ORBIT SPACES OF UNIMODULAR ROWS OVER SMOOTH REAL AFFINE ALGEBRAS

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To

My Teachers and Parents

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Contents

Chapter 1

Introduction

Let R be a commutative, Noetherian ring of (Krull) dimension d. It is well known that the set of isomorphism classes of (oriented, if d is even) stably free R-modules of rank d carries the structure of an abelian group. This group can be identified with the orbit space of unimodular rows namely, $Um_{d+1}(R)/SL_{d+1}(R)$. The prime objective of this thesis is to provide the complete computation of this group, when $X = \text{Spec}(R)$ be a smooth real affine variety of dimension $d \geq 2$ (with the assumption that the set of real points of X is non-empty and orientable). In order to achieve our goal, we first carry out the computation of the $''$elementary orbit space$''$ $Um_{d+1}(R)/E_{d+1}(R)$, when $X = \text{Spec}(R)$ be as above. We also prove a structure theorem for the Mennicke symbols of length $d+1$ ($MS_{d+1}(R)$).

These results will be discussed in Chapter 4. These results have been obtained in a joint work with Mrinal Kanti Das and Md. Ali Zinna. This thesis is based primarily on our paper [DTZ1].

We now give brief introductions to the problems tackled in this thesis and the statements of the main results that we obtain.

Let R be as above. It follows from a classical result of Bass [Ba] that the stably free R-modules of rank at least $d+1$ are all free. Let R be an affine algebra of dimension d over a field k. If k is algebraically closed, or more generally, if the cohomological dimension of k is at most one, Suslin then proved that a stably free R-module of rank d

is free (see [Su 2, Su 4]). These results of Suslin do not extend to any arbitrary k. For example, if $d \neq 1, 3, 7$, the tangent bundle of a real d-sphere is stably free but not free. These examples also show that the aforementioned result of Bass is the best possible. Therefore, it is certainly of interest to understand the stably free R-modules of rank $d \geq 2$ when R is the coordinate ring of an affine variety over the field of real numbers. Among other results, we prove the following: *Let $X = Spec(R)$ be a smooth real affine variety of even dimension d, whose real points $X(\mathbb{R})$ constitute an orientable manifold. Then the set of isomorphism classes of (oriented) stably free R-modules of rank d is a free abelian group of rank equal to the number of compact connected components of $X(\mathbb{R})$.* In contrast, if $d \geq 3$ is odd, then the set of isomorphism classes of stably free R-modules of rank d is a $\mathbb{Z}/2\mathbb{Z}$-vector space (possibly trivial). We elaborate below.

The rings considered in this thesis are assumed to have (Krull) dimension at least two, unless mentioned otherwise. Recall that for any ring R of dimension d, a stably free R-module P of rank d corresponds to a *unimodular row* $(a_0, \cdots, a_d) \in R^{d+1}$ (meaning, there exist $b_0, \cdots, b_d \in R$ such that $\sum_0^d a_i b_i = 1$). The module P is free if and only if $(a_0, \cdots, a_d)$ is the last row of a matrix in $SL_{d+1}(R)$. Let $Um_{d+1}(R)$ be the set of unimodular rows of length $d+1$ over R. The preceding discussion inspires one to study the action of $SL_{d+1}(R)$ on $Um_{d+1}(R)$. The group $SL_{d+1}(R)$ and its elementary subgroup $E_{d+1}(R)$ act naturally on this set by multiplication from right. Thanks to the foundational works due to Vaserstein [SuVa, Section 5] (for $d = 2$) and van der Kallen [vdK 1] (for $d \geq 2$), the orbit space $Um_{d+1}(R)/E_{d+1}(R)$ carries the structure of an abelian group (inducing a group structure on $Um_{d+1}(R)/SL_{d+1}(R)$ as well [vdK 1]). Due to Jean Fasel's work [F 1], we now also have a modern-day interpretation of $Um_{d+1}(R)/E_{d+1}(R)$ in terms of cohomology. In this thesis we compute this group and its quotient $Um_{d+1}(R)/SL_{d+1}(R)$, when R is a smooth affine domain over the reals of dimension $d \geq 2$. We now present our results one by one. But first, let us set up some notations.

Notation. Let $X = \mathrm{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. We always assume that the set of real points $X(\mathbb{R})$ of X is non-empty, and therefore under the Euclidean topology, it is a smooth real manifold of dimension d. Let $\mathbb{R}(X)$

denote the ring obtained from R by inverting all the functions having no real zeros. Note that $\dim(R) = \dim(\mathbb{R}(X))$. Let $\mathcal{C}$ be the (finite) set of connected components of $X(\mathbb{R})$ which are compact. In this article we always assume that $X(\mathbb{R})$ is orientable.

The following is an accumulation of various results (Theorem 4.2.8, Corollary 4.3.5, Theorem 4.3.6, Corollary 4.3.9, Corollary 4.7.4).

Theorem 1.0.1. *Let $X = Spec(R)$ be as above. Then, we have the following assertions:*

(i) $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \xrightarrow{\sim} \bigoplus_{C\in\mathcal{C}} \mathbb{Z}$.

(ii) *The canonical map $\beta : Um_{d+1}(R)/E_{d+1}(R) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is a surjective group homomorphism and $K = ker(\beta)$ is the unique maximal divisible subgroup of $Um_{d+1}(R)/E_{d+1}(R)$. Consequently, $Um_{d+1}(R)/E_{d+1}(R) \xrightarrow{\sim} K \oplus (\bigoplus_{C\in\mathcal{C}} \mathbb{Z})$.*

(iii) *Precisely, K consists of those elementary orbits which can be represented by a unimodular row whose one entry is a square. Further, for any $[(a_0, \cdots, a_d)] \in K$ and any $r \geq 1$, $[(a_0, \cdots, a_d)]^r = [(a_0, \cdots, a_d^r)]$.*

(iv) *If $d \geq 3$, then K is torsion-free.*

Let $[v] \in K$ be arbitrary. Using divisibility of K and taking $r = d!$ in (iii) above, it immediately follows from a celebrated result of Suslin [Su 2] that v is the last row of a matrix in $SL_{d+1}(R)$. This observation makes the computation of $Um_{d+1}(R)/SL_{d+1}(R)$ quite easy. Note that the group $Um_{d+1}(R)/SL_{d+1}(R)$ is in bijection with the set of isomorphism classes of (oriented[1], if d is even) stably free R-modules of rank d. We prove the following result (in Chapter 4, Theorems 4.4.1 and 4.4.5):

Theorem 1.0.2. *Let $X = Spec(R)$ be as above. If the dimension d is <u>even</u>, then we have:*

$$\frac{Um_{d+1}(R)}{SL_{d+1}(R)} \xrightarrow{\sim} \frac{Um_{d+1}(\mathbb{R}(X))}{SL_{d+1}(\mathbb{R}(X))} \xrightarrow{\sim} \frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))} \xrightarrow{\sim} \bigoplus_{C\in\mathcal{C}} \mathbb{Z}.$$

[1]For any ring R of dimension d, a unimodular row $(a_0, \cdots, a_d)$ gives rise to a stably free R-module P together with a canonical orientation $\chi : R \xrightarrow{\sim} \wedge^d(P)$. In this article, for d even, stably free modules are always chosen with an orientation. See the discussion in section 2.3 preceding Theorem 2.3.1 . We refer to [BRS 3, Page 214] for further details.

If d is odd, then $Um_{d+1}(R)/SL_{d+1}(R)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space of rank $\leq |\mathcal{C}|$.

If d is odd, $Um_{d+1}(R)/SL_{d+1}(R)$ can be trivial. For example, by [F 1, Proposition 5.13], it is trivial when R is the coordinate ring of the real 3-sphere or the 7-sphere. We also touch on this in Section 4.5 using simpler arguments. For the other spheres of odd dimension, it follows from our results that this group is $\mathbb{Z}/2\mathbb{Z}$.

We now turn our attention to Mennicke symbols of Suslin. In [Su 3], Suslin used them to prove that the Milnor K-theory of a field injects into the Quillen K-theory modulo torsion. Our interest is in its connection with the group structure defined on $Um_{d+1}(A)/E_{d+1}(A)$, where A is a commutative Noetherian ring of dimension $d \geq 2$. In [vdK 2] van der Kallen introduced *weak Mennicke symbols* and showed that the universal weak Mennicke symbol $(wms, WMS_{d+1}(A))$ is in bijection with $Um_{d+1}(A)/E_{d+1}(A)$, thus giving the latter the structure of an abelian group. As a Mennicke symbol is also a weak Mennicke symbol, the universal Mennicke symbol $MS_{d+1}(A)$ is a quotient of $Um_{d+1}(A)/E_{d+1}(A)$. We prove the following results in Section 4.6.

Theorem 1.0.3. *Let $X = Spec(R)$ be as in Theorem 1.0.1. Then,*

(i) $MS_{d+1}(\mathbb{R}(X)) \xrightarrow{\sim} \bigoplus_{C\in\mathcal{C}} \mathbb{Z}/2\mathbb{Z}$.

(ii) *The kernel L of the canonical surjection $\beta_0 : MS_{d+1}(R) \twoheadrightarrow MS_{d+1}(\mathbb{R}(X))$ is the unique maximal divisible subgroup of $MS_{d+1}(R)$. Consequently, $MS_{d+1}(R) \xrightarrow{\sim} L \oplus (\bigoplus_{C\in\mathcal{C}} \mathbb{Z}/2\mathbb{Z})$.*

(iii) *The kernel of the canonical surjection $Um_{d+1}(R)/E_{d+1}(R) \twoheadrightarrow MS_{d+1}(R)$ is a free abelian group of rank $|\mathcal{C}|$.*

(iv) *If $d \geq 3$, then L is torsion-free.*

We now spend a few words on our methods. As it turns out, the computation of $Um_{d+1}(R)/E_{d+1}(R)$, with explicit description of its maximal divisible subgroup K, is the key. Such computations become easier if there is another related group to compare with, whose structure is well-understood. Recall from [BRS 3, DZ, vdK 4] that if A is a Noetherian ring of dimension $d \geq 2$, there is an exact sequence

$$Um_{d+1}(A)/E_{d+1}(A) \xrightarrow{\phi_A} E^d(A) \longrightarrow E_0^d(A) \longrightarrow 0, \qquad (*)$$

where $E^d(A)$ is the d-th Euler class group of A and $E_0^d(A)$ is the d-th weak Euler class group of A. We recall the definition of $E^d(A)$ in Chapter 2. We do not use $E_0^d(A)$ in this thesis.

For smooth affine real varieties the following structure theorem was proved in [BRS 2].

Theorem 1.0.4. *[BRS 2] Let R be as in Theorem 1.0.1. Then, $E^d(R) \xrightarrow{\sim} E^d(\mathbb{C}) \oplus E^d(\mathbb{R}(X))$, where $E^d(\mathbb{C})$ is the subgroup generated by all those Euler cycles in $E^d(R)$, which are supported on complex maximal ideals of R. Further, $E^d(\mathbb{C})$ is uniquely divisible and $E^d(\mathbb{R}(X))$ is free abelian of rank $|\mathcal{C}|$.*

We compare the elementary orbit spaces with the Euler class groups. As mentioned in $(*)$ above, we have group homomorphisms $\phi_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \to E^d(\mathbb{R}(X))$, and $\phi_R : Um_{d+1}(R)/E_{d+1}(R) \to E^d(R)$. But we found these maps to be insufficient for our purposes. To have more leverage, we take a reverse path, as follows.

Let A be a regular domain of dimension $d \geq 2$, which is essentially of finite type over an infinite perfect field k such that $2A = A$. Based on the formalism developed in [DTZ2], in Chapter 3 we introduce a map $\delta_A : E^d(A) \to Um_{d+1}(A)/E_{d+1}(A)$. When $k = \mathbb{R}$, this map gives us a lot of control.

Again let R be as in Theorem 1.0.1. In Section 4.2 we prove that $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is an isomorphism. In Section 4.3 we prove that $\delta_R : E^d(R) \to Um_{d+1}(R)/E_{d+1}(R)$ is a group homomorphism which is trivial on the divisible component $E^d(\mathbb{C})$. This enables us to analyze the kernel K of the natural map $\beta : Um_{d+1}(R)/E_{d+1}(R) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ and deduce our main results. Composing with the canonical projection $\epsilon : Um_{d+1}(R)/E_{d+1}(R) \to Um_{d+1}(R)/SL_{d+1}(R)$, we also have a group homomorphism $\delta'_R : E^d(R) \to Um_{d+1}(R)/SL_{d+1}(R)$, which turns out to be surjective (Theorem 4.4.2). Finally, the relation between the Euler class group and the elementary orbit space can be summed up in the form of the following exact sequence:

$$0 \to E^d(\mathbb{C}) \to E^d(R) \xrightarrow{\delta_R} Um_{d+1}(R)/E_{d+1}(R) \to K \to 1$$

Evidently, the maps δ_R and $\delta_{\mathbb{R}(X)}$ make a lot of arguments remarkably easier, which can be seen in Sections 4 and 4.3.

When the real variety $X = \mathrm{Spec}(R)$ is rational, Jean Fasel carried out some computation of the orbit spaces and Mennicke symbol using cohomological methods in [F 1, F 2]. Results in [F 1, Section 5] inspired us to take up this project.

1.1 layout

The layout of this thesis is as follows:-

In Chapter 2, we record some useful definitions and some general results. In this chapter,we also give a brief introduction to the orbit spaces of unimodular rows and their group structure. Further, the notion of Euler class group is recalled in this chapter with specific emphasis on the structure of the Euler class groups $E(R)$, when R is a smooth affine algebra over $\mathbb{R}$.

In Chapter 3, we focuss on the the maps between the orbit spaces of unimodular rows and the Euler class group and investigating their important properties.

In Chapter 4, we establish our main results.

In Chapter 5 we prove the map from Euler class group to orbit space of unimodular rows is bijective.

Chapter 2

Preliminaries

All the rings considered in this thesis are commutative and Noetherian. By the dimension of a ring we mean its Krull dimension. Modules are assumed to be finitely generated.

2.1 Definitions and general results

In this section we shall collect some definitions and preliminary results which will be used throughout this thesis. We begin with the following definition.

Definition 2.1.1. Let R be a ring. An R-module P is said to be *projective* if there exists another R-module Q such that $P \oplus Q \simeq R^n$ for some positive integer n. In other words, $P \oplus Q$ is free .

For this thesis, projective modules are always assumed to have constant rank.

The proof of the following lemma can be found in [BRS 3, Corollary 2.13]. This is a consequence of a result of Eisenbud-Evans [EE], as stated in [Pl, p. 1420]. Recall that for a projective R-module P, the R-module $\mathrm{Hom}_R(P, R)$ is denoted by P^*.

Lemma 2.1.2. *Let R be a ring and P be a projective R-module of rank n. Let $(\alpha, a) \in (P^* \oplus R)$. Then there exists an element $\beta \in P^*$ such that $\mathrm{ht}\,(I_a) \geq n$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$ then $\mathrm{ht}\, I \geq n$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of R, then $\mathrm{ht}\, I = n$.*

When R is a geometrically reduced affine algebra we have the following version of Swan's Bertini theorem, as stated in [BRS 2, Theorem 2.11], which is a refinement of above Lemma (2.1.2). This version can be deduced from [Sw, Theorems 1.3 and 1.4].

Lemma 2.1.3. *Let R be a geometrically reduced affine ring over an infinite field and P be a projective R-module of rank n. Let $(\alpha, a) \in (P^* \oplus R)$. Then there exists an element $\beta \in P^*$ such that if $I = (\alpha + a\beta)(P)$ then,*

(i) *Either $I_a = R_a$ or $\mathrm{ht}\,(I_a) = n$ such that $(R/I)_a$ is a geometrically reduced ring .*

(ii) *If $n <$ dim R and R_a is geometrically integral, then $(R/I)_a$ is also geometrically integral.*

(iii) *If R_a is smooth, then $(R/I)_a$ is also smooth.*

In particular, if we consider free modules instead of projective modules, then the proof of lemma 2.1.2 is an easy application of the *prime avoidance lemma* (for a proof, see [IR, Lemma 7.1.4]). We state this version below.

Lemma 2.1.4. *Let R be a ring and $(a_1, \cdots, a_n, a) \in R^{n+1}$. Then there exist $\mu_1, \cdots, \mu_n \in R$ such that $ht(I_a) \geq n$, where $I = (a_1 + a\mu_1, \cdots, a_n + a\mu_n)$. In other words, if $\mathfrak{p} \in Spec(R)$ such that $I \subset \mathfrak{p}$ and $a \notin \mathfrak{p}$, then $ht(\mathfrak{p}) \geq n$.*

Remark 2.1.5 If R is a geometrically reduced affine algebra over an infinite field then Swan's Bertini theorem, as stated above, implies that $\mu_1, \cdots, \mu_n$ can be so chosen that the ideal $I = (a_1 + a\mu_1, \cdots, a_n + a\mu_n)$ has the additional property that $(R/I)_a$ is a geometrically reduced ring.

The following lemma also follows from the *prime avoidance lemma* and standard general position arguments (see for example [RS, Lemma 3] or [BRS 1, Lemma 4.4]). Here $E_n(R)$ is the subgroup of $SL_n(R)$ generated by elementary matrices. For the definition of $E_n(R)$ see definition 2.2.3.

Lemma 2.1.6. *Let R be a ring. Let J and K be ideals of R such that $J + K = R$. Assume that $ht((J)) = n \geq dim(R/K) + 2$ and that J is generated by n elements $a_1, \ldots, a_n$. Then, there exists an element $\sigma \in E_n(R)$ such that*

(i) $[a_1, \ldots, a_n]\sigma = [b_1, \ldots, b_n]$.

(ii) $ht(b_1, \ldots, b_{n-1}) = n - 1$.

(iii) $(b_1, \ldots, b_{n-1}) + K = R$.

The following lemma is an application of Nakayama lemma. We give a proof from [BRS 3] for the sake of completeness.

Lemma 2.1.7. *Let R be a ring and $J \subset R$ be an ideal of R. Let $K \subset J$ and $L \subset J^2$ be two ideals of R such that $K + L = J$. Then $J = K + (e)$ for some $e \in L$ with $e(1-e) \in K$. Further, there is an ideal J' such that $J' + L = R$ and $K = J \cap J'$.*

Proof. Let bar denote reduction modulo the ideal K. Since $\overline{J}^2 = \overline{J}$, by Nakayama lemma there exists $\overline{e} \in \overline{J}$ such that $(1 - \overline{e})\overline{J} = 0$. It then follows that $\overline{J} = (\overline{e})$ and $\overline{e}^2 = \overline{e}$. Since the map $L \to \overline{J}$ is surjective, we may assume that $e \in L$. Now $J = K + (e)$. Since $\overline{e}^2 = \overline{e}$, we have $e - e^2 \in K$. Take $J' = K + (1 - e)$. Then $L + J' = R$, since $e \in L$. As $\overline{J} \cap \overline{J'} = (\overline{e})(1 - \overline{e}) = (\overline{0})$, it follows that $K = J \cap J'$. $\square$

In this context, we recall the next lemma, which is an application of Lemmas 2.1.2 and 2.1.7. This is a synthesis of [BRS 3, Corollary 2.14] and [BRS 4, Corollary 2.4] and we give a proof for the sake of completeness. We shall call this lemma as the "Moving lemma 1".

Lemma 2.1.8. (Moving Lemma 1) *Let R be a ring of dimension $d \geq 2$ and let P be a projective R-module of rank d. Let $J \subset R$ be an ideal of height d and let $\widetilde{\alpha} : P/JP \twoheadrightarrow J/J^2$ be a surjection. Then there exists an ideal $J' \subset R$ and a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:*

(i) $J + J' = R$.

(ii) $\beta \otimes R/J = \widetilde{\alpha}$.

(iii) $ht(J') \geq d$.

(iv) *Given finitely many ideals $J_1, \cdots, J_r$ of R, each of height ≥ 1, the ideal J' can be chosen with the additional property that it is comaximal with J_i for each $i = 1, \cdots, r$.*

Proof. Let $K = J^2 \cap J_1 \cap \cdots \cap J_r$. Then by the assumption, $\mathrm{ht}(K) \geq 1$. Therefore, there exists an element $a \in K \subset J^2$ such that $\mathrm{ht}(R/aR) \leq d-1$. Note that $(J/aR)^2 = J^2/aR$. Let bar denote reduction modulo the ideal (a).

As P is a projective module, $\widetilde{\alpha}$ can be lifted to an R-linear map $\delta : \overline{P} \longrightarrow \overline{J}$. Then $\delta(\overline{P}) + \overline{J}^2 = \overline{J}$. By Lemma 2.1.7, there exists $\overline{c} \in \overline{J}^2$ such that $\delta(\overline{P}) + (\overline{c}) = \overline{J}$. Now applying Lemma 2.1.2 to the element $(\delta, \overline{c}) \in \overline{P}^* \oplus \overline{R}$, we see that there exists $\gamma \in \overline{P}^*$ such that if $N = (\delta + \overline{c}\gamma)(\overline{P})$ then $\mathrm{ht}(N_{\overline{c}}) \geq d$. Note that $\dim(\overline{R}) \leq d - 1$. This implies that $(\overline{c})^r \in N$ for some positive integer r. Therefore, as $N + (\overline{c}) = \overline{J}$ and $\overline{c} \in \overline{J}^2$, we have $N = \overline{J}$. Therefore we get $(\delta + \overline{c}\gamma)(P/aP) = J/aJ$.
Let $\theta : P \longrightarrow J$ be a lift of $\delta + \overline{c}\gamma$, then we have $\theta(P) + (a) = J$. By Lemma 2.1.2, replacing θ by $\theta + a\theta_1$ for some $\theta_1 \in P^*$, we may assume that $\theta(P) = J \cap J'$, where $\mathrm{ht}(J') \geq d$ and $J' + (a) = R$. This proves the lemma. $\square$

Before finishing this section let us recall the definition of a special type of projective modules, which constitutes one of the main themes in this thesis.

Definition 2.1.9. Let R be a ring and P be projective R-module. Then P is said to be *stably free*, if there exist $m, n \in \mathbb{N}$ such that $P \oplus R^m \simeq R^n$, and rank of module is $n - m$.(Such a module is of course projective.)

It is easy to see that any stably free module of rank one is free. It follows from a classical result of Bass [Ba] that if R is a ring of dimension d, then any stably free R-module of rank at least $d + 1$ is free. There are examples of rings R of dimension d, and stably free modules of rank d which are not free.

2.2 Unimodular rows and related results

Definition 2.2.1. let R be a ring. A row $(a_1, \ldots, a_n) \in R^n$ is said to be a *unimodular row* of length n if there exists another row $(b_1, \ldots, b_n) \in R^n$ such that $\sum_{i=1}^n a_i b_i = 1$, (equivalently, the ideal generated by $a_1, \ldots, a_n$ is the whole ring R.)

The set of unimodular rows of length n is denoted by $Um_n(R)$.

Example 2.2.2. (i) Consider the following ring

$$R = \frac{\mathbb{R}[X_1, \cdots, X_n]}{(X_1^2 + \cdots + X_n^2 - 1)} = \mathbb{R}[x_1, \cdots, x_n]$$

where x_i is the image of X_i in R. Then $(x_1, \cdots, x_n)$ is unimodular row, as $x_1^2 + \cdots + x_n^2 = 1$.

(ii) Any row of an invertible matrix over R is a unimodular row. In particular any row of a matrix in $SL_n(R)$ is a unimodular row of length n.

Definition 2.2.3. Recall that e_{ij} denotes the $n \times n$ matrix whose only non-zero entry is 1 at the (i, j)-th place, and define $E_{ij}(\lambda) := I_n + \lambda e_{ij}$ for $\lambda \in R$ and $i \neq j$. Such $E_{ij}(\lambda)$ are called the *elementary matrices.* Let $E_n(R)$ denote the subgroup of $SL_n(R)$ generated by all elementary matrices $E_{ij}(\lambda)$ (where $i \neq j$, and $\lambda \in R$). The group $E_n(R)$ is called the elementary group.

Remark 2.2.4 The group $SL_n(R)$ acts on $Um_n(R)$ as follows:

$$Um_n(R) \times SL_n(R) \to Um_n(R) \text{ by } ((a_1, \ldots, a_n), M) \to (a_1, \ldots, a_n)M$$

The orbit space will be denoted by $Um_n(R)/SL_n(R)$.

Notation. Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in Um_n(R)$. We write

$$(a_1, \cdots, a_n) \overset{SL_n(R)}{\sim} (b_1, \cdots, b_n)$$

if there exists a matrix $M \in SL_n(R)$ such that $(a_1, \ldots, a_n)M = (b_1, \ldots, b_n)$.
The orbit of $(a_1, \ldots, a_n)$ will be denoted by $[(a_1, \ldots, a_n)]$. (Or $[a_1, \ldots, a_n]$ when there is no ambiguity.)

Remark 2.2.5 Note that $E_n(R)$ acts on $Um_n(R)$ in a similar manner. The orbit space will be denoted by $Um_n(R)/E_n(R)$.

Definition 2.2.6. A unimodular row $(a_1, \ldots, a_n) \in R^n$ is said to be *completable* if it is (last) row of some matrix $M \in SL_n(R)$. Or equivalently if it belongs to the orbit of the unimodular row $(0, \ldots, 0, 1)$ in $Um_n(R)/SL_n(R)$.

Remark 2.2.7 It can be easily deduced that if $(a_1, \ldots, a_n) \in R^n$ is completable then it can be any row of a matrix in $SL_n(R)$.

Remark 2.2.8 A unimodular row $(a_1, \ldots, a_n) \in R^n$ is said to be *elementarily completable* if

$$(a_1, \ldots, a_n) \overset{E_n(R)}{\sim} (0, \cdots, 0, 1)$$

Remark 2.2.9 It can be easily shown that any unimodular row of length 2 is completable. It is well known that for odd $n \neq 1, 3, 7$, the unimodular row given by previous example [2.2.2 (i)] is not completable (for details, see [IR, proposition 3.1.10]).

In [SwT] R. G. Swan and J. Towber observed the remarkable fact that for any ring R and any $(a, b, c) \in Um_3(R)$, the unimodular row $(a^2, b, c) \in Um_3(R)$ is completable. Suslin [Su 2] generalized the result of Swan-Towber by proving the following theorem which will be crucial for this thesis.

Theorem 2.2.10. *[Su 2] Let $(a_0, \ldots, a_n)$ be a unimodular row and let $m_0, \ldots, m_n$ be positive integers such that $\prod_{i=0}^{i=n} m_i$ is divisible by $n!$. Then $(a_0^{m_0}, \ldots, a_n^{m_n})$ is completable. In particular $(a_0, \ldots, a_{n-1}, a_n^{n!})$ is completable.*

Remark 2.2.11 Let P be an R-module such that $P \oplus R \simeq R^{n+1}$. In other words, P is a projective R-module of of rank n (which is stably free of type 1). One can then regard P as $ker(\alpha)$, where $\alpha : R^{n+1} \to R$ is some suitable surjection. Let $e_1, \ldots, e_{n+1}$ be the canonical basis of R^{n+1} and also let $\alpha(e_i) = a_i$, for all $i = 1, \ldots, n+1$. Since α is surjective, there exists $(b_1, \ldots, b_{n+1}) \in R^{n+1}$ such that $\sum_{i=1}^{n+1} a_i b_i = 1$. Therefore, $(a_1, \ldots, a_{n+1}) \in Um_{n+1}(R)$.

Conversely, Let $v = (b_1, \cdots, b_{n+1}) \in Um_{n+1}(R)$, then v gives a surjective map $\beta : R^{n+1} \to R$ such that $\beta(e_i) = b_i$, for all $i = 1, \ldots, n+1$. Let $P = ker(\beta)$. Then clearly, $P \oplus Rv \simeq P \oplus R \simeq R^{n+1}$. Let us denote P as P_v.

The following is a standard result. The proof can be found in [IR].

Proposition 2.2.12. *[IR, Proposition 3.1.6] Let $v \in Um_{n+1}(R)$. Then P_v is free if and only if v is completable.*

2.3 Group structure on orbit space of unimodular rows

Let R be a ring of dimension $d \geq 2$. By a simple application of the *prime avoidance lemma*, it can be shown that for $n \geq d+2$, the group $E_n(R)$ acts transitively on $Um_n(R)$ (see [La, Chapter II, Theorem 7.3] for a proof). There are counterexamples to show that this is the best possible result. For $d = 2$, in [SuVa, Section 5] Vaserstein shows that, $Um_3(R)/E_3(R)$ has bijective correspondence with a certain Witt group. This abelian group structure can be pulled back on $Um_3(R)/E_3(R)$ to give the latter the structure of an abelian group. Later, van der Kallen [vdK 1] derived from this (inductively) the abelian group structure on the orbit space $Um_{d+1}(R)/E_{d+1}(R)$ (when $dimR = d$). This induces a group structure on $Um_{d+1}(R)/SL_{d+1}(R)$ as well.

Let P be a projective R-module of rank d with $R \xrightarrow{\sim} \wedge^d(P)$. An isomorphism $\chi : R \xrightarrow{\sim} \wedge^d(P)$ is called an *orientation* of P, and P is called oriented if it comes with a chosen χ. A pair of oriented projective modules (P_1, χ_1) and (P_2, χ_2) are said to be isomorphic if there exists an isomorphism $\phi : P_1 \xrightarrow{\sim} P_2$ such that $(\wedge^d \phi)\chi_1 = \chi_2$.

Let $v = (a_0, \cdots, a_d) \in Um_{d+1}(R)$ and P_v be the kernel of the surjection $\theta : R^{d+1} \twoheadrightarrow R$ given by $\theta(e_i) = a_i$, $0 \leq i \leq d$ (here $\{e_0, \cdots, e_d\}$ is the canonical basis of R^{d+1}). In [BRS 3, Page 214], it is shown in detail that P_v is oriented, endowed with a natural orientation. We give a sketch. Let $b_0, \cdots, b_d \in R$ be such that $a_0b_0 + \cdots + a_db_d = 1$. Then the map $\psi : R \longrightarrow R^{d+1}$ given by $x \mapsto (xb_0, \cdots, xb_d)$ is the injection which splits θ, and we have an isomorphism $R^{d+1} \xrightarrow{\sim} P_v \oplus R$. Let $f = b_0e_0 + \cdots + b_de_d$, and $p_i = a_if - e_i$ for $0 \leq i \leq d$. It is then easy to check that under the isomorphism, p_i are images of e_i and therefore they generate P_v. Eventually, it is shown in [BRS 3] that if $\omega_i = p_0 \wedge \cdots p_{i-1} \wedge p_{i+1} \cdots p_d$ $(0 \leq i \leq d)$, and $\chi := \sum_0^d (-1)^i a_i \omega_i$, then χ is an orientation of P_v. Thus, every unimodular row v corresponds to an oriented stably free module.

The following result from [vdK 1] (see Theorem 4.8 and the discussion surrounding it in [vdK 1]) inspires one to study the action of $SL_{d+1}(R)$ on $Um_{d+1}(R)$.

Theorem 2.3.1. *Let R be ring of dimension $d \geq 2$. If d is even, then the set of isomorphism classes of oriented stably free R-modules of rank d is in bijection with $Um_{d+1}(R)/SL_{d+1}(R)$ and therefore carries the structure of an abelian group. If d is odd, then $Um_{d+1}(R)/SL_{d+1}(R)$ is the set of isomorphism classes of stably free R-modules of rank d (without orientation).*

We now recall the group structure on the orbit space $Um_{d+1}(R)/E_{d+1}(R)$ from [vdK 1, vdK 2] and record some useful results. The defining relation goes as follows: If $(q, v_1, \cdots, v_d)$ and $(1+q, v_1, \cdots, v_d)$ are both unimodular and if $r(1+q) \equiv q$ modulo $v_1R + \cdots + v_dR$, then

$$[q, v_1, \cdots, v_d] = [r, v_1, \cdots, v_d][1+q, v_1, \cdots, v_d]$$

The following 'useful formulas' from [vdK 2, Lemma 3.5] are extensively used in this thesis.

Theorem 2.3.2. *Let R be a ring of dimension $d \geq 2$. Let $(a, v_1, \ldots, v_d)$, $(b, v_1, \cdots, v_d)$, and (a, r) be unimodular. Choose $p \in R$ such that $ap \equiv 1$ modulo $v_1R + \cdots + v_dR$. Then we have the followings assertions:*

(i) $[b, v_1, \cdots, v_d][a, v_1, \ldots, v_d] = [a(b+p) - 1, (b+p)v_1, \ldots, v_d]$.

(ii) $[a, v_1, \ldots, v_d] = [a, r^2v_1, v_2, \ldots, v_d] = [a, rv_1, rv_2, \ldots, v_d]$.

(iii) $[a, v_1, \ldots, v_d][b^2, v_1, \cdots, v_d] = [ab^2, v_1, \cdots, v_d]$.

(iv) $[a, v_1, \ldots, v_d]^{-1} = [-p, v_1, \ldots, v_d]$.

(v) $[a^2, v_1, \cdots, v_d] = [a, v_1^2, v_2, \cdots, v_d]$.

Much later, van der Kallen gives a simpler relation for the group structure on $Um_{d+1}(R)/E_{d+1}(R)$ in [vdK 3]. The following is a useful lemma for that purpose.

Lemma 2.3.3. *(Mennicke-Newman) [vdK 3, Lemma 3.2] Let $v, w \in Um_{d+1}(R)$. Then there exist $\alpha, \beta \in E_{d+1}(R)$ such that $v\alpha = (x, a_1, \ldots, a_d)$ and $w\beta = (y, a_1, \ldots, a_d)$ such that $x + y = 1$.*

The product rule then goes as follows: If $(x, a_1, \ldots, a_d), (y, a_1, \ldots, a_d)$ are unimodular with $x + y = 1$, then

$$[x, a_1, \ldots, a_d][y, a_1, \ldots, a_d] = [xy, a_1, \ldots, a_d].$$

2.4 The Euler class group and related results

In this section we quickly recall the generalities of the Euler class group theory. We first accumulate some basic definitions, namely, the definitions of the Euler class group, the Euler class of a projective module, and then state some results which are relevant to this thesis. Detailed accounts of these topics can be found in [BRS 1, BRS 3].

Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. We recall definitions of the Euler class groups from [BRS 1]. Our emphasis will also be on the definition of the Euler class group given by M. V. Nori in terms of homotopy (as appeared in [BRS 1]). In [DK], the authors investigated in detail the relation between these two equivalent definitions and their consequences. We reproduce some of those results in one place for completeness.

The Euler class group $E^d(R)$:

Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. Let B be the set of pairs (m, ω_m) where m is a maximal ideal of R and $\omega_m : (R/m)^d \twoheadrightarrow m/m^2$. Let G be the free abelian group generated by B. Let $J = m_1 \cap \cdots \cap m_r$, where m_i are distinct maximal ideals of R. Any $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ induces surjections $\omega_i : (R/m_i)^d \twoheadrightarrow m_i/m_i^2$ for each i. We associate $(J, \omega_J) := \sum_1^r (m_i, \omega_i) \in G$.

Definition 2.4.1. (Nori) Let S be the set of elements $(I(1), \omega(1)) - (I(0), \omega(0))$ of G where (i) $I \subset R[T]$ is a local complete intersection ideal of height d; (ii) Both $I(0)$ and $I(1)$ are reduced ideals of height d; (iii) $\omega(0)$ and $\omega(1)$ are induced by $\omega : (R[T]/I)^d \twoheadrightarrow I/I^2$. Let H be the subgroup generated by S. The d-th Euler class group $E^d(R)$ is defined as

$E^d(R) := G/H$.

Definition 2.4.2. (Bhatwadekar-Sridharan) Let S_1 be the set of elements (J, ω_J) of G for which ω_J has a lift to a surjection $\theta : R^d \twoheadrightarrow J$ and H_1 be the subgroup of G generated by S_1 . The Euler class group $E^d(R)$ is defined as $E^d(R) := G/H_1$.

Remark 2.4.3 We shall refer to the elements of the Euler class group as *Euler cycles.*

Remark 2.4.4 The above definitions appear to be slightly different than the ones given in [BRS 1]. However, note that if $(J, \omega_J) \in S$ (resp. S_1) and if $\overline{\sigma} \in E_d(R/J)$, then the element $(J, \omega_J\overline{\sigma})$ is also in S (resp. S_1). For details, see [DK, Remark 5.4] and [DZ, Proposition 2.2].

Remark 2.4.5 Bhatwadekar-Sridharan proved (see [BRS 1, Remark 4.6]) that $H = H_1$ and therefore the above definitions of the Euler class group are equivalent.

The following theorem collects a few results in one place (see [BRS 1, 4.11], [K, 4.2], [DK, Theorem 5.13] for details).

Theorem 2.4.6. *Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. Let $J \subset R$ be a reduced ideal of height d and $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ be a surjection. Then, the following are equivalent:*

(i) *The image of $(J, \omega_J) = 0$ in $E^d(R)$*

(ii) *ω_J can be lifted to a surjection $\theta : R^d \twoheadrightarrow J$.*

(iii) *$(J, \omega_J) = (I(0), \omega(0)) - (I(1), \omega(1))$ in G where (i) $I \subset R[T]$ is a local complete intersection ideal of height d; (ii) Both $I(0)$ and $I(1)$ are reduced ideals of height d, and (iii) $\omega(0)$ and $\omega(1)$ are induced by $\omega : (R[T]/I)^d \twoheadrightarrow I/I^2$.*

A series of remarks are in order.

Remark 2.4.7 Let $J \subset R$ be an ideal of height d which is not necessarily reduced and let $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ be a surjection. Then also one can associate an element (J, ω_J) in $E^d(R)$ and prove the above theorem for (J, ω_J). See [BRS 1, Remark 4.16] for details.

Remark 2.4.8 ([BRS 1], Remark 4.14) An arbitrary element of $E^d(R)$ can be represented by a single Euler cycle (J, ω_J), where J is a reduced ideal of height d and $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ is a surjection.

The following notation will be used in the rest of this thesis.

Notation. Let $\dim(R) = d$. Let $(J, \omega_J) \in E^d(R)$ and $u \in R$ be a unit modulo J. Let σ be any matrix in $GL_d(R/J)$ with determinant $\overline{u}$ (bar means modulo J). We shall denote the composite surjection

$$(R/J)^d \overset{\sigma}{\underset{\sim}{\to}} (R/J)^d \overset{\omega_J}{\twoheadrightarrow} J/J^2$$

by $\overline{u}\omega_J$. It is easy to check that the element $(J, \overline{u}\omega_J) \in E^d(R)$ is independent of σ (the key fact used here is that $SL_d(R/J) = E_d(R/J)$ as $\dim(R/J) = 0$).

The following remark is useful for the next section of this thesis.

Remark 2.4.9 ([BRS 3, Remark 5.0, Lemma 5.4]) Let R and J be as above and also let (J, ω_1) and (J, ω_2) in $E^d(R)$. Then it can be obtained from [BRS 3, Lemma 2.2] that $(J, \omega_2) = (J, \bar{u}\omega_1)$, for some unit $\bar{u} \in R/J$. If $\bar{u} \in (R/J)^*$ is a square say, $\bar{u} = \bar{v^2}$. Then $(J, \omega_1) = (J, \bar{v^2}\omega_1)$.

Remark 2.4.10 [BDM, Lemma 3.4, 3.6] Let R and J be as above. Let ω_J be induced by $J = (a_1, \cdots, a_d) + J^2$. Consider $K = (a_1, \cdots, a_{d-1}) + J^2$. Then it is proved in [BDM, Lemma 3.4, 3.6] that $(K, \omega_K) = (J, \omega_J) + (J, -\omega_J)$ in $E^d(R)$, where ω_K corresponds to $K = (a_1, \cdots, a_{d-1}, a_d^2) + K^2$. In fact, (K, ω_K) is independent of ω_K.

Definition 2.4.11. (*The Euler class of a projective module*): Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. Let P be a projective R-module of rank d such that $R \simeq \wedge^d(P)$ and let $\chi : R \overset{\sim}{\to} \wedge^d P$ be an isomorphism.

Let $\varphi : P \twoheadrightarrow J$ be a surjection where J is an ideal of height d. Therefore we obtain an induced surjection $\overline{\varphi} : P/JP \twoheadrightarrow J/J^2$. Let $\overline{\gamma} : (R/JR)^d \xrightarrow{\sim} P/JP$ be an isomorphism such that $\wedge^d(\overline{\gamma}) = \overline{\chi}$. Let ω_J be the composite surjection $\overline{\varphi}\,\overline{\gamma} : (R/JR)^d \twoheadrightarrow J/J^2$. Let $e(P,\chi)$ be the image in $E^d(R)$ of the element (J,ω_J) of G. Then it is proved in [BRS 3] that the assignment sending the pair (P,χ) to the element $e(P,\chi)$ of $E^d(R)$ is well defined. The *Euler class* of (P,χ) is defined to be $e(P,\chi)$.

We record the following results from [BRS 3] for later use.

Theorem 2.4.12. *Let R be a be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k such that $dimR = d \geq 2$. Let P be a projective R-module of rank d with $R \simeq \wedge^d(P)$ and let $\chi : R \xrightarrow{\sim} \wedge^d P$ be an isomorphism. Let $J \subset R$ be an ideal of height n and $\omega_J : (R/JR)^d \twoheadrightarrow J/J^2$ be a surjection.*

(i) *Let $e(P,\chi) = (J,\omega_J)$ in $E^d(R)$. Then there exists a surjective map $\alpha : P \twoheadrightarrow J$ such that (J,ω_J) is induced by (α,χ).*

(ii) *$P \simeq P_1 \oplus R$ for some projective R-module P_1 of rank $d-1$ if and only if $e(P,\chi) = 0$ in $E^d(R)$.*

2.5 Results on the Euler class groups of smooth affine domain over $\mathbb{R}$

Let $X = \mathrm{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Also, let $\mathfrak{m}$ be a maximal ideal of R. If $R/\mathfrak{m} \simeq \mathbb{R}$, then we call it a real maximal ideal and if $R/\mathfrak{m} \simeq \mathbb{C}$, then we call it a complex maximal ideal. Let $X(\mathbb{R})$ denote the set of all real points of X. In this thesis we assume that $X(\mathbb{R}) \neq \emptyset$. Let S denote the multiplicatively closed subset of R consisting of all functions which do not have any real zeros (which do not belong to any real maximal ideal of R). Let $\mathbb{R}(X) := R_S$. (The ring $\mathbb{R}(X)$ informally dubbed as the "real" coordinate ring of the variety). Then $\dim(R) = \dim(\mathbb{R}(X))$. Under the Euclidean topology, $X(\mathbb{R})$ is a smooth real manifold of dimension d. Let $\mathcal{C}$ be the (finite) set of connected components of $X(\mathbb{R})$ which are compact.

Moreover, as $\mathbb{R}(X)$ is a direct limit of smooth affine domains then it can be verified that Euler class group $E^d(\mathbb{R}(X))$ can be taken as direct limits of the respective Euler class groups of smooth affine domains. Hence all the analogous results concerning the Euler class groups of smooth affine domains hold for $\mathbb{R}(X)$.

Some essential observations to get the structure theorem of $E^d(R)$:

Here we list some results from [BRS 2] that we shall need in this thesis.

(i) Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{R}(X)$ and $(\mathfrak{m}, \omega_1)$, $(\mathfrak{m}, \omega_2) \in E^d(\mathbb{R}(X))$ where $\omega_1, \omega_2 : (\mathbb{R}(X)/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ be two surjections. From Remark 2.4.9, $(\mathfrak{m}, \omega_2) = (\mathfrak{m}, \overline{u}\omega_1)$ for some $\overline{u} \in (R/\mathfrak{m})^* = \mathbb{R}^*$. Then either $\overline{u}$ or $-\overline{u}$ is a square. If $\overline{u} = \overline{v}^2$, Then $(\mathfrak{m}, \omega_2) = (\mathfrak{m}, \overline{v}^2\omega_1) = (\mathfrak{m}, \omega_1)$, by Remark 2.4.9. Otherwise, $-\overline{u} = \overline{v}^2$ and $(\mathfrak{m}, \omega_2) = (\mathfrak{m}, -\overline{v}^2\omega_1) = (\mathfrak{m}, -\omega_1)$. Therefore, $(\mathfrak{m}, \omega_2) = (\mathfrak{m}, \omega_1)$ or $(\mathfrak{m}, \omega_2) = (\mathfrak{m}, -\omega_1)$.

(ii) Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{R}(X)$ such that the real point associated to $\mathfrak{m}$ does not belong to any compact connected component of $X(\mathbb{R})$. Then it follows from [BRS 2, Corollary 4.9] that $(\mathfrak{m}, \omega_{\mathfrak{m}}) = 0$, for any surjection $\omega_{\mathfrak{m}} : (\mathbb{R}(X)/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$.

(iii) If $\mathfrak{m}$ is maximal ideal of $\mathbb{R}(X)$ such that the real point associated to $\mathfrak{m}$ belongs to a compact connected component of $X(\mathbb{R})$, then it follows from the [BRS 2, proof of Theorem 4.13] that for any surjection $\omega_{\mathfrak{m}} : (\mathbb{R}(X)/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$, $(\mathfrak{m}, \omega_{\mathfrak{m}}) + (\mathfrak{m}, -\omega_{\mathfrak{m}}) = 0$ in $E^d(\mathbb{R}(X))$.

Moreover, by [BRS 2, proof of Theorem 4.13] , if $\mathfrak{m}'$ in $\mathbb{R}(X)$ is another maximal ideal such that the real point associated to $\mathfrak{m}'$ belongs to the same compact connected component of $X(\mathbb{R})$, then it follows from (i) and the above discussion that for any other surjection $\omega_{\mathfrak{m}'} : (\mathbb{R}(X)/\mathfrak{m}')^d \twoheadrightarrow \mathfrak{m}'/\mathfrak{m}'^2$, either $(\mathfrak{m}', \omega_{\mathfrak{m}'}) = (\mathfrak{m}, \omega_{\mathfrak{m}})$ or $(\mathfrak{m}', \omega_{\mathfrak{m}'}) = -(\mathfrak{m}, \omega_{\mathfrak{m}})$.

Structure theorem of $E^d(R)$

In[[BRS 2], [BDM]] the structure of $E^d(\mathbb{R}(X))$ has been extensively studied. Our interest is in the case when $X(\mathbb{R})$ is orientable. By [BRS 2, Theorem 4.12, 4.13] and [BDM, Theorem 4.21] $E^d(\mathbb{R}(X))$ is a free abelian group of rank equal to the number of compact connected components of $X(\mathbb{R})$:

$$E^d(\mathbb{R}(X)) \xrightarrow{\sim} \bigoplus_{|\mathcal{C}|} \mathbb{Z},$$

where $\mathcal{C}$ is the set of all compact connected components of $X(\mathbb{R})$.

Since $\mathbb{R}(X)$ is a localization of R and is equidimensional, then there is a canonical surjective group homomorphism $\Gamma : E^d(R) \twoheadrightarrow E^d(\mathbb{R}(X))$ (for details see [BRS 2, page 307]).

For smooth affine real varieties the following structure theorem was proved in [BRS 2, Theorem 4.14].

Theorem 2.5.1. *Let R be as above and let $X(\mathbb{R})$ be orientable. Then, $E^d(R) \xrightarrow{\sim} E^d(\mathbb{C}) \oplus E^d(\mathbb{R}(X))$, where $E^d(\mathbb{C})$ is the subgroup generated by all those Euler cycles in $E^d(R)$, which are supported on complex maximal ideals of R. As mentioned above, $E^d(\mathbb{R}(X))$ is free abelian of rank $|\mathcal{C}|$.*

It can also be deduced from [BDM, 4.26,4.25] that $E^d(\mathbb{C})$ is divisible and torsion free. We give an outline of the proof.

Remark 2.5.2 Let $X = \mathrm{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$ and $CH_0(X)$ denote the group of zero cycles of X modulo rational equivalence. Let $J \subset R$ be an ideal of height d. By abuse of notation we will denote the cycles associated to the module R/J by $[J]$. The assignment $(J, \omega_J) \in E^d(R)$ to $[J] \in CH_0(X)$ gives rise to a well defined surjective group homomorphism say, ϕ.

Following [BDM, Remark 4.24] let us consider $R_{\mathbb{C}} := R \otimes_{\mathbb{R}} \mathbb{C}$. Let $Y := \mathrm{Spec}(R_{\mathbb{C}})$. Then $R_{\mathbb{C}}$ is a smooth affine domain of dimension d over $\mathbb{C}$ and hence $CH_0(Y)$ is a divisible group. Let $\pi : Y \to X$ be the canonical finite group homomorphism. Then we have induced maps

$$\pi_* : CH_0(Y) \to CH_0(X) \text{ and } \pi^* : CH_0(X) \to CH_0(Y),$$

such that $\pi_*\pi^*$ is multiplication by 2.

Write $G := \pi_*(CH_0(Y)) \subset CH_0(X)$. Then by [BDM, Lemma 4.25], G is a divisible group and torsion free for $d \geq 2$. It is also proved in [BDM, Lemma 4.26], that $G = \phi(E^d(\mathbb{C}))$. Now consider the following commutative diagram:

$$\begin{tikzcd}
E^d(R) \arrow[r, "\phi"] & CH_0(X) \\
E^d(\mathbb{C}) \arrow[u, "i"] \arrow[r, "\phi|_{E^d(\mathbb{C})}"] & G \arrow[u, "i"']
\end{tikzcd}$$

where i denotes the inclusion map. From the proof of [BDM, proposition 4.29] it follows that ϕ is injective on $(E^d(\mathbb{C}))$. Therefore we have $(E^d(\mathbb{C})) \simeq G$. Since G is a divisible torsion free group $(E^d(\mathbb{C}))$ is also divisible and torsion free group for $d \geq 2$.

Chapter 3

Objects, Maps and Homotopy

In this chapter we collect some basic definitions and useful results. We also establish a map δ_R from the Euler class group $E^d(R)$ to the group $Um_{d+1}(R)/E_{d+1}(R)$, which is one of the main tools of this thesis. This definition involves homotopy orbits of certain objects. By *'homotopy'* we mean *'naive homotopy'*, as defined below.

3.1 Homotopy

Definition 3.1.1. Let F be a functor originating from the category of rings to the category of sets. For a given ring R, two elements $F(u_0), F(u_1) \in F(R)$ are said to be homotopic if there is an element $F(u(T)) \in F(R[T])$ such that $F(u(0)) = F(u_0)$ and $F(u(1)) = F(u_1)$.

Definition 3.1.2. Let F be a functor from the category of rings to the category of sets. Let R be a ring. Consider the equivalence relation on $F(R)$ *generated by* homotopies (the relation is easily seen to be reflexive and symmetric but is not transitive in general). The set of equivalence classes thus obtained will be denoted by $\pi_0(F(R))$ and an equivalence class will be called a *homotopy orbit*.

Example 3.1.3. Let R be a ring. Two matrices $\sigma, \tau \in GL_n(R)$ are *homotopic* if there is a matrix $\theta(T) \in GL_n(R[T])$ such that $\theta(0) = \sigma$ and $\theta(1) = \tau$. Of particular interest are the matrices in $GL_n(R)$ which are *homotopic to identity*.

Remark 3.1.4 Any $\theta \in E_n(R)$ is homotopic to identity. To see this, let $\theta = \prod E_{ij}(\lambda_{ij})$. Define $\Theta(T) := \prod E_{ij}(T\lambda_{ij})$. Then, clearly $\Theta(T) \in E_n(R[T])$ and we observe that $\Theta(1) = \theta$, $\Theta(0) = I_n$.

In this context, we record below a remarkable result of Vorst.

Theorem 3.1.5. *[Vo, Theorem 3.3] Let R be a regular ring which is essentially of finite type over a field k. Let $n \geq 3$ and $\theta(T) \in GL_n(R[T])$ be such that $\theta(0) = I_n$ (θ is thus a homotopy between I_n and $\theta(1) \in GL_n(R)$). Then $\theta(T) \in E_n(R[T])$.*

Remark 3.1.6 Using a result of Popescu [Po], the above theorem can be extended to the case when R is a regular ring containing a field.

3.2 Homotopy orbits of unimodular rows

For a ring R, consider the set

$$Um_{n+1}(R) := \{(a_1, \cdots, a_{n+1}) \in R^{n+1} \mid \sum_{i=1}^{n+1} a_i b_i = 1 \text{ for some } b_1, \cdots, b_{n+1} \in R\}$$

of *unimodular rows* of length $n+1$ in R. Then $F_{n+1}(R) := Um_{n+1}(R)$ is a functor. Two unimodular rows $(a_1, \cdots, a_{n+1})$ and $(a'_1, \cdots, a'_{n+1})$ are homotopic if there exists $(f_1(T), \cdots, f_{n+1}(T)) \in Um_{n+1}(R[T])$ such that $f_i(0) = a_i$ and $f_i(1) = a'_i$ for $i = 1, \cdots, n+1$. The set $Um_{n+1}(R)$ has a base point, namely, $(0, \cdots, 0, 1)$.

We shall need the following theorem later. See also [F 1, Theorem 2.1] for a more general version.

Theorem 3.2.1. *Let R be a regular ring containing a field k. Then, for any $n \geq 2$ there is a bijection $\eta_R : \pi_0(Um_{n+1}(R)) \xrightarrow{\sim} Um_{n+1}(R)/E_{n+1}(R)$.*

Proof. Let $v \in Um_{n+1}(R)$. We define η_R by sending the homotopy orbit of v to the elementary orbit of v. But we have to ensure that η_R is well-defined. Let $u \in Um_{n+1}(R)$ be such that v is homotopic to u. Then, by definition, there exists $f(T) \in Um_{n+1}(R[T])$ such that $f(0) = v$ and $f(1) = u$. As R is a regular ring containing a field k, it follows from [Li, Po] that $f(T)$ is extended from R. In other words, there exists

$\sigma(T) \in GL_{n+1}(R[T])$ such that $f(T)\sigma(T) = f(0)$. Therefore, $f(0)\sigma(0) = f(0)$. It then follows that $f(T)\sigma(T)\sigma(0)^{-1} = f(0)$. Writing $\tau = \sigma(T)\sigma(0)^{-1}$ we see that $\tau \in GL_{n+1}(R[T])$ and $\tau(0) = I_{n+1}$. By Theorem 3.1.5 (and Remark 3.1.6) above, we actually have $\tau \in E_{n+1}(R[T])$. As $u\tau(1) = f(1)\tau(1) = f(0) = v$, we are done proving that η_R is well-defined.

Injectivity of η_R is clear because elementary matrices are homotopic to identity. Surjectivity is trivial. $\square$

3.3 The pointed set $Q'_{2n}(R)$ and its homotopy orbits

Let R be any commutative Noetherian ring. Let $n \geq 2$ and consider the set

$$Q'_{2n}(R) = \{(x_1, \cdots x_n, y_1, \cdots, y_n, z) \in R^{2n+1} \mid \sum_{i=1}^{n} x_i y_i + z^2 = 1\}$$

with a base point $(0, \cdots, 0, 0, \cdots, 0, 1)$. Assume that $2R = R$ and let $O_{2n+1}(R)$ be the group of orthogonal matrices preserving the quadratic form $\sum_{i=1}^{n} X_i Y_i + Z^2$. Then there is a natural action of $O_{2n+1}(R)$ and its subgroup $SO_{2n+1}(R)$ on the set $Q'_{2n}(R)$. Let $EO_{2n+1}(R)$ be the elementary subgroup of $SO_{2n+1}(R)$ as defined in [VaP, Section 3], [Va, p. 1503]. As $n \geq 2$, the subgroup $EO_{2n+1}(R)$ is normal in $SO_{2n+1}(R)$ (see [VaP, Lemma 4]). Indeed, the group $EO_{2n+1}(R)$ also naturally acts on the set $Q'_{2n}(R)$. For the convenience of the reader, we recall the definition of $EO_{2n+1}(R)$ from [Va, VaP] below. We explicitly describe the generators of this group by writing out their actions on a vector $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$. The first three correspond to the *long root unipotents*, while the last two correspond to the *short root unipotents*, as mentioned in [Va, VaP].

Definition 3.3.1. The elementary subgroup $EO_{2n+1}(R)$: The group $EO_{2n+1}(R)$ is the subgroup of $SO_{2n+1}(R)$ generated by the following *elementary orthogonal transvections:*

$(1 \leq i, j \leq n$, $i \neq j$, and $\lambda \in R)$

(i) $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$
$\mapsto (x_1, \cdots, x_{i-1}, x_i + \lambda x_j, x_{i+1}, \cdots, y_{j-1}, y_j - \lambda y_i, y_{j+1}, \cdots, y_n, z)$

(ii) $(x_1, \cdots, x_n, y_1, \cdots, y_n, z) \mapsto (x_1, \cdots, x_{i-1}, x_i + \lambda y_j, \cdots, x_j - \lambda y_i, x_{j+1}, \cdots, y_n, z)$

(iii) $(x_1, \cdots, x_n, y_1, \cdots, y_n, z) \mapsto (x_1, \cdots, y_{i-1}, y_i + \lambda x_j, \cdots, y_j - \lambda x_i, y_{j+1}, \cdots, y_n, z)$

(iv) $(x_1, \cdots, x_n, y_1, \cdots, y_n, z) \mapsto (x_1, \cdots, x_{i-1}, x_i + 2\lambda z - \lambda^2 y_i, x_{i+1}, \cdots, y_n, z - \lambda y_i)$

(v) $(x_1, \cdots, x_n, y_1, \cdots, y_n, z) \mapsto (x_1, \cdots, y_{i-1}, y_i + 2\lambda z - \lambda^2 x_i, y_{i+1}, \cdots, y_n, z - \lambda x_i)$

Remark 3.3.2 Any reader consulting [Va, VaP] should be cautioned that we are following a different scheme of notations here. The 0^{th} coordinate of [Va, VaP] is the last coordinate here. Also, they use negative indices for the y-coordinates. More precisely, their x_0 is our z, and their x_{-i} is our y_i. The reader may also note that the description of $EO_{2n+1}(R)$ by Calmes-Fasel [CF] as mentioned in [F3, p. 320] is concurrent with the above definition.

Remark 3.3.3 It is easy to see from the description of $EO_{2n+1}(R)$ above that if $\sigma \in EO_{2n+1}(R)$, then σ is homotopic to the identity matrix I_{2n+1}.

The following result is a consequence of a much more general result proved by Stavrova in [St, Theorem 1.3]. This is an analogue of [Vo, Theorem 3.3].

Theorem 3.3.4 (Stavrova). *Let R be a regular ring containing a field k with Char$(k) \neq 2$. Let $n \geq 2$ and $\tau(T) \in O_{2n+1}(R[T])$ be such that $\tau(0) = I_{2n+1}$. Then, $\tau(T) \in EO_{2n+1}(R[T])$.*

The following result has been proved by Mandal-Mishra [MaMi, Theorem 4.2].

Theorem 3.3.5 (Mandal-Mishra). *Let R be a regular ring containing a field k with Char$(k) \neq 2$. Let $H(T) \in Q'_{2n}(R[T])$. Then, there exists $\tau(T) \in O_{2n+1}(R[T])$ such that $\tau(0) = I_{2n+1}$ and $H(T)\tau(T) = H(0)$.*

We now prove the following theorem. Recall from the beginning of this section that two elements $(x_1, \cdots x_n, y_1, \cdots, y_n, z)$ and $(x'_1, \cdots x'_n, y'_1, \cdots, y'_n, z')$ from $Q'_{2n}(R)$ are homotopic if there is $(f_1(T), \cdots, f_n(T), g_1(T), \cdots, g_n(T), h(T)) \in Q'_{2n}(R[T])$ such that $f_i(0) = x_i,\ \ f_i(1) = x'_i,\ \ g_i(0) = y_i,\ \ g_i(1) = y'_i$ for $i = 1, \cdots, n$, and $h(0) = z$ and $h(1) = z'$. Also, $\pi_0(Q'_{2n}(R))$ is the set of homotopy orbits of $Q'_{2n}(R)$.

Theorem 3.3.6. *Let R be a regular ring containing a field k with Char$(k) \neq 2$. Then, for any $n \geq 2$ there is a bijection*

$$\eta : \pi_0(Q'_{2n}(R)) \xrightarrow{\sim} Q'_{2n}(R)/EO_{2n+1}(R).$$

Proof. Let $u := (x_1, \cdots x_n, y_1, \cdots, y_n, z) \in Q'_{2n}(R)$. Let η be the map which takes the homotopy orbit of u to its $EO_{2n+1}(R)$-orbit. We have to check first that η is well-defined.

Let u and v be two representatives of the same homotopy orbit. By definition, they will be connected by a finite sequence of homotopies. We first consider the simple case, namely, when u and v are homotopic. In that case, there exists $H(T) \in Q'_{2n}(R[T])$ such that $H(0) = u$ and $H(1) = v$. By Theorem 3.3.5, there exists $\tau(T) \in O_{2n+1}(R[T])$ such that $\tau(0) = I_{2n+1}$ and $H(T)\tau(T) = H(0)$. Applying Theorem 3.3.4 we conclude that $\tau(T) \in EO_{2n+1}(R[T])$. Then, $\sigma := \tau(1) \in EO_{2n+1}(R)$ and we have

$$v\sigma = H(1)\tau(1) = H(0) = u.$$

Therefore u and v are connected by an element of $EO_{2n+1}(R)$ (namely, σ), and they define the same $EO_{2n+1}(R)$-orbit. To tackle the general case, we can take the product of those σ obtained for each homotopy.

The map η is clearly surjective. To prove the injectivity, let $u, v \in Q'_{2n}(R)$ be such that there exists $\sigma \in EO_{2n+1}(R)$ with $v = u\sigma$. By Remark 3.3.3, σ is homotopic to I_{2n+1}. Therefore, there exists $\Theta(T) \in EO_{2n+1}(R[T])$ such that $\Theta(0) = I_{2n+1}$ and $\Theta(1) = \sigma$. Define

$$H(T) := u\Theta(T) \in Q'_{2n}(R[T]).$$

Then, $H(0) = u$ and $H(1) = v$, showing that u and v are homotopic. □

The following corollary is now obvious.

Corollary 3.3.7. *Let R be a regular ring containing a field k with Char$(k) \neq 2$. Then, for $n \geq 2$, the relation induced by homotopy on $Q'_{2n}(R)$ is also transitive, and hence an equivalence relation.*

Remark 3.3.8 Observe that there is an obvious map from $Q'_{2n}(R)$ to $Um_{n+1}(R)$ taking $(x_1, \cdots x_n, y_1, \cdots, y_n, z)$ to $(x_1, \cdots x_n, z)$, which will induce a set-theoretic map $\zeta_R : \pi_0(Q'_{2n}(R)) \to \pi_0(Um_{n+1}(R))$ between the homotopy orbit spaces.

3.4 The pointed set $Q_{2n}(R)$ and its homotopy orbits

Let R be any commutative Noetherian ring. Let $n \geq 2$ and consider the set

$$Q_{2n}(R) = \{(x_1, \cdots x_n, y_1, \cdots, y_n, z) \in R^{2n+1} \mid \sum_{i=1}^{n} x_i y_i = z - z^2\}$$

with a base point $(0, \cdots, 0, 0, \cdots, 0, 0)$. It is proved in [F3] that if $\frac{1}{2} \in R$, then there is a bijection $\beta_n : Q_{2n}(R) \to Q'_{2n}(R)$ and its inverse $\alpha_n : Q'_{2n}(R) \to Q_{2n}(R)$ given by

- $\beta_n(x_1, \cdots x_n, y_1, \cdots, y_n, z) = (2x_1, \cdots 2x_n, 2y_1, \cdots, 2y_n, 1 - 2z)$
- $\alpha_n(x_1, \cdots x_n, y_1, \cdots, y_n, z) = \frac{1}{2}(x_1, \cdots x_n, y_1, \cdots, y_n, 1 - z)$

Note that both α_n and β_n preserve the base points of the respective sets. They induce bijections between the sets $\pi_0(Q_{2n}(R))$ and $\pi_0(Q'_{2n}(R))$ (will use the same notations). By transporting the action of $EO_{2n+1}(R)$ on $Q'_{2n}(R)$ through the bijections given above one sees that $EO_{2n+1}(R)$ also acts on $Q_{2n}(R)$ in the following way:

$$Mv := \alpha_n((\beta_n(v))M),$$

for $v \in Q_{2n}(R)$ and $M \in EO_{2n+1}(R)$ (with the assumption $\frac{1}{2} \in R$). Further, note that the bijections α_n, β_n induce bijections between the sets $\pi_0(Q_{2n}(R))$ and $\pi_0(Q'_{2n}(R))$. Combining these with Theorem 3.3.6, one obtains the following result.

Theorem 3.4.1. *Let R be a regular ring containing a field k with $Char(k) \neq 2$. Then, for any $n \geq 2$ there is a bijection*

$$\pi_0(Q_{2n}(R)) \xrightarrow{\sim} Q_{2n}(R)/EO_{2n+1}(R).$$

We shall require the following corollary later.

Corollary 3.4.2. *Let R be a regular ring containing a field k with $Char(k) \neq 2$. Then, for $n \geq 2$, the relation induced by homotopy on $Q_{2n}(R)$ is also transitive, and hence an equivalence relation.*

In the next three sections we shall discuss about the maps between the Euler class group $E^d(R)$ and $Um_{d+1}(R)/E_{d+1}(R)$, when R is a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k.

3.5 The map $\phi_R : Um_{d+1}(R)/E_{d+1}(R) \to E^d(R)$

Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. We now recall the definition of a group homomorphism $\phi_R : Um_{d+1}(R)/E_{d+1}(R) \to E^d(R)$. When d is even, ϕ_R has been defined in [BRS 3]. The extension to general d is available in [DZ, vdK 4]. We urge the reader to look at [DZ, Section 4] for the details.

Definition 3.5.1. Let $v = (a_1, \cdots, a_{d+1}) \in Um_{d+1}(R)$. Applying elementary transformations if necessary, we may assume that $\text{ht}(a_1, \cdots, a_d) \geq d$. Write $J = (a_1, \cdots, a_d)$ and let $\omega_J : R^d \twoheadrightarrow J$ be the surjection induced by $(a_1, \cdots, a_d)$. As a_{d+1} is a unit modulo J, we have $J = (a_1, \cdots, a_d a_{d+1}) + J^2$ and the corresponding element in $E^d(R)$ is $(J, \overline{a_{d+1}}\omega_J)$. Let $[v]$ denote the orbit of v in $Um_{d+1}(R)/E_{d+1}(R)$. Define $\phi_R([v]) = (J, \overline{a_{d+1}}\omega_J)$. It is proved in [DZ, vdK 4] that ϕ_R is a group homomorphism.

Remark 3.5.2 When d is even, the above definition coincides with the one given in [BRS 3]. A short remark on the definition given in [BRS 3] is in order. Note that the unimodular row v gives rise to a stably free R-module, say, P of rank d together with a canonical orientation $\chi : R \xrightarrow{\sim} \wedge^d(P)$. Bhatwadekar-Sridharan defines $\phi_R([v])$ to be the *Euler class* of the pair (P, χ) which resides in $E^d(R)$. The computation of this Euler class in [BRS 3, Page 214] shows that it turns out to be exactly the one given above, namely, $(J, \overline{a_{d+1}}\omega_J)$.

The rest of this chapter is devoted to establishing a (set-theoretic) map from the Euler class group $E^d(R)$ to $Um_{d+1}(R)/E_{d+1}(R)$. In order to do so, we have to factor through $\pi_0(Q_{2d}(R))$ which we take up in the next section.

3.6 The map $\theta_R : E^d(R) \longrightarrow \pi_0(Q_{2d}(R))$

We first recall the definition of a set-theoretic map from the Euler class group $E^d(R)$ to $\pi_0(Q_{2d}(R))$ from [DTZ2], where R is a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k.

By Remark 2.4.8, we know that an arbitrary element of $E^d(R)$ can be represented by a single Euler cycle (J, ω_J), where J is a reduced ideal of height d. Now $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ is given by $J = (a_1, \cdots, a_d) + J^2$, for some $a_1, \cdots, a_d \in J$. Applying Nakayama Lemma one obtains $s \in J^2$ such that $J = (a_1, \cdots, a_d, s)$ with $s - s^2 = a_1b_1 + \cdots + a_db_d$ for some $b_1, \cdots, b_d \in R$ (see Lemma 2.1.7 for a proof). We associate to (J, ω_J) the homotopy class $[(a_1, \cdots, a_d, b_1, \cdots, b_d, s)]$ in $\pi_0(Q_{2d}(R))$.

The following proposition from [DTZ2], has also been proved in [AF, MaMi] with the additional assumption that $\mathrm{Char}(k) \neq 2$. We do not need that assumption and our line of proof is entirely different.

Proposition 3.6.1. *Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. The association $(J, \omega_J) \mapsto [(a_1, \cdots, a_d, b_1, \cdots, b_d, s)]$ is well defined and it gives rise to a set-theoretic map $\theta_R : E^d(R) \to \pi_0(Q_{2d}(R))$. The map θ_R takes the trivial Euler cycle to the homotopy orbit of the base point $(0, \cdots, 0)$ of $Q_{2d}(R)$.*

Proof. We need to check the following:

(i) If ω_J is also given by $J = (\alpha_1, \cdots, \alpha_d) + J^2$ and if $\tau \in J^2$ is such that $\tau - \tau^2 = \alpha_1\beta_1 + \cdots + \alpha_d\beta_d$, then $[(a_1, \cdots, a_d, b_1, \cdots, b_d, s)] = [(\alpha_1, \cdots, \alpha_d, \beta_1, \cdots, \beta_d, \tau)]$ in $\pi_0(Q_{2d}(R))$.

(ii) If $\overline{\sigma} \in E_d(R/J)$, then the image of $(J, \omega_J\overline{\sigma})$ in $Q_{2d}(R)$ is homotopic to the image of (J, ω_J).

(iii) If (J, ω_J) is also represented by $(J', \omega_{J'})$ in $E^d(R)$, then their images are homotopic in $Q_{2d}(R)$.

Proof of (i) : This has been proved in [F3, Theorem 2.0.2].
Proof of (ii) : Suppose that (J, ω_J) is given by $J = (a_1, \cdots, a_d) + J^2$, and $s \in J^2$ be such that $J = (a_1, \cdots, a_d, s)$ with $s - s^2 = a_1b_1 + \cdots + a_db_d$ for some $b_1, \cdots, b_d \in R$. Let $\sigma \in$

$E_d(R)$ be a lift of $\overline{\sigma}$ and write $(a_1, \cdots, a_d)\sigma = (\alpha_1, \cdots, \alpha_d)$. Then $J = (\alpha_1, \cdots, \alpha_d) + J^2$ and $J = (\alpha_1, \cdots, \alpha_d, s)$. If we write $(b_1, \cdots, b_d)(\sigma^{-1})^t = (\beta_1, \cdots, \beta_d)$ (here t stands for transpose), then it is easy to see that $s(1-s) = \alpha_1\beta_1 + \cdots + \alpha_d\beta_d$. Now, note that

$$\lambda = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & (\sigma^{-1})^t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in E_{2d+1}(R)$$

and $(a_1, \cdots, a_d, b_1, \cdots, b_d, s)\lambda = (\alpha_1, \cdots, \alpha_d, \beta_1, \cdots, \beta_d, s)$. Since elementary matrices are homotopic to identity, we are done in this case.

Proof of (iii) : We break this proof into two steps.
Step 1. We have $(J, \omega_J) = (J', \omega_{J'}) \in E^d(R) = G/H_1$, where H_1 is as in Definition 2.4.2. It then follows that

$$(J, \omega_J) + \sum_{i=1}^{r}(K_i, \omega_{K_i}) = (J', \omega_{J'}) + \sum_{j=r+1}^{s}(K'_j, \omega_{K'_j})$$

in G, where all the (K_i, ω_{K_i}) and $(K'_j, \omega_{K'_j})$ are in S_1 (where S_1 is as in Definition 2.4.2).

Adapting the proof of [BRS 1, 4.11], if necessary, we can change the above equation to obtain a new one where the ideals appearing on the left are mutually comaximal (consequently, so are the ideals on the right). Therefore, without loss of generality, we may assume that $J, K_1, \cdots, K_r$ are mutually comaximal (and so are $J', K'_{r+1}, \cdots, K'_s$) and we have $J \cap K_1 \cap \cdots \cap K_r = J' \cap K'_{r+1} \cap \cdots \cap K'_s$. Let us write $K = K_1 \cap \cdots \cap K_r$ and $K' = K'_{r+1} \cap \cdots \cap K'_s$. Also, let ω_K be the surjection $(R/K)^d \twoheadrightarrow K/K^2$ induced by $\omega_{K_1}, \cdots, \omega_{K_r}$. Similarly, let $\omega_{K'}$ be the surjection $(R/K')^d \twoheadrightarrow K'/K'^2$ induced by $\omega_{K'_{r+1}}, \cdots, \omega_{K'_s}$. Summing up, we have:

(1) $(J, \omega_J) + (K, \omega_K) = (J', \omega_{J'}) + (K', \omega_{K'})$ in G;

(2) $J + K = R = J' + K'$;

(3) $J \cap K = J' \cap K'$.

Step 2. Assume that ω_J is induced by $J = (a_1, \cdots, a_d) + J^2$ and let ω_K be induced by

$K = (c_1, \cdots, c_d)$. They will together induce $\omega_{J\cap K} : (R/J \cap K)^d \twoheadrightarrow (J \cap K)/(J \cap K)^2$, say, given by $J \cap K = (\beta_1, \cdots, \beta_d) + (J \cap K)^2$.

Because of (ii) above, we are now free to apply elementary transformations. Applying elementary transformations on $(c_1, \cdots, c_d)$, if necessary, we may assume by Lemma 2.1.6 that $\mathrm{ht}(c_1, \cdots, c_{d-1}) = d - 1$ and $J + (c_1, \cdots, c_{d-1}) = R$ (Note that if we apply $\sigma \in E_d(R)$ on $(c_1, \cdots, c_d)$, we have to apply σ on $(a_1, \cdots, a_d)$ as well to retain the relations and equations). Consider the ideal $L = (c_1, \cdots, c_{d-1}, (1 - c_d)T + c_d)$ in $R[T]$. Write $I = L \cap J[T]$. Using the Chinese Remainder Theorem we can then find $f_1, \cdots, f_d \in I$ such that:

(a) $I = (f_1, \cdots, f_d) + I^2$.

(b) $f_i = c_i \bmod L^2$ for $i = 1, \cdots, d-1$ and $f_d = (1 - c_d)T + c_d \bmod L^2$.

(c) $f_i = a_i \bmod J[T]^2$ for $i = 1, \cdots, d$.

Let $\omega : (R[T]/I)^d \twoheadrightarrow I/I^2$ be the surjection corresponding to $f_1, \cdots, f_d$. We then have, $I(0) = J \cap K$, $I(1) = J$. From (b) we get $f_i(0) = c_i \bmod K^2$. On the other hand, from (c) we get $f_i(0) = a_i \bmod J^2$. Combining, we observe that $f_i(0) \equiv \beta_i \bmod (J \cap K)^2$. In other words, $\omega_{J\cap K}$ is the same as $\omega(0)$. Also, from (c), we obtain $f_i(1) \equiv a_i \bmod J^2$, implying that ω_J is the same as $\omega(1)$. Therefore, by Lemma 3.6.2 proved below, the images of (J, ω_J) and $(J \cap K, \omega_{J\cap K})$ are the same in $\pi_0(Q_{2d}(R))$.

Following exactly the same procedure, as above, we can see that the images of $(J', \omega_{J'})$ and $(J' \cap K', \omega_{J'\cap K'})$ are the same in $\pi_0(Q_{2d}(R))$. Since $J \cap K = J' \cap K'$, and $\omega_{J\cap K} = \omega_{J'\cap K'}$, it follows that (J, ω_J) and $(J', \omega_{J'})$ have the same image in $\pi_0(Q_{2d}(R))$. This completes the proof of the proposition. $\square$

Lemma 3.6.2. *Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. Let $I \subset R[T]$ be an ideal of height d such that both $I(0)$ and $I(1)$ are ideals of height d in R. Assume that there is a surjection $\omega : (R[T]/I)^d \twoheadrightarrow I/I^2$. Then, the images of $(I(0), \omega(0))$ and $(I(1), \omega(1))$ in $\pi_0(Q_{2d}(R))$ are the same.*

Proof. Let $\omega : (R[T]/I)^d \twoheadrightarrow I/I^2$ be given by $I = (f_1(T), \cdots, f_d(T)) + I^2$. Then, $\omega(0)$ is given by $I(0) = (f_1(0), \cdots, f_d(0)) + I(0)^2$ and $\omega(1)$ is given by $I(1) = (f_1(1), \cdots, f_d(1)) + I(1)^2$.

There exist $h(T) \in I^2$ and $g_1(T), \cdots, g_d(T) \in R[T]$ such that:

(i) $I = (f_1(T), \cdots, f_d(T), h(T))$;

(ii) $h(T) - h(T)^2 = f_1(T)g_1(T) + \cdots + f_d(T)g_d(T)$.

Then the $2d+1$-tuple $(f_1(T), \cdots, f_d(T), g_1(T), \cdots, g_d(T), h(T)) \in Q_{2d}(R[T])$.

Now $h(0) \in I(0)^2$ with $h(0) - h(0)^2 = f_1(0)g_1(0) + \cdots + f_d(0)g_d(0)$. Similarly, we have $h(1) \in I(1)^2$ with $h(1) - h(1)^2 = f_1(1)g_1(1) + \cdots + f_d(1)g_d(1)$. Therefore it is easy to see that $(f_1, \cdots, f_d, g_1, \cdots, g_d, h) \in Q_{2d}(R[T])$ is the required homotopy for the images of $(I(0), \omega(0))$ and $(I(1), \omega(1))$ in $Q_{2d}(R)$. This concludes the proof. $\square$

Remark 3.6.3 So far we have worked with Euler cycles represented by reduced ideals. Now let J be an ideal which is not reduced and $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ be a surjection. As indicated in [BRS 1, 4.16], using Swan's Bertini theorem we can find a reduced ideal K of height d and elements $a_1, \cdots, a_d$ such that: (i) $J \cap K = (a_1, \cdots, a_d)$; (ii) $J + K = R$; (iii) the images of $a_1, \cdots, a_d$ induce ω_J (for a proof, see [DRS, Lemma 2.7, Remark 2.8]). Let $\omega_K : (R/K)^d \twoheadrightarrow K/K^2$ be the surjection induced by $a_1, \cdots, a_d$. We may apply the same procedure again and find a reduced ideal L of height d such that: (iv) $K \cap L = (b_1, \cdots, b_d)$; (v) $L + K \cap J = R$; (vi) $b_i - a_i \in K^2$ for $i = 1, \cdots, d$. Let $\omega_L : (R/L)^d \twoheadrightarrow L/L^2$ be the surjection induced by $b_1, \cdots, b_d$. One then associates $(J, \omega_J) := (L, \omega_L)$ in $E^d(R)$. It can be easily checked that this association is well-defined and (J, ω_J) also satisfies the calculus of Euler cycles represented by reduced ideals. Now, to the data $J = (a_1, \cdots, a_d) + J^2$ we can associate an element of $\pi_0(Q_{2d}(R))$ (exactly as we did in Definition 3.6). On the other hand, from $L = (b_1, \cdots, b_d) + L^2$, we shall obtain $\theta_R((L, \omega_L)) \in \pi_0(Q_{2d}(R))$. We now prove:

Proposition 3.6.4. *With notations as above, the element of $\pi_0(Q_{2d}(R))$ associated to $J = (a_1, \cdots, a_d) + J^2$ is the same as $\theta_R((L, \omega_L))$.*

Proof. We first note that statements (i) and (ii) of Proposition 3.6.1 are also true for the pair (J, ω_J).

The idea of proof of this proposition is essentially contained in the proof of Proposition 3.6.1 (iii) (*Step 2*). Therefore, we shall only give a sketch. As $K \cap L = (b_1, \cdots, b_d)$, we can easily construct an ideal $I \subset R[T]$ and a surjection $\omega : (R[T]/I)^d \twoheadrightarrow I/I^2$ such that $I(0) = J$, $I(1) = J \cap K \cap L$, $\omega(0) = \omega_J$, and $\omega(1) = \omega_{J\cap K\cap L}$. On the other hand, as $J \cap K = (a_1, \cdots, a_d)$, we can construct an ideal $I' \subset R[T]$ and a surjection $\omega' : (R[T]/I)^d \twoheadrightarrow I'/I'^2$ such that $I'(0) = L$, $I'(1) = J \cap K \cap L$, $\omega'(0) = \omega_L$, and $\omega'(1) = \omega_{J\cap K\cap L}$. We can now apply Lemma 3.6.2 to conclude the proof. $\square$

Remark 3.6.5 It has been proved in [DTZ2] that the map $\theta_R : E^d(R) \to \pi_0(Q_{2d}(R))$ is in fact a bijection. We provide the proof in the appendix of this thesis.

Now we are ready to define our desired map from $E^d(R)$ to $Um_{d+1}(R)/E_{d+1}(R)$. We carry this out in the next section where we also record some useful properties of this map.

3.7 The map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$

Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k with $\mathrm{Char}(k) \neq 2$.

From Proposition 3.6.1 we get a well defined map $\theta_R : E^d(R) \to \pi_0(Q_{2d}(R))$. In Section 3.4 we discussed about the set theoretic bijection $\beta_d : \pi_0(Q_{2d}(R)) \simeq \pi_0(Q'_{2d}(R))$. Therefore composition of these two maps yields a set-theoretic map $\beta_d \circ \theta_R$ from $E^d(R)$ to $\pi_0(Q'_{2d}(R))$ whose description goes as follows. Let $(J, \omega_J) \in E^d(R)$, where J is a reduced ideal of height d. Now $\omega_J : (R/J)^d \twoheadrightarrow J/J^2$ is given by $J = (a_1, \cdots, a_d) + J^2$, for some $a_1, \cdots, a_d \in J$. Applying Nakayama Lemma one obtains $s \in J^2$ such that $J = (a_1, \cdots, a_d, s)$ with $s - s^2 = a_1b_1 + \cdots + a_db_d$ for some $b_1, \cdots, b_d \in R$.

The assignment of (J, ω_J) to

$$\beta_d \circ \theta_R(J, \omega_J) = \beta_d([(a_1, \cdots, a_d, b_1, \cdots, b_d, s)]) = [(2a_1, \cdots, 2a_d, 2b_1, \cdots, 2b_d, 1 - 2s)]$$

in $\pi_0(Q'_{2d}(R))$ is a well-defined set-theoretic map. The following composite

$$E^d(R) \xrightarrow{\theta_R} \pi_0(Q_{2d}(R)) \xrightarrow[\sim]{\beta_d} \pi_0(Q'_{2d}(R)) \xrightarrow{\zeta_R} \pi_0(Um_{d+1}(R)) \xrightarrow[\sim]{\eta_R} Um_{d+1}(R)/E_{d+1}(R)$$

gives a set-theoretic map from $E^d(R)$ to $Um_{d+1}(R)/E_{d+1}(R)$. Let us call it δ_R. (For the description of the last two maps see Remark 3.3.8 and Theorem 3.2.1.)

Remark 3.7.1 Thus $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ takes (J, ω_J) (where ω_J is induced by $a_1, \cdots, a_d, s$, as above) to the orbit $[(2a_1, \cdots 2a_d, 1-2s)] \in Um_{d+1}(R)/E_{d+1}(R)$, where $(1-2s)^2 \equiv 1$ modulo the ideal $(2a_1, \cdots 2a_d)$. Conversely, let an orbit $[v] = [(x_1, \cdots, x_d, z)] \in Um_{d+1}(R)/E_{d+1}(R)$ be such that the ideal $(x_1, \cdots, x_d)$ is reduced of height d, and $z^2 \equiv 1$ modulo $(x_1, \cdots, x_d)$, then $[v]$ is in the image of δ_R.

Notation. An orbit $[(x_1, \cdots, x_d, z)] \in Um_{d+1}(R)/E_{d+1}(R)$ will be written as $[x_1, \cdots, x_d, z]$.

We now compute the composite map $\phi_R\delta_R : E^d(R) \longrightarrow E^d(R)$. The description of this composite will play a very important role in the next chapter.

Theorem 3.7.2. *Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k with Char$(k) \neq 2$. For any $(J, \omega_J) \in E^d(R)$, we have*

$$\phi_R\delta_R((J, \omega_J)) = (J, 2^d\omega_J) - (J, -2^d\omega_J).$$

Consequently, if d is even or if $\sqrt{2} \in R$, then $\phi_R\delta_R((J, \omega_J)) = (J, \omega_J) - (J, -\omega_J)$.

Proof. Suppose that ω_J is given by $J = (a_1, \cdots, a_d) + J^2$. Using some standard arguments we may assume that $\mathrm{ht}(a_1, \cdots, a_d) = d$. There is $s \in J^2$ with $s - s^2 \in (a_1, \cdots, a_d)$. Now, $s - s^2 = a_1b_1 + \cdots + a_db_d$, for some $b_1, \cdots, b_d \in R$. Then $\delta_R((J, \omega_J)) = [2a_1, \cdots, 2a_d, 1 - 2s] \in Um_{d+1}(R)/E_{d+1}(R)$. Write $K = (2a_1, \cdots, 2a_d) = (a_1, \cdots, a_d)$ (as $\frac{1}{2} \in R$).

If we write $J' = (a_1, \cdots, a_d, 1-s)$, then it is easy to see that $K = J \cap J'$, and $J' = (a_1, \cdots, a_d) + J'^2$. Therefore, we write $0 = (K, \omega_K) = (J, \omega_J) + (J', \omega_{J'})$ in $E^d(R)$, where ω_K is induced by the generators $a_1, \cdots, a_d$ of K and $\omega_{J'}$ is induced from the data $J' = (a_1, \cdots, a_d) + J'^2$.

Now, from the definition of ϕ_R it follows that $\phi_R\delta_R((J, \omega_J)) = (K, \overline{(1-2s)2^d}\omega_K)$. We write $u = (1-2s)$. Then, we have (here 'tilde' means modulo J^2, and so on),

$$\phi_R\delta_R((J, \omega_J)) = (K, \overline{u2^d}\omega_K) = (J, \widetilde{u2^d}\omega_J) + (J', \overline{\overline{u2^d}}\omega_{J'})$$

As $1-2s \equiv 1 \bmod J$ and $1-2s \equiv -1 \bmod J'$, we have $\phi_R\delta_R((J,\omega_J)) = (J, 2^d\omega_J) + (J', -2^d\omega_{J'})$. Further, note that $(J, -2^d\omega_J) + (J', -2^d\omega_{J'}) = 0$. Therefore, finally we have,

$$\phi_R\delta_R((J,\omega_J)) = (J, 2^d\omega_J) - (J, -2^d\omega_J).$$

If d is even or $\sqrt{2} \in R$, then 2^d is a square and it follows from Remark 2.4.9 that $\phi_R\delta_R((J,\omega_J)) = (J,\omega_J) - (J,-\omega_J)$. $\square$

The proof of the following corollary is routine and we omit the proof.

Corollary 3.7.3. *The composite $\phi_R\delta_R : E^d(R) \longrightarrow E^d(R)$ is a morphism of groups.*

We shall also need the following proposition in the next chapter.

Proposition 3.7.4. *Let R be a regular domain of dimension $d \geq 2$ which is essentially of finite type over an infinite perfect field k with $Char(k) \neq 2$. Let $(J,\omega_J) \in E^d(R)$. Then $\delta_R((J,\omega_J) + (J,-\omega_J))$ is the trivial orbit in $Um_{d+1}(R)/E_{d+1}(R)$.*

Proof. Let ω_J be given by $J = (a_1, \cdots, a_d) + J^2$. Then by Remark 2.4.10 it follows that $(K,\omega_K) = (J,\omega_J) + (J,-\omega_J)$ in $E^d(R)$. Where

$$K = (a_1, \cdots, a_{d-1}) + J^2.$$

Here we are taking $K = (a_1, \cdots, a_{d-1}, a_d^2) + K^2$ which corresponds to ω_K.

Now there exists $t \in K^2$ such that $K = (a_1, \cdots, a_{d-1}, a_d^2, t)$ with

$$t - t^2 \in (a_1, \cdots, a_{d-1}, a_d^2).$$

We have, $\delta_R(K,\omega_K) = [2a_1, \cdots, 2a_{d-1}, 2a_d^2, 1-2t] = [2a_1, \cdots, 2a_{d-2}, 4a_{d-1}, a_d^2, 1-2t]$ (applying [Theorem 2.3.2,(ii)] here). But if we move the square, we remain in the same elementary orbit [Theorem 2.3.2,(v)], implying that

$$[2a_1, \cdots, 2a_{d-2}, 4a_{d-1}, a_d^2, 1-2t] = [2a_1, \cdots, 2a_{d-2}, 4a_{d-1}, a_d, (1-2t)^2].$$

But $(1-2t)^2$ is 1 modulo $(a_1, \cdots, a_{d-1}, a_d^2)$ and therefore it is also 1 modulo the ideal $(2a_1, \cdots, 2a_{d-2}, 4a_{d-1}, a_d)$. As a consequence, this orbit is trivial. $\square$

Chapter 4

Structure Theorem of Orbit Spaces Of Unimodular Rows

As we mentioned in introduction, in this chapter we concentrate on the group structure of $Um_{d+1}(R)/E_{d+1}(R)$ over smooth real affine algebras.

4.1 A key result

We first prove a key lemma which is inspired by the proof of [OPS, Proposition 2.1].

Lemma 4.1.1. *Let R be a smooth affine domain over $\mathbb{R}$ of dimension $d \geq 2$. Let $v \in Um_{d+1}(R)$. Then there is some $t \in R$ and $(x_1, \cdots, x_d, z) \in Um_{d+1}(R)$ such that:*

(i) $[v] = [x_1, \cdots, x_d, z]$ *in $Um_{d+1}(R)/E_{d+1}(R)$ (and hence in $Um_{d+1}(R)/SL_{d+1}(R)$);*

(ii) $(zt^2)^2 \equiv 1$ *modulo the (reduced) ideal $(x_1, \cdots, x_d)$;*

(iii) $[x_1, \cdots, x_d, zt^2] = [x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^2]$ *in $Um_{d+1}(R)/E_{d+1}(R)$ (and hence in $Um_{d+1}(R)/SL_{d+1}(R)$);*

(iv) *The orbit $[x_1, \cdots, x_d, zt^2]$ is in the image of $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$.*

Proof. Let $v = (y_1, \cdots, y_d, w)$. We can use Swan's Bertini Theorem [Remark 2.1.5] and find $\alpha_1, \cdots, \alpha_d \in R$ such that the ideal $I = (y_1 + \alpha_1 w, \cdots, y_d + \alpha_d w)$ is a reduced ideal of height d. We write $x_i = y_i + \alpha_i w$ for $i = 1, \cdots, d$, and we rename w as z. Then note that $[v] = [x_1, \cdots, x_d, z]$ in $Um_{d+1}(R)/E_{d+1}(R)$ (and hence in $Um_{d+1}(R)/SL_{d+1}(R)$).

As $(x_1, \cdots, x_d, z)$ is unimodular, there exist $b_1, \cdots, b_d, b \in R$ such that $x_1 b_1 + \cdots + x_d b_d + zb = 1$. Now, R/I is a finite direct product of $\mathbb{R}$ or $\mathbb{C}$. Therefore, the unit $\bar{b}^2 \in (R/I)^*$ is a fourth power, say, $\bar{b}^2 = \bar{t}^4$. Let $t \in R$ be a lift of $\bar{t}$. Then $z^2 t^4 \equiv 1$ modulo I. Therefore, there exist $a_1, \cdots, a_d \in R$ such that $x_1 a_1 + \cdots + x_d a_d + (zt^2)^2 = 1$. It is then easy to see that the orbit $[x_1, \cdots, x_d, zt^2]$ is in the image of $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ (see Remark 3.7.1). Statement (iii) simply follows from Theorem [2.3.2,(iii)]. $\square$

4.2 The "real" coordinate ring

We set up some notations.

Notation. Let $X = \mathrm{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the set of real points of X. We assume that $X(\mathbb{R}) \neq \emptyset$. Therefore, under the Euclidean topology, $X(\mathbb{R})$ is a smooth real manifold of dimension d. Let $\mathbb{R}(X)$ denote the ring obtained from R by inverting all functions which do not have any real zeroes. Informally the ring $\mathbb{R}(X)$ dubbed as the "real" coordinate ring of the variety. Since $\mathbb{R}(X)$ is a localization of R and $\dim(R) = \dim(\mathbb{R}(X))$, there is a canonical surjective group homomorphism $\Gamma : E^d(R) \twoheadrightarrow E^d(\mathbb{R}(X))$ (see [BRS 2, page 307], as we mentioned in Chapter 2.)

Theorem 4.2.1. *The map $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \longrightarrow Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is surjective.*

Proof. Take any orbit $[v] \in Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$. Note that Lemma 4.1.1 applies to $\mathbb{R}(X)$ as well. Therefore, we have

$$[x_1, \cdots, x_d, zt^2] = [x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^2] \text{ in } Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)),$$

such that $[v] = [x_1, \cdots, x_d, z]$. The row $(x_1, \cdots, x_d, t^2)$ can be taken to $(x_1, \cdots, x_d, x_1^2 + \cdots + x_d^2 + t^2)$ using elementary transformations. Since $x_1^2 + \cdots + x_d^2 + t^2$ does not vanish at any real point, it is a unit in $\mathbb{R}(X)$. Consequently, $[x_1, \cdots, x_d, t^2]$ is trivial in $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ and the proof is complete by Lemma 4.1.1 (iv). $\square$

For the rest of this thesis we will assume that $X(\mathbb{R})$ is orientable. In this case, the real line bundle on $X(\mathbb{R})$ induced by the canonical bundle $K_R := \wedge^d(\Omega_{R/\mathbb{R}})$ is trivial and therefore, by [BDM, Theorem 4.21], $E^d(\mathbb{R}(X))$ is torsion-free. We use only this piece of (nontrivial) information to prove that $\delta_{\mathbb{R}(X)}$ is bijective in Theorem (4.2.4) below. But before that we record a crucial result (which is discussed in Section [2.5,(iii)]) in the form of the following proposition and a corollary (as they are implicit in [BRS 2]).

Proposition 4.2.2. *Let $\mathfrak{m}$ be a real maximal ideal of R. Assume that the real point corresponding to $\mathfrak{m}$ belongs to a compact connected component of $X(\mathbb{R})$. Then, for any $\omega_{\mathfrak{m}} : (R/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$, one has $(\mathfrak{m}, \omega_{\mathfrak{m}}) + (\mathfrak{m}, -\omega_{\mathfrak{m}}) = 0$ in $E^d(\mathbb{R}(X))$.*

Proof. See toward the end of the proof of [BRS 2, Theorem 4.13]. $\square$

Corollary 4.2.3. *Let $J \subset \mathbb{R}(X)$ be a reduced ideal and $\omega_J : (\mathbb{R}(X)/J)^d \twoheadrightarrow J/J^2$ be a surjection. Then $(J, \omega_J) + (J, -\omega_J) = 0$ in $E^d(\mathbb{R}(X))$.*

Proof. Let $J = \mathfrak{m}_1 \cap \cdots \mathfrak{m}_r \cap \mathfrak{m}_{r+1} \cap \cdots \mathfrak{m}_s$. Assume that the real points corresponding to the maximal ideals $\mathfrak{m}_{r+1}, \cdots, \mathfrak{m}_s$ do not belong to any compact connected component of $X(\mathbb{R})$. Now ω_J will induce $\omega_i : (\mathbb{R}(X)/\mathfrak{m}_i)^d \twoheadrightarrow \mathfrak{m}_i/\mathfrak{m}_i^2$ for $i = 1, \cdots, s$ and we have: $(J, \omega_J) = \sum_{i=1}^{s}(\mathfrak{m}_i, \omega_i)$. By (the proof of) [BRS 2, Theorem 4.13] or as discussed in section 2.5(ii), $(\mathfrak{m}_i, \omega_i) = 0$ for $i = r+1, \cdots, s$. The corollary now follows from the above proposition. $\square$

Theorem 4.2.4. *The map $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is a bijection.*

Proof. We proved above that $\delta_{\mathbb{R}(X)}$ is surjective. To prove that $\delta_{\mathbb{R}(X)}$ is injective, it is enough to prove that $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}$ is injective. Since $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}$ is a morphism of groups by Corollary 3.7.3, we pick $(J, \omega_J) \in E^d(\mathbb{R}(X))$ (with J reduced) such that $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}((J, \omega_J)) = 0$ and prove that $(J, \omega_J) = 0$.

By the assumption, from Theorem 3.7.2 we have $(J, \omega_J) - (J, -\omega_J) = 0$. But as J is reduced, by Corollary 4.2.3 we also have $(J, \omega_J) + (J, -\omega_J) = 0$ in $E^d(\mathbb{R}(X))$. Therefore,

$2(J, \omega_J) = 0$. But under the assumptions on $X(\mathbb{R})$, the group $E^d(\mathbb{R}(X))$ has no nontrivial torsion. Therefore, $(J, \omega_J) = 0$. □

Corollary 4.2.5. $\phi_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \longrightarrow E^d(\mathbb{R}(X))$ *is injective.*

Proof. As $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}$ is injective and $\delta_{\mathbb{R}(X)}$ is a surjection, the result follows. □

The set-theoretic map $\delta_{\mathbb{R}(X)}$ turns out to be a group homomorphism.

Theorem 4.2.6. *The group homomorphism* $\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \to Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ *is in fact an isomorphism of groups, where the group structure on* $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ *is the one given in [vdK 1].*

Proof. Let us denote the group composition in $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ by $*$. In this proof the actual representation of elements would not matter. Therefore, let $\alpha, \beta \in E^d(\mathbb{R}(X))$. Our aim is to show that $\delta_{\mathbb{R}(X)}(\alpha + \beta) = \delta_{\mathbb{R}(X)}(\alpha) * \delta_{\mathbb{R}(X)}(\beta)$ (here $+$ is the group composition of the Euler class group). As $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)} : E^d(R) \longrightarrow E^d(R)$ is a group homomorphism,

$$\begin{aligned}\phi_{\mathbb{R}(X)}(\delta_{\mathbb{R}(X)}(\alpha+\beta)) &= (\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)})(\alpha+\beta) = (\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)})(\alpha) + (\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)})(\beta) \\ &= \phi_{\mathbb{R}(X)}(\delta_{\mathbb{R}(X)}(\alpha)) + \phi_{\mathbb{R}(X)}(\delta_{\mathbb{R}(X)}(\beta)) = \phi_{\mathbb{R}(X)}(\delta_{\mathbb{R}(X)}(\alpha) * \delta_{\mathbb{R}(X)}(\beta))\end{aligned}$$

As $\phi_{\mathbb{R}(X)}$ is injective, we have $\delta_{\mathbb{R}(X)}(\alpha + \beta) = \delta_{\mathbb{R}(X)}(\alpha) * \delta_{\mathbb{R}(X)}(\beta)$. □

Let $X(\mathbb{R})$ be connected but not compact. Then we know from [BRS 2, Corollary 4.9] that the Euler class group $E^d(\mathbb{R}(X))$ is trivial. The same conclusion is now immediate for the group $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$.

Corollary 4.2.7. *Let R as above and set of real points, $X(\mathbb{R})$ be connected but not compact. The group $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is then trivial.*

Proof. We have $\delta_{\mathbb{R}(X)}$ surjective and under the assumptions, $E^d(\mathbb{R}(X))$ is trivial by [BRS 2, Corollary 4.9]. □

As a consequence of the results obtained in this section and the structure theorem for the Euler class groups as established in [BRS 2, BDM], we obtain the following structure theorem. (See the Section 2.5 for the structure theorem for $E^d(\mathbb{R}(X))$.)

Theorem 4.2.8. *Let $X = Spec(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. Let $\mathcal{C}$ be the set of compact connected components of $X(\mathbb{R})$. Then,*

$$Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \xrightarrow{\sim} \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$$

Corollary 4.2.9. *The composite group homomorphism $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)} : E^d(\mathbb{R}(X)) \longrightarrow E^d(\mathbb{R}(X))$ is multiplication by 2.*

Proof. It is clearly enough to consider the case when $X(\mathbb{R})$ is compact and connected. Then we know from [BRS 2] that $E^d(\mathbb{R}(X))$ is generated by $(\mathfrak{m}, \omega)$, where $\mathfrak{m}$ is any real maximal ideal and $\omega : (\mathbb{R}(X)/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is any surjection.

Now, from Theorem 3.7.2, we have $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}((\mathfrak{m}, \omega)) = (\mathfrak{m}, \omega) - (\mathfrak{m}, -\omega)$. But by Proposition 4.2.2, $(\mathfrak{m}, \omega) + (\mathfrak{m}, -\omega) = 0$. Therefore, it follows that $\phi_{\mathbb{R}(X)}\delta_{\mathbb{R}(X)}((\mathfrak{m}, \omega)) = 2(\mathfrak{m}, \omega)$. □

The following theorem will be useful in the next section.

Theorem 4.2.10. *Let $X = Spec(R)$ be a smooth affine variety of <u>even</u> dimension d over $\mathbb{R}$. Then,*

$$Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \xrightarrow{\sim} Um_{d+1}(\mathbb{R}(X))/SL_{d+1}(\mathbb{R}(X)).$$

Proof. It suffices to prove that the canonical projection $\epsilon : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \to Um_{d+1}(\mathbb{R}(X))/SL_{d+1}(\mathbb{R}(X))$ is injective. Recall from the Definition(3.5.1) and the subsequent remark (or [DZ, Section 4]), that the group homomorphism $\phi_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \longrightarrow E^d(\mathbb{R}(X))$ is such that when d is even, then $\phi_{\mathbb{R}(X)}([v])$ is precisely the Euler class of the stably free module associated to the unimodular row v in a canonical way.

Now let $v = (a_1, \cdots, a_{d+1}) \in Um_{d+1}(\mathbb{R}(X))$ be such that it is completable to a matrix in $SL_{d+1}(\mathbb{R}(X))$. It is enough to show that this unimodular row is elementarily completable. As v is completable in $SL_{d+1}(\mathbb{R}(X))$, the Euler class of the stably free module associated to v is trivial, and therefore, $\phi_{\mathbb{R}(X)}([v]) = 0$ in $E^d(\mathbb{R}(X))$. As $\phi_{\mathbb{R}(X)}$ is injective, $[v]$ is trivial in $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$. □

4.3 The elementary orbit space

Let $X = \operatorname{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over reals. As before, we always assume that $X(\mathbb{R})$ is orientable. As in the previous sections, we are treating the orbit spaces of unimodular rows as a multiplicative groups.

Recall from [BRS 2, page 307] that there is a canonical surjective group homomorphism $\Gamma : E^d(R) \twoheadrightarrow E^d(\mathbb{R}(X))$. Bhatwadekar-Sridharan denotes the kernel of this map by $E^d(\mathbb{C})$. They prove that $E^d(\mathbb{C})$ is the subgroup of $E^d(R)$ generated by all $(\mathfrak{m}, \omega_{\mathfrak{m}})$, where $\mathfrak{m}$ runs over the complex maximal ideals of R, and $\omega_{\mathfrak{m}} : (R/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is any surjection. Let $\beta : Um_{d+1}(R)/E_{d+1}(R) \longrightarrow Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ be the canonical map. The group $E^d(\mathbb{C})$ is torsion-free and divisible. We have discussed this in detail in Chapter 2, Remark 2.5.2.

We have the following commutative diagram with exact rows. As $\delta_{\mathbb{R}(X)}$ is an isomorphism and Γ is surjective, it follows that β is a surjective group homomorphism. Write $K = \ker(\beta)$.

$$\begin{tikzcd}
0 \arrow[r] & E^d(\mathbb{C}) \arrow[r] \arrow[d] & E^d(R) \arrow[r, "\Gamma"] \arrow[d, "\delta_R"] & E^d(\mathbb{R}(X)) \arrow[r] \arrow[d, "\delta_{\mathbb{R}(X)}"] & 0 \\
1 \arrow[r] & K \arrow[r] & \frac{Um_{d+1}(R)}{E_{d+1}(R)} \arrow[r, "\beta"] & \frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))} \arrow[r] & 1
\end{tikzcd}$$

Proposition 4.3.1. *The restriction of δ_R on the subgroup $E^d(\mathbb{C})$ is trivial.*

Proof. Let $(J, \omega) \in E^d(\mathbb{C})$, where J is a product of complex maximal ideals. It follows from Remark 2.5.2 that $E^d(\mathbb{C})$ is a torsion-free divisible group.

As $E^d(\mathbb{C})$ is divisible, there is some $(I, \omega_I) \in E^d(\mathbb{C})$ such that $(J, \omega) = 2(I, \omega_I)$. By Remark 2.4.9 note that $(I, \omega_I) = (I, -\omega_I)$. Therefore, $(J, \omega) = (I, \omega_I) + (I, -\omega_I)$, and consequently, by Proposition 3.7.4, $\delta_R((J, \omega)) = 0$. □

Proposition 4.3.2. *The group homomorphism β is injective on the image of the map δ_R.*

Proof. Let $(J, \omega_J) \in E^d(R)$ be such that $\beta\delta_R((J, \omega_J)) = [0, \cdots, 0, 1]$. Then, from the diagram we have, $\delta_{\mathbb{R}(X)}\Gamma((J, \omega_J)) = [0, \cdots, 0, 1]$. Since $\delta_{\mathbb{R}(X)}$ is an isomorphism, we have

$\Gamma((J,\omega_J)) = 0$ in $E^d(\mathbb{R}(X))$. By exactness of the top row, $(J,\omega_J) \in E^d(\mathbb{C})$. But then $\delta_R((J,\omega_J)) = [0,\cdots,0,1]$ by the above proposition. □

Theorem 4.3.3. *The map δ_R is a group homomorphism.*

Proof. We have to prove that if (J,ω_J) and (I,ω_I) are two elements of $E^d(R)$ such that J, I are both reduced ideals and $J + I = R$, then

$$\delta_R((J,\omega_J) + (I,\omega_I)) = \delta_R((J,\omega_J))\delta_R((I,\omega_I)),$$

where the multiplication on the right is that of $Um_{d+1}(R)/E_{d+1}(R)$. There are three cases to consider.

Case 1. Both J and I are contained only in complex maximal ideals of R. Then, both (J,ω_J) and (I,ω_I) are from $E^d(\mathbb{C})$. This case follows trivially from Proposition 4.3.1.

Case 2. Both J and I are contained only in real maximal ideals of R.

Note that the exact sequence in the top row of the above diagram splits. There is a split group homomorphism $\theta_1 : E^d(\mathbb{R}(X)) \to E^d(R)$. Define

$$\theta_2 := \delta_R\,\theta_1\,(\delta_{\mathbb{R}(X)})^{-1} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \to Um_{d+1}(R)/E_{d+1}(R).$$

It is easy to see that θ_2 is a split map (for the bottom row) and $\delta_R\theta_1 = \theta_2\delta_{\mathbb{R}(X)}$. We then have the following diagram, where $\overline{\delta_R}$ denotes the restriction of δ_R. The same for $\overline{\beta}$ and $\overline{\Gamma}$.

$$\begin{tikzcd}
\theta_1(E^d(\mathbb{R}(X)) \arrow[r, "\sim", "\overline{\Gamma}"'] \arrow[d, "\overline{\delta_R}"] & E^d(\mathbb{R}(X)) \arrow[d, "\delta_{\mathbb{R}(X)}"] \\
\theta_2\left(\frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))}\right) \arrow[r, "\overline{\beta}", "\sim"'] & \frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))}
\end{tikzcd}$$

We can treat the elements (J,ω_J) and (I,ω_I) as elements of $\theta_1(E^d(\mathbb{R}(X))$. It is therefore enough to prove that $\overline{\delta_R}$ is a group homomorphism. This is clear from the diagram.

Case 3. In this case we assume that J is contained only in complex maximal ideals and I is contained only in real maximal ideals of R.

For convenience, we write $x = (J,\omega_J)$ and $y = (I,\omega_I)$. Note that $\delta_R(x)$ is trivial.

It is therefore enough to show that $\delta_R(x+y) = \delta_R(y)$. We compute: $\beta(\delta_R(x+y)) = \delta_{\mathbb{R}(X)}\Gamma(x+y) = \delta_{\mathbb{R}(X)}\Gamma(y) = \beta(\delta_R(y))$. By Proposition 4.3.2, β is injective on the image of δ_R. We are done. $\square$

Proposition 4.3.4. *The group K is divisible.*

Proof. Let $[x_1, \cdots, x_d, z] \in K$ and n be any integer. Let $\alpha_1 x_1 + \cdots + \alpha_d x_d + bz = 1$. Recall from Lemma 4.1.1 that t is chosen so that $b^2 \equiv t^{4n} \bmod (x_1, \cdots, x_d)$. Therefore $(t^{2n}z)^2 \equiv 1$ modulo $(x_1, \cdots, x_d)$. We have $[x_1, \cdots, x_d, t^{2n}z] = [x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^{2n}]$.

Clearly, $\beta([x_1, \cdots, x_d, t^{2n}]) = [0, \cdots, 0, 1]$, and therefore it follows that $[x_1, \cdots, x_d, t^{2n}z] \in K$. But $[x_1, \cdots, x_d, t^{2n}z] = \delta_R((J, \omega_J))$ for some $(J, \omega_J) \in E^d(R)$. Then $\delta_{\mathbb{R}(X)}\Gamma((J, \omega_J)) = [0, \cdots, 0, 1]$. As $\delta_{\mathbb{R}(X)}$ is an isomorphism, we see that $\Gamma((J, \omega_J)) = 0$ and thus $(J, \omega_J) \in E^d(\mathbb{C})$. Therefore, $[x_1, \cdots, x_d, t^{2n}z] = \delta_R((J, \omega_J)) = [0, \cdots, 0, 1]$ by Proposition 4.3.1. So $[x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^{2n}] = [0, \cdots, 0, 1]$. Then

$$[x_1, \cdots, x_d, z] = ([x_1, \cdots, x_d, t^2]^n)^{-1} = ([x_1, \cdots, x_d, t^2]^{-1})^n.$$

Note that $([x_1, \cdots, x_d, t^2]^{-1}) \in K$ (any unimodular row with a square entry is in K and K is a subgroup of $Um_{d+1}(R)/E_{d+1}(R)$). Therefore K is a divisible subgroup. $\square$

$\square$

Before the next corollary let us recall that an abelian group is said to be *reduced* if its only divisible subgroup is $\{0\}$.

Corollary 4.3.5. *K is the unique maximal divisible subgroup of $Um_{d+1}(R)/E_{d+1}(R)$.*

Proof. Let D be the unique maximal divisible subgroup of $Um_{d+1}(R)/E_{d+1}(R)$. Then $K \subseteq D$. Write $H = Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$. Now, we have $Um_{d+1}(R)/E_{d+1}(R) = K \oplus H$ and H is reduced. It follows that

$$D = (D \cap K) \oplus (D \cap H) = K \oplus (D \cap H).$$

But then $(D \cap H)$ is a direct summand of the divisible group D and is contained in the reduced group H, implying that $(D \cap H)$ is trivial. Therefore, $D = K$. $\square$

Combining Theorem 4.2.8 and the results proved above, we have the following:

Theorem 4.3.6. *Let $X = Spec(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. Let $\mathcal{C}$ be the set of compact connected components of $X(\mathbb{R})$. Then,*

$$Um_{d+1}(R)/E_{d+1}(R) \tilde{\to} K \oplus (\bigoplus_{C \in \mathcal{C}} \mathbb{Z}),$$

where K is the unique maximal divisible subgroup of $Um_{d+1}(R)/E_{d+1}(R)$.

In Section 4.7, we shall prove that K is torsion-free if $d \geq 3$. A summary of our conclusions above fits in an exact sequence, as given below.

Theorem 4.3.7. *The sequence $0 \to E^d(\mathbb{C}) \to E^d(R) \xrightarrow{\delta_R} Um_{d+1}(R)/E_{d+1}(R) \to K \to 1$ is an exact sequence of abelian groups.*

We now analyze the subgroup K in intricate detail. This is in fact, a preparation for the next section.

Theorem 4.3.8. *Let $[x_1, \cdots, x_d, z] \in K$. Then $[x_1, \cdots, x_d, z] = [x_1, \cdots, x_d, -z]$ and as a consequence, $[x_1, \cdots, x_d, z]^n = [x_1, \cdots, x_d, z^n]$ for any $n \geq 1$.*

Proof. Let $\alpha_1 x_1 + \cdots + \alpha_d x_d + bz = 1$. Recall from Lemma 4.1.1 that t is chosen so that $b^2 \equiv t^4 \bmod (x_1, \cdots, x_d)$ and then $[x_1, \cdots, x_d, t^2 z] = [x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^2]$. As $\alpha_1 x_1 + \cdots + \alpha_d x_d + (-b)(-z) = 1$, in a similar manner we have, $[x_1, \cdots, x_d, -t^2 z] = [x_1, \cdots, x_d, -z][x_1, \cdots, x_d, t^2]$. As $[x_1, \cdots, x_d, z] \in K$, the argument as in Proposition 4.3.4 shows that $[x_1, \cdots, x_d, t^2 z] = [0, \cdots, 0, 1]$. As $(t^2 z)^2 \equiv 1$ modulo $(x_1, \cdots, x_d)$, by [Theorem 2.3.2,(iv)], $[x_1, \cdots, x_d, -t^2 z]$ is nothing but the inverse of $[x_1, \cdots, x_d, t^2 z]$, and hence trivial. Therefore, $[x_1, \cdots, x_d, z] = [x_1, \cdots, x_d, -z]$ and by [Ra, Lemma 1.3.1], $[x_1, \cdots, x_d, z]^n = [x_1, \cdots, x_d, z^n]$ for any $n \geq 1$. $\square$

It is obvious that any unimodular row over R with one square entry is in K. The following easy corollary is the converse.

Corollary 4.3.9. *Any element in K is of the form $[x_1, \cdots, x_d, w^2]$.*

Proof. Let $[v] \in K$. As K is 2-divisible, $[v] = [x_1, \cdots, x_d, w]^2$ for some $[x_1, \cdots, x_d, w] \in K$. Then, $[v] = [x_1, \cdots, x_d, w]^2 = [x_1, \cdots, x_d, w^2]$ by the above theorem. $\square$

Corollary 4.3.10. *Let $v \in Um_{d+1}(R)$ be such that $[v] \in K$. Then the row v can be completed to a matrix in $SL_{d+1}(R)$.*

Proof. As K is divisible, $[v] = [x_1, \cdots, x_d, w]^{d!}$ for some $[x_1, \cdots, x_d, w] \in K$. Then, $[v] = [x_1, \cdots, x_d, w]^{d!} = [x_1, \cdots, x_d, w^{d!}]$ by the above theorem. Under the canonical group homomorphism $Um_{d+1}(R)/E_{d+1}(R) \twoheadrightarrow Um_{d+1}(R)/SL_{d+1}(R)$, the image of $[v]$ is trivial by a celebrated theorem of Suslin [Su 2] recorded in Theorem 2.2.10. $\square$

4.4 Stably free modules

Our aim in this section is to prove the following: *Let $X = Spec(R)$ be a smooth real affine variety of <u>even</u> dimension d, whose real points $X(\mathbb{R})$ constitute an orientable manifold. Then the set of isomorphism classes of (oriented) stably free R-modules of rank d is a free abelian group of rank equal to the number of compact connected components of $X(\mathbb{R})$. In contrast, if $d \geq 3$ is odd, then the set of isomorphism classes of stably free R-modules of rank d is a $\mathbb{Z}/2\mathbb{Z}$-vector space (possibly trivial)..*

We now proceed to compute the group $Um_{d+1}(R)/SL_{d+1}(R)$. In order to do so, we consider the following composite group homomorphisms. We shall call the first composite as δ'_R, and the second one as $\delta'_{\mathbb{R}(X)}$.

(i) $E^d(R) \overset{\delta_R}{\to} Um_{d+1}(R)/E_{d+1}(R) \overset{\epsilon_R}{\to} Um_{d+1}(R)/SL_{d+1}(R)$, and

(ii) $E^d(\mathbb{R}(X)) \overset{\delta_{\mathbb{R}(X)}}{\to} Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \overset{\epsilon_{\mathbb{R}(X)}}{\to} Um_{d+1}(\mathbb{R}(X))/SL_{d+1}(\mathbb{R}(X))$.

Note that by results proved in the previous section, $\delta'_{\mathbb{R}(X)}$ is a group homomorphism. We shall refer to the following commutative diagram with exact rows. Since $\epsilon_{\mathbb{R}(X)}$ and β are both surjective, it follows that γ is also a surjective group homomorphism.

$$\begin{tikzcd}
0 \arrow[r] & E^d(\mathbb{C}) \arrow[r] \arrow[d] & E^d(R) \arrow[r, "\Gamma"] \arrow[d, "\delta_R"] & E^d(\mathbb{R}(X)) \arrow[r] \arrow[d, "\delta_{\mathbb{R}(X)}"] & 0 \\
1 \arrow[r] & K \arrow[r] \arrow[d, "\bar{\epsilon}"] & \frac{Um_{d+1}(R)}{E_{d+1}(R)} \arrow[r, "\beta"] \arrow[d, "\epsilon_R"] & \frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))} \arrow[r] \arrow[d, "\epsilon_{\mathbb{R}(X)}"] & 1 \\
1 \arrow[r] & \ker(\gamma) \arrow[r] & \frac{Um_{d+1}(R)}{SL_{d+1}(R)} \arrow[r, "\gamma"] & \frac{Um_{d+1}(\mathbb{R}(X))}{SL_{d+1}(\mathbb{R}(X))} \arrow[r] & 1
\end{tikzcd}$$

We now prove the following theorem.

Theorem 4.4.1. *Let d be <u>even</u>. Then, $Um_{d+1}(R)/SL_{d+1}(R) \xrightarrow{\sim} \bigoplus_{C\in\mathcal{C}} \mathbb{Z}$, where $\mathcal{C}$ is the set of all compact connected components of $X(\mathbb{R})$.*

Proof. We have observed that γ is surjective. As ϵ_R is surjective, and $\epsilon_{\mathbb{R}(X)}$ is an isomorphism by Theorem 4.2.10, it follows that the induced map $\overline{\epsilon} : K \to \ker(\gamma)$ is also surjective. It is immediate from Corollary 4.3.10 that $\overline{\epsilon}$ is the trivial group homomorphism, implying that $\ker(\gamma)$ is trivial. Now apply Theorem 4.2.10. □

Theorem 4.4.2. *The group homomorphism $\delta'_R : E^d(R) \to Um_{d+1}(R)/SL_{d+1}(R)$ is surjective.*

Proof. From the above theorem, this is obvious when d is even. For general d we need additional arguments. Recall from the proof of Theorem 4.3.3 that there is a split group homomorphism $\theta_1 : E^d(\mathbb{R}(X)) \to E^d(R)$ for the top row. Also, there is a split group homomorphism $\theta_2 : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \to Um_{d+1}(R)/E_{d+1}(R)$. We checked that the restriction of δ_R on $\theta_1(E^d(\mathbb{R}(X)))$ is an isomorphism onto $\theta_2(Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)))$.

Note that ϵ_R is surjective and it is trivial on K. As $Um_{d+1}(R)/E_{d+1}(R) = K \oplus \theta_2(Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)))$, it follows that for any element in $Um_{d+1}(R)/SL_{d+1}(R)$, there is a preimage in $\theta_2(Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)))$, which further has a preimage in $\theta_1(E^d(\mathbb{R}(X))) \subset E^d(R)$ under δ_R. □

We record the following corollary which will be used soon.

Corollary 4.4.3. *Let $X(\mathbb{R})$ be orientable, compact and connected. Then $Um_{d+1}(R)/SL_{d+1}(R)$ is generated by $\delta'_R((\mathfrak{m}, \omega_\mathfrak{m}))$, where $\mathfrak{m}$ is any real maximal ideal of R and $\omega_\mathfrak{m} : (R/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is any surjection.*

Proof. By the discussion in Section 2.5, $E^d(\mathbb{R}(X)) = \mathbb{Z}$, and it is generated by any $(\mathfrak{m}, \omega_\mathfrak{m})$ as in the statement of this corollary. By the proof of the above theorem, $\delta'_R((m, \omega_\mathfrak{m}))$ generates $Um_{d+1}(R)/SL_{d+1}(R)$. □

Remark 4.4.4 Let d be even. If $E^d(\mathbb{C}) = 0$ (for example, when R is the coordinate ring of a real sphere, or when $\mathrm{Spec}(R)$ is a rational variety), then it follows that $E^d(R)$ is isomorphic to $Um_{d+1}(R)/SL_{d+1}(R)$. Consequently, under this assumption, a stably free R-module P of rank d is free if and only if it has a unimodular element (see also [F 1, Theorem 5.10]).

We are now ready to compute $Um_{d+1}(R)/SL_{d+1}(R)$ when d is odd.

Theorem 4.4.5. *Let $d \geq 3$ be odd. Then $Um_{d+1}(R)/SL_{d+1}(R)$ is an $\mathbb{F}_2$-vector space of rank $\leq |\mathcal{C}|$, where $\mathcal{C}$ is the set of all compact connected components of $X(\mathbb{R})$.*

Proof. By [vdK 1, 4.3], the group $Um_{d+1}(R)/SL_{d+1}(R)$ satisfies Mennicke relations. In particular, for any orbit $[x_1, \cdots, x_d, z]$, and for any $r \geq 1$, one has $[x_1, \cdots, x_d, z]^r = [x_1, \cdots, x_d, z^r]$. Let us keep this in mind.

Recall that we proved that δ'_R is a surjective group homomorphism. We actually proved that for any $[v] \in Um_{d+1}(R)/SL_{d+1}(R)$, there is $(J, \omega_J) \in \theta_1(E^d(\mathbb{R}(X)))$ such that $\delta'_R((J, \omega_J)) = [v]$. Let ω_J be induced by $J = (a_1, \cdots, a_s, s)$ where $s - s^2 \in (a_1, \cdots, a_d)$. Then, by the definition of δ'_R, it follows that $[v] = [a_1, \cdots, a_d, 1 - 2s]$. Since Mennicke relations hold in $Um_{d+1}(R)/SL_{d+1}(R)$, $[v]^2 = [a_1, \cdots, a_d, 1 - 2s]^2 = [a_1, \cdots, a_d, (1-2s)^2] = [0, \cdots, 0, 1]$. It shows that every element of $Um_{d+1}(R)/SL_{d+1}(R)$ is 2-torsion.

As $\theta_1(E^d(\mathbb{R}(X)))$ is isomorphic to $E^d(\mathbb{R}(X))$, and $E^d(\mathbb{R}(X)) = \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$, the result follows. $\square$

4.5 Computations on spheres

Let us now apply the above computations on real spheres. Consider the coordinate ring of the d-dimensional real sphere $S^d(\mathbb{R})$ for $d \geq 2$ (no further assumption on d):

$$R = \frac{\mathbb{R}[X_1, \cdots, X_{d+1}]}{(X_1^2 + \cdots + X_{d+1}^2 - 1)} = \mathbb{R}[x_1, \cdots, x_{d+1}]$$

We now have the following result (see also [F 1, Corollary 5.12]).

Theorem 4.5.1. *Let R be the coordinate ring of $S^d(\mathbb{R})$. Then $Um_{d+1}(R)/SL_{d+1}(R)$ is generated by the orbit of the tangent bundle.*

Proof. By Corollary 4.4.3, $Um_{d+1}(R)/SL_{d+1}(R)$ is generated by $\delta'_R((\mathfrak{m}, \omega_{\mathfrak{m}}))$ (see notations therein), where $\mathfrak{m}$ is any real maximal ideal of R and $\omega_{\mathfrak{m}} : (R/\mathfrak{m})^d \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is any surjection.

We now concentrate on the orbit $[x_1, \cdots, x_{d+1}]$ of the tangent bundle. We have the following relations among the ideals involved:

$$(x_1, \cdots, x_d) = (x_1, \cdots, x_d, 1 - x_{d+1}) \cap (x_1, \cdots, x_d, 1 + x_{d+1}) = \mathfrak{m}_1 \cap \mathfrak{m}_2,$$

and $\mathfrak{m}_1$, $\mathfrak{m}_2$ are both real maximal ideals. Let $s = \frac{1}{2}(1 - x_{d+1})$. Then, $s - s^2 = \frac{1}{4}(1-x_{d+1}^2) \in (x_1, \cdots, x_d)$. Therefore, $\mathfrak{m}_1 = (x_1, \cdots, x_d, \frac{1}{2}(1-x_{d+1}))$ will induce $\omega_{\mathfrak{m}_1}$ and by definition, $\delta'_R((\mathfrak{m}_1, \omega_{\mathfrak{m}_1})) = [x_1, \cdots, x_d, x_{d+1}]$. This shows that $Um_{d+1}(R)/SL_{d+1}(R)$ is generated by the orbit of the tangent bundle. $\square$

The following corollary is now obvious.

Corollary 4.5.2. *All stably free modules of top rank on $S^3(\mathbb{R})$ and $S^7(\mathbb{R})$ are free. For odd $d \neq 1, 3, 7$, the set of isomorphism classes of stably free modules of rank d over $S^d(\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Proof. For $S^3(\mathbb{R})$ and $S^7(\mathbb{R})$, the orbit of the tangent bundle in each case is trivial. For $d \neq 3, 7$, the orbit of the tangent bundle is non-trivial. $\square$

Remark 4.5.3 The assertion on $S^3(\mathbb{R})$ and $S^7(\mathbb{R})$ was first proved in [F 1, Proposition 5.13].

Remark 4.5.4 Most of the results of this section can be extended to smooth affine varieties over $\mathbf{R}$, where $\mathbf{R}$ is any real closed field. One needs to use the structure theorem for the Euler class group in this case which is available in [BS].

4.6 Mennicke symbols

In this section we are going to describe a similar kind of structure theorem for Mennicke symbols of length $d+1$ over the ring R, with the same notation of previous sections. At first we briefly recall the definition of Mennicke symbols.

Definition 4.6.1. Let B be a ring. A *Mennicke symbol* of length $n+1 \geq 3$ is a pair (ψ, G), where G is a group and $\psi : Um_{n+1}(B) \to G$ is a map such that:

ms1. $\psi((0, \cdots, 0, 1)) = 1$ and $\psi(v) = \psi(v\sigma)$ for any $\sigma \in E_{n+1}(B)$;

ms2. $\psi((b_1, \cdots, b_n, x))\psi((b_1, \cdots, b_n, y)) = \psi((b_1, \cdots, b_n, xy))$ for any two unimodular rows $(b_1, \cdots, b_n, x)$ and $(b_1, \cdots, b_n, y)$.

Clearly, a universal Mennicke symbol $(ms, MS_{n+1}(B))$ exists.

W. van der Kallen introduced the weak Mennicke symbol in [vdK 2]. Now let $\dim(B) = n \geq 2$. It was proved in [vdK 2] that the universal weak Mennicke symbol $(wms, WMS_{n+1}(B))$ is in bijective correspondence with $Um_{n+1}(B)/E_{n+1}(B)$, giving the latter a group structure. A Mennicke symbol of length $n+1$ is also a weak Mennicke symbol of length $n+1$ and there is a unique surjective group homomorphism $WMS_{n+1}(B) \to MS_{n+1}(B)$. So,we have the following commutative diagram:

$$
\begin{tikzcd}
(wms, WMS_{n+1}(B)) \arrow[rr] \arrow[dr, two heads] & & Um_{n+1}(B)/E_{n+1}(B) \arrow[dl, two heads] \\
& (ms, MS_{n+1}(B)) &
\end{tikzcd}
$$

Summing up, we have a surjective group homomorphism $f_B : Um_{n+1}(B)/E_{n+1}(B) \to MS_{n+1}(B)$, whose kernel is generated by all elements of the following form:

$$[b_1, \cdots, b_n, x][b_1, \cdots, b_n, y][b_1, \cdots, b_n, xy]^{-1}.$$

Let $X = \operatorname{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. To compute $MS_{d+1}(R)$, we first focus on $MS_{d+1}(\mathbb{R}(X))$. We shall consider the following diagram. Here L denotes the kernel of the natural map

$\beta_0 : MS_{d+1}(R) \to MS_{d+1}(\mathbb{R}(X))$. As $f_{\mathbb{R}(X)}\beta = \beta_0 f_R$ is surjective, it follows that β_0 is surjective.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & \frac{Um_{d+1}(R)}{E_{d+1}(R)} & \xrightarrow{\beta} & \frac{Um_{d+1}(\mathbb{R}(X))}{E_{d+1}(\mathbb{R}(X))} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f_R} & & \downarrow{\scriptstyle f_{\mathbb{R}(X)}} & & \\ 1 & \longrightarrow & L & \longrightarrow & MS_{d+1}(R) & \xrightarrow{\beta_0} & MS_{d+1}(\mathbb{R}(X)) & \longrightarrow & 1 \end{array}$$

Theorem 4.6.2. *$MS_{d+1}(\mathbb{R}(X))$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space.*

Proof. Take any element $ms((x_1, \cdots, x_d, z)) \in MS_{d+1}(\mathbb{R}(X))$, where $(x_1, \cdots, x_d, z)$ is a unimodular row over $\mathbb{R}(X)$. It is clear that

$$(ms((x_1, \cdots, x_d, z)))^2 = ms((x_1, \cdots, x_d, z^2)) = (f_{\mathbb{R}(X)}([x_1, \cdots, x_d, z^2])).$$

But $[x_1, \cdots, x_d, z^2] = [0, \cdots, 0, 1]$ in $Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$. It follows that every element in $MS(\mathbb{R}(X))$ is 2-torsion and therefore it is a $\mathbb{Z}/2\mathbb{Z}$-vector space. $\square$

It follows from the above theorem and Theorem 4.2.8 that $MS_{d+1}(\mathbb{R}(X))$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space of dimension $\leq |\mathcal{C}|$. We now claim that it is actually of dimension $|\mathcal{C}|$. To prove this, we first need the following easy lemma.

Lemma 4.6.3. *Let $(\mathfrak{m}, \omega_\mathfrak{m}) \in E^d(\mathbb{R}(X))$ and let $\lambda, \mu \in (\mathbb{R}(X)/\mathfrak{m})^* = \mathbb{R}^*$. Then, $(\mathfrak{m}, \lambda\omega_\mathfrak{m}) + (\mathfrak{m}, \mu\omega_\mathfrak{m}) - (\mathfrak{m}, \lambda\mu\omega_\mathfrak{m})$ is equal to:*

(i) $(\mathfrak{m}, \omega_\mathfrak{m})$ *if $\lambda > 0$, $\mu > 0$;*

(ii) $-3(\mathfrak{m}, \omega_\mathfrak{m})$ *if $\lambda < 0$, $\mu < 0$;*

(iii) $(\mathfrak{m}, \omega_\mathfrak{m})$ *if λ and μ have opposite signs.*

Proof. This is nothing but a straightforward computation. From Section 2.5 we can use the following two facts:

- If $\lambda > 0$ then $(\mathfrak{m}, \lambda\omega_\mathfrak{m}) = (\mathfrak{m}, \omega_\mathfrak{m})$ and if $\lambda < 0$ then $(\mathfrak{m}, \mu\omega_\mathfrak{m}) = (\mathfrak{m}, -\omega_\mathfrak{m})$ in $E^d(\mathbb{R}(X))$.

- $(\mathfrak{m}, \omega_\mathfrak{m}) + (\mathfrak{m}, -\omega_\mathfrak{m}) = 0$ in $E^d(\mathbb{R}(X))$.

□

Corollary 4.6.4. *Let $J = (a_1, \cdots, a_d)$ be a reduced ideal of height d in $\mathbb{R}(X)$. Let $\omega_J : (\mathbb{R}(X)/J)^d \twoheadrightarrow J/J^2$ be the surjection induced by $a_1, \cdots, a_d$. Let λ, μ be units modulo J. Then $(J, \lambda\omega_J) + (J, \mu\omega_J) - (J, \lambda\mu\omega_J) \in 4E^d(\mathbb{R}(X))$.*

Proof. We have $(J, \omega_J) = 0$ in $E^d(\mathbb{R}(X))$. Let $J = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$, where each $\mathfrak{m}_i$ is a maximal ideal. We then have,

$$0 = (J, \omega_J) = (\mathfrak{m}_1, \omega_{\mathfrak{m}_1}) + \cdots + (\mathfrak{m}_k, \omega_{\mathfrak{m}_k}), \qquad (*)$$

where $\omega_{\mathfrak{m}_i} : (\mathbb{R}(X)/\mathfrak{m}_i)^d \twoheadrightarrow \mathfrak{m}_i/\mathfrak{m}_i^2$ is the surjection induced by ω_J.

Let us write λ as the tuple $(\lambda_1, \cdots, \lambda_k)$, where λ_i is the image of λ in $\mathbb{R}(X)/\mathfrak{m}_i$. Similarly, $\mu = (\mu_1, \cdots, \mu_k)$. By renaming if necessary, we may assume that λ_i and μ_i are both negative for $i = 1, \cdots, r$, for some r $(0 \leq r \leq k)$. Then an easy verification using the lemma above will show that

$$(J, \lambda\omega_J) + (J, \mu\omega_J) - (J, \lambda\mu\omega_J) = (\mathfrak{m}_{r+1}, \omega_{\mathfrak{m}_{r+1}}) + \cdots + (\mathfrak{m}_k, \omega_{\mathfrak{m}_k}) - 3\left((\mathfrak{m}_1, \omega_{\mathfrak{m}_1}) + \cdots (\mathfrak{m}_r, \omega_{\mathfrak{m}_r})\right),$$

which equals $(J, \omega_J) - 4\left((\mathfrak{m}_1, \omega_{\mathfrak{m}_1}) + \cdots (\mathfrak{m}_r, \omega_{\mathfrak{m}_r})\right)$, and we are done by the relation $(*)$ above. □

We are now ready to prove:

Theorem 4.6.5. *$MS_{d+1}(\mathbb{R}(X))$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space of dimension $|\mathcal{C}|$.*

Proof. Recall from Corollary 4.2.5 that $\phi_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \longrightarrow E^d(\mathbb{R}(X))$ is injective, and in fact, it is an isomorphism onto $2E^d(\mathbb{R}(X))$.

Consider the kernel of the map $f_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \twoheadrightarrow MS_{d+1}(\mathbb{R}(X))$. We know that $\ker(f_{\mathbb{R}(X)})$ is generated by elements of the form

$$[w] = [a_1, \cdots, a_d, \lambda][a_1, \cdots, a_d, \mu][a_1, \cdots, a_d, \lambda\mu]^{-1}.$$

Adding suitable multiples of $\lambda\mu$ to $a_1, \cdots, a_d$, we may assume that $J = (a_1, \cdots, a_d)$ is a reduced zero-dimensional ideal. Then we have $\phi_{\mathbb{R}(X)}([w]) = (J, \lambda\omega_J) +$

$(J, \mu\omega_J) - (J, \lambda\mu\omega_J) \in 4E^d(\mathbb{R}(X))$, by the corollary proved above. As a consequence, we have an induced surjective group homomorphism $\overline{\phi}_{\mathbb{R}(X)} : MS_{d+1}(\mathbb{R}(X)) \twoheadrightarrow 2E^d(\mathbb{R}(X))/4E^d(\mathbb{R}(X))$. As the target object is a $\mathbb{Z}/2\mathbb{Z}$-vector space of dimension $|\mathcal{C}|$, combining with Theorem 4.6.2 we are done. □

From the above theorem and Theorem 4.2.8, we have the following easy corollary.

Corollary 4.6.6. *Any element $[v]$ in the kernel of $f_{\mathbb{R}(X)} : Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X)) \to MS_{d+1}(\mathbb{R}(X))$ is a square.*

Theorem 4.6.7. *The map $\overline{f} : K \longrightarrow L$ is surjective and therefore L is divisible. In fact, L is the unique maximal divisible subgroup of $MS_{d+1}(R)$.*

Proof. We take any element from L. As f_R is surjective, we will have a preimage of the form $[v]\,\theta_2([w])$ in $Um_{d+1}(R)/E_{d+1}(R)$, where $[w] \in Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ and $[v] \in K$. But $[w] \in \ker(f_{\mathbb{R}(X)})$ and by Corollary 4.6.6, it is a square, say, $[w] = [w_1]^2$ for some $[w_1]$. Let $[w_1] = \delta_{\mathbb{R}(X)}((J, \omega_J))$ for some $(J, \omega_J) \in E^d(\mathbb{R}(X))$. As Mennicke relations hold in $MS_{d+1}(R)$, exactly the same argument as in the proof of Theorem 4.4.5 will show that $f_R\theta_2([w])$ is trivial. This shows that $\overline{f} : K \longrightarrow L$ is surjective. Thus L is divisible and therefore, $L \oplus (\oplus_{C\in\mathcal{C}}(\mathbb{Z}/2\mathbb{Z})) \xrightarrow{\sim} MS_{d+1}(R)$. We can argue as in Corollary 4.3.5 to prove that L is the maximal divisible subgroup. □

Theorem 4.6.8. *The map $\overline{f} : K \longrightarrow L$ is an isomorphism.*

Proof. Let $[v] \in K$ be such that $\overline{f}([v])$ is trivial. So, $[v] \in \ker(f_R)$ and therefore $[v]$ is a finite product of elements of the form $[w] = [x_1, \cdots, x_d, zu][x_1, \cdots, x_d, z]^{-1}[x_1, \cdots, x_d, u]^{-1}$.

We apply the method of Lemma 4.1.1 once again. We choose appropriate $t, \lambda \in R$ such that $(t^2z)^2 \equiv (\lambda^2u)^2 \equiv 1$ modulo $(x_1, \cdots, x_d)$. We have:

(i) $[x_1, \cdots, x_d, t^2z] = [x_1, \cdots, x_d, z][x_1, \cdots, x_d, t^2]$;

(ii) $[x_1, \cdots, x_d, \lambda^2u] = [x_1, \cdots, x_d, u][x_1, \cdots, x_d, \lambda^2]$;

(iii) $[x_1, \cdots, x_d, (t\lambda)^2zu] = [x_1, \cdots, x_d, zu][x_1, \cdots, x_d, t^2\lambda^2]$.

Then, writing $\mathbf{x}$ for $x_1, \cdots, x_d$ and regrouping, we have

$$[w] = \left([\mathbf{x}, t^2\lambda^2]^{-1}[\mathbf{x}, t^2][\mathbf{x}, \lambda^2]\right)\left([\mathbf{x}, (t\lambda)^2zu][\mathbf{x}, t^2z]^{-1}[\mathbf{x}, \lambda^2u]^{-1}\right)$$

Note that the first bunch is trivial by Theorem [2.3.2,(iii)] and each term in the second bunch is in the image of δ_R. Then, it follows that $[v]$ is in fact a finite product of elements, each of which is in the image of δ_R. Therefore, $[v]$ is in K as well as in the image of δ_R, implying that $[v]$ is trivial. $\square$

Corollary 4.6.9. *The kernel of $f_R : Um_{d+1}(R)/E_{d+1}(R) \twoheadrightarrow MS_{d+1}(R)$ is a free abelian group of rank $|\mathcal{C}|$.*

Proof. Easy to see, as we now have $K \xrightarrow{\sim} L$. $\square$

4.7 Cohomological methods.

We now prepare ourselves to prove that L is torsion-free if $d \geq 3$. For this purpose, we shall require some cohomological interpretation of $MS_{d+1}(R)$ from [F 2]. We shall freely use various terms and notations from [F 1, F 2], without explicitly recalling their definition. In the result below K_{d+1} is the sheafification of the pre-sheaf arising out of Milnor K-theory groups and $H^d(A, K_{d+1})$ is the K-cohomology group.

Theorem 4.7.1. *[F 2, Theorem 1.4] Let A be a smooth affine algebra of dimension $d \geq 3$ over a perfect field of characteristic unequal to 2. Then $MS_{d+1}(A)$ is isomorphic to $H^d(A, K_{d+1})$.*

Let $X = \mathrm{Spec}(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. Consider $R_{\mathbb{C}} := R \otimes_{\mathbb{R}} \mathbb{C}$. Let $Y := \mathrm{Spec}(R_{\mathbb{C}})$ and $\pi : Y \to X$ be the canonical finite group homomorphism. We then have induced maps

$$\pi_* : H^d(R_{\mathbb{C}}, K_{d+1}) \to H^d(R, K_{d+1}) \text{ and } \pi^* : H^d(R, K_{d+1}) \to H^d(R_{\mathbb{C}}, K_{d+1})$$

such that $\pi_*\pi^*$ is multiplication by 2. This result follows from the *Projection Formula* as available in [Ro] (or see [EKM, Proposition 56.9]). By a slight abuse of notation, we record the following reformulation to suit our needs. Note that in this section we are treating the groups as additive groups.

Proposition 4.7.2. *Let $d \geq 3$. The finite group homomorphism $\pi : Y \to X$ induces group homomorphisms*

$$\pi_* : MS_{d+1}(R_{\mathbb{C}}) \to MS_{d+1}(R) \text{ and } \pi^* : MS_{d+1}(R) \to MS_{d+1}(R_{\mathbb{C}})$$

such that $\pi_*\pi^*$ is multiplication by 2.

We are now ready to prove the following theorem.

Theorem 4.7.3. *Let $X = Spec(R)$ be a smooth affine variety of dimension $d \geq 3$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. Then, the divisible group $L = ker(\beta_0)$ is torsion-free.*

Proof. We have already proved that L is divisible. Let $\alpha \in MS_{d+1}(R)$ be a torsion element, say, $n\alpha = 0$. Then $0 = \pi^*(n\alpha) = n\pi^*(\alpha)$ in $MS_{d+1}(R_{\mathbb{C}})$. By [F 2, Theorem 2.2], $MS_{d+1}(R_{\mathbb{C}})$ is a torsion-free divisible group. Therefore $\pi^*(\alpha) = 0$, implying that $2\alpha = \pi_*\pi^*(\alpha) = 0$. This shows that any torsion element of $MS_{d+1}(R)$ is 2-torsion. The same is true for the subgroup L. As L is divisible, it is now easy to deduce that L is torsion-free. □

Corollary 4.7.4. *Let $d \geq 3$. The kernel K of the canonical surjection β : $Um_{d+1}(R)/E_{d+1}(R) \twoheadrightarrow Um_{d+1}(\mathbb{R}(X))/E_{d+1}(\mathbb{R}(X))$ is torsion-free.*

Proof. K and L are isomorphic by Theorem 4.6.8. □

Theorems 4.6.5, 4.6.7, and 4.7.3, yield the following structure theorem for $MS_{d+1}(R)$.

Theorem 4.7.5. *Let $X = Spec(R)$ be a smooth affine variety of dimension $d \geq 2$ over $\mathbb{R}$. Assume that $X(\mathbb{R})$ is orientable. Then $MS_{d+1}(R) \xrightarrow{\sim} L \oplus (\oplus_{C \in \mathcal{C}} \mathbb{Z}/2\mathbb{Z})$, where L is the unique maximal divisible subgroup of $MS_{d+1}(R)$. Further, if $d \geq 3$, then L is torsion-free.*

Remark 4.7.6 To prove that L is torsion-free, we rely on Fasel's cohomological interpretation of $MS_{d+1}(R)$, which in turn depends on the work of Fabien Morel (see [F 1, 4.5, 4.6, 4.7] and [F 2, 1.4]). The restriction $d \geq 3$ stems from there. At the moment we do not know how to extend the final statement of the above theorem to $d = 2$.

4.8 An auxiliary result

In this thesis, we saw that the map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ served us well when the base field is $\mathbb{R}$. However, it is completely useless if the base field is algebraically closed, as we show now. But so is its counter-part ϕ_R.

Theorem 4.8.1. *Let R be a smooth affine domain of dimension $d \geq 2$ over an algebraically closed field k of characteristic $\neq 2$. Then, the map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ is the trivial group homomorphism.*

Proof. Under the assumptions, the Euler class group is isomorphic to the Chow group $CH^d(R)$ of 0-cycles. Let $(I, \omega_I) \in E^d(R)$. As $CH^d(R)$ is uniquely divisible, it follows that there exists $(J, \omega_J) \in E^d(R)$ such that $(I, \omega_I) = 2(J, \omega_J)$. As k is algebraically closed, -1 is a square and therefore, applying Remark 2.4.9 we have $(J, \omega_J) = (J, -\omega_J)$ in $E^d(R)$. Therefore, $(I, \omega_I) = (J, \omega_J) + (J, -\omega_J)$. The proof is now complete by Proposition 3.7.4. □

After reading an earlier version of our paper[DTZ1], Jean Fasel suggested us this improvement, also indicating a proof.

Theorem 4.8.2. (Fasel) *Let k be an infinite perfect field of cohomological dimension ≤ 1 and of characteristic unequal to 2. Let R be a smooth affine domain of dimension $d \geq 3$ over k. Then the map $\delta_R : E^d(R) \longrightarrow Um_{d+1}(R)/E_{d+1}(R)$ is the trivial group homomorphism.*

Proof. Under the assumptions of the theorem, by [GRa, F 2], the group $Um_{d+1}(R)/E_{d+1}(R)$ is isomorphic to $MS_{d+1}(R)$. Therefore, by [F 2, Theorem 2.2], it is uniquely 2-divisible. Consequently, the map $\kappa : [v] \mapsto [v]^2$ is an isomorphism of $Um_{d+1}(R)/E_{d+1}(R)$. As the group structure is Mennicke-like, κ is actually Vaserstein's square operation, taking an orbit $[x_1, \cdots, x_d, z]$ to $[x_1, \cdots, x_d, z^2]$.

Now let $(J, \omega_J) \in E^d(R)$ and let ω_J be induced by $(a_1, \cdots, a_d, s)$ with $s(1-s) \in (a_1, \cdots, a_d)$. Then $\kappa\delta_R((J, \omega_J)) = [a_1, \cdots, a_d, (1-2s)^2]$ and the image is clearly the trivial orbit, as $(1-2s)^2 \equiv 1$ modulo $(a_1, \cdots, a_d)$. As κ is an isomorphism, the result follows. □

Chapter 5

Appendix: A bijection

We now prove that the set-theoretic map $\theta_d : E^d(R) \to \pi_0(Q_{2d}(R))$ is a bijection.

We shall need another "moving lemma" for $\pi_0(Q_{2n}(A))$ (Lemma 5.0.1 below), where A is any commutative Noetherian ring. This has been proved in [AF]. However, we reprove it here using the *prime avoidance lemma*, which is perhaps a bit easier to follow.

Lemma 5.0.1. *(***Moving Lemma 2)** *Let A be a commutative Noetherian ring. Let $(a, b, s) = (a_1, \cdots, a_n, b_1, \cdots, b_n, s) \in Q_{2n}(A)$. Then there exists $\mu = (\mu_1, \cdots, \mu_n) \in A^n$ such that*

(i) *The row $(a', b', s') = (a_1 + \mu_1(1-s)^2, \cdots, a_n + \mu_n(1-s)^2, b_1(1-\mu b^t), \cdots, b_n(1-\mu b^t), s + \mu b^t(1-s)) \in Q_{2n}(A)$,*

(ii) *$[(a, b, s)] = [(a', b', s')]$ in $\pi_0(Q_{2n}(A))$ and*

(iii) *$ht(K) \geq n$, where $K = (a_1 + \mu_1(1-s)^2, \cdots, a_n + \mu_n(1-s)^2, s + \mu b^t(1-s))$.*

Proof. We consider the row $(a_1, \cdots, a_n, (1-s)^2) \in A^{n+1}$. By Lemma 2.1.4 there exist $\mu_1, \cdots, \mu_n \in A$ such that $\mathrm{ht}(I_{(1-s)^2}) \geq n$, where $I = (a_1 + \mu_1(1-s)^2, \cdots, a_n + \mu_n(1-s)^2)$. In other words, if $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $I \subset \mathfrak{p}$ and $(1-s) \notin \mathfrak{p}$, then $\mathrm{ht}(\mathfrak{p}) \geq n$.

Set $A = a + T(1-s)^2\mu \in A[T]^n$, then an easy computation yields that

$$Ab^t(1 - T\mu b^t) = (1-s)(1-T\mu b^t) - (1-s)^2(1-T\mu b^t)^2.$$

Setting $B = (1 - T\mu b^t)b$, it is easy to check that $(A, B, (1-s)(1-T\mu b^t)) \in Q_{2n}(A[T])$.

Then it follows that $(A, B, 1-(1-s)(1-T\mu b^t)) = (A, B, s+T\mu b^t(1-s)) \in Q_{2n}(A[T])$. Thus (i) and (ii) are proved.

Now we have the following relations among the ideals:

$$\begin{aligned} I &= \big(a_1+\mu_1(1-s)^2, \cdots, a_n+\mu_n(1-s)^2\big) \\ &= \big(a+\mu(1-s)^2, (1-s)(1-\mu b^t)\big) \cap \big(a+\mu(1-s)^2, s+\mu b^t(1-s)\big) \\ &= \big(a+\mu(1-s)^2, (1-s)(1-\mu b^t)\big) \cap K \end{aligned}$$

Let $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $K \subset \mathfrak{p}$. As $s+\mu b^t(1-s) \in K \subset \mathfrak{p}$, it follows that $(1-s)(1-\mu b^t) \notin \mathfrak{p}$ and therefore, $1-s \notin \mathfrak{p}$. Note that $I \subset K \subset \mathfrak{p}$. Therefore, by the first paragraph, $\operatorname{ht}(\mathfrak{p}) \geq n$. This proves (iii). $\square$

Remark 5.0.2 If A is a geometrically reduced affine algebra over an infinite perfect field then using Swan's Bertini theorem or Remark 2.1.5 in place of Lemma 2.1.4, one can choose K to have the additional property that either $K = A$ or K is a reduced ideal.

Theorem 5.0.3. *Let R be a smooth affine domain of dimension $d \geq 2$ over an infinite perfect field k. The set-theoretic map $\theta_d : E^d(R) \to \pi_0(Q_{2d}(R))$ is a bijection.*

Proof. Let $v = (a_1, \cdots, a_d, b_1, \cdots, b_d, s) \in Q_{2d}(R)$. Then the ideal $I(v) := (a_1, \cdots, a_d, s)$ of R need not be of height d. However, we may apply Lemma 5.0.1 to obtain $v' = (a'_1, \cdots, a'_d, b'_1, \cdots, b'_d, s')$ in the same homotopy class of v such that the ideal $K = (a'_1, \cdots, a'_d, s')$ has height $\geq d$. Assume that K is proper. We have $K = (a'_1, \cdots, a'_d) + K^2$. If $\omega_K : (R/K)^d \twoheadrightarrow K/K^2$ is the corresponding map, then it follows that the image of (K, ω_K) under θ_d is $[v'] = [v]$ in $\pi_0(Q_{2d}(R))$. On the other hand, if $K = R$, then the row $(a'_1, \cdots, a'_d, s')$ is unimodular and therefore there exist $\alpha_1, \cdots, \alpha_d, \beta \in R$ such that

$$\alpha_1 a'_1 + \cdots + \alpha_d a'_d + \beta s' = 1, \text{ and therefore,}$$

$$(1-s')(\alpha_1 a'_1 + \cdots + \alpha_d a'_d) + \beta(s' - s'^2) = 1 - s'.$$

As $s' - s'^2$ is in the ideal $(a'_1, \cdots, a'_d)$, it follows that there exist $\lambda_1, \cdots, \lambda_d \in R$ such that $\lambda_1 a'_1 + \cdots + \lambda_d a'_d = 1 - s'$. We can apply elementary orthogonal transformation of type

(v) in Definition 3.3.1 and change $(a'_1, \cdots, a'_d, b'_1, \cdots, b'_d, s')$ to $(a'_1, \cdots, a'_d, b''_1, \cdots, b''_d, 1)$. The latter is clearly homotopic to $(0, \cdots, 0, 0, \cdots, 0, 1)$. By [MaMi, Lemma 5.3], the orbits $[(0, \cdots, 0, 0, \cdots, 0, 1)]$ and $[(0, \cdots, 0, 0, \cdots, 0, 0)]$ are the same in $\pi_0(Q_{2d}(R))$. This trivial orbit has preimage in $E^d(R)$. Therefore, θ_d is surjective.

The rest of the proof is devoted to proving that θ_d is injective. Let (J, ω_J) and $(J', \omega_{J'})$ be elements of $E^d(R)$ be such that $\theta_d((J, \omega_J)) = \theta_d((J', \omega_{J'}))$. Let ω_J be given by $J = (a_1, \cdots, a_d) + J^2$. As $\mathrm{ht}(J) = d$, applying Lemma 2.1.4 if necessary, we may assume that $\mathrm{ht}(a_1, \cdots, a_d) = d$. Now there exists $s \in J^2$ such that $J = (a_1, \cdots, a_d, s)$ with $s - s^2 = a_1 b_1 + \cdots + a_d b_d$ for some $b_1, \cdots, b_d \in R$. Similarly, $\omega_{J'}$ is given by $J' = (a'_1, \cdots, a'_d) + J'^2$ with $\mathrm{ht}(a'_1, \cdots, a'_d) = d$. There exists $s' \in J'^2$ be such that $J' = (a'_1, \cdots, a'_d, s')$ with $s' - s'^2 = a'_1 b'_1 + \cdots + a'_d b'_d$ for some $b'_1, \cdots, b'_d \in R$.

We now assume that

$$\theta_d((J, \omega_J)) = [(a_1, \cdots, a_d, b_1, \cdots, b_d, s)] = [(a'_1, \cdots, a'_d, b'_1, \cdots, b'_d, s')] = \theta_d((J', \omega_{J'}))$$

in $\pi_0(Q_{2d}(R))$. Applying Corollary 3.4.2 we have $V = (f_1, \cdots, f_d, g_1, \cdots, g_d, h) \in Q_{2d}(R[T])$ such that $V(0) = (a_1, \cdots, a_d, b_1, \cdots, b_d, s)$ and $V(1) = (a'_1, \cdots, a'_d, b'_1, \cdots, b'_d, s')$. If we consider the ideal $I = (f_1, \cdots, f_d, h)$ of $R[T]$ then we have $I = (f_1, \cdots, f_d) + I^2$. Let $\omega_I : (R[T]/I)^d \twoheadrightarrow I/I^2$ denote the corresponding surjection. However, the height of I need not be d, although both $I(0)$ $(= J)$ and $I(1)$ $(= J')$ have height d.

As both $\mathrm{ht}((a_1, \cdots, a_d) = d = \mathrm{ht}(a'_1, \cdots, a'_d)$, it follows that

$$\mathrm{ht}(f_1, \cdots, f_d, T(T-1)) = d + 1 \qquad (*)$$

Consider $(f_1, \cdots, f_d, (T^2 - T)h^2) \in R[T]^{d+1}$. By Lemma 2.1.4, there exist $\mu_1, \cdots, \mu_d \in R[T]$ such that $\mathrm{ht}((F_1, \cdots, F_d)_{h^2(T^2-T)}) \geq d$, where $F_i = f_i + \mu_i h^2(T^2 - T)$, for $i = 1, \cdots, d$. Note that we have $I = (F_1, \cdots, F_d) + (h)$, and $(h) \subset I^2$. Applying [BRS 3, 2.11], there exists $e \in (h)$ such that $I = (F_1, \cdots, F_d, e)$ where $e - e^2 \in (F_1, \cdots, F_d)$. We now take $K = (F_1, \cdots, F_d, 1 - e)$ and write $\omega_K : (R[T]/K)^d \twoheadrightarrow K/K^2$ for the corresponding surjection. We record that $I \cap K = (F_1, \cdots, F_d)$ in $R[T]$.

Let $P \in \mathrm{Spec}(R[T])$ be such that $K \subseteq P$. Then, as $e \in (h)$ and $1 - e \in K$, we see that $h \notin P$. If $T^2 - T \notin P$, then $\mathrm{ht}(P) \geq d$. If $T^2 - T \in P$, then by $(*)$ above, $\mathrm{ht}(P) \geq d+1$. In any case, $\mathrm{ht}(K) \geq d$. Note that

$$K(0) \cap I(0) = K(0) \cap J = (F_1(0), \cdots, F_d(0)) = (a_1, \cdots, a_d),$$

$$K(1) \cap I(1) = K(1) \cap J' = (F_1(1), \cdots, F_d(1)) = (a'_1, \cdots, a'_d).$$

As the height of each of the ideals involved here is d, we have

$$(J, \omega_J) + (K(0), \omega_{K(0)}) = 0 = (J', \omega_{J'}) + (K(1), \omega_{K(1)}) \text{ in } E^d(R),$$

where $\omega_{K(0)}$ is induced by $a_1, \cdots, a_d$, and $\omega_{K(1)}$ is induced by $a'_1, \cdots, a'_d$.

Therefore, $(J, \omega_J) - (J', \omega_{J'}) = (K(1), \omega_{K(1)}) - (K(0), \omega_{K(0)}) \in H$ (where H is as in (2.4.1)) and consequently, $(J, \omega_J) = (J', \omega_{J'})$ in $E^d(R)$. This completes the proof. $\square$

Remark 5.0.4 In Proposition 3.6.1 and Theorem 5.0.3 we can take R to be a regular domain of dimension d which is essentially of finite type over an infinite perfect field k.

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