Higher Chow Cycles on the Jacobian of curves

Subham Sarkar

Indian Statistical Institute

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Author:
Subham Sarkar

Supervisor:
Prof. Ramesh Sreekantan

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Chapter 1

Introduction

The following formula, usually called Beilinson's formula — though independently due to Deligne as well — describes the motivic cohomology group of a smooth projective variety X over a number field as the group of extensions in a conjectured abelian category of mixed motives, $\mathcal{MM}_{\mathbb{Q}}$. If i and n are two integers then [Sch93],

$$\mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}}(\mathbb{Q}(-n), h^i(X)) = \begin{cases} CH^n_{\mathrm{hom}}(X) \otimes \mathbb{Q} & \text{if } i+1 = 2n \\ H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n)) & \text{if } i+1 \neq 2n \end{cases}$$

Hence, if one had a way of constructing extensions in the category of mixed motives by some other method, this would provide a way of constructing motivic cycles.

One way of doing so is by considering the group ring of the fundamental group of an algebraic variety $\mathbb{Z}[\pi_1(X,P)]$. If J_P is its the augmentation ideal, the kernel of the map from $\mathbb{Z}[\pi_1(X,P)] \to \mathbb{Z}$, then the graded pieces J_P^a/J_P^b with $a < b$ are expected to have a motivic structure. These give rise to natural extensions of motives. So one could hope that these extensions could be used to construct natural motivic cycles.

Understanding the motivic structure on the fundamental group is appears difficult. However, the Hodge structure on the fundamental group is well understood [Hai87]. The regulator of a motivic cohomology cycle can be thought of as the realisation of the corresponding extension of motives as an extension in the category of mixed Hodge structures. So while we may not be able to construct motivic cycles as extensions of *motives* coming from the fundamental group - we can hope to construct their regulators as extensions of *mixed Hodge structures* (MHS) coming from the fundamental group.

The aim of this thesis is to describe this construction in the case of the motivic cohomology group of the Jacobian of a curve. The first work in this direction is due to Harris [Har83] and Pulte [Pul88], [Hai87]. They showed that the Abel-Jacobi image of the modified diagonal cycle on the triple product of a pointed curve (C, P), or alternatively the Ceresa cycle in the Jacobian $\mathrm{Jac}(C)$ of the curve, is the same as an extension class coming from J_P/J_P^3, where J_P is the augmentation ideal in the group ring of the fundamental group of C based at P.

In [Col02], Colombo extended this theorem to show that the regulator of a cycle in the motivic cohomology of a Jacobian of a *hyperelliptic* curve, discovered by Collino [Col97], can be realised as an extension class coming from J_P/J_P^4, where J_P is the augmentation ideal of a related curve. In this thesis we extend Colombo's result to more general curves.

Let C be a smooth projective curve of genus g with a function f on it whose divisor is of the form $div(f) = NQ - NR$ for some points Q and R and some integer N and $f(P) = 1$ for some other point P. Then there is a motivic cohomology cycle $Z_{QR,P}$ in $H^{2g-1}_{\mathcal{M}}(\mathrm{Jac}(C), \mathbb{Z}(g))$ discovered by Bloch [Blo00]. We show that the regulator of this cycle can be expressed in terms of an extensions coming from J_P/J_P^4. When C is hyperelliptic and Q and R are ramification points of the canonical map to $\mathbb{P}^1$, this recovers Colombo's result.

A crucial step in Colombo's work is the fact that the modified diagonal cycle is *torsion* in the Chow group $CH^2_{\mathrm{hom}}(C^3)$ when C is a hyperelliptic curve. This means the extension coming from J_P/J_P^3 splits and hence does not depend on the base point P. This allows her to consider the extension for J_P/J_P^4. For general curves modified diagonal cycle is not torsion. In fact the known examples of non-torsion modified diagonal cycles come from the curves we consider - namely modular and Fermat curves. Our main contribution is to use an idea of Rabi [Rab01] to show that Colombo's arguments can be extended to work in our case as well. As a result we have a more general situation — which has some arithmetical applications.

Colombo's paper had some errors in Propositions 3.2 and 3.3 which was pointed out by a referee of an earlier version of this thesis. Hence we had to make some revisions.

As it turns out the statement of the main result continues to hold under some restricted conditions.

1.0.1 Main Theorem

We have the following theorem (**Theorem** (4.2.13)):

Theorem 1.0.1. *Let C be a smooth projective curve of genus g over $\mathbb{C}$. Let P, Q and R be three distinct points such that there is a function f_{QR} with $div(f_{QR}) = NQ - NR$ for some integer N and $f_{QR}(P) = 1$. Let $Z_{QR} = Z_{QR,P}$ be the element of the motivic cohomology group $H^{2g-1}_{\mathcal{M}}(\mathrm{Jac}(C), \mathbb{Z}(g))$ constructed by Bloch [Blo00]. There exists an extension class $\epsilon^4_{QR,P}$ in $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), \wedge^2 H^1(C))$ constructed from the mixed Hodge structures associated to the fundamental groups $\pi_1(C\backslash Q, P)$ and $\pi_1(C\backslash R, P)$ such that*

$$\epsilon^4_{QR,P} = (2g+1)N \,\mathrm{reg}_{\mathbb{Z}}(Z_{QR})$$

in $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), \wedge^2 H^1(C))$.

In other words the regulator of a natural cycle in the motivic cohomology group of a Jacobian of a curve, being thought of as an extension class, is same as the extension class of a natural extension of mixed Hodge structures coming from the fundamental group of the curve.

Our primary motivation are the conjectures relating regulators of the motivic cycles to special values of L-functions. One application we have is to the case of modular curves. Beilinson [Beĭ84] constructed a cycle in the group $H^3_{\mathcal{M}}(X_0(N) \times X_0(N), \mathbb{Q}(2))$ and showed that its regulator is related to a special value of the L-function. We construct the extension of MHS coming from the fundamental group which corresponds to the regulator of the image of this cycle in the Jacobian of $X_0(N)$. In other words, this is the projection on to the sub-motive $\wedge^2 H^1(C)$ of $\otimes^2 H^1(C)$.

Since the mixed Hodge structure associated to the fundamental group is related to iterated integrals we also get an expression for the regulator as an iterated integral. In a subsequent paper we apply this in the case of Fermat curves to get an explicit expression for the regulator in terms of hypergeometric functions analogous to the works of Otsubo [Ots12],[Ots11].

Darmon-Rotger-Sols [DRS12] have used the modified diagonal cycle to construct points on Jacobians of the curves and used the iterated integral approach to find a formula for the Abel-Jacobi image of these points. Starting with Bloch [Blo84] and later Collino [Col97] and Colombo [Col02] it has been known that these null homologous cycles degenerate to higher Chow cycles on related varieties. Recently Iyer and Müller-Stach [IMS14] have shown that the modified diagonal cycle degenerates to the kind of cycles we consider in some special cases. This degeneration can be understood from the point of view of extensions and we make a few remarks on that.

The thesis is organised as follows. In §2, we begin by recalling the definition of motivic cohomology, Deligne cohomology groups and the regulator map for a graded part of K_1. In §3, we discuss mixed Hodge structures (MHS). The main theme of this section is to understand certain interesting extensions in the category of MHS. Here we discuss the result of Pulte which explains the Abel-Jacobi image of the Ceresa cycles as an extension class coming from fundamental group of C. In §4, we discuss the generalisation of the result of Pulte by Colombo [Col02] which expresses the regulator image of Collino cycle as an extension coming from fundamental group of an open curve. In the Appendix we describe the Baer sum which is the addition in the Ext-group and a generalisation due to Rabi [Rab01].

Notations and Abbreviations

- $X := X(\mathbb{C})$, the set of $\mathbb{C}$ valued points of an algebraic variety X.
- $H^i(X) :=$ The singular cohomology with $\mathbb{Z}$ coefficients.
- $C :=$ a smooth projective curve of genus g.
- $\mathrm{Jac}(C) :=$ The Jacobian of C.
- $MHS :=$ category of *integral mixed Hodge structures*.
- $H^i_c(X) :=$ The cohomology group of forms with compact support on X.
- $J_P := \mathrm{Ker}(\mathbb{Z}\pi_1(C, P) \longrightarrow \mathbb{Z})$ and $J_{\bullet,P} := \mathrm{Ker}(\mathbb{Z}\pi_1(C \setminus \{\bullet\}, P) \longrightarrow \mathbb{Z})$, where $\bullet \in C \setminus \{P\}$.
- $H^i_{\mathcal{D}}(X, A(n)) :=$ the Deligne cohomology group with coefficients in a $\mathbb{Z}$ module A
- $H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) :=$ the integral Motivic cohomology group of X.
- $C \setminus \gamma := C \setminus U(\gamma)$ be the manifold with boundary $C \setminus \gamma$, where $U(\gamma)$ is an open tubular neighborhood of γ in C.

Chapter 2

Motivic cohomology, Deligne cohomology and Regulator map

Beilinson formulated a set of conjectures relating the values at integers of L-functions of an algebraic variety defined over a number field to algebraic invariants coming from motivic cohomology groups of the variety. He defined a regulator map from motivic cohomology groups to certain real vector spaces called Real Deligne cohomology groups. He then conjectured that the image of the regulator map is a full sub-lattice of this space. Further, he conjectured that the covolume of this lattice is related to the first non-zero value of the Taylor expansion of the L-function. Beilinson conjectures on special values of L-function are discussed in the first chapter of the book [Sch88]. However, as we are not going to address the conjectures directly we will describe the objects only over $\mathbb{C}$, though much of what we do can be done over number field. In this section we introduce the objects involved especially in the particular case of curves and related objects.

2.1 Motivic cohomology

Let X be a smooth projective variety defined over $\mathbb{C}$ and $K_i(X)$ be the i^{th} higher algebraic K-group introduced by Quillen [Qui10]. The motivic cohomology groups of X are defined to be, for integer $n \geq 0$,

$$H_{\mathcal{M}}^{2n-1}(X, \mathbb{Z}(n)) := K_1(X)^{(n)},$$

where $K_1(X)^{(n)}$ is the Adams eigenspace of weight n [Sch88]. This has the following alternative description. Let $Z^i(X)$ be the free abelian group generated by irreducible subvarieties of X of codimension i. Let Z be an irreducible subvariety of codimension $n-1$ in X. Let $j : \tilde{Z} \to Z$ be a normalization of Z. Let k_Z be the field of rational functions on Z and k_Z^* be the set of all nonzero elements of k_Z. Let us denote $div_Z(f) := j_* div_{\tilde{Z}}(f) \in Z^n(X)$. Then one has

$$H^{2n-1}_{\mathcal{M}}(X, \mathbb{Z}(n)) := \frac{\operatorname{Ker}\left(\bigoplus_{Z \in Z^{n-1}(X)} k_Z^* \overset{\oplus div_Z(f)}{\longrightarrow} Z^n(X)\right)}{\operatorname{Im}\left(K_2(X) \overset{T}{\longrightarrow} \bigoplus_{Z \in Z^{n-1}(X)} k_Z^*\right)},$$

where T is the Tame symbol map. An element of the above group can be represented by $Z = \sum_{i=1}^t (Z_i, f_i)$ such that $Z_i \in Z^{n-1}(X)$ and $f_i \in k(Z_i)^*$ such that $\sum_{i=1}^t div_{Z_i}(f_i) = 0$.

Beilinson defined motivic cohomology groups as a group of extensions in the conjectured category $\mathcal{MM}_{\mathbb{Q}}$ of mixed motives [Beĭ87]. In particular one expects

$$\begin{aligned} \operatorname{Ext}_{\mathcal{MM}_{\mathbb{Q}}}(\mathbb{Q}(-n), h^{i-1}(X)) \simeq & H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) \text{ for } 2n \neq i \\ \simeq & \operatorname{CH}^n_{\text{hom}}(X)_{\mathbb{Q}} \text{ for } 2n = i. \end{aligned}$$

There are various categories of mixed motives constructed by Levine [Lev13], Deligne-Jannsen. Thus an element in the motivic cohomology group can be interpreted as an extension class in those respective categories. The regulator map which we discuss here is the realisation of that extension class in the category of Mixed Hodge Structures.

2.1.1 Elements in the motivic cohomology group $H^{2g-1}_{\mathcal{M}}(X, \mathbb{Z}(g))$

Let C be a smooth projective curve of genus g over $\mathbb{C}$. Let $X = \operatorname{Jac}(C)$ be its Jacobian. In this section we will construct an element $Z_{QR,P} \in H^{2g-1}_{\mathcal{M}}(X, \mathbb{Z}(g))$ under the added assumption that there exist two distinct points Q, $R \in C$ and a function f_{QR} with

$$div(f_{QR}) = N(Q - R)$$

for some integer N. To determine the function precisely we have to choose another distinct point $P \in C$ and require that $f_{QR}(P) = 1$.

The first examples of such curves and functions are hyperelliptic curves. In this case the element was constructed by Collino [Col97]. Other examples of curves and functions are modular curves with the points being cusps ([Man72]) and Fermat curves with the points being the 'trivial solutions' of Fermats Last Theorem.

Let C_Q be the image of the map $i_Q : C \to \mathrm{Jac}(C)$ defined by $x \to x - Q$ and C^R be the image of the map $i^R : C \to \mathrm{Jac}(C)$ defined by $x \to R - x$. Let f_Q, f^R denote the function f_{QR} considered as a function on C_Q and C^R respectively.

Consider the cycle in $\mathrm{Jac}(C)$ given by

$$Z_{QR,P} := (C_Q, f_Q) + (C^R, f^R) \tag{2.1.1}$$

$$div_{C_Q}(f_Q) + div_{C^R}(f^R) = N(0) - N(R-Q) - N(0) + N(R-Q) = 0 \tag{2.1.2}$$

This implies that $Z_{QR,P}$ gives an element of $H^{2g-1}_{\mathcal{M}}(\mathrm{Jac}(C), \mathbb{Z}(g))$. Consider the natural map

$$\begin{aligned} \eta : C \times C &\to \mathrm{Jac}(C) \\ (x, y) &\mapsto (x - y). \end{aligned} \tag{2.1.3}$$

It induces a functorial homomorphism

$$\eta_* : H^3_{\mathcal{M}}(C \times C, \mathbb{Z}(2)) \to H^{2g-1}_{\mathcal{M}}(\mathrm{Jac}(C), \mathbb{Z}(g)).$$

Let us define the element

$$Z_{\Delta^{QR},P} := (C \times Q, 1/f^Q) + (\Delta, f^\Delta) + (R \times C, 1/f^R), \tag{2.1.4}$$

in $H^3_{\mathcal{M}}(C \times C, \mathbb{Z}(2))$. Bloch studied $Z_{\Delta^{QR},P}$ when $C = X_0(37)$, with Q, R cusps. In fact $Z_{\Delta^{QR},P}$ maps to Collino cycle –

$$\eta_*(Z_{\Delta^{QR},P}) = Z_{QR,P} \in H^{2g-1}_{\mathcal{M}}(\mathrm{Jac}(C), \mathbb{Z}(g)).$$

2.2 Deligne cohomology

Let $X = \mathrm{Jac}(C)$. The Deligne cohomology group of X with $\mathbb{Z}$ coefficients is defined to be the hypercohomology of certain complex, known as Deligne complex (Page 7, in [Sch88]). In the case, one has the following identification

$$\begin{aligned} H^{2g-1}_{\mathcal{D}}(X, \mathbb{Z}(g)) &\cong \frac{H^{2g-2}(X, \mathbb{C})}{F^g H^{2g-2}(X, \mathbb{C}) + H^{2g-2}(X, \mathbb{Z}(g))} \\ &\cong \frac{(F^1 H^2(X, \mathbb{C}))^*}{H_2(X, \mathbb{Z}(2))}, \end{aligned}$$

where second isomorphism is induced by Poincaré duality.

The Deligne cohomology with $\mathbb{Z}$ coefficients is thus a generalised complex torus. In other words it is the $\mathbb{C}$-vector space of linear functionals on the cohomology group $F^1H^2(X)$ modulo the lattice $H_2(X, \mathbb{Z}(2))$. In next chapter we identify the Deligne cohomology group with the group of Extensions of Mixed Hodge structures. The Deligne cohomology with $\mathbb{R}$ coefficients is obtained by considering $\mathbb{R} - MHS$s. Deligne cohomology with $\mathbb{R}$ coefficients is

$$H^{2g-1}_{\mathcal{D}}(X, \mathbb{R}(g)) = \frac{(F^1 H^2(X, \mathbb{C}))^*}{H_2(X, \mathbb{R}(2))} \cong (F^1 H^2(X, \mathbb{R}(1)))^*.$$

2.3 Regulator Maps

The Regulator map is a map from motivic cohomology group to Deligne cohomology group. Conjecturally an element $Z \in H^{2g-1}_{\mathcal{M}}(X, \mathbb{Z}(g))$ corresponds to an extension in $\mathcal{MM}_{\mathbb{Q}}$. The Regulator map is the realisation of such extensions in the category of MHSs. In other words for any Z one obtains an extension of MHS. By the Carlson isomorphism (see Theorem 3.2.1) such an extension class can be evaluated as a functional on the cohomology group $F^1 \wedge^2 H^1(C)$. An example of such extension class was discussed in §6 in [KLMS06].

Beilinson defined a regulator map

$$\mathrm{reg}_{\mathbb{Z}} : H^{2n-1}_{\mathcal{M}}(X, \mathbb{Z}(n)) \longrightarrow H^{2n-1}_{\mathcal{D}}(X, \mathbb{Z}(n)).$$

In the particular case when $n = g$, the motivic cohomology group is $H^{2g-1}_{\mathcal{M}}(X, \mathbb{Z}(g))$. One has the following explicit formula: Let

$$Z = \sum_i (C_i, f_i)$$

be an element of the motivic cohomology group, where C_i and f_i satisfy the conditions (2.1.2). Let $[0, \infty]$ be the path from 0 to ∞ along the real axis in $\mathbb{P}^1(\mathbb{C})$. Let $\mu_i : \tilde{C}_i \to C_i$ be a resolution of singularities. We can think of f_i as a map from $\tilde{C}_i$ to $\mathbb{P}^1$. Let

$$\gamma_i = \mu_{i*}(f_i^{-1}[0, \infty]).$$

From the co-cycle condition and the fact that $H_2(X)$ does not have torsion, we have

$$\sum_{i=1}^{i=t} \gamma_i = \partial(D),$$

for some 2-cycle D on X. The regulator map is defined to be

$$\operatorname{reg}_{\mathbb{Z}}(Z)(\omega) = \left(\sum_{i=1}^{i=t} \int_{C_i \setminus \gamma_i} \log(f_i)\omega + 2\pi i \int_D \omega \right). \tag{2.3.1}$$

where $\omega \in F^1 H^2(X, \mathbb{C})$. Here $C_i \setminus \gamma_i$ is the Riemann surface with boundary obtained by removing an open tubular neighbourhood of γ_i from C_i. It is a closed subset of C_i with the structure of a manifold with boundary. The boundary $\partial(C_i \setminus \gamma_i)$ is made up of two copies of γ_i with opposite orientation.

When C is a hyperelliptic curve and Z is Collino's element constructed above, Colombo [Col02] constructed an extension of mixed Hodge structures coming from the fundamental group of C which corresponds to the regulator of Z. In this thesis we generalise her construction to get an extension class corresponding to the more general elements we have discussed above.

Chapter 3

Extensions of Mixed Hodge Structures and the Regulator map

Let X be a smooth projective variety defined over $\mathbb{C}$. Its cohomology group $H^i(X)$ with complex coefficients has a bidegree decomposition known as *Hodge decomposition*. More generally Deligne showed in [Del71] that cohomology groups (modulo the torsion elements) of a variety possess a *Mixed Hodge structure* and natural maps between them are example of morphisms of MHS. Mixed Hodge Structures form an abelian category with tensor product which contains the category of pure Hodge structures as a full subcategory. Certain extensions in the category of MHS are of our interest. The image of a null homologous cycle under the Abel Jacobi maps gives an example of such an extension. More generally the image of an element of a motivic cohomology group under the regulator map gives examples of such extensions. The Carlson representative can be used to understand such extensions. In this chapter we discuss such objects.

3.1 Mixed Hodge Structures

We recall some definitions and constructions due to Griffiths and Deligne. The book of Voisin is a good reference [Voi02].

Definition 3.1.1. A *integral pure Hodge structure of weight* l is a pair $V = (V_{\mathbb{Z}}, F^{\bullet})$ where $V_{\mathbb{Z}}$ is a $\mathbb{Z}$-module and $F^{\bullet}$ is a decreasing filtration on $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$, called the Hodge filtration, such that for all $p, q \in \mathbb{Z}$ with $p + q = l + 1$,

$$F^p V_{\mathbb{C}} \oplus \overline{F^q V_{\mathbb{C}}} = V_{\mathbb{C}},$$

$$F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}} = \emptyset,$$

where $-$ indicates complex conjugation.

An *integral mixed Hodge structure (MHS)* is a triple $V = (V_{\mathbb{Z}}, W_{\bullet}, F^{\bullet})$ where $\bullet \in \mathbb{Z}$ such that

- $V_{\mathbb{Z}}$ is an integral lattice.
- $W_{\bullet}$ is an increasing filtration on $V_{\mathbb{Q}}$ called the *weight* filtration.
- $F^{\bullet}$ is a decreasing filtration on $V_{\mathbb{C}}$ called the *Hodge* filtration.
- The weight and Hodge filtration are compatible in the sense that $F^{\bullet}$ induces a pure Hodge structure of weight l on each of the graded pieces $Gr_l^W = W_l / W_{l-1} \otimes \mathbb{C}$.

Let $V' = (V'_{\mathbb{Z}}, W_{\bullet}, F^{\bullet})$ be another object in the category of MHS. Then a morphism of weight $2m$ of mixed Hodge structures is a map $\phi : V \longrightarrow V'$ such that

- $\phi : V_{\mathbb{Z}} \longrightarrow V'_{\mathbb{Z}}$ is a group homomorphism.
- $\phi(\mathrm{F}^p\, V_{\mathbb{C}}) \subset \mathrm{F}^{p+m}\, V'_{\mathbb{C}}$.
- $\phi(W_n V) \subset W_{n+2m} V'$.

Let us formulate few properties of the category of MHS.

- *MHS on* Hom: For two MHS V_1, V_2, $\mathrm{Hom}(V_1, V_2)$ has a natural Mixed Hodge structure. The Hodge filtration $F^{\bullet}$ is defined by

$$F^p \operatorname{Hom}(V_1, V_2)_{\mathbb{C}} = \{f \in \operatorname{Hom}(V_1, V_2)_{\mathbb{C}} : f(F^i V_{1\mathbb{C}}) \subset F^{p+i} V_{2\mathbb{C}}, i \geq 0\}.$$

Weight filtration $W_\bullet$ is defined by

$$W_p \operatorname{Hom}(V_1, V_2)_{\mathbb{Q}} = \{f \in \operatorname{Hom}(V_1, V_2)_{\mathbb{Q}} : f(W_i V_{1\mathbb{Q}}) \subset W_{p+i} V_{2\mathbb{Q}}, \text{ for all } i \geq 0\}.$$

- *Direct Sum*: Let V_1 and V_2 be two MHS. Their direct sum $V = V_1 \oplus V_2$ is a MHS with $W_m V = W_m V_1 \oplus W_m V_2$ and $F^m(V_{\mathbb{C}}) = F^m V_{1\mathbb{C}} \oplus F^m V_{2\mathbb{C}}$.
- *Tensor product*: For two MHS V_1 and V_2, Hodge and weight filtration the tensor product $V_1 \otimes V_2$ is defined by

$$F^p(V_1 \otimes V_2)_{\mathbb{C}} = \oplus_i F^i V_{1\mathbb{C}} \otimes F^{p-i} V_{2\mathbb{C}}$$

$$W_p\,(V_1 \otimes V_2) = \oplus_i W_i V_{1\mathbb{Q}} \otimes W_{p-i} V_{2\mathbb{Q}}.$$

- If f is a morphism of MHS, then $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are also have a mixed Hodge structure as sub and quotient groups with induced filtrations. Deligne has shown in [Del74] that MHS on the cokernel of the kernel coincides with the MHS on the kernel of the cokernel. Hence MHS form an abelian category.

3.1.1 Examples of MHS

Let X be a variety defined over $\mathbb{C}$. The i^{th} cohomology group of X, $H^i(X)$, is endowed with a functorial *mixed Hodge structure*. That has weight filtration of length $2i$

$$\{0\} = W_{-1} \subset W_0 \subset ... \subset W_{2i} = H^i(X)$$

where Gr_k^W is a pure Hodge structure of weight k. When X is smooth and projective then $W_l = 0$ for $l \neq i$ and $W_i = H^i(X)$. In other words $H^i(X)$ has a pure Hodge structure of weight i. If X is smooth but not necessarily projective then $W_l = 0$ for $l < i$ and for X projective but not necessarily smooth then $W_l = H^i(X)$ for $l \geq i$.

Another source of MHS are the homotopy groups of a complex algebraic variety [Hai87]. A first non trivial case in this direction is MHS on the subquotients of $\mathbb{Z}\pi_1(X, P)$ by its augmentation ideals where P be a fixed closed point in $X(\mathbb{C})$. Let

$$J_P = \operatorname{Ker}\left[\mathbb{Z}\pi_1(X, P) \to \mathbb{Z}\right]$$

be the augmentation ideal of the group-ring $\mathbb{Z}\pi_1(X(\mathbb{C}), P)$. Let $\gamma : [0,1] \to X$ be piecewise smooth path on X and ω_i are smooth $\mathbb{C}$-valued 1-form on X. The *iterated integral of length* n of $\omega_1\omega_2..\omega_n$ is defined by

$$\int_\gamma \omega_1\omega_2...\omega_n = \int_{0\le t_1\le t_2\le ...\le t_n=1} ... \int f_1(t_1)..f_i(t_i)..f_n(t_n)dt_1dt_2...dt_n,$$

where $\gamma^*(\omega_i) = f_i(t_i)dt_i$. An iterated integral of length $\le n$ is linear combination of iterated integrals of length $l \le n$. It is a functional on the space of paths on $X(\mathbb{C})$. An iterated integral is said to be a homotopy functional if it depends only on the homotopy class of a path. Let us recall a few properties of iterated integrals.

Lemma 3.1.2. *Let ω_1 and ω_2 be smooth 1-forms on X and α, β be two piecewise smooth paths on X with $\alpha(1) = \beta(0)$. Then*

1. $\int_{\alpha.\beta} \omega_1.\omega_2 = \int_\alpha \omega_1.\omega_2 + \int_\beta \omega_1.\omega_2 + \int_\alpha \omega_1 . \int_\beta \omega_2$
2. $\int_\alpha \omega_1.\omega_2 + \int_\alpha \omega_2.\omega_1 = \int_\alpha \omega_1 . \int_\alpha \omega_2$
3. $\int_\alpha df\omega_1 = \int_\alpha f\omega_1 - f(\alpha(0)) \int_\alpha \omega_1$
4. $\int_\alpha \omega_1 df = f(\alpha(1)) \int_\alpha \omega - \int_\alpha f\omega_1$

Proof. Proposition 1.3 in [Hai87]. □

Let $H^0(B_s(X, P))$ be the $\mathbb{C}$ vector space generated by iterated integrals which are homotopy functionals of length less than or equal to s. Chen's π_1-de Rham theorem states the following-

Theorem 3.1.3. *[Hai87] For $s \ge 0$, there is an isomorphism*

$$\pi_{DR} : H^0(B_s(X), P)) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1(X, P)/J_P^{s+1}, \mathbb{C}).$$

Proof. Theorem 4.1 in [Hai87]. □

In [Hai87] Hain defined a canonical mixed Hodge structure on $H^0(B_s(X), P)$. Hence from the above isomorphism, one has.

Theorem 3.1.4. *If X is an algebraic variety over $\mathbb{C}$ and $P \in X$, there is a MHS on* $\mathbb{Z}\pi_1(X,P)/J_P^{s+1}$.

Proof. Theorem (5.1) in [Hai87]. □

3.2 Group of extensions of Mixed Hodge Structures

An extension H of B by A in the category MHS is represented by a short exact sequence

$$0 \to A \to H \to B \to 0,$$

where A, B and H are Mixed Hodge structures. An extension is called separated if the highest weight of A is less than the lowest weight of B. Let $\mathrm{Ext}^1_{MHS}(B,A)$ be the set of congruence classes of separated extensions of B by A. It is an abelian group, where addition is defined by *Baer Sum*, discussed in the Appendix. In [Car80], Carlson showed that $\mathrm{Ext}^1_{MHS}(B,A)$ has an alternative description which is more amenable to computation.

For a mixed Hodge structure V of negative weight, let

$$J^0(V) := \frac{V_{\mathbb{C}}}{F^0V_{\mathbb{C}} + V_{\mathbb{Z}}}.$$

When A and B are separated extensions of MHS, $\mathrm{Hom}(B,A)$ is of negative weight. Carlson showed

Theorem 3.2.1. *[Car80] Let A and B be two MHS such that the highest weight of A is less than the lowest weight of B. Then the group*

$$\mathrm{Ext}^1_{MHS}(B,A) \cong J^0\,\mathrm{Hom}(B,A),$$

Proof. See Proposition 2 in [Car80]. □

From this one can see that the Deligne cohomology group $H^{2g-1}_{\mathcal{D}}(\mathrm{Jac}(C), \mathbb{Z}(g)) \cong \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), \wedge^2 H^1(C))$.

3.3 Examples from Geometry

In this section we discuss the regulator map in two situations. The first is the Abel-Jacobi map from $K_0(\mathrm{Jac}(C))^{g-1}$ and the second is the higher regulator map from $K_1(\mathrm{Jac}(C)^g$.

3.3.1 Abel-Jacobi map as an extension of MHS

Let $P \in C$. Consider the map $i_P : C \to \mathrm{Jac}(C)$, defined by $x \to x - P$. Let C_P be the image of C. Let C^P be the image of C_P under the involution $(-1)^*$ induced by multiplication by -1 on $\mathrm{Jac}(C)$. $(-1)^*$ acts by the identity on $H^{2k}(\mathrm{Jac}(C))$. Hence, for two distinct points P and Q, the cycle $C_P - C^Q$ is a codimension $(g-1)$ homologically trivial cycle. If $P = Q$ such cycles are called Ceresa cycles.

Let J_P be the augmentation ideal as in Section 3.1.1. Consider the extension

$$e_P^3 : \ 0 \to (J_P/J_P^2)^* \to (J_P/J_P^3)^* \to (J_P^2/J_P^3)^* \to 0.$$

One knows $(J_P/J_P^2) = H_1(C, \mathbb{Z})$. Let $K = \mathrm{Ker}(\otimes^2 H^1(C) \xrightarrow{\cup} H^2(C))$. It turns out (Section 6 in [Hai87]) $(J_P^2/J_P^3)^* = K$ as MHS. Both $H^1(C)$ and K do not depend on the base point P. Hence the extension e_P^3 determines a class $m_P^3 \in \mathrm{Ext}^1_{MHS}(K, H^1(C))$. Generalising a result of Harris, Pulte obtained an expression for the Abel-Jacobi image of the Ceresa cycle in terms of the class m_P^3.

Theorem 3.3.1. *[Pul88] Let C be a smooth projective curve of genus g. Let P and Q be points on C and $C_P - C^Q$ be null homologous cycles defined above. Then the extension class corresponding to the image of $C_P - C^Q$ under the Abel-Jacobi map is given by $m_P^3 + m_Q^3$.*

$$\begin{aligned} \mathrm{CH}^{g-1}_{\mathrm{hom}}(\mathrm{Jac}\, C) &\xrightarrow{Aj} J^0 H_3(\mathrm{Jac}(C)) \xrightarrow{\Phi} \mathrm{Ext}^1_{MHS}(K, H^1(C)) \\ C_P - C^Q &\longrightarrow Aj(C_P - C^Q) \longrightarrow m_P^3 + m_Q^3. \end{aligned}$$

Proof. See Theorem 3.9 and 4.9 in [Pul88]. □

In particular, if $P = Q$ the Ceresa cycle $C_P - C^P$ corresponds to the extension class $2m_P^3$. As an application, if C is a Fermat curve of degree ≥ 5, Otsubo used this theorem to express the Abel-Jacobi image of certain Ceresa cycles in terms of values of hypergeometric series (Theorem 4.8, 5.3 [Ots12]).

3.3.2 Beilinson Regulators as Extensions of Mixed Hodge Structures

Collino [Col97] constructed a related cycle in the group $H^{2g-1}_{\mathcal{M}}(\mathrm{Jac}(C), \mathbb{Z}(g))$, where C is a hyper-elliptic curve of genus g. Colombo [Col02] extended Pulte's result to this cycle as follows. Collino's cycle depends on a choice of Weierstrass points Q and R on the hyper-elliptic curve.

Let us consider $C \setminus \bullet$, where $\bullet \in \{Q, R\}$ and $J_{\bullet,P} := \ker\{\mathbb{Z}[\pi_1(C \setminus \{\bullet\}, P)] \to \mathbb{Z}\}$. For $r \geq 3$ one has extensions,

$$e^r_{\bullet P} : 0 \to (J_{\bullet,P}/J^{r-1}_{\bullet,P})^* \to (J_{\bullet,P}/J^r_{\bullet,P})^* \to (J^{r-1}_{\bullet,P}/J^r_{\bullet,P})^* \to 0.$$

Let

$$m^r_{\bullet P} \in \mathrm{Ext}^1_{MHS}\left((J^{r-1}_{\bullet,P}/J^r_{\bullet,P})^*), (J_{\bullet,P}/J^{r-1}_{\bullet,P})^*\right)$$

be the extension class associated to the extension $e^r_{\bullet P}$. One knows $(J^{r-1}_{\bullet,P}/J^r_{\bullet,P})^* = \otimes^{r-1}H^1(C)$ so it does not depend on the base point P or the points Q and R. However, $(J_{\bullet,P}/J^r_{\bullet,P})^*$ could. In the case $r = 4$ when C is hyperelliptic, since $(J_{\bullet,P}/J^3_{\bullet,P})^*$ is related to the class of the Ceresa cycle. We know from [Col02] Proposition 2.1 that this class is $2 - torsion$. Hence the extension splits rationally. Hence it turns out that the classes m^4_{QP} and m^4_{RP} lie in $\mathrm{Ext}^1_{MHS}(\otimes^3 H^1(C), \otimes^2 H^1(C) \oplus H^1(C))$. Colombo shows that the class of the Collino cycle is given by the extension

$$e^4_{QP} \ominus_B e^4_{RP},$$

where $\ominus_B$ denotes the Baer difference of the two extensions.

In general, the class of the Ceresa cycle need not be torsion. For instance, if C is a modular curve or a Fermat curve, one has examples when it is known to be non-torsion. Under some conditions one can still construct a cycle similar to Collino's cycle and we extend the result of Colombo to this more general case. In order to do this, we need to use the work of Kaenders in which he describes the extension class corresponding to $(J_{\bullet,P}/J^3_{\bullet,P})^*$.

Consider the extension $e^3_{\bullet P}$. From Hain, one has that $(J^2_{\bullet,P}/J^3_{\bullet,P})^* = \mathrm{Ker}(\otimes^2 H^1(C \setminus \{\bullet\}) \xrightarrow{\cup} H^2(C \setminus \{\bullet\}))$. One has $H^1(C \setminus \{\bullet\}) = H^1(C)$. Since $C \setminus \{\bullet\}$ is non-compact, $H^2(C \setminus \{\bullet\}) = 0$, so $(J^2_{\bullet,P}/J^3_{\bullet,P})^* = \otimes^2 H^1(C)$. Hence the extension class $m^3_{\bullet P}$ lies in $\mathrm{Ext}^1_{MHS}(\otimes^2 H^1(C), H^1(C))$.

As above K is the kernel of the cup product map $\otimes^2 H^1(C) \to H^2(C) = \mathbb{Z}(-1)$. The exact sequence of Hodge structures

$$0 \to K \to \otimes^2 H^1(C) \xrightarrow{\cup} \mathbb{Z}(-1) \to 0$$

splits over $\mathbb{Q}$ but not over $\mathbb{Z}$. This happens as follows: There is a bilinear form [Kae01]

$$b : \otimes^2 H^1(C) \times \otimes^2 H^1(C) \longrightarrow \mathbb{Z}$$

defined by

$$b(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1 \cup y_2) \cdot (x_2 \cup y_1).$$

Let S denote the orthogonal complement of K in $\otimes^2 H^1(C)$ with respect to this bilinear form. Then, under the cup product S projects to $2g\mathbb{Z}(-1)$ where g is the genus of C and $\otimes^2 H^1(C)_{\mathbb{Q}} = K_{\mathbb{Q}} \oplus S_{\mathbb{Q}}$ as $\mathbb{Q}$-Hodge structures.

It is well known that $\mathrm{Ext}^1_{MHS}(S, H^1(C)) = \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1), H^1(C)) = \mathrm{Pic}^0(C)$. From the work of Hain and Pulte described in the previous section, the other term in $\mathrm{Ext}^1_{MHS}(K, H^1(C))$ is the class m^3_P of the Ceresa cycle or the extension e^3_P. Kaenders and independently Rabi, have the following explicit description of the class $m^3_{\bullet P}$.

Proposition 3.3.2. *The image of the class $m^3_{\bullet P}$ corresponding to the extension $e^3_{\bullet P}$ with respect to the above splitting is given by*

$$\mathrm{Ext}^1_{MHS}(\otimes^2 H^1(C), H^1(C)) \xrightarrow{\phi} \mathrm{Ext}^1_{MHS}(K_{\mathbb{Q}}, H^1(C)) \times \mathrm{Ext}^1_{MHS}(\mathbb{Q}(-1), H^1(C))$$
$$m^3_{\bullet P} \to (m^3_P, 2g \bullet -2P - \kappa_C),$$

where κ_C is the Canonical divisor of C.

Proof. See [Kae01] Theorem 1.2. $\square$

Let C, Q, R, P and $f_{QR,P}$ as in the section 2.1.1. Then one has $m^3_{QP} - m^3_{RP} = (0, 2g(Q-R))$. As $Q-R$ is torsion in $\mathrm{Jac}(C)$, this is torsion. It means the mixed Hodge structure $m^3_{QP} - m^3_{RP}$ splits rationally. Hence

$$(m^3_{QP} - m^3_{RP}) = H^1(C)_{\mathbb{Q}} \oplus \otimes^2 H^1(C)_{\mathbb{Q}},$$

Suppose we are able to construct an extension in MHS

$$e^4_{QR,P}: \; 0 \to (J_{QP}/J^3_{QP})^* \ominus (J_{RP}/J^3_{RP})^* \to A \to \otimes^3 H^1(C) \to 0.$$

Then projecting $e^4_{QR,P}$ to $H^1(C)$ will give a class in $\mathrm{Ext}^1_{MHS}(\otimes^3 H^1(C), H^1(C))$. Then a standard pull back and push forward argument along the lines of Colombo ([Col02]) will produce a class

$$\epsilon^4_{QR,P} \in \mathrm{Ext}^1_{MHS}(\mathbb{Q}(-2), \wedge^2 H^1(C)).$$

It turns out that this class is related to the regulator of the cycle $Z_{QR,P}$. In the next chapter we explain how one can do this.

Chapter 4

An Explicit Formula for the Regulator map

Let C, Q, R, P, f_{QR} and $Z_{QR,P}$ be as in § 2.1.1. In this chapter we prove our main theorem which relates the regulator of $Z_{QR,P}$ with an extension class coming from the fundamental groups of the curves $C \setminus Q$ and $C \setminus R$. We have the following theorem.

Theorem 4.0.1. *Let C be a smooth projective curve of genus g over $\mathbb{C}$. Let P, Q and R be three distinct points such that there is a function f_{QR} with $div(f_{QR}) = NQ - NR$ for some integer N and $f_{QR}(P) = 1$. Let $Z_{QR,P}$ be the element of the motivic cohomology group $H^{2g-1}_{\mathcal{M}}(\mathrm{Jac}(C), \mathbb{Z}(g))$ constructed in § 2.1.1 ([Blo00]). There exists an extension class $\epsilon^4_{QR,P}$ in $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), \wedge^2 H^1(C))$ constructed from the mixed Hodge structures associated to the fundamental groups $\pi_1(C\backslash Q, P)$ and $\pi_1(C\backslash R, P)$ such that*

$$\epsilon^4_{QR,P} = (2g+1)N\,\mathrm{reg}_{\mathbb{Z}}(Z_{QR})$$

in $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), \wedge^2 H^1(C))$.

4.1 The regulator of $Z_{QR,P}$

Let $Z_{QR,P}$ be the element of $H^{2g-1}_{\mathcal{M}}(\mathrm{Jac}(C), \mathbb{Z}(g))$ constructed in § 2.1.1. In this section we obtain a more explicit description of the regulator map (2.3.1) which will allow us to relate it to an extension class. To that end we obtain a more explicit description of the 2-cycle D.

Recall that $f_{QR} : C \to \mathbb{P}^1$ is a finite map of degree N. Let $[0, \infty]$ be the positive real line in $\mathbb{P}^1$ and $\gamma = f_{QR}^{-1}[0, \infty]$. Thus γ is the union of N paths γ^i which lie in different sheets having Q and R in common, where $1 \leq i \leq N$. Let f_Q, f^R be as in §2.1.1. Let γ_Q and ${}_R\gamma$ be the corresponding paths on C_Q and C^R respectively. From co-cycle condition (2.1.2) one has $\gamma_Q^-.{}_R\gamma^- = \partial(D)$, where $\alpha^-(t)$ is the inverse of the path $\alpha : [0, 1] \to C$. It is defined by $\alpha^-(t) := \alpha(1 - t)$. We parametrize $\gamma : [0, 1] \to [0, \infty] \subset C$ so that $f_{QR}(\gamma(t)) = \frac{t}{1-t}$.

Lemma 4.1.1. *Let* $a(s, t) = t$ *and* $b(s, t) = \dfrac{t(1-s)}{1-s(1-t)}$, *where* $s, t \in [0, 1]$. *Define* $F_i : [0, 1] \times [0, 1] \to \mathrm{Jac}(C)$ *by*

$$F_i(s, t) = \gamma^i(a(s, t)) - \gamma^i(b(s, t)),$$

for $1 \leq i \leq N$ *and let* $D_i = \mathrm{Im}(F_i)$.

Then $\partial(D_i) = \gamma_Q^{i-}.{}_R\gamma^{i-}$. *In particular if* $D = \cup_{i=1}^N D_i$ *then* $\partial(D) = \gamma_Q^-.{}_R\gamma^-$.

Proof. The oriented boundary of D_i is

$$\partial(D_i) = F(0, t) \cup F_i(s, 1) \cup F_i(1, 1 - t) \cup F_i(1 - s, 0).$$

Restricting F_i to the boundary

$$\begin{aligned}
F_i(0, t) &= \{\gamma^i(t) - \gamma^i(t)\} = 0 \\
F_i(s, 1) &= \{\gamma^i(1) - \gamma^i(1 - s)\} = ({}_R\gamma^i)^- \\
F_i(1, 1 - t) &= \{\gamma^i(1 - t) - \gamma^i(0)\} = \gamma_Q^{i-} \\
F_i(1 - s, 0) &= \{\gamma^i(0) - \gamma^i(0)\} = 0.
\end{aligned}$$

Therefore,

$$\partial(D_i) = \gamma_Q^{i-}.({}_R\gamma^i)^-.$$

Hence the proof follows. $\square$

Lemma 4.1.2. *Let ϕ and ψ be harmonic 1-forms on $\mathrm{Jac}(C)$ and D_i be a disc as in the above lemma. Then*

$$\int_{D_i} \phi \wedge \psi = \int_{\gamma_Q^{i-}} \phi\psi - \int_{R\gamma^{i-}} \psi\phi,$$

where the right hand side is an iterated integral.

Proof. Proof is similar to Lemma 1.3, [Col02]. □

Combining Lemma 4.1.1 and 4.1.2, expression (2.3.1) reduces to the following theorem.

Theorem 4.1.3. *Let $Z_{QR,P}$ be the element of $H^{2g-1}_{\mathcal{M}}(\mathrm{Jac}(C), \mathbb{Z}(g))$ and ϕ, ψ are two harmonic 1-forms in $\mathrm{Jac}(C)$ with ψ holomorphic and $f_{QR} = f$. Then*

$$\mathrm{reg}_{\mathbb{Z}}(Z_{QR,P})(\phi \wedge \psi) = 2 \int_{C \setminus \gamma} \log(f) \phi \wedge \psi + 2\pi i \int_{\gamma} (\phi\psi - \psi\phi)$$

Proof. Since ϕ and ψ are harmonic forms on $\mathrm{Jac}(C)$, they are translation invariant. Further, $(-1)^*$ acts by (-1) on 1-forms. Hence $\phi \wedge \psi$ is invariant.

Since γ_Q is a translate of γ and γ_R is a translate followed by the action of (-1), the integral in Lemma 4.1.2 becomes

$$\int_{D_i} \phi \wedge \psi = \int_{\gamma_Q^{i-}} \phi\psi - \int_{R\gamma^{i-}} \psi\phi = \int_{\gamma_Q^{i-}} = \int_{\gamma^i} (\phi\psi - \psi\phi).$$

Taking sum over all D_i, we obtain the following iterated integral expression

$$\int_D \phi \wedge \psi = \int_{\gamma} (\phi\psi - \psi\phi)$$

Recall that (2.3.1) states

$$\mathrm{reg}_{\mathbb{Z}}(Z_{QR,P})(\phi \wedge \psi) = \left(\int_{C^R} \log(f^R) \phi \wedge \psi + \int_{C_Q} \log(f_Q) \phi \wedge \psi + 2\pi i \int_D \phi \wedge \psi \right).$$

Since $\phi \wedge \psi$ is also invariant under translation and $(-1)^*$, this expression becomes

$$\mathrm{reg}_{\mathbb{Z}}(Z_{QR,P})(\phi \wedge \psi) = \left(2 \int_{C \setminus \gamma} \log(f) \phi \wedge \psi + 2\pi i \int_{\gamma} (\phi\psi - \psi\phi). \right)$$

$\square$

4.1.1 Elements in $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), \wedge^2 H^1(C))$

As we observed, in order to get an extension class similar to Colombo, the main obstruction is one cannot add extensions lying in different Ext groups. In order to do this we will use a homological algebra lemma which can be found in [Rab01]. Then a standard pushforward and pullback argument will produce the desired extension class $\epsilon^4_{QR,P} \in \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), \wedge^2 H^1(C))$.

Recall that for $\bullet \in \{Q, R\}$, we have extension of MHS

$$e^3_{\bullet P} : 0 \to (J_{\bullet P}/J^2_{\bullet P})^* \to (J_{\bullet P}/J^3_{\bullet P})^* \to \otimes^2 H^1(C) \to 0,$$

$$e^4_{\bullet P} : 0 \to (J_{\bullet P}/J^3_{\bullet P})^* \to (J_{\bullet P}/J^4_{\bullet P})^* \to \otimes^3 H^1(C) \to 0.$$

Let us consider the following diagram constructed from above extensions with all the rows and columns are exact and $\bullet \in \{Q, R\}$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & H^1(C) \arrow[r] \arrow[d] & H^1(C) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (J_{\bullet P}/J^3_{\bullet P})^* \arrow[r] \arrow[d, "\pi_\bullet"] & (J_{\bullet P}/J^4_{\bullet P})^* \arrow[r] \arrow[d] & \otimes^3 H^1(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \otimes^2 H^1(C) \arrow[r] \arrow[d] & \frac{(J_{\bullet P}/J^4_{\bullet P})^*}{H^1(C)} \arrow[r] \arrow[d] & \otimes^3 H^1(C) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now using Lemma (5.1.1) in the appendix we obtain the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \otimes^2 H^1(C) \arrow[d] & \\
0 \arrow[r] & \mathcal{B}_1 \arrow[r, "l"] & \mathcal{B}_2 \arrow[r, "\pi"] & F \arrow[r] \arrow[d] & 0 \\
 & & & \otimes^3 H^1(C) \arrow[d] & \\
 & & & 0 &
\end{tikzcd},$$

where

$$\begin{aligned}
\mathcal{B}_1 =&(J_{QP}/J^3_{QP})^* \ominus_B (J_{RP}/J^3_{RP})^*,\\
\mathcal{B}_2 =&(J_{QP}/J^4_{QP})^* \tilde{\ominus}_B (J_{RP}/J^4_{RP})^*\\
&\text{and}\\
F =&(J_{QP}/J^4_{QP})^* \tilde{\ominus}_B (J_{RP}/J^4_{RP})^* / (J_{QP}/J^3_{QP})^* \ominus_B (J_{RP}/J^3_{RP})^*\\
=&\frac{(J_{QP}/J^4_{QP})^*}{H^1(C)} \ominus_B \frac{(J_{RP}/J^4_{RP})^*}{H^1(C)}
\end{aligned}$$

and $\tilde{\ominus}_B$ is generalised Baer sum defined in the Appendix.

Let $m_\bullet \in \mathrm{Ext}^1_{MHS}(\otimes^3 H^1(C), \otimes^2 H^1(C))$ be the extension class obtained from the push foroward of the extension $e^4_{\bullet P}$ by $\pi_\bullet$. From Corollary 5.1.2 in Appendix the extension class m_{QR} corresponds to the extension

$$0 \to \otimes^2 H^1(C) \to F \to \otimes^3 H^1(C) \to 0.$$

where

$$m_{QR} = m_Q - m_R,$$

Lemma 4.1.4. *m_{QR} is N-torsion in $\mathrm{Ext}(\otimes^3 H^1(C), \otimes^2 H^1(C))$. Namely,*

$$N \cdot F \cong \otimes^2 H^1(C) \oplus \otimes^3 H^1(C) \quad \textit{as MHS.}$$

Proof. From Rabi [Rab01], Corollary 3.3, one has that the class m_Q and $m_R \in \mathrm{Ext}(\otimes^3 H^1(C), \otimes^2 H^1(C))$ are represented by extension whose middle term are

$$H^{23}_{Q,P} = H^1(C) \otimes e^3_{QP} \oplus_B e^3_{QP} \otimes H^1(C)$$

and

$$H^{23}_{R,P} = H^1(C) \otimes e^3_{RP} \oplus_B e^3_{RP} \otimes H^1(C).$$

Taking their difference gives

$$H^{23}_{R,P} \ominus_B H^{23}_{Q,P} = H^1(C) \otimes (e^3_{R,P} \ominus_B e^3_{Q,P}) \oplus_B (e^3_{RP} \ominus_B e^3_{QP}) \otimes H^1(C)$$

From Lemma 3.3.2 we have $\left[e^3_{R,P} \ominus_B e^3_{Q,P}\right] = (0, 2g(Q-R)) \in \mathrm{Ext}_{MHS}(\otimes^2 H^1(C), H^1(C))$. As $Q - R$ is N-torsion, we have

$$N \cdot F = N \cdot (H^{23}_{R,P} \ominus_B H^{23}_{Q,P}) \cong H^1(C) \oplus \otimes^2 H^1(C).$$

□

The middle term $\mathcal{B}_1$ corresponds to the extension $e^3_{QR,P} = e^3_Q \ominus_B e^3_R$ which is N torsion by Proposition 3.3.2. Thus we have the following extension

$$Ne^4_{QR,P} : 0 \to \otimes^2 H^1(C) \oplus H^1(C) \to \mathcal{B}_2 \to \otimes^3 H^1(C) \oplus \otimes^2 H^1(C) \to 0.$$

In other words, the extension class corresponding to the $Ne^4_{QR,P}$ is

$$\begin{aligned}[Ne^4_{QR,P}] &\in \mathrm{Ext}^1_{MHS}\left(\otimes^3 H^1(C) \oplus \otimes^2 H^1(C), \otimes^2 H^1(C) \oplus H^1(C)\right) \\ &= \prod_{i,j} \mathrm{Ext}^1_{MHS}(\otimes^i H^1(C), \otimes^j H^1(C)),\end{aligned}$$

where $i \in \{2, 3\}$ and $j \in \{1, 2\}$. Projecting to the Kunneth component we have

$$[Ne^4_{QR,P}] \in \mathrm{Ext}^1_{MHS}\left(\otimes^3 H^1(C), H^1(C)\right).$$

We use the same notation $e^4_{QR,P}$ for the projection. Let $\Omega \in \otimes^2 H^1(C)$ be a polariazation which induces a monomorphism

$$J_\Omega : H^1(C)(-1) \to \otimes^3 H^1(C).$$

Pulling back the extension under the morphism J_Ω we get

$$N[J^*_\Omega e^4_{QR,P}] \in \mathrm{Ext}^1_{MHS}(H^1(C)(-1), H^1(C)).$$

Tensoring with $H^1(C)$ we get the class

$$N[H^1(C) \otimes J^*_\Omega e^4_{QR,P}] \in \mathrm{Ext}^1_{MHS}(\otimes^2 H^1(C)(-1), \otimes^2 H^1(C)).$$

Let β be the section of the cup product map

$$\beta : \mathbb{Z}(-1) \to \otimes^2 H^1(C),$$

and $i : \wedge^2 H^1(C) \to \otimes^2 H^1(C)$ be the inclusion map. Pulling back by the map i and pushing forward by β we get

$$\epsilon^4_{QR,P} = i^* \beta_*([N J^*_\Omega e^4_{QR,P} \otimes H^1(C)]) \in \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), \wedge^2 H^1(C)).$$

In the following section we will compute the Carlson representative of this extension class $\epsilon^4_{QR,P}$. We will conclude by comparing the expression for the regulator of $Z_{QR,P}$ and and the Carlson representative of the class $\epsilon^4_{QR,P}$.

4.2 Carlson representative of the extension $\epsilon^4_{QR,P}$

This section is an application of the Theorem 3.2.1. It identifies an extension group in the category of mixed Hodge structures with a generalised torus. In order to relate our extension class with the regulator of the motivic cycle we need to compute its image under this isomorphism. This is similar to the case of the Abel-Jacobi image of the Ceresa cycle.

We first describe the Carlson representative of the extension

$$e^4_{QR,P} \in \mathrm{Ext}_{MHS}(\otimes^3 H^1(C), H^1(C)).$$

From the Theorem 3.1.2, this is an element of

$$J^0(\mathrm{Hom}(\otimes^3 H^1(C), H^1(C)) = \frac{\mathrm{Hom}(\otimes^3 H^1(C)_{\mathbb{C}}, H^1(C)_{\mathbb{C}})}{F^0\,\mathrm{Hom}(\otimes^3 H^1(C)_{\mathbb{C}}, H^1(C)_{\mathbb{C}}) \oplus \mathrm{Hom}(\otimes^3 H^1(C), H^1(C))}.$$

Thus, given an element of $\otimes^3 H^1(C)_{\mathbb{C}}$ we get an element of $H^1(C)_{\mathbb{C}}$ which we can think of as a functional on $H_1(C)_{\mathbb{C}}$.

Let C_{QR} denote the open curve $C \setminus \{Q, R\}$. In fact we will describe the functional as an iterated integral made up of forms in $H^1(C_{QR})_{\mathbb{C}}$ and will naturally be a functional on $H_1(C_{QR})$. We have a natural inclusion

$$i : C_{QR} \hookrightarrow C$$

which induces i_* on homology and i^* on cohomology. In order to consider the iterated integral as a functional on $H_1(C)$ we have to make a choice of an embedding $H_1(C) \hookrightarrow H_1(C_{QR})$ which splits the map i_*. There are many ways of doing this, but for our formula to work, we need to make a particular choice. In this section we first construct a 'natural' splitting of the map i_* – namely a subgroup of $H_1(C_{QR})$ which maps isomorphically to $H_1(C)$ under i_*.

Consider the group $\pi_1(C_{QR}; P)$. This is a free group on $2g + 1$ generators. The generators have the following description. The fundamental polygon of C is a $4g$ sided polygon with the edges e_i and e_{i+g} identified. The end points of the edges are identified and so they give $2g$ loops α'_i in C_{QR} which we consider as loops based at P. Let β_Q be a small simple loop around Q based at P. Then $\pi_1(C_{QR}; P) =< \alpha'_1, \ldots, \alpha'_{2g}, \beta_Q >$. Without loss of generality we assume that f is unramified at P.

The map $f : C \to \mathbb{P}^1$ restricts to give $f : C_{QR} \to \mathbb{P}^1 - \{0, \infty\}$ and this induces

$$f_* : \pi_1(C_{QR}; P) \longrightarrow \pi_1(\mathbb{P}^1 - \{0, \infty\}; 1).$$

One knows $\pi_1(\mathbb{P}^1 - \{0, \infty\}) \simeq \mathbb{Z}$. Let β_0 denote the generator. Let $H = \mathrm{Ker}(f_*)$. $f_*(\pi_1(C_{QR}; P))$ is a subgroup of $\mathbb{Z}$. In a deleted neighbourhood of 0 the map looks like

$z \to z^N$ where N is the degree. Hence the loop β_Q is taken to $N\beta_0$. Let $f_*(\alpha'_i) = m_i\beta_0$ for some $m_i \in \mathbb{Z}$. Then $\alpha_i = {\alpha'_i}^N \beta_Q^{-m_i}$ satisfies $f_*(\alpha_i) = 0$. Let G denote the subgroup of $H = \ker(f_*)$ generated by the $\{\alpha_i\}$.

The inclusion map i also induces i_* on the fundamental groups. Since $i_*(\beta_Q) = 0$, $i_*(\alpha_i) = i_*(\alpha'_i)^N$. The fundamental group of C is $\pi_1(C;P) = \{< i_*(\alpha'_1), \ldots, i_*(\alpha'_{2g}) > / \prod[i_*(\alpha'_i), i_*(\alpha'_{i+g})] = 0\}$. Hence one has a map $G \to \pi_1(C;P)$ whose image is the subgroup generated by the N^{th}-powers of α'_i.

Lemma 4.2.1. *The abelianization of G is isomorphic to the subgroup of index N^{2g} of the abelianization of $\pi_1(C;P)$.*

$$G/[G,G] \simeq N \cdot \pi_1(C)/[\pi_1(C), \pi_1(C)],$$

where $N\cdot$ denotes multiplication by N.

Proof. Let $\alpha = \prod \alpha_{a_i}^{b_i}$ be a word in G. For a generator α_i of G define

$$\mathrm{ord}_{\alpha_i}(\alpha) = \sum_{a_i = i} b_i$$

namely, the number of times α_i appears in the word. Define

$$\Psi : G \to \mathbb{Z}^{2g}$$

$$\Psi(\alpha) = \big(\mathrm{ord}_{\alpha_i}(\alpha), \ldots, \mathrm{ord}_{\alpha_{2g}}(\alpha)\big)$$

Let $K = \ker(\Psi)$. Clearly $[G,G] \subset K$. Further, the map Ψ factors through i_* and is surjective. We claim $K = [G,G]$. To see this, observe that if $a, b \in G$

$$ab \equiv ba\,(mod[G,G]).$$

Repeatedly applying this one can see that any word

$$\alpha = \prod \alpha_{a_i}^{b_i} \equiv \prod_{i=1}^{2g} \alpha_i^{\mathrm{ord}_{\alpha_i}(\alpha)}\, mod([G,G]).$$

In particular, if $\mathrm{ord}_{\alpha_i}(\alpha) = 0$ for all i, $\alpha \in [G, G]$. Hence $K = [G, G]$. Hence

$$G/[G, G] \simeq \mathbb{Z}^{2g}$$

The map i_* takes α_i to ${\alpha'}_i^N$. One has a similar map $\Psi' : \pi_1(C; P) \to \mathbb{Z}^{2g}$ using α'_i instead of α_i which shows that the abelianization of $\pi_1(C; P)$ is $\mathbb{Z}^{2g}$ as well. However, under this map $\Psi'(\alpha_i) = N$ and hence $G/[G, G]$ is carried to the subgroup $N \cdot \mathbb{Z}^{2g}$. Multiplication by N is an isomorphism so the map

$$i_N := \frac{1}{N} \circ i_* : G/[G, G] \longrightarrow \pi_1(C)/[\pi_1(C), \pi_1(C)]$$

is an isomorphism between the two abelianizations.

$\square$

Let $V = V_{\mathbb{Z}} = G/[G, G]$. The abelianization of the fundamental group of C_{QR} is $H_1(C_{QR})$ and so V is a subgroup of $H_1(C_{QR})$. The abelianization of $\pi_1(C)$ is $H_1(C)$. Hence the map i_N is an isomorphism between V and $H_1(C)$. Let $j_N : H_1(C) \longrightarrow V$ be the inverse isomorphism. This gives an embedding of $H_1(C)$ in $H_1(C_{QR})$. As discussed above, the Carlson representative is a functional on $H_1(C)$. However, we will obtain a functional on $H_1(C_{QR})$ which will be the Carlson representative of the extension when considered as a functional on V.

Let $[\alpha]$ denote the homology class of a loop α. The collection $\{[\alpha'_i]\}$ has the property that their images $\{[i_*(\alpha'_i)]\}$ in $H_1(C)$ form a symplectic basis. Since $i_*([\beta_Q]) = 0$, $i_*([\alpha_i]) = N i_*([\alpha'_i])$. Hence under the isomorphism, $i_N([\alpha_i]) = [i_*(\alpha'_i)]$. Let $\{dx_i\}$ be the dual basis of harmonic form in $H^1(C, \mathbb{C})$ satisfying $\int_{[i_*(\alpha'_i)]} dx_j = \delta_{ij}$, where δ_{ij} is the Kronecker Delta function. With this choice of $\{[\alpha'_i]\}s$ and $\{dx_i\}s$, the volume form on $H^2(C)$ can be expressed as follows. Let

$$c(i) = \begin{cases} 1 & \text{if } i \leq g \\ -1 & \text{if } i > g \end{cases}$$

and $\sigma(i) = i + c(i)g$. The volume form is

$$\sum_{i=1}^{g} c(i) dx_i \wedge dx_{\sigma(i)}.$$

From that one gets that a Poincaré dual of $[\alpha'_i]$ is $c(i)dx_{\sigma(i)}$.

The cohomology group $H^1(C_{QR}, \mathbb{C})$ is generated by the $i^*(dx_i) = dx_i$ along with the logarithmic form $\frac{df}{f} = f^*(\frac{dz}{z})$ where $\frac{dz}{z}$ generates the cohomology group of $H^1(\mathbb{C}P^1 - \{0, \infty\})$.

Since we are dealing with the non-compact manifold C_{QR}, recall that Poincaré duality states that

$$H^1_c(C_{QR}) \simeq H_1(C_{QR}),$$

where $H^1_c(C_{QR})$ is the compactly supported cohomology of C_{QR}. This group has mixed Hodge structure determined by identifying it with relative cohomology group $H^1(C_{QR}, \{Q, R\})$. Unlike $H^1(C_{QR})$ which has non trivial weight 1 and weight 2 pieces, cohomology with compact support has weight 0 and weight 1 pieces and is covariant. However

$$Gr^W_1 H^1_c(C_{QR})_{\mathbb{Q}} \simeq Gr^W_1 H^1(C_{QR})_{\mathbb{Q}} \simeq H^1(C)_{\mathbb{Q}}$$

Here the first isomorphism is induced by identity and the second by i^*.

The space V determines a splitting of the Hodge structure on $H_1(C_{QR})$. The space V^* of Poincaré duals of element of V is a subspace of $H^1_c(C_{QR})$ which determines a splitting of the Hodge structure on $H^1_c(C_{QR})$. Further, V^* is isomorphic to $H^1(C)$. Hence if η is a form in $H^1(C)$ it is cohomologous in C_{QR} to a compactly supported form in $V^* \subset H^1_c(C_{QR})$. One has

$$H^1_c(V_{QR})_{\mathbb{Q}} = V_{\mathbb{Q}} \oplus \mathbb{Q} \cdot \omega_Q$$

where ω_Q is a Poincaré dual of β_Q. $\mathbb{Q} \cdot \omega_Q \simeq \mathbb{Q}(0)$. Note that

$$\int_{[\alpha_j]} \omega_Q = \int_{C_{QR}} i^*(c(j)dx_{\sigma(j)}) \wedge \omega_Q =$$

$$= -\int_{C_{QR}} \omega_Q \wedge i^*(c(j)dx_{\sigma(j)}) = -\int_{\beta_Q} i^*(c(j)dx_{\sigma(j)}) = -\int_{i_*([\beta_Q])} c(j)dx_{\sigma(j)} = 0$$

since $i_*(\beta_Q) = 0$.

Further

$$\int_{[\alpha_i]} i^*(dx_j) = \int_{i_*([\alpha_i]} dx_j = \int_{Ni_*([\alpha'_i])} dx_j = N\delta_{ij}.$$

Hence the dual of $[\alpha_i]$ is $\frac{i^*(dx_i)}{N}$ and under the dual map dx_i is taken to $\frac{dx_i}{N}$ in $V^* = \mathrm{Hom}(V,\mathbb{Z})$. Further, note that

$$\int_{[\alpha_i]} \frac{df}{f} = 0$$

since $[\alpha_i] \in Ker(f_*)$.

From the calculation of Ω, a Poincaré dual of $\frac{dx_k}{N}$ is $c(k)\alpha_{\sigma(k)}$. Finally

$$\int_{C_{QR}} \frac{dx_k}{N} \wedge \frac{df}{f} = \int_{c(k)\alpha_{\sigma(k)}} \frac{df}{f} = \int_{f_*(c(k)\alpha_{\sigma(k)})} \frac{dz}{z} = 0.$$

Hence $dx_k \wedge \frac{df}{f}$ is exact.

We now construct a cover of C_{QR} which has the property that its homology group is $G/[G,G]$ and the form $\frac{df}{f}$ is *exact*. Further, the loops α_i lift to loops on this cover. We do that as follows. Let $u : X \to C_{QR}$ denote the universal cover of C_{QR}. The group G acts on X as a group of deck transformations. Let $\tilde{C} = X/G$ denote the quotient and $q : \tilde{C} \to C_{QR}$ denote the covering map. This is a cover

$$q : (\tilde{C}, \tilde{P}) \longrightarrow (C_{QR}, P) \tag{4.2.1}$$

such that $\pi_1(\tilde{C}; \tilde{P}) = G$, where $\tilde{P}$ is a point in $q^{-1}(P)$. Now by homotopy lifting ([Hat02], Proposition 1.31), loops based at P whose homotopy class lie in $G \subset \pi_1(C_{QR})$ will lift to loops in $\tilde{C}$ based at $\tilde{P}$. Thus $\alpha_i \in G$ will lift to a unique, upto homotopy loop $\tilde{\alpha_i}$ based at $\tilde{P}$ such that $q_*(\tilde{\alpha_i}) = \alpha_i$. The covering space $\tilde{C}$ is not an algebraic variety.

Proposition 4.2.2. *$q^*(\frac{df}{f}) = 0$ in $H^1(\tilde{C})$. Hence there is a function, which we call $\log(q^*(f))$, defined on $\tilde{C}$ such that $d\log(q^*(f)) = q^*(\frac{d(f)}{f})$.*

Proof. From Lemma 4.2.1,

$$H_1(\tilde{C}) \simeq G/[G,G] \simeq V \simeq N \cdot \pi_1(C)/[\pi_1(C;P), \pi_1(C;P)] \simeq N \cdot H_1(C) \simeq H_1(C)$$

By the de Rham isomorphism, $H^1(\tilde{C}) \simeq H^1(C)$.

The maps q^* and q_* are adjoint with respect to the de Rham isomorphism. If $\sigma \in H_1(\tilde{C})$ and $\omega \in H^1(C_{QR}, \mathbb{C})$ then

$$\int_{q_*(\sigma)} \omega = \int_{\sigma} q^*(\omega).$$

Further, $q^*(\omega)$ is 0 in $H^1(\tilde{C})$ if and only if $\int_{q_*(\sigma)} \omega = 0$ for all $\sigma \in H_1(\tilde{C})$. Applying this to $\omega = \frac{df}{f}$ and using the fact that $[\tilde{\alpha}_i], 1 \leq i \leq 2g$ give a basis for $H_1(\tilde{C})$, we have

$$q^*(\frac{df}{f}) = 0 \in H^1(\tilde{C}) \Leftrightarrow \int_{[\tilde{\alpha}_i]} q^*(\frac{df}{f}) = 0 \text{ for all } i \Leftrightarrow \int_{[\alpha_i]} \frac{df}{f} = 0 \text{ for all } i.$$

The map f induces

$$f_* : H_1(C_{QR}) \longrightarrow H_1(\mathbb{C}P^1 - \{0, \infty\}).$$

The form $\frac{df}{f} = f^*(\frac{dz}{z})$. Hence, one has

$$\int_{[\alpha_i]} \frac{df}{f} = \int_{[\alpha_i]} f^*(\frac{dz}{z}) = \int_{f_*([\alpha_i])} \frac{dz}{z}.$$

However, since $\alpha_i \in G$ and by choice $G \subset \ker(f_*)$, we have $f_*(\alpha_i) = 0$ so $f_*([\alpha_i]) = 0$ and finally

$$\int_{f_*([\alpha_i])} \frac{dz}{z} = 0.$$

Hence $q^*(\frac{df}{f}) = 0 \in H^1(\tilde{C})$. Therefore integration of $\frac{df}{f}$ is path independent and we have a well defined function

$$\log(q^*(f))(x) = \int_{\tilde{P}}^{x} q^*(\frac{df}{f})$$

on $\tilde{C}$. Note that $\log(q^*(f)(\tilde{P})) = 0$.

□

Hence the space V can be understood as the homology of the space $\tilde{C}$ and the map q_* gives a rational splitting of the map i_*. We also have the following description of $V_{\mathbb{C}}$.

Lemma 4.2.3. *Let $f : C_{QR} \longrightarrow \mathbb{C}P^1 - \{0, \infty\}$ be the map with divisor $div(f) = NQ - NR$ and $f(P) = 1$ and $V = G/[G, G]$ as above. Let $W_{\mathbb{Q}} = \mathrm{Ker}(f_* : H^1(C_{QR})_{\mathbb{Q}} \longrightarrow H^1(\mathbb{P}^1 - \{0, \infty\}))$. Then $V_{\mathbb{Q}} = W_{\mathbb{Q}}$.*

Proof. Since $V \subset \mathrm{Ker}(f_*)$, $V_{\mathbb{C}} \subset W_{\mathbb{C}}$. However, both $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ are subspaces of codimension 1 in $H^1(C_{QR})_{\mathbb{C}}$. Hence they are isomorphic. □

Note that it does not appear to be true that $V = \mathrm{Ker}(f_*)$ as $\mathbb{Z}$-modules. Intrinsically, the reason why there is such a V is the following. If C and C_{QR} are as above, there is an exact sequence of mixed Hodge structures

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow H_1(C_{QR}) \longrightarrow H_1(C) \longrightarrow 0$$

induced by the inclusion map. Hence $H_1(C_{QR})$ determines a class in $\mathrm{Ext}(H_1(C), \mathbb{Z}(1))$. From the Carlson isomorphism one knows

$$\mathrm{Ext}_{MHS}(H_1(C), \mathbb{Z}(1)) \simeq \mathrm{Ext}_{MHS}(\mathbb{Z}(-1), H^1(C)) \simeq CH^1_{hom}(C)$$

and the class determined by $H_1(C_{QR})$ is nothing but the class of $Q - R$ in $CH^1_{hom}(C)$. Since there exists a function f with $div(f) = NQ - NR$ it implies that this sequence splits rationally. Hence there is a map

$$p : H_1(C_{QR})_{\mathbb{Q}} \longrightarrow \mathbb{Q}(1)$$

which splits the exact sequence. This map can be seen to be

$$p(\sigma) = \int_\sigma \frac{df}{f} = \int_{f_*(\sigma)} \frac{dz}{z}$$

and if $V_{\mathbb{Q}}$ is the kernel, then $V_{\mathbb{Q}} \simeq H_1(C)_{\mathbb{Q}}$. Clearly $\sigma \in \mathrm{Ker}(p) \Leftrightarrow \sigma \in \mathrm{Ker}(f_*)$. Hence $\mathrm{Ker}(f_*)$ is isomorphic to $H_1(C)_{\mathbb{Q}}$. The V defined above is only contained in $\mathrm{Ker}(f_*)$ but is a subgroup of the integral homology $H_1(C)$ – so has a little more information.

4.2.1 The Carlson representative of $e^4_{QR,P}$

The Carlson representative of $e^4_{QR,P}$ is given by

$$p_1 \circ r_{\mathbb{Z}} \circ s_F \circ i_3,$$

where

- p_1 is the projection of $N \cdot \mathcal{B}_1 \simeq H^1(C) \oplus \otimes^2 H^1(C) \xrightarrow{p_1} H^1(C)$.
- i_3 is the inclusion map $\otimes^3 H^1(C) \xrightarrow{i_3} \otimes^3 H^1(C) \oplus \otimes^2 H^1(C)$.

To describe s_F we need a little more. Let us consider $C \setminus \{\bullet\}$ for $\bullet \in \{Q, R\}$. The inclusion map

$$i_\bullet : C \setminus \{\bullet\} \hookrightarrow C$$

induces isomorphisms on the first homology and cohomology groups and in we will identify elements of $H_1(C \setminus \{\bullet\})$ with their images in $H_1(C)$ and similarly elements of $H^1(C)$ with their images in $H^1(C \setminus \{\bullet\})$.

Recall $\tilde{\ominus}_B$ denotes the generalised Baer difference. Let

$$s_F \circ i_3 : \otimes^3 H^1(C) \longrightarrow N \cdot \mathcal{B}_2 \simeq N \cdot \big((J_{Q,P}/J^4_{Q,P})^* \tilde{\ominus}_B (J_{R,P}/J^4_{R,P})^*\big)$$

be the section preserving the Hodge filtration given by

$$s_F(dx_i \otimes dx_j \otimes dx_k) = (I^{ijk}_Q, I^{ijk}_R).$$

Here $I^{ijk}_\bullet \in (J_{\bullet,P}/J^4_{\bullet,P})^*$ for $\bullet \in \{Q, R\}$ are iterated integrals with

$$I^{ijk}_\bullet = N\left(\int dx_i dx_j dx_k + dx_i \mu_{jk,\bullet} + \mu_{ij,\bullet} dx_k + \mu_{ijk,\bullet}\right). \tag{4.2.2}$$

where $\mu_{ij,\bullet}$, $\mu_{jk,\bullet}$ and $\mu_{ijk,\bullet}$ are smooth, logarithmic 1-forms on $C \setminus \{\bullet\}$ such that

1. $d\mu_{jk,\bullet} + dx_j \wedge dx_k = 0$
2. $d\mu_{ij,\bullet} + dx_i \wedge dx_j = 0$
3. $dx_i \wedge \mu_{jk,\bullet} + \mu_{ij,\bullet} \wedge dx_k + d\mu_{ijk,\bullet} = 0.$

There are inclusion maps of C_{QR} into $C \setminus Q$ and $C \setminus R$ and we can pull back the forms dx_i, $\mu_{ij,\bullet}$ and $\mu_{ijk,\bullet}$ to C_{QR} and consider all the forms as forms on C_{QR}. To compute the element of $\mathrm{Hom}(\otimes^3 H^1(C)_{\mathbb{C}}, H^1(C)_{\mathbb{C}})$ obtained as the projection under p_1, we describe it as an element of $H_1(C)^*_{\mathbb{C}} = \mathrm{Hom}(H_1(C), \mathbb{C}) = H^1(C, \mathbb{C})$. The integrands $I^{ijk}_\bullet$ are made up of forms on C_{QR} and so to compute it on elements of $H_1(C)$ we have to choose an embedding of $H_1(C)$ in $H_1(C_{QR})$. This is precisely what the subgroup V gives us. Hence from now on if α is a homology class in $H_1(C)$ we think of it as an element of $H_1(C_{QR})$ by identifying it with its image under the map j_N.

The map from

$$H^1(C) \longrightarrow (H^1(C) \oplus H^1(C))/\Delta_{H^1(C)}$$

is given by

$$x \longrightarrow (x, -x).$$

Further, if α is a loop based at P on C_{QR}, the class in $H_1(C) = J_{\bullet,P}/J^2_{\bullet,P}$ corresponding to it is $1-\alpha$. So one has $p_1 \circ r_{\mathbb{Z}} \circ s_F \circ i_3 \in \mathrm{Hom}(\otimes^3 H^1(C)_{\mathbb{C}}, H^1(C)_{\mathbb{C}})$. As an integral, it is

$$p_1 \circ r_{\mathbb{Z}} \circ s_F \circ i_3(dx_i \otimes dx_j \otimes dx_k)(\alpha) = \int_{1-\alpha} I_Q^{ijk} - \int_{1-\alpha} I_R^{ijk}$$

where the first $1-\alpha$ is the class in $H_1(C_Q)$ and the second is the class in $H_1(C_R)$. They are both carried to the same class in V under the isomorphism, so we can take the difference of the integrals when we consider α as a loop in C_{QR} whose corresponding homology class lies in V. This resulting expression is

$$\int_{1-\alpha} I_Q^{ijk} - \int_{1-\alpha} I_R^{ijk} = N\left(\int_{1-\alpha} dx_i\,(\mu_{jk,Q} - \mu_{jk,R}) + (\mu_{ij,Q} - \mu_{ij,R})\,dx_k + (\mu_{ijk,Q} - \mu_{ijk,R})\right).$$

We can choose the logarithmic forms $\mu_{ij,\bullet}$ and $\mu_{ijk,\bullet}$, for $\bullet \in \{Q, R\}$, satisfying the following

- $\mu_{ij,\bullet} = -\mu_{ji,\bullet}$.
- For $|i-j| \neq g$, $\mu_{ij,\bullet}$ is smooth on $C\backslash\bullet$, as $d\mu_{ij,\bullet} = dx_j \wedge dx_i = 0$. As $H^2(C_{QR}, \mathbb{Z}) = 0$ and $\mu_{ij,\bullet}$ is smooth, it is orthogonal to all closed forms, that is, $\mu_{ij,\bullet} \wedge dx_k$ is exact.
- $\mu_{i\sigma(i),\bullet}$ has a logarithmic singularity at $\bullet$ with residue $c(i)$.
- $\mu_{ij,Q} - \mu_{ij,R} = 0$ if $|i-j| \neq g$ as forms on C_{QR}.
- $\mu_{i\sigma(i),Q} - \mu_{i\sigma(i),R} = \frac{c(i)}{N}\frac{df}{f}$, where $f = f_{QR}$ is a function such that $div(f_{QR}) = NQ - NR$. We can normalise f_{QR} once again by requiring that $f_{QR}(P) = 1$.

In terms of the basis of forms of $H^1(C)$, $\Omega \in \otimes^2 H^1(C)$ is

$$\Omega = \sum_{i=1}^{g} dx_i \otimes dx_{(i+g)} - dx_{(i+g)} \otimes dx_i = \sum_{i=1}^{2g} c(i) dx_i \otimes dx_{\sigma(i)}.$$

With these choices of $\mu_{ij,\bullet}$ and $\mu_{ijk,\bullet}$, we have the following theorem:

Theorem 4.2.4. *Let* $\mathbf{G}_{QR,P} \in \mathrm{Hom}(H^1(C)(-1)_{\mathbb{C}}, H^1(C)_{\mathbb{C}})$ *be the Carlson representative corresponding to the extension class* $J^*_\Omega(e^4_{QR,P})$*. It is given by*

$$\mathbf{G}_{QR,P}(dx_k)(\alpha_j) = p_1 \circ r_{\mathbb{Z}} \circ s_F \circ i_3(dx_k \otimes \Omega)(\alpha_j) = (2g+1)\int_{\alpha_j} \frac{df}{f} dx_k - N\int_{\alpha_j} W(dx_k).$$

in $J^0(\mathrm{Hom}(H^1(C)(-1), H^1(C))$*, where*

$$W(dx_k) = \sum_{i=1}^{2g} c(i)(\mu_{ki\sigma(i),Q} - \mu_{ki\sigma(i),R})$$

is a 1*-form on* C_{QR} *which satisfies*

$$dW(dx_k) = (2g+1)\frac{dx_k}{N} \wedge \frac{df}{f}.$$

Proof. Let S_F denote the map $S_F = s_F \circ i_3 \circ J_\Omega : H^1(C)(-1) \to N \cdot H^4_{QR,P}$. This is given by

$$S_F(dx_k) = \sum_{i=1}^{2g} c(i) s_F(dx_k \otimes dx_i \otimes dx_{\sigma(i)})$$

From (4.2.2) one has

$$S_F(dx_k) = \left(\sum_{i=1}^{2g} c(i) \int I_Q^{ki\sigma(i)}, \sum_{i=1}^{2g} c(i) \int I_R^{ki\sigma(i)} \right)$$

Evaluating on a loop α_j based at P using the maps described above, this is

$$\sum_{i=1}^{2g} \int_{1-\alpha_j} c(i) \left(I_Q^{ki\sigma(i)} - I_R^{ki\sigma(i)} \right)$$

$$\sum_{i=1}^{2g} N \left(\int_{1-\alpha_j} c(i) dx_k(\mu_{i\sigma(i),Q} - \mu_{i,\sigma(i),R}) + (\mu_{ki,Q} - \mu_{ki,R}) dx_{\sigma(i)} + (\mu_{ki\sigma(i),Q} - \mu_{ki\sigma(i),R}) \right)$$

From the choice of the forms $\mu_{ij,\bullet}$ and $\mu_{ijk,\bullet}$ above, the leading terms and several of the lower order terms cancel out and

$$\mu_{ki,Q} - \mu_{ki,R} = c(k)\delta_{k\sigma(i)} \frac{1}{N} \frac{df}{f}$$

and

$$\mu_{i\sigma(i),Q} - \mu_{i\sigma(i),R} = c(i) \frac{1}{N} \frac{df}{f}.$$

Since $c(i)^2 = 1$ what remains is

$$\sum_{i=1}^{2g} \int_{1-\alpha_j} dx_k \frac{df}{f} - \int_{1-\alpha_j} \frac{df}{f} dx_k + N \sum_{i=1}^{2g} c(i) \int_{1-\alpha_j} \left(\mu_{ki\sigma(i),Q} - \mu_{ki\sigma(i),R}\right).$$

Let

$$W(dx_k) = \sum_{i=1}^{2g} c(i) \left(\mu_{ki\sigma(i),Q} - \mu_{ki\sigma(i),R}\right).$$

Since integration over a point, which corresponds to the constant loop 1, is 0 and $\int_{\alpha_j} \frac{df}{f} = 0$ by choice of α_j, using Lemma 3.1.2 (2) the integral becomes

$$\mathbf{G}_{QR,P}(dx_k)(\alpha_j) = 2g \int_{1-\alpha_j} dx_k \frac{df}{f} - \int_{1-\alpha_j} \frac{df}{f} dx_k + N \int_{1-\alpha_j} W(dx_k).$$

$$= -(2g+1) \int_{\alpha_j} dx_k \frac{df}{f} - N \int_{\alpha_j} W(dx_k).$$

Now consider

$$dW(dx_k) = \sum_{i=1}^{2g} c(i) d\left(\mu_{ki\sigma(i),Q} - \mu_{ki\sigma(i),R}\right).$$

From the choice of $\mu_{ijk,\bullet}$, one has

$$d\mu_{ijk,\bullet} = -dx_i \wedge \mu_{jk,\bullet} - \mu_{ij,\bullet} \wedge dx_k.$$

So the sum becomes

$$dW(dx_k) = \sum_{i=1}^{2g} -c(i) \left(\left(dx_k \wedge \mu_{i\sigma(i),Q} + \mu_{ki,Q} \wedge dx_{\sigma(i)}\right) - \left(dx_k \wedge \mu_{i\sigma(i),R} + \mu_{ki,R} \wedge dx_{\sigma(i)}\right)\right)$$

$$= \sum_{i=1}^{2g} -c(i) \left(dx_k \wedge (\mu_{i\sigma(i),Q} - \mu_{i\sigma(i),R}) + (\mu_{ki,Q} - \mu_{ki,R}) \wedge dx_{\sigma(i)}\right).$$

In the second sum, only one term survives and one has

$$= -c(\sigma(k))(\mu_{k\sigma(k),Q} - \mu_{k\sigma(k),R}) \wedge dx_k + \sum_{i=1}^{2g} -c(i) \left(dx_k \wedge \frac{c(i)}{N} \frac{df}{f}\right)$$

$$= -c(\sigma(k))(\frac{c(\sigma(k))}{N} \frac{df}{f}) \wedge dx_k + \sum_{i=1}^{2g} -c(i) \left(dx_k \wedge \frac{c(i)}{N} \frac{df}{f}\right)$$

$$= -\frac{(2g+1)}{N} \frac{df}{f} \wedge dx_k = \frac{(2g+1)}{N} dx_k \wedge \frac{df}{f}.$$

□

We have computed the Carlson representative $\mathbf{G_{QR,P}}$ of our class in $\mathrm{Ext}(H^1(C)(-1), H^1(C))$. We now tensor with $H^1(C)$ and pull back using the map $\otimes\Omega : \mathbb{Z}(-1) \longrightarrow \otimes^2 H^1(C)$. This gives us an element of $\mathrm{Ext}(\mathbb{Z}(-2), \otimes^2 H^1(C))$. We denote its Carlson representative by $\mathbf{F}_{QR,P}$.

Lemma 4.2.5. *The Carlson representative of the class in* $\mathrm{Ext}(\mathbb{Z}(-2), \otimes^2 H^1(C))$ *is given by*

$$\mathbf{F}_{QR,P} = (\mathbf{G}_{QR,P} \otimes Id) \circ \otimes\Omega$$

in $(\otimes^2 H^1(C)_{\mathbb{C}})^*$*. On an element* $\alpha_j \otimes \alpha_k$ *it is given by*

$$\mathbf{F}_{QR,P}(\Omega)(\alpha_j \otimes \alpha_k) = c(\sigma(k))N \left(\int_{\alpha_j} (2g+1)\frac{df}{f} dx_{\sigma(k)} - NW(dx_{\sigma(k)}) \right) \tag{4.2.3}$$

Proof. Recall that

$$\Omega = \sum_1^{2g} c(i) dx_i \otimes dx_{\sigma(i)}.$$

From above we have

$$(\mathbf{G}_{QR,P} \otimes Id)(\Omega)(\alpha_j \otimes \alpha_k) = \sum_1^{2g} c(i)\mathbf{G}_{QR,P}(dx_i)(\alpha_j) \cdot Id(dx_{\sigma(i)})(\alpha_k).$$

From the choice of α_k one has

$$Id(dx_{\sigma(i)})(\alpha_k) = N\delta_{k\sigma(i)}.$$

Hence, in the sum above, precisely one term survives, at $i = \sigma(k)$. Therefore

$$(\mathbf{G}_{QR,P} \otimes Id)(\Omega)(\alpha_j \otimes \alpha_k) = Nc(\sigma(k))\mathbf{G}_{QR,P}(dx_{\sigma(k)})(\alpha_j).$$

In particular

$$\begin{aligned} \mathbf{F}_{QR,P}(\Omega)(\alpha_j \otimes \alpha_k) =& Nc(\sigma(k))\mathbf{G}_{QR,P}(dx_{\sigma(k)})(\alpha_j) \\ =& Nc(\sigma(k)) \left(\int_{\alpha_j} (2g+1)\frac{df}{f} dx_{\sigma(k)} - NW(dx_{\sigma(k)}) \right) \end{aligned}$$

□

We now use Proposition 4.2.2 to convert the iterated integral in to an ordinary integral. The iterated integral term in (4.2.3) is

$$Nc(\sigma(k))(2g+1)\int_{\alpha_j}\frac{df}{f}dx_k$$

which we can evaluate using Lemma 3.1.2(3) if $\frac{df}{f}$ is exact. However, $\frac{df}{f}$ is *not* exact on C_{QR} but it is exact on $\tilde{C}$ using Proposition 4.2.2. So we do the integration on $\tilde{C}$. Precisely, we do that as follows.

Let α be a loop such that $[\alpha] \in q_*(H_1(\tilde{C}))$, where $q : \tilde{C} \longrightarrow C_{QR}$ is the cover. Let $\alpha = q_*(\tilde{\alpha})$, where $\tilde{\alpha}$ is a loop based at $\tilde{P}$ lying over the basepoint P of α. In other words $\tilde{\alpha}$ be a lift of α. Let ψ be another compactly supported 1 form on C_{QR} whose Poincare dual lies in $q_*(H_1(\tilde{C}))$. We have

$$\int_\alpha \frac{df}{f}\psi = \int_{\tilde{\alpha}} q^*(\frac{df}{f})q^*(\psi)$$

From Proposition 4.2.2, $q^*(\frac{df}{f})$ is exact on $\tilde{\alpha}$. In other words $q^*(\frac{df}{f}) = d\log(q^*(f))$. Choose a primitive $\log(q^*(f))$ such that $\log(q^*(f)(\tilde{P})) = 0$. Using Lemma 3.1.2(3) and the fact that we have chosen $\log(q^*(f))$ with $\log(q^*(f)(\tilde{P})) = 0$,

$$\int_{\tilde{\alpha}} q^*(\frac{df}{f})q^*(\psi) = \int_{\tilde{\alpha}} \log(q^*(f))q^*(\psi)$$

Hence we have

$$\int_\alpha \frac{df}{f}\psi = \int_{\tilde{\alpha}} \log(q^*(f))q^*(\psi)$$

Applying this to the case at hand we have

$$\mathbf{F}_{QR,P}(\Omega)(\alpha_j\otimes\alpha_k) = Nc(\sigma(k))\left(\int_{\tilde{\alpha}_j}(2g+1)\log(q^*(f))q^*(dx_\sigma(k)) - Nq^*(W(dx_{\sigma(k)}))\right). \tag{4.2.4}$$

We have made a choice of $\tilde{\alpha_j}$. If we chose a different basepoint, the value of $\log(q^*(f))$ will change by $2\pi iM$ for some $M \in \mathbb{Z}$. This will change the integral by $2\pi iM\int_{\alpha_j} dx_{\sigma(k)}$. This does not affect the class in the intermediate Jacobian.

We would like to connect the expression above which is the Carlson representative of the extension class $\epsilon^4_{QR,P}$ – to the regulator of an explicit cycle on the Jacobian of the curve. To that end, we have the following lemma.

Proposition 4.2.6 (Colombo Proposition 3.3)**.** *Let $f = f_{QR}$ be as before and ψ a closed 1-form C_{QR}. Let $W(\psi)$ be a 1-form such that $dW(\psi) = \frac{df}{f} \wedge \psi$ such that $\Theta = \log(f)\psi + \omega(\psi)$ closed 1-form on the manifold with boundary $C \setminus \gamma$. Let α be a loop in C_{QR} such that $[\alpha] \in V$ and $\eta_\alpha \in H^1_c(C_{QR})$ be the Poincaré dual of $[\alpha]$ constructed below. Then we have*

$$\int_\alpha \frac{df}{f}\psi + \omega(\psi) = \int_{C\setminus\gamma} \eta_\alpha \wedge \Theta + 2\pi i \int_\gamma \eta_\alpha \psi \left(mod\ 2\pi i \int_\alpha \psi \mathbb{Z} \right)$$

Proof. The subgroup V is generated by the classes of $\alpha_i = {\alpha'}_i^N \beta_Q^{-m_i}$ where α'_i is one of the 'standard' generators of $\pi_1(C)$ coming from the edges of the fundamental polygon and β_Q is a small simple loop around Q. These loops satisfy $f_*(\alpha_i) = 0$.

It suffices to prove the theorem for the α_i and extend linearly, so from this point on we let $\alpha = \alpha_i$, $\alpha' = \alpha'_i$. Let η_α be the compactly supported 1-form which is the Poincaré dual of $[\alpha]$ constructed as in [FK80] as follows: Suppose δ is a simple closed curve in C_{QR}. Let $D = D_\delta$ be a tubular neighbourhood of δ. We can write $D_\delta - \delta = D^+_\delta \cup D^-_\delta$ with D^-_δ to the left and D^+_δ to the right of δ. Let D_0 be a sub-tubular neighbourhood of δ in D and $D^\pm_0 = D_0 \cap D^\pm_\delta$. Let G_δ be a function such that is smooth on $C_{QR} - \delta$ and

$$G_\delta \equiv \begin{cases} 1 & \text{on } D^-_0 \cup \delta \\ 0 & \text{outside } D^-_\delta \end{cases}$$

Define

$$\eta_\delta = \begin{cases} dG_\delta & \text{on } D_\delta - \delta \\ 0 & \text{elsewhere} \end{cases}$$

so the support $\mathrm{Supp}(\eta_\delta) \subset D^-_\delta$. One can then see that if ψ is a closed 1-form on C_{QR}

$$\int_{C_{QR}} \eta_\delta \wedge \psi = \int_{D^-_\delta} dG_\delta \wedge \psi = \int_{D^-_\delta} dG_\delta \psi = \int_{\partial(D^-_\delta)} G_\delta \psi = \int_{[\delta]} \psi$$

since $G_\delta \equiv 1$ on δ and with this choice of orientation $\partial(D^-_\delta) = \delta$.

In our case, in general $\alpha = \alpha_i = {\alpha'}_i^N \beta_Q^{-m_i}$, is not a *simple* closed curve. However, α' and β_Q are. Let $D_\alpha^- = D_{\alpha'}^- \cup D_{\beta_Q}^-$. Define

$$\eta_\alpha = N\eta_{\alpha'} - m_i \eta_{\beta_Q}.$$

η_α is supported in D_α^- and is the Poincaré dual of $[\alpha]$ as, for a 1-form ψ,

$$\int_{C_{QR}} \eta_\alpha \wedge \psi = N \int_{C_{QR}} \eta_{\alpha'} \wedge \psi - m_i \int_{C_{QR}} \eta_{\beta_Q} \wedge \psi = \int_{N[\alpha'] - m_i[\beta_Q]} \psi = \int_{[\alpha]} \psi$$

Let $\tilde{\Theta} = q^*(\Theta) = \log(q^*(f))q^*(\psi) + q^*(\omega(\psi))$. From the discussion after Lemma 4.2.5

$$\int_\alpha \frac{df}{f}\psi + \omega(\psi) = \int_{\tilde{\alpha}} \tilde{\Theta}.$$

where $\tilde{\alpha}$ is a lifting of α to a loop in $\tilde{C}$ such that it is based at $\tilde{P}$ and $\log(q^*(f))$ is chosen such that $\log(q^*(f)(\tilde{P}) = 0$. We would like to compute this integral. However, $\tilde{\Theta}$ is a form on the manifold with boundary $\tilde{C}^B = \tilde{C} - q^{-1}(U(\gamma))$ so we cannot simply use Poincaré duality.

We have $\alpha = \alpha'^N \beta_Q^{-m_i}$. Let $\tilde{\alpha}'$ be the lift of α' and $\tilde{\beta}_Q$ the lift of β_Q such that $\tilde{\alpha}' - \tilde{m_i \beta_Q}_Q = \tilde{\alpha}$.

The restriction $\alpha'|_{C\setminus\gamma} = \bigcup_{j=0}^M \alpha'^j$ is a union of paths α'^j. The covering map q induces a homeomorphism from each α'^j to a path $\tilde{\alpha}'^j$ such that $\bigcup \tilde{\alpha}'^j = \tilde{\alpha}'|_{\tilde{C}^B}$. Let $D_{\alpha'^j}$ denote the restriction of the tubular neighbourhood of α' to a tubular neighbourhood of the path α'^j. We have

$$D_{\alpha'}^- - \gamma = \bigcup_j D_{\alpha'^j}^-$$

Hence we have

$$\int_{C_{QR}\setminus\gamma} \eta_{\alpha'} \wedge \Theta = \int_{D_{\alpha}'^- - \gamma} \eta_{\alpha'} \wedge \Theta = \sum_j \int_{D_{\alpha'^j}^-} \eta_{\alpha'} \wedge \Theta$$

The boundary of the restricted tubular neighbourhood $D_{\alpha'^j}^-$ is

$$\partial(D_{\alpha'^j}^-) = \alpha'^j \cup I_0^j \cup -I_1^j \cup (\gamma_1 \cap \overline{D_{\alpha'^j}^-}) \cup (-\gamma_2 \cap \overline{D_{\alpha'^j}^-})$$

Here I_0^j and I_1^j are the half intervals at the endpoints. By construction $I_1^j = -I_0^{j+1}$ except at the final stage when $I_1^M = -I_0^0$. Applying Stokes' Theorem we get

$$\int_{D^-_{\alpha'j}} \eta_{\alpha'} \wedge \Theta = \int_{D^-_{\alpha'j}} dG_{\alpha'}\Theta = \int_{\partial(D^-_{\alpha'j})} G_{\alpha'}\Theta$$

$$\int_{\alpha'j} \Theta + \int_{I_0^j} G_{\alpha'}\Theta - \int_{I_1^j} G_{\alpha'}\Theta + \int_{\gamma_1 \cap \overline{D^-_{\alpha'j}}} G_{\alpha'}\Theta - \int_{\gamma_2 \cap \overline{D^-_{\alpha'j}}} G_{\alpha'}\Theta$$

Summing up over j we have $\sum_j \int_{I_0^j} G_{\alpha'}\Theta + \int_{I_1^{j+1}} G_{\alpha'}\Theta = 0$ as $I_1^j = -I_0^{j+1}$ and the terms cancel.

$$\sum_j \left(\int_{\gamma_1 \cap \overline{D^-_{\alpha'j}}} G_{\alpha'}\Theta - \int_{\gamma_2 \cap \overline{D^-_{\alpha'j}}} G_{\alpha'}\Theta \right) = \int_{\gamma_1} G_{\alpha'}\Theta - \int_{\gamma_2} G_{\alpha'}\Theta$$

as $G_{\alpha'}$ is supported in $D^-_{\alpha'}$. Similar to the case of dg, as $G_{\alpha'}(Q) = 0$, this simplifies to

$$\int_{\gamma_1} G_{\alpha'}\Theta - \int_{\gamma_2} G_{\alpha'}\Theta = -2\pi i \int_\gamma G_{\alpha'}\psi = -2\pi i \int_\gamma \eta_{\alpha'}\psi$$

So we get

$$\int_{C_{QR}\setminus\gamma} \eta_{\alpha'} \wedge \Theta = \sum_j \int_{\alpha'j} \Theta - 2\pi i \int_\gamma \eta_{\alpha'}\psi$$

We can make a similar argument for β_Q.

Finally combining these two we get

$$N \sum_j \int_{\alpha'j} \Theta - m_i \sum_s \int_{\beta_Q^s} \Theta = \int_{C_{QR}\setminus\gamma} \eta_\alpha \wedge \Theta + 2\pi i \int_\gamma \eta_\alpha \psi$$

Since the support η_α is outside Q and R, thus we can replace $C_{QR} \setminus \gamma$ with $C \setminus \gamma$. In particular one has

$$N \sum_j \int_{\alpha'j} \Theta - m_i \sum_s \int_{\beta_Q^s} \Theta = \int_{C\setminus\gamma} \eta_\alpha \wedge \Theta + 2\pi i \int_\gamma \eta_\alpha \psi$$

To link this to the integral over $\tilde{\alpha}$ we observe the following. The loop α'^N lifts to a path in $\widetilde{\alpha'^N}$ in $\tilde{C}$ which is made up of copies of $\tilde{\alpha}'$. Let $\tilde{\alpha}'_k$ denote the lift of the k^{th} copy of α' so $\tilde{\alpha}'_k(1) = \tilde{\alpha}'_{k+1}(0)$. We can choose the homeomorphisms between α'^j and $\tilde{\alpha}'^j$ such that the k^{th} copy of α'^j is homeomorphic to a path $\tilde{\alpha}'^j_k$ in $\tilde{\alpha}'_k$. So we have homeomorphisms

$$\bigcup_{k=1}^N \bigcup_j \alpha'^j \simeq \bigcup_{k=1}^N \bigcup_j \tilde{\alpha}'^j_k \simeq \widetilde{\alpha'^N} - \partial(\tilde{C} \setminus q^{-1}(\gamma))$$

A similar situation holds for β_Q. Via these homeomorphisms

$$N\sum_j \int_{\alpha'^j} \Theta - m_i \sum_s \int_{\beta_Q^s} \Theta = \sum_{k=1}^{N}\sum_j \int_{\tilde{\alpha}_k'^j} \tilde{\Theta} - \sum_{r=1}^{m_i}\sum_s \int_{\tilde{\beta}_{Q,r}^s} \tilde{\Theta}$$

which is

$$\int_{\widetilde{\alpha'^N \beta_Q^{-m_i}} - \partial(\tilde{C} - q^{-1}U(\gamma))} \tilde{\Theta} = \int_{\tilde{\alpha} - \partial(\tilde{C} - q^{-1}(U(\gamma))} \tilde{\Theta} = \int_{\tilde{\alpha}} \tilde{\Theta}$$

as the $\tilde{\alpha} \cap \partial(\tilde{C} - q^{-1}(\gamma))$ is a set of measure 0. Therefore

$$\int_{\tilde{\alpha}} \tilde{\Theta} = \int_{C\setminus\gamma} \eta_\alpha \wedge \Theta - 2\pi i \int_\gamma \eta_\alpha \psi$$

We have made a choice of a lifting $\tilde{\alpha}$ of α. A different choice of basepoint $\tilde{P}'$ would change the value of $\log(q^*(f))$ by $2\pi i M$ for some $M \in \mathbb{Z}$. This would change the integral by $2\pi i M \int_\alpha \psi$. Hence this equality holds only up to $2\pi i \int_\alpha \psi\mathbb{Z}$.

□

We have the following useful corollary to the above proposition, which says that in fact, we can replace η_α by any form on C_{QR} which is cohomologous to $i_*[\eta_\alpha] \in H^1_c(C_{QR})$ and compactly supported in $C \setminus \gamma$.

Corollary 4.2.7. *Let f, ψ, and $W(\psi)$ be as above and $\phi = \phi_\alpha$ a closed 1-form on C which is compactly supported in the manifold with boundary $C \setminus \gamma$ and is cohomologous in $H^1(C)$ to $i_*[\eta_\alpha]$ for some α in V. Then*

$$\int_\alpha \frac{df}{f}\psi + W(\psi) = \int_{C\setminus\gamma} \phi_\alpha \wedge \Theta + 2\pi i \int_\gamma \phi_\alpha \psi \left(mod\, 2\pi i \int_\alpha \psi\mathbb{Z}\right)$$

Proof. Any closed form ϕ on C is compactly supported in manifold with boundary $C \setminus \gamma$ as manifold with boundary $C \setminus \gamma$ is a closed subset of C. Hence we can apply the Corollary to $i^*(\phi)$ in $H^1(C_{QR})$, which, by abuse of notation, we will continue to denote by ϕ. Further, by choice of V it will be cohomologous to η_α for some $\alpha \in V$. In particular we know that $c(i)\frac{dx_{\sigma(i)}}{N}$ is cohomologous to a Poincaré dual of $[\alpha_i]$ in $H^1(C_{QR})$.

Let ϕ_α denote the form above. Then $\phi_\alpha - \eta_\alpha = dg$. One has

$$\int_{C\setminus\gamma} \phi_\alpha \wedge \Theta = \int_{C\setminus\gamma} dg \wedge \Theta + \int_{C\setminus\gamma} \eta_\alpha \wedge \Theta$$

So it suffices to compute the two terms separately.

Since both ϕ_α and η_α are compactly supported on the manifold with boundary $C\setminus\gamma$, so is dg and hence g and $dg\wedge\Theta$ are compactly supported as well. From Stokes' Theorem we get

$$\int_{C\setminus\gamma} dg \wedge \Theta = \int_{C\setminus\gamma} d(g\Theta) = \int_{\partial(C\setminus\gamma)} g\Theta$$

$\partial(C\setminus\gamma) = \gamma_1 \cup \gamma_2$, where γ_1 and γ_2 are two copies of γ oriented oppositely. So

$$\int_{\partial(C\setminus\gamma)} g(\log(f)\psi + W(\psi)) = \int_{\gamma_1} g(\log(f)\psi + W(\psi)) - \int_{\gamma_2} g(\log(f)\psi + W(\psi))$$

The value of $\log(f)$ on the γ_i differ by $2\pi i$. The values of $gW(\psi)$ agree on γ_1 and γ_2, since both g and $W(\psi)$ are defined on C. Further $gW(\psi)$ has compact support. Therefore $\int_{\gamma_1} gW(\psi) - \int_{\gamma_2} gW(\psi) = 0$. Hence, keeping track of the orientation, the integral simplifies to

$$\int_{\partial(C\setminus\gamma)} g\Theta = -2\pi i \int_\gamma g\psi$$

Finally, note that $\int_\gamma dg\psi = \int_\gamma (g - g(Q))\psi$. Since dg is compactly supported on $C\setminus\gamma$, g can not have singularity at Q or R. As Θ is a closed 1-form on $C\setminus\gamma$, using Stokes theorem one obtains

$$\int_{\partial(C\setminus\gamma)} \Theta = \int_{C\setminus\gamma} d\Theta = 0.$$

As above the left hand side simplifies to

$$\int_{\partial(C\setminus\gamma)} \Theta = -2\pi i \int_\gamma \psi = 0.$$

Hence the integral deos not depend on the choice of primitive dg and we have $\int_\gamma dg\psi = \int_\gamma g\psi$. Thus

$$\int_{C\setminus\gamma} dg \wedge \Theta = -2\pi i \int_\gamma dg\psi.$$

From Proposition 4.2.6 we have

$$\int_\alpha \frac{df}{f}\psi + W(\psi) = \int_{C\backslash\gamma} \eta_\alpha \wedge \Theta + 2\pi i \int_\gamma \eta_\alpha \psi \left(mod\, 2\pi i \int_\alpha \psi \mathbb{Z} \right).$$

Since the integral of $dg \wedge \Theta$ cancels the iterated integral term, we have

$$\int_{C\backslash\gamma} \phi_\alpha \wedge \Theta + 2\pi i \int_\gamma \phi_\alpha \psi = \int_{C\backslash\gamma} (\eta_\alpha \wedge \Theta + dg \wedge \Theta) + 2\pi i \int_\gamma (\eta_\alpha \psi + dg\psi)$$

$$= \int_{C\backslash\gamma} \eta_\alpha \wedge \Theta + 2\pi i \int_\gamma \eta_\alpha \psi = \int_\alpha \frac{df}{f}\psi + W(\psi) \left(mod\, 2\pi i \int_\alpha \psi \mathbb{Z} \right)$$

□

Remark 4.2.8. We will apply this to compute the Carlson representative of an extension class. This is an element of the intermediate Jacobian associated to the extension, hence the two expressions, while they possibly differ by and element of $(2\pi i \int_\alpha \psi)\mathbb{Z}$, will have the same class in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(-2), \otimes^2 H^1(C))$ keeping in mind isomorphism of $H_1(C)$ with $q_*(H_1(\tilde{C})) \subset H_1(C_{QR})$. We will use $\equiv$ to keep in mind the fact that all equalities hold only in the intermediate jacobian.

We now apply this in the case of interest to us.

Corollary 4.2.9. *Let α_j be as above. Then modulo $2\pi i \int_{\alpha_j} dx_{\sigma(k)}\mathbb{Z}$*

$$\mathbf{F}_{QR,P}(\Omega)(\alpha_j \otimes \alpha_k) \equiv N(2g+1)c(j)c(\sigma(k)) \left(\int_{C\backslash\gamma} dx_{\sigma(j)} \wedge \left(\log(f) dx_{\sigma(k)} - \frac{N}{(2g+1)} W(dx_{\sigma(k)}) \right) \right.$$
$$\left. + \, 2\pi i \int_\gamma dx_{\sigma(j)} dx_{\sigma(k)} \right)$$

Proof. It is a straightforward application of Corollary 4.2.7 to the expression in Lemma 4.2.5.

□

$\mathbf{F}_{QR,P}(\Omega)$ determines an element of the intermediate Jacobian of $(\otimes^2 H^1(C)^*_{\mathbb{C}})$

$$J(\otimes^2 H^1(C)^*_{\mathbb{C}}) \simeq \frac{F^1(\otimes^2 H^1(C)^*_{\mathbb{C}})}{(\otimes^2 H^1(C)^*)}$$

so to determine $\mathbf{F}_{QR,P}(\Omega)$ it suffices to evaluate it on elements of $F^1(\otimes^2 H^1(C)^*_{\mathbb{C}})$. We can choose the basis dz_i of the space of holomorphic 1-forms on C such that

$$\int_{\alpha_i} dz_j = N\delta_{ij} \qquad\qquad 1 \leq i \leq g,$$

where $\{\alpha_i\}$ is the basis of V. We have

$$i^*[dz_j] = [dx_j] + \sum_{i=1}^{g} A_{ji}[dx_{i+g}] \qquad\qquad \text{where } A_{ji} = \frac{1}{N}\int_{\alpha_{i+g}} dz_j,$$

obtained from the fact that $c(j)\frac{dx_{\sigma(j)}}{N}$ is dual to α_j. Then

$$dz_j = dx_j + \sum_{i=1}^{g} A_{ji} dx_{i+g},$$

when we think of it as form on C_{QR}. Let $\zeta_j = c(\sigma(j))\alpha_{\sigma(j)} + \sum_{1\leq i\leq g} A_{ji}c(i)\alpha_i$, where $j \leq g$. Then dz_j is cohomologous to the Poincaré dual of ζ_j in C_{QR}. Thus we can apply Corollary 4.2.10. We then have the following proposition.

Proposition 4.2.10. *The map $\mathbf{F}_{QR,P}(\Omega)$ evaluated on elements of the form $\zeta_i \otimes \alpha_j$ is*

$$\mathbf{F}_{QR,P}(\Omega)(\zeta_i \otimes c(\sigma(j))\alpha_{\sigma(j)}) \equiv (2g+1)N\left(\int_{C\setminus\gamma} \log(f) dz_i \wedge dx_j + 2\pi i \int_{\gamma} dz_i dx_j\right)$$

In other words

$$dz_i \wedge W(dx_j) = 0.$$

Proof. See Proposition 3.4 in [Col02]. □

In fact, the theorem holds for the other term as well.

Proposition 4.2.11. *For a suitable choice of $\mu_{ijk,Q}$ and $\mu_{ijk,R}$ one has*

$$W(dz_i) := W(dx_i) + \sum_k A_{ki} W(dx_{i+g}) = 0$$

Proof. [Col02] Lemma 3.1. □

Hence we have

Proposition 4.2.12.

$$\mathbf{F}_{QR,P}(\Omega)(c(\sigma(j))\alpha_{\sigma(j)} \otimes \zeta_i) \equiv (2g+1)N \left(\int_{C\setminus\gamma} \log(f) dx_j \wedge dz_i + 2\pi i \int_\gamma dx_j dz_i \right).$$

Comparing this with the regulator term in Theorem 4.1.3 we get

Theorem 4.2.13. *Let $Z_{QR,P}$ be the motivic cohomology cycle constructed above and $\epsilon^4_{QR,P}$ the extension in $\mathrm{Ext}_{MHS}(\mathbb{Z}(-2), \wedge^2 H^1(C))$. We use $\epsilon^4_{QR,P}$ to denote its Carlson representative as well. Then one has*

$$\epsilon^4_{QR,P}(\omega) \equiv (2g+1)N \operatorname{reg}_{\mathbb{Z}}(Z_{QR,P})(\omega)$$

where $\omega \in F^1 \wedge^2 H^1(C)$.

Proof. It suffices to check this on $dz_i \wedge dx_j = dz_i \otimes dx_j - dx_j \otimes dz_i$. The result then follows by comparing the formula for the Carlson representative $\mathbf{F}_{QR,P}$ in Lemma 4.2.5 with the expression for the regulator in Theorem 4.1.3 using Proposition 4.2.10.

From Theorem 4.2.12 and Lemma 4.2.5 we have

$$\mathbf{F}_{QR,P}(\Omega)(c(\sigma(j))\alpha_{\sigma(j)} \otimes \zeta_i) \equiv (2g+1)N \left(\int_{C\setminus\gamma} \log(f) dx_j \wedge dz_i + 2\pi i \int_\gamma dx_j dz_j \right)$$

On the other hand, from Propostion 4.2.12 one has

$$\begin{aligned}\mathbf{F}_{QR,P}(\Omega)(\zeta_i \otimes c(\sigma(j))\alpha_{\sigma(j)}) &= (2g+1)N \left(\int_{C\setminus\gamma} \log(f) dz_i \wedge dx_j + 2\pi i \int_\gamma dz_i dx_j \right) \\ &= (2g+1)N \left(- \int_{C\setminus\gamma} \log(f) dx_j \wedge dz_i + 2\pi i \int_\gamma dz_i dx_j \right)\end{aligned}$$

Therefore we get

$$\begin{aligned}\mathbf{F}_{QR,P}(\Omega)(c(\sigma(j))\alpha_{\sigma(j)} \wedge \zeta_i) \equiv& \mathbf{F}_{QR,P}(\Omega)(c(\sigma(j))\alpha_{\sigma(j)} \otimes \zeta_i) - \mathbf{F}_{QR,P}(\Omega)(\zeta_i \otimes c(\sigma(j))\alpha_{\alpha_{\sigma(j)}}) \\ \equiv& (2g+1)N \left(2 \int_{C\setminus\gamma} \log(f) dx_j \wedge dz_i + 2\pi i \int_\gamma (dx_j dz_i - dz_i dx_j) \right)\end{aligned}$$

On the other hand, from Theorem 4.1.3

$$(2g{+}1)N\,\mathrm{reg}_{\mathbb{Z}}(Z_{QR})(dx_j{\wedge}dz_i) = (2g{+}1)N\left(2\int_{C\setminus\gamma}\log(f)dx_j\wedge dz_i + 2\pi i\int_\gamma(dx_jdz_i - dz_idx_j)\right)$$

$\square$

Recall that we have assumed in both cases that $f_{QR}(P) = 1$. If we do not make that assumption, then one has a term corresponding to a decomposable element that one has to account for. However, if we work modulo the decomposable cycles we can ignore that term.

As a result of this theorem, we get the following expression of the regulator as an integral over a loop - which is more amenable to computation.

Corollary 4.2.14. *Let $Z_{QR,P}$ be the element of $H^{2g-1}_{\mathcal{M}}(\mathrm{Jac}(C),\mathbb{Z}(g))$ and let η and ω be two closed, 1-forms on C with ω holomorphic. Let α be a loop in C_{QR} based at P, such that $\alpha\in V$ and a Poincare dual of $[\alpha]$ is homologous to η in $H^1_c(C_{QR})$. Let $\tilde{\alpha}$ be a lift of α to a loop in $\tilde{C}$ based at $\tilde{P}$ where $\tilde{P}$ is chosen so that $\log(q^*(f))(\tilde{(P)}) = 0$. Then*

$$\mathrm{reg}_{\mathbb{Z}}(Z_{QR,P})(\eta\wedge\omega) \equiv 2(2g+1)N\int_{\tilde{\alpha}}\log(q^*(f_{QR}))q^*(\omega)$$

Proof. It suffices to check this for $\eta = dx_i$ and $\omega = dz_j$. From Theorem 4.2.13, one has

$$\mathbf{F}_{QR,P}(\Omega)(c(\sigma(j))\alpha_{\sigma(j)}{\wedge}\zeta_i) \equiv (2g{+}1)N\left(2\int_{C\setminus\gamma}\log(f)dx_j\wedge dz_i + 2\pi i\int_\gamma(dx_jdz_i - dz_idx_j)\right).$$

From Lemma 3.1.2 we have

$$\int_\gamma dx_jdz_i + \int_\gamma dz_idx_j = \int_\gamma dx_j\int_\gamma dz_i.$$

From Corollary 4.2.7 $\int_\gamma dz_i = 0$. Thus we have $\int_\gamma dx_jdz_i = -\int_\gamma dz_idx_j$.

Therefore we have

$$\mathrm{reg}_{\mathbb{Z}}(Z_{QR,P})(dx_j\wedge dz_i) = 2(2g+1)N\left(\int_{C\setminus\gamma}\log(f)dx_j\wedge dz_i + 2\pi i\int_\gamma dx_jdz_i\right)$$

Using Corollary 4.2.7 this becomes

$$\mathrm{reg}_{\mathbb{Z}}(Z_{QR,P})(dx_j \wedge dz_i) = 2(2g+1)c(j)N \int_{\tilde{\alpha}_{\sigma(j)}} \log(q^*(f))q^*(dz_i)$$

$\square$

Chapter 5

Appendix

5.1 Extensions in an Abelian category

Let A and B be elements in an abelian category $\mathcal{C}$ of R-modules over a commutative ring R. An extension of B by A is an exact sequence

$$E: \ 0 \to A \to H \to B \to 0.$$

Two extensions are said to be congruent if there is an isomorphism Λ such that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i_1} & H_1 & \xrightarrow{\pi_1} & B & \longrightarrow & 0 \\ & & \Big\downarrow I_A & & \Big\downarrow \Lambda & & \Big\downarrow I_B & & \\ 0 & \longrightarrow & A & \xrightarrow{i_2} & H_2 & \xrightarrow{\pi_2} & B & \longrightarrow & 0 \end{array}$$

The set of congruence classes of extensions in $\mathcal{C}$ is denoted by $\mathrm{Ext}_{\mathcal{C}}(B, A)$. It can be given an abelian group structure using the Baer sum.

In this Appendix we recall the Baer sum and describe a generalisation due to Rabi which we need for our purposes.

5.1.1 Baer Sum and its Generalisation

For an extension E let $[E]$ denote its extension class. Given two extensions E_1 and E_2, we will define a third extension E such that $[E]$ defines an addition structure on $\mathrm{Ext}_{\mathcal{C}}(B, A)$. One defines the Baer sum $[E_1] \oplus_B [E_2] = [E]$. Let E_j, where $j \in \{1, 2\}$ be extensions

$$0 \to A \overset{i_j}{\to} H_j \overset{\pi_j}{\to} B \to 0.$$

Let E denote the extension

$$0 \to A \to H \to B \to 0,$$

where H is defined as follows. Let $Y = \{(h_1, h_2) \in H_1 \oplus H_2,\ \pi_1(h_1) = \pi_2(h_2)\}$ and $D = (i_1(a), -i_2(a))$. Let $H = Y/D$. The class $[E]$ is defined to be the Baer sum of $[E_1]$ and $[E_2]$. We will use $\oplus_B$ to denote the Baer sum. Using $D' = (i_1(a), i_2(a))$ in the place of D gives an extension congruent to the Baer difference $[E_1] \ominus_B [E_2]$.

We now define the generalised Baer sum which exists under certain circumstances. Now suppose we have diagrams of the following type:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & A_1 \arrow[d, "i_j"] & & & \\
0 \arrow[r] & B_1^j \arrow[r, "f_j"] \arrow[d, "\pi_j"] & B_2^j \arrow[r, "p_j"] & B_3 \arrow[r] & 0 \\
 & C_1 \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

where the vertical and horizontal sequences are exact for $j \in \{1, 2\}$. Let E_j denote the horizontal exact sequences:

$$E_j : 0 \longrightarrow B_1^j \xrightarrow{f_j} B_2^j \xrightarrow{p_j} B_3 \longrightarrow 0.$$

We would like to take the Baer difference of the E_j — but since they do not lie in the same Ext group we cannot quite do that. However, we can still salvage something.

One gets two types of extension classes in Ext groups which do not depend on j. The vertical exact sequences give classes in $\mathrm{Ext}(C_1, A_1)$. We can form their Baer difference to get an exact sequence

$$0 \longrightarrow A_1 \longrightarrow \mathbf{B}_1 \longrightarrow C_1 \longrightarrow 0.$$

The horizontal exact sequences give extensions in $\mathrm{Ext}(B_3, B_1^j)$. These depend on j but their push forward under π_j give classes $\mathbf{f}_{B_2^j}$ in $\mathrm{Ext}(B_3, C_1)$.

Define $\mathbf{B}_2$ as follows: Let $H_2 = \mathrm{Ker}(\psi)$, where ψ is the 'difference' map

$$\psi : B_2^1 \oplus B_2^2 \longrightarrow B_3$$

$$\psi((b_2^1, b_2^2)) = (p_1(b_2^1) - p_2(b_2^2))$$

Let D_2 be the image of the map

$$A_1 \longrightarrow B_1^1 \oplus B_1^2 \longrightarrow H_2$$

$$a \longrightarrow (f_1(i_1(a)), f_2(i_2(a)))$$

Define $\mathbf{B}_2 = H_2/D_2$. We call this the *generalised* Baer difference of E_1 and E_2 and denote it by $\tilde{\ominus}_B$. Observe that this is almost the Baer difference of E_1 and E_2 in the sense that if $B_1 = B_1^1 = B_1^2$, then we could take the difference in $\mathrm{Ext}(B_3, B_1)$. Since that is not the case, we do the best we can — we take the difference of the *inexact* sequences

$$0 \longrightarrow A_1 \longrightarrow B_2^j \longrightarrow B_3 \longrightarrow 0.$$

As a result of this one has a complex

$$0 \longrightarrow \mathbf{B}_1 \xrightarrow{f_1 \oplus f_2} \mathbf{B}_2 \xrightarrow{p_1 (or\ p_2)} B_3 \longrightarrow 0$$

However, this complex is *not* exact since $\mathrm{Ker}(p_1)$ is larger than $(f_1 \oplus f_2)(\mathbf{B}_1)$. The next lemma describes this difference.

Lemma 5.1.1 (Rabi[Rab01]). *Let* $\mathbf{F} = \mathbf{F}_{B_2^1 \tilde{\ominus}_B B_2^2} = \mathbf{B}_2/\mathbf{B}_1$. *Then one has the following diagram, in which the horizontal and vertical sequences are exact.*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & C_1 \arrow[d, "\phi"] & \\
0 \arrow[r] & \mathbf{B}_1 \arrow[r, "f"] & \mathbf{B}_2 \arrow[r, "\eta"] & \mathbf{F} \arrow[r] \arrow[d, "\bar{p}"] & 0 \\
 & & & B_3 \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

Proof. [Rab01], Appendix B. We repeat the proof here as that is unpublished. The horizontal sequence is exact by definition. To show the vertical sequence is exact we have to first describe be map ϕ. It is defined as follows. One has maps $\pi_j : B_1^j \longrightarrow C_1$. Consider the natural map

$$\tilde{\phi} : C_1 \oplus C_1 \longrightarrow (B_1^1 \oplus B_1^2)/\Delta_{A_1} \xrightarrow{(f_1,f_2)} \mathbf{B}_2 = H_2/D_2$$

$$\tilde{\phi}(c_1, c_2) \to (\pi_1^{-1}(c), \pi_2^{-1}(c)) \to (f_1(\pi_1^{-1}(c_1)), f_2(\pi_2^{-1}(c_2)))$$

where $\Delta_{A_1} = \{(i_1(a), i_2(a)) | a \in A_1\}$. ϕ gives a well defined map

$$(C_1 \oplus C_1)/\Delta_{C_1} \longrightarrow \mathbf{B}_2/\tilde{\phi}(\Delta_{C_1})$$

where $\Delta_{C_1} = \{(c, -c) | c \in C_1\}$ is the anti-diagonal. This is well defined as if (b_1, b_2) and (b_1', b_2') are in $(\pi_1^{-1}(c_1), \pi_2^{-1}(c_2))$ we have to show

$$(f_1(b_1), f_2(b_2)) \equiv (f_1(b_1'), f_2(b_2'))\ mod\ \tilde{\phi}(\Delta_{C_1})$$

or

$$(f_1(b_1 - b_1'), f_2(b_2 - b_2')) \in \tilde{\phi}(\Delta_{C_1}).$$

From exactness, we have $b_1 - b_1' = i_1(a_1)$ and $b_2 - b_2' = i_2(a_2)$ with $a_i \in A_1$. The image of Δ_{C_1} under (π_1^{-1}, π_2^{-1}) consists of (b, b') such that $\pi_1(b) = \pi_2(b')$. $(i_1(a_1), i_2(a_2))$ lie in this image, hence

$$(f_1(i_1(a_1)), f_2(i_2(a_2))) = (f_1(b_1 - b_1'), f_2(b_2 - b_2')) \in \tilde{\phi}(\Delta_{C_1}).$$

Note that the pre-image $(\pi_1^{-1}, \pi_2^{-1})(\Delta_{C_1})$ in $B_1^1 \oplus B_1^2)/\Delta_{A_1}$ is the Baer difference $\mathbf{B}_1$. Further, $(C_1 \oplus C_1)/\Delta_{C_1}) \simeq C_1$. Hence one has a map $\phi : C_1 \to \mathbf{F} = \mathbf{B}_2/\mathbf{B}_1$ and we get a exact sequence

$$0 \longrightarrow C_1 \xrightarrow{\phi} \mathbf{F}_{B_2^1 \tilde{\ominus}_B B_2^2} \xrightarrow{\bar{p}} B_3 \longrightarrow 0$$

This sequence is exact as if $b = (b_2^1, b_2^2)$ is in $\mathbf{F}_{B_2^1 \tilde{\ominus}_B B_2^2}$ and $\bar{p}(b) = 0$, then $p_1(b_2^1) = p_2(b_2^2) = 0$. So b_2^1 and b_2^2 lie in the image of $B_1^1 \oplus B_1^2$ — say $b_2^1 = f_1(b_1^1)$ and $b_2^2 = f_2(b_1^2)$. Let $c_i = \pi_1(b_1^1)$ and $c_2 = \pi_2(b_1^2)$. Then

$$b = \phi(c_1, c_2)$$

so it lies in the image of ϕ.

$\square$

In general, for any $\mathbb{Z}$-linear combination $m \cdot B_2^1 \tilde{\ominus}_B \; n \cdot B_2^2$ of B_2^1 and B_2^2 we get an extension class $\mathbf{f}_{m \cdot B_2^1 \tilde{\ominus}_B \; n \cdot B_2^2}$ in $\operatorname{Ext}(B_3, C_1)$ corresponding to $\mathbf{F}_{m \cdot B_2^1 \tilde{\ominus}_B \; n \cdot B_2^2}$. The relation between this and the extension classes constructed above is given as follows:

Corollary 5.1.2. *Let* $\mathbf{f}_{B_2^j}$ *and* $\mathbf{f}_{m \cdot B_2^1 \tilde{\ominus}_B \; n \cdot B_2^2}$ *be the extensions in* $\operatorname{Ext}(B_3, C_1)$ *described above. Then,*

$$\mathbf{f}_{m \cdot B_2^1 \tilde{\ominus}_B \; n \cdot B_2^2} = m \cdot \mathbf{f}_{B_2^1} \ominus_B \; n \cdot \mathbf{f}_{B_2^2}.$$

Proof. This follows from the construction of the map ϕ. $\square$

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