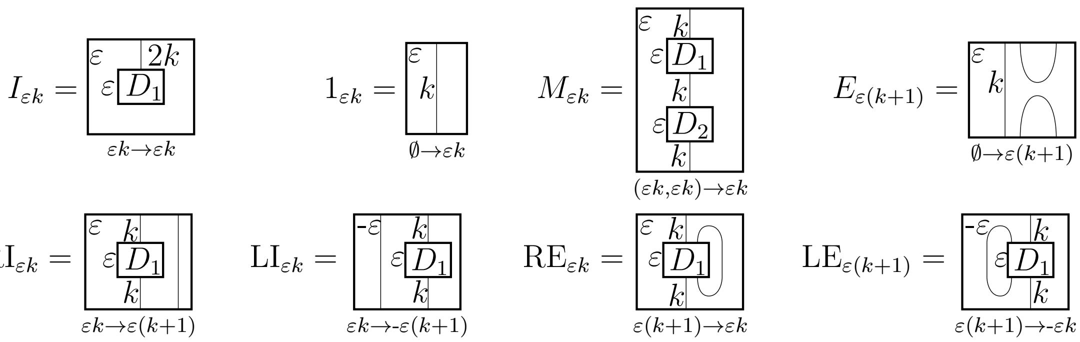
3. **positive**, if P is a connected $*$-planar algebra and the sesquilinear form $P_{\mathrm{IP}_{\varepsilon k}} \circ (\mathrm{id}_{P_{\varepsilon k}} \times *_{\varepsilon k})$ is positive definite for every $(\varepsilon, k) \in$ **Col**.

4. **spherical** if P is connected and actions of any two spherically isotopic tangles are identical.

In a connected planar algebra, we have P $= \delta_\varepsilon P$ $= \delta_\varepsilon P_{1_{\varepsilon 0}} = \delta_\varepsilon 1_{P_{\varepsilon 0}}$, for $\varepsilon \in \{+, -\}$. The tuple (δ_+, δ_-) is called the *modulus* of the planar algebra. It is said to be *unimodular* if $\delta_+ = \delta_-$. Note that sphericality automatically implies unimodularity. A *subfactor planar algebra* is the one which is connected, finite dimensional, positive and spherical. We are mainly interested in subfactor planar algebras in this thesis.

1.1.5 Subfactor planar algebras and extremal subfactors

Before we get into the connections between subfactor planar algebras and extremal subfactors, we make a note of the observation about "generating" tangles. We have the following set of tangles which generate the whole operad of planar tangles under tangle composition (see [KS04]).



Then, we have the following landmark result which assigns a unique planar algebra to every extremal subfactor.

Theorem 1.1.13 ([Pop95, Jon99, GJS08, JSW10, KS09]). *If*

$$(M_{-1} =)N \subseteq M(= M_0) \subseteq^{e_1} M_1 \subseteq^{e_2} \subseteq \cdots \subseteq^{e_k} M_k \subseteq^{e_{k+1}} \cdots$$

is the tower of the basic construction associated to an extremal subfactor with $[M : N] = \delta^2 < \infty$*, then there exists a unique (upto isomorphism) subfactor planar algebra* $P =$

$P^{N\subseteq M}$ of modulus δ such that, the grid of relative commutants,

$$\begin{array}{cccccc} \mathbb{C} = N'\cap N \subseteq & N'\cap M \subseteq & N'\cap M_1 \subseteq & N'\cap M_2 \subseteq & \cdots\cdots \\ & \cup & \cup & \cup & \\ & \mathbb{C} = M'\cap M \subseteq & M'\cap M_1 \subseteq & M'\cap M_2 \subseteq & \cdots\cdots \end{array}$$

is isomorphic to

$$\begin{array}{ccccc} \mathbb{C} = P_{+0} \subseteq & P_{+1} \subseteq & P_{+2} \subseteq & P_{+3} \subseteq & \cdots\cdots \\ & \cup & \cup & \cup & \\ & \mathbb{C} = P_{-0} \subseteq & P_{-1} \subseteq & P_{-2} \subseteq & \cdots\cdots \end{array}$$

such that,

(i) the multiplication in the relative commutants correspond to the action of multiplication tangle ($M_{\varepsilon k}$),

(ii) the horizontal inclusions are given by the action of right inclusion tangle ($RI_{\varepsilon k}$),

(iii) the vertical inclusions correspond to the action of left inclusion tangle ($LI_{\varepsilon k}$),

(iv) the horizontal conditional expectations, $E^{N'\cap M_k}_{N'\cap M_{k-1}}$ and $E^{M'\cap M_k}_{M'\cap M_{k-1}}$ correspond to the action of the right conditional expectation tangle $\delta.RE_{+k}$ and $\delta.RE_{-(k-1)}$ respectively,

(v) the vertical conditional expectation $E^{N'\cap M_k}_{M'\cap M_k}$ is given by the action of $\delta.LE_{+k}$, and

(vi) the element $P_{E_{+(k+1)}} \in P_{+(k+1)}$ corresponds to δe_k.

Conversely, any subfactor planar algebra P with modulus δ arises from an extremal subfactor of index δ^2 in this way. □

The above theorem coupled with our earlier discussion of relation between relative commutants and intertwiners (Section 1.1.3) gives an idea of how one can go from one to another and thus standard invariant can also be expressed in terms of planar algebras.

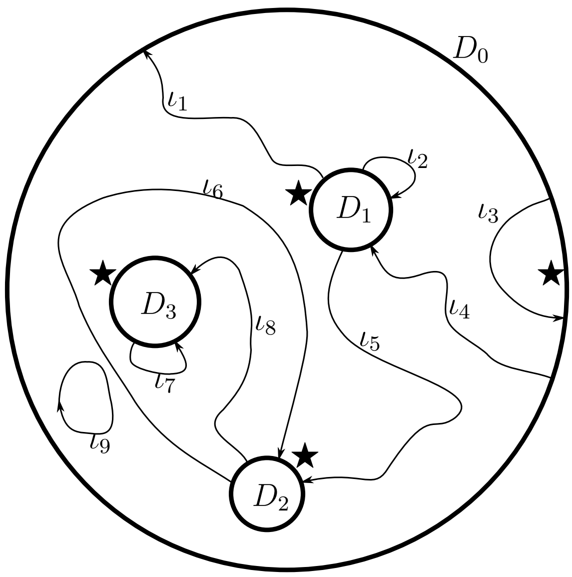
1.1.6 Oriented planar algebras

Next we move onto "oriented" version of planar algebras, which are generalization of the planar algebras discussed till now. While many of the results focus on subfactor planar algebras, the arguments apply much more generally. See [Jon11] for a more general perspective. For explanations closely related to the explanations we give below, see [BHP12]. To make comparision easy, we define along the same lines as in Section 1.1.4.

Let Λ be a non-empty set. Define another disjoint copy of the set Λ via $\overline{\Lambda} := \{\overline{\lambda} : \lambda \in \Lambda\}$. Consider the free semigroup W_Λ (or simply W) of words with letters in $\Lambda \sqcup \overline{\Lambda}$. There is an obvious involution $\Lambda \sqcup \overline{\Lambda} \ni \iota \overset{-}{\longmapsto} \overline{\iota} \in \Lambda \sqcup \overline{\Lambda}$ by defining $\overline{\overline{\lambda}} := \lambda$. This gives rise to the involution $W \ni w = (\iota_1, \ldots, \iota_n) \overset{*}{\longmapsto} w^* := (\overline{\iota}_n, \ldots, \overline{\iota}_1) \in W$.

Definition 1.1.14. A *Λ-oriented planar tangle diagram* consists of

- a subset D_0 of $\mathbb{R}^2$ homeomorphic[1]to the closed unit disc (this is referred to as the 'external disc'), and finitely many, mutually non-intersecting subsets, $D_1, \ldots, D_n$, of $\text{int}(D_0)$, each of which is homeomorphic[1] to the closed unit disc (referred as 'internal discs'),
- each disc has finitely many marked points on its boundary dividing it into finitely many segments,
- each disc has a distinguished boundary segment denoted by ★,
- finitely many oriented paths in $D_0 \setminus \bigcup_{i=1}^{n} \text{int}(D_i)$ (referred as 'strings') each of which is either a closed loop or has end points at two distinct marked points,
- the strings exhaust all the marked points (as end points) and are labelled by elements of Λ (and not $\Lambda \sqcup \overline{\Lambda}$).



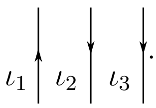
Two planar tangle diagrams T_1, T_2 are said to be *planar isotopic* if there exists a continuous map $\varphi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi_0 = \text{id}_{\mathbb{R}^2}$, φ_t is a homeomorphism for all $t \in [0,1]$ and $\varphi_1(T_1) = T_2$ preserving the labeling and the orientation of each string. The planar isotopy class of a Λ-oriented planar tangle diagram is called *Λ-oriented planar tangle.*

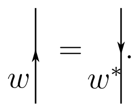
[1]If we use diffeomorphisms instead of homeomorphisms (which had been usually done in the literature), the techinical complications in defining the tangle, planar isotopies or even composition of tangles outweigh the actual purpose of introducing tangles. Hence, we contend ourselves with homeomorphisms.

Given such a tangle diagram, to each marked point on the internal disc, there is a string labeled by λ which meets this point. To this point we will assign λ if the string is oriented away from the internal disc and $\overline{\lambda}$ is the string is oriented into the internal disc. For each marked point on the external disc, we have the opposite convention: if the string meeting this point is labeled λ, then we assign the label λ if the string is oriented towards the exterior of the external disc and $\overline{\lambda}$ is it is oriented towards the interior.

Once this is done, each disc (internal and external) has a unique word in $\Lambda \sqcup \bar{\Lambda}$ attached to it by reading off the letters assigned to marked points starting from ★ and moving clockwise along the boundary of the disc. We call this unique word the *color* of the corresponding disc. The colors of D_0, D_1, D_2, and D_3 in the above tangle diagram are $w_0 = (\iota_3, \bar{\iota}_4, \iota_1, \bar{\iota}_3)$, $w_1 = (\iota_1, \iota_2, \bar{\iota}_2, \bar{\iota}_4, \iota_5)$, $w_2 = (\bar{\iota}_5, \iota_6, \iota_8, \bar{\iota}_6)$ and $w_3 = (\bar{\iota}_8, \bar{\iota}_7, \iota_7)$ respectively. Note that planar isotopy does not affect the colors of the disc. If tangle T has internal discs with colors $v_1, v_2, \dots v_n$ and external disc with color v_0, then we denote this situation by $T : (v_1, v_2, \dots v_n) \to v_0$; the set of all such tangles will be denoted by $\mathscr{T}_{(v_1,\dots,v_n);v_0}$. For example, in the tangle shown above we have $T : (w_1, w_2, w_3) \to w_0$. If there is no internal disc in T, then it is denoted by $T : \varnothing \to v_0$; the set of all such tangles will be referred as $\mathscr{T}_{\varnothing;v_0}$. Further, $\mathscr{T}_{v_0} = \mathscr{T}^{\Lambda}_{v_0}$ will denote the set of all tangles each of whose external disc has color v_0. Composition of two Λ-oriented tangles is defined in exactly in the same way as in Jones' shaded planar tangles. (See [Jon99]).

In order to draw a picture of an oriented tangle, it will be convenient to represent a collection of parallel strings in any portion of the tangle by a single oriented string labelled by the word in W, constructed from the letters labelling the individual strings along with their orientations. For example, if $w = (\iota_1, \bar{\iota}_2, \bar{\iota}_3)$ where $\iota_1, \iota_2, \iota_3 \in \Lambda$, then 

will represent 

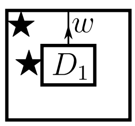
. With this convention, note that 

.

Before we proceed to more definitions, we describe some useful tangles. Here we will often replace discs by rectangles, so there is a natural "source" and "target" associated to these diagrams.

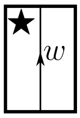
Let $w, w_1, w_2 \in W$.

- *Identity tangle:*

$I_w :=$ 

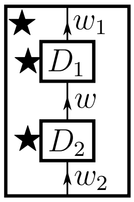
$: w \to w$

- *Unit tangle:*

$$1_w :=$$ 

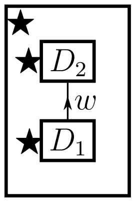
$$: \varnothing \to ww^*$$

- *Multiplication tangle:*

$$M^w_{w_1,w_2} :=$$ 

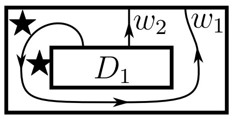
$$: (w_1 w^*\ ,\ ww_2^*) \to w_1 w_2^*$$

- *Inner product tangle:*

$$H_w :=$$ 

$$: (w, w^*) \to \varnothing$$

- *Rotation tangle:*

$$\rho_{w_1,w_2} :=$$ 

$$: (w_1 w_2) \to (w_2 w_1)$$

When $w = w_1 = w_2$, we denote the multiplication tangle simply by M_w.

Definition 1.1.15. A *Λ-oriented planar algebra P* consists of collection of complex vector spaces $\{P_w\}_{w \in W}$ and for every tangle $T : (w_1, \dots, w_n) \to w_0$, we have a multi-linear map $P_T : P_{w_1} \times \cdots \times P_{w_n} \to P_{w_0}$ satisfying the following conditions:

1. For tangles $S : (w_1, \dots, w_n) \to w_0$ and $T : (u_1, \dots, u_m) \to w_j$ we have

$$P_{S \underset{D_j}{\circ} T} = P_S \circ (\mathrm{id}_{P_{w_1}} \times \cdots \times \mathrm{id}_{P_{w_{j-1}}} \times P_T \times \mathrm{id}_{P_{w_{j+1}}} \times \cdots \times \mathrm{id}_{P_{w_n}})$$

2. $P_{I_w} = \mathrm{id}_{P_w}$ for all words w

3. For $T : (w_1, \dots, w_n) \to w_0$, any collection of $x_j \in P_{w_j}$ for $1 \le j \le n$, and $\sigma \in S_n$, we have

$$P_{\sigma(T)}(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}) = P_T(x_1, \dots, x_n),$$

where the tangle $\sigma(T) : (w_{\sigma^{-1}(1)}, \dots, w_{\sigma^{-1}(n)}) \to w_0$ is obtained by renaming the j-th internal disc D_j in T as $D_{\sigma(j)}$ in $\sigma(T)$ for $1 \le j \le n$.

In the above definition, we adopt the convention that $T : \varnothing \to w$ should give a map $P_T : \mathbb{C} \to P_w$. This is consistent with the convention that the tensor power of vectors spaces over the empty set is the scalar field. Note that $\varnothing$ is a valid word, so there is a distinction between a tangle $T : (\varnothing) \to w$ and $T : \varnothing \to w$. The first has an internal disc with no strings attached to it, while the second has no internal disc at all.

As in Jones' planar algebras, the multiplication and unit tangles equip $\{P_{vv^*}\}_{v\in W}$ unital associative algebras. We often write $x \cdot y$ (or simply xy) for $P_{M^v_{v_1,v_2}}(x, y)$ whenever x and y are in appropriate spaces to make sense of the action.

There is also a $*$-structure on the space of tangles. For $T : (w_1, \ldots, w_n) \to w_0$ the tangle $T^* : (w_1^*, \ldots, w_n^*) \to w_0^*$ is obtained by reflecting T along any straight line where the numbering of the internal discs (if any) and Λ-labels of the strings are induced by the reflection from T whereas the orientation of each string is reversed after reflection. Clearly $*$ is an involution.

Definition 1.1.16. A Λ-oriented planar algebra P is said to be an *oriented $*$-planar algebra* if there exists conjugate linear involutions $\{*_w : P_w \to P_{w^*}\}_{w\in W}$ such that $[P_T(x_1, \ldots, x_n)]^* = P_{T^*}(x_1^*, \ldots, x_n^*)$.

Definition 1.1.17. A Λ-oriented planar algebra P is said to be

1. *connected* if $\dim(P_\varnothing) = 1$.
2. *finite dimensional* if $\dim(P_w) < \infty$ for every $w \in W$.
3. *positive*, if P is a connected $*$-planar algebra and the sesquilinear form $P_{H_w} \circ (\mathrm{id}_{P_w} \times *_w)$ is positive definite (and thereby gives an inner product on P_w) for every $w \in W$.
4. *spherical* if P is connected and actions of any two spherically isotopic tangles are identical.

In this article, we mainly focus on Λ-oriented planar algebra which are connected, finite-dimensional, positive and spherical; we will refer these as Λ- *oriented factor planar algebra*. Note that this is more flexible than the definition given in [BHP12], where they assume $\lambda = \bar{\lambda}$ for every $\lambda \in \Lambda$.

The reason we call ours factor planar algebras as well is as follows: a II_1 factor and a collection of bifinite bimodules Λ, one can construct a Λ-oriented planar algebra as described in the next section. By the results of [BHP12], every Λ-oriented planar algebra can be realized this way for some II_1 factor.

We now proceed to discuss morphisms of planar algebras.

Definition 1.1.18. Let P (resp., P') be a Λ- (resp., Λ'-) oriented planar algebra and $\varphi : \Lambda \to \Lambda'$ be any map. If W_Λ and $W_{\Lambda'}$ denote the free semigroups of words with letters in $\Lambda \sqcup \overline{\Lambda}$ and $\Lambda' \sqcup \overline{\Lambda'}$ respectively, then φ extends to a homomorphism $\varphi : W_\Lambda \to W_{\Lambda'}$ by setting $\varphi(\overline{\lambda}) := \overline{\varphi(\lambda)}$ for $\lambda \in \Lambda$. If T is a Λ-oriented w-tangle, then replacing labels assigned to strings by its corresponding image under φ, we get a unique Λ'-oriented tangle $\varphi(T)$. Thus φ can also be seen as a map from all Λ-oriented tangles to Λ'-oriented tangles which preserves composition and identity. A *homomorphism* $\varphi : P \to P'$ consists of a map $\varphi : \Lambda \to \Lambda'$ along with a collection of linear maps $\varphi_w : P_w \to P'_{\varphi(w)}$ for each $w \in W_\Lambda$ such that the action of oriented tangles is preserved i.e., for every Λ-oriented tangle $T : (w_1, \ldots, w_n) \to w$, we have $\varphi_w(P_T(x_1, \ldots, x_n)) = P'_{\varphi(T)}(\varphi_{w_1}(x_1), \ldots, \varphi_{w_n}(x_n))$ for every $x_i \in P^\Lambda_{w_i}$, $i = 1, 2, \ldots, n$. It is said to be an *isomorphism* if all the maps $\varphi, \varphi_w, w \in W$ are bijections.

If P and P' are $*$-planar algebras, then φ is called a $*$-*homomorphism* if each φ_w preserves the $*$-structure.

Remark 1.1.19. Any $*$-homomorphism between two oriented factor planar algebras will be automatically injective (cf. [Jon99]).

1.2 The Category Language

Contents of this section are more or less directly taken from [Mac71, Kas95, LR97, NT13] and [EGNO15].

1.2.1 Categories

Definition 1.2.1. A **category** $\mathcal{C}$ consists of two classes $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Hom}(\mathcal{C})$ called the objects and morphisms of the category respectively, such that

(i) For each X and Y in $\mathrm{Obj}(\mathcal{C})$, there is a sub-class $\mathcal{C}(X,Y) = \mathrm{Hom}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ of $\mathrm{Hom}(\mathcal{C})$ called the morphisms from X to Y. For $f \in \mathcal{C}(X,Y)$ we say that f has *source* X and *target* Y. We write it as $f : X \to Y$ or $X \xrightarrow{f} Y$.

(ii) For $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$ (notice that source of g is same as target of f) there exists a unique $g \circ f \in \mathcal{C}(X,Z)$ called the *composition* of f and g with the property that for any three morphism f, g and h, composible appropriately, $(h \circ g) \circ f = h \circ (g \circ f)$.

(iii) For each $X \in \text{Obj}(\mathcal{C})$, there exists unique $1_X \in \mathcal{C}(X, X)$ called the *identity morphism of* X, such that for all $X, Y \in \text{Obj}(\mathcal{C})$ and every $f \in \mathcal{C}(X, Y)$, we have $f \circ 1_X = 1_Y \circ f = f$.

We write $\text{End}(X)$ for $\text{Hom}(X, X)$ for any object X.

Remark 1.2.2. Most of the categories we deal with have the class of objects and morphisms to be sets. Such categories are called *small* categories.

For a any category $\mathcal{C}$, we have a distinct category, called *opposite category* of $\mathcal{C}$, $\mathcal{C}^{op}$, whose objects are exactly the objects of $\mathcal{C}$ but $\mathcal{C}^{op}(X, Y) := \mathcal{C}(Y, X)$. It is clear that $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

Given any two categories $\mathcal{C}, \mathcal{D}$, by $\mathcal{C} \times \mathcal{D}$, we mean the category with objects as pairs (X, Y) with $X \in \text{Obj}(\mathcal{C})$ and $Y \in \text{Obj}(\mathcal{D})$ and morphisms given by

$$(\mathcal{C} \times \mathcal{D})\left((X, Y), (X', Y')\right) = \mathcal{C}(X, X') \times \mathcal{D}(Y, Y').$$

A *subcategory* $\mathcal{C}$ of a category D consists of subclass $\text{Obj}(\mathcal{C})$ of $\text{Obj}(\mathcal{D})$ and a subclass $\text{Hom}(\mathcal{C})$ of $\text{Hom}(\mathcal{D})$ such that

- whenever $X \in \text{Obj}(\mathcal{C})$, $1_X \in Mor(\mathcal{C})$,
- if $f \in \text{Hom}(\mathcal{C})$, then the source and target of f are objects of $\mathcal{C}$, and
- if $f, g \in \text{Hom}(\mathcal{C})$ and are composible, then $g \circ f \in \text{Hom}(\mathcal{C})$.

Example 1.2.3. Basic examples include:

(i) Category of sets, **Sets** with set maps as morphisms.

(ii) Category of vector spaces, **Vec**, with linear transformations as morphisms. We denote the category of finite dimensional vector spaces by $\mathbf{Vec}_f$. Both of these are subcategories of **Sets**.

(iii) Category of modules over any fixed algebra $\mathcal{A}$, $\boldsymbol{\mathcal{A}}$**-mod** with intertwiners as morphisms.

(iv) Category of all representations of a fixed group G in Hilbert spaces, $\boldsymbol{Rep(G)}$, with intertwiners as morphisms. We denote by $Rep(G)$ its finite dimensional version.

(v) For two groups G, A, let $\mathcal{C}_G = \mathcal{C}_G(A)$ be the category with objects $g \in G$ and $\text{Hom}(g_1, g_2) = \emptyset$ if $g_1 \neq g_2$ and $\text{Hom}(g, g) = A$.

Example 1.2.4. ***G*-graded vector spaces - Vec(G), Vec$_f$(G).** Let G be any group and $\mathbf{Vec}(G)$ be the category with objects as G-graded vector spaces, i.e., vector spaces V with a decomposition $V = \bigoplus_{g \in G} V_g$ and morphisms which are just linear transformations between these vector spaces which preserve grading. Similarly, one has the category $\mathbf{Vec}_f(G)$ of finite dimensional G-graded vector spaces.

A morphism $f \in \mathcal{C}(X, Y)$ is said to be an *isomorphism* if there exists $g \in \mathcal{C}(Y, X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. We say X and Y to be *isomophic* if there is an isomorphism from X to Y and denote it by $X \cong Y$.

Definition 1.2.5. A **functor** $F : \mathcal{C} \to \mathcal{C}'$ from the category $\mathcal{C}$ to the category $\mathcal{C}'$ consists of a map $F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C}')$ and of a map $F : \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}')$ such that

(a) for any $X \in \mathrm{Obj}(\mathcal{C})$, we have $F(1_X) = 1_{F(V)}$,

(b) for any $f \in \mathcal{C}(X, Y)$, we have $F(f) \in \mathcal{C}'(F(X), F(Y))$,

(c) if f, g are composible morphisms in $\mathcal{C}$, then $F(g \circ f) = F(g) \circ F(f)$.

A **contravariant functor** $G : \mathcal{C} \to \mathcal{C}'$ is a functor from $\mathcal{C}^{op} \to \mathcal{C}'$.

Definition 1.2.6. Let F, G be functors from $\mathcal{C}$ to $\mathcal{C}'$. A **natural transformation** η from F to G, written as $\eta : F \to G$, is a family of morphisms $\eta(X) : F(X) \to G(X)$ in $\mathcal{C}'$ indexed by the $X \in \mathrm{Obj}(\mathcal{C})$ such that, for any $f : X \to Y$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta(X)} & G(X) \\ \Big\downarrow F(f) & & \Big\downarrow G(f) \\ F(Y) & \xrightarrow{\eta(Y)} & G(Y) \end{array}$$

commutes.

Further, if $\eta(X)$ is an isomorphism of $\mathcal{C}'$ for every $X \in \mathrm{Obj}(\mathcal{C})$, then η is said to be a **natural isomorphism**.

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *essentially surjective* if, for any object Y of $\mathcal{D}$, there exists $X \in \mathrm{Obj}(\mathcal{C})$ such that $F(X) \cong Y$ in $\mathcal{D}$. It is said to be *faithful* (resp. *fully faithful*) if, for any pair of objects (X, X') of $\mathcal{C}$, the map $F : \mathcal{C}(X, X') \to \mathcal{D}(F(X), F(X'))$ is injective (resp. bijective).

Definition 1.2.7. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be **equivalence** of categories if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : id_{\mathcal{D}} \to FG$ and $\theta : GF \to id_{\mathcal{C}}$, where $id_{\mathcal{C}}$ and $id_{\mathcal{D}}$ are identity functors on $\mathcal{C}$ and $\mathcal{D}$ respectively.

The following criterion becomes very handy in determining whether a functor is an equivalence, proof of which can be found in [Kas95].

Proposition 1.2.8. *A functor is an equivalence of categories if and only if it is essentially surjective and fully faithful.* □

Definition 1.2.9. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors and suppose there are natural isomorphisms $\eta : id_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to id_{\mathcal{D}}$ such that both of the composites

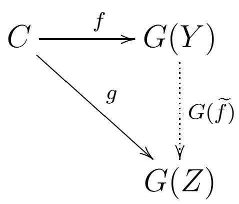
$$G(Y) \xrightarrow{\eta_{G(Y)}} GFG(Y) \xrightarrow{G(\varepsilon_Y)} G(Y) \text{ and } F(X) \xrightarrow{F(\eta_X)} FGF(X) \xrightarrow{\varepsilon_{F(X)}} F(X)$$

are identity morphisms for all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$. Then, F is said to be a **left adjoint** of G or equivalently, G is said to be a **right adjoint** of F.

Example 1.2.10. For any set X, let $\mathbb{C}[X]$ be the free algebra over $\mathbb{C}$ associated to X. Then the functor $X \mapsto \mathbb{C}[X]$ is a left adjoint functor to the forgetful functor which assigns to any algebra its underlying set.

In most of the cases its pretty straightforward to check whether two functors are adjoints of each other using the above definition. An equivalent condition for checking adjointness, due to Mac Lane, is found to be more useful in our case (in Chapter 3) which uses the notion of universal arrows.

Definition 1.2.11. Let $G : \mathcal{D} \to \mathcal{C}$ be any functor and $C \in \text{Obj}(\mathcal{C})$. A **universal arrow** from C to G consists of a pair (Y, f), $Y \in \text{Obj}(\mathcal{D}), f \in \mathcal{C}(C, G(Y))$ such that, for any other pair (Z, g) with $Z \in \text{Obj}(\mathcal{D}), g \in \mathcal{C}(C, G(Z))$, there exists unique $\widetilde{f} \in \mathcal{D}(Y, Z)$ with $G(\widetilde{f}) \circ f = g$. That is,



The proof of the following thoerem can be found in the book of Mac Lane ([Mac71, Chap IV, Theorem 2]).

Theorem 1.2.12 (Mac Lane)**.** *Let F, G be functors as above. Then F is a left adjoint to G if and only if there exists a natural transformation $\eta : id_{\mathcal{C}} \to GF$ such that the pair $(F(C), \eta_C)$ is a universal arrow from C to G for every $C \in Obj(\mathcal{C})$.*

1.2.2 Tensor Categories

Let $\mathcal{C}$ be a category and $\otimes$ be a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$. Thus,

- for each pair of objects (X, Y) we have an object $X \otimes Y$,
- for each pair of morphisms (f, g) we have a morphism $f \otimes g$,
- for any two objects X, Y, $1_{X \otimes Y} = 1_X \otimes 1_Y$, and,
- if $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(A, B)$, $f' \in \mathcal{C}(Y, Z)$ and $g' \in \mathcal{C}(B, C)$, then

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g) \tag{1.2.1}$$

Example 1.2.13. Consider the category of vector spaces **Vec**. Then, the tensor product of vector spaces defines a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$.

Any functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is called a **tensor product**. An *associativity constraint* for $\otimes$ is a natural isomorphism $a : \otimes(\otimes \times id) \to \otimes(id \times \otimes)$. That is, for each triple (X, Y, Z) of objects there exists an isomorphism $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ such that the square

$$\begin{array}{ccc} (X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) \\ \downarrow{\scriptstyle (f \otimes g) \otimes h} & & \downarrow{\scriptstyle f \otimes (g \otimes h)} \\ (X' \otimes Y') \otimes Z' & \xrightarrow{a_{X',Y',Z'}} & X' \otimes (Y' \otimes Z') \end{array}$$

commutes whenever f, g, h are morphisms in the category.

The associativity constraint a is said to satisfy the *Pentagon Axiom* if the following pentagonal diagram commutes for all objects X, Y, Z, W of $\mathcal{C}$:

$$\begin{array}{ccccc} & & (X \otimes (Y \otimes Z)) \otimes W & & \\ & \swarrow{\scriptstyle a_{X,Y \otimes Z,W}} & & \nwarrow{\scriptstyle a_{X,Y,Z} \otimes 1_W} & \\ X \otimes ((Y \otimes Z) \otimes W) & & & & ((X \otimes Y) \otimes Z) \otimes W \\ \downarrow{\scriptstyle 1_X \otimes a_{Y,Z,W}} & & & & \downarrow{\scriptstyle a_{X \otimes Y,Z,W}} \\ X \otimes (Y \otimes (Z \otimes W)) & & \xleftarrow{a_{X,Y,Z \otimes W}} & & (X \otimes Y) \otimes (Z \otimes W) \end{array}$$

For a fixed object I, a *left unit constraint* (resp. a *right unit constraint*) with respect to I is a natural isomorphism $l : \otimes(I \times id) \to id$ (resp. $r : \otimes(id \times I) \to id$). This means that for any object X, there exists an isomorphism $l_X : I \otimes X \to X$ (resp. $r_X : X \otimes I \to X$) such that:

$$\begin{array}{ccc} I\otimes X & \xrightarrow{l_X} & X \\ {\scriptstyle 1_I\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ I\otimes X' & \xrightarrow{l'_X} & X' \end{array} \quad \text{(resp.} \quad \begin{array}{ccc} X\otimes I & \xrightarrow{r_X} & X \\ {\scriptstyle f\otimes 1_I}\downarrow & & \downarrow{\scriptstyle f} \\ X'\otimes I & \xrightarrow{r'_X} & X' \end{array} \text{)}$$

commutes for any morphism f.

Given an associativity constraint a and left and right unit constraints l, r with respect to an object I, they are said to satisfy the *Triangle axiom* if the triangle,

$$\begin{array}{ccc} (X\otimes I)\otimes Y & \xrightarrow{a_{X,I,Y}} & X\otimes(I\otimes Y) \\ & {\scriptstyle r_X\otimes 1_Y}\searrow \quad \swarrow{\scriptstyle 1_X\otimes l_Y} & \\ & X\otimes Y & \end{array}$$

commutes for all pairs (X, Y) of objects.

Definition 1.2.14. A **tensor category** $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ is a category $\mathcal{C}$ which is equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, with an object $\mathbb{1}$, called the unit of the tensor category, with an associativity constraint a, a left unit constraint l, and a right unit constraint r with respect to I such that the Pentagon Axiom and the Triangle axiom are satisfied.

The tensor category $\mathcal{C}$ is said to be **strict** if the associativity and unit constraints a, l, r are all identities of the category.

If $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ is a tensor category, recall that $\mathcal{C}^{op}$ is the category with objects same as that of $\mathcal{C}$ but morphisms are reversed. Then $\mathcal{C}^{op}$ is also a tensor category in a natural way.

Remark 1.2.15. For a tensor category $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$, there is also the *monoidal opposite category*, which we denote by $\mathcal{C}^{\vee}$. As a category $\mathcal{C}^{\vee} = \mathcal{C}$, its tensor product is given by $X \overset{\vee}{\otimes} Y := Y \otimes X$ and the associativity constraint of $\mathcal{C}^{\vee}$ is $a^{\vee}_{X,Y,Z} := a^{-1}_{Z,Y,X}$.

Example 1.2.16. The category $\mathcal{C} = \mathbf{Vec}$ is equipped with a tensor structure for which $\otimes$ is the usual tensor product of the vector spaces, the unit object I is the ground field ($\mathbb{C}$, most of the times) and the associativity and unit constraints are given by

$$a((U \otimes V) \otimes W) = U \otimes (V \otimes W) \text{ and } l(I \otimes V) = V = r(V \otimes I).$$

It is easy to see that the pentagon and triangle axioms are satisfied.

Example 1.2.17. Now consider the category $\boldsymbol{Rep(G)}$ for a group G. Being a subcategory of **Vec**, it has a tensor product structure with the G-action on the tensor product $U \otimes V$given by $g \cdot (u \otimes v) = g \cdot u \otimes g \cdot v$ for $g \in G, u \in U, v \in V$. The tensor unit is $\mathbb{C}$ with the trivial action of G.

Example 1.2.18. Let G be a group and A be an abelian group with an action $\pi : G \to \mathrm{Aut}(A)$. Let $\mathcal{C}_G(A, \pi)$ be defined in the same way as $\mathcal{C}_G$ (Example 1.2.3(v)) with tensor product of objects given by $g \otimes h = gh$ whereas tensor product of morphisms is defined as follows: if $a : g \to g$ and $b : h \to h$, then $a \otimes b := a\pi(g)b$. It is routine to check that $\mathcal{C}_G(A, \pi)$ is a tensor category.

If G is non-abelian, this category also serves an example for $X \otimes Y$ not being isomorphic to $Y \otimes X$.

Example 1.2.19. In $\mathbf{Vec}(G)$, one can define the tensor product as

$$(U \otimes V)_g = \bigoplus_{x,y \in G,\ xy=g} U_x \otimes V_y$$

and unit object $\mathbb{1}$ by $\mathbb{1}_e = \mathbb{K}$ (the base field) and $\mathbb{1}_g = 0$ for $g \neq e$. Then defining a, l and r in the obvious way, $\vec{(}G)$ can be seen as a tensor category.

Let $\delta_g, g \in G$ be the objects of $\mathbf{Vec}_f(G)$ defined by $(\delta_g)_x = \mathbb{K}$, if $x = e$ and $(\delta_g)_x = 0$ otherwise. We then have $\delta_g \otimes \delta_h \cong \delta_{gh}$. Thus the category $\mathcal{C}_G(\mathbb{K}^*)$ is a subcategory of $\mathbf{Vec}_f(G)$ and this subcategory can be viewed as a "basis" of $\mathcal{V}ec_f(G)$ as any object of $\mathbf{Vec}_f(G)$ is isomorphic to a direct sum (Definition 1.2.38) of objects δ_g with non-negative integer multiplicities.

Example 1.2.20. Here is a generalization of the above examples in which the associativity constraint is not trivial.

Let G be a group and A be an abelian group with an action $\pi : G \to \mathrm{Aut}(A)$. ω be a 3-cocycle for this action of G on A. That is, $\omega : G \times G \times G \to A$ is map which satisfies the equation

$$\omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2g_3, g_4)\pi(g_1)\left(\omega(g_2, g_3, g_4)\right)$$

for all $g_i \in G, i = 1, 2, 3, 4$.

We define a new tensor category $\mathcal{C}_G^\omega = \mathcal{C}_G^\omega(A)$ which, as a category is same as $\mathcal{C}_G$. The functor $\otimes$ and $l, r, \mathbb{1}$ are also the same as those in $\mathcal{C}_G$. The only difference is in the associativity constraint a^ω, which is not identity as in $\mathcal{C}^G$ but is defined by the formula

$$a^\omega_{\delta_g,\delta_h,\delta_k} = \omega(g, h, k)1_{\delta_{ghk}} : (\delta_g \otimes \delta_h) \otimes \delta_k \to \delta_g \otimes (\delta_h \otimes \delta_k),$$

where $g, h, k \in G$.

Properties of ω will help in establishing the pentagon axiom and hence $\mathcal{C}_G^\omega$ is a tensor category.

Example 1.2.21. Similarly, for a 3-cocycle ω with values in $\mathbb{K}^*$ on can define the category $\mathbf{Vec}(G, \omega)$ which differs from $\mathbf{Vec}(G)$ just by the associativity constraint. This is done by extending the associativity constraint of $\mathcal{C}_G^\omega$ by additivity to arbitrary direct sums (Definition 1.2.38) of objects δ_g. This category contains a subcategory $\mathcal{V}ec(G, \omega)$ of finite dimensional G-graded vector spaces with associativity defined by ω.

Example 1.2.22. The category $\mathbf{End}(\mathcal{C})$, of endofunctors of a category is a strict tensor category with $\otimes$ given by composition of functors. The unit object is the identity functor.

1.2.3 Tensor Functors and The Theorem of Mac Lane

Definition 1.2.23. Let $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ and $(\mathcal{C}', \otimes', \mathbb{1}', a', l', r')$ be tensor categories. A **tensor functor** from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (F, J) where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, and $J_{X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y)$ is a family of natural isomorphisms indexed by pairs (X, Y) of objects of $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc}
((F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{a'_{F(X),F(Y),F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\
\downarrow{\scriptstyle J_{X,Y} \otimes' 1_{F(Z)}} & & \downarrow{\scriptstyle 1_{F(X)} \otimes' J_{Y,Z}} \\
F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\
\downarrow{\scriptstyle J_{X \otimes Y, Z}} & & \downarrow{\scriptstyle J_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X \otimes (Y \otimes Z))
\end{array}$$

commute for all objects (X, Y, Z) in $\mathcal{C}$ ("the tensor structure axiom").

A tensor functor (F, J) is said to be *strict* if the isomorphism J is an identity of $\mathcal{C}'$.

Definition 1.2.24. (a) A **natural tensor transformation** $\eta : (F, J) \to (F', J')$ between tensor functors from $\mathcal{C}$ to $\mathcal{C}'$ is a natural transformation $\eta : F \to F'$ such that the following diagram commutes for each pair (X, Y) of objects of $\mathcal{C}$:

$$\begin{array}{ccc}
F(X) \otimes' F(Y) & \xrightarrow{J_{X,Y}} & F(X \otimes Y) \\
\downarrow{\scriptstyle \eta(X) \otimes \eta(Y)} & & \uparrow{\scriptstyle \eta(X \otimes Y)} \\
F'(X) \otimes' F'(Y) & \xrightarrow{J'_{X,Y}} & F'(X \otimes Y)
\end{array}$$

A natural tensor isomorphism is a natural tensor transformation which is also a natural isomorphism.

(b) **A tensor equivalence** between tensor categories is a tensor functor $F : \mathcal{C} \to \mathcal{D}$ such that there exists a tensor functor $G : \mathcal{D} \to \mathcal{C}$ and natural tensor isomorphisms $\eta : id_{\mathcal{D}} \xrightarrow{\cong} FG$ and $\theta : GF \xrightarrow{\cong} id_{\mathcal{C}}$.

In such a case, we say that $\mathcal{C}$ and $\mathcal{D}$ are **tensor equivalent**.

Remark 1.2.25. If (F, J) is a tensor functor, then there is a canonical isomorphism $\varphi : \mathbb{1}' \to F(\mathbb{1})$ defined by the commutative diagram,

$$\begin{array}{ccc} \mathbb{1}' \otimes' F(\mathbb{1}) & \xrightarrow{l'_{F(\mathbb{1})}} & F(\mathbb{1}) \\ \Big\downarrow{\scriptstyle \varphi \otimes' 1_{F(\mathbb{1})}} & & \Big\uparrow{\scriptstyle F(l_{\mathbb{1}})} \\ F(\mathbb{1}) \otimes' F(\mathbb{1}) & \xrightarrow{J_{\mathbb{1},\mathbb{1}}} & F(\mathbb{1} \otimes \mathbb{1}) \end{array}$$

Proposition 1.2.26. *For any tensor functor $(F, J) : \mathcal{C} \to \mathcal{C}'$, the diagrams*

$$\begin{array}{ccc} \mathbb{1}' \otimes' F(X) & \xrightarrow{l'_{F(X)}} & F(X) \\ \Big\downarrow{\scriptstyle \varphi \otimes 1_{F(X)}} & & \Big\uparrow{\scriptstyle F(l_X)} \\ F(\mathbb{1}) \otimes' F(X) & \xrightarrow{J_{\mathbb{1},X}} & F(\mathbb{1} \otimes X) \end{array}$$

and

$$\begin{array}{ccc} F(X) \otimes' \mathbb{1}' & \xrightarrow{r'_{F(X)}} & F(X) \\ \Big\downarrow{\scriptstyle 1_{F(X)} \otimes \varphi} & & \Big\uparrow{\scriptstyle F(r_X)} \\ F(X) \otimes' F(\mathbb{1}) & \xrightarrow{J_{X,\mathbb{1}}} & F(X \otimes \mathbb{1}) \end{array}$$

commute for every object X of $\mathcal{C}$.

By Proposition 1.2.26, a tensor functor can be equivalently defined by saying that it is a triple (F, φ, J) which satisfy the tensor structure axiom and Proposition 1.2.26. This is indeed the traditional definition of a tensor functor.

Example 1.2.27. An important class of examples of tensor functors arise from "forgetful" functors which are functors that forget some or the other structure. A forgetful functor from the category of topological spaces/ vector spaces/ groups to the category of sets is the one which takes each of the space to its underlying set and morphisms to the basic set maps. A more important example in the context of this thesis is the forgetful functor $Rep(G) \to$ **Vec** from the representation category of a group to the category of vector spaces. More generally, if $H \subset G$ is a subgroup, then we have the restriction functor $Rep(G) \to Rep(H)$. Further, if $\phi : H \to G$ is a group homomorphism, then we have the pullback functor $\phi^* : Rep(G) \to Rep(H)$. All these functors are tensor functors.

Example 1.2.28. Suppose $\phi : H \to G$ is a group homomorphism, then any H-graded vector space is naturally a G-graded vector spaces. Thus we have a natural tensor functor $\phi_* : \mathbf{Vec}(H) \to \mathbf{Vec}(G)$.

Example 1.2.29. Let G_1, G_2 be groups, A be an abelian group, and $\omega_i \in Z^3(G_i, A)$, $i = 1, 2$ be 3-cocycles (with actions of G_i on A taken to be trivial). Let $\mathcal{C}_i = \mathcal{C}_{G_i}^{\omega_i}$, $i = 1, 2$ be the tensor categories of graded vector spaces described in Example 1.2.20.

Any tensor functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ defines a group homomorphism $f : G_1 \to G_2$. By the tensor structure axiom, it is easy to see that a tensor structure on F is given by

$$J_{g,h} = \mu(g, h)1_{\delta_{f(gh)}} : F(\delta_g) \otimes F(\delta_h) \tilde{\to} F(\delta_{gh}),\ g, h \in G_1, \tag{1.2.2}$$

where $\mu : G_1 \times G_1 \to A$ is a function satisfying

$$\omega_1(g, h, l)\mu(gh, l)\mu(g, h) = \mu(g, hl)\mu(h, l)\omega_2(f(g), f(h), f(l)), \tag{1.2.3}$$

for all $g, h, l \in G_1$. Thus, $\omega_1 = f^*\omega_2 \cdot d_3(\mu)$. In otherwords, ω_1 and $F^*\omega_2$ are cohomologous in $Z^3(G_1, A)$.

Conversely, given a group homomorphism $F : G_1 \to G_2$ and any function $\mu : G_1 \times G_2 \to A$ satisfying (1.2.3) gives rise to a tensor functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ defined by $F(\delta_g) = \delta_{f(g)}$ with the tensor structure given by (1.2.2). This functor is an equivalence if and only if f is an isomorphism.

In particular if $G = G_1 = G_2$, then $\boldsymbol{\mathcal{V}ec(G, \omega_1)} \overset{\otimes}{\cong} \boldsymbol{\mathcal{V}ec(G, \omega_2)}$(tensor equivalent) if and only if ω_1 and ω_2 are cohomologous.

The following theorem of Mac Lane implies that in practice we may always assume that a tensor category to be strict.

Theorem 1.2.30. *Any tensor category is tensor equivalent to a strict tensor category.* □

Caution must be exercised in interpreting the Mac Lane thoerem. It says that every tensor category is *tensor equivalent* to a strict one not isomorphic! If ω is cohomologically non-trivial, then $\mathcal{C}_G^\omega(A)$ is clearly not isomorphic to $\mathcal{C}_G(A)$. However, by the above theorem, it is equivalent to some strict category $\mathcal{C}$.

1.2.4 Rigid Semisimple C*-tensor categories

A category is said to be $\mathbb{C}$-*linear* if every morphism space is a complex vector space. A $\mathbb{C}$-linear category is said to be a $*$-*category* if it is equipped with an anti-linear map $* : \mathcal{C}(X, Y) \to \mathcal{C}(Y, X)$ for all objeccts X, Y. It must satisfy the axioms $(f^*)^* = f$ and

$(f \circ g)^* = g^* \circ f^*$ for composable morphisms f and g. This also implies that $1_X^* = 1_X$ for all objects X of $\mathcal{C}$. Further, a morphism $f : X \to Y$ in a $*$-category is said to be *unitary* if $ff^* = 1_Y$ and $f^*f = 1_X$.

Definition 1.2.31. A $*$-category $\mathcal{C}$ is said to be a **C*-category** if

(i) $\mathcal{C}(X, Y)$ is a Banach Space for every $X, Y \in \mathrm{Obj}(\mathcal{C})$, the map

$$\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z),\ (f, g) \mapsto f \circ g$$

is bilinear and $\|fg\| \leq \|f\| \, \|g\|$;

(iii) the $*$ satisfies the following properties:

(a) $\|f^*f\| = \|f\|^2$ for every morphism f in $\mathcal{C}$. In particular, $\mathrm{End}(X)$ is a unital C*-algebra for every object X;

(b) for any $f \in \mathcal{C}(X, Y)$, the element f^*f is positive in the C*-algebra $\mathrm{End}(X)$.

If $\mathcal{C}$ is a tensor category as well as C*-category, then it called a C*-*tensor category* if the associativity and unit constraints are unitary and $(f \otimes g)^* = f^* \otimes g^*$ for every $f, g \in \mathrm{Hom}(\mathcal{C})$.

Definition 1.2.32. A tensor functor $(F, J) : \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}, \mathcal{C}'$ are C*-tensor categories, is said to be **unitary** if it satisfies $F(f^*) = F(f)^*$ for every $f \in \mathcal{C}(X, Y)$ and $J_{X,Y}$ is unitary for all objects X, Y in $\mathcal{C}$.

Remark 1.2.33. A W*-*category* is a C*-category such that each morphism space has a predual ([GLR85]). Although the norms on the spaces appear as additional structure, being a C* (or W*)-category is actually a property of a $*$-category. Indeed, one can take the semi-norms given by the spectral radius, and ask if they satisfy the conditions listed above. In particular it makes sense to say a $*$-category *is* a C* (or W*)-category without specifying extra structure.

Example 1.2.34. The most basic example of a C*-tensor category is the category $\mathbf{Hilb}_f$ of finite dimensional Hilbert spaces. The $*$ of a morphism is the usual adjoint as a linear operator between Hilbert spaces.

Example 1.2.35. The category *Rep*(G) is a C* -tensor category as it is a subcategory of $\mathbf{Hilb}_f$

Now, let $\mathcal{C}$ be a C*-tensor category.

Definition 1.2.36. An object $\bar{X}$ is said to be **conjugate** or **dual** to an object X if there exist morphisms $R_X \in \mathrm{Hom}(\mathbb{1}, \bar{X} \otimes X)$ and $\bar{R}_X \in \mathrm{Hom}(\mathbb{1}, X \otimes \bar{X})$ such that

$$(1_{\bar{X}} \otimes \bar{R}_X^*)(R_X \otimes 1_{\bar{X}}) = 1_{\bar{X}} \text{ and } (1_X \otimes R_X^*)(\bar{R}_X \otimes 1_X) = 1_X \tag{1.2.4}$$

The above identities are called **conjugate equations** and the pair $(R_X, \bar{R}_X)$ is called a **solution** of the conjugate quations for X.

If every object of $\mathcal{C}$ has a conjugate object, then $\mathcal{C}$ is said to be a **rigid** C*-tensor category.

Rigidity gets its origin from group representations. It is known that for every representation ρ_V of a group of a group G on a vector space V, there is a contragredient representation on the dual space V^* defined by $\rho^*_{V^*}(g) := (\rho_V(g^{-1}))^*$. The category *Rep*$(G)$ is rigid with this dual structure and a solution of the conjugate equation given by $R_V(1) = \sum_i e_i^* \otimes e_i$ and $\bar{R}_V(1) = \sum_i e_i \otimes e_i^*$, where $\{e_i\}_i$ is a basis of V and $\{e_i^*\}_i$ its corresponding dual basis.

One defines the *statistical dimension* of an object X by

$$d(X) := inf_{(R_X,\bar{R}_X)} \|R_X\| \left\|\bar{R}_X\right\|$$

where the infimum is taken over all solutions to the conjugate equations for X. The function $d(.) : \mathrm{Obj}(\mathcal{C}) \to \mathbb{R}_+$ depends on objects only up to unitary isomorphism. It is multiplicative and satisfies $d(X) = d(\bar{X})$ for any dual of X. We call solutions to the conjugate equations *standard* if $\|R_X\| = \left\|\bar{R}_X\right\| = d(X)^{\frac{1}{2}}$. It turns out that such solutions exist and are essentially unique. From now on, by solutions of the conjugate equations we always mean the standard ones. It turns out that, for any object X, its conjugate, if exists, is unique upto a unique unitary isomorphism (see [Yam04, Pen18]). The following is a standard result, proof of which can be found in any standard category theory book, say, [NT13], for instance.

Proposition 1.2.37 (**Frobenius reciprocity**). *For any three objects X, Y and Z,*

$$Hom(X \otimes Y, Z) \cong Hom(Y, \bar{X} \otimes Z) \text{ and } Hom(X \otimes Y, Z) \cong Hom(X, Z \otimes \bar{Y}) \qquad \square$$

For a more detailed study of duality we refer the reader to [LR97] and [Yam04].

We say that an object X in a rigid C^*-tensor category is *simple* or *irreducible* if $\mathrm{End}(X) \cong \mathbb{C}$. By the reciprocity, it is easy to see that if X is simple, then so is $\bar{X}$ and the spaces $\mathrm{Hom}(\mathbb{1}, \bar{X} \otimes X), \mathrm{Hom}(\mathbb{1}, X \otimes \bar{X})$ are one dimensional. We always assume that

$\mathbb{1}$ is simple i.e., $\mathrm{End}(\mathbb{1}) \cong \mathbb{C}$. Often, we denote by $\mathrm{Irr}(\mathcal{C})$, a set of representatives of isomorphism classes of simple objects in $\mathcal{C}$. What follows is the notion of semi-simplicity in rigid C*-tensor categories.

Definition 1.2.38. A $*$- category is said to **semi-simple** if

(i) for any two objects X_1 and X_2, there exist an object Z (called **direct sum** of X_1 and X_2) and morphisms $i_1 \in Mor(X_1, Z)$, $i_2 \in Mor(X_2, Z)$ such that $i_1^* i_1 = 1_{X_1}, i_2^* i_2 = 1_{X_2}$ and $i_1 i_1^* + i_2 i_2^* = 1_Z$, and

(ii) every object in the category is a direct sum of finitely many simple objects.

(iii) the category has subobjects, i.e, for every projection p in $\mathrm{End}(X)$, there exists an object Y and an ismometry $v \in \mathrm{Hom}(Y, X)$ such that $vv^* = p$. We say that Y is a **subobject** of X.

In (ii) of Definition 1.2.38, the object Z is unique upto a unique unitary isomorphism and is denoted by $X_1 \oplus X_2$. Since the zero projection belongs to $\mathrm{End}(X)$ for every object X, the object defined by it is denoted by $\mathbf{0}$ and is called the *zero object*. Note that $\mathrm{Hom}(\mathbf{0}, X) = \mathrm{Hom}(X, \mathbf{0}) = 0$ for every object X.

Remark 1.2.39. In a semi-simple category though every object is a direct sum of finitely many simple objects, the number of isomorphism classes of simple objects may not be finite. Rigid categories in which the number of isomorphism classes of simple objects is finite are called *fusion categories*.

Remark 1.2.40. In a semi-simple category, all morphism spaces are finite dimensional, and so a semi-simple $*$-category is C* if and only if it is W*.

For a semi-simple tensor category $\mathcal{C}$, the *fusion algebra* is the complex linear span of isomorphism classes of simple objects, with product given by the linear extension of $[X] \cdot [Y] := \sum_{Z \in \mathrm{Irr}(\mathcal{C})} N_{XY}^Z [Z]$, where $N_{XY}^Z = \dim\left(\mathcal{C}(X \otimes Y, Z)\right)$. This is an associative, unital algebra. When $\mathcal{C}$, in addition, is rigid, there is a $*$-structure on this algebra, given by the conjugate linear extension of $[X]^* := [\bar{X}]$. This associative $*$-algebra is denoted by $\mathrm{Fus}(\mathcal{C})$.

In the presence of direct sums, the functions $d(.)$ turns out to be additive as well. We have a well defined trace Tr_X on endomorphism spaces $\mathrm{End}(X)$ given by

$$\mathrm{Tr}_X(f) := R_X^*(1_{\bar{X}} \otimes f) R_X = \bar{R}_X^*(f \otimes 1_{\bar{X}}) \bar{R}_X \ \in \mathrm{Hom}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$$

This trace does not depend on the choice of dual for X or on the choice of standard solutions. Moreover, we have $\mathrm{Tr}_X(1_X) = d(X)$.

Example 1.2.41. Suppose G is a compact quantum group and $\widetilde{F} : \mathit{Rep}(G) \to \mathbf{Hilb}_f$ be the obvious functor which sends every unitary representation of G to the underlying finite dimensional Hilbert space. Then F is a faithful "exact" functor. In fact, any functor from a rigid semisimple C*-tensor categories to $\mathbf{Hilb}_f$ is automatically so. What is more interesting is that, given any rigid semisimple C*-tensor category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathbf{Hilb}_f$ (such functors are called *fiber functors*), there exists a compact quantum group G and a unitary tensor equivalence $E : \mathcal{C} \to \mathit{Rep}(G)$ such that F is naturally tensor isomorphic to $\widetilde{F} \circ E$. Moreover, for such a G, the "Hopf *-algebra" $(\mathbb{C}[G], \Delta)$ is uniquely determined upto isomorphism. This is known famously as *Woronowicz's Tannaka-Krein duality.*

1.2.5 Some basic facts of C*-categories

Let $\mathcal{C}$ be a C*-category. For $X, Y, X_i \in \mathrm{Obj}(\mathcal{C})$, $1 \leq i \leq n$, recall that :
(i) Y *is a subobject of* X if the morphism space $\mathcal{C}(Y, X)$ contains an isometry,
(ii) a projection $p \in \mathcal{C}(X, X)$ *factors through* Y if there exists an isometry $u \in \mathcal{C}(Y, X)$ such that $p = uu^*$,
(iii) X *is a direct sum of* $\{X_i\}_{i=1}^n$ if for all $1 \leq i \leq n$, there exists an isometry $u_i \in \mathcal{C}(X_i, X)$ such that $1_X = \sum_{i=1}^n u_i u_i^*$.

In general, $\mathcal{C}$ may neither be closed under direct sums nor have every projection factoring through a subobject. However, we have a C*-category $\mathcal{K}(\mathcal{C})$ which we refer as the *unitary Karoubi envelope of* $\mathcal{C}$, such that the following holds:

(1) there exists a fully faithful $*$-functor $\iota : \mathcal{C} \to \mathcal{K}(\mathcal{C})$ which is isometry on morphisms,

(2) every projection in $\mathcal{K}(\mathcal{C})$ factors through a subobject,

(3) $\mathcal{K}(\mathcal{C})$ is closed under direct sums, and

(4) every $Z \in \mathrm{Obj}(\mathcal{K}(\mathcal{C}))$ appears as a subobject of direct sum of objects of the form $\iota(X)$, $X \in \mathrm{Obj}(\mathcal{C})$.

Moreover, the pair $(\mathcal{K}(\mathcal{C}), \iota)$ satisfies the universal property:
for every pair $(\mathcal{D}, \sigma)$ where $\mathcal{D}$ is a C*-category closed under direct sums and having every projection factoring through a subobject, and $\sigma : \mathcal{C} \to \mathcal{D}$ is a C*-functor, there exists a C*-functor $\widetilde{\sigma} : \mathcal{K}(\mathcal{C}) \to \mathcal{D}$ such that σ is equivalent to $\widetilde{\sigma} \circ \iota$ via a unique natural unitary.

It is rather easy to achieve conditions (1) and (2) by considering the *unitary idempotent completion* of $\mathcal{C}$, denoted by $\mathrm{proj}(\mathcal{C})$, whose objects are pairs (X, p) with $X \in \mathrm{Obj}(\mathcal{C})$ and projection $p \in \mathcal{C}(X, X)$. The morphism space from (X, p) to (Y, q) consists of those $f \in \mathcal{C}(X, Y)$ satisfying $q \circ f \circ p = f$. The *-structure on $\mathrm{proj}(\mathcal{C})$ is simply induced from $\mathcal{C}$. However, $\text{proj}(\mathcal{C})$ might still be far from being closed under direct sums. We denote the canonical functor $X \mapsto (X, 1_X)$ for $X \in \text{Obj}(\mathcal{C})$ by $\alpha^{\mathcal{C}} : \mathcal{C} \to \text{proj}(\mathcal{C})$. Any C*-functor $F : \mathcal{C} \to \mathcal{D}$ between C*-categories, induces a canonical C*-functor between the unitary idempotent completions by simply applying F; we denote this by $\text{proj}(F)$:$\text{proj}(\mathcal{C})$ →$\text{proj}(\mathcal{D})$. Further, if F is monoidal, then $\text{proj}(F)$ is also.

Fact 1.2.42.

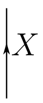
1. If $\mathcal{C}$ is a strict C*-tensor category, then $\text{proj}(\mathcal{C})$ also inherits this structure by extending the tensor product appropriately. Note that $\alpha^{\mathcal{C}}(X \otimes Y) = \alpha^{\mathcal{C}}(X) \otimes \alpha^{\mathcal{C}}(Y)$, and indeed $\alpha^{\mathcal{C}}$ is trivially monoidal.
2. If $\mathcal{C}$ is a semi-simple C*-category, then so is $\text{proj}(\mathcal{C})$, and $\alpha^{\mathcal{C}} : \mathcal{C} \to \text{proj}(\mathcal{C})$ is an equivalence.
3. If $F : \mathcal{C} \to \mathcal{D}$ is a C*-functor, then $\text{proj}(F) \circ \alpha^{\mathcal{C}} = \alpha^{\mathcal{D}} \circ F$.

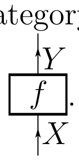
Throughout this thesis, we use monoidal C*-functors between C*-tensor categories where the tensor preserving properties of the functors are implemented by natural unitaries; we refer such functors as *unitary tensor functors.*

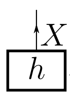
Now, we briefly go over a constructive description of the Karoubi envelope; for details, see [JP17, GLR85]. Let $\mathcal{C}$ be a C*-category which is not necessarily semisimple. By the *unitary direct sum completion of* $\mathcal{C}$, we mean a category $\mathcal{C}^{\oplus}$ with objects as formal unitary (finite) direct sums of objects of $\mathcal{C}$. Morphisms between formal direct sums are matrices whose entries are morphisms between the corresponding objects. The axioms of a C*-category guarantee this category is again C*. If $\mathcal{C}$ is a C*-tensor category, then $\mathcal{C}^{\oplus}$ also inherits this structure by extending the tensor product additively, and thereafter, if $\mathcal{C}$ is rigid, then so is $\mathcal{C}^{\oplus}$. Furthermore, any unitary tensor functor $\mathcal{C}$ to $\mathcal{D}$ canonically extends to unitary tensor functor from $\mathcal{C}^{\oplus}$ to $\mathcal{D}$.

Remark 1.2.43. $\text{proj}(\mathcal{C}^{\oplus})$ turns out to be a unitary Karoubi envelope of the C*-category $\mathcal{C}$.

1.2.6 Graphical Calculus

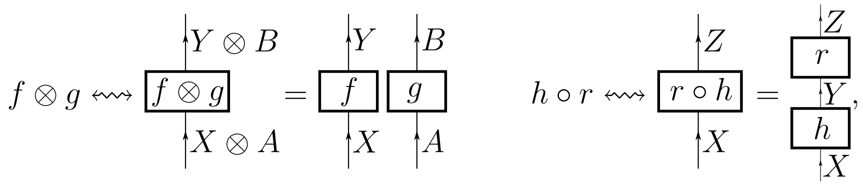
We present a way of representing morphisms of a rigid strict tensor category by planar diagrams. Let $\mathcal{C}$ be a strict tensor category. We denote the identity morphism, 1_X by 

and a morphism $f \in \mathcal{C}(X, Y)$ by 

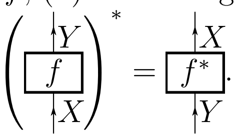
. The convention is to read the picture from bottom to top. The tensor unit $\mathbb{1}$ is denoted by an empty strand as in 

.

The tensor product and composition of morphisms are represented by horizontal and vertical stacking of diagrams:



where $f, h \in \mathcal{C}(X, Y), g \in \mathcal{C}(A, B)$ and $r \in \mathcal{C}(Y, Z)$.

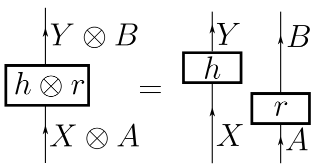
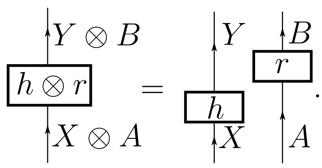
Given a morphism $f \in \mathcal{C}(X, Y)$, one can get the picture associated to f^* by (i) reflecting the picture of f, (ii) relabelling the box by f^* and (iii) reversing the directions of the strings. That is, 

Using these techniques one can translate any expression involving morphisms in a strict tensor category into planar diagrams involving boxes labelled by morphisms and strings with directions labelled by objects.

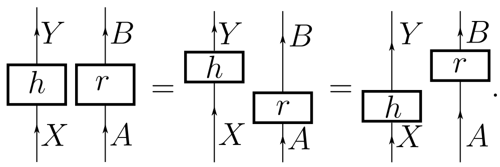
Given any planar diagram representing a morphism g which itself is an expression involving several morphisms, then g^* is obtained simply by reflecting the diagram, relabeling all the boxes with the corresponding dual morphisms and finally, reversing the directions of the strings.

To illustrate this we will now consider the property of tensor product as a functor stated in Equation (1.2.1). If we take $f = h \in \mathcal{C}(X, Y)$, $g = 1_B$, $f' = 1_X$ and $g' = r \in \mathcal{C}(A, B)$, then the identity becomes

$$h \otimes r = (h \otimes 1_B) \circ (1_X \otimes r) \tag{1.2.5}$$

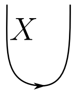
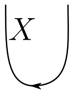
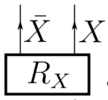
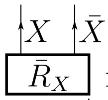
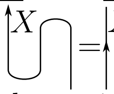
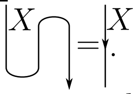
Pictorially,  and similarly, 

Thus we have,

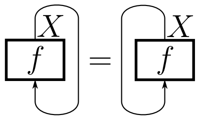


What this says is that, vertical sliding of boxes is legal while presenting morphisms in terms of pictures.

Now, we will look at the conjugate equations (Equation (1.2.4))for an object X with dual $\bar{X}$ and solutions to conjugate equations $(R_X, \bar{R}_X)$. It is customary to represent

R_X and $\bar{R}_X$ by  and  instead of  and  respectively. With this convention the conjugate equations will become  and . Thus conjugate equations essentially allow us to straighten a wiggle or vice versa as and when required.

The trace defined earlier on endomorphism spaces when seen pictorially will be

$\mathrm{Tr}_X(f) =$  for any $f \in \mathrm{End}(X)$.

1.2.7 2-categories

Definition 1.2.44. A **2-category** $\mathcal{C}$ consists of :

- a class $\mathcal{C}_0$ called the *objects or 0-cells* of $\mathcal{C}$
- for each $\alpha, \beta \in \mathcal{C}_0$, a category $\mathcal{C}(\alpha, \beta)$ whose objects are called *1-cells* and morphisms are called *2-cells*
- for each $\alpha, \beta, \gamma \in \mathcal{C}_0$, a functor $\otimes : \mathcal{C}(\beta, \gamma) \times \mathcal{C}(\alpha, \beta) \to \mathcal{C}(\alpha, \gamma)$
- a 1-cell $1_\alpha : \alpha \to \alpha$ called the *identity on* α for each 0-cell α
- *associativity constraint:* for each triple $\alpha \xrightarrow{X} \beta$, $\beta \xrightarrow{Y} \gamma$, $\gamma \xrightarrow{Z} \delta$ of 1-cells, an isomorphism $(Z \otimes Y) \otimes X \xrightarrow{\alpha_{Z,Y,X}} Z \otimes (Y \otimes X)$ in $\mathrm{Hom}(\mathcal{C}(\alpha, \delta))$
- *unit constraints:* for each $\alpha \xrightarrow{X} \beta$, isomorphisms $1_\beta \otimes X \xrightarrow{\lambda_X} X$ and $X \otimes 1_\alpha \xrightarrow{\rho_X} X$ in $\mathrm{Hom}(\mathcal{C}(\alpha, \beta))$

such that $\alpha_{Z,Y,X}$, λ_X and ρ_X are natural in Z, Y, X and satisfy the pentagon and triangle axioms (which are exactly similar to the ones in the definition of a tensor category).

A *strict 2-category* is the one in which the associativity and the unit constraints are identities.

Example 1.2.45. A 2-category with only one 0-cell is simply a tensor category.

Example 1.2.46. A 2-category can be obtained by taking rings as 0-cells, for any two 0-cells A, B, 1-cells $A \to B$ as (B, A)-bimodules and 2-cells as bimodule maps. The tensor functor is given by the obvious tensor product over a ring.

Unlike usual categories, a 2-category $\mathcal{C}$ has three different "opposites":

- The *1-cell dual* $\mathcal{C}^{op}$ in which 1-cells are reversed but not 2-cells.

- The *2-cell dual* $\mathcal{C}^{co}$ in which 2-cells are reversed but not 1-cells.

- The *bidual* $\mathcal{C}^{coop}$ in which both 1-cells and 2-cells are reversed.

That is, $\mathcal{C}_0^{op} = \mathcal{C}_0 = \mathcal{C}^{co}$ and $\mathcal{C}^{op}(\alpha, \beta) = \mathcal{C}(\beta, \alpha) = (\mathcal{C}^{co}(\alpha, \beta))^{op}$ as categories.

Remark 1.2.47. Notion of multi-categories do exist in the literature (see [Gho11], for instance).

Remark 1.2.48. $\mathcal{C}(\alpha, \alpha)$ is a tensor category and $\mathcal{C}(\alpha, \beta)$ is a $(\mathcal{C}(\beta, \beta), \mathcal{C}(\alpha, \alpha))$-bimodule category for 0-cells α, β (see [Ost03, EGNO15] for definition of module category).

Definition 1.2.49. Let $\mathcal{C}$, $\mathcal{C}'$ be 2-categories. A **weak functor** $F = (F, \varphi) : \mathcal{C} \to \mathcal{C}'$ consists of:

- a function $F : \mathcal{C} \to \mathcal{C}'$,

- for all $\alpha, \beta \in \mathcal{C}_0$, there exists a functor $F^{\alpha,\beta} : \mathcal{C}(\alpha, \beta) \to \mathcal{C}'(F(\alpha), F(\beta))$,

- for all $\alpha, \beta, \gamma \in \mathcal{C}_0$, there exists a natural isomorphism $\varphi : \otimes' \circ (F^{\beta,\gamma} \times F^{\alpha,\beta}) \to F^{\alpha,\gamma} \circ \otimes$ written simply as φ (where $\otimes$ and $\otimes'$ are the tensor functors of $\mathcal{C}$ and $\mathcal{C}'$ respectively),

- for all $\alpha \in \mathcal{C}_0$, there exists an invertible (with respect to composition) 2-cell $\varphi_\alpha : 1_{F(\alpha)} \to F(1_\alpha)$,

satisfying commutativity of certain diagrams (consisting of 2-cells) which are analogous to the diagrams appearing in the definition of a tensor functor.

Analogous to natural transformation two functors in the context of categories, we have weak transformation two weak functors defined as below.

Definition 1.2.50. Let $F = (F, \varphi), G = (G, \psi) : \mathcal{C} \to \mathcal{C}'$ be weak functors. A weak transformation $\sigma : F \to G$ consists of:

- for all $\alpha \in \mathcal{C}_0$, there exists a 1-cell $\sigma_\alpha \in \text{Obj}(\mathcal{C}'(F(\alpha), G(\alpha)))$,

- for all $\alpha, \beta \in \mathcal{C}_0$, there exists a natural transformation $\sigma^{\alpha,\beta} : (\sigma_\beta \otimes' F^{\alpha,\beta}) \to (G^{\alpha,\beta} \otimes' \sigma_\alpha)$ written simply as σ (where $(\sigma_\beta \otimes' F^{\alpha,\beta})$, $(G^{\alpha,\beta} \otimes' \sigma_\alpha)$ are defined in the obvious way), satisfying the following:

For all $X \in \mathrm{Obj}(\mathcal{C}(\beta,\gamma)), Y \in \mathrm{Obj}(\mathcal{C}(\alpha,\beta))$ where $\alpha,\beta,\gamma \in \mathcal{C}_0$, the following two diagrams commute:

$$\begin{array}{ccccc} & & G(X)\otimes'\sigma_\beta\otimes' F(Y) & & \\ & \nearrow^{\sigma_X\otimes' id_{F(Y)}} & & \searrow^{id_{G(X)}\otimes'\sigma_Y} & \\ \sigma_\gamma\otimes' F(X)\otimes' F(Y) & & & & G(X)\otimes' G(Y)\otimes'\sigma_\alpha \\ \downarrow{\scriptstyle id_{\sigma_\gamma}\otimes'\varphi_{X,Y}} & & & & \downarrow{\scriptstyle \psi_{X,Y}\otimes' id_{\sigma_\alpha}} \\ \sigma_\gamma\otimes' F(X\otimes Y) & & \xrightarrow{\sigma_{X\otimes Y}} & & G(X\otimes Y)\otimes'\sigma_\alpha \end{array}$$

$$\begin{array}{ccccc} & & \sigma_\alpha & & \\ & \nearrow^{\rho'_{\sigma_\alpha}} & & \nwarrow^{\lambda'_{\sigma_\alpha}} & \\ \sigma_\alpha\otimes' id_{F(\alpha)} & & & & id_{G(\alpha)}\otimes'\sigma_\alpha \\ \downarrow{\scriptstyle id_{\sigma_\alpha}\otimes'\varphi_\alpha} & & & & \downarrow{\scriptstyle \psi_\alpha\otimes' id_{\sigma_\alpha}} \\ \sigma_\alpha\otimes' F(1_\alpha) & & \xrightarrow{\sigma_{1_\alpha}} & & G(1_\alpha)\otimes'\sigma_\alpha \end{array}$$

where λ' and ρ' are the left and right unit constraints of $\mathcal{C}'$ respectively.

When such a weak transformation exists, we say that F and G are *weakly isomorphic*. Analogous to Theorem 1.2.30, we have the following *coherence theorem* for 2-categories proof of which can be found in [Lei08].

Theorem 1.2.51. *Every 2-category $\mathcal{C}$ is equivalent to some strict 2-category $\mathcal{C}'$, i.e., there exist weak functors $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}$ such that $id_{\mathcal{C}}$ (resp., $id_{\mathcal{C}'}$) is weakly isomorphic to $G \circ F$ (resp., $F \circ G$).* ☐

In view of this result, we mostly deal with strict 2-categories and hence will not talk about associativity and unit constraints.

C*-2-categories. We say that a 2-category $\mathcal{C}$ is $\mathbb{C}$-linear if $\mathcal{C}(\alpha,\beta)$ is a $\mathbb{C}$-linear category for all 0-cells α,β. A $*$-*structure* on $\mathcal{C}$ is a weak functor $* : \mathcal{C} \to \mathcal{C}$ such that:

(1) it is identity on 0-cells and 1-cells,

(2) the functor $*^{\alpha,\beta} : \mathcal{C}(\alpha,\beta) \to \mathcal{C}(\alpha,\beta)$ is a contravariant functor for every α and β in $\mathcal{C}_0$.

(3) it is an involution, i.e., $* \circ * = id_{\mathcal{C}}$.

(4) $*$ is compatible with the tensor structure, i.e., for 0-cells α, β, γ and 2-cells $f \in \text{Hom}(\mathcal{C}(\alpha, \beta)), g \in \text{Hom}(\mathcal{C}(\beta, \gamma))$, $(g \otimes f)^* = f^* \otimes g^*$ in $\text{Hom}(\mathcal{C}(\alpha, \gamma))$.
For a 2-cell $X \xrightarrow{f} Y$, we denote $*(f)$ simply by f^*.

Definition 1.2.52. A $*$-2-category $\mathcal{C}$ is said to be a **C*-2-category** if

(i) $\text{Hom}_{\mathcal{C}(\alpha,\beta)}(X, Y)$ is a Banach Space for all 1-cells $\alpha \xrightarrow{X} \beta, \alpha \xrightarrow{Y} \beta$ and 0-cells α, β;

(ii) the map

$$\text{Hom}_{\mathcal{C}(\alpha,\beta)}(Y, Z) \times \text{Hom}_{\mathcal{C}(\alpha,\beta)}(X, Y) \to \text{Hom}_{\mathcal{C}(\alpha,\beta)}(X, Z),\ (f, g) \mapsto f \circ g$$

is bilinear and $\|fg\| \leq \|f\|\,\|g\|$ for all 0-cells α, β and 1-cells $\alpha \xrightarrow{X} \beta, \alpha \xrightarrow{Y} \beta, \alpha \xrightarrow{Z} \beta$;

(ii) the $*$ satisfies the following properties:

(a) $\|f^*f\| = \|f\|^2$ for every 2-cell $X \xrightarrow{f} Y$ where $\alpha \xrightarrow{X} \beta, \alpha \xrightarrow{Y} \beta$ are 1-cells. In particular, $\text{End}_{\mathcal{C}(\alpha,\beta)}(X)$ is a unital C*-algebra for every 1-cell $\alpha \xrightarrow{X} \beta$;

(b) for any 2-cell $f \in \text{Hom}_{\mathcal{C}(\alpha,\beta)}(X, Y)$, the element f^*f is positive in the C*-algebra $\text{End}_{\mathcal{C}(\alpha,\beta)}(X)$;

(iii) the unit and associativity constraints are unitary.

Example 1.2.53. Any C*-tensor category is a C*-2-category $\mathcal{C}$ with single 0-cell.

Remark 1.2.54. In a C*-2-category $\mathcal{C}$, for any fixed 0-cell α, the category $\mathcal{C}(\alpha, \alpha)$ is a C*-tensor category.

Definition 1.2.55. A C*-2-category $\mathcal{C}$ is said to be **semi-simple** if $\mathcal{C}(\alpha, \beta)$ is a semi-simple category for all $\alpha, \beta \in \mathcal{C}_0$ and $\otimes : \mathcal{C}(\beta, \gamma) \times \mathcal{C}(\alpha, \beta) \to \mathcal{C}(\alpha, \gamma)$ is distributive over the direct sums of $\mathcal{C}(\beta, \gamma)$ and $\mathcal{C}(\alpha, \beta)$ for all 0-cells α, β, γ.

Rigidity in 2-categories. Let $\alpha \xrightarrow{X} \beta$ be a 1-cell in a 2-category $\mathcal{C}$. By a *right dual* (resp., *left dual*) of X we mean a 1-cell $\beta \xrightarrow{X^\#} \alpha$ (resp., $\beta \xrightarrow{\#X} \alpha$) such that there exists 2-cells $X^\# \otimes X \xrightarrow{e_X} 1_\alpha$ and $1_\beta \xrightarrow{c_X} X \otimes X^\#$ (resp., $X \otimes X^\# \xrightarrow{{}_Xe} 1_\alpha$ and $1_\beta \xrightarrow{{}_Xc} {}^\#X \otimes X$) which satisfy the following equations (similar to the conjugate equations mentioned in Equation (1.2.4)):

$$(1_X \otimes e_X) \circ (c_X \otimes 1_X) = 1_X \text{ and } (e_X \otimes 1_{X^\#}) \circ (1_{X^\#} \otimes c_X) = 1_{X^\#}$$

$$(\text{resp., } (1_X \otimes {}_Xe) \circ ({}_Xc \otimes 1_{\#X}) = 1_{\#X} \text{ and } ({}_Xe \otimes 1_{\#X}) \circ 1_X \otimes {}_Xc = 1_X)$$

Where e stands for evaluation and c stands for coevaluation.

One can show that two right (resp., left) duals are isomorphic via an isomorphism which is compatible with the evaluation and coevaluation maps. A 2-category is said to be *right* (resp., *left*) *rigid* if right (resp., left) dual exists for every 1-cell.

Further, in a rigid 2-category $\mathcal{C}$, one can consider dual as an invertible weak functor $\# = (\#, s) : \mathcal{C} \to \mathcal{C}^{coop}$ in the following way:

- for each 1-cell X, we fix a triplet $(X^{\#}, e_X, c_X)$ so that when $X = 1_{\alpha}$ for a 0-cell α, then $X^{\#} = 1_{\alpha}$, $e_X = \lambda_{1_{\alpha}}\ (= \rho_{1_{\alpha}})$ and $c_X = \lambda_{1_{\alpha}}^{-1} = \rho_{1_{\alpha}}^{-1}$ (see [Kas95] for proof of such a possibility);

- $\#$ induces identity map on $\mathcal{C}_0$;

- for each pair of 0-cells α and β, define the contravariant functor $\# : \mathcal{C}(\alpha, \beta) \to \mathcal{C}(\beta, \alpha)$ as follows:

  for each $X, Y \in \mathrm{Obj}(\mathcal{C}(\alpha, \beta))$ and 2-cell $X \xrightarrow{f} Y$, set $\#(X) = X^{\#}$ and $\#(f)$, denoted by $f^{\#}$, be given by the following composition

$$\begin{array}{ccccc} Y^{\#} & \xrightarrow{\rho_{Y^{\#}}^{-1}} & Y^{\#} \otimes 1_{\alpha} & \xrightarrow{id_{Y^{\#}} \otimes c_X} & Y^{\#} \otimes X \otimes X^{\#} \\ {\scriptstyle f^{\#}} \downarrow & & & & \downarrow {\scriptstyle id_{Y^{\#}} \otimes f \otimes id_{X^{\#}}} \\ X^{\#} & \xleftarrow[\lambda_{X^{\#}}]{} & 1_{\beta} \otimes X^{\#} & \xleftarrow[e_Y \otimes id_{X^{\#}}]{} & Y^{\#} \otimes Y \otimes X^{\#} \end{array}$$

- $\alpha, \beta, \gamma \in \mathcal{C}_0$, the natural isomorphism $s : \otimes \circ (\mathit{flip}) \circ \left(\#^{\beta,\gamma} \times \#^{\alpha,\beta}\right) \to (\#^{\alpha,\gamma} \circ \otimes)$ is defined by:

  for $X \in \mathrm{Obj}(\mathcal{C}(\alpha, \beta)), Y \in \mathrm{Obj}(\mathcal{C}(\beta, \gamma))$, the invertible 2-cell $s_{X,Y}$ is given by the composition

$$\begin{array}{ccc} X^{\#} \otimes Y^{\#} & \xrightarrow{id_{(X^{\#} \otimes Y^{\#})} \otimes c_{(Y \otimes X)}} & X^{\#} \otimes Y^{\#} \otimes (Y \otimes X) \otimes (Y \otimes X)^{\#} \\ {\scriptstyle s_{X,Y}} \downarrow & & \downarrow {\scriptstyle id_{X^{\#}} \otimes e_Y \otimes id_X \otimes id_{(Y \otimes X)^{\#}}} \\ (Y \otimes X)^{\#} & \xleftarrow[e_X \otimes id_{(Y \otimes X)^{\#}}]{} & (X^{\#} \otimes X) \otimes (Y \otimes X)^{\#} \end{array}$$

- for all $\alpha \in \mathcal{C}_0$ the invertible 2-cell $s_{\alpha} : 1_{\alpha} \to 1_{\alpha}$ is given by identity morphism on 1_{α}.

Note that the above prescription of the dual functor $(\#, s)$ carries forward almost verbatim to another weak functor $(\widetilde{\#}, \widetilde{s}) : \mathcal{C}^{coop} \to (\mathcal{C}^{coop})^{coop} = \mathcal{C}$. This allows us to consider the composition $(\widetilde{\#}, \widetilde{s}) \circ (\#, s) : \mathcal{C} \to \mathcal{C}$. This is again a weak functor and we abuse notation to denote it by $(\#\#, t)$ and call it the bi-dual functor.

Definition 1.2.56. A 2-category $\mathcal{C}$ is said to be **pivotal** if $\mathcal{C}$ is (right) rigid and there exists a weak transformation $a : id_{\mathcal{C}} \to \#\#$ such that $a_\alpha = 1_\alpha$ for all $\alpha \in \mathcal{C}_0$.

Remark 1.2.57. Suppose $\mathcal{C}$ is a rigid semi-simple C*-2-tensor category. Then for any 0-cell α and every 1-cell $\alpha \overset{X}{\to} \alpha$, left dual of X exists and will coincide with the right dual. Thus $\mathcal{C}(\alpha,\alpha)$ is a rigid semi-simple C*-tensor category.

Example 1.2.58. 2-category associated to an extremal subfactor. Given a extremal subfactor $N \subseteq M$, $L^2(M)$ is obtained by the GNS construction with respect to the canonical tr and it can be seen as an N-M bimodule as described earlier in Section 1.1.1.

Consider the 2-category $\mathcal{C} = \mathcal{C}^{N\subseteq M}$ with 0-cells $\{N, M\}$ and $\mathcal{C}_{NN} := \mathcal{C}(N,N)$ (resp., $\mathcal{C}_{MM} := \mathcal{C}(M,M)$) be the tensor category of N-N (resp.,M-M) -bimodules appearing as submodules of tensor powers of $X\bar{X}$ (resp., $\bar{X}X$). Also, let $\mathcal{C}_{NM} := \mathcal{C}(N,M)$ (resp., $\mathcal{C}_{MN} := \mathcal{C}(M,N)$) be the category of N-M (resp., M-N) -bimodules appearing as submodules of $X(\bar{X}X)^{\otimes^k}$ (resp., $(\bar{X}X)^{\otimes^k}\bar{X}$) for $k \in \mathbb{N}\cup\{0\}$. The tensor functor is given by the usual relative tensor of bimodules and the tensor units in $\mathcal{C}_{NN}$ (resp., $\mathcal{C}_{MM}$) is the canonical N-N-bimodule $L^2(N)$ (resp., M-M-bimodule $L^2(M)$). There is a natural associativity constraint for relative tensor product of bimodules. For the following let $A, B \in \{N, M\}$. For an A-B bimodule $\mathcal{H}$, the unit constraints are given by the canonical isomorphisms $L^2(A)\underset{A}{\otimes}\mathcal{H} \overset{A-B}{\cong} \mathcal{H}$ and $\mathcal{H}\underset{B}{\otimes}L^2(B) \overset{A-B}{\cong} \mathcal{H}$. Thus, $\mathcal{C}$ has a natural 2-category structure.

For the (right) rigid structure on $\mathcal{C}$, for each A-B-bimodule $\mathcal{H}$, we set $({}_A\mathcal{H}_B)^\# = {}_B\bar{\mathcal{H}}_A$ and define the evaluation and coevaluation maps $e_{\mathcal{H}} \in {}_B\mathcal{L}_B(\bar{\mathcal{H}}\underset{A}{\otimes}\mathcal{H}, L^2(B))$ and $c_{\mathcal{H}} \in {}_A\mathcal{L}_A(L^2(A), \mathcal{H}\underset{B}{\otimes}\bar{\mathcal{H}})$ respectively, (on bounded vectors,) by $e_{\mathcal{H}}(\bar{\xi}\underset{A}{\otimes}\eta) := \langle \xi, \eta\rangle_B$ and $c_{\mathcal{H}}(\hat{a}) := \sum\limits_i a\cdot(\eta_j \underset{B}{\otimes}\bar{\eta}_j)$ for all $\xi, \eta \in \mathcal{H}^0, a \in A, b \in B$, where $\{\eta_j\}_j$ is a basis for the right B-module $\mathcal{H}_B$. Thus $\mathcal{C}$ inherits a canonical rigid structure.

Finally, the canonical isomorphism ${}_A\mathcal{H}_B \overset{A-B}{\cong} {}_A\bar{\bar{\mathcal{H}}}_B$ for any such A-B-bimodule ${}_A\mathcal{H}_B$, equips $\mathcal{C}$ with a pivotal structure. Note that, for $T \in {}_A\mathcal{L}_B(\mathcal{H},\mathcal{K})$, it can be shown that $T^\#(\bar{\xi}) = \overline{T^*(\xi)}$ and hence $T^{\#\#}(\bar{\bar{\xi}}) = \overline{\overline{T(\xi)}}$ for all $\xi \in \mathcal{H}^0$, where T^* is the usual adjoint of the intertwiner T. Further, this usual adjoint gives a canonical C*-structure on $\mathcal{C}^{N\subseteq M}$ making it a rigid semi-simple C*-2-category.

1.3 The Interconnections

1.3.1 Subfactor planar algebras and rigid 2-categories

We narrow down our focus on the particular type of 2-categories arising out of extremal subfactors mentioned in Example 1.2.58. Abstractly, what we have is the following:

- a rigid semi-simple C*-2-category $\mathcal{C}$ with only two 0-cells, say $\{+,-\}$,
- simple tensor units in $\mathcal{C}_{++} := \mathcal{C}(+,+)$ and $\mathcal{C}_{--} := \mathcal{C}(+,+)$, and
- a given generating object $\rho \in \mathcal{C}_{+-} := \mathcal{C}(+,-)$, i.e., $\mathcal{C}_{++}$ (resp., $\mathcal{C}_{--}$, $\mathcal{C}_{+-}$, $\mathcal{C}_{-+}$) is same as the full subcategory generated by subobjects of $(\rho\bar{\rho})^k$ (resp., $(\bar{\rho}\rho)^k$, $\rho(\bar{\rho}\rho)^k$, $(\bar{\rho}\rho)^k\bar{\rho}$) for $k \in \mathbb{N} \cup \{0\}$.

We have seen in Section 1.1.5 that from a subfactor planar algebra one can get an extremal subfactor of finite index and hence a 2-category (Example 1.2.58) with the above properties. The obvious question now is whether the converse holds. We here give a construction of a subfactor planar algebra from such a 2-category.

We just give an outline of the contruction by mentioning the underlying vector spaces and the action of tangles. For the action of tangles, graphical calculus for 2-cells in a 2-category will be extensively used which is very similar to the one discussed in Section 1.2.6. For a more detailed description we refer the reader to [Gho11, Bur15].

To start with, we have a rigid semi-simple C*-2-category with two 0-cells and a 1-cell ρ which generates whole 2-category as described above. We define a planar algebra P^ρ whose underlying vector spaces are as follows:

$$P^\rho_{\epsilon k} := \begin{cases} \mathcal{C}_{++}\left(\mathbb{1}, (\rho\bar{\rho})^k\right), & \text{if } \epsilon = + \\ \mathcal{C}_{--}\left(\mathbb{1}, (\bar{\rho}\rho)^k\right), & \text{if } \epsilon = - \end{cases}$$

In order to specify the action of any tangle $T : (\varepsilon_1 k_1, \varepsilon_2 k_2, \ldots, \varepsilon_n k_n) \to \varepsilon k$, we choose a standard form $\widetilde{T}$ of T. So $\widetilde{T}$ is divided into horizontal stripes each of which contain exactly one local maxima or local minima or an internal rectangle. We label each string in the tangle by ρ.

To define $P^\rho_T : P^\rho_{\varepsilon_1 k_1} \times \cdots \times P^\rho_{\varepsilon_n k_n} \to P^\rho_{\varepsilon k}$, we fix 2-cells $x_j \in P^\rho_{\varepsilon k_j}$ for $1 \leq j \leq n$. We label the internal rectangle in $\widetilde{T}$ associated to the j^{th} internal disc in T, by $x_j \in P_{\varepsilon k_j}$. Now, each horizontal stripe makes sense as a 2-cell according to the graphical calculus already set up. We define $P_T(x_1, x_2, \ldots, x_n)$ to be the composition of these 2-cells read from bottom to top. This completes the definition of the planar algebra. The connectedness of P^ρ

follows from the fact that tensor units of $\mathcal{C}_{++}$ and $\mathcal{C}_{--}$ are simple. Finite dimesionality, positivity and sphericality of P^ρ are trivial to see. Thus P^ρ is a subfactor planar algebra.

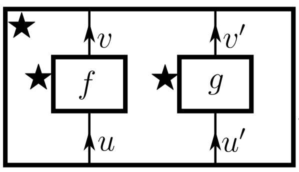
Before proceeding further, we exhibit the connection between semi-simple, rigid C*-tensor categories and oriented factor planar algebras. We will just highlight the main points of this correspondence which is a kind of a folklore in the C*-tensor category, quantum group and subfactor communities. We setup the following definition which will be useful throughout this thesis.

Definition 1.3.1. Let $\mathcal{C}$ be a rigid semi-simple C*-tensor category and $\mathscr{X} = \{X_\alpha\}_\alpha$ be a family of objects in $\mathcal{C}$. The full subcategory of $\mathcal{C}$ obtained from all possible direct sums of simple objects which appear as a sub-object of a finite tensor-fold of elements from the family and their duals, will be referred as the *full subcategory tensor-generated by* $\mathscr{X}$ and denoted by $\langle \mathscr{X} \rangle$. When this subcategory is the whole of $\mathcal{C}$, we simply say $\mathscr{X}$ *tensor-generates* $\mathcal{C}$.

1.3.2 C*-tensor category associated to an oriented planar algebra

Let P be a Λ-oriented factor planar algebra. We define a C*-tensor category $\mathcal{C}^P$ as follows:

- Objects are words $w \in W$.
- For two objects v, w, the morphism space $\mathcal{C}^P(v, w) := P_{wv^*}$.
- For objects u, v, w and morphisms $f \in \mathcal{C}^P(v, w), g \in \mathcal{C}^P(u, v)$, composition is given by the action of multiplication tangle. That is, $f \circ g := P_{M^v_{w,u}}(f, g)$.
- $*$-structure on the category is given by the $*$-structure of the planar algebra P.
- Tensor product of objects is just the concatenation of the words and for two morphism $f \in \mathcal{C}^P(u, v), f' \in \mathcal{C}^P(u', v')$, the tensor product $f \otimes g$ is given by the action of the tangle



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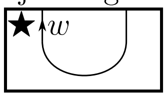
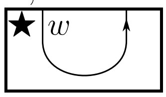
- The unit object is given by the empty word.

- For each word $w \in W$, a dual object is given by w^*, with evaluation and coevaluation given by the action of tangles  and .

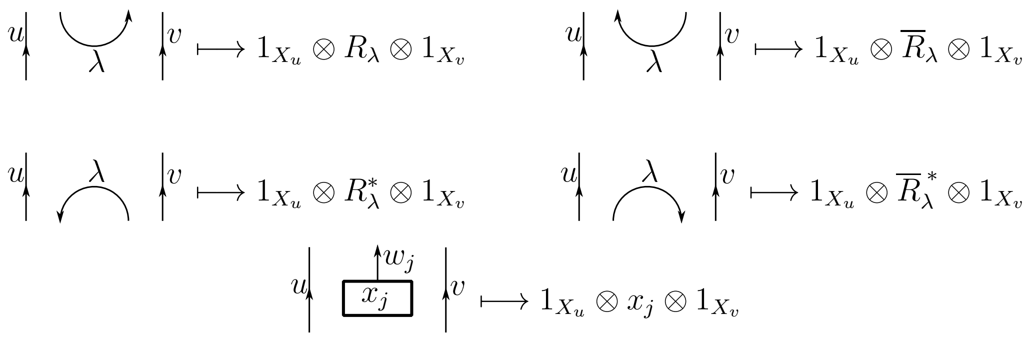
While this is a C*-tensor category with duals, it does not have direct sums and subobjects. The unitary Karaubi envelope will allow us to remedy this problem and give us a semisimple C*-tensor category (see Section 1.2.5).

Definition 1.3.2. For a Λ-oriented factor planar algebra P, its *projection category* is the rigid C*-tensor category $\mathcal{K}(\mathcal{C}^P)$.

Thus starting from a Λ-oriented factor planar algebra P, we get a rigid semisimple C*-tensor category $\mathcal{K}(\mathcal{C}^P)$ as described above. Note that $\mathcal{K}(\mathcal{C}^P)$ is tensor-generated by $\{1_{\lambda\overline{\lambda}}\}_{\lambda\in\Lambda}$.

1.3.3 Oriented planar algebra associated to a C*-tensor category

Conversely, starting from a strict, rigid, semi-simple C*-tensor category $\mathcal{C}$ with the trivial object $\mathbb{1}$ being simple, and a family of objects $\mathscr{X} = \{X_\lambda\}_{\lambda\in\Lambda}$ tensor-generating $\mathcal{C}$, one can define a Λ-oriented planar algebra $P^{\mathscr{X}}$ (or simply P) whose associated projection category is equivalent to $\mathcal{C}$. For this, we first fix normalized standard solutions $(R_\lambda, \overline{R}_\lambda)$ to conjugate equations for each X_λ in the family. For $\lambda \in \Lambda$, set $X_{\overline{\lambda}} := \overline{X_\lambda}$, and for a word $w := (\alpha_1, \ldots, \alpha_n) \in W = W_\Lambda$, let X_w denote the object $X_{\alpha_1} \otimes \cdots \otimes X_{\alpha_n}$. Define $P_w := \mathcal{C}(\mathbb{1}, X_w)$ for all $w \in W$. Now, for a Λ-oriented tangle $T : (w_1, \ldots, w_n) \to w_0$, one needs to define its action, that is, a multi-linear map $P_T : P_{w_1} \times \cdots \times P_{w_n} \to P_{w_0}$. Let $x_j \in P_{w_j}$ for $1 \le j \le n$. In the isotopy class of T, choose a tangle diagram $\widetilde{T}$ in *standard form*, namely, (i) all discs (internal and external) are rectangles with sides parallel to the coordinate axes in $\mathbb{R}^2$, (ii) all strings are smooth with finitely many local maximas or minimas, (iii) all marked points are on the top side of every rectangle (internal or external) where the strings end transversally, and (iv) it is possible to slice $\widetilde{T}$ into horizontal strips which contains exactly one local maxima or local minima or an internal rectangle. For $1 \le j \le n$, we label the internal rectangle in $\widetilde{T}$ associated to the j^{th} internal disc in T, by $x_j \in P_{w_j} = \mathcal{C}(\mathbb{1}, X_{w_j})$. To each horizonatal strip of $\widetilde{T}$, we assign a morphism prescribed by



Define $P_T(x_1, \ldots, x_n)$ as the composition of the morphisms associated to each horizontal strip. The action is indeed well defined and the planar algebra is a factor planar algebra. Observe that we use the strictness of the tensor structure in $\mathcal{C}$ at every step of this construction, starting from the definition of X_w till the action of tangles and its invariance under isotopy. One can possibly extend this construction in the non-strict case as well by introducing appropriate associators.

Remark 1.3.3. We summarize the above discussion. Any rigid semi-simple C*-tensor category tensor-generated by $\mathscr{X}$ is equivalent to the $\mathcal{K}(\mathcal{C}^{P^{\mathscr{X}}})$. Conversely, the projection category of any Λ-oriented factor planar algebra P, is tensor-generated by $\mathscr{X} := \{1_{\lambda\overline{\lambda}}\}_{\lambda \in \Lambda}$; further, P is isomorphic to the Λ-oriented factor planar algebra $P^{\mathscr{X}}$.

Remark 1.3.4. Given any Λ-oriented factor planar algebra P, there exists a II_1-factor N and a family $\{X_\lambda\}_{\lambda \in \Lambda}$ of extremal, bifinite N-N-bimodules such that the Λ-oriented planar algebra associated to this family is isomorphic to P. To see this, one considers the projection category $\mathcal{K}(\mathcal{C}^P)$ associated to P. Now by [BHP12], any rigid, semi-simple C*-tensor category, in particular $\mathcal{C}^P$, is equivalent to a full subcategory of extremal, bifinite bimodules over a II_1-factor. Applying the converse above, one gets the required result.

1.4 Free Product of Categories

We will give the definition of free product of two semi-simple C*-tensor categories with simple tensor units. This notion, due to Bisch and Jones, arises from the free composition of finite index subfactors (see [BJ97]). It also appears in the study of free products of compact quantum groups [Wan95]. Our approach to free products closely follows the construction of Bisch and Jones as elaborated by [IMS16], except we do not require duals in our categories. To proceed with this construction, we will first define a certain C*-category involving the two given categories, and controlled by non-crossing partitions. The free product will be the resulting unitary idempotent completion described in Section 1.2.5.

Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be two semi-simple C*-categories with simple tensor units $\mathbb{1}_+$ and $\mathbb{1}_-$ respectively. In our construction, we pick a strict model of $\mathcal{C}_\pm$. Let Σ be the set of words with letters in $\mathrm{Obj}(\mathcal{C}_+) \cup \mathrm{Obj}(\mathcal{C}_-)$. For $\sigma \in \Sigma$, the length of σ will be denoted by $|\sigma|$. To a word $\sigma \in \Sigma$, we associate the subword (whose letters are not necessarily adjacent) $\sigma_+ \in \mathrm{Obj}(\mathcal{C}_+)$ (resp., $\sigma_- \in \mathrm{Obj}(\mathcal{C}_-)$) consisting of all the letters in σ coming from $\mathrm{Obj}(\mathcal{C}_+)$ (resp., $\mathrm{Obj}(\mathcal{C}_-)$). The object obtained by tensoring the letters in $\sigma_\pm$ will be denoted by $t(\sigma_\pm)$ with the convention $t(\emptyset) = \mathbb{1}_\pm$ where appropriate. For instance, if $\sigma = a_1^+ a_2^- a_3^+ a_4^- a_5^-$, then $\sigma_+ = a_1^+ a_3^+$, $t(\sigma_+) = a_1^+ \otimes a_3^+$, $\sigma_- = a_2^- a_4^- a_5^-$ and $t(\sigma_-) = a_2^- \otimes a_4^- \otimes a_5^-$.

Definition 1.4.1. Let $\sigma, \tau \in \Sigma$. A '(σ,τ)-*NCP*' consists of:

- a *non-crossing partitioning* of the letters in σ and τ arranged at the bottom and on the top edges of a rectangle respectively moving from left to right, such that each partition block consists only of objects from $\mathcal{C}_+$ or only of objects $\mathcal{C}_-$,
- every block gives a pair of (possible empty) subwords of σ and τ, say, (σ_0, τ_0), where σ_0 (resp. τ_0) consists of letters in the partition coming from σ (resp. τ). For each such block, seen as a rectangle with the bottom labeled by σ_0 and the top labeled by τ_0, we choose a morphism from $t(\sigma_0)$ to $t(\tau_0)$ in the appropriate category.

We give an example of a (σ,τ)-NCP in Figure 1.2 where $\sigma = a_1^+ a_2^+ a_3^+ a_4^- a_5^+ a_6^+ a_7^- a_8^+$ and $\tau = b_1^+ b_2^- b_3^- b_4^+ b_5^+$ with $a_i^\varepsilon, b_j^\varepsilon \in \mathcal{C}_\varepsilon$, $\varepsilon \in \{+,-\}$.

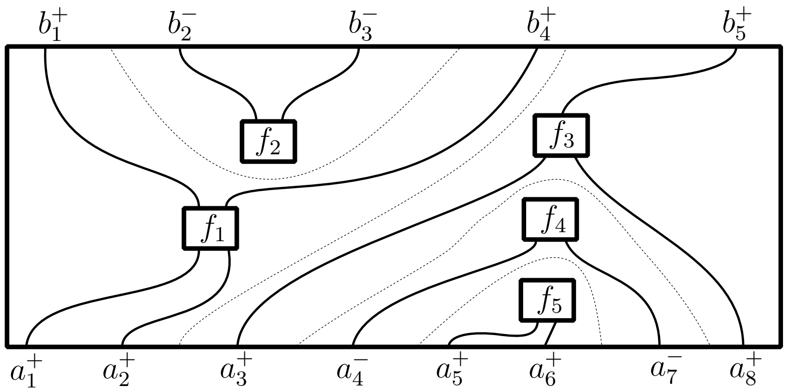


Figure 1.2: (σ,τ)-NCP

Here, the pair of subwords corresponding to the partition blocks are $\rho_1 = (a_1^+ a_2^+, b_1^+ b_4^+)$, $\rho_2 = (\emptyset, b_2^- b_3^-)$, $\rho_3 = (a_3^+ a_8^+, b_5^+)$, $\rho_4 = (a_4^- a_7^-, \emptyset)$, and $\rho_5 = (a_5^+ a_6^+, \emptyset)$. Note that each letter of ρ_i either belongs $\mathrm{Obj}(\mathcal{C}_+)$ alone or $\mathrm{Obj}(\mathcal{C}_-)$ alone, for every i and each of ρ_i is assigned a morphism from the corresponding category. For instance, all the letters of ρ_3 are objects of $\mathcal{C}_+$ and is assigned the morphism $f_3 \in \mathcal{C}_+(a_3^+ \otimes a_8^+, b_5^+)$.

We denote the set of such (σ, τ)-NCPs by $NCP(\sigma, \tau)$. Now, to every $T \in NCP(\sigma, \tau)$, we can associate unique $T_\pm \in NCP(\sigma_\pm, \tau_\pm)$ by deleting all blocks whose letters are labeled by the opposite sign.

Since all letters in $\sigma_\pm$ and $\tau_\pm$ come from either $\mathcal{C}_+$ or $\mathcal{C}_-$ only, the non-crossing partitions $T_\pm$ give rise to unique morphisms $Z_{T_\pm} \in \mathcal{C}_\pm\left(t(\sigma_\pm), t(\tau_\pm)\right)$ using the standard graphical calculus for monoidal categories.

So, for any $\sigma, \tau \in \Sigma$ and $T \in NCP(\sigma, \tau)$, we have morphisms $Z_{T_\pm} \in \mathcal{C}_\pm\left(t(\sigma_\pm), t(\tau_\pm)\right)$. We write $Z_T \coloneqq Z_{T_+} \otimes Z_{T_-} \in \mathcal{C}_+\left(t(\sigma_+), t(\tau_+)\right) \otimes \mathcal{C}_-\left(t(\sigma_-), t(\tau_-)\right)$. For example, for the NCP T in Figure 1.2,

$$Z_{T_+} = (f_1 \otimes f_3) \circ (1_{a_1^+ \otimes a_2^+ \otimes a_3^+} \otimes f_5 \otimes 1_{a_8^+}) \text{ and } Z_{T_+} = f_2 \circ f_4$$

We define the category $\mathcal{NCP}$ as follows:

- Objects in $\mathcal{NCP}$ are given by Σ.

- For $\sigma, \tau \in \Sigma$, the morphism space is defined by

$$\mathcal{NCP}(\sigma, \tau) \coloneqq \operatorname{span}\left\{Z_T : T \in NCP(\sigma, \tau)\right\} \subset \mathcal{C}_+\left(t(\sigma_+), t(\tau_+)\right) \otimes \mathcal{C}_-\left(t(\sigma_-), t(\tau_-)\right).$$

Composition of morphisms is given by composing the tensor components, which is obviously bilinear, and associative. However, one needs to verify whether the morphism spaces of $\mathcal{NCP}$ are closed under such composition. Let $S \in NCP(\sigma, \tau)$ and $T \in NCP(\tau, \kappa)$. Consider the 'composed' rectangle obtained by gluing T on the top of S matching along the letters of τ. The non-crossing partitions of S and T induce a non-crossing partition on the composed rectangle with σ at the bottom and τ on the top; each partition is then labeled by composing the corresponding morphisms in S and T. We call this $T \circ S \in NCP(\sigma, \kappa)$. In this process of composing two NCPs, we have ignored certain partitions of S (staying only on its top) and T (staying only at its bottom) which cancel each other and do not contribute towards the non-crossing partitioning of the composed rectangle. Since the tensor units $\mathbb{1}_\pm$ are assumed to be simple, composing the morphisms associated to these partitions simply yield a scalar. Suppose $\lambda(T, S)$ denote the product of all such scalars. Then, $\left(Z_{T_+} \circ Z_{S_+}\right) \otimes \left(Z_{T_-} \circ Z_{S_-}\right) = \lambda(T, S)\, Z_{(T\circ S)_+} \otimes Z_{(T\circ S)_-} \in \mathcal{NCP}(\sigma, \kappa)$.

Clearly, $\mathcal{NCP}$ is a $\mathbb{C}$-linear category. There is also a $*$-structure given by applying $*$ on each of the tensor components. To see whether the morphism spaces of $\mathcal{NCP}$ is closed under $*$, we define an involution $\left(NCP(\sigma, \tau) \ni T \overset{*}{\longmapsto} T^* \in NCP(\tau, \sigma)\right)_{\sigma, \tau \in \Sigma}$ where we reflect T about any horizontal line to obtain T^* with a non-crossing partitioning and their corresponding morphisms being induced by the reflection of the initial partitioning and $*$ of the assigned morphisms in T respectively.

Indeed, $Z_T^* = Z_{T^*} \in \mathcal{NCP}(\tau, \sigma)$ for all $T \in NCP(\sigma, \tau)$. Thus, $\mathcal{NCP}$ is a $*$-category. Note that by construction, $\mathcal{NCP}$ is equipped with a canonical faithful $*$-functor to the Deligne tensor product $\mathcal{C}_+ \boxtimes \mathcal{C}_-$, which sends σ to $\sigma_+ \boxtimes \sigma_- \in \mathcal{C}_+ \boxtimes \mathcal{C}_-$. Since $\mathcal{C}_\pm$ are both semi-simple, the Deligne tensor product is again a C*-category with finite dimensional morphism spaces. But any (not necesarily full) *-subcategory of a C*-category with finite dimensional morphism spaces is easily seen to be C* itself. Since our canonical functor is faithful, this implies $\mathcal{NCP}$ is a C*-category.

For the tensor structure, define $\sigma \otimes \tau$ as the concatenated word $\sigma\tau$. If $f = \sum\limits_i a_i \otimes b_i \in \mathcal{NCP}(\sigma, \tau) \subset \mathcal{C}_+(\sigma_+, \tau_+) \otimes \mathcal{C}_-(\sigma_-, \tau_-)$ and $g = \sum\limits_j c_j \otimes d_j \in \mathcal{NCP}(\kappa, \nu) \subset \mathcal{C}_+(\kappa_+, \nu_+) \otimes \mathcal{C}_-(\kappa_-, \nu_-)$, then $f \otimes g := \sum\limits_{i,j} (a_i \otimes^+ c_j) \otimes (b_i \otimes^- d_j)$ where $\otimes^\pm$ denote for the tensor functor of $\mathcal{C}^\pm$. It is easy to check $f \otimes g \in \mathcal{NCP}(\sigma \otimes \kappa, \tau \otimes \nu)$ and $(f \otimes g)^* = f^* \otimes g^*$. This implies $\mathcal{NCP}$ is a C*-tensor category. Note that $\mathcal{C}_\pm$ sit inside $\mathcal{NCP}$ as full $*$-subcategories.

Definition 1.4.2. A (σ, τ)-NCP T will be called *elementary* if $\sigma = (\sigma_1, \dots, \sigma_n)$, $\tau = (\tau_1, \dots, \tau_n)$ for some n, and the only block partitions of T are (σ_i, τ_i) for $1 \leq i \leq n$ where at most one of $\sigma_1, \dots, \sigma_n, \tau_1, \dots, \tau_n$ is empty.

Remark 1.4.3. Any morphism in $\mathcal{NCP}$ can be expressed as a linear combination of composition of Z_T's for T being elementary NCP.

Definition 1.4.4. The *free product of the categories* $\mathcal{C}_+$ *and* $\mathcal{C}_-$ (as above) is defined as the unitary idempotent completion proj($\mathcal{NCP}$) which we denote by $\mathcal{C}_+ * \mathcal{C}_-$.

From Fact 1.2.42, it is clear that $\mathcal{C}_+ * \mathcal{C}_-$ is a C*-tensor category with simple tensor unit containing $\mathcal{C}_\pm$ as full tensor subcategories via the fully faithful unitary tensor functors

$$\iota_\pm \ : \ \mathcal{C}_\pm \xrightarrow{\gamma_\pm} \mathcal{NCP} \xrightarrow{\alpha^{\mathcal{NCP}}} \mathcal{C}_+ * \mathcal{C}_- \ .$$

Definition 1.4.5. Let $\mathrm{Irr}(\mathcal{C}_\pm)$ denote a choice of object from each isomorphism class of simple objects, such that the tensor units are chosen to represent their isomorphism class.

$$\Sigma_0 := \{\emptyset\} \cup \{a_1^{\varepsilon_1} \dots a_k^{\varepsilon_k} \ : \ k \in \mathbb{N},\ \varepsilon_i \in \{\pm\},\ a_i^{\varepsilon_i} \in \mathrm{Irr}(\mathcal{C}_{\varepsilon_i}) \setminus \{\mathbb{1}_{\varepsilon_i}\},\ \varepsilon_i = -\varepsilon_{i+1} \text{ for } 1 \leq i \leq k\}$$

With the above notations, we have the following proposition

Proposition 1.4.6. *The free product $\mathcal{C}_+ * \mathcal{C}_-$ is a strict, semi-simple C^*-tensor category with $\mathcal{C}_\pm$ as full tensor subcategories (via $\iota_\pm$) which tensor-generate $\mathcal{C}_+ * \mathcal{C}_-$. Moreover, the fully faithful unitary tensor functor $\alpha^{\mathcal{NCP}} : \mathcal{NCP} \to \mathcal{C}_+ * \mathcal{C}_-$ gives rise to a bijection between Σ_0 and isomorphism classes of simple objects in $\mathcal{C}_+ * \mathcal{C}_-$. Further, if $\mathcal{C}_\pm$ are rigid, then so is $\mathcal{C}_+ * \mathcal{C}_-$.*

Proof. First we show that the objects $\sigma \in \Sigma_0$ form a distinct set of irreducible objects in $\mathcal{NCP}$.

Let $\sigma \in \Sigma_0$ be a nonempty word, and T a (σ, σ)-NCP. If T has a block which connects only letters on the top or only letters on bottom, then T necessarily also has a singleton block and its associated morphism turns out to be zero (since $\sigma \in \Sigma_0$ is non-empty and the tensor units $\mathbb{1}_\pm$ are simple) which implies $Z_T = 0$. Thus every partition in T consists of letters in the top as well as bottom. Since the letters in σ come alternatively from $\mathcal{C}_+$ and $\mathcal{C}_-$, and the partitions are non-crossing, the partition blocks should be of the form $(a_1^{bottom}, a_1^{top})$, $(a_2^{bottom}, a_2^{top}), \ldots$, where $\sigma = a_1 a_2 \ldots$. The assigned morphisms of these blocks are then scalars since a_i's are simple. This says that Z_T has to be a scalar multiple of 1_σ. Hence, $\mathcal{NCP}(\sigma, \sigma)$ is one-dimensional implying σ is simple for all $\sigma \in \Sigma_0$. Similar arguments will tell us that $\mathcal{NCP}(\sigma, \tau)$ is zero for two distinct $\sigma, \tau \in \Sigma_0$.

We now show Σ_0 is complete, in the sense that any object $\sigma \in \mathcal{NCP}$ is isomorphic to a direct sum of objects from Σ_0. Observe that if $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that the letters in each σ_i come from $\mathcal{C}_+$ alone or $\mathcal{C}_-$ alone, then $\sigma_1 \ldots \sigma_n$ is isomorphic to the word $t(\sigma_1) \ldots t(\sigma_n)$. Moreover, a quick sketch of non-crossing partitions shows that $\sigma \mathbb{1}_\pm \tau \cong \sigma\tau$. It is also easy to see that if $a \cong b_1 \oplus b_2$ in $\mathcal{C}_\pm$ via decomposition isometries $v_i \in \mathcal{C}_\pm(b_i, a)$, then the word $\sigma a \tau \cong \sigma b_1 \tau \bigoplus \sigma b_2 \tau$ via decomposition isometries given by the $(\sigma b_1 \tau, \sigma a \tau)$-non-crossing partitions T_i defined as follows: The underlying non-crossing partition has pairings which connect elements vertically, and for each block ending in σ or τ, we have the identity morphism, while the block connecting b_i with a is assigned the isometry v_i. Taken together, these observations imply that any object can be decomposed as a finite direct sum of words in Σ_0.

$\mathcal{C}_+ * \mathcal{C}_-$ inherits all the above properties from $\mathcal{NCP}$. In particular, since every object $\sigma \in \mathcal{NCP}$ is isomorphic to a direct sum of simple objects in Σ_0, this will be true for any subobject. Hence in the projection category, every object (τ, p) is isomorphic to a direct sum of objects of the form $(\sigma, 1_\sigma)$ for $\sigma \in \Sigma_0$.

Thus to show that $\mathcal{C}_+ * \mathcal{C}_-$ has direct sums, it suffices to show that for $\sigma, \tau \in \Sigma_0$, there exists an object $(\sigma, 1_\sigma) \oplus (\tau, 1_\tau) \in \mathcal{C}_+ * \mathcal{C}_-$ satisfying direct condition.

Let α_i and ε_j be the signs given by $a_i \in \mathcal{C}_{\alpha_i}$ and $b_j \in \mathcal{C}_{\varepsilon_j}$. Consider $\widehat{a}_i := a_i \oplus$

$\mathbb{1}_{\alpha_i}$ implemented by the isometries $u_i \in \mathcal{C}_{\alpha_i}(a_i, \widehat{a}_i)$ and $e_i \in \mathcal{C}_{\alpha_i}(\mathbb{1}_{\alpha_i}, \widehat{a}_i)$. Similarly, pick $\widehat{b}_j := bj \oplus \mathbb{1}_{\varepsilon_j}$ and implementing isometries $v_j \in \mathcal{C}_{\varepsilon_j}(b_j, \widehat{b}_j)$ and $f_j \in \mathcal{C}_{\varepsilon_j}(\mathbb{1}_{\varepsilon_j}, \widehat{b}_j)$. Set

$$\begin{cases} \widehat{\sigma} := \widehat{a}_1 \dots \widehat{a}_m \text{ and } \sigma' := a_1 \dots a_m \mathbb{1}_{\varepsilon_1} \dots \mathbb{1}_{\varepsilon_n}, \\ \widehat{\tau} := \widehat{b}_1 \dots \widehat{b}_n \text{ and } \tau' := \mathbb{1}_{\alpha_1} \dots \mathbb{1}_{\alpha_m} b_1 \dots b_n \ , \\ \gamma := \widehat{\sigma}\widehat{\tau}. \end{cases}$$

We have already seen that $\sigma' \cong \sigma$ and $\tau' \cong \tau$ in $\mathcal{NCP}$. Consider the isometries $u := u_1 \otimes \cdots \otimes u_m \otimes \mathbb{1}_{\varepsilon_1} \otimes \cdots \otimes \mathbb{1}_{\varepsilon_n} \in \mathcal{NCP}(\sigma', \gamma)$ and $v := \mathbb{1}_{\alpha_1} \otimes \cdots \otimes \mathbb{1}_{\alpha_m} \otimes v_1 \otimes \cdots \otimes v_n \in \mathcal{NCP}(\tau', \gamma)$. Note that projections uu^* and vv^* are mutually orthogonal in $\mathcal{N}CP(\gamma, \gamma)$ (since $(u_i u_i^* \ , \ e_i e_i^*)$ and $(v_j v_j^* \ , \ f_j f_j^*)$ are pairs of mutually orthogonal projections). So, we have a projection in $\mathcal{NCP}(\gamma, \gamma)$, namely $(uu^* + vv^*) \cong 1_{\sigma'} \oplus 1_{\tau'} \cong 1_\sigma \oplus 1_\tau$ in $\mathcal{C}_+ * \mathcal{C}_-$. $\square$

Often times, when our work is confined only to objects of the form $(\sigma, 1_\sigma)$, we will simply do the calculations in $\mathcal{NCP}$ via the identification of $\sigma \in \mathrm{Obj}(\mathcal{NCP})$ and $(\sigma, 1_\sigma) \in \mathrm{Obj}(\mathcal{C}_+ * \mathcal{C}_-)$.

The construction of the free product given above is explicit, but was tailored to graphical calculus considerations. There are many other possible candidates for a free product construction. To justify calling it *the* free product, we need to verify that it satisfies a universal property.

Theorem 1.4.7. *Let $\mathcal{C}$, $\mathcal{C}_+$ and $\mathcal{C}_-$ be a strict, rigid, semi-simple C*-tensor categories and $F_\pm : \mathcal{C}_\pm \to \mathcal{C}$ be unitary tensor functors. Then, there exists a triplet $\left(\widetilde{F} \ , \ \kappa_+ \ , \ \kappa_-\right)$ where $\widetilde{F} : \mathcal{C}_+ * \mathcal{C}_- \to \mathcal{C}$ is a unitary tensor functor and $\kappa_\pm : F_\pm \to \widetilde{F} \circ \iota_\pm$ are unitary monoidal natural isomorphisms. Moreover, $\left(\widetilde{F} \ , \ \kappa_+ \ , \ \kappa_-\right)$ is unique up to a unique unitary monoidal natural isomorphism compatible with $\kappa_\pm$.*

Proof. We will first construct a unitary tensor functor $G : \mathcal{NCP} \to \mathcal{C}$ such that $F_\pm = G \circ \gamma_\pm$. Suppose such a G exists. Since $\mathcal{C}$ is a semi-simple, strict C*-tensor category, by Fact 1.2.42(1) and (2), $\alpha^{\mathcal{C}} : \mathcal{C} \to \mathrm{proj}(\mathcal{C})$ is a monoidal C*-equivalence. Choose $\beta : \mathrm{proj}(\mathcal{C}) \to \mathcal{C}$ such that $\alpha^{\mathcal{C}} \circ \beta$ and $\beta \circ \alpha^{\mathcal{C}}$ are monoidally equivalent to the corresponding identity functors via natural unitaries. In particular, let $\lambda : \mathrm{id}_{\mathcal{C}} \to \beta \circ \alpha^{\mathcal{C}}$ be natural monoidal unitary. Then, $\kappa_\pm := \lambda_{F_\pm} : F_\pm \to \beta \circ \alpha^{\mathcal{C}} \circ F_\varepsilon$ is also a natural monoidal unitary. Define $\widetilde{F} := \beta \circ \mathrm{proj}(G) \ : \ \mathcal{C}_+ * \mathcal{C}_- \to \mathcal{C}$ which is a unitary tensor functor by Fact 1.2.42(3). For $\varepsilon = \pm$, the restriction of $\widetilde{F}$ to $\mathcal{C}_\varepsilon$ is given by

$$\widetilde{F} \circ \iota_\varepsilon \ = \ \beta \circ \mathrm{proj}(G) \circ \alpha^{\mathcal{NCP}} \circ \gamma_\varepsilon = \beta \circ \alpha^{\mathcal{C}} \circ G \circ \gamma_\varepsilon \ = \ \beta \circ \alpha^{\mathcal{C}} \circ F_\varepsilon \xleftarrow{\kappa_\varepsilon} F_\varepsilon$$

where the second last equality comes from Fact 1.2.42(3).

We will now construct G. While applying the functor $F_\pm$ on an object or a morphism, we will often drop the sign in the suffix and simply write F; the sign can be read off from the category $\mathcal{C}_\pm$ in which the object or the morphism belongs. For $\sigma = X_1 \ \ldots \ X_m \in \mathrm{Obj}(\mathcal{NCP})$, define

$$\begin{aligned} G\sigma &:= FX_1 \otimes \cdots \otimes FX_m \in \mathrm{Obj}(\mathcal{C}), \text{ and} \\ F\sigma &:= \text{ the word } F(X_1) \ \ldots \ F(X_m) \ . \end{aligned}$$

To define G at the level of morphisms, consider $\sigma, \tau \in \mathrm{Obj}(\mathcal{NCP})$ and a (σ,τ)-NCP T. For $\varepsilon = \pm$, suppose $J^\varepsilon : \otimes \circ (F_\varepsilon \times F_\varepsilon) \to F_\varepsilon \circ \otimes$ (resp., $\eta_\varepsilon : F_\varepsilon \mathbb{1}_\varepsilon \to \mathbb{1}$) is a natural unitary (resp., unitary) which implements the tensor- (resp., unit-) preserving property of F_ε. Again, applying F on the morphisms assigned to the partition blocks of T and composing with appropriate J^ε's and η_ε's, we get a $(F\sigma, F\tau)$-NCP which we denote by FT. Note that the letters of $F\sigma$ and $F\tau$, and the morphisms assigned to the partition blocks of FT all belong to the strict tensor category $\mathcal{C}$. Using graphical calculus of morphisms in $\mathcal{C}$, the NCP FT yield a unique morphism $Z^{\mathcal{C}}_{FT} \in \mathcal{C}(G\sigma, G\tau)$. The obvious choice of G would be

$$\mathcal{NCP}(\sigma,\tau) \in Z_T = Z_{T_+} \otimes Z_{T_-} \overset{G}{\longmapsto} Z^{\mathcal{C}}_{FT} \in \mathcal{C}(G\sigma, G\tau)$$

and extending it linearly.

We need to show that G is well-defined at the level of morphisms. Consider two objects $\sigma = X_1 \ \ldots \ X_m$ and τ in $\mathcal{NCP}$ and two (σ,τ)-NCPs S and T. For $1 \le i \le m$, choose a dual $\overline{X}_i$ of X_i, and a normalized standard solution $(R_i, \overline{R}_i)$ to the conjugate equation for the duality of $(X_i, \overline{X}_i)$. Generate the positive, faithful 'left' trace $\mathrm{tr}_{t(\sigma_\pm)}$ on the endomorphism space $\mathcal{C}_\pm\left(t(\sigma_\pm), t(\sigma_\pm)\right)$ using the R_i's. We then have the following inner product on the space $\mathcal{NCP}(\sigma,\tau)$:

$$\langle Z_S, Z_T \rangle := \mathrm{tr}_{t(\sigma_+)} \otimes \mathrm{tr}_{t(\sigma_-)} \left(Z_{T^* \circ S}\right) = \mathrm{tr}_{t(\sigma_+)} \left(Z_{(T^*\circ S)_+}\right) \quad \mathrm{tr}_{t(\sigma_-)} \left(Z_{(T^*\circ S)_-}\right) .$$

Suppose X_i lies in $\mathcal{C}_{\varepsilon_i}$ for $1 \le i \le m$. Define $R'_i := \left(J^{\varepsilon_i}_{\overline{X}_i, X_i}\right)^* \circ F(R_i) \circ \eta^*_{\varepsilon_i}$ and $\overline{R}'_i := \left(J^{\varepsilon_i}_{X_i, \overline{X}_i}\right)^* \circ F(\overline{R}_i) \circ \eta^*_{\varepsilon_i}$. One can easily check (using the equations satisfied by J^ε and η_ε in making F_ε a monoidal functor) that $(R'_i, \overline{R}'_i)$ is a solution to the conjugate equation (possibly not standard) for the duality of $(FX_i, F\overline{X}_i)$. Using these solutions in an obvious way, we generate the solution $(R', \overline{R}')$ to the conjugate equation for $\left(G\sigma, F\overline{X}_m \otimes \cdots \otimes F\overline{X}_1\right)$; for instance, $R' := \left(1_{F\overline{X}_m \otimes \cdots \otimes F\overline{X}_2} \otimes R'_1 \otimes 1_{FX_2 \otimes \cdots \otimes FX_m}\right) \cdots \left(1_{F\overline{X}_m} \otimes R'_{m-1} \otimes 1_{FX_m}\right) R'_m$. Consider the positive, faithful (possibly not tracial) functional

$$\varphi_{G\sigma} := R'^* \left(1_{F\overline{X}_m \otimes \cdots \otimes F\overline{X}_1} \otimes \bullet\right) R'$$

on the endomorphism space $\mathcal{C}(G\sigma, G\sigma)$. A careful observation and some straight-forward calculations will tell us

$$\langle Z_S, Z_T \rangle = \varphi_{G\sigma}\left(Z^{\mathcal{C}}_{FT^*\circ FS}\right) = \varphi_{G\sigma}\left(\left(Z^{\mathcal{C}}_{FT}\right)^* \circ Z^{\mathcal{C}}_{FS}\right) = \varphi_{G\sigma}\left((G(Z_T))^* \circ G(Z_S)\right)$$

where the first equality can be derived by crucially using the fact that any NCP T' can be expressed as a non-crossing overlay of T'_+ and T'_-. Thus, $G : \mathcal{NCP}(\sigma,\tau) \to \mathcal{C}(G\sigma, G\tau)$ is well-defined and injective as well.

The remaining properties for G being a unitary tensor functor is routine to verify, and so is the condition $F_\pm = G \circ \gamma_\pm$.

We are now left to establish the uniqueness part. Let $\left(H^i, \kappa^i_+, \kappa^i_-\right)$ for $i = 1, 2$ be two triplets satisfying the conditions in the statement of this theorem. We need to find a unitary natural monoidal isomorphism $\mu : H^1 \to H^2$ such that $\mu_{\iota_\varepsilon} \circ \kappa^1_\varepsilon = \kappa^2_\varepsilon$ for $\varepsilon = \pm$, and show that such a μ is unique.

The compatibility condition forces us to set

$$\mu_{\iota_\varepsilon(X)} \;\coloneqq\; \kappa^2_{\varepsilon,X}\left(\kappa^1_{\varepsilon,X}\right)^* \;\in\; \mathcal{C}\left(H^1(\iota_\varepsilon(X))\;,\; H^2(\iota_\varepsilon(X))\right) \text{ for } X \in \operatorname{Obj}(\mathcal{C}_\varepsilon).$$

Next, we intend to define μ one level higher, namely, for objects in $\mathcal{NCP}$. Consider $\sigma = (X_1, \ldots, X_m) \in \operatorname{Obj}(\mathcal{NCP})$ for $X_j \in \operatorname{Obj}(C_{\varepsilon_j})$. For this we set up the following convention which will come handy in the rest of the proof.

Notation: Let $A : \mathcal{D} \to \mathcal{E}$ be a tensor functor between strict tensor categories where $J : \otimes \circ (A \times A) \to A \circ \otimes$ is the natural transformation implementing the tensor preserving property of A. For a nonempty word $\sigma = X_1 \ \ldots \ X_m$ with letters in $\operatorname{Obj}(\mathcal{D})$, using the $J_{\bullet,\bullet}$'s iteratively, we may obtain a morphism in $\mathcal{E}$

$$J_\sigma \;:\; A(X_1) \otimes \cdots \otimes A(X_m) \;\longrightarrow\; A(X_1 \otimes \cdots \otimes X_m)$$

which is independent of any iterative algorithm by the commuting hexagonal diagram (in fact, a square due to strictness of $\mathcal{D}$ and $\mathcal{E}$) satisfied by $J_{\bullet,\bullet}$. If σ has length one, then set $J_\sigma \coloneqq 1_{A(X_1)}$. Note that J_σ is natural in the letters of σ. For $\sigma = \sigma_1 \ \ldots \ \sigma_n$ (where σ_j's are nonempty subwords of σ), we have the formula:

$$J_\sigma = J_{t(\sigma_1)\ \ldots\ t(\sigma_n)}\left(J_{\sigma_1} \otimes \cdots \otimes J_{\sigma_n}\right) \tag{1.4.1}$$

where $t(\cdot)$ continues to denote tensoring the letters from left to right.

For $\varepsilon = \pm$, $i = 1, 2$, let $J^\varepsilon_{\bullet,\bullet}$ and $J^i_{\bullet,\bullet}$ denote the natural unitaries implementing the tensor preserving property of F_ε and H^i respectively. As objects in $\mathcal{NCP}$, we have

$\sigma = \gamma_{\varepsilon_1}(X_1) \otimes \cdots \otimes \gamma_{\varepsilon_m}(X_m)$. On both sides, applying $\alpha^{\mathcal{NCP}} : \mathcal{NCP} \to \mathcal{C}_+ * \mathcal{C}_-$ (which we denote simply by α for notational convenience), we get $\alpha(\sigma) = \iota_{\varepsilon_1}(X_1) \otimes \cdots \otimes \iota_{\varepsilon_m}(X_m)$ since α is trivially monoidal. Since we are looking for a monoidal $\mu : H^1 \to H^2$, the only potential candidate must be defined as:

$$\mu_{\alpha(\sigma)} := J^2_{\iota_{\varepsilon_1}(X_1) \ \ldots \ \iota_{\varepsilon_m}(X_m)} \left(\mu_{\iota_{\varepsilon_1}(X_1)} \otimes \cdots \otimes \mu_{\iota_{\varepsilon_m}(X_m)}\right) \left(J^1_{\iota_{\varepsilon_1}(X_1) \ \ldots \ \iota_{\varepsilon_m}(X_m)}\right)^* \in \mathcal{C}\left(H^1(\alpha(\sigma)), H^2(\alpha(\sigma))\right)$$

for $\sigma \in \mathrm{Obj}(\mathcal{NCP})$. Clearly, if length of σ is 1, $\mu_{\alpha(\sigma)}$ matches with the one defined before, that is, $\mu_{\alpha\gamma_\varepsilon(\bullet)} = \mu_{\iota_\varepsilon(\bullet)}$.

It is clear (using Equation (1.4.1)) from the definition that $\mu_\alpha : H^1 \circ \alpha \to H^2 \circ \alpha$ is a monoidal unitary but we still need to check naturality of μ_α, that is, for $T \in NCP(\sigma, \tau)$, we need to prove $\mu_{\alpha(\tau)} \circ H^1(\alpha(Z_T)) = H^2(\alpha(Z_T)) \circ \mu_{\alpha(\sigma)}$. By Remark 1.4.3, it is enough to obtain $\mu_{\alpha(\tau)} \circ H^1(\alpha(Z_T)) = H^2(\alpha(Z_T)) \circ \mu_{\alpha(\sigma)}$ for elementary $T \in NCP(\sigma, \tau)$.

Let T be as in Definition 1.4.2 where the block partition (σ_j, τ_j) is labelled by $f_j \in \mathcal{C}_{\varepsilon_j}(t(\sigma_j), t(\tau_j))$ for $1 \leq j \leq n$. Let T_j denote the (σ_j, τ_j)-NCP with a single block partition labeled with f_j. Then, $Z_T = Z_{T_1} \otimes \cdots \otimes Z_{T_n}$ implying $\alpha(Z_T) = \alpha(Z_{T_1}) \otimes \cdots \otimes \alpha(Z_{T_n})$. Suppose $\sigma_j = (X^j_1, \ldots, X^j_{k_j})$ and $\tau_j = (Y^j_1, \ldots, Y^j_{l_j})$ for $1 \leq j \leq n$; here we are assuming that all σ_j's and τ_j's are nonempty. Thus,

$$H^1(\alpha(Z_T)) = J^1_{\alpha(\tau_1) \ \ldots \ \alpha(\tau_n)} \left[H^1(\alpha(Z_{T_1})) \otimes \cdots \otimes H^1(\alpha(Z_{T_n}))\right] \left(J^1_{\alpha(\sigma_1) \ \ldots \ \alpha(\sigma_n)}\right)^* .$$

Using Equation (1.4.1), $\mu_{\alpha(\tau)} \circ H^1\alpha(Z_T)$ can be expressed as

$$J^2_{\iota_{\varepsilon_1}(Y^1_1) \ \ldots \ \iota_{\varepsilon_n}(Y^n_{l_n})} \left[\bigotimes_{j=1}^{n} \underline{\left(\mu_{\iota_{\varepsilon_j}(Y^j_1)} \otimes \cdots \otimes \mu_{\iota_{\varepsilon_j}(Y^j_{l_j})}\right) \left(J^1_{\iota_{\varepsilon_j}(Y^j_1) \ \ldots \ \iota_{\varepsilon_j}(Y^j_{l_j})}\right)^* H^1(\alpha(Z_{T_j}))}\right] \left(J^1_{\alpha(\sigma_1) \ \ldots \ \alpha(\sigma_n)}\right)^* . \tag{1.4.2}$$

Now, $Z_{T_j} = \left(J^{\gamma_{\varepsilon_j}}_{Y^j_1 \ \ldots \ Y^j_{l_j}}\right)^* \ \gamma_{\varepsilon_j}(f_j) \ J^{\gamma_{\varepsilon_j}}_{X^j_1 \ \ldots \ X^j_{k_j}}$. Thus, the j-th tensor component (underlined) in the middle of the expression 1.4.2 becomes

$$\left(\kappa^2_{\varepsilon_j, Y^j_1} \otimes \cdots \otimes \kappa^2_{\varepsilon_j, Y^j_{l_j}}\right) \left(\kappa^1_{\varepsilon_j, Y^j_1} \otimes \cdots \otimes \kappa^1_{\varepsilon_j, Y^j_{l_j}}\right)^* \left(J^1_{\iota_{\varepsilon_j}(Y^j_1) \ \ldots \ \iota_{\varepsilon_j}(Y^j_{l_j})}\right)^* \left[H^1\alpha\left(J^{\gamma_{\varepsilon_j}}_{Y^j_1 \ \ldots \ Y^j_{l_j}}\right)\right]^* H^1\left(\iota_{\varepsilon_j}(f_j)\right) \ H^1\alpha\left(J^{\gamma_{\varepsilon_j}}_{X^j_1 \ \ldots \ X^j_{k_j}}\right) \tag{1.4.3}$$

From the monoidal property of $\kappa^i_\varepsilon : F_\varepsilon \to H^i \circ \iota_\varepsilon$, we get another formula

$$H^i\alpha\left(J^{\gamma_\varepsilon}_{Z_1 \ \ldots \ Z_n}\right) \ J^i_{\iota_\varepsilon(Z_1) \ \ldots \ \iota_\varepsilon(Z_n)} \ \left(\kappa^i_{\varepsilon, Z_1} \otimes \cdots \otimes \kappa^i_{\varepsilon, Z_n}\right) = \kappa^i_{\varepsilon, Z_1 \otimes \cdots \otimes Z_n} \ J^\varepsilon_{Z_1 \ \ldots \ Z_n} . \tag{1.4.4}$$

Applying Formula 1.4.4 twice (namely, for $i = 1, 2$) on the expression 1.4.3, we get

$$\left(J^2_{\iota_{\varepsilon_j}(Y^j_1)\ \dots\ \iota_{\varepsilon_j}(Y^j_{l_j})}\right)^* \left[H^2\alpha\left(J^{\gamma_{\varepsilon_j}}_{Y^j_1\ \dots\ Y^j_{l_j}}\right)\right]^* \left(\kappa^2_{\varepsilon_j,t(\tau_j)}\right)\left(\kappa^1_{\varepsilon_j,t(\tau_j)}\right)^* H^1\left(\iota_{\varepsilon_j}(f_j)\right) H^1\alpha\left(J^{\gamma_{\varepsilon_j}}_{X^j_1\ \dots\ X^j_{k_j}}\right)$$

$$= \left(J^2_{\iota_{\varepsilon_j}(Y^j_1)\ \dots\ \iota_{\varepsilon_j}(Y^j_{l_j})}\right)^* \left[H^2\alpha\left(J^{\gamma_{\varepsilon_j}}_{Y^j_1\ \dots\ Y^j_{l_j}}\right)\right]^* H^2\left(\iota_{\varepsilon_j}(f_j)\right)\ \left(\kappa^2_{\varepsilon_j,t(\sigma_j)}\right)\left(\kappa^1_{\varepsilon_j,t(\sigma_j)}\right)^* H^1\alpha\left(J^{\gamma_{\varepsilon_j}}_{X^j_1\ \dots\ X^j_{k_j}}\right)$$

where the last equality follows from naturality of $\kappa^i_{\varepsilon_j}$. Again applying Formula 1.4.4 twice and the equation $\mu_{\iota_\varepsilon(Z)} = \kappa^2_{\varepsilon,Z}\left(\kappa^1_{\varepsilon,Z}\right)^*$, the last expression becomes

$$\left(J^2_{\iota_{\varepsilon_j}(Y^j_1)\ \dots\ \iota_{\varepsilon_j}(Y^j_{l_j})}\right)^* \left[H^2\alpha\left(J^{\gamma_{\varepsilon_j}}_{Y^j_1\ \dots\ Y^j_{l_j}}\right)\right]^* H^2\left(\iota_{\varepsilon_j}(f_j)\right)\ H^2\alpha\left(J^{\gamma_{\varepsilon_j}}_{X^j_1\ \dots\ X^j_{k_j}}\right)\ J^2_{\iota_{\varepsilon_j}(X^j_1)\ \dots\ \iota_{\varepsilon_j}(X^j_{k_j})} \left(\mu_{\iota_{\varepsilon_j}(X^j_1)}\otimes\cdots\otimes\mu_{\iota_{\varepsilon_j}(X^j_{k_j})}\right)\left(J^1_{\iota_{\varepsilon_j}(X^j_1)\ \dots\ \iota_{\varepsilon_j}(X^j_{k_j})}\right)^*$$

which turns out to be

$$\left(J^2_{\iota_{\varepsilon_j}(Y^j_1)\ \dots\ \iota_{\varepsilon_j}(Y^j_{l_j})}\right)^* H^2\alpha\left(Z_{T_j}\right)\ \mu_{\alpha(\sigma_j)}\ . \tag{1.4.5}$$

Replacing the underlined part in expression 1.4.2 by 1.4.5, we get

$$J^2_{\iota_{\varepsilon_1}(Y^1_1)\ \dots\ \iota_{\varepsilon_n}(Y^n_{l_n})}\left[\bigotimes_{j=1}^n \left(J^2_{\iota_{\varepsilon_j}(Y^j_1)\ \dots\ \iota_{\varepsilon_j}(Y^j_{l_j})}\right)^* H^2\alpha\left(Z_{T_j}\right)\ \mu_{\alpha(\sigma_j)}\right]\ \left(J^1_{\alpha(\sigma_1)\ \dots\ \alpha(\sigma_n)}\right)^*$$

$$= J^2_{\alpha(\tau_1)\ \dots\ \alpha(\tau_n)}\left[\bigotimes_{j=1}^n H^2\alpha\left(Z_{T_j}\right)\ \mu_{\alpha(\sigma_j)}\right]\ \left(J^1_{\alpha(\sigma_1)\ \dots\ \alpha(\sigma_n)}\right)^* \quad \text{(using Formula 1.4.1)}$$

$$= J^2_{\alpha(\tau_1)\ \dots\ \alpha(\tau_n)}\left[\bigotimes_{j=1}^n H^2\alpha\left(Z_{T_j}\right)\right]\ \left[\bigotimes_{j=1}^n \mu_{\alpha(\sigma_j)}\right]\ \left(J^1_{\alpha(\sigma_1)\ \dots\ \alpha(\sigma_n)}\right)^*$$

$$= J^2_{\alpha(\tau_1)\ \dots\ \alpha(\tau_n)}\left[\bigotimes_{j=1}^n H^2\alpha\left(Z_{T_j}\right)\right]\ \left(J^2_{\alpha(\sigma_1)\ \dots\ \alpha(\sigma_n)}\right)^*\ \mu_{\alpha(\sigma)} \quad \text{(since } \mu_\alpha \text{ is monoidal)}$$

$$= H^2\alpha\left(Z_{T_1}\otimes\cdots\otimes Z_{T_n}\right)\ \mu_{\alpha(\sigma)}\ =\ H^2\alpha\left(Z_T\right)\ \mu_{\alpha(\sigma)}\ .$$

Finally, we have obtained a unitary natural monoidal isomorphism $\mu_\alpha : H^1\alpha \to H^2\alpha$ such that $\mu_{\alpha\gamma_\varepsilon}\ \kappa^1_\varepsilon = \kappa^2_\varepsilon$. For defining μ in the general form, consider $(\sigma, p) \in \mathrm{Obj}(\mathcal{NCP})$ for $\sigma \in \mathrm{Obj}(\mathcal{NCP})$ and projection $p \in \mathcal{NCP}(\sigma,\sigma)$. Here also, there is only one choice (by naturality), namely

$$\mu_{(\sigma,p)}\ :=\ H^2(p)\ \mu_\sigma\ H^1(p)\ \in\ \mathcal{C}\left(H^1(\sigma,p)), H^2(\sigma,p)\right)\ .$$

Since α is trivially monoidal, it almost comes for free that μ is a unitary natural monoidal isomorphism compatible with κ_ε^i for $i = 1, 2$, $\varepsilon = \pm$. Note that in our construction of μ, there is a unique choice at each stage. Hence, μ has to be unique. □

1.5 Annular Representations

Let $\mathcal{C}$ be a rigid C*-tensor category $\mathcal{C}$ and $\mathrm{Irr}(\mathcal{C})$ denote a set of representatives of isomorphism classes of simple objects in $\mathcal{C}$. We assume that $\mathbb{1} \in \mathrm{Irr}(\mathcal{C})$ is chosen to represent its isomorphism class. For each object, choose a dual object along with a standard solution to the conjugate equations in such a way that $\bar{\bar{a}} = a$ for every object a. Such a choice is always possible by a result of Yamagami ([Yam04]).

Definition 1.5.1. Let Λ be any subset of the set representatives of isomorphism classes of all objects in $\mathcal{C}$. Then the **annular algebra with weight set** Λ is defined as a vector space

$$\mathcal{A}\Lambda := \bigoplus_{b,c\in\Lambda, a\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a\otimes b, c\otimes a)$$

.

An element $f \in \mathcal{A}\Lambda$ is given by a sequence $f^a_{b,c} \in \mathcal{C}(a\otimes b, c\otimes a)$ with only finitely many terms non-zero. For a simple object $a \in \mathcal{C}$ and an arbitrary object $b \in \mathcal{C}$, we naturally have an inner product on $\mathcal{C}(a, b)$ given by $g^* f = \langle f, g\rangle 1_a$. This inner product differs by the tracial inner product by a factor of $d(a)$.

For $f \in \mathcal{C}(a_1 \otimes b_1, b_2 \otimes a_1)$ and $g \in \mathcal{C}(a_2 \otimes b_3, b_4 \otimes a_2)$, multiplication in $\mathcal{A}\Lambda$ is given by

$$f \cdot g \;:=\; \delta_{b_1=b_4} \sum_{c\in\Lambda} \sum_{u\in\mathrm{onb}(\mathcal{C}(c,a_1\otimes a_2))} (1_{b_2} \otimes u^*)(f \otimes 1_{b_2})(1_{a_1} \otimes g)(u \otimes 1_{b_3})$$

where onb denotes an orthonormal basis with respect to the inner product defined above and we may have $\mathrm{onb}(c, a_1\otimes a_2) = \emptyset$, if c is not equivalent to any sub-object of $a_1\otimes a_2$. This multiplication is associative and is independent of choice of representatives of isomorphism classes of simple objects and choice of onb in consideration.

$\mathcal{A}\Lambda$ has a $*$-structure, which we denote by $\#$, defined by

$$f^\# := (R_a^* \otimes 1_{b_1} \otimes 1_{\bar{a}})(1_{\bar{a}} \otimes f^* \otimes 1_{\bar{a}})(1_{\bar{a}} \otimes 1_{b_2} \otimes \bar{R}_a) \in \mathcal{C}(\bar{a} \otimes b_2, b_1 \otimes \bar{a})$$

for $f \in \mathcal{C}(a \otimes b_1, b_2 \otimes a)$, where R_a and $\bar{R}_a$ are solutions to the conjugate equations for a.

The associative $*$-algebra $\mathcal{A}\Lambda$ is unital if and only if $\mathrm{Irr}(\mathcal{C}) < \infty$. This algebra has a canonical trace defined by $\Omega(f) := \delta_{b=c}\ \delta_{a=\mathbb{1}}\ tr(f)$ for all $f \in \mathcal{C}(a \otimes b, c \otimes a)$, where tr

is the unnormalized categorical trace on $\mathcal{C}(b,b)$, $tr(f) := R_b^*(1_{\overline{b}} \otimes f)R_b = \overline{R}_b^*(f \otimes 1_b)\bar{R}_b$. Further, we have a normalized trace $\omega(f) := \frac{1}{d(b)}\Omega(f)$ for any such f.

We denote the subspaces

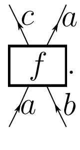
$$\mathcal{A}\Lambda^a_{b,c} := \mathcal{C}(a \otimes b, c \otimes a) \subset \mathcal{A}\Lambda \text{ and } \mathcal{A}\Lambda_{b,c} := \bigoplus_{a \in \mathrm{Irr}(\mathcal{C})} \mathcal{A}\Lambda^a_{b,c} \subseteq \mathcal{A}\Lambda$$

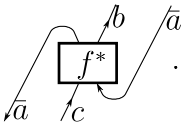
The associative $*$-algebra $\mathcal{A}\Lambda_{b,b}$ is called the **weight** b **centralizer algebra**. We call $\mathcal{A}\Lambda_{\mathbb{1},\mathbb{1}}$ the **weight** 0 **centralizer algebra**, primarily for historical reasons in connection with planar algebras. It turns out that the fusion algebra of $\mathcal{C}$, Fus($\mathcal{C}$), is $*$-isomorphic to $\mathcal{A}\Lambda_{\mathbb{1},\mathbb{1}}$ (See [GJ16, Proposition 3.1]). For each $b \in \Lambda$, if we denote the projection $p_b := 1_b \in \mathcal{C}(\mathbb{1} \otimes b, b \otimes \mathbb{1})$, then $\mathcal{A}\Lambda_{b_1,b_2} = p_{b_2}\mathcal{A}\Lambda p_{b_1}$.

The **annular category with weight set** Λ is the category with objects space as Λ and the morphism space from b to c as $\mathcal{A}\Lambda_{b,c}$. Composition is given by the multiplication defined above. Both the algebra as well as category are often denoted by $\mathcal{A}\Lambda$. Since both of these essentially contain the same information, they are used interchangeably.

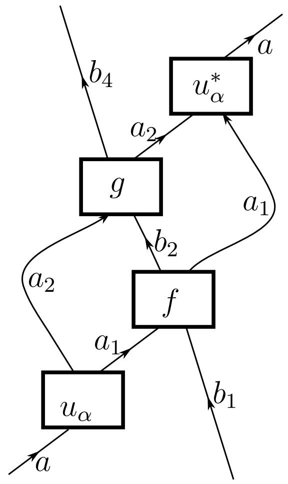
The **tube algebra**, $\mathcal{AC}$ is (by a slight abuse of notation) the annular algebra with weight set Irr($\mathcal{C}$). This algebra was first introduced by Ocneanu ([Ocn94]). A weight set $\Lambda \subseteq$ is said to be **full** if every simple object is equivalent to subobject of some $b \in \Lambda$. We have the following from [GJ16], which says that any annular algebra with full weight set is strongly Morita equivalent to the tube algebra.

Proposition 1.5.2. *If Λ is full, then $F(I) \otimes \mathcal{AC} \cong F(I) \otimes \mathcal{A}\Lambda$ as $*$-algebras, where $F(I)$ denotes the $*$-algebra spanned by the system of matrix units $\{E_{ij} \in \mathcal{L}(l^2(I)) : i,j \in I\}$ for any countable set I.*

Before we proceed further, we shall see how the graphical calculus for tensor categories extends to the context of annular algebras. For $f \in \mathcal{A}\Lambda^a_{b,c}$, we denote it by 

. Thus, $f^\# \in \mathcal{A}\Lambda^{\bar{a}}_{c,b}$ will look like 

. For $f \in \mathcal{C}(a_1 \otimes b_1, b_2 \otimes a_1), g \in \mathcal{C}(a_2 \otimes b_2, b_3 \otimes a_2), a \in$ Irr($\mathcal{C}$), the a^{th} component of the product, $(f \cdot g)_a$, is given pictorially by:



where $u_\alpha \in \text{onb}\,(\mathcal{C}(a,(a_2 \otimes a_1)))$.

Definition 1.5.3. A **non-degenerate representation** of annular algebra $\mathcal{A}\Lambda$ is a $*$-homomorphism $\pi : \mathcal{A}\Lambda \to \mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ with the property that $\pi(\mathcal{A}\Lambda)\xi = 0$ for $\xi \in \mathcal{H}$ implies $\xi = 0$. We denote the category of non-degenerate repre-sentations with bounded intertwiners $Rep(\mathcal{A}\Lambda)$.

Note that $Rep(\mathcal{A}\Lambda)$ is a W*-category. By Proposition 1.5.2, it turns out that whenever Λ is full, we have $Rep(\mathcal{A}\Lambda) \cong Rep(\mathcal{A})$ as W*-categories ([GJ16, Theorem 4.2]), and thus it makes sense to talk about *the category* of annular representations, which can be realized as the representation category of any annular algebra with full weight set. We shall see in Section 2.1 that the weight set can further be reduced in some cases without affecting the resulting category of annular representations.

Definition 1.5.4. For $b \in \Lambda$, a linear functional $\varphi : \mathcal{A}\Lambda_{b,b} \to \mathbb{C}$ is called a **weight** b **annular state** if

(i) $\varphi(p_b) = 1$,

(ii) $\varphi(f^{\#} \cdot f) \geq 0$ for every $f \in \mathcal{A}\Lambda_{b,c}$ and $c \in \Lambda$.

The collection of all weight b annular states is denoted by $\Phi\Lambda_b$ (or simply by Φ_b if we are dealing with the tube algebra).

The only difficulty in this context to have a generalization of the GNS construction is that $\mathcal{A}\Lambda$ does not have a natural norm. Thus the boundedness of the action cannot be asserted by positivity as in the realm of C*-algebras. Nevertheless, we have the following lemma from [GJ16].

Lemma 1.5.5. *Let $g \in \mathcal{A}\Lambda^a_{c,d}$ for $a \in Irr(\mathcal{C})$. Then $\varphi(f^\# \cdot g^\# \cdot g \cdot f) \leq d(a)^2 \omega(g \cdot g^\#)\varphi(f^\# \cdot f)$ for all $\varphi \in \Phi\Lambda_b$ and $f \in \mathcal{A}\Lambda_{b,c}$.*

If $\varphi \in \Phi\Lambda_b$, we define a sesquilinear form on $\widehat{\mathcal{H}}_\varphi := \bigoplus_{c \in \Lambda} \mathcal{A}\Lambda_{b,c}$ by $\langle f, g \rangle_\varphi := \varphi(g^\# \cdot f)$. This is a positive definite form and the vector space has a natural left action of $\mathcal{A}\Lambda$ given by multiplication. Quotienting with the kernel of the form, we get a Hilbert space which we denote by $\mathcal{H}_\varphi$ and the action is denoted by π_φ. By virtue of Lemma 1.5.5, π_φ acts in a bounded fashion and hence extending linearly we get $\pi_\varphi : \mathcal{A}\Lambda \to \mathcal{L}(\mathcal{H})$, a non-degenerate $*$-representation of $\mathcal{A}\Lambda$. Thus we have the following.

Corollary 1.5.6. *A linear functional $\varphi : \mathcal{A}\Lambda_{b,b} \to \mathbb{C}$ is in $\Phi\Lambda_b$ if and only if there exists a non-degenerate $*$-representation $(\pi, \mathcal{H})$ of $\mathcal{A}\Lambda$ and a unit vector ξ in $\pi(p_b)\mathcal{H}$, such that $\varphi(f) = \langle \pi(f)\xi, \xi \rangle$. Furthermore, the sub-representation $\mathcal{H}_\xi := [\pi(\mathcal{A}\Lambda)\xi] \subseteq \mathcal{H}$ is unitarily equivalent to the representation $\mathcal{H}_\varphi$ described above.*

Since an arbitrary element in the tube algebra will have its norm bounded by the constant in Lemma 1.5.5 in any representation, we can take arbitrary direct sums of representations. This allows us to define a universal representation, and a corresponding universal C*-algebra.

Definition 1.5.7. (1) The **universal representation** of the annular algebra $\mathcal{A}\Lambda$ is given by $(\pi_u, \mathcal{H}_u) := \bigoplus_{b \in \Lambda, \varphi \in \Phi\Lambda_b} (\pi_\varphi, \mathcal{H}_\varphi)$.

(2) The **universal norm** on $\mathcal{A}\Lambda$ is given by $\|x\|_u := \|\pi_u(x)\|$.

(3) The **universal C*-algebra** is the completion $\mathcal{C}^*_u(\mathcal{A}\Lambda) := \overline{\pi_u(\mathcal{A}\Lambda)}^{\|\|_u}$.

Remark 1.5.8. Non-degenerate $*$-representations of $\mathcal{A}\Lambda$ are in 1-1 correspondence with non-degenerate, bounded $*$-representations of $C^*_u(\mathcal{A}\Lambda)$ and finiteness of the universal norm follows from Lemma 1.5.5. Moreover, we have $Rep(\mathcal{A}\Lambda) \cong Rep(C^*_u(\mathcal{A}\Lambda))$

Thus one way to access the category $Rep(\mathcal{A}\Lambda)$ is to understand the representation theory of $C^*_u(\mathcal{A}\Lambda)$. In this direction, we have the following corollary ([GJ16, Corollary 4.8]) proof of which is an application of GNS type construction described earlier.

Corollary 1.5.9. *For $b \in \Lambda$ and $(\pi_b, \mathcal{H}_b)$ a non-degenerate $*$-representation of $\mathcal{A}\Lambda_{b,b}$, the following are equivalent:*

(i) Every vector state in $(\pi_b, \mathcal{H}_b)$ is a weight b annular state.

(ii) $\|\pi_b(f)\| \leq \|f\|_u$ for all $f \in \mathcal{A}\Lambda_{b,b}$.

(iii) $(\pi_b, \mathcal{H}_b)$ *extends to a representation of the unital C*-algebra* $p_b C_u^*(\mathcal{A}\Lambda)p_b$.

(iv) There exists a representation $(\pi, \mathcal{H})$ *of* $\mathcal{A}\Lambda$ *whose restriction* $(\pi, \mathcal{H})|_{\mathcal{A}\Lambda_{b,b}}$ *to* $\mathcal{A}\Lambda_{b,b}$ *is unitarily equivalent to* $(\pi_b, \mathcal{H}_b)$.

Definition 1.5.10. A representation satisfying the equivalent conditions of the previous corollary is called a **weight b admissible representation**.

Admissible representations can be seen simply as representations of the centralizer algebras which are restrictions of representations of the whole tube algebra. Alternatively, they are representations of the corner algebras which induce representations of the whole tube algebra. Further, by virtue of Corollary 1.5.6, they are in 1-1 correspondence with annular states. Understanding admissible representations for all weights allows us to understand representations of the whole tube algebra. Since the norm in weight b admissible representations is bounded by the universal norm for $\mathcal{A}\Lambda_{b,b}$ one can construct a universal C*-algebra completion $C_u^*(\mathcal{A}\Lambda_{b,b})$. From the above corollary, it is clear that $C_u^*(\mathcal{A}\Lambda_{b,b}) \cong p_b C_u^*(\mathcal{A}\Lambda)p_b$.

At this point its useful to make note of two canonical examples of a non-degenerate $*$-representation of $\mathcal{A}\Lambda$ that always exists for all categories.

Definition 1.5.11.

(i) The **left regular representation** of $\mathcal{A}\Lambda$ has Hilbert space $L^2(\mathcal{A}\Lambda, \omega)$ and action of π_ω is given by the left multiplication. Boundedness of the action follows from the fact that (i) $\omega|_{\mathcal{A}\Lambda_{b,b}}$ is an annular weight b state, hence every vector state in $\pi_\omega(p_b)\left(L^2(\mathcal{A}\Lambda, \omega)\right)$ is in $\Phi\Lambda_b$, and (ii) Lemma 1.5.5.

(ii) The one dimensional representation of $\mathcal{A}\Lambda_{\mathbb{1},\mathbb{1}}$ defined by the character $1_{\mathcal{C}}(a) = d(a)$ for all $a \in \mathrm{Irr}(\mathcal{C})$, is a weight $\mathbb{1}$ annular state (see [GJ16, Theorem 6.6]). If $(\pi_{1_{\mathcal{C}}}, \mathcal{H}_{1_{\mathcal{C}}})$ is the representation of $\mathcal{A}\Lambda$ given by this state, then $\pi_{1_{\mathcal{C}}}(a) = 0$ whenever $a \neq \mathbb{1}$ and hence all "higher weight" spaces are 0, so that $1_{\mathcal{C}}$ is a character on $\mathcal{A}\Lambda$. The **trivial representation** of $\mathcal{A}\Lambda$ is the one dimensional representation $1_{\mathcal{C}}$ of $\mathcal{A}\Lambda$.

The trivial representation will play a similar role in our representation theory to the trivial representation in the theory of groups. It can be used to define approximation and rigity properties for rigid C*-tensor categories, which we will a bit later.

Remark 1.5.12. We already know that $\mathcal{A}\Lambda_{\mathbb{1},\mathbb{1}} \cong \mathrm{Fus}(\mathcal{C}) = \mathbb{C}\left[\mathrm{Irr}(\mathcal{C})\right]$ for any weight set Λ. Moreover, it is true that if $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$, then for any full Λ, $\varphi \in \Phi\Lambda_{\mathbb{1}}$ if and only if $\varphi \in \Phi_{\mathbb{1}}$. See [GJ16, Lemma 6.1] for a proof.

We have notion of *multipliers* on rigid C*-tensor categories defined by [PV15] as below.

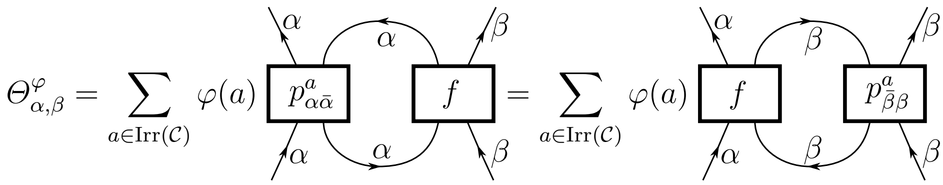
Definition 1.5.13. A **multiplier** on a rigid C*-tensor category $\mathcal{C}$ is a family of linear maps $\Theta_{\alpha,\beta} : \mathrm{End}(\alpha \otimes \beta) \to \mathrm{End}(\alpha \otimes \beta)$ for all $\alpha, \beta \in \mathrm{Obj}(\mathcal{C})$ such that

(i) Each $\Theta_{\alpha,\beta}$ is $\mathrm{End}(\alpha) \otimes \mathrm{End}(\beta)$-bimodular, and

(ii) $\Theta_{\alpha_1\otimes\alpha_2,\beta_1\otimes\beta_2}(1 \otimes f \otimes 1) = 1 \otimes \Theta_{\alpha_2,\beta_1}(f) \otimes 1$ for all $\alpha_i, \beta_i \in \mathcal{C}$, $f \in \mathrm{End}(\alpha_2 \otimes \beta_1)$.

A multiplier is a **cp-multiplier** if each $\Theta_{\alpha,\beta}$ is completely positive.

In [PV15, Proposition 3.6] it was shown that multipliers are in one-one correspondence with functions $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$. Moreover, any such φ gives a multiplier $\Theta^{\varphi}_{\alpha,\beta}$ as follows: For $f \in \mathrm{End}(\alpha \otimes \beta)$,



where $p^a_{\alpha\bar{\alpha}} := \sum_{u \in \mathrm{onb}(\mathcal{C}(\alpha\bar{\alpha},a))} u^*u$, a central projection in $\mathrm{End}(\alpha \otimes \bar{\alpha})$.

Popa and Vaes show that every multiplier is of this form and that if $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ is a cp-multiplier, then $d(\cdot)\varphi(\cdot) : \mathbb{C}\,[\mathrm{Irr}(\mathcal{C})] \to \mathbb{C}$ is a state on the fusion algebra. By a slight abuse of notation, in [GJ16], they call a function φ is a cp-multiplier if Θ^{φ} is a cp-multiplier. With this setup, Popa-Vaes' ([PV15]) definition of admissibility is as below.

Definition 1.5.14.

(1) A function $\varphi : \mathrm{Irr}(\mathcal{C}) \to \mathbb{C}$ is called an **admissible state** if $\varphi(\cdot)d(\cdot)$ is a cp-multiplier.

(2) A (non-degenerate) $*$-representation π of $\mathcal{A}\Lambda_{\mathbb{1},\mathbb{1}} = \mathbb{C}\,[\mathrm{Irr}(\mathcal{C})]$ is called **admissible** if every vector state in the representation is admissible.

(3) Define $\|\cdot\|_u := \sup_{\text{admissible } \pi} \|\cdot\|_{\pi_u}$.

Remark 1.5.15. $C^*_u(\mathcal{C})$ is defined as the completion of $A\Lambda_{\mathbb{1},\mathbb{1}} = \mathbb{C}\,[\mathrm{Irr}(\mathcal{C})]$ with respect to this universal norm. It is shown in [PV15] that this is finite and a C*-norm.

Then we have the following result from [GJ16] which links both the notions of admissibility.

Theorem 1.5.16. *(i) φ is a weight $\mathbb{1}$ annular state if and only if φ is admissible in the sense of Definition 1.5.14.*

(ii) $(\pi, \mathcal{H})$ be a $*$-representation of the fusion algebra $\mathbb{C}\left[Irr(\mathcal{C})\right]$. Then the following are equivalent:

(a) $(\pi, \mathcal{H})$ is admissible in the sense of Definition 1.5.4, namely, there exists a non-degenerate $*$-representation of $\mathcal{A}$ which restricted to $\mathcal{A}_{\mathbb{1},\mathbb{1}}$ is unitarily equivalent to $(\pi, \mathcal{H})$.

(b) $(\pi, \mathcal{H})$ is admissible in the sense of Popa and Vaes, Definition 1.5.14.

(iii) $C_u^*(\mathcal{C}) \cong C_u^*(\mathcal{A}_{\mathbb{1},\mathbb{1}}) \cong C_u^*(\mathcal{A}\Lambda_{\mathbb{1},\mathbb{1}})$ for any full Λ.

This allows import of Popa-Vaes' ([PV15]) definitions of approximation and rigidity properties into the world of annular representations.

Definition 1.5.17 ([PV15])**.** A rigid C*-tensor category (with $\mathrm{Irr}(\mathcal{C})$ countable) is said

(i) to be **amenable** if there exists a sequence of finitely supported weight $\mathbb{1}$ annular states φ_n that converges to $1_{\mathcal{C}}$ pointwise on $\mathrm{Irr}(\mathcal{C})$.

(ii) to have **property (T)** if for every sequence of annular states φ_n which converges pointwise to $1_{\mathcal{C}}$, the sequence of functions $\varphi(\cdot)d(\cdot)$ converges uniformly to 1 on $\mathrm{Irr}(\mathcal{C})$.

(iii) to have the **Haagerup property** if there exists a sequence of annular states φ_n each of which vanish at ∞ (for every ε, there exists a finite subset $K \subseteq \mathrm{Irr}(\mathcal{C})$ such that $\left|\frac{\varphi(a)}{d(a)}\right| < \varepsilon$ for all $a \in K^c$), which converge to $1_{\mathcal{C}}$ pointwise.

It is shown in [PV15] that these definitions are equivalent to the usual ones given in terms of symmetric enveloping algebras in the case where $\mathcal{C}$ is the even part of some subfactor standard invariant. Popa and Vaes also give several very interesting examples of categories with each of these approximation properties.

Chapter 2

Annular Representations of Free Product Categories

2.1 Annular Algebra of Free Product of Categories

We will characterize the annular algebra of $\mathcal{C} * \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are rigid, semi-simple C*-tensor categories with simple unit objects. We note that while providing definitions of the free product $\mathcal{C}_{\pm}$ was more convenient to distinguish the two categories, while in this section, using $\mathcal{C}$ and $\mathcal{D}$ seems better. By [GJ16], the annular representation category can be obtained from representations of any annular algebra with respect to any full weight set in $\mathrm{Obj}(\mathcal{C} * \mathcal{D})$ (in particular, a set of representatives of the isomorphism classes of simple objects).

However, in our case, we can actually work with a smaller, non-full weight set, and still capture the entire category. To describe this weight set, let $\mathbf{I}_{\mathcal{C}}$ (respectively $\mathbf{I}_{\mathcal{D}}$) be a set of representatives of the isomorphism classes of simple objects in $\mathcal{C}$ (respectively $\mathcal{D}$) *excluding the isomorphism class of the unit object*. Recall that the set of words (including the empty one) with letters coming alternatively from $\mathbf{I}_{\mathcal{C}}$ and $\mathbf{I}_{\mathcal{D}}$ is in bijective correspondence with the set of isomorphism classes of simple objects $\mathrm{Irr}(\mathcal{C} * \mathcal{D})$, where the empty word corresponds to the tensor unit in $\mathcal{C} * \mathcal{D}$. We define $\mathbf{W}$ to be the subset of these words with *strictly positive and even* length, such that the first letter comes from $\mathbf{I}_{\mathcal{C}}$. We will say a positive length word is a $\mathcal{C}$-$\mathcal{D}$ word if it starts with a letter of $\mathcal{C}$ and ends with a letter of $\mathcal{D}$, and extend this terminology in the obvious way. We define the weight set $\Lambda := \{\emptyset\} \cup \mathbf{I}_{\mathcal{C}} \cup \mathbf{I}_{\mathcal{D}} \cup \mathbf{W}$, which we note is not full. Indeed, the alternating words of odd length and the alternating words of even length starting with a letter from $\mathbf{I}_{\mathcal{D}}$ do not appear in Λ. Nevertheless, we have the following result:

Lemma 2.1.1. *Rep($\mathcal{A}\Lambda$) and the representation category of the tube algebra $\mathcal{A}$ of $\mathcal{C} * \mathcal{D}$, are unitarily equivalent as linear $*$-categories.*

Proof. Clearly, the restriction functor $Res : Rep(\mathcal{A}) \longrightarrow Rep(\mathcal{A}\Lambda)$ is a linear $*$-functor. We begin by showing that Res is essentially surjective.

Given a representation (π, V) of $\mathcal{A}\Lambda$ and $w \in \text{Irr}(\mathcal{C} * \mathcal{D})$, we consider the vector space $\bigoplus_{v \in \Lambda} \{\mathcal{A}_{v,w} \bigotimes V_v\}$. We define a sesquilinear form $\langle \cdot, \cdot \rangle$ on this vector space by $\langle y_1 \otimes \xi_1, y_2 \otimes \xi_2 \rangle_w := \langle \pi(y_2^{\#} \cdot y_1)\xi_1, \xi_2 \rangle_{v_2}$, where $y_i \in \mathcal{A}_{v_i,w}$ and $\xi_i \in V_{v_i}$.

We first want to show that $\langle x, x \rangle_w \geq 0$ for any vector $x = \sum_{i=1}^{n} y_i \otimes \xi_i$. But we have $\langle x, x \rangle_w = \langle T\xi, \xi \rangle$, where $T = \left(\pi(y_i^{\#} \cdot y_j)\right)_{i,j} : \bigoplus_{i=1}^{n} V_{v_i} \to \bigoplus_{i=1}^{n} V_{v_i}$, and $\xi = (\xi_i)_i \in \bigoplus_{i=1}^{n} V_{v_i}$.

If $w \in \Lambda$, then T is clearly a positive operator and hence we have non-negativity of $\langle x, x \rangle_w$. Suppose now that w has even length and its first letter is in $\mathbf{I}_{\mathcal{D}}$, say $w = d_1 c_1 d_2 c_2 \ldots d_k c_k$. Consider the word $w' = c_1 d_2 c_2 \ldots d_k c_k d_1 \in \Lambda$. Let $\rho \in \mathcal{A}_{w',w}$ be the canonical rotation unitary. Then, for any $y \in \mathcal{A}_{v,w}$, there is a unique $y' \in \mathcal{A}_{v,w'}$ such that $y = \rho \cdot y'$. Thus we have

$$T = \left(\pi(y_i^{\#} \cdot y_j)\right)_{i,j} = \left(\pi\left((\rho \cdot y_i')^{\#} \cdot (\rho \cdot y_j')\right)\right)_{i,j} = \left(\pi(y_i'^{\#} \cdot y_j')\right)_{i,j},$$

hence positivity follows from the previous case. Defining $\overline{\Lambda}$ to be the union of Λ and the set of words of even length (regardless of starting character), we have just shown positivity for weights in $\overline{\Lambda}$.

Now suppose w has odd length; say $w = a_{-k} \ldots a_{-1} a_0 a_1 \ldots a_k$. Note that the a_{2l}'s are either all in $\mathbf{I}_{\mathcal{C}}$ or all in $\mathbf{I}_{\mathcal{D}}$, and similarly for the odd letters. Now define the word $w' = a_0 a_1 \ldots a_k a_{-k} \ldots a_{-1}$. This word no longer represents an isomorphism class of simple object, however the object it represents is isomorphic to a direct sum of simple objects, all of which have even length, i.e., $w' \cong \oplus_s u_s$, where $u_s \in \overline{\Lambda}$. Let $p_s \in (\mathcal{C} * \mathcal{D})(u_s, w')$ be isometries such that $\sum_s p_s p_s^* = 1_{w'}$ (which automatically implies $p_s^* p_t = \delta_{s,t} \, 1_{u_s}$).

Let $\mathcal{A}\mathbf{Obj}$ denote the annular algebra whose weight set consists of all isomorphism classes of objects in $\mathcal{C} * \mathcal{D}$, and pick any rotation $\rho \in \mathcal{A}\mathbf{Obj}_{w',w}$ (which is automatically unitary). Then any element $y_i \in \mathcal{A}_{v_i,w}$ can be written $y_i = \sum_s \rho \cdot p_s \cdot y'_{s,i}$, where $y'_{s,i} := p_s^* \cdot \rho^{\#} \cdot y_i \in \mathcal{A}_{v_i,u_s}$. Observe that

$$\begin{aligned} T = \left(\pi(y_i^{\#} \cdot y_j)\right)_{i,j} &= \left(\pi\left(\left[\sum_s \rho \cdot p_s \cdot y'_{s,i}\right]^{\#} \cdot \left[\sum_t \rho \cdot p_t \cdot y'_{t,j}\right]\right)\right)_{i,j} \\ &= \sum_t \left(\pi\left([y'_{t,i}]^{\#} \cdot y'_{t,j}\right)\right)_{i,j} \end{aligned}$$

which is positive as all u_t's are in $\overline{\Lambda}$ and hence our argument is complete.

Now that we have shown $\langle x, x\rangle_w \geq 0$, we can define $\mathrm{Ind}(V)_w$ as the Hilbert space obtained by the completion of the quotient of our vector space over the null space of the inner product. Before quotienting and completing, our vector space has the obvious action of $\mathcal{A}$. Our above argument shows that $\langle \pi(\cdot)x, x\rangle_w$ is a positive annular functional. Thus by [GJ16, Lemma 4.4], we have a well-defined, bounded, $*$-action of the tube algebra $\mathcal{A}$ on $\mathrm{Ind}(V)$. It is now easy to verify that $Res \circ \mathrm{Ind}(V) \cong V$ via the interwiner defined by sending $\sum_i y_i \otimes \xi_i$ to $\pi(y_i)\xi_i$.

Now to prove that Res is fully faithful, we first claim that any representation $(\theta, \mathcal{H}) \in \mathrm{Rep}(\mathcal{A})$ is generated by $\bigcup\limits_{w\in\Lambda} \mathcal{H}_w$. We need to check

$$\mathcal{H}^0_w := \mathrm{span}\left\{\theta(x)\xi : x \in \mathcal{A}_{v,w},\ \xi \in \mathcal{H}_v,\ v \in \Lambda\right\}$$

is dense in $\mathcal{H}_w$ for all $w \in \mathrm{Irr}(\mathcal{C} * \mathcal{D}) \setminus \Lambda$; we will, in fact, show $\mathcal{H}^0_w = \mathcal{H}_w$. Now, $w \in \mathrm{Irr}(\mathcal{C} * \mathcal{D}) \setminus \Lambda$ implies $|w| \geq 2$. Suppose w is of $\mathcal{D}$-$\mathcal{C}$ type, so that $w = du$ for some u of $\mathcal{C}$-$\mathcal{C}$ type. We have the unitary rotation

$$\rho := 1_d \otimes 1_u \otimes 1_d \in (\mathcal{C} * \mathcal{D})\,(dw', wd) = \mathcal{A}^d_{w',w} \subset \mathcal{A}_{w',w},$$

where $w' = ud \in \Lambda$, whose θ-action implements a unitary from $\mathcal{H}_{w'}$ to $\mathcal{H}_w$; so, $\mathcal{H}^0_w = \mathcal{H}_w$.

The remaining elements of $\mathrm{Irr}(\mathcal{C} * \mathcal{D}) \setminus \Lambda$ are words of types $\mathcal{C}$-$\mathcal{C}$ or $\mathcal{D}$-$\mathcal{D}$ type, which neccessarily have odd length ≥ 3. Consider such a w, say $w = a_{-k} \ldots a_{-1}a_0a_1 \ldots a_k$. As above, the even a_i's are either all in $\mathbf{I}_\mathcal{C}$ or all in $\mathbf{I}_\mathcal{D}$. Let $w' := a_0a_1 \ldots a_k \otimes a_{-k} \ldots a_{-1}$ or $a_1 \ldots a_k \otimes a_{-k} \ldots a_{-1}a_0$ depending on whether $a_0 \in \mathbf{I}_\mathcal{C}$ or $\mathbf{I}_\mathcal{D}$, and ρ' be the rotation unitary from w to w'. Note that w' may no longer be simple; however, it decomposes into a direct sum of simple objects all of which either have even length or lie in Λ (using the fusion rule). Suppose $w' \cong \underset{i}{\oplus} u_i$ is the simple object decomposition. Let $p_i \in (\mathcal{C} * \mathcal{D})\,(u_i, w')$ be isometries such that $\sum_s p_ip_i^* = 1_{w'}$. Set $x_i := (\rho')^{\#} \cdot p_i \in \mathcal{A}_{u_i,w}$. Clearly, $\sum_i x_i \cdot x_i^{\#} = 1_w$ (in $\mathcal{A}_{w,w}$). Since the u_i's belong to Λ, any $\xi \in \mathcal{H}_w$ can be expressed as $\sum_i \theta(x_i)[\theta(x_i^{\#})\xi] \in \mathcal{H}^0_w$.

Thus our claim that any representation is generated by the Λ weight spaces is proven. This immediately implies that the restriction functor is faithful. It also shows that Res is full. Indeed, consider a morphism $f : Res(\pi, \mathcal{H}) \to Res(\gamma, \mathcal{K})$ in $Rep(\mathcal{A}\Lambda)$. For $w \in \mathrm{Irr}(\mathcal{C}*\mathcal{D}) \setminus \Lambda$, if an $\mathcal{A}$-linear extension of f exists we see that $f(\sum \pi(y_i)\xi_i) = \sum \gamma(y_i)f(\xi_i)$, for $y_i \in \mathcal{A}_{v,w}$, $v \in \Lambda$, and $\xi \in \mathcal{H}_v$. Indeed, this will serve as a definition of the extension, but we must show it is well defined. Suppose $\sum\limits_i \pi(y_i)\xi_i = 0$. Then for any fixed j, $\sum\limits_i \pi(y_j^{\#} \cdot y_i)\xi_i = 0$. Since $y_j^{\#} \cdot y_i \in \mathcal{A}\Lambda$, we have

$$\sum_{i,j} \langle \gamma(y_i) f(\xi_i), \gamma(y_j) f(\xi_j) \rangle_{\mathcal{K}} = \langle \gamma(y_j^{\#} \cdot y_i) f(\xi_i), f(\xi_j) \rangle_{\mathcal{K}}$$

$$= \sum_j \sum_i \langle \pi(y_j^{\#} \cdot y_i) \xi_i, f^* f(\xi_j) \rangle_{\mathcal{H}} = 0$$

It is easy to see that the extension of f remains bounded. This concludes the proof. □

We proceed to the study of the $*$-algebra $\mathcal{A}\Lambda$. We divide this into subsections corresponding to the length (denoted by $|\cdot|$) of the words in Λ. Since the empty word (that is, zero length word) stands for the tensor unit of $\mathcal{C} * \mathcal{D}$, the centralizer algebra $\mathcal{A}\Lambda_{\emptyset,\emptyset}$ is isomorphic to the fusion algebra, and we will be able to describe admissible representations of these in terms of representations of free product C*-algebras. Thus in this section, we will focus on the structure of $\mathcal{A}\Lambda_{v,w}$ for words $v, w \in \Lambda$ of positive length. By $\mathcal{AC}$ (resp., $\mathcal{AD}$) we mean the tube algebra/category of $\mathcal{C}$ (resp. $\mathcal{D}$).

2.1.1 Words of length at least 2

Define a relation on $\mathbf{W}$ by $w_1 \sim w_2$ if and only if $w_1 = uv$ and $w_2 = vu$ for some subwords u, v. Clearly, $\sim$ defines an equivalence relation on $\mathbf{W}$. Obviously if $w_1 \sim w_2$, then $|w_1| = |w_2|$.

Lemma 2.1.2. *For $w_1, w_2 \in \boldsymbol{W}$, $\mathcal{A}\Lambda_{w_1.w_2} \neq \{0\}$ if and only if $w_1 \sim w_2$.*

Proof. Suppose $w_1 \sim w_2$ so that $w_1 = uv$ and $w_2 = vu$. Consider the rotation $\rho := (1_v \otimes \bar{R}_u)(R_u^* \otimes 1_v) \in (\mathcal{C} * \mathcal{D})\,(\bar{u} w_1, w_2 \bar{u}) \subseteq \mathcal{A}_{w_1,w_2}$ for any standard solution $(R_u, \overline{R}_u)$ to the conjugate equation for $(u, \overline{u})$. It is non-zero (since it is unitary) and hence $\mathcal{A}\Lambda_{w_1.w_2} \neq \{0\}$.

Now suppose $\mathcal{A}\Lambda_{w_1.w_2} \neq \{0\}$ and without loss of generality, let $w_1 \neq w_2$. Then there exists $v \in \mathrm{Irr}(\mathcal{C} * \mathcal{D})$ (of length, say, $m > 0$) such that $\mathcal{A}\Lambda^v_{w_1.w_2} \neq \{0\}$. Suppose m is odd. Then v is either of $\mathcal{C}$-$\mathcal{C}$ type or $\mathcal{D}$-$\mathcal{D}$ type. If v is of $\mathcal{C}$-$\mathcal{C}$ type (resp. $\mathcal{D}$-$\mathcal{D}$ type), then $w_2 v$ (resp. $v w_1$) is simple and is of odd length, whereas $v w_1$ (resp. $w_2 v$) is not simple and any simple subobject will be of length strictly smaller than that of $v w_1$. Hence $\mathcal{A}\Lambda^v_{w_1.w_2} = \{0\}$ which is a contradiction. So m cannot be odd.

Thus m must be even, so v can be of $\mathcal{C}$-$\mathcal{D}$ or $\mathcal{D}$-$\mathcal{C}$ type. It is enough to consider the case where v is of $\mathcal{C}$-$\mathcal{D}$ type, since the other case will follow by taking #. As $w_1, w_2 \in \mathbf{W}$, $v w_1$ and $w_2 v$ are simple. Therefore, $\mathcal{A}\Lambda^v_{w_1.w_2} \neq \{0\}$ implies the equality

$$v w_1 = w_2 v \tag{2.1.1}$$

In particular, we see that w_1 and w_2 have the same length, say n.

If $m = n$, then Equation 2.1.1 implies $w_1 = v = w_2$ which is not possible by assumption. Suppose $m < n$. By Equation 2.1.1, there exists a word u such that $w_2 = vu$. So, $vw_1 = vuv$ implying $w_1 = uv$, and thus $w_1 \sim w_2$.

We are left with the case when $m > n$. Equation 2.1.1 tells us that v starts with the subword w_2; say $v = w_2 v'$. Plugging this into Equation 2.1.1, we get $v'w_1 = w_2 v'$. Note that $|v'| = n - m$. If length of v' is not less than or equal to n, then we repeat the above argument with v'. Since $|v'| < |v|$, we will eventually find some tail-end subword of v, say v_0, such that $v_0 w_1 = w_2 v_0$ with $|v_0| \leq n$. Then we apply the previous cases. □

Using similar techniques, we also have the following lemma:

Lemma 2.1.3. *Let $w \in \boldsymbol{W}$. For any $v \in \Lambda \setminus \boldsymbol{W}$, $\mathcal{A}_{v,w} = \{0\}$.*

Proof. First we consider the case $v = \emptyset$. In general, $\mathcal{A}_{\emptyset,w} \neq \{0\}$ implies that w is an object in the adjoint sub-category of $\mathcal{C} * \mathcal{D}$, or in other words, w is isomorphic to a sub-object of $u\bar{u}$ for some simple object $u \in \mathcal{C} * \mathcal{D}$. If u is length 0, then obviously $|w| = 0$, a contradiction. If u has length greater than or equal to 1, as every word that appears as a sub-object of $v\bar{v}$ is of $\mathcal{C}$-$\mathcal{C}$ or $\mathcal{D}$-$\mathcal{D}$ type, w cannot be a sub-object of $u\bar{u}$, which implies that $\mathcal{A}_{\emptyset,w} = \{0\}$.

Now we consider the case that v has length 1. First assume $v \in \mathcal{C}$. If $\mathcal{A}_{v,w} \neq \{0\}$, then there is some word u so that $(\mathcal{C} * \mathcal{D})(uv, wu) \neq \{0\}$, which is equivalent to $(\mathcal{C} * \mathcal{D})(v\bar{u}, \bar{u}w) \neq \{0\}$. First suppose $|u|$ is odd. If it is of $\mathcal{C}$-$\mathcal{C}$ type, then wu is simple, and uv is isomorphic to a direct sum of simple objects each of which have length strictly smaller than the length of wu, so the morphism space must be 0. Similarly if u is of $\mathcal{D}$-$\mathcal{D}$ type, then so is $\bar{u}$, and our hypothesis implies $(\mathcal{C} * \mathcal{D})(v\bar{u}, \bar{u}w) \neq \{0\}$. In this case, both words are simple, but $|v\bar{u}| < |\bar{u}w|$, and thus the morphism space must be $\{0\}$.

Thus we are left to consider the case when $|u|$ is even. If u is $\mathcal{C}$-$\mathcal{D}$ type, then wu is simple, and the length is strictly greater than the length of any subobject of uv (since $|v| = 1$) a contradiction. If u is $\mathcal{D}$-$\mathcal{C}$ type, then $\bar{u}w$ is simple with length strictly greater than the length of any simple sub-object of $v\bar{u}$.

The case with $v \in \mathcal{D}$ is entirely analogous. □

Lemma 2.1.4. *For $w \in \boldsymbol{W}$, the centralizer algebra $\mathcal{A}\Lambda_{w,w}$ is isomorphic to the group algebra $\mathbb{C}[\mathbb{Z}]$ as $*$-algebras.*

Proof. Let v be a subword of w such that $w = v^k = \underbrace{vv \ldots v}_{k\text{-times}}$, for largest possible positive integer k. We will say that w is *maximally periodic with respect to* v. Note that v must be of $\mathcal{C}$-$\mathcal{D}$ type. Consider the (unitary) rotation

$$\rho^v_{w,w} := 1_{v^{k+1}} \in (\mathcal{C} * \mathcal{D})(vw, wv) = \mathcal{A}\Lambda^v_{w,w}$$

whose inverse is given by

$$\left(\rho^v_{w,w}\right)^{\#} = (1_{v^{k-1}} \otimes \overline{R}_v)(R_v^* \otimes 1_{v^{k-1}}) \in (\mathcal{C} * \mathcal{D})(\overline{v}w, w\overline{v}) = \mathcal{A}\Lambda^{\overline{v}}_{w,w}$$

for any standard solution $(R_v, \overline{R}_v)$ of the conjugate equation for $(v, \overline{v})$.

Note that for any $n \in \mathbb{Z}$, $\left(\rho^v_{w,w}\right)^n \in \mathcal{A}\Lambda^{v^n}_{w,w}$ with the convention $v^{-1} = \bar{v}$ and $v^0 := \mathbb{1}$. Since $v^m \neq \mathbb{1}$ for any non-zero m, we have $\Omega\left(\left(\rho^v_{w,w}\right)^m\right) = 0$ for any such m. Thus, $\left\{\left(\rho^v_{w,w}\right)^n : n \in \mathbb{Z}\right\}$ is an orthogonal sequence in $\mathcal{A}\Lambda_{w,w}$ with respect to the canonical trace. Hence, we have an injective homomorphism from $\mathbb{C}[\mathbb{Z}]$ to $\mathcal{A}\Lambda$ sending the generator of $\mathbb{Z}$, which we denote g, to $\rho^v_{w,w}$. It remains to show that the homomorphism is surjective.

We now claim that if $u \in \text{Irr}(\mathcal{C} * \mathcal{D})$, then $\mathcal{A}\Lambda^u_{w,w} = (\mathcal{C} * \mathcal{D})(uw, wu) \neq \{0\}$ if and only if $u = v^n$ for some $n \in \mathbb{Z}$.

By the same argument as in proof of "if" part of Lemma 2.1.2, it is easy to deduce that u must be any one of $\mathcal{C}$-$\mathcal{D}$ or $\mathcal{D}$-$\mathcal{C}$ types if $\mathcal{A}\Lambda^u_{w,w} = (\mathcal{C} * \mathcal{D})(uw, wu) \neq \{0\}$. It suffices to consider the case of $\mathcal{C}$-$\mathcal{D}$ type u, since the other case will follow from this by applying $\#$.

Since both u and w are of $\mathcal{C}$-$\mathcal{D}$ type, both uw and wu are simple, $(\mathcal{C} * \mathcal{D})(uw, wu) \neq \{0\}$ implies $uw = wu$. Now, consider the bi-infinite word $\ldots uwuwuw \ldots$. If $m = |u|$ and $n = |w|$, then by the commutation of u and w, we may conclude that the infinite word is both m- and n-periodic, and thereby, $l := \gcd(m, n)$-periodic. So, there exists a word v' of length l such that both u and w are integral powers of v'. Since w is maximally periodic with respect to v, $|v| \leq |v'|$, which will then imply that v' is an integral power of v. Hence, u is an integral power of v.

We will be done if we can show $\mathcal{A}\Lambda^{v^n}_{w,w} = \mathbb{C}\rho^{v^n}_{w,w}$ for $n \in \mathbb{Z}$. Again, it is enough to show for $n \geq 0$ since the other cases follow by taking $\#$. Now for $n \geq 0$, we have $\mathcal{A}\Lambda^{v^n}_{w,w} = (\mathcal{C} * \mathcal{D})(v^{k+n}, v^{k+n})$. Since $w = vv \ldots v$, v must be an even length word with first letter from $\mathbf{I}_{\mathcal{C}}$. This implies that, any power of v is simple. In particular, v^{k+n} is simple and hence $\mathcal{A}\Lambda^{v^n}_{w,w} = (\mathcal{C} * \mathcal{D})(v^{k+n}, v^{k+n})$ is one-dimensional. □

Via the inclusion $\mathbf{W} \subset \Lambda$, we may consider $\mathcal{A}\mathbf{W}$ as a $*$-subalgebra of $\mathcal{A}\Lambda$. In fact, by Lemma 2.1.3, we see that $A\mathbf{W}$ is actually a summand of $A\Lambda$. The above lemma now allows us to identify $A\mathbf{W}$. Let $\mathbf{W}_0 = \mathbf{W}/\sim$, the set of equivalence classes of words in $\mathbf{W}$ modulo the cyclic relation $\sim$ defined in the beginning of this section. Recall that $M_n(\mathbb{C})$ denotes the algebra of $n \times n$ matrices.

Corollary 2.1.5. *$\mathcal{A}\boldsymbol{W}$ is a direct summand of the algebra $\mathcal{A}\Lambda$. Moreover, as $*$-algebras*

$$\mathcal{A}\boldsymbol{W} \cong \bigoplus_{[w]\in \boldsymbol{W}_0} M_{|w|}(\mathbb{C}) \otimes \mathbb{C}[\mathbb{Z}].$$

Proof. As explained above, the first statement follows from Lemma 2.1.3.

For the second one, we pick a representative $w \in [w] \in \mathbf{W}_0$. Then for any other $v \in [w]$, it is clear from Lemma 2.1.4 that $\mathcal{A}\mathbf{W}_{w,v} \cong \mathbb{C}[\mathbb{Z}]$ as a vector space, where $\mathbb{Z}$ is identified with powers of unitary rotation operators $\sigma_v \in \mathcal{A}\Lambda_{w,v}$ for all $v \in [w]$. Note that $\mathcal{A}\mathbf{W}_{w,v} = \{0\}$ for $v \notin [w]$ by Lemma 2.1.2.

The required isomorphism is given by the map defined for $w_1, w_2 \in [w]$ and $x \in \mathcal{A}S_{w_1,w_2}$ by

$$x \longmapsto E_{w_1,w_2} \otimes \sigma_{w_2}\, x\, \sigma_{w_1}^{\#} \in M_{|w|}(\mathbb{C}) \otimes \mathcal{A}\Lambda_{w,w} \cong M_{|w|}(\mathbb{C}) \otimes \mathbb{C}[\mathbb{Z}].$$

□

2.1.2 Words of length 1

For a rigid C*-tensor category $\mathcal{C}$, we let $\mathbf{S}(\mathcal{C}) := \{[a] \in \mathrm{Irr}(\mathcal{C}) \ : \ N_{b\bar{b}}^{a} \neq 0 \text{ for some } [b] \in \mathrm{Irr}(\mathcal{C})\}$. Observe that $\mathbf{S}(\mathcal{C})$ tensor generates the *adjoint subcategory* of $\mathcal{C}$, which is the trivial graded component with respect to the universal grading group, but in general $\mathbf{S}(\mathcal{C})$ gives a proper subset of the simple objects in the adjoint subcategory.

Lemma 2.1.6. *Let $w \in \boldsymbol{I}_{\mathcal{C}}$. Then $\mathcal{A}\Lambda_{\emptyset,w} \neq \{0\}$ if and only if w belongs to $\boldsymbol{S}(\mathcal{C})$. The same holds replacing $\mathcal{C}$ with $\mathcal{D}$.*

Proof. Suppose $w \in \mathbf{S}(\mathcal{C})$. Then there is a simple v such that $\{0\} \neq (\mathcal{C} * \mathcal{D})\,(v, wv) = \mathcal{A}\Lambda^{v}_{\emptyset,w}$ implying, $\mathcal{A}\Lambda_{\emptyset,w} \neq \{0\}$.

Now suppose $\mathcal{A}\Lambda_{\emptyset,w} \neq \{0\}$. Choose $v \in \mathrm{Irr}(\mathcal{C} * \mathcal{D}) \setminus \{\mathbb{1}\}$ such that $\mathcal{A}\Lambda^{v}_{\emptyset,w} = (\mathcal{C} * \mathcal{D})\,(v, wv) \neq \{0\}$. By arguments as in the proof of Lemma 2.1.2, one can see that v must be of $\mathcal{C}$-$\mathcal{C}$ or $\mathcal{C}$-$\mathcal{D}$ type for the morphism space to be non-zero. Let $v = cv'$ with $c \in \mathbf{I}_{\mathcal{C}}$. If $v' = \mathbb{1}$, then we are done. Suppose $|v'| \geq 1$; so, v' starts in $\mathbf{I}_{\mathcal{D}}$. Consider the simple objects $\{u_i : i = 0, 1, \dots n\} \subset \mathrm{Irr}(\mathcal{C} * \mathcal{D})$ that appear as subobjects in the decomposition of $v'\bar{v}'$, with $u_0 = \mathbb{1}$. Note that, for $i \geq 1$, u_i is non-trivial and is of $\mathcal{D}$-$\mathcal{D}$ type (since v' is simple). Thus, for all $i \geq 1$, $cu_i\bar{c}$ is simple and is of length greater than 1, implying $(\mathcal{C} * \mathcal{D})\,(w, cu_i\bar{c}) = \{0\}$. Since $\{0\} \neq (\mathcal{C} * \mathcal{D})\,(v, wv) \cong (\mathcal{C} * \mathcal{D})\,(c\,v'\,\bar{v}'\,\bar{c}, w)$, we must have $\mathcal{C}(c\bar{c}, w) = (\mathcal{C} * \mathcal{D})\,(c\bar{c}, w) \neq \{0\}$. So $w \in \mathbf{S}(C)$. □

For the statement of the next lemma, for $c \in \mathbf{I}_{\mathcal{C}}$, note that since $\mathcal{C}$ is a full subcategory of $\mathcal{C} * \mathcal{D}$, we can view $\mathcal{AC}_{c,\mathbb{1}} \subseteq \mathcal{A}\Lambda_{c,\emptyset}$. Similar observation can be made for $d \in \mathbf{I}_{\mathcal{D}}$.

Lemma 2.1.7. *If $c \in \boldsymbol{I}_{\mathcal{C}} \subseteq \Lambda$ and $d \in \boldsymbol{I}_{\mathcal{D}} \subseteq \Lambda$, then $\mathcal{A}\Lambda_{c,d} \neq 0$ if and only if $c \in \boldsymbol{S}(\mathcal{C})$ and $d \in \boldsymbol{S}(\mathcal{D})$. Furthermore $\mathcal{A}\Lambda_{c,d} = \mathcal{AD}_{d,\mathbb{1}} \cdot \mathcal{A}\Lambda_{\emptyset,\emptyset} \cdot \mathcal{AC}_{c,\mathbb{1}}$.*

Proof. If $c \in \mathbf{I}_{\mathcal{C}}$ and $d \in \mathbf{I}_{\mathcal{D}}$, choose $a \in \mathbf{I}_{\mathcal{C}}$ and $b \in \mathbf{I}_{\mathcal{D}}$ such that c and d are subobjects of $\bar{a}a$ and $b\bar{b}$ in $\mathcal{C}$ and $\mathcal{D}$ respectively. Let $0 \neq y_1 \in \mathcal{C}(ac, a)$, $0 \neq y_2 \in \mathcal{D}(b, db)$. Note that $(y_2 \otimes 1_{v_1})(1_{v_2} \otimes y_1) \in (\mathcal{C} * \mathcal{D})(bac, dba) = \mathcal{A}\Lambda^{ba}_{c,d} \subset \mathcal{A}\Lambda_{c,d}$ is nonzero.

Conversely, let $\mathcal{A}\Lambda_{c,d} \neq \{0\}$. Then there exists a non-unit simple object $v \in \mathrm{Irr}(\mathcal{C} * \mathcal{D})$ such that $(\mathcal{C} * \mathcal{D})(vc, dv) = \mathcal{A}\Lambda^{v}_{c,d} \neq \{0\}$. If v is of $\mathcal{C}$-$\mathcal{C}$ (resp. $\mathcal{D}$-$\mathcal{D}$) type, then $(\mathcal{C} * \mathcal{D})(vc, dv) = \{0\}$ as dv (resp. vc) is simple of $\mathcal{D}$-$\mathcal{C}$ type, and any simple subobject of vc (resp. dv) in $\mathcal{C} * \mathcal{D}$ has length smaller than that of dv (resp. vc). Now suppose v is of $\mathcal{C}$-$\mathcal{D}$ type; then, both vc and dv are simple with the same length but are of different types, hence $(\mathcal{C} * \mathcal{D})(vc, dv) = \{0\}$. Thus v can only be of $\mathcal{D}$-$\mathcal{C}$ type. Also since $v \neq \mathbb{1}$, length of v is at least 2.

Let $v = d'v'c'$, where $d' \in \mathbf{I}_{\mathcal{D}}, c' \in \mathbf{I}_{\mathcal{C}}$ and $v' \in \mathrm{Irr}(\mathcal{C} * \mathcal{D})$ is either trivial or $\mathcal{C}$-$\mathcal{D}$ type. Consider $\bar{v}d'v = \bar{c}'\,\bar{v}'\,\bar{d}'\,d\,d'\,v'\,c'$. If $\bar{d}'dd'$ does not contain $\mathbb{1}$ as a subobject, then the length of every simple subobject of $\bar{v}dv$ is strictly greater than 1, and thereby $(\mathcal{C} * \mathcal{D})(vw_1, w_2v) \cong (\mathcal{C} * \mathcal{D})(w_1, \bar{v}w_2v) = \{0\}$ which is a contradiction. Thus, $\mathbb{1}$ appears as a subobject of $\bar{d}'dd'$ and hence $d \in \mathbf{S}(\mathcal{D})$. Similarly, by considering $vc\bar{v}$, one may deduce that $c \in \mathbf{S}(\mathcal{C})$.

For the last part, let $v = d'v'c'$ be as above. Then $vc = d'v'c'c$ and $dv = dd'v'c'$. Since v' is a word of $\mathcal{C}$-$\mathcal{D}$ type of length at least 2 whose letters are all simple, by the definition of the free product category, any morphism $x \in (\mathcal{C} * \mathcal{D})(vc, dv)$ factorizes as $x_1 \otimes 1_{v'} \otimes x_2$, where $x_1 \in \mathcal{D}(d', dd')$ and $x_2 \in \mathcal{C}(c', cc')$. The result then follows. ☐

Lemma 2.1.8. *Suppose $c_1, c_2 \in \boldsymbol{I}_{\mathcal{C}}$. If $v \in Irr(\mathcal{C} * \mathcal{D})$ and $|v| \geq 1$, then the space $\mathcal{A}\Lambda^{v}_{c_1,c_2} \neq \{0\}$ implies v is of $\mathcal{C}$-$\mathcal{C}$ type. Furthermore, we have*

(i) If $|v| = 1$, then $v \in \boldsymbol{I}_{\mathcal{C}}$ and $\mathcal{A}\Lambda^{v}_{c_1,c_2} = \mathcal{AC}^{v}_{c_1,c_2}$.

(ii) If $|v| \geq 2$ then $\mathcal{A}\Lambda^{v}_{c_1,c_2} \neq 0$ implies both c_1 and c_2 lie in $\boldsymbol{S}(\mathcal{C})$. Furthermore, $\mathcal{A}\Lambda^{v}_{c_1,c_2} = \mathcal{AC}_{\mathbb{1},c_2} \cdot \mathcal{A}\Lambda_{\emptyset,\emptyset} \cdot \mathcal{AC}_{c_1,\mathbb{1}}$.

The same statement holds, replacing $\mathcal{C}$ with $\mathcal{D}$.

Proof. Let $c_1, c_2 \in \mathbf{I}_{\mathcal{C}}$. And suppose $\mathcal{A}\Lambda^{v}_{c_1,c_2} \neq \{0\}$, for $|v| \geq 1$.

If v is of $\mathcal{C}$-$\mathcal{D}$ type or $\mathcal{D}$-$\mathcal{C}$ type, then vc_1 (respectively, c_2v) is simple, and any simple subobject of c_2v (respectively vc_1) will have length strictly smaller than that of vc_1

(respectively c_2v). Hence $\mathcal{A}\Lambda^v_{c_1,c_2} = (\mathcal{C} * \mathcal{D})(vc_1, c_2v) = \{0\}$. Again, we can rule out v being $\mathcal{D}$-$\mathcal{D}$ type by comparison of the two simple objects vc_1 and c_2v, which cannot be equal since one starts with $\mathcal{D}$ while the other starts with $\mathcal{C}$.

For (i), note that for $|v| = 1$ and $\mathcal{A}\Lambda^v_{c_1,c_2} \neq \{0\}$, we must have $v \in \mathbf{I}_{\mathcal{C}}$ and in this case we see that $\mathcal{A}\Lambda^v_{c_1,c_2} = (\mathcal{C} * \mathcal{D})(vc_1, c_2v) = \mathcal{C}(vc_1, c_2v) = \mathcal{A}\mathcal{C}^v_{c_1,c_2}$.

For (ii), suppose we have $\mathcal{A}\Lambda^v_{c_1,c_2} \neq \{0\}$, with $|v| \geq 2$. By the first part of the lemma, v is of $\mathcal{C}$-$\mathcal{C}$ type, and hence we have $v = c_1'v'c_2'$, where v' is a simple word of $\mathcal{D}$-$\mathcal{D}$ type of length ≥ 1. Thus we see that for any $x \in (\mathcal{C} * \mathcal{D})(vc_1, c_2v) = (\mathcal{C} * \mathcal{D})(c_1'v'c_2'c_1, c_2c_1'v'c_2')$, from the definition of the free product category we must have $x_1 \in \mathcal{C}(c_1', c_2c_1')$ and $x_2 \in \mathcal{C}(c_2'c_1, c_2')$ so that x factorizes as $x = x_1 \otimes 1_{v'} \otimes x_2$. This gives us (ii). □

2.2 Annular Representations of Free Product of Categories

Let $\mathcal{C}$ be an arbitrary rigid C^*-tensor category, and $\Gamma \subseteq [\text{Obj } \mathcal{C}]$ be an arbitrary weight set containing $\mathbb{1}$, which is sufficiently full to generate a universal C*-algebra. Consider the ideal $\mathcal{J}\Gamma_0 := \mathcal{A}\Gamma \cdot \mathcal{A}\Gamma_{\mathbb{1},\mathbb{1}} \cdot \mathcal{A}\Gamma$ in $\mathcal{A}\Gamma$ generated by $\mathcal{A}\Gamma_{\mathbb{1},\mathbb{1}}$. In the particular case of $\Gamma = \text{Irr}(\mathcal{C})$, we write $\mathcal{J}\mathcal{C}_0$ for $\mathcal{J}\Gamma_0$.

Given any bounded $*$-representation of $\mathcal{J}\Gamma_0$ one can get a bounded $*$-representation of $\mathcal{A}\Gamma$ by a construction very similar to the one done in the proof of Lemma 2.1.1. By going along the similar lines, one can also show that the induction functor $\text{Ind}_0 : Rep(\mathcal{J}\Gamma_0) \to Rep(\mathcal{A}\Gamma)$ is fully faithful. Let $Rep_0(\mathcal{A}\Gamma)$ be the image of $Rep(\mathcal{J}\Gamma_0)$ under Ind. Thus, $Rep_0(\mathcal{A}\Gamma)$ is the full subcategory of representations generated by their weight $\mathbb{1}$ space. As every representation of $\mathcal{A}\Gamma$ which belongs to this subcategory is generated by its weight $\mathbb{1}$ space, it is easy to see that $Rep_0(\mathcal{A}\Gamma)$ is precisely the category of *admissible representations of the fusion algebra* with respect to Γ.

Consider the W*-category $Rep_+(\mathcal{A}\Gamma) := Rep(\mathcal{A}\Gamma/\mathcal{J}\Gamma_0)$ of representations of $\mathcal{A}\Gamma$ which contain $\mathcal{J}\Gamma_0$ in their kernel. $Rep_+(\mathcal{A}\Gamma)$ is referred to as the category of *higher weight representations*. It consists of precisely the representations of $\mathcal{A}\Gamma$ such that the projection $p_{\mathbb{1}} \in \mathcal{A}\Gamma_{\mathbb{1},\mathbb{1}}$ acts by 0.

Then, for any non-degenerate $*$-representation of $(\pi, \mathcal{H}) \in Rep(\mathcal{A}\Gamma)$, we can decompose $\mathcal{H}$ as direct sum of subrepresentations $\mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, where $\mathcal{H}_0 := [\pi(\mathcal{J}\Gamma_0)\mathcal{H}]$ and $\mathcal{H}_0^\perp$ is its orthogonal complement. We can view $\mathcal{H}_0 \in Rep_0(\mathcal{A}\Gamma)$ and $\mathcal{H}_0^\perp \in Rep_+(\mathcal{A}\Gamma)$. Any representation of $\mathcal{J}\Gamma_0$ and any representation of $\mathcal{A}\Gamma_+$ are disjoint as representations of $\mathcal{A}\Gamma$. This discussion gives us the following proposition:

Proposition 2.2.1. *For any sufficiently full weight set, $Rep(\mathcal{A}\Gamma) \cong Rep_0(\mathcal{A}\Gamma) \oplus Rep_+(\mathcal{A}\Gamma)$.*

Thus, the problem of understanding $Rep(\mathcal{A}\Lambda)$ decomposes into the problem of understanding the admissible representations of the fusion algebra, and the higher weight structure. In the particular case of free products, what we will see is that the weight 0 part is controlled by a free product C*-algebra, while the higher weight parts can be read off in terms of the higher weight parts of $\mathcal{C}$ and $\mathcal{D}$. There are also some additional copies of the category $Rep(\mathbb{Z})$ that appear at higher weights.

We first turn our attention to the weight 0 case. Let $\mathrm{Fus}(\mathcal{C})$ be the fusion algebra of $\mathcal{C}$ with the distinguished basis $\mathrm{Irr}(\mathcal{C})$. Recall there exists a universal C*-algebra completion of the fusion algebra, denoted by $C^*_u(\mathcal{C})$, first introduced by Popa and Vaes [PV15], which is universal with respect to *admissible representations*. In [GJ16], it was shown that $\mathcal{A}\mathcal{C}_{\mathbb{1},\mathbb{1}} \cong \mathrm{Fus}(\mathcal{C})$ and admissible representations are precisely those that induce bounded $*$-representations of the tube algebra, and thus $C^*_u(\mathcal{C})$ can be viewed as the weight 0 corner (or centralizer algebra) of the universal C*-algebra of the tube algebra.

Via the inclusions of $\mathcal{C}$ and $\mathcal{D}$ into $\mathcal{C}*\mathcal{D}$, $\mathrm{Fus}\,(\mathcal{C}*\mathcal{D})$ contains the fusion algebras $\mathrm{Fus}(\mathcal{C})$ and $\mathrm{Fus}(\mathcal{D})$ as unital $*$-subalgebras. Indeed, we have a canonical $*$-algebra isomorphism between $\mathrm{Fus}(\mathcal{C}*\mathcal{D})$ and the (algebraic) free product $\mathrm{Fus}(\mathcal{C}) * \mathrm{Fus}(\mathcal{D})$.

We briefly recall the definition of (universal) free product of C*-algebras:

Definition 2.2.2. If A_1 and A_2 are unital C*-algebras, a *free product* is a unital C*-algebra $A_1 * A_2$, together with unital $*$-homomorphisms $\iota_i : A_i \to A_1 * A_2$ satisfying the following universal property: for any unital C*-algebra C and unital $*$-homomorphisms $\gamma_i : A_i \to C$ there exists a unique $*$-homomorphism $\gamma_1 * \gamma_2 : A_1 * A_2 \to C$ such that $(\gamma_1 * \gamma_2) \circ \iota_i = \gamma_i$.

Any two free products of two C*-algebras are $*$-isomorphic if they exists by the universal property. Furthermore, free products *do* exist, so it makes sense to talk about *the* free product C*-algebra, which we will denote by $A_1 * A_2$.

The main result of this section is the following:

Proposition 2.2.3. $C^*_u(\mathcal{C}*\mathcal{D}) \cong C^*_u(\mathcal{C}) * C^*_u(\mathcal{D})$.

To prove this, we already know that $\mathcal{A}\mathcal{C}_{\mathbb{1},\mathbb{1}}$, $\mathcal{A}\mathcal{D}_{\mathbb{1},\mathbb{1}}$ and $\mathcal{A}\Lambda_{\emptyset,\emptyset}$ are isomorphic to the fusion algebras $\mathrm{Fus}(\mathcal{C})$, $\mathrm{Fus}(\mathcal{D})$ and $\mathrm{Fus}(\mathcal{C}*\mathcal{D}) \cong \mathrm{Fus}(\mathcal{C})*\mathrm{Fus}(\mathcal{D})$ respectively. Using these isomorphisms, any representation of the weight zero centralizer algebra $\mathcal{A}\Lambda_{\emptyset,\emptyset}$ can also be viewed as representations of $\mathcal{A}\mathcal{C}_{\mathbb{1},\mathbb{1}}$ and $\mathcal{A}\mathcal{D}_{\mathbb{1},\mathbb{1}}$ by restricting π to the corresponding subalgebras. We have the following lemma:

Lemma 2.2.4. *A representation $(\pi, \mathcal{H})$ of Fus$(\mathcal{C} * \mathcal{D})$ is admissible if and only if its restrictions $(\pi^c, \mathcal{H})$ and $(\pi^d, \mathcal{H})$ to Fus$(\mathcal{C})$ and Fus$(\mathcal{D})$ are admissible respectively.*

Proof. If $(\pi, \mathcal{H})$ be admissible then, $(\pi^c, \mathcal{H})$ and $(\pi^d, \mathcal{H})$ are clearly admissible.

Suppose $(\pi^c, \mathcal{H})$ and $(\pi^d, \mathcal{H})$ are admissible. Set $\widehat{\mathcal{H}}_w := \mathcal{A}\Lambda_{\emptyset,w} \otimes \mathcal{H}$ for $w \in \Lambda$. By Lemma 2.1.3 and Lemma 2.1.6, $\widehat{\mathcal{H}}_w$ is nonzero only when $w = \emptyset$ or w has length 1 and is in $\mathbf{S}(\mathcal{C}) \cup \mathbf{S}(\mathcal{D})$. As usual, we define a sesquilinear form on $\widehat{\mathcal{H}}_w$ by

$$\langle y_1 \otimes \xi_1, y_2 \otimes \xi_2 \rangle_w := \langle \pi(y_2^{\#} \cdot y_1)\xi_1, \xi_2 \rangle$$

for $y_1, y_2 \in \mathcal{A}\Lambda_{\emptyset,w}$ and $\xi_1, \xi_2 \in \mathcal{H}$.

By the definition of admissibility and [GJ16], it suffices to show that this form is positive semi-definite. Further, it is enough to show

$$\sum_{i,j=1}^{n} \langle \pi(x_j^{\#} \cdot x_i)\xi_i, \xi_j \rangle \geq 0$$

for $x_i \in \mathcal{A}\Lambda_{\emptyset,w}^{v_i}$, $v_i \in \mathrm{Irr}(\mathcal{C} * \mathcal{D})$, $\xi_i \in \mathcal{H}$. When $w = \emptyset$, the sum becomes $\sum_{i=1}^{n} \|\pi(x_i)\xi_i\|_{\mathcal{H}}^2 \geq 0$. It remains to consider the case $w \in \mathbf{S}(\mathcal{C}) \cup \mathbf{S}(\mathcal{D})$. Suppose $w \in \mathbf{S}(\mathcal{C})$. In order to have $\mathcal{A}\Lambda_{\emptyset,w}^{v_i} = (\mathcal{C} * \mathcal{D})(v_i, wv_i)$ nonzero, v_i must be one of $\mathcal{C}$-$\mathcal{C}$ or $\mathcal{C}$-$\mathcal{D}$ type. Let $v_i = c_i u_i$ where $c_i \in \mathbf{I}_{\mathcal{C}}$ and u_i is either $\emptyset$ or of $\mathcal{D}$-$\mathcal{C}$ or $\mathcal{D}$-$\mathcal{D}$ type. Note that $wv_i = wc_i u_i$. As $w \in \mathcal{C}$, any morphism $x_i \in (\mathcal{C} * \mathcal{D})(c_i u_i, wc_i u_i)$ is of the form $x_i = z_i \otimes 1_{u_i}$, where $z_i \in \mathcal{C}(c_i, wc_i)$.

One may express this in another useful way: $x_i = z_i \cdot 1_{u_i}$ where we view $z_i \in \mathcal{AC}_{\mathbb{1},w}^{c_i} \subset \mathcal{A}\Lambda_{\emptyset,w}$, and $1_{u_i} \in \mathcal{A}\Lambda_{\emptyset,\emptyset}^{u_i}$. Setting $\zeta_i := \pi(1_{u_i})\xi_i$, $1 \leq i \leq n$, we have

$$\sum_{i,j=1}^{n} \langle \pi(x_j^{\#} \cdot x_i)\xi_i, \xi_j \rangle = \sum_{i,j=1}^{n} \langle \pi^c(z_j^{\#} \cdot z_i)\zeta_i, \zeta_j \rangle \geq 0$$

where the last inequality follows from admissibility of $(\pi^c, \mathcal{H})$. An entirely analogous argument holds for the case $w \in \mathbf{S}(\mathcal{D})$. $\square$

Proof of Proposition 2.2.3. Let $i_{\mathcal{C}}$ (resp., $i_{\mathcal{D}}$) be the canonical $*$-inclusion of Fus$(\mathcal{C})$ (resp., Fus$(\mathcal{D})$) into Fus$(\mathcal{C} * \mathcal{D})$.

If $(\pi, \mathcal{H})$ is any admissible representation of Fus$(\mathcal{C} * \mathcal{D})$, then $(\pi \circ i_{\mathcal{C}}, \mathcal{H})$ and $(\pi \circ i_{\mathcal{D}}, \mathcal{H})$ are admissible representations of Fus$(\mathcal{C})$ and Fus$(\mathcal{D})$ respectively by Lemma 2.2.4. Therefore, for any $x \in$ Fus$(\mathcal{C})$,

$$||i_{\mathcal{C}}(x)||_{\pi} = ||x||_{\pi \circ i_{\mathcal{C}}} \leq ||x||_{C_u^*(\mathcal{C})} \ .$$

By the definition of the universal norm,

$$||i_{\mathcal{C}}(x)||_{C^*_u(\mathcal{C}*\mathcal{D})} = \sup_{\pi'} ||i_{\mathcal{C}}(x)||_{\pi'}$$

where the supremum is taken over all admissible representations of $\mathrm{Fus}(\mathcal{C}*\mathcal{D})$. Thus the map $i_{\mathcal{C}}$ extend to $*$-homomorphisms $\iota_{\mathcal{C}} : C^*_u(\mathcal{C}) \to C^*_u(\mathcal{C}*\mathcal{D})$. The same argument applies to $\mathcal{D}$, yielding an extension $\iota_{\mathcal{D}} : C^*_u(\mathcal{D}) \to C^*_u(\mathcal{C}*\mathcal{D})$.

Let A be any C*-algebra with *-homomorphisms $\gamma_{\mathcal{C}} : C^*_u(\mathcal{C}) \to A$ and $\gamma_{\mathcal{D}} : C^*_u(\mathcal{D}) \to A$. By the universal property of free product of ordinary $*$-algebras, there is a unique $*$-homomorphism $h : \mathrm{Fus}(\mathcal{C}*\mathcal{D}) \to A$ such that $h \circ i_{\mathcal{C}} = \gamma_{\mathcal{C}}|_{\mathrm{Fus}(\mathcal{C})}$ and $h \circ i_{\mathcal{D}} = \gamma_{\mathcal{D}}|_{\mathrm{Fus}(\mathcal{D})}$. By density of the fusion algebras in their universal C*-algebras, to conclude the proof it suffices to show that h extends to a $*$-homomorphism $\gamma_{\mathcal{C}} * \gamma_{\mathcal{D}} : C^*_u(\mathcal{C}*\mathcal{D}) \to A$, which is equivalent to showing $||h(x)||_A \leq ||x||_{C^*_u(\mathcal{C}*\mathcal{D})}$.

Without loss of generality, assume $A \subset B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. Since $||\gamma_{\mathcal{C}}(y)||_A \leq ||y||_{C^*_u(\mathcal{C})}$ for every $y \in \mathrm{Fus}(\mathcal{C})$, $(\gamma_{\mathcal{C}}|_{\mathrm{Fus}(\mathcal{C})}, \mathcal{K})$ is admissible and similarly, $(\gamma_{\mathcal{D}}|_{\mathrm{Fus}(\mathcal{D})}, \mathcal{K})$ is also admissible. Thus, by Lemma 2.2.4, $(h, \mathcal{K})$ is an admissible representation of $\mathrm{Fus}\,(\mathcal{C}*\mathcal{D})$. Therefore, $||x||_h = ||h(x)||_A \leq ||x||_{C^*_u(\mathcal{C}*\mathcal{D})}$. □

This immediately implies the following corollary:

Corollary 2.2.5. *The category of $Rep_0(\mathcal{A}\Lambda)$ is equivalent as a W*-category to $Rep(C^*_u(\mathcal{C}) * C^*_u(\mathcal{D}))$.*

On one hand, it is well known that representation categories of free product algebras are wild and uncontrollable, and thus this answer for describing $Rep_0(\mathcal{A}\Lambda)$ is somewhat unsatisfactory, compared to descriptions of other representation categories such as $Rep(\mathcal{A}TLJ)$ ([GJ16]). On the other hand, there are a plethora of ways to produce examples of representations of free products, so these categories are quite flexible. For example, given two states ψ, ϕ on C*-algebras A and B, one can construct the free convolution state $\psi * \phi$ on the C*-algebra $A * B$ ([Avi82]). Alternatively one simply has to take a representation of A and one of B, and identify their underlying Hilbert space.

We now move on to describing the higher weight categories, which, depending on $\mathcal{C}$ and $\mathcal{D}$, can be more manageable. As described in the beginning of the section $Rep_+(\mathcal{A}\Lambda) = Rep(\mathcal{A}\Lambda/\mathcal{J}\Lambda_0)$. We have the following lemma:

Lemma 2.2.6. *As $*$-algebras, $\mathcal{A}\Lambda/\mathcal{J}\Lambda_0 \cong \mathcal{A}\mathcal{C}/\mathcal{J}\mathcal{C}_0 \oplus \mathcal{A}\mathcal{D}/\mathcal{J}\mathcal{D}_0 \oplus \mathcal{A}\boldsymbol{W}$.*

Proof. Recall that $\mathcal{A}\Lambda \cong \mathcal{A}[\Lambda \setminus \mathbf{W}] \oplus \mathcal{A}\mathbf{W}$. From Lemma 2.1.3, we see that $\mathcal{J}\Lambda_0 \subseteq \mathcal{A}[\Lambda \setminus \mathbf{W}]$, and thus

$$\mathcal{A}\Lambda/\mathcal{J}\Lambda_0 \cong \mathcal{A}[\Lambda \setminus \mathbf{W}]/\mathcal{J}\Lambda_0 \oplus \mathcal{A}\mathbf{W}$$

Thus we consider the spaces $\mathcal{A}\Lambda^v_{w_1,w_2}$ with $w_1, w_2 \in \mathbf{S}(\mathcal{C}) \cup \mathbf{S}(\mathcal{D})$, and $v \in \mathrm{Irr}(\mathcal{C} * \mathcal{D})$. By Lemma 2.1.7 and Lemma 2.1.8, the image of these spaces under the quotient is 0 unless w_1 and w_2 are either both in $\mathbf{S}(\mathcal{C})$ and $v \in \mathrm{Irr}(\mathcal{C})$ or both w_1 and w_2 are in $\mathbf{S}(\mathcal{D})$ and $v \in \mathrm{Irr}(\mathcal{D})$. Since $\mathcal{J}\mathcal{C}_0, \mathcal{J}\mathcal{D}_0 \subseteq \mathcal{J}\Lambda_0$, it is now clear that the quotient map assembles into an isomorphism $\mathcal{A}[\Lambda \setminus \mathbf{W}]/\mathcal{J}\Lambda_0 \cong \mathcal{A}\mathcal{C}/\mathcal{J}\mathcal{C}_0 \oplus \mathcal{A}\mathcal{D}/\mathcal{J}\mathcal{D}_0$, concluding the proof. □

Finally, we recall that $\mathbf{W}_0$ is the set of cyclic equivalence classes of words in $\mathbf{W}$, and note that $Rep(\mathcal{A}\mathbf{W}) \cong Rep(\mathbb{Z})^{\oplus \mathbf{W}_0}$. The above results imply the following, which is the main result of this chapter.

Theorem 2.2.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be rigid C^*-tensor categories. Then as W^*-categories,*

$$Rep(\mathcal{A}(\mathcal{C} * \mathcal{D})) \cong Rep(C^*_u(\mathcal{C}) * C^*_u(\mathcal{D})) \oplus Rep_+(\mathcal{A}\mathcal{C}) \oplus Rep_+(\mathcal{A}\mathcal{D}) \oplus Rep(\mathbb{Z})^{\oplus \mathbf{W}_0}$$

2.3 Examples

In this section, we apply the main result to several examples. First, we show how this matches another known result.

Example 2.3.1. Free products of group categories. In particular, for any countable group G, we consider the rigid C*-tensor category $\mathbf{Hilb}_{f.d.}(G)$ of finite dimensional G-graded Hilbert spaces. Let Λ denote the set of conjugacy classes of G. For each $\lambda \in \Lambda$ we can define $C_\lambda(G)$ to be the centralizer subgroup of some element $g \in \lambda$. We note that different choices of $g \in \Lambda$ yield conjugate subgroups, and so $C_\lambda(G)$ is well defined up to isomorphism. Then, from [GJ16], the category of annular representations

$$Rep(\mathcal{A}) \cong \bigoplus_{\lambda \in \Lambda} Rep(C_\lambda(G))$$

Now, for any two countable groups G and H, its easy to see that $\mathbf{Hilb}_{f.d.}(G) * \mathbf{Hilb}_{f.d.}(H)$ is equivalent as a C*-tensor category to $\mathbf{Hilb}_{f.d.}(G*H)$. Thus we can compare our result for $\mathbf{Hilb}_{f.d.}(G) * \mathbf{Hilb}_{f.d.}(H)$ to the above result for $\mathbf{Hilb}_{f.d.}(G * H)$.

Since $C^*_u(\mathbf{Hilb}_{f.d.}(G))$ is isomorphic to the universal group C*-algebra $C^*_u(G)$, and $C^*_u(G * H) \cong C^*_u(G) * C^*_u(H)$, we can identify the first component in the main theorem (Theorem 2.2.7) with $Rep(G * H)$.

Note that there is always distinguished conjugacy class $[1] \in \Lambda$, the conjugacy class of the unit 1. We have $C_{[1]}(G) = G$. It is easy to see that

$$Rep_+(\mathcal{A}\mathbf{Hilb}_{f.d.}(G)) \cong \bigoplus_{\lambda \in \Lambda \setminus [1]} C_\lambda(G)$$

This helps us identify the second two components, while the last component needs no identification.

Now, consider the group $G*H$. This group has 4 types of conjugacy classes: $\{[1]\}, \{[g] : g \in G\}, \{[h] : h \in H\}$ and $\{[g_1h_1 \cdots g_kh_k] : g_i \in G, h_i \in H, k \geq 1\}$. It is also easy to see that $C_{[1]}(G * H) = G * H$, $C_{[g]}(G * H) = G$, $C_{[h]}(G * H) = H$ and $C_{[g_1h_1 \cdots g_kh_k]} = \{(g_1h_1 \cdots g_kh_k)^n : n \in \mathbb{Z}\} \cong \mathbb{Z}$. It is now easy to see the equivalence of the two descriptions.

Example 2.3.2. Fuss-Catalan representations. Bisch and Jones introduced the *Fuss-Catalan* subfactor planar algebras $\mathcal{FC}(\alpha, \beta)$, where $\alpha, \beta \in \{2\cos(\frac{\pi}{n}) : n \geq 3\} \cup [2, \infty)$ [BJ97]. These planar algebras are universal for intermediate subfactors. For a subfactor planar algebra, the category of affine annular representations in the sense of Jones-Reznikoff [JR06] is equivalent to the category of annular representations of the even part of the subfactor (see, for example, [DGG14a, Remark 3.6] or [NY18, Corollary 4.4]). The even part of the Fuss-Catalan can be realized as a full subcategory of the free product category $\mathcal{TLJ}(\alpha) * \mathcal{TLJ}(\beta)$. In particular, if $a \in \mathcal{TLJ}(\alpha)$ is the standard tensor generating object with dimension α and $b \in \mathcal{TLJ}(\beta)$ is the standard tensor generating object with dimension β, then the full subcategory generated by $abba \in \mathcal{TLJ}(\alpha) * \mathcal{TLJ}(\beta)$ is equivalent to the even part of $\mathcal{FC}(\alpha, \beta)$. Thus to determine the annular representation category of $\mathcal{FC}(\alpha, \beta)$, it suffices to determine the annular representations of the full subcategory $\mathcal{TLJ}(\alpha) * \mathcal{TLJ}(\beta)$ generated by $abba$. Let $\mathcal{TLJ}_0(\alpha)$ denote the adjoint subcategory, generated by aa. This can also be realized as the even part of the usual Temperley-Lieb-Jones subfactor planar algebras.

We recall briefly that two rigid C*-tensor categories $\mathcal{C}$ and $\mathcal{D}$ are *weakly Morita equivalent* if there is a rigid C*-2 category with two objects 0 and 1, such that the tensor category $\text{End}(0) \cong \mathcal{C}$ and the tensor category $\text{End}(1) \cong \mathcal{D}$ (see [NY18] for further details). The two even parts of a subfactor planar algebra are weakly Morita equivalent, but weak Morita equivalence is more general. If we have two full subcategories of a tensor category, to show they are weakly Morita equivalent, it suffices to find an object $x \in \mathcal{C}$ so that $x\overline{x}$ tensor generates one and $\overline{x}x$ tensor generates the other, since one can, using the usual subfactor approach, construct a rigid C*-2 category whose two even parts are as desired. We apply this in the free product case to obtain the following proposition:

Proposition 2.3.3. *The tensor category generated by abba is weakly Morita equivalent to $\mathcal{TLJ}_0(\alpha) * \mathcal{TLJ}_0(\beta)$.*

Proof. It suffices to find an object $x \in \mathcal{TLJ}(\alpha) * \mathcal{TLJ}(\beta)$ such that $\langle x\overline{x}\rangle = \langle abba\rangle$ and $\langle \overline{x}x\rangle = \mathcal{TLJ}(\alpha) * \mathcal{TLJ}(\beta)$. Choose $x := abb$. Then since both aa and bb contain the tensor unit as a subobject, we see $\langle abbbba\rangle = \langle abba\rangle$. On the other hand, $bbaabb$ contains aa and bb as subobjects, and so clearly $\langle bbaabb\rangle = \langle aa, bb\rangle$. □

Again, by [DGG14a, Remark 3.6] or [NY18, Corollary 4.4], the above proposition implies the following:

Corollary 2.3.4. *The category of affine annular representations of the subfactor planar algebra $\mathcal{FC}(\alpha, \beta)$ is equivalent as a W*-category to the annular representation category of $\mathcal{TLJ}_0(\alpha) * \mathcal{TLJ}_0(\beta)$.*

This category $\mathcal{TLJ}_0(\alpha)$ is fully described in [JR06], and thus combining those results with ours leads to a description of the representations of Fuss-Catalan categories.

Chapter 3

Free oriented extensions of subfactor planar algebras

3.1 The free oriented extension of subfactor planar algebras

We will call a Λ-oriented planar algebra simply an *oriented planar algebra* if Λ is singleton. This not only simplifies the terminology, but also agrees with the definition presented in [Jon11, Definition 1.2.7]. In this section we will study oriented factor planar algebras and their relation to subfactor planar algebras. Throughout this section, whenever we talk about oriented planar algebra, we assume $\Lambda := \{+\}$ and $\overline{\Lambda} := \{-\}$. So, $W = W_\Lambda$ will be the set of words with letters from $\{\pm\}$ (note this $\pm$ has nothing to do with $\pm$ discussed in the preliminaries concerning free products). Since Λ is a singleton, we do not label any of the (oriented) strings of any Λ-oriented tangle; rather, we assume that each string is labeled with $+$. With this convention, a marked point on the external disc is assigned $+$ or $-$ according as the string attached to it has orientation towards or away from the point; for marked points on the internal discs, the convention is just the opposite.

In this context, it is important to talk about Jones' *subfactor planar algebras* ([Jon99]). We briefly recall the definitions. Let W_{alt} denote the set of words having even length with $+$ and $-$ appearing alternately. Then, one can consider oriented tangles where the colors of internal and external discs must belong to W_{alt} such that it is possible to put a checker-board shading; such tangles are called *shaded tangles*. If the color of the external or an internal disc in a shaded tangle is $\varnothing$, then the region attached to boundary of the disc could be unshaded or shaded; we specify this by renaming the color of the disc as $+\varnothing$ or $-\varnothing$ respectively. Let W_{alt} contain the elements $\pm\varnothing$ as well. A subfactor planar algebra consists of a family of vector spaces $\{P_w\}_{w\in W_{\text{alt}}}$ on which the shaded tangles act satisfying properties analogous to that in Definitions 1.1.15 and 1.1.17. In the original definition, Jones indexed the vector spaces by $\{\varepsilon k : \varepsilon \in \{\pm\}, k \in \mathbb{N}\}$ instead of W_{alt} (where εk corresponds to the word of length $2k$ with alternate letters $\pm$, beginning with ε).

Given a oriented factor planar algebra Q, since shaded tangles can be thought of as oriented tangles by simply forgetting the shading, we can canonically construct a subfactor planar algebra called its *shaded part of* Q, denoted $\mathcal{S}(Q)$. For $w \in W_{alt}$, $\mathcal{S}(Q)_w := Q_w$. The action of shaded tangles is simply the action of the associated oriented tangle.

Let $\mathcal{P}_{or}$ denote the category whose objects are oriented factor planar algebras and whose morphisms are $*$-planar algebra morphisms. Similarly, let $\mathcal{P}_{sh}$ denote the category whose objects are subfactor planar algebras, and whose morphisms are $*$-planar algebra morphisms. Obviously the assignment $Q \mapsto \mathcal{S}(Q)$ induces a functor $\mathcal{S} : \mathcal{P}_{or} \to \mathcal{P}_{sh}$.

Definition 3.1.1. The functor $\mathcal{S} : \mathcal{P}_{or} \to \mathcal{P}_{sh}$, $Q \mapsto \mathcal{S}(Q)$ is called the *shading functor.*

To understand this functor on the level of von Neumann algebras, an oriented factor planar algebra Q corresponds to the rigid C*-tensor category generated by a single bimodule $\mathcal{H}$ of a II_1 factor N. Taking alternating tensor powers of $\mathcal{H}$ and $\overline{\mathcal{H}}$ gives the standard invariant for the subfactor $N \subseteq M$, where M is the II_1 factor associated to the Q-system $\mathcal{H} \otimes_N \overline{\mathcal{H}} \in \mathrm{Bim}(N)$. The standard invariant $\mathcal{S}(Q)$ is precisely the standard invariant of this subfactor. Note that we cannot recover tensor powers of $\mathcal{H}$ from this information. In other words, the sufactor standard invariant forgets information. We are led naturally to the following definition.

Definition 3.1.2. Let P be a subfactor planar algebra. An *oriented extension of* P is a oriented factor planar algebra Q such that $\mathcal{S}(Q)$ is $*$-isomorphic to P.

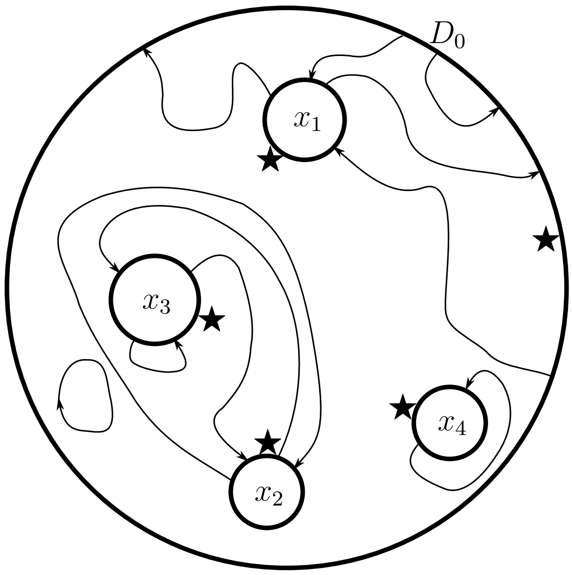
The set of (isomorphism classes of) oriented extensions of a subfactor planar algebra is precisely its pre-image under the functor $\mathcal{S}$.

Remark 3.1.3. Note that subfactor planar algebras correspond to rigid, semisimple C*-2-categories $\mathcal{B}$ with two 0-cells $\{+, -\}$ such that (i) tensor units in $\mathcal{B}_{++}$ and $\mathcal{B}_{--}$ are simple and (ii) there is a 1-cell $\rho \in \mathcal{B}_{+-}$ which tensor-generates the whole of 2-category $\mathcal{B}$. If we call such 2-categories as *singly generated*, with this correspondence one can also define oriented extension of the singly generated 2-category $\mathcal{B}$ as a singly generated C*-tensor category $\mathcal{C}$, generated by σ, such that the underlying 2-category generated by σ is equivalent to $\mathcal{B}$.

3.1.1 The free oriented extension

The first obvious question is whether an oriented extension always exists. We answer this affirmatively, by constructing a canonical one, called the *free oriented extension*.

To proceed with this construction, let P be a subfactor planar algebra. For every word $w \in W$ let D_w be the set of oriented tangles in which the color of the external disc is w and all the internal discs (if any) have their colors in W_{alt} along with a labelling of each internal disc with an element in the corresponding P-space. Note that the tangles are arbitrary oriented tangles such that the boundary conditions along any disc are alternating, but these tangles *do not* need to admit a checker-board shading. A typical element of D_w will be denoted by $T(x_1, \ldots, x_n)$ where T is the unlabelled tangle with internal discs (if any) $D_1, \ldots, D_n$ having colors $w_1, \ldots, w_n$, labelled with $x_1 \in P_{w_1}, \ldots, x_n \in P_{w_n}$ respectively. An example of such an element with colors of internal discs $w_1 = +-+-, w_2 = +-+-, w_3 = -+-+, w_4 = -+$ and $x_i \in P_{w_i}, i = 1, 2, 3, 4$ is given in the following figure. Note that, in this example, the external disc has 6 marked points on it and its color is $w_0 = -+--++$.



Set $\mathcal{D}_w := \mathbb{C}\text{-span}(D_w)$. Note that we have an involutive map $D_w \ni T(x_1, \ldots, x_n) \overset{*}{\longmapsto} T^*(x_1^*, \ldots, x_n^*) \in D_{w^*}$; we extend this conjugate linearly to get an involution $* : \mathcal{D}_w \to \mathcal{D}_{w^*}$. Observe that $\mathcal{D} := \{\mathcal{D}_w\}_{w \in W}$ is an oriented $*$-planar algebra where the action of tangles on labelled tangles comes simply from composition. However, $\mathcal{D}$ is far from being a oriented factor planar algebra, since at this point, the spaces $\mathcal{D}_w$ are all infinite dimensional

In order to define a sesquilinear form on each of $\mathcal{D}_w$, we first define an evaluation map associated to an oriented tangle with external disc having color $\varnothing$ and all internal discs having colors in W_{alt} labelled by elements of P; such labeled tangles will be referred as 'networks' (see [Jon99]). Topologically, a network N is a disjoint union of its connected components, and each component necessarily consists of closed, shaded, P-labelled tangles. These can be nested in the disc in complicated ways. However, for each connected component, one can forget the rest of the network and think of the tangle as being a closed shaded tangle with elements from P. Now, $P_{\pm\varnothing}$ are one dimensional algebras, and thus we can associate a scalar to each closed network. For a network N, define $Z(N)$ to be the product of the scalars arising from each connected component shaded tangle. We call this the *partition function.* Note that by construction, since P is a spherical subfactor planar algebra this partition function is also spherical.

We define a sesquilinear form on $\mathcal{D}_w$ by

$$[X, Y]_w := Z\left(H_w \circ (S, T^*)(x_1, \ldots, x_m, y_1^*, \ldots, y_n^*)\right)$$

for $X = S(x_1, \ldots, x_m),\ Y = T(y_1, \ldots, y_n) \in D_w \subseteq \mathcal{D}_w$, where H_w is the inner product tangle defined in Section 1.1.6. The following lemma is a crucial step in our construction.

Lemma 3.1.4. *For all $w \in W$, $[X, X]_w \geq 0$ for all $X \in \mathcal{D}_w$.*

Proof. Without loss of generality, we may assume that in the decomposition $X = \sum\limits_{i=1}^{k} c_i X_i$ with respect to the canonical basis D_w, none of the X_i's contain any non-empty network, that is, the union of the strings and the boundary of the discs (internal and external) is connected in each X_i. This automatically settles the case of $w = \varnothing$. Moreover, if $\varnothing \neq w \in W_{\text{alt}}$, then all X_i are shaded and thereby the positivity of the P-action implies $[\cdot, \cdot]_w$ is positive semi-definite.

From now on, we will assume $w \in W \setminus W_{\text{alt}}$. Recall that D_w is the set of oriented tangles in which the color of external disc is w and all the internal discs (if any) have their colors in W_{alt}. Since all the internal discs (a) have even number of marked points, and (b) have colors with equal number of $+$ signs and $-$ signs (as they belong to W_{alt}), in order to have $\mathcal{D}_w \neq \{0\}$ (that is, $D_w \neq \emptyset$,) the word w must be of even length with the same number of $+$ signs and $-$ signs. Again, since w has same number of $+$ signs and $-$ signs and $w \in W \setminus W_{\text{alt}}$, w must have a sub-word (in the non-consecutive sense) of the form $(+, +)$ or $(-, -)$, so it will be enough to consider the case when w starts and end with same sign. This is because the sesquilinear form is invariant under the action of rotation tangle. More precisely, if $w = (w_1, w_2)$ and $\rho_{w_1,w_2} : (w_1 w_2) \to (w_2 w_1)$ denotes the rotation tangle as described in the preliminaries, then $[X, Y]_w = [\rho_{w_1,w_2}(X), \rho_{w_1,w_2}(Y)]_{(w_2,w_1)}$ for all $X, Y \in \mathcal{D}_w$. Let

$$W' := \{w \in W \setminus W_{\text{alt}} : w \text{ starts and ends with the same sign and } D_w \neq \emptyset\}.$$

Every $w \in W'$ can be expressed as a unique concatenation $w_1 w_2 \dots w_k$ of consecutive sub-words, where each w_i is a word with $\pm$ appearing alternately such that the last sign of the sub-word w_i matches with the first one of w_j; we will refer these special sub-words as 'MAS' (which stands for *maximally alternately signed*).

Observe that (i) each MAS sub-word of even length has equal number of $+$ signs and $-$ signs and (ii) each MAS sub-word of odd length starting and ending in $+$ (resp. $-$) has a $+$ (resp. $-$) more in number than that of $-$ (resp. $+$). Since w has the same number of $+$ signs and $-$ signs, it follows that

1. the number of MAS sub-words of w with odd length starting and ending with $+$ must be the same as that with $-$; in particular, the number of odd length MAS sub-words must be even.

2. there must be at least one MAS sub-word of even length.

We will now prove that the total number of MAS sub-words of any $w \in W'$ must be even. This is clearly true if all the MAS sub-words of w have even length since w starts and end with the same sign. So, let us assume $w \in W'$ has both odd and even length MAS sub-words. It will be useful to take a disc and arrange the signs in w as marked points on the boundary moving clockwise. Note that if the last sign of any odd length MAS sub-word (a) differs from or (b) matches with the first sign of the very next odd length MAS sub-word moving clockwise, then the number of even length MAS sub-words in between must be (a) odd or (b) even respectively. By (1) above, the number of instances of the case (a) is even. Thus, in the end, the total number of even length MAS sub-words is even and so is the number of MAS sub-words.

Now, fix a $w \in W'$. Let $w = w_1, \dots, w_{2k}$ be the MAS sub-word decomposition. Set $w_{\text{odd}} := w_1 w_3 \dots w_{2k-1}$ and $w_{\text{even}} := w_2 w_4 \dots w_{2k}$. We have the following assertion.

Assertion. Every $X = T(x_1, \dots, x_n) \in D_w$ which does not contain any non-empty network, can be expressed uniquely as an overlay of labelled tangles $X_{\text{odd}} \in D_{w_{\text{odd}}}$ and $X_{\text{even}} \in D_{w_{\text{even}}}$.

Proof of the assertion. First we consider the case in which there is no internal disc inside $X = T \in D_w$. Then T is a Temperley-Lieb (TL) diagram with color of the external disc being $w \in W'$. Note that any such TL diagram induces a non-crossing pairing of opposite signs in w exhausting all the signs; this puts a further restriction that two opposite signs coming from two MAS sub-words which are adjacent around the disc, can never be paired.

What we need to show is that two opposite signs can be paired only if either both belong to two even-indexed MAS sub-words or two odd-indexed ones. We use induction on the length of w. The minimum length of elements in W' is 4 and there are exactly two words, namely, $+--+$ and $-++-$. In both instances, there are exactly two MAS sub-words. Thus each MAS sub-word should be paired within itself.

For the inductive step, suppose w has length $2n$. Let $w = w_1 \dots w_{2k}$ be the MAS sub-word decomposition. In w there must exist two consecutive signs (namely, $+-$ or $-+$) which are paired by the TL diagram T; let us denote this sub-word by v. Clearly, this v must appear in a MAS sub-word, say w_j. Let w' (resp., w'_j) denote the word obtained by removing v from w (resp. w_j), and T' be the corresponding w'-TL diagram obtained from T. If v is strictly smaller that w_j, then we have the MAS sub-word decomposition $w' = w_1 \dots w_{j-1} w'_j w_{j+1} \dots w_{2k}$. Since $|w'| < |w|$, using the inductive hypothesis, we can express T' as an overlay; we simply attach the pairing of v at the appropriate place to get the overlay of T. If $v = w_j$, then the MAS sub-word decomposition becomes $w' = w_1 \dots w_{j-2} w''_{j-1} w_{j+2} \dots w_{2k})$ where $w''_{j-1} = w_{j-1} w_{j+1}$. By inductive hypothesis on T', we see that no pairing can occur between an even-indexed MAS sub-word of w' and an odd-indexed one; so the same holds for T as well. For the case when v is w_1 or w_{2k}, we simply apply a rotation to make v interior and use the same argument.

For the general case, we assume $X = T(x_1, \dots, x_n)$ has internal disc(s). We say two internal discs D_i and D_j of T are *related* if there is a sequence of internal discs starting with D_i and ending with D_j such that any two consecutive internal discs in the sequence are connected by a string. Since there is no non-empty network in X, this clearly becomes an equivalence relation. Fixing an equivalence class, we could use isotopy to bring all the internal discs in the class along with the strings connecting them inside a new disc whose boundary is intersected by the strings connecting these internal discs with the external one. The interior of this new disc is a labelled shaded tangle. Without loss of generality, we may assume that all strings emanating from every internal disc in T go to the external one which has color w. By composing T with TL diagrams in all its internal disc, we get a TL diagram. Since the assertion holds for TL diagrams, we may conclude that if a string from an internal disc connects to a sign in an odd (resp., even) indexed MAS sub-word of w, then all other strings from the same disc should go to only odd (resp., even) indexed MAS sub-word. That is all one needs to obtain the overlay mentioned in the statement of the assertion.

The uniqueness of the overlay holds because there is no network inside X. In particular, one obtains X_{even} (resp., X_{odd}) simply by erasing all the marked points on the external disc corresponding to the odd (resp., even) indexed MAS sub-words along with the strings and internal discs connected to these marked points. □

We return to the proof of the lemma. For every $w \in W$ and $X \in D_w$, let λ_X denote the product of the scalars corresponding to the P-action of every connected networks in X, and $X' \in D_w$ be the element obtained by removing all networks in X.

In order to establish positivity of $[\cdot,\cdot]_w$ for $w \in W'$, consider the linear map defined by

$$\mathcal{D}_w \supset D_w \ni X \overset{\Phi_w}{\longmapsto} \lambda_X \left(X'_{\text{odd}} \otimes X'_{\text{even}} \right) \in \mathcal{D}_{w_{\text{odd}}} \otimes \mathcal{D}_{w_{\text{even}}}.$$

Observe that if $[\cdot,\cdot]_{w,\otimes}$ denotes the sesquilinear form on $\mathcal{D}_{w_{\text{odd}}} \otimes \mathcal{D}_{w_{\text{even}}}$ obtained from the product of $[\cdot,\cdot]_{w_{\text{odd}}}$ and $[\cdot,\cdot]_{w_{\text{even}}}$, then $[X,Y]_w = [\Phi_w(X), \Phi_w(Y)]_{w,\otimes}$ for all $X, Y \in \mathcal{D}_w$. Since the lengths of both w_{odd} and w_{even} are strictly smaller than that of w, a simple induction on the length of w yields the required result as the tensor product of positive sesquilinear forms is again positive. □

For $w_1, w_2 \in W$, a *P-labelled annular tangle from w_1 to w_2* is an oriented tangle in which the color of the external disc is w_2, there is an unlabelled distinguished internal disc with color w_1 and all other internal discs have colors in W_{alt} and labels from the corresponding P-spaces. Any such annular tangle $A : w_1 \to w_2$ induces a linear map from $\mathcal{D}_{w_1}$ to $\mathcal{D}_{w_2}$ via composition; moreover, one can define an annular tangle $A^{\#} : w_2 \to w_1$ which is obtained by (i) reflecting A around the external disc so that the external (resp. distinguished internal) disc becomes the distinguished internal (resp. external) disc after reflection, (ii) reversing the orientation of every string after reflection, and (iii) replacing the label of each internal disc by its $*$. It is easy to see that $\#$ is an involution and $[A(X), Y]_{w_2} = [X, A^{\#}(Y)]_{w_1}$ for all $X \in \mathcal{D}_{w_1}$, $Y \in \mathcal{D}_{w_2}$ (here we use sphericality of the partition function Z.)

Following [Jon99], we define $\mathcal{J}_w \subseteq \mathcal{D}_w$ by $\mathcal{J}_w := \{x \in \mathcal{D}_w \ : \ Z(A(x)) = 0 \text{ for all } A : w \to \varnothing\}$. By [Jon99, Proposition 1.24], this is a planar ideal of $\mathcal{D}$, and clearly in our context this is a *-ideal. We claim that $X \in \mathcal{J}_w$ if and only if X is in the kernel of $[\cdot,\cdot]_w$. Certainly if $\mathcal{J}_w$ is in the kernel of our form. Suppose $[X, X]_w = 0$. Let $A : w \to \varnothing$ be a P-labelled annular tangle. Then by Cauchy-Schwartz, we have

$$|Z(A(X))| := |[A(X), \varnothing]_{\varnothing}| = |[X, A^{\#}(\varnothing)]_w| \leq [X, X]_w^{\frac{1}{2}} [A^{\#}(\varnothing), A^{\#}(\varnothing)]_w^{\frac{1}{2}} = 0.$$

proving the claim. Therefore we can define the planar algebra $\mathcal{F}(P)_w := \mathcal{D}_w / \mathcal{J}_w$. This is non-zero, since for $w \in W_{alt}$, $\mathcal{D}_w / \mathcal{J}_w \cong P_w \neq \{0\}$.

We need to show that $\mathcal{F}(P)_w$ is finite dimensional for every $w \in W$. We already have $\mathcal{F}(P)_w \cong P_w$ for every $w \in W_{\text{alt}}$ and hence $\mathcal{F}(P)_w$ is finite dimensional whenever $w \in W_{\text{alt}}$. For $w \in W \setminus W_{\text{alt}}$, by the proof of Lemma 3.1.4, we have $\dim(\mathcal{F}(P)_w) \leq \dim(\mathcal{F}(P)_{w_{\text{odd}}}) \cdot \dim(\mathcal{F}(P)_{w_{\text{even}}})$. By similar induction argument used towards the end of proof of Lemma 3.1.4, the required finite dimensionality of $\mathcal{F}(P)_w$ follows. Further, by [Jon99, Proposition 1.33], this is a C*-planar algebra.

Definition 3.1.5. The oriented factor planar algebra $\mathcal{F}(P)$ is called *free oriented extension* of the subfactor planar algebra P. We denote the (subfactor planar algebra) isomorphism between $\iota : P \longrightarrow \mathcal{S}(\mathcal{F}(P))$ by $(\iota_w : P_w \longrightarrow \mathcal{F}(P)_w)_{w\in W_{\text{alt}}}$.

Note that the free oriented extension $\mathcal{F}$ is actually a functor. Namely, if $\varphi : P \to P'$ is a planar *-homomorphism, then the obvious definition $\mathcal{F}(\varphi) : \mathcal{F}(P) \to \mathcal{F}(P')$ works. Simply define $\widetilde{\varphi} : \mathcal{D}^P_w \to \mathcal{D}^{P'}_w$, and check that it preserves the partition function. One of the motivations for studying this functor is that it is a left adjoint to the shading functor $\mathcal{S}$. We express this via the following universal property.

Theorem 3.1.6. *Let P be a subfactor planar algebra and Q an oriented factor planar algebra. For any $*$-homomorphism $\psi : P \to \mathcal{S}(Q)$, there exists a unique $*$-homomorphism $\widetilde{\psi} : \mathcal{F}(P) \to Q$ such that $\widetilde{\psi} \circ \iota = \psi$.*

Proof. Let $\psi = (\psi_w : P_w \to Q_w)_{w\in W_{\text{alt}}}$. We will use the notations set up in the construction of the free oriented extension. Define

$$\mathcal{D}_w \supset D_w \ni T(x_1, \dots, x_n) \overset{\widehat{\psi}_w}{\longmapsto} Q_T(\psi_{w_1}x_1, \dots, \psi_{w_n}x_n) \in Q_w$$

where $T : (w_1, \dots, w_n) \to w$ and $x_i \in P_{w_i}$, $i = 1, \dots, n$ and extend $\widetilde{\psi}_w$ linearly to $\mathcal{D}_w$. From the very definition of the positive semi-definite form $[\cdot,\cdot]_w$ on $\mathcal{D}_w$, the map $\widehat{\psi}_w$ takes it to the inner product in Q_w induced by the Q-action of the inner product tangle H_w (since the shaded part of Q is P via ψ). So, $\widehat{\psi}_w$ factors through the quotient $\mathcal{F}(P)_w$ producing the map $\widetilde{\psi}_w : \mathcal{F}(P)_w \to Q_w$. Moreover, $\widetilde{\psi}$ preserving the action is almost immediate.

To check the equation $\widetilde{\psi}_w \circ \iota_w = \psi_w$ for all $w \in W_{\text{alt}}$, take the element $T(x_1, \dots, x_n)$ in the previous paragraph with the extra assumption $w \in W_{\text{alt}}$. We may assume T has no non-empty network (which anyway gives scalar), and thereby T becomes a shaded tangle. Consider the equivalence class $[T(x_1, \dots, x_n)] \in \mathcal{F}(P)_w$; note that $\iota P_T(x_1, \dots, x_n) = [T(x_1, \dots, x_n)]$. On the other hand,

$$\widetilde{\psi}_w[T(x_1, \dots, x_n)] = Q_T(\psi_{w_1}x_1, \dots, \psi_{w_n}x_n) = \psi_w P_T(x_1, \dots, x_n).$$

Since elements of the form $P_T(x_1, \dots, x_n)$ span P_w, we have the desired equation.

For uniqueness, consider a $*$-homomorphism $\varphi : \mathcal{F}(P) \to Q$ satisfying $\varphi_v \circ \iota_v = \psi_v$ for all $v \in W_{\text{alt}}$. Again, consider $T(x_1, \dots, x_n)$ as above with w possibly not in W_{alt}. Observe that

$$\varphi_w[T(x_1, \dots, x_n)] = \varphi_w \mathcal{F}(P)_T(\iota_{w_1} x_1, \dots, \iota_{w_n} x_n)] = Q_T(\psi_{w_1} x_1, \dots, \psi_{w_n} x_n) = \widetilde{\psi}_w[T(x_1, \dots, x_n)]$$

where the first equality follows from the definition of $\mathcal{F}(P)$, second follows from $\varphi : \mathcal{F}(P) \longrightarrow Q$ being a $*$-homomorphism and the equation satisfied by φ and ψ, and the final equality comes from the definition of $\widetilde{\psi}$. □

The above universal property makes the functor $\mathcal{F} : \mathcal{P}_{sh} \to \mathcal{P}_{or}$ into a left adjoint of this functor $\mathcal{S}$ (see [Mac71, Chap IV, Theorem 2]).

3.1.2 Concrete realization

Above, we constructed the free oriented extension from scratch whereas Thoerem 3.1.6 gives us a universal property for an oriented extension being isomorphic to the free one. In the next theorem, we will provide another useful construction of the free oriented extension. Given some oriented extension, we will show that one can find the *free oriented extension* inside a certain free product category; we use techniques similar to those appeared in [MPS17]. We will make use of this in the next section when we study hyperfinite realizability.

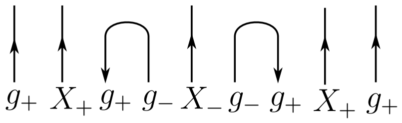
First, we describe a general construction for producing new oriented extensions from a given one. Let $\mathcal{C}$ be a strict rigid semisimple C*-tensor category with simple tensor unit and tensor-generated by X_+; suppose $Q := P^{X_+}$ and $P := \mathcal{S}(Q)$. Let g_+ be an element in a group G and $\mathcal{B} := \text{Hilb}_{f.d.}(G)$ be the rigid C*-tensor category of G-graded finite dimensional Hilbert spaces. We look at the following objects in the free product $\mathcal{B} * \mathcal{C}$

$$X_- := \overline{X}_+, \quad g_- := g_+^{-1}, \quad Y_+ := g_+ \; X_+ \; g_+, \quad Y_- := g_- \; X_- \; g_- = \overline{Y}_+.$$

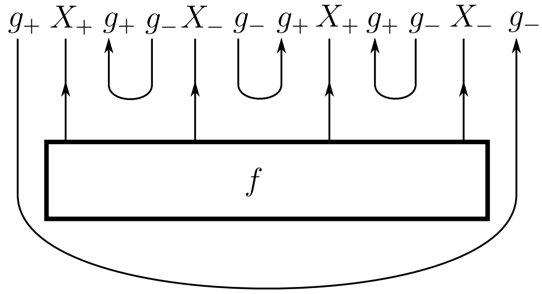
Appealing to Proposition 1.4.6, although we will continue using the notation $\mathcal{B}*\mathcal{C}$, all our objects and morphisms in the rest of this subsection will come from the corresponding full subcategory $\mathcal{NCP}$. For $w = (\varepsilon_1, \dots, \varepsilon_n) \in W$, suppose X_w (resp., Y_w) denotes the object $X_{\varepsilon_1} \otimes \cdots \otimes X_{\varepsilon_n}$ (resp., $Y_{\varepsilon_1} \otimes \cdots \otimes Y_{\varepsilon_n}$). Choose unitaries $R_- : \mathbb{1} \to g_- \otimes g_+$, $R_+ : \mathbb{1} \to g_+ \otimes g_-$ solving the conjugate equations for (g_+, g_-). Note that, since $g_\pm$ are invertible (and hence simple), these solutions are automatically standard, and they being unitaries, are normalized. If an alternately signed word u (possibly of odd length) starts with ε and ends with ν, then we say that u *is of* (ε, ν)*-type*; moreover, for such u, we have the unitary $\alpha_u \in (\mathcal{B} * \mathcal{C})\,(Y_u\ ,\ g_\varepsilon\ X_u\ g_\nu)$ applying the unitaries R_-^* (resp., R_+^*) on each $g_- \otimes g_+$ (resp., $g_+ \otimes g_-$) appearing between X_- and X_+ (resp., X_+ and X_-) which appear in the tensor expansion of Y_u. For $w \in W_{\text{alt}}$ of $(\varepsilon, -\varepsilon)$-type, define the map $\gamma_w : P_w \to P_w^{Y_+}$ by

$$P_w = \mathcal{C}(\mathbb{1}, X_w) \ni f \overset{\gamma_w}{\longmapsto} \alpha_w^*\ (1_{g_\varepsilon} \otimes f \otimes 1_{g_{-\varepsilon}})\ R_\varepsilon\ \in P_w^{Y^+}.$$

For instance, if $u = + - +$, then, graphically,

$\alpha_u =$ 

and, if $w = + - + -$, then for $f \in P_w = \mathcal{C}(\mathbb{1}, X_w)$, $\gamma_w(f)$ looks like



In order to build the planar algebra $Q = P^{X_+}$ (and hence P too), we need to fix a normalized standard solution $(R_{X_+}, \overline{R}_{X_+})$ to the conjugate equations for (X_+, X_-) (see Section 1.3.3). Choosing such solutions for (Y_+, Y_-) involving $(R_{X_+}, \overline{R}_{X_+})$ and (R_+, R_-) in the most obvious way, one can easily derive that $\left(\gamma_w : P_w \longrightarrow P_w^{Y_+}\right)_{w \in W_{\text{alt}}}$ is indeed a subfactor planar algebra isomorphism. As we will see in the next lemma, for *any oriented extension* Q, this oriented extension is isomorphic to the free oriented extension in the case g_+ has infinite order.

Theorem 3.1.7. *Let Q be any oriented extension of a subfactor planar algebra P. Then the projection category of $\mathcal{F}(P)$ (that is, $\mathcal{K}(\mathcal{C}^{\mathcal{F}(P)})$) is equivalent to a full tensor-subcategory of the free product* $\text{Hilb}_{f.d.}(\mathbb{Z}) * \mathcal{K}(\mathcal{C}^Q)$.

Proof. Without loss of generality (via the converse part of Remark 1.3.3), we may assume $Q \coloneqq P^{X_+}$ and $P \coloneqq \mathcal{S}(Q)$ for some strict, semisimple, rigid C*-tensor category $\mathcal{C}$ with simple tensor unit and tensor-generated by X_+. We borrow the notations described right before the statement of this theorem by setting $G = \mathbb{Z}$ and $g_+ = 1 \in \mathbb{Z}$. It is enough to show that P^{Y_+} is isomorphic to $\mathcal{F}(P)$.

Applying Theorem 3.1.6, we get a unique $*$-homomorphism $\widetilde{\gamma} : \mathcal{F}(P) \longrightarrow P^{Y_+}$ such that $\widetilde{\gamma}_w \circ \iota_w = \gamma_w$ for all $w \in W_{\text{alt}}$. It is enough to show $\widetilde{\gamma}_w$ is surjective for all $w \in W$ (which implies $\mathcal{F}(P)$ and P^{Y_+} are isomorphic since $*$-homomorphisms between oriented factor planar algebras are automatically injective). We are already done with the cases when $w \in W_{\text{alt}}$ (since γ_w is surjective) or $P^{Y_+}_w$ is zero.

For $w, w' \in W$, we say *w is a rotation of w'* if there exists $w_1, w_2 \in W$ such that $w = w_1 w_2$ and $w' = w_2 w_1$. For any such $w, w' \in W$, if $\widetilde{\gamma}_w$ is surjective, then so is $\widetilde{\gamma}_{w'}$. To see this, pick any rotation implementing oriented tangle $\rho : w \longrightarrow w'$; note that

$$P^{Y_+}_{w'} = P^{Y_+}_\rho(P^{Y_+}_w) = P^{Y_+}_\rho\ \widetilde{\gamma}_w(\mathcal{F}(P)_w) = \widetilde{\gamma}_{w'}\ \mathcal{F}(P)_\rho(\mathcal{F}(P)_w) = \widetilde{\gamma}_{w'}(\mathcal{F}(P)_{w'}).$$

Clearly, the rotation class of every word in $W \setminus W_{\text{alt}}$ whose P^{Y_+}-space is non-zero, must have at least one word belonging to the set

$$W_0 \coloneqq \left\{ w \in W \setminus W_{\text{alt}} : w \text{ starts and ends with the same sign, and } P^{Y_+}_w \neq \{0\} \right\}.$$

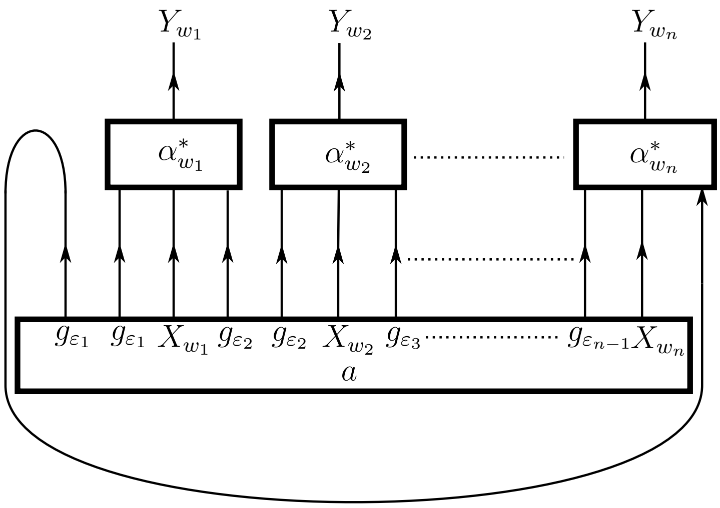
Our goal boils down to establishing surjectivity of $\widetilde{\gamma}_w$ for all $w \in W_0$. For this, we use induction on the number of MAS sub-words.

Let $h_+ \coloneqq g_+ \otimes g_+$ and $h_- \coloneqq g_- \otimes g_- = \overline{h}_+$. Suppose $w \in W_0$. If $w = w_1 w_2 \dots w_k$ is the MAS sub-word decomposition and w_i has $(\varepsilon_i, \varepsilon_{i+1})$-type for $1 \le i \le n$ (and thereby $\varepsilon_1 = \varepsilon_{n+1}$), then we have an isomorphism

$$(\mathcal{B} * \mathcal{C})\left(\mathbb{1}\ ,\ h_{\varepsilon_1} X_{w_1} h_{\varepsilon_2} X_{w_2} \cdots h_{\varepsilon_n} X_{w_n}\right) \ni a \overset{\sigma}{\longmapsto}$$

$$(R^*_{-\varepsilon_1} \otimes \alpha^*_{w_1} \otimes \cdots \otimes \alpha^*_{w_n})(1_{g_{-\varepsilon_1}} \otimes a \otimes 1_{g_{\varepsilon_1}}) R_{-\varepsilon_1} \in (\mathcal{B} * \mathcal{C})\left(\mathbb{1}\ ,\ Y_{w_1} \otimes \cdots \otimes Y_{w_n}\right) = P^{Y_+}_w.$$

Graphically, $\sigma(a)$ is given by



We now use the description of morphism spaces of $\mathcal{B} * \mathcal{C}$ (in fact, $\mathcal{NCP}$) in ??; the morphism space $(\mathcal{B} * \mathcal{C})\left(\mathbb{1}\ ,\ h_{\varepsilon_1} X_{w_1} h_{\varepsilon_2} X_{w_2} \cdots h_{\varepsilon_n} X_{w_n}\right)$ by definition is a subspace of

$$\mathcal{B}(\mathbb{1}_\mathcal{B}, h_{\varepsilon_1} \otimes \cdots \otimes h_{\varepsilon_n})\ \otimes\ \mathcal{C}(\mathbb{1}_\mathcal{C}, X_{w_1} \otimes \cdots \otimes X_{w_n}) = \mathcal{B}(\mathbb{1}_\mathcal{B}, h_{\varepsilon_1} \otimes \cdots \otimes h_{\varepsilon_n})\ \otimes\ \mathcal{C}(\mathbb{1}_\mathcal{C}, X_w).$$

Let f be a $(\varnothing\ ,\ h_{\varepsilon_1}X_{w_1}h_{\varepsilon_2}X_{w_2}\dots h_{\varepsilon_n}X_{w_n})$-NCP such that Z_f is non-zero. Observe that f can viewed as an non-crossing overlay of a pair of oriented tangles S and $T(x_1,\dots,x_m)$ such that

1. S connects to the points on the boundary of the rectangle marked by h_{ε_i}'s, and has internal discs labelled by non-zero elements of $P_\bullet^{h_+}$,

2. T connects to the points marked by X_{w_j}'s with internal discs labeled by x_k's coming from $P_\bullet^{X_+}$.

Since (h_+, h_-) is a dual pair of invertible objects and none of their non-zero finite tensor powers is equivalent to $\mathbb{1}$, we may replace S (up to a non-zero scalar) by a Temperley-Lieb diagram where h_+ can be joined by a string only to h_-. In other words, the partitions in f consisting of h_{ε_i}'s can be assumed to be pair-partitions of h_+ and h_-.

Now recall the definition of D_w from the free oriented extension construction in the previous section. We claim that $T(x_1,\dots,x_m)\in D_w$. To see this, we use induction on the number of MAS-sub-words. First of all, note that n has to be even since S is given by pair-partitions, each consisting of h_+ and h_-, implying

$$\{i\in\{1,\dots n\}: h_{\varepsilon_i}=+\}=\{i\in\{1,\dots n\}: h_{\varepsilon_i}=-\}\,.$$

So, the smallest n is 2 in which case $(h_{\varepsilon_1}, h_{\varepsilon_2})$ is either $(+,-)$ or $(-,+)$. This implies both w_1 and w_2 has to be even and thereby lie in W_{alt}. Now, the non-crossing nature of the partitioning forces X_{w_1} and X_{w_2} to be separate singleton partitions because $(h_{\varepsilon_1}, h_{\varepsilon_2})$ forms a partition. Thus T has exactly 2 internal discs with colors $w_1, w_2\in W_{\text{alt}}$. Hence, $T(x_1,x_2)\in D_w$.

Suppose our $T(x_1,\dots,x_m)\in D_w$ holds for all T, x_i, and $w\in W_0$ with number of MAS sub-words at most $2n$. Pick $w\in W_0$ with $2n+2$ MAS sub-words. In the Temperley-Lieb diagram S, we can find consecutive elements h_{ε_i} and $h_{\varepsilon_{i+1}}$ which are pair partitioned, implying $\varepsilon_{i+1}=-\varepsilon_i$. Further, we may assume $i>1$ since $2n+2>4$. As a result, w_i must have even length and thereby belong to W_{alt}. The non-crossing partitioning forces X_{w_i} to become a singleton partition. So, T has an internal disc with the color $w_i\in W_{\text{alt}}$, connected to the MAS sub-word w_i of w on the boundary of the external disc, and labelled with $\widetilde{x}\in P_{w_i}^{X_+}=\mathcal{C}(\mathbb{1}, X_{w_i})$. Set $w' := w_{i-1}w_{i+1}$ and $w'' := w_1,\dots w_{i-2}, w', w_{i+2},\dots w_{2n+2}$. Clearly, the word w' is alternately signed and the defining equation of w'' gives its MAS sub-word decomposition. In the non-crossing partitioning of f, we erase the partitions (X_{w_i}) and $(h_{\varepsilon_i}, h_{\varepsilon_{i+1}})$ and their associated morphisms, to get a new one, say f'. Note that

$Z_{f'} \in (\mathcal{B} * \mathcal{C})\,(\mathbb{1}, h_{\varepsilon_1} X_{w_1} \cdots X_{w_{i-2}} h_{\varepsilon_{i-1}} X_{w'} h_{\varepsilon_{i+2}} X_{w_{i+2}} \cdots X_{w_{2n+2}})$. We have the formula

$$Z_f = \left(1_{h_{\varepsilon_1} X_{w_1} \cdots X_{w_{i-2}} h_{\varepsilon_{i-1}} X_{w_{i-1}}} \otimes \widetilde{x} \otimes 1_{X_{w_{i+1}} h_{\varepsilon_{i+2}} X_{w_{i+2}} \cdots X_{w_{2n+2}}}\right) \circ Z_{f'}.$$

Therefore $Z_{f'} \neq 0$ (Z_f is assumed to be nonzero). Let S' and $T'(x'_1, \ldots, x'_{m'})$ be the corresponding tangles coming from f'. As w'' has $2n$ MAS sub-words, by induction hypothesis, we have $T'(x'_1, \ldots, x'_{m'}) \in D_{w''}$. Now, $T(x_1, \ldots, x_m)$ can be obtained from $T'(x'_1, \ldots, x'_{m'})$ first by splitting w' on the boundary of the external disc into w_{i-1} and w_{i+1} and inserting the word $w_i \in W_{\text{alt}}$ in between, and then attaching an internal disc of color w_i labelled with $\widetilde{x}$ to this inserted word w_i with strings. Hence, $T(x_1, \ldots, x_m) \in D_w$ proving the claim.

To complete the proof, it will suffice to show σ sends the non-zero Z_f to

$$P_T^{Y_+}(\gamma_{v_1} x_1, \ldots, \gamma_{v_m} x_m) = \widetilde{\gamma}[T(x_1, \ldots, x_m)] \in P_w^{Y_+} = (\mathcal{B} * \mathcal{C})\,(\mathbb{1}, Y_w) \subset \mathcal{B}(\mathbb{1}_{\mathcal{B}}, h_w) \otimes \mathcal{C}(\mathbb{1}_{\mathcal{C}}, X_w).$$

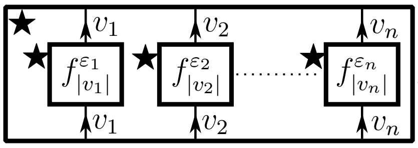
Note that $\mathcal{B}(\mathbb{1}_{\mathcal{B}}, h_w)$ is one-dimensional. Just as $Z_f = P^{h_+} \otimes P_T(x_1, \ldots, x_m)$, a its straightforward to see that σZ_f can also be expressed as $P_{S'}^{g_+} \otimes P_T^{X_+}(x_1, \ldots, x_m)$ where S' is a pair-partitioning of the $g_\pm$ appearing in Y_w.

Next, we look at the non-crossing partitioning $P_T^{Y_+}(\gamma_{v_1} x_1, \ldots, \gamma_{v_m} x_m)$. The way $T(x_1, \ldots, x_m)$ is read off from the non-crossing partition view of f, we can say that the colors $v_1, \ldots, v_m$ (lying in W_{alt}) correspond to the partitions consisting of X_{w_i}'s. At this point, it is useful to work with a standard form representative of T (as described in Section 1.3.3); here, we do have a standard form, where the v_j's are connected straight up to the top of the external rectangle by strings without any local maxima or minima. Let us look at the j-th internal rectangle; suppose $v_j \in W_{\text{alt}}$ is of $(\varepsilon, -\varepsilon)$-type. By the action of tangles of the oriented planar algebra P^{Y_+} defined in Section 1.3.3, the unitary $\alpha_{v_j}^*$ appearing in the label $\gamma_{v_j} x_j = \alpha_{v_j}^* \left(1_{g_\varepsilon} \otimes x_j \otimes 1_{g_{-\varepsilon}}\right) R_\varepsilon$, slides straight up to the top of the external rectangle of T. Moreover, the g_ε and $g_{-\varepsilon}$ at the two extremes of Y_{v_j} (which are also part of the external rectangle), are pair-partitioned by the R_ε appearing in $\gamma_{v_j} x_j$. This lets us to express $P_T^{Y_+}(\gamma_{v_1} x_1, \ldots, \gamma_{v_m} x_m)$ as $P_{S''}^{g_+} \otimes P_T^{X_+}(x_1, \ldots, x_m)$ where S'' is a pair-partitioning of the $g_\pm$ appearing in Y_w and thus the proof is complete. □

3.1.3 Free oriented extension of the Temperley-Lieb Planar algebra

The foremost example of oriented extensions comes from the Temperley-Lieb planar algebra. For $\delta \geq 2$, TL^δ will denote the subfactor planar algebra with modulus δ.

We consider the free oriented Temperley-Lieb planar algebra, $\mathcal{F}(TL^\delta)$. For a word w with letters in $\{+,-\}$, the vector space $\mathcal{F}(TL^\delta)_w$ is the complex span of w-tangles without any internal discs and loops, that is, oriented Temperley-Lieb diagrams. The oriented planar tangles act on $\mathcal{F}(TL^\delta)$ exactly same way as in the ordinary TL^δ. As in TL^δ, we have a double sequence of Jones-Wenzl idempotents $f_n^+ \in TL^\delta_{+n} = \mathcal{F}(TL^\delta)_{(+-)^{2n}}$ and $f_n^- \in TL^\delta_{-n} = \mathcal{F}(TL^\delta)_{(-+)^{2n}}$ for $n \geq 1$. When $n = 0$, f_0^+ and f_0^- get identified in $\mathcal{F}(TL^\delta)$ which we denote by f_0. The projection category of $\mathcal{F}(TL^\delta)$, $\mathcal{K}(\mathcal{C}^{\mathcal{F}(TL^\delta)})$, is generated by the projection f_1^+ and has f_0 as the tensor unit. It will be interesting to look at the irreducible objects of $\mathcal{C}^\delta_{\text{free}}$. For this, we use the MAS sub-word decomposition of words with letters in $\{+,-\}$ (possibly starting and ending with the same sign). Let $v = v_1 v_2 \cdots v_n$ be the MAS sub-word decomposition of v where each v_i starts in $\varepsilon_i \in \{+,-\}$. If we set f_{vv^*} to be the projection,



then, with little effort, one can prove that f_{vv^*} is a minimal projection of $\mathcal{F}(TL^\delta)_{vv^*}$ and thereby a simple object $\mathcal{K}(\mathcal{C}^{\mathcal{F}(TL^\delta)})$. We now need to see whether for all w, every simple object in $\mathcal{K}(\mathcal{C}^{\mathcal{F}(TL^\delta)})$ (same as a minimal projection in $\mathcal{F}(TL^\delta)_w$) is equivalent to f_0 or one of these f_{vv^*}'s. For projections to exist, we should necessarily have $w = u\tilde{u}$. Using the standard trick of 'through strings' and 'middle pattern analysis' (see [BJ97]), one can show that this is indeed the case.

From [Ban97] and [BRV06], one can see that the (co-)representation category of free unitary quantum group $A_u(F)$ for $F \geq 0$ and $\mathcal{F}(TL^\delta)$ are equivalent.

One more TL^δ are the so called *unshaded Temperley-Lieb* denoted by $USTL^\delta$. Define $USTL^\delta_w$ as the span of set of all non-crossing pairings of letters in w (irrespective of the signs, that is, pairing of like signs is allowed). This automatically puts the restriction $USTL^\delta_w$ is zero if length of w is odd. Given any oriented tangle, removing all the directions and labels of the strings gives an unshaded TL tangle and hence can act on $USTL^\delta$ (action of tangles with any of the discs having color of odd length is taken to be zero). Note that, irrespective of the sequence of letters in words, if the length of two words are same, then the corresponding vector spaces are identical. Clearly, $USTL^\delta$ is a $\{+\}$-oriented factor planar algebra for $\delta \geq 2$ and $\mathcal{F}(TL^\delta)$ sits inside it in a canonical way (as proved in Theorem 3.1.6). Under this inclusion, the projection f_{vv^*} is no longer minimal if there is at least two MAS sub-words in v. In fact, the irreducibles in the projection category $\mathcal{K}(\mathcal{C}^{USTL^\delta})$ of $USTL^\delta$, come from those f_{vv^*} 's for which all the letters in v are alternately signed. Now, note that $USTL^{\delta}_{++}$ and $USTL^{\delta}_{--}$ are one dimensional; this shows f_{+-} and f_{-+} are isomorphic and thereby, $f_{(+-)^k}$ and $f_{(-+)^k}$ become isomorphic. Hence simple objects of $\mathcal{K}(\mathcal{C}^{USTL^{\delta}})$ can be identified with $\mathbb{N} \cup \{0\}$.

This category can be realized as the representation category of the free orthogonal quantum groups $A_o(F)$, where F is a matrix with $\mathrm{Tr}(F^*F) = \delta$ and $\overline{F}F = 1$. There is another case, namely when $\overline{F}F = -1$, which corresponding to $Rep(SU_q(2))$ with $q + q^{-1} = \delta$. These planar algebras cannot actually be "unshaded" since the generating object is not symmetrically self-dual, but nevertheless provide oriented extensions TL^{δ} (see [Ban96]).

We propose the following natural problem.

Problem 3.1.8. Classify all oriented extension of the TL^{δ} for $\delta \geq 2$.

We expect the corresponding problem for $\delta < 2$ to actually be more difficult, and relate closely to the extension theory of fusion categories [ENO10]. The free product of the "even parts" of this subfactor planar algebra are very likely to have a large number of quotients, and each of these will likely have a large number of invertible bimodules, making classifications of extensions difficult. However, in the case $\delta \geq 2$, we expect the number quotients of the even part to be manageable, and the number of invertible bimodules to be small, making this problem feasible.

3.2 Hyperfinite constructions

In this section, we make use of a result of Vaes about existence and uniqueness of subfactor standard invariants in the hyperfinite II_1 factor to provide some more examples. In Section 3.1, we have seen that every subfactor planar algebra has a canonical oriented extension, namely the free one. However, as described in this introduction, if we know that there exists a hyperfinite subfactor whose standard is given by the subfactor planar algebra (which we refer as *hyperfinite realizable*), we can produce many oriented extensions. We describe the procedure below.

Suppose $N \subset M$ is an extremal, finite index subfactor such that there is an isomorphism $\varphi : N \to M$. Consider the extremal bifinite N-N bimodule $\mathcal{H}^{\varphi}$ given by: $\mathcal{H}^{\varphi} := L^2(M)$ and $n_1 \cdot \hat{x} \cdot n_2 := [\widehat{n_1 y \varphi(n_2)}]$ for all $x \in M$ and $n_1, n_2 \in N$. As discussed in Section 1.3.3, we can associate a singly generated oriented planar algebra, say OP^{φ}, to the rigid C^*-tensor category generated by $\mathcal{H}^{\varphi}$. The shaded part of OP^{φ} indeed turns out to be isomorphic to the subfactor planar algebra $P^{N \subset M}$ associated to $N \subset M$; thereby,

OP^{φ} becomes an oriented extension of $P^{N\subset M}$. To see this, one has to use the isomorphism between the grid of relative commutants and intertwiner spaces (as in [JS97]) and the decomposition ${}_N\mathcal{H}^{\varphi}_N \cong {}_N L^2(M)_M \underset{M}{\otimes} {}_M\mathcal{H}^{\varphi}_N$ (where ${}_M\mathcal{H}^{\varphi}_N$ is an invertible bimodule); a more explicit isomorphism can be found in [Bur15] or [DGG14a, Theorem 5.2]. In other words, starting from a subfactor planar algebra P such that it is known that the P comes from a subfactor $N \subset M$ where N and M are isomorphic, then every isomorphism between N and M gives rise to an oriented extension of P. In particular, if the subfactor planar algebra corresponds to a hyperfinite one, then one can easily obtain many of its oriented extensions by picking different isomorphisms from N to M.

We next deal with the question whether the free oriented extension of a hyperfinite realizable subfactor planar algebra is hyperfinite realizable. For this, recall the following definitions from [Vae08] regarding freeness of two fusion subalgebras of bifinite bimodules over a II_1 factor.

Definition 3.2.1 (Vaes)**.** Let M be a II_1-factor and $\mathcal{F}_1, \mathcal{F}_2$ be two fusion subalgebras of the fusion algebra of bifinite bimodules over M with basis χ_1 and χ_2 respectively. Then, $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be *free* if:

(i) every tensor product of non-trivial irreducible bimodules, with factors alternatingly from χ_1 and χ_2, is irreducible,

(ii) two tensor products of non-trivial irreducible bimodules, with factors alternatingly from χ_1 and χ_2, are equivalent if and only if they are factor by factor equivalent.

Proposition 3.2.2. *The free oriented extension of a hyperfinite realizable subfactor planar algebra is isomorphic to a oriented factor planar algebra singly generated by an extremal bifinite bimodule over the hyperfinite* II_1*-factor* R*.*

Proof. Suppose $\mathrm{Bim}_{\mathrm{ext}}(R)$ denotes the category of extremal bifinite bimodules over R. Since subfactor planar algebras are assumed to be spherical, without loss of generality, we may start with a subfactor planar algebra P associated to ${}_R\mathcal{H}_R \in \mathrm{Obj}(\mathrm{Bim}_{\mathrm{ext}}(R))$. Let Q be the singly generated oriented factor planar algebra associated to ${}_R\mathcal{H}_R$, and $\mathcal{C}$ be the full subcategory of $\mathrm{Bim}_{\mathrm{ext}}(R)$, tensor-generated by ${}_R\mathcal{H}_R$. From Remark 1.3.3, $\mathcal{C}$ is monoidally equivalent to the projection category $\mathcal{K}(\mathcal{C}^Q)$ (associated to Q) as C*-categories.

Consider an outer action κ of $\mathbb{Z}$ on R. For $n \in \mathbb{Z}$, let ${}_R\mathcal{K}_{n\,R}$ be the invertible bimodule $L^2(\kappa_n)$, that is, $\mathcal{K}_n := L^2(R)$ on which the left (resp., right) action of R is the usual one (resp., twisted by κ_n). Suppose $\mathcal{D}$ denotes the subcategory of $\mathrm{Bim}_{\mathrm{ext}}(R)$, whose irreducible sub-modules are isomorphic to $\mathcal{K}_n$'s. Clearly, $\mathcal{D}$ is equivalent to $\mathrm{Hilb}_{f.d.}(\mathbb{Z})$.

Let $\mathcal{G}_{\mathcal{C}}$ and $\mathcal{G}_{\mathcal{D}}$ be the fusion sub-algebras of the fusion algebra of $\mathrm{Bim}_{\mathrm{ext}}(R)$ corresponding to $\mathcal{C}$ and $\mathcal{D}$ respectively. Now, [Vae08, Theorem 5.1] tells us that there exists $\theta \in \mathrm{Aut}(R)$ such that the fusion sub-algebras corresponding to $\mathcal{C}$ and $L^2(\theta) \underset{R}{\otimes} \mathcal{D} \underset{R}{\otimes} \overline{L^2(\theta)}$ are free. So, replacing the outer action κ by $\mathrm{Ad}\theta \circ \kappa$, we may assume that $\mathcal{G}_{\mathcal{C}}$ and $\mathcal{G}_{\mathcal{D}}$ are free. We claim that the full subcategory $\mathcal{E}$ of $\mathrm{Bim}_{\mathrm{ext}}(R)$ tensor-generated by $\mathcal{C}$ and $\mathcal{D}$ is monoidally equivalent to $\mathcal{C} * \mathcal{D}$.

Applying Theorem 1.4.7, we get a monoidal C*-functor $A : \mathcal{C} * \mathcal{D} \to \mathcal{E}$ such that the restriction of A to $\mathcal{C}$ (resp., $\mathcal{D}$) is equivalent to the containment of $\mathcal{C}$ (resp., $\mathcal{D}$) in $\mathcal{E}$ as a full subcategory. Since $\mathcal{C} * \mathcal{D}$ and $\mathcal{E}$ are semi-simple, rigid C*-tensor categories, A being a monoidal C*-functor, must be faithful. Definition 3.2.1 (i) and Proposition 1.4.6 implies that A must send simple objects to simple ones; Definition 3.2.1 (ii) implies that A must induce a bijection on isomorphism classes of simple objects. Any monoidal C*-functor between semi-simple rigid C*-tensor categories which induces a bijection between isomorphism classes of simple objects is an equivalence.

Thus we have a copy of $\mathcal{C} * \mathrm{Hilb}_{f.d.}(\mathbb{Z})$ as a full subcategory of $\mathrm{Bim}_{\mathrm{ext}}(R)$, so the result follows by Theorem 3.1.7. □

Given a subfactor planar algebra, it would be interesting to find out all possible oriented extensions (up to isomorphism).

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