Essays on Collective Contests and Bargaining

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Thesis submitted to the Indian Statistical Institute

in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

“To Ma and Baba”

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July, 2020

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Acknowledgements

First of all I would like to mention my advisor Dr. Prabal Roy Chowdhury, for his support during the many years of my PhD tenure. A big thank you to him.

In the same vein, I would like to thank other faculty members of the Economics and Planning Unit, ISI for their influence on me. I would like to specially mention Dr. Tridip Ray, for his intellectual and moral support over the years.

When it comes to life at ISI, it is incomplete without the camaraderie of the fellow students there. Many of them have become really close friends and I hope these friendships will last the test of time. The list is too long to be stated in full. I would, however, like to mention Anup, Dushyant, Anirban, Soham, Komal, Manas, Aditya, Pranabesh, Rohan, Abhinaba, Saikat, Abdul, Pinaki, Dyotona, Priyanka, Gaurav, Sabyasachi, Arka, Amlan and Ashok da for making my time at ISI a special one.

When it comes to family, again the list is too long to be stated in full. But I would like to thank my Ma and Baba for just being what they are. They are the reason I am where I am. My sisters Mumun and Jhumi, who I spent my childhood with. They are and always will be very close to my heart. I would also like to thank Bapu Mama, Tublu Mami, Medha, Laila Mashi, Captain, Tima, Tibaba, Ingi Di, Puchku Di and Tamal da for being the family everyone would want to have. I cannot thank you guys enough.

I would also like to thank those friends, who have not been a part of the PhD journey but who I really treasure. Sourjya, Anik, Bubla, Paroma, Sambuddha, Debarati, Abhinav, Emon, Sayan, Ananya and Ionna. I feel really lucky to have come across you guys. You have only helped enrich my existence. Thanks a lot!!

Now for the two most important people who have been a part of this long journey. Firstly, Soumendu da for being more than family. He has been a coauthor, a friend, and a guide in many situations. I hope our collaboration, both on the professional and personal front, continues for years to come. Thank You!!

There is not much I can say about my much better half, Prachi Singh, who has been through thick and thin, all ups and downs and not wavered in her support towards me. Her liveliness and sense of humour have been a great source of comfort for me at all times. To be honest, this thesis is as much hers as it is mine.

Finally, I would like to make special mention of three people who have shaped me by their constructive influence since childhood. My father for always pushing and encouraging me to do my best, specially with regard to physical fitness. Bapu Mama for being the one, who in his inimitable way, nudged me towards intellectual activities. Finally, Tibaba, who was the calming influence in my life. A big thank you to all three of them for always wishing the best for me and lending a helping hand whenever it was required.

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Chapter 1

Introduction

This thesis consists of three theoretical essays in Microeconomics. The first two essays analyze the properties of a particular class of prize sharing rules groups may employ in a situation of a collective contest with another group, over a private good. The third essay studies a situation of multi-lateral bargaining, where a buyer wants to successfully bargain with multiple sellers, who own an input each, so that he can implement a grand project. The main focus of the thesis is to theorize about and generate hypotheses of the situations summarized above.

In the first chapter, we consider a situation of a collective contest between two groups of different sizes and pick for analysis a prize sharing rule groups may employ to share the prize within the group in case of success. The rule being analyzed was introduced in Nitzan (1991) and subsequently became the standard in the collective contests literature. Despite its popularity the rule is ad hoc. In this chapter, we provide a robust strategic basis to these prize sharing rules.

In the second chapter once again we deal with the same context as the first chapter, i.e., a collective contest over a private prize between two differently sized groups. We analyze in depth the prize sharing rules introduced in Nitzan (1991). We posit a restriction on the rule which can be interpreted as a group specific norm of competitiveness. We then go on to analyze how the posited social norms affect the outcomes of the contest. In particular, we analyze how these social norms affect the welfare of the groups participating in the contest.

In the third chapter we consider a situation of multi-lateral bargaining between a buyer and multiple sellers, who own and input each. The buyer needs to successfully bargain with multiple sellers in order to implement a project. We embed the sellers in a graph and study how the underlying structure of the graph, which determines the locations of the sellers, affects the outcomes of the ensuing bargaining game. Specifically, we show how the presence of indispensable sellers turns out to be crucial to the way the surplus is divided in equilibrium.

In the following sections we take up one chapter at a time and provide a brief description outlining the research questions, the theoretical approach taken and the main findings.

Chapter 1

Prize Sharing Rules in Collective Contests: Towards Strategic Foundations

In this chapter, we consider a situation where two differently sized groups engage in a contest over a private prize. In such situations the assumption made is contracts cannot be written between groups and any conflict has to be solved via a contest. But, within groups contracts are possible. The main issue a group confronts in such situations is to decide how to divide the prize within the group if the group succeeds. Given that the amount of promised rewards affects the effort choice of an individual member of a group, the group has to carefully choose a sharing scheme, as that will affect the chances of its success. To that end, Nitzan (1991) proposed a sharing scheme, which has non-cooperative features.

The sharing scheme introduced in Nitzan (1991) is a weighted average of an egalitarian sharing scheme, which discourages individual effort and a competitive sharing scheme, which boosts individual effort. The problem with boosting efforts is that it eats into the prize. So a leader essentially faces a trade-off. If he tries to boost efforts then his group will win the contest more often but the size of the prize which can be enjoyed *ex post* is low. If the group chooses to be more egalitarian then prize dissipation is low but the group does not win the contest often. Given this trade-off inherent in the rule, it has been subject to substantial analysis.

Despite its intuitive properties, the rule is ad hoc. Why choose this rule and not some other rule? That is the question we pose. To answer this, we introduce a rule which represents intra-group cooperation and ask the following question: Will the prize sharing rule of Nitzan (1991) ever be chosen given that cooperative options are also available? We construct a two stage game where group leaders choose between the cooperative rule and prize sharing rules simultaneously in the first stage, followed by individuals putting in efforts in the second stage. We analyze the subgame perfect Nash equilibrium of the game .

We find that the prize sharing rules may be chosen in equilibrium by both groups under certain conditions. But, the game is a Coordination game where both groups choosing the cooperative option is a also a Nash equilibrium. Moreover, the equilibrium with the cooperative option is Pareto superior, thereby satisfying the payoff dominance criterion of equilibrium selection. However, when we subject the equilibria to the selection criterion of risk dominance and the security principle, we find that the equilibrium involving the non-cooperative prize sharing rules may indeed be selected. Based on these results we claim a robust strategic basis to the prize sharing rules introduced by Nitzan (1991).

Chapter 2

Prize Sharing Rules in Collective Contests: When Does Group Size Matter?

In this chapter, we again consider a situation where two differently sized groups contest over a private prize. Like the previous chapter, there is no possibility of a writing a contract between groups. But a group needs to decide how to share the prize among group members in case of success. To that end, we take up the prize sharing rule introduced by Nitzan (1991). The rule is a weighted average of an egalitarian component, which commits to divide the prize equally among group members in case of success and a competitive component, in which an individual's reward depends on the amount of effort he has put in relation to the aggregate group effort.

The egalitarian component discourages individuals from putting in effort, which lowers the chances of the group winning the contest but leaves behind a lot of surplus to be consumed in the case of success. The competitive component, on the other hand, encourages individual efforts, thereby increasing chances of the group winning the contest. However, most of the surplus is dissipated in effort provision. The literature on strategic choice of sharing rules focuses on how to resolve this trade off by optimally choosing the weight to be be put on each component. The main takeaway of this literature is that smaller groups generally put all the wight on the competitive component to incentivize efforts given the disadvantages of a smaller size in a collective contest. The larger group, however, chooses to put some weight on the egalitarian component, compromising on chances of winning to save some surplus for consumption in case of success.

But, the result critically depends on the restrictions which are imposed on the rule. One strand of the literature assumes that the leader can reward effort at most proportionally, i.e., there cannot be any transfers between individual members of the group. Another part of the literature does away with the assumption of proportional rewards by allowing such transfers between members. The two strands agree on the fact that at least one group will choose to put a positive weight on the egalitarian component. But practically, this seems to be unlikely behavior in the situation of a contest. So we impose restrictions on the rule which generalize the above literature, i.e., both strands of literature are special cases of our model. The restriction limits the amount competition that can be induced within a particular group. We treat it as a group specific norm of competition and consider it as a parameter in the model. Besides the generality such an assumption allows us, we are able to find precise conditions under which both groups try to maximize their chances of winning the contest by putting all the weight on the competitive component. We show that the result critically depends on the norm of the larger group. If the larger group has sufficiently egalitarian norms then cases arises, where both groups focus exclusively on winning the contest by putting all the weight on the competitive component. This is an important observation as trying to maximize chances of winning is the natural thing to do in a situation of conflict.

We also take up the question of welfare of the groups in the contest in the presence of such group specific norms. Specifically, we take up the question of when a larger group will fare worse, a phenomenon called Group Size Paradox (GSP) in the literature. We are able to provide precise conditions on the norms, which will lead to GSP.

Remark: We focus exclusively on analyzing the Nitzan rule in the first two chapters because the literature on general sharing schemes in collective contests is sparse. There are a few recent studies on general sharing schemes like Nitzan and Ueda (2014b), Trevisan (2020) and Kobayashi and Konishi (2020), but they all assume that the sharing scheme of a particular group is its private information, whereas we assume complete information about sharing rules. In fact, characterization of general sharing schemes when information is complete is still an open question.

Chapter 3

Bargaining for Assembly

In this chapter, we consider a situation where a single buyer has to bargain successfully with multiple sellers to implement a project. Examples of such situations include land assembly, production of new drugs etc. The inherent problem is such situations is that once the buyer reaches an agreement with all sellers except one, the remaining seller gets too much bargaining power and may demand too much. *Ex ante* all sellers have that power. Therefore, it may turn out to be the case that agreements involve significant delay or bargaining may simply not take off ,i.e., either efficient projects are not implemented or are implemented with significant delay. This problem is recognized as the problem of hold out in the bargaining literature.

In an important contribution to the literature on the hold out problem, Roy Chowdhury and Sengupta (2012) showed that such situations may lead to the buyer getting approximately zero surplus in equilibrium in absence of outside options. What underlies their stark result is the assumption of perfect complementarity between the inputs, that need to be acquired. We were skeptical of the assumption of perfect complementarity and try to weaken it.

In order to model different degrees of complementarity between inputs we propose special production processes. We assume sellers are located on a graph. The nodes are the sellers and an edge exits between two sellers if they own complementary inputs. The buyer needs to pick up a path of a certain size in the graph to implement the project. The existence of multiple such paths allows the possibility of input substitutability.

Using the same bargaining protocol as Roy Chowdhury and Sengupta (2012), we are able to show the importance of the assumption of perfect complementarity that was made in their paper. We show that unless there exist sellers who belong to every path, i.e., an indispensable seller, the buyer can extract full surplus from the sellers with minimal delay in equilibrium.

Chapter 2

Prize Sharing Rule in Collective Contests: Towards Strategic Foundations

2.1 Introduction

Collective contests are situations where agents organize into groups to compete over a given prize. Such situations are quite common: funds to be allocated among different departments of an organization, team sports, projects to be allocated among different divisions of a firm, regions within a country vying for shares in national grants, party members participating in pre-electoral campaigns, disputes between tribes over scarce resources.

Prizes in such contests may be purely private, e.g. money. Or the prizes may have some public characteristic like reputation or glory for the winning team. In this chapter we focus on purely private prizes. For prizes with public characteristics the reader may refer to Baik (2008), Balart et al. (2016).

One essential feature of collective contests is that a groups' performance depends on the individual contribution of its members. Departments in universities usually receive funds depending on the publication record of the department, which in turn depends on the individual publication of its members. So the group needs to coordinate and establish some rules regarding its internal organization, in particular how to share the prize in case of success in a contest. In this study we focus on two such important sharing rules. One such prize sharing rule, which was proposed by Nitzan (1991) suggests the following way of sharing the prize within the group in case of success:

$$(1-\alpha_i)\frac{x_{ki}}{X_i}+\alpha_i\frac{1}{n_i}, \tag{2.1}$$

where x_{ki} is the effort put in by the k^{th} member of group i, X_i is the total effort of group i and n_i is the size of group i. Further, α_i is weight put on egalitarian sharing of the prize within the group and $1-\alpha_i$ is the weight put on a sharing rule, which rewards higher efforts within the group, thereby inducing intra-group competition. We call this scheme prize sharing rule N. Rule N introduces intra-group externalities by making each member's reward depend on efforts of all other members of the group.

This prize sharing rule has been extensively studied in the literature on collective contests, see e.g. Flamand et al. (2015). The popularity of this rule lies in its intuitive appeal. It combines two extreme forms of internal organization, capturing the tension between intra-group competition and the tendency to free ride on efforts of other group members. Despite its popularity the rule is ad hoc. In this chapter we try to provide strategic foundations to these prize sharing rules N.

In order to do that, we introduce another rule E, which represents cooperative behavior within a group. According to this rule, the net expected group payoff is divided equally among all group members, thereby aligning individual and group interests. In other words, using rule E helps to internalize all intra-group externalities. It is defined as follows:

$$\frac{1}{n_i}(P_i(X_i,X_j)-X_i), \tag{2.2}$$

where $P_i(X_i,X_j)$ is the probability with which group i wins the prize and X_i is aggregate effort of the group i.

We consider a situation in which a group has access to these two prize sharing rules E and N. We construct a two stage game where the groups choose between the rules simultaneously in the first stage. The rules having been chosen, the individual group members simultaneously put in efforts in the second stage. The question we ask is whether this game has any subgame perfect Nash equilibrium in which rule N is chosen by any group.

We find that both groups choosing E always constitutes a subgame perfect Nash equilibrium in pure strategies. However, we also uncover a class of games, that we call Coordination games, in which both groups choosing N is also a subgame perfect Nash equilibrium in pure strategies.

The reason why such Coordination games arise is that, when the weight on intra-group competition is high enough in both groups, a situation of strategic uncertainty is created between the groups. In these cases rule N is a powerful instrument to increase chances of winning the contest. If a particular group chooses N, it generates high efforts and helps win the contest with a high probability. The other group should, in that case, choose N to increase its own efforts to counter the first group and keep its probability of winning from falling too much. The upward spiral in efforts comes at the cost of a vastly reduced net surplus [1], which harms both groups in terms of net payoffs.

In fact, we go on to show that the Nash equilibrium in which E is chosen payoff dominates the one in which both groups choose N. So it does not survive the equilibrium selection criterion of payoff dominance, as suggested in Harsanyi et al. (1988).

However, when we consider criteria of equilibrium selection, which are based on the "riskiness" of the equilibrium point, the results change. First, we consider the notion of risk dominance, as suggested in Harsanyi (1995). We are able to provide necessary and sufficient conditions for equilibrium profile NN to risk dominate EE. We go on to show the existence of such games by considering a special subclass of coordination games we call symmetric coordination games.

We also consider a equilibrium selection criterion called the Security Principle. According to it the players choose the strategy that maximizes their minimum possible payoff, see e.g.

[1] Surplus minus total efforts put in the contest

Van Huyck et al. (1990). We show that equilibrium profile NN is always selected by this criterion.

Even though different equilibrium selection criterion make different prescriptions, the fact that equilibrium NN is selected by some of them helps us establish that there exists a strategic basis to the prize sharing rules N introduced by Nitzan (1991).

The chapter is structured as follows. In Section 2 we discuss the relevant literature. In Section 3 we describe the model. In Section 4 we analyze the second stage of the game, where individuals make effort choices. In Section 5 we analyze the first stage of the game where the group leaders make their choice between E and N. In Section 6 we study the robustness of the equilibria to equilibrium refinement criteria of Payoff Dominance and Risk Dominance and the Security Principle. Section 7 contains a discussion of the results and things left out of the main body. Section 8 provides a few extensions of the basic model. Section 9 concludes. All proofs can be found in the Appendix 1.

2.2 Literature

The literature on prize sharing rules in collective contests started with an influential paper by Nitzan (1991). Thereafter, this class of rules have been widely applied to the analysis of group competition. The popularity of this class of rules owes to the fact that it very nicely captures effects of intra-group competition on the welfare of the groups in the collective contest. For an extensive survey the reader can look at Flamand et al. (2015).

These rules have been used to study two very important features of collective contests, *(a) Monopolization* and *(b) Group Size Paradox (GSP).*

In two group contests Davis and Reilly (1999) uses the term monopolization to refer to a situation where one group withdraws from the competition. Ueda (2002) extended the idea of monopolization to multi-group contests. In our analysis monopolization is possible but plays a supplementary role with regard to the main aim of this chapter.

Group Size Paradox (GSP) is a situation where a smaller group outperforms a larger one in terms of payoffs. The notion dates back to the seminal work by Olson (1965), who focused on the detrimental effects of free riding within large groups. Our focus not being on GSP, the interested reader is referred to Flamand et al. (2015).

There is an extensive literature on strategic choice of sharing rules under different restrictions on publicness of the prize and the sharing rule itself. One part of the literature (Baik (1994), Lee (1995), Noh (1999), Ueda (2002)) focuses on the case where the prize can be shared at most proportionally to individual contributions. Another part of the literature weakens this assumption (Baik and Shogren (1995), Lee and Kang (1998), Baik and Lee (1997), Baik and Lee (2001), Lee and Kang (1998), Gürtler (2005)) and allows transfers from worse performing group members to better performing group members. A recent strand of literature, (Nitzan and Ueda (2014a), Vázquez-Sedano (2014)) has studied cost sharing schemes with purely public prizes, where prize sharing is not possible.

There are a few other papers, which study the effect of publicness of the prize on group welfare. The purely public prizes case, where the prize sharing rules do not apply, has been analyzed by (Baik (1993), Baik (2008), Bag and Mondal (2014)). Esteban and Ray (2001) considers the case of a mixed private-pubic goods, with exogenous and fully egalitarian sharing rules, which was later endogenized in a private information framework in Nitzan and Ueda (2011). Balart et al. (2016) analyze the case of a mixed public-private prize with strategic choice of sharing rules in a complete information setting.

This chapter differs in focus from all the strands of literature cited above, in that it attempts to provide non-cooperative foundations to these prize sharing rules N instead of studying its effects on group welfare. We assume the prize to be fully private. Moreover, we do not allow any group the freedom to choose what weight to assign to different components of rule N. Instead, we provide the groups a strategic choice between an exogenous and intra-group non-cooperative prize sharing rule N and an intra-group cooperative prize sharing rule E and ask whether a group chooses rule N in any subgame perfect Nash equilibrium of an appropriately defined two stage game.

There are two papers, which analyze the choice between E and N, when both options are available. Cheikbossian (2012) questions the validity of GSP, by giving individual members of the groups a choice between N with $\alpha_i = 1$, which captures maximal internal non-cooperation and cooperative rule E. He goes on to show that it is easier to sustain E as a subgame perfect Nash equilibrium within the larger group, where the punishment used for a group member deviating from E is that other group members deviate to N thenceforth.

The focus of our chapter is different. We focus on how the presence of different options creates strategic uncertainty between the groups and why that may lead to N being chosen by both groups in equilibrium. In our model, individuals cannot deviate from the sharing rule chosen by their leaders. Cheikbossian (2012), on the other hand, focuses on the question of the ease of maintaining cooperation within a group, given that non-cooperative options are present for each individual member.

To the best of our knowledge, the only other paper that seeks to develop a strategic foundation for rule N is Ursprung (2012). He considers two groups of the same size. He gives the groups a choice among E, and the two extreme points of rule N, i.e. $\alpha_i = 0$ and $\alpha_i = 1$. He goes onto show that in an evolutionary game, N with $\alpha_i = 0$ crowds out E in the long run. In our model, there is no choice between different points of rule N. Also, groups can be of different sizes. Besides, our study does not take the evolutionary game route. Instead, we try to characterize which parts of rule N can arise in equilibrium of an appropriately constructed two stage game. As our study differs on important features from Ursprung (2012), our analysis can be considered to be complementary to theirs.

2.3 Model

There are two groups A and B, of size n_i , $i = \{A, B\}$, where $n_i \in \{2, 3, ...\}$. We assume without loss of generality that group B is at least as large as A, i.e. $n_B \geqslant n_A$. We denote the total number of agents as N, so that $N = n_B + n_A$. All agents are assumed to be risk neutral.

Both groups compete for a purely private prize, the size of which we normalize to 1. The groups cannot write binding contracts among themselves regarding sharing the prize. Instead they indulge in a rent-seeking Tullock contest spending efforts trying to win the contest. The outcome of this contest depends on the aggregate effort spent by the two groups. Let x_{ki} denote the effort level of individual k belonging to group i, where effort costs are $C(x_{ki})$. In particular $C(x_{ki}) = x_{ki}$. The aggregate effort of group i is $X_i = \sum_{k=1}^{n_i} x_{ki}$. The aggregate effort of the groups in the contest is denoted X, i.e., $X = X_1 + X_2$.

Efforts do not add to productivity, and only determine the probability $P_i(X_i, X_j)$ that group i wins the contest. We assume that $P_i(X_i, X_j)$ takes the ratio form, i.e.

$$P_i(X_i, X_j) = \begin{cases} \frac{X_i}{X_i + X_j}, & \text{if } X_i > 0 \text{ or } X_j > 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{2.3}$$

Every group has a leader, who has the authority to enforce a sharing rule that specifies how the spoils are to be shared within the group. Both leaders are benevolent, maximizing the expected group payoff while making their decisions.

The leaders can choose between two alternative sharing rules, either a *cooperative sharing rule* denoted E, or a *non-cooperative sharing rule* denoted N. We next turn to discussing these two rules.

■ **Cooperative Sharing Rule E:** The cooperative sharing rule E, introduced in (2.2), involves the group leader committing to share the net expected group payoff equally among all its members. Given $P_i(X_i, X_j)$ takes the ratio form in (2.3), that is equivalent to the leader committing to divide the surplus net of aggregate efforts, i.e., $1 - X$, equally among all members in case of success[2]. It is important to note that rule E implies that group effort levels X_i are contractible, i.e., verifiable. The expected net utility of member k of group i is

[2] $P_i(X_i, X_j) - X_i = \frac{X_i}{X_i + X_j} - X_i = \frac{X_i}{X_i + X_j}(1 - X_i - X_j) = P_i(X_i, X_j)(1 - X)$

as follows:

$$EU_{ki}(\mathrm{E}) = \frac{1}{n_i}(P_i(X_i, X_j) - X_i) = P_i(X_i, X_j)\Big(\frac{1 - X}{n_i}\Big). \tag{2.4}$$

Individual k in group i chooses effort x_{ki} to maximize equation (2.4).

As this scheme gives each member a fixed share in the net group payoff, each individual's interest gets aligned with group interest. Therefore, rule E allows the leader to implement the first best effort levels within the group. That is why we call the rule E *cooperative*. The equal sharing assumption is of course not necessary for perfect alignment of individual and group interests. Any asymmetric sharing scheme which gives all members a fixed positive share in the net group payoff will also work. We fix it at equal shares because it has natural appeal in a setting where all agents are symmetric. More importantly, the equal sharing assumption makes the leader a representative agent of his group, which makes concerns about his identity irrelevant.

■ **Non-cooperative Rule N:** The group leader can instead opt for the prize sharing rules introduced by Nitzan (1991). We denote this prize sharing rule by N. If group i leader chooses Rule N, then in case of success, the share of the k^{th} member of group i (s_{ki}) is as follows:

$$s_{ki}(x_{ki}, X_i; \alpha_i, n_i) = (1 - \alpha_i)\frac{x_{ki}}{X_i} + \frac{\alpha_i}{n_i}, \tag{2.5}$$

where $\alpha_i \in [0, 1]$. α_i is fixed for a group and cannot be manipulated by the leaders [3]. N is feasible as $\sum_{k \in n_i} s_{ki} = 1$. It is important to note that rule N implies that only the ratio of individual to group efforts, i.e., $\frac{x_{ki}}{X_i}$, needs to be verifiable [4].

Note that this rule is a weighted average of an egalitarian component $\frac{1}{n_i}$ and a competitive component $\frac{x_{ki}}{X_i}$. The egalitarian part tends to reduce group effort because individual members of a group free ride on effort provision, given that his share is independent of his efforts. The

[3] We endogenize the choice of α_i in Chapter 2

[4] One way to look at our problem is when the leader makes a choice between E and N he cannot see anything. If he chooses E then X_i becomes verifiable and if he chooses N then $\frac{x_{ki}}{X_i}$ becomes verifiable. He cannot see both X_i and $\frac{x_{ki}}{X_i}$. So the problem can be imagined to be one of endogenous verifiability.

competitive component, on the other hand, tends to increase group efforts because individual members compete internally to get a larger share of the prize in case of success.

It should be noted that a change in group efforts has two countervailing effects. On the one hand, an increase in groups efforts increases the chances that the group wins the contest. On the other hand, higher group efforts also dissipates the prize leaving a lower *ex post* surplus.

This is the trade off, which the literature on strategic choice of prize sharing rules focuses on, see e.g. Flamand et al. (2015). While abstracting from this trade-off in this chapter by fixing the weights α_i, we focus on a qualitatively similar trade-off which is generated when the groups choose between E and N.

When group i leader chooses N, individual k in group i chooses efforts x_{ki} to maximize his expected utility, which is as follows:

$$EU_{ki}(N) = \begin{cases} s_{ki}(x_{ki}, X_i; \alpha_i, n_i) P_i(X_i, X_j) - x_{ki} & \text{if } X_i > 0, X_j \geqslant 0, \\ \frac{1}{2n_i} & \text{if } X_i = X_j = 0, \\ 0 & \text{if } X_i = 0, X_j > 0. \end{cases} \tag{2.6}$$

■ **Leader's Objective:** Recall that the leader of both groups are benevolent social planners. The strategy of the leader of group i is denoted $\sigma_i \in \{E, N\}$, $i \in \{A, B\}$. The leader chooses σ_i, i.e., either the cooperative rule E or non-cooperative rule N, to maximize the net group payoffs. The maximization problem of leader of group i is as follows:

$$\max_{\sigma_i \in \{E,N\}} P_i\Big(X_i(\sigma_i, \sigma_j), X_j(\sigma_i, \sigma_j)\Big)\Big(1 - X(\sigma_i, \sigma_j)\Big) \tag{2.7}$$

where $X(\sigma_i, \sigma_j) = X_i(\sigma_i, \sigma_j) + X_j(\sigma_i, \sigma_j)$.

The payoff representation in equation (2.7) is intuitive, and captures the trade-off inherent in the group leader's maximization problem. X measures the amount of prize dissipated in the competition between the two groups. Therefore, $1 - X$ is the surplus net of efforts, which remains for *ex post* consumption. The probability with which group i wins this net surplus is $P_i(X_i, X_j)$. If leader of group i wants to win the contest with a higher probability she has to take measures, which increase group efforts X_i. But when X_i goes up so does X, which reduces the size of the net surplus.

■ **Description of the Game:** Our game consists of two stages. In the first stage the two leaders simultaneously choose between E and N. Having observed the choice of the sharing rules, in stage two all agents simultaneously decide on their own effort levels.

We denote an equilibrium strategy profile of the game $\sigma^* = (\sigma_A^*, \sigma_B^*)$.

We solve for the Subgame Perfect Nash equilibrium (SPNE) of the game described above.

2.4 Choice of Individual Efforts

In this section we characterize the Nash equilibrium effort choices of individual members of the groups taking as given the sharing rules, which are chosen by the group leaders in the first stage.

Before stating the results we define the phenomenon of Monopolization of a group in the contest, which is well recognized in the collective contest literature, see e.g. Davis and Reilly (1999).

Definition 1 ***Monopolization***

A SPNE $\langle \sigma_A^*, \sigma_B^* \rangle$ is said to involve monopolization of group i, if group i does not put in any effort in the contest.

Convention: In what follows we denote generic efforts as X_A and X_B. But when we talk about equilibrium efforts, surpluses and probabilities of winning we use superscripts. We fix the first component of the superscripts to be the strategy chosen by group A and the second component to be the strategy chosen by group B in the first stage.

2.4.1 Equilibrium Net Surplus and Probabilities of Success

In the following proposition we report only the surplus net of effort S, which remains for consumption, i.e. $S = 1 - X$, and the probabilities with which each group wins the net surplus, P_i and P_j. Such a choice was made to keep the discussion in line with the basic trade-off in the model. In the Appendix 1 we provide all the details. Before proceeding we introduce the following notation:
For $i, j \in \{A, B\}$ and $j \neq i$ we define

$$\chi_i = n_i + n_i(n_j - 1)\alpha_j - n_j(n_i - 1)\alpha_i. \tag{2.8}$$

χ_i can be interpreted as a measure of the competitiveness of group i relative to group j. In fact, when both groups choose N, the probability that group i wins the contest P_i is directly proportional to χ_i. Note that χ_i is increasing in α_j and decreasing in α_i. When α_j is large relative to α_i, group j is relatively less competitive, which gives group i an advantage in the contest. On the other hand when α_i large relative to α_j, group j wins the contest more often.

In Proposition 1 we report the net surplus and probabilities of winning in an equilibrium of the second stage of our game. For features of the best response functions the readers are encouraged to go to Appendix 2. There we do a detailed analysis of individual and aggregate best response functions and analyze when aggregate efforts are strategic substitutes and when they are strategic complements.

Proposition **1**

(A) If both groups choose E then in any Nash equilibrium of the effort subgame

(a) The net surplus in the contest is $S^{EE} = \frac{1}{2}$.

(b) The probabilities of winning are $(P_i^{EE}, P_j^{EE}) = (\frac{1}{2}, \frac{1}{2})$.

(B) If group i chooses E and group j chooses N, $i, j \in \{A, B\}$ and $j \neq i$, then in any Nash equilibrium of the effort subgame

(a) The net surplus in the contest is $S^{\sigma_A \sigma_B} = 1 - \frac{1+(1-\alpha_j)(n_j-1)}{n_j+1}$.

(b) The probabilities of winning are $(P_i^{\sigma_A \sigma_B}, P_j^{\sigma_A \sigma_B}) = (\frac{1+\alpha_j(n_j-1)}{(n_j+1)}, 1 - \frac{1+\alpha_j(n_j-1)}{(n_j+1)})$.

(C) If both groups choose N *then*

(1) If $\chi_i \leqslant 0$, $i, j \in \{A, B\}$ *and* $j \neq i$ [5], *then group* i *is monopolized by group* j. *In the unique intra-group symmetric Nash Equilibrium of the effort subgame*

(a) The net surplus in the contest is $S^{NN} = 1 - \frac{(1-\alpha_j)(n_j-1)}{n_j}$.

(b) The probabilities of winning are $(P_i^{NN}, P_j^{NN}) = (0, 1)$.

(2) If $\chi_i > 0$ *and* $\chi_j > 0$, $i, j \in \{A, B\}$ *and* $j \neq i$, *then neither group is monopolized and in the unique intra-group symmetric Nash equilibrium of the effort subgame*

(a) The net surplus in the contest is $S^{NN} = 1 - \frac{1+(1-\alpha_i)(n_i-1)+(1-\alpha_j)(n_j-1)}{N}$.

(b) The probabilities of winning are $(P_i^{NN}, P_j^{NN}) = (\frac{\chi_i}{N}, 1 - \frac{\chi_i}{N})$.

We next discuss the results summarized in Proposition 1.

■ **Both groups choose E:** When both groups choose E in the first stage, there exists a continuum of Nash equilibria in individual efforts in all of which $X^{EE} = \frac{1}{2}$ and so the net surplus is $S^{EE} = \frac{1}{2}$. Both groups win with equal probabilities $P_i^{EE} = P_j^{EE} = \frac{1}{2}$. Therefore, $X_i^{EE} = X_j^{EE} = \frac{1}{4}$, but the individual effort choices can be asymmetric. Given the fact that aggregate effort choices are all that matters, we find that the equilibrium levels of aggregate efforts are independent of group sizes. We will treat this case as our benchmark for comparison as it represents full cooperation within both the groups.

■ **Group i chooses E, group j chooses N:** Here we analyze the individual effort choices of group members when group i has chosen E and group j has chosen N in the first stage. For ease of exposition, let us assume that group $i = A$ and $j = B$. Just as in the benchmark case, the individual effort choices in the Nash equilibrium is not unique but the aggregate efforts X_A^{EN} and X_B^{EN} are. The Nash equilibrium levels of net surplus S^{EN} and the probability of

[5]If $\chi_i \leqslant 0$ then $\chi_j > 0$ as $\chi_i + \chi_j = N$

group A winning, P_A^{EN} are stated in Proposition 1. In Figure 2.1, we make a comparison to the benchmark case.

The total effort X^{EN} monotonically decreases and net surplus S^{EN} monotonically increases in α_B, equaling the benchmark level of $\frac{1}{4}$ at $\alpha_B = \frac{1}{2}$. For $\alpha_B > \frac{1}{2}$ aggregate effort costs X^{EN} is lower compared to the benchmark case, and hence the net surplus, S^{EN} is higher.

On the other hand, the probability that group A wins the contest, P_A^{EN}, monotonically increases in α_B, equaling the benchmark level at $\alpha_B = \frac{1}{2}$. As α_B rises, free riding increases within group B, thereby not only creating a larger net surplus but also reducing the probability that group B wins the contest.

■ **Both groups choose N:** When both groups choose N in the first stage, we may have Monopolization of one group by the other, in that the equilibrium effort level of the other group is zero,(see Figure 2.2). It is clear that the probability with which group i wins the contest is zero when $\chi_i \leqslant 0$, which happens when α_i is large relative to α_j.

We now focus on the more interesting case, where neither group is Monopolized, which happens when $\chi_i > 0$. From Proposition 1,

The net surplus $S^{NN} > \frac{1}{2}$ if:

$$(n_i - 1)(1 - 2\alpha_i) + (n_j - 1)(1 - 2\alpha_j) < 0 \tag{2.9}$$

whereas probability that group i wins $P_i^{NN} > \frac{1}{2}$ if:

$$\chi_i > \frac{N}{2} \tag{2.10}$$

The equations are represented in Figure 2.3. For relatively low levels of both α_A and α_B the effort expended in the contest is more than the benchmark level of $\frac{1}{2}$, which makes the net surplus less than $\frac{1}{2}$. The probability of group i winning is lower the closer we are to the line where it is monopolized.

The total effort X^{NN} is monotonically decreasing and the net surplus S^{NN} is monotonically increasing in both α_A and α_B. When α_A goes up free riding goes up within group A reducing the total effort put in the contest, thereby increasing the net surplus. Similarly for α_B.

The probability that group i wins, P_i^{NN}, goes up as α_j rises as free riding goes up within group j. But, P_i^{NN} falls with α_i, as now there is more free riding among its own members.

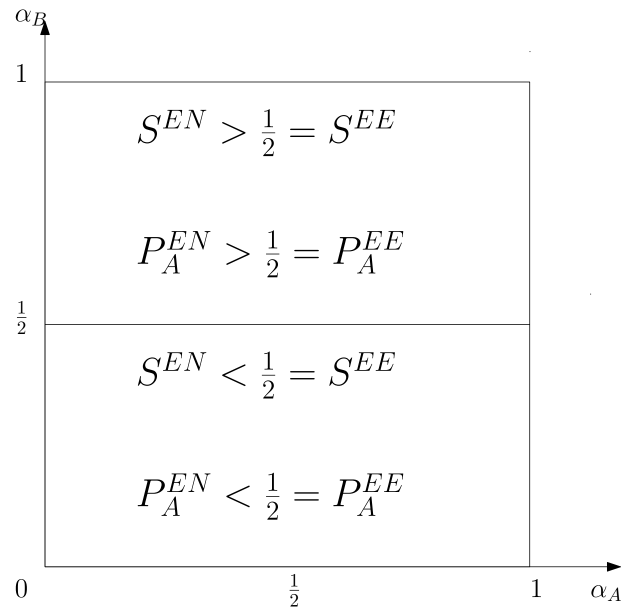

Figure 2.1: Comparison of EN to EE

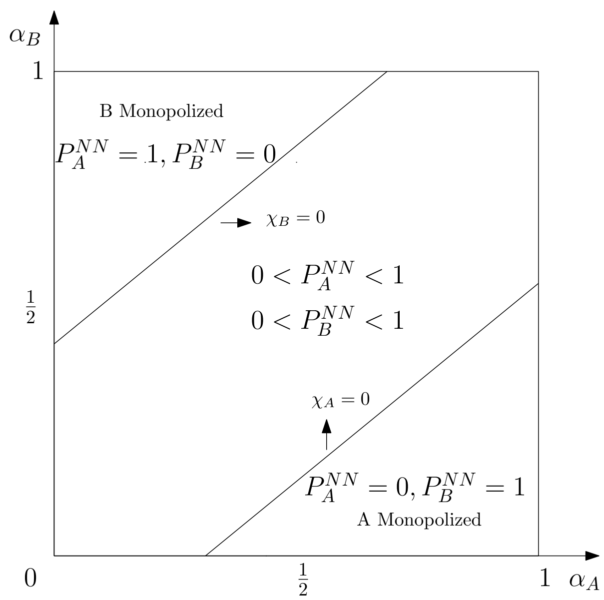

Figure 2.2: Probabilities of winning under NN

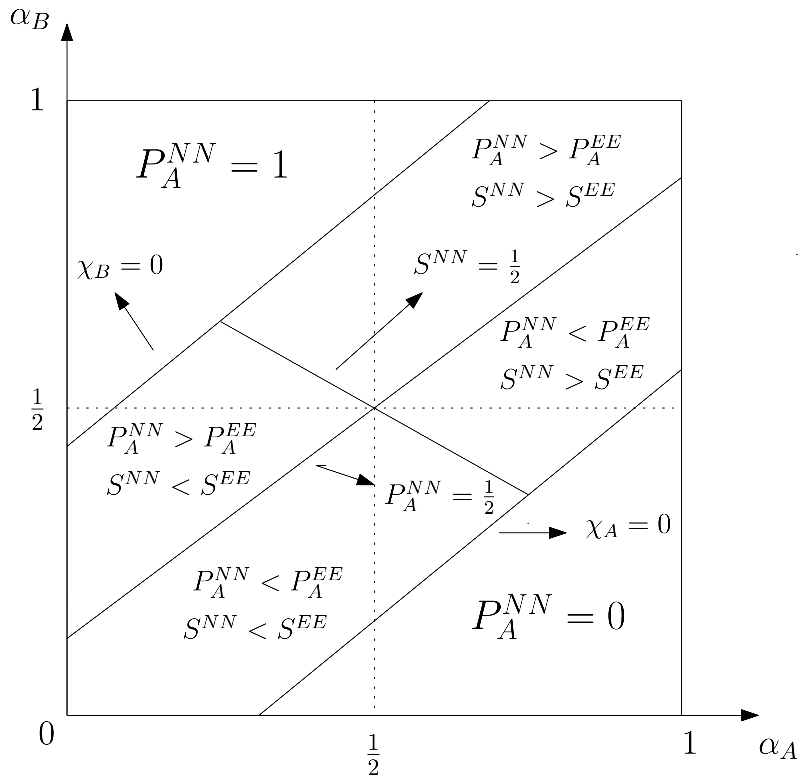

Figure 2.3: Comparison of NN to EE

2.4.2 Group Payoff Functions

In the previous subsection we analyzed properties of the equilibrium in the second stage of our game, specifically focusing on the associated net surplus and the probabilities of winning. But given that we are primarily interested in group payoffs instead of its individual components, we next we analyze what happens to the group payoffs when the parameters in the model are changed.

As mentioned at the beginning under any strategy profile the payoff of group i can be expressed as follows:

$$\Pi_i = P_i S \tag{2.11}$$

where P_i is the probability with which group i wins the contest and S is the surplus net of efforts of the groups.

So, the growth rate of group payoffs with respect to a particular parameter, will just be the sum of the growth rate of the probability of winning and the growth rate of the net surplus with respect to that parameter. Suppose we change parameter K, then the following will be true

$$g_K^{\Pi_i} = g_K^{P_i} + g_K^S \tag{2.12}$$

where $g_K^Y = \frac{1}{Y}\frac{dY}{dK}$, for any variable Y.

In the previous subsection we analyzed $\frac{dP_i}{d\alpha_i}$ and $\frac{dS}{d\alpha_i}$. Here we analyze the composition of the two effects when α_i is changed. Given that there exists a trade-off between P_i and S, analyzing the composition of the two separate growth rates helps us pin down the growth rate of group payoffs. Obviously, the growth rate of group payoffs will be of the same sign as $\frac{d\Pi_i}{d\alpha_i}$.

Changing α_i

Here we will change α_A and α_B and see how it affects group payoffs. The following Proposition contains the information.

Before stating the proposition we introduce the following notation:

$$\alpha_B^o = \frac{(n_B - n_A)(1 + \alpha_A(n_A - 1))}{2n_A(n_B - 1)} \tag{2.13}$$

α_B^o is the value of α_B, which maximizes the payoff of group B, Π_B^{NN}.

Proposition **2**

(A) If group i chooses E and group j chooses N, $i, j \in \{A, B\}$ and $j \neq i$, then

- *(a) $\Pi_j^{\sigma_A \sigma_B}$ is strictly increasing (decreasing) in α_j iff $\alpha_j < (>)\frac{1}{2}$ and achieves global maximum at $\alpha_j = \frac{1}{2}$.*
- *(b) $\Pi_i^{\sigma_A \sigma_B}$ is strictly increasing in α_j.*

(B) If both groups choose N and neither group is monopolized, then

(a) Π_A^{NN} *is strictly decreasing in* α_A.

(b) Π_A^{NN} *is strictly increasing in* α_B.

(c) Π_B^{NN} *is strictly increasing in* α_A.

(d) Π_B^{NN} *is strictly increasing (decreasing) in* α_B *iff* $\alpha_B < (>)\alpha_B^o$ *and achieves global maximum at* $\alpha_B = \alpha_B^o$.

■ **Group A chooses E, Group B chooses N:**

▶ **Case 1:** $\alpha_B < \frac{1}{2}$.

In this case the payoffs of the groups depend only on α_B. When $\alpha_B < \frac{1}{2}$, we have $P_B^{EN} > S^{EN}$, so that the base probability of winning for group B is higher than the base net surplus.

It is also true that X_A and X_B are strategic substitutes in this case [6]. An increase in α_B reduces X_B^{EN} as free riding increases within group B. But, X_A^{EN} increases as the strategies are substitutes. This causes X_B^{EN} to fall farther. The net surplus S^{EN} rises as X_B^{EN} falls more than X_A^{EN} rises, thereby reducing aggregate efforts X^{EN}.

As X_A^{EN} increases so does the probability of winning for group A, P_A^{EN}. As the growth rates of both S^{EN} and P_A^{EN} are positive, Π_A^{EN} is increasing with α_B.

The payoff of group B, Π_B^{EN}, also rises in this case as the base probability of winning P_B^{EN} is quite high and S^{EN} is low to start with. So, the growth in S^{EN} dominates the deceleration in probability of success P_B^{EN}, causing group B payoffs to increase with α_B.

▶ **Case 2:** $\alpha_B > \frac{1}{2}$.

In this case, we have $P_B^{EN} < S^{EN}$, so that the base net surplus higher than the base probability of winning for group B.

As α_B rises, X_B^{EN} falls due to increased free riding in group B. But, X_A^{EN} also declines as X_A is a strategic complement to X_B. But, X_B^{EN} falls more than X_A^{EN}, so that P_A^{EN} is still increasing. Again, as the growth rates of both S^{EN} and P_A^{EN} are positive, Π_A^{EN} keeps on increasing with α_B

[6] See Appendix 2

For group B, on the other hand, the deceleration in P_B^{EN} is now more than positive growth the in net surplus S^{EN}, by the base effect. So, the payoff of group B declines as α_B increases.

■ **Both groups choose N:** In this case it is easier to clarify part (b) and (c) of the proposition. As α_B rises, S^{NN} rises and so does P_A^{NN}. The growth rates of both are positive and so Π_A^{NN} also grows with α_B. Similarly, as α_A goes up, Π_B^{NN} is increasing.

To understand part (a) of the proposition, notice that as α_A goes up so does S^{NN}. Therefore, the growth rate of the net surplus, S^{NN}, is positive. But, the growth rate of P_A^{NN} is negative when α_A rises. Given that group A is the smaller group, when α_A increases, a small number of agents reduce their efforts, causing a minute growth of net surplus. However, decreased efforts contribute more to a reduction of the group's chances of victory. So, the growth rate in net surplus is always outdone by the slowdown in winning probabilities for group A. So, Π_A^{NN} is decreasing in α_A.

To understand part (d), notice that when α_B goes up, S^{NN} goes up but P_B^{NN} falls. When, $\alpha_B < \alpha_B^o$, the growth rate of net surplus dominates the deceleration in chances of winning for group B. This happens because, at such a low level of α_B the larger group B is also very competitive. It generates a lot of effort X_B^{NN}, causing a lot of the rent to be dissipated. This makes the base net surplus S^{NN} lower than the base P_B^{NN} in this case. When α_B rises, the growth rate in net surplus dominates the deceleration in probability of winning due to a lower base. So, the payoffs of group B is rising here.

When, $\alpha_B > \alpha_B^o$, the bases switch and therefore the deceleration in probabilities of winning dominates the growth in net surplus and the payoffs of group B start to fall.

2.5 Choice of Sharing Rules by Group Leaders

In this section we consider the choice made by the group leaders in the first stage. Given the effort choices made by individual group members in the second stage, the group leaders play a normal form game in the first stage. A strategy profile is a Nash equilibrium of the normal

		Group B	
		E	N
Group A	E	Π_A^{EE}, Π_B^{EE}	Π_A^{EN}, Π_B^{EN}
	N	Π_A^{NE}, Π_B^{NE}	Π_A^{NN}, Π_B^{NN}

Table 2.1: Game Γ

form game, if both leaders choose strategies, which maximize (2.7), taking the other groups strategy choice as fixed.

Given any configuration of parameters $(\alpha_A, \alpha_B, n_A, n_B)$, we have a normal form game we denote $\Gamma(\alpha_A, \alpha_B, n_A, n_B)$. We denote the set of all such normal form games Γ. Games in Γ are bi-matrix games as represented in Table 2.1.

Proposition **3**

Consider any game $G \in \Gamma$. EE is a pure strategy Nash equilibrium of G.

This result is quite convenient and serves as a benchmark for us. The fact that E constitutes mutual best responses means that the only way we can generate N as a part of a Nash Equilibrium of any $G \in \Gamma$ is when both groups choose N, which takes the structure of a Coordination Game. To prove that EE is a Nash Equilibrium we have to show that

For $i, j \in \{A, B\}$ and $j \neq i$ and $\forall G \in \Gamma$

$$\Pi_i^{\sigma_A \sigma_B}(\sigma_i = N, \sigma_j = E) \leqslant \Pi_i^{EE}$$

Using Proposition 1 the inequality can be written as follows:

$$\left(\frac{1 + (1-\alpha_i)(n_i - 1)}{n_i + 1}\right)\left(1 - \frac{1 + (1-\alpha_i)(n_i - 1)}{n_i + 1}\right) \leqslant \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \tag{2.14}$$

where the first term in brackets is the probability that group i wins the contest P_i and the second term in brackets is the net surplus $1 - X$. But (2.14) follows directly from part (A) of Proposition 2 and the fact that $\Pi_i^{EE} = \Pi_i^{EE} = \frac{1}{4}$.

When group j chooses E, group i can guarantee a payoff of $\frac{1}{4}$ by responding with E. At profile EE, the net surplus is $S^{EE} = \frac{1}{2}$ and each group wins it with $P_i^{EE} = P_j^{EE} = \frac{1}{2}$. On the other hand, if group i responds with N it can get a maximum of $\frac{1}{4}$ when $\alpha_i = \frac{1}{2}$. Otherwise, it gets a lower payoff. Therefore E is always a best response for group i when group j plays E. Look at Figure 2.4, where we plot Π_i^{EE} and $\Pi_i^{\sigma_A \sigma_B}(\sigma_i = N, \sigma_j = E)$.

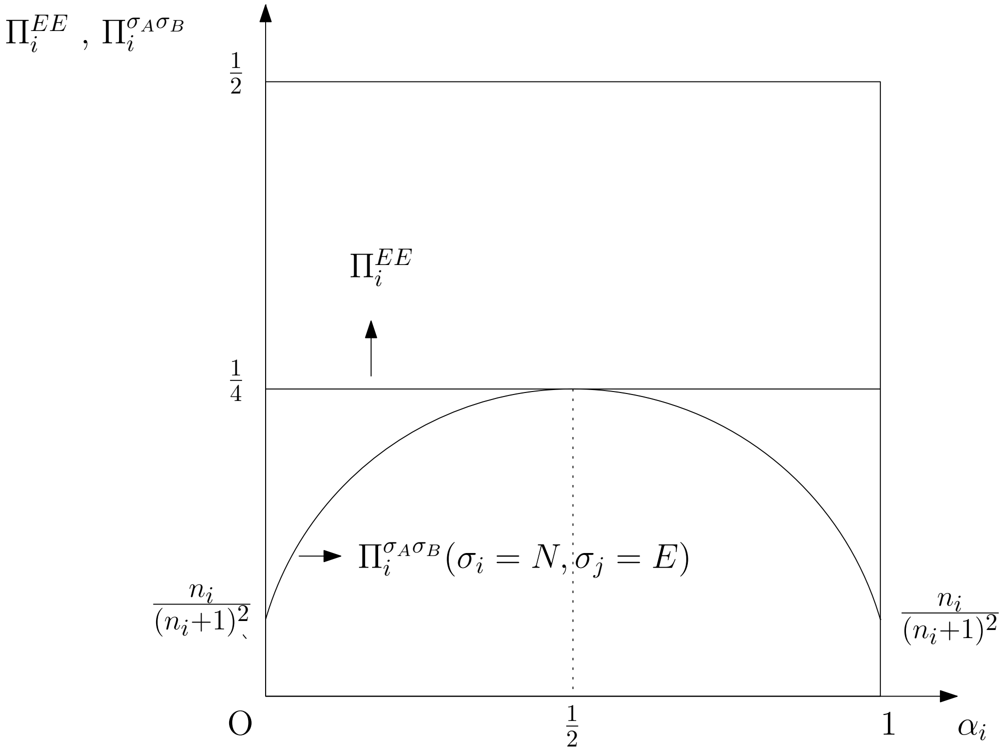

Figure 2.4: Payoff Comparison of EE and EN

► **Case 1:** $\alpha_i < \frac{1}{2}$

Consider $i = A$ and $j = B$. In this case, we know that $P_A^{NE} > S^{NE}$. We also know that $P_A^{NE} > P_A^{EE} = \frac{1}{2}$ and $S^{NE} < S^{EE} = \frac{1}{2}$, so that group A gets a larger share of a smaller net surplus. As, X_A and X_B are strategic substitutes in this case, as α_A increases, X_A falls and X_B increases. S^{NE} increases but P_A^{NE} falls. This means that the incremental net surplus, which is a public good created by a reduction in efforts by group A, is mostly captured by group B. Even, though the payoff of group A is increasing due to a lower base S^{NE}, choosing

N cannot be an optimal response because group A could switch to E, where both groups contribute equally to the net surplus and take away an equal share of it.

▶ **Case 2:** $\alpha_i > \frac{1}{2}$

Consider $i = A$ and $j = B$. In this case, we know that $P_A^{NE} < S^{NE}$. It is also true that $P_A^{NE} < P_A^{EE} = \frac{1}{2}$ and $S^{NE} > S^{EE} = \frac{1}{2}$. Here, as α_A increases X_A falls but so does X_B as it is strategic complement to X_A. But X_A falls more and P_A^{NE} keeps on decreasing. So, again group A gets a smaller share of the public good it largely creates. It would be better for group A to switch to E, and get an equal share in a lower net surplus, which both groups have contributed to equally.

Given that EE is a Nash equilibrium of any $G \in \Gamma$, we need to check when games in Γ also have as Nash equilibrium the strategy profile NN.

DEFINITION **2** ***Coordination game***

Consider any game $G \in \Gamma$. G will be called a Coordination game iff $\Pi_A^{EE} > \Pi_A^{NE}$, $\Pi_A^{NN} > \Pi_A^{EN}$, $\Pi_B^{EE} > \Pi_B^{EN}$ and $\Pi_B^{NN} > \Pi_B^{NE}$. The set of Coordination games is denoted Γ^C.

For $i = A, B$ and $j \neq i$, we introduce the following notations:

$$\overline{\alpha}_i = \frac{1 + \alpha_j(n_j - 1)}{n_j + 1}, \tag{2.15}$$

and

$$\underline{\alpha}_i = \frac{(1 + \alpha_j(n_j - 1))(n_i - n_j^2)}{n_j(n_j + 1)(n_i - 1)}, \tag{2.16}$$

where $\overline{\alpha}_i$ is the larger and $\underline{\alpha}_i$ is the smaller root of the following quadratic equation [7]

$$\Pi_i^{\sigma_A \sigma_B}(\sigma_i = E, \sigma_j = N) = \Pi_i^{NN}$$

We are now in a position to state and analyze the main result of the chapter. Proposition 4 confirms the existence and helps us clearly identify the Coordination games we are looking

[7] Notice that $\underline{\alpha}_i = \frac{n_i - n_j^2}{n_j(n_i-1)}\overline{\alpha}_i$. So the roots are multiples of each other, i.e., $\underline{\alpha}_i = C\overline{\alpha}_i$, where $C < 1$.

for. This result helps us establish strategic foundations of the prize sharing rules N, which have been subjected to extensive analysis in the collective contests literature, see e.g. Flamand et al. (2015).

Proposition 4

Consider any game $G \in \Gamma$

(A) EE and NN are pure strategy Nash equilibria of G iff $\alpha_A \in [0, \overline{\alpha}_A]$ and $\alpha_B \in [max\{0, \underline{\alpha}_B\}, \overline{\alpha}_B]$.

(B) Otherwise, G is dominance solvable and EE is its unique pure strategy Nash equilibrium.

This is the main result of this chapter. We have been able to show, that there exist games $G \in \Gamma$ such that NN is a Nash equilibrium outcome, thereby providing strategic foundations to the prize sharing rules N.

G belongs to the set of Coordination games Γ^C when $\alpha_A \in [0, \overline{\alpha}_A)$ and $\alpha_B \in (\underline{\alpha}_B, \overline{\alpha}_B)$ if $\underline{\alpha}_B \geqslant 0$. On the other hand, when $\underline{\alpha}_B < 0$ then G belongs to the set of Coordination games Γ^C, if $\alpha_A \in [0, \overline{\alpha}_A)$ and $\alpha_B \in [0, \overline{\alpha}_B)$. Under the conditions specified above N is a strict best response to N for both the groups and hence satisfies the requirements for any $G \in \Gamma$ to be a Coordination game.

If $\alpha_A = \overline{\alpha}_A$, then N is a weak best response to N for group A. The cooperative strategy E weakly dominates the non-cooperative strategy N for group A. This follows from Proposition 3. Similarly, E weakly dominates N for group B, when $\alpha_B = \overline{\alpha}_B$ or $\alpha_B = \underline{\alpha}_B$, in case $\underline{\alpha}_B$ is positive [8]. Even though we can see in Part (A) of Proposition 4 that both EE and NN are Nash equilibria in such cases [9], we ignore them while considering Coordination games Γ^C as they are defined to have N as a strict best response to N for both groups.

To check when N is a best response to N for group i we need to check the following inequality:

[8] Of course in these cases equilibrium NN will be lost if we apply Iterated Elimination of Weakly Dominated Strategies (IEWDS)

[9] It can also be easily verified that $\overline{\alpha}_A$ and $\overline{\alpha}_B$ intersect at $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$. At $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$ all strategy profiles, i.e., EE, NN, EN and NE are Nash equilibria of Γ.

For $i, j \in \{A, B\}$ and $j \neq i$

$$P_i^{NN} S^{NN} \geqslant [P_i^{\sigma_A \sigma_B} S^{\sigma_A \sigma_B}](\sigma_i = E, \sigma_j = N) \tag{2.17}$$

It can be easily verified that $S^{NN} > S^{\sigma_A \sigma_B}(\sigma_i = E, \sigma_j = N)$ iff $\alpha_i > \overline{\alpha}_i$. Similarly, it can be verified that $P_i^{NN} > P_i^{\sigma_A \sigma_B}(\sigma_i = E, \sigma_j = N)$ iff $\alpha_i < \overline{\alpha}_i$. At, $\alpha_i = \overline{\alpha}_i$, the strategies N and E are equivalent for group i both in terms of net surplus and probabilities of winning the contest.

Let us first consider group A and refer to Figure 2.5. Let us start from $\alpha_A = \overline{\alpha}_A$, where $\Pi_A^{NN} = \Pi_A^{EN}$. Now from Proposition 2 we know that Π_A^{NN} is strictly decreasing in α_A. So, starting from, $\alpha_A = \overline{\alpha}_A$, if we reduce α_A, then Π_A^{NN} will strictly increase, while Π_A^{EN}, being independent of α_A, will remain unchanged. Given that the smaller root $\underline{\alpha}_A$ is negative, it follows that for all $\alpha_A \in [0, \overline{\alpha}_A)$, N is a strict best response to N for group A.

The story for group B is slightly different and can be seen in Figures 2.6 and 2.7. If $\alpha_B = \overline{\alpha}_B$, then $\Pi_B^{NN} = \Pi_B^{NE}$. From Proposition 2 we know that Π_B^{NN} is decreasing in α_B if $\alpha_B > \alpha_B^o$. So, starting from $\alpha_B = \alpha_B^1$, if we reduce α_B, Π_B^{NN} first increases upto α_B^o and then decreases. Π_B^{NE}, being independent of α_B is unchanged. Given that Π_B^{NN} decreases when we reduce α_B below α_B^o, gives rise to the possibility that the smaller root of $\Pi_B^{NN} = \Pi_B^{NE}$, which we denote $\underline{\alpha}_B$, is positive.

It can be easily verified that $\underline{\alpha}_B$ is negative when $n_B < n_A^2$. So in this case N is a strict best response to N for group B when $\alpha_B \in [0, \overline{\alpha}_B)$ (see Figure 2.6).

In the other case, when $n_B \geqslant n_A^2$, the smaller root $\underline{\alpha}_B$ is non negative and N is a strict best response to for group B when $\alpha_B \in (\underline{\alpha}_B, \overline{\alpha}_B)$. This is captured in Figure 2.7.

In Figures 2.8 and 2.9 we represent the Coordination games for the two different cases in the $\alpha_A \alpha_B$ plane. The case, where $\underline{\alpha}_B$ is negative is captured in Figure 2.8. The case where $\underline{\alpha}_B$ is non-negative is captured in Figure 2.9. The Coordination games are marked in blue.

■ **Intuition:** To see why NN turns out to be a Nash equilibrium when $G \in \Gamma^C$, we have to understand how presence of the non-cooperative rule N creates a situation of strategic uncertainty for the groups. The main feature of this rule N is that it allows the groups a chance to enhance its probability of winning at the expense of the other group, when $G \in \Gamma^C$. Even, though the net surplus is lower a group wins with a higher chance by choosing N. If both groups believe that the other is going to choose N to increase its chances of winning the contest, both end up choosing N, so as not to give up a substantial winning advantage to the other group. Of course, coordinating on NN comes at the cost of a substantially reduced net surplus.

For example, consider the case where group B chooses N. If group A were to choose E, then it gives up the option of increasing its chances of winning the contest. If α_B is sufficiently low, then group B puts in a lot of effort and wins with a very high probability a net surplus, which is lower. But, group A has no way to counter group B. However, if group A were to respond with N, then it would be able to stop its probability of winning from falling too much. The result is similar in flavor to the results in Baliga and Sjöström (2004) and Baliga and Sjöström (2008).

Therefore, in the race to keep its probability of winning high, a group may choose N if it believes the other group will also do so. These kind of perverse incentives of groups results from the fact the net surplus behaves exactly like a public good between the groups, leading to free riding on its maintenance by both groups. Instead, both groups have an incentive to increase their winning chances by putting in more effort. Therefore, if one group believes that the other is trying to enhance its chances of winning by choosing N, it should respond by doing the same to maintain parity. Given efforts eat into the prize, none of the groups ideally want to end up in this spiral of higher efforts. But, given the strategic uncertainty embodied in the normal form game $G \in \Gamma^C$, NN turns out to be an equilibrium outcome. This result essentially has the flavor of a failure to coordinate on the Pareto efficient outcome EE.

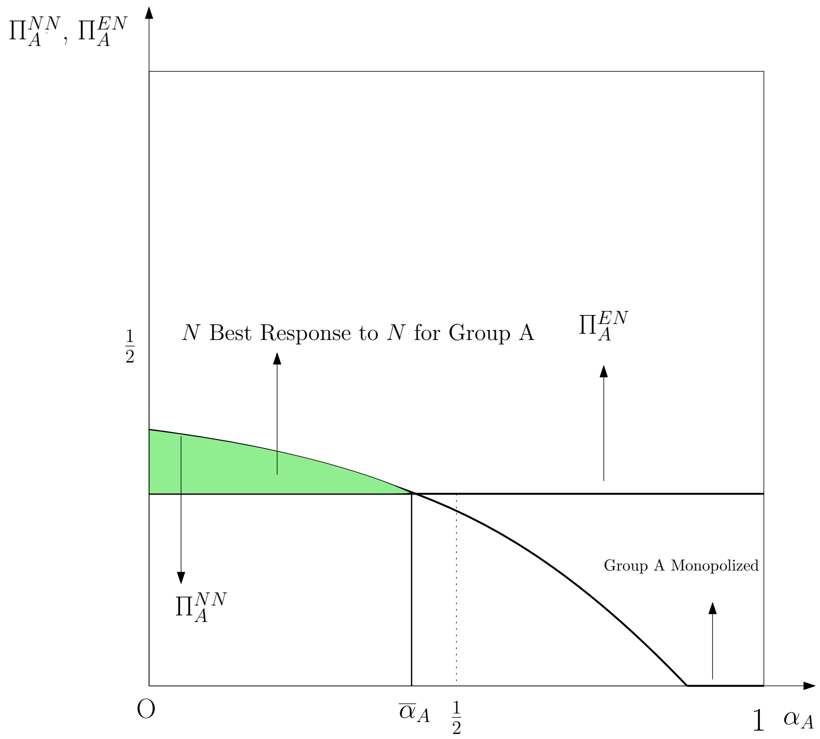

Figure 2.5: N best response to N for group A

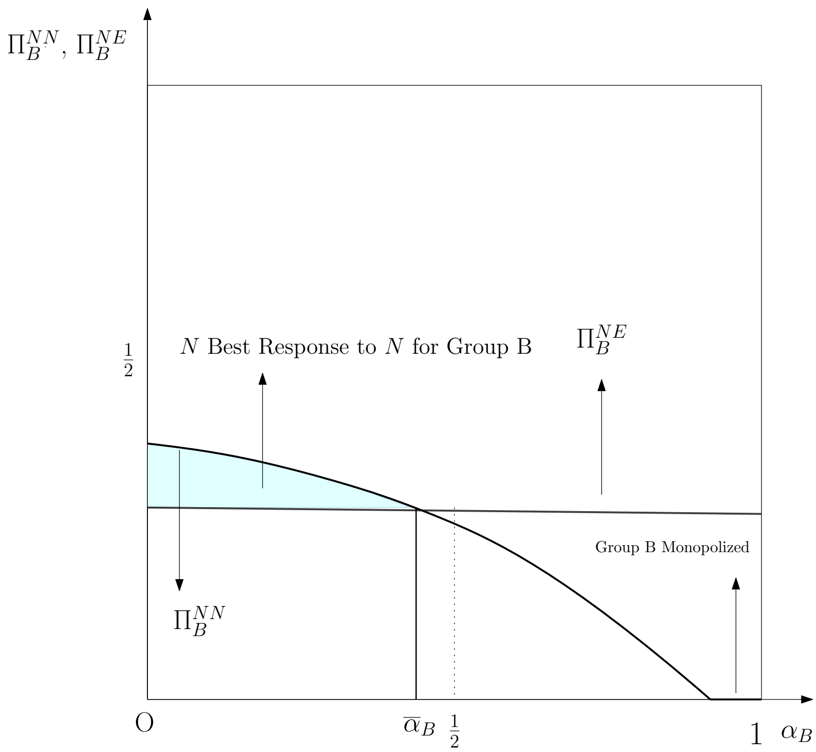

Figure 2.6: N best response to N for group B when $n_B < n_A^2$

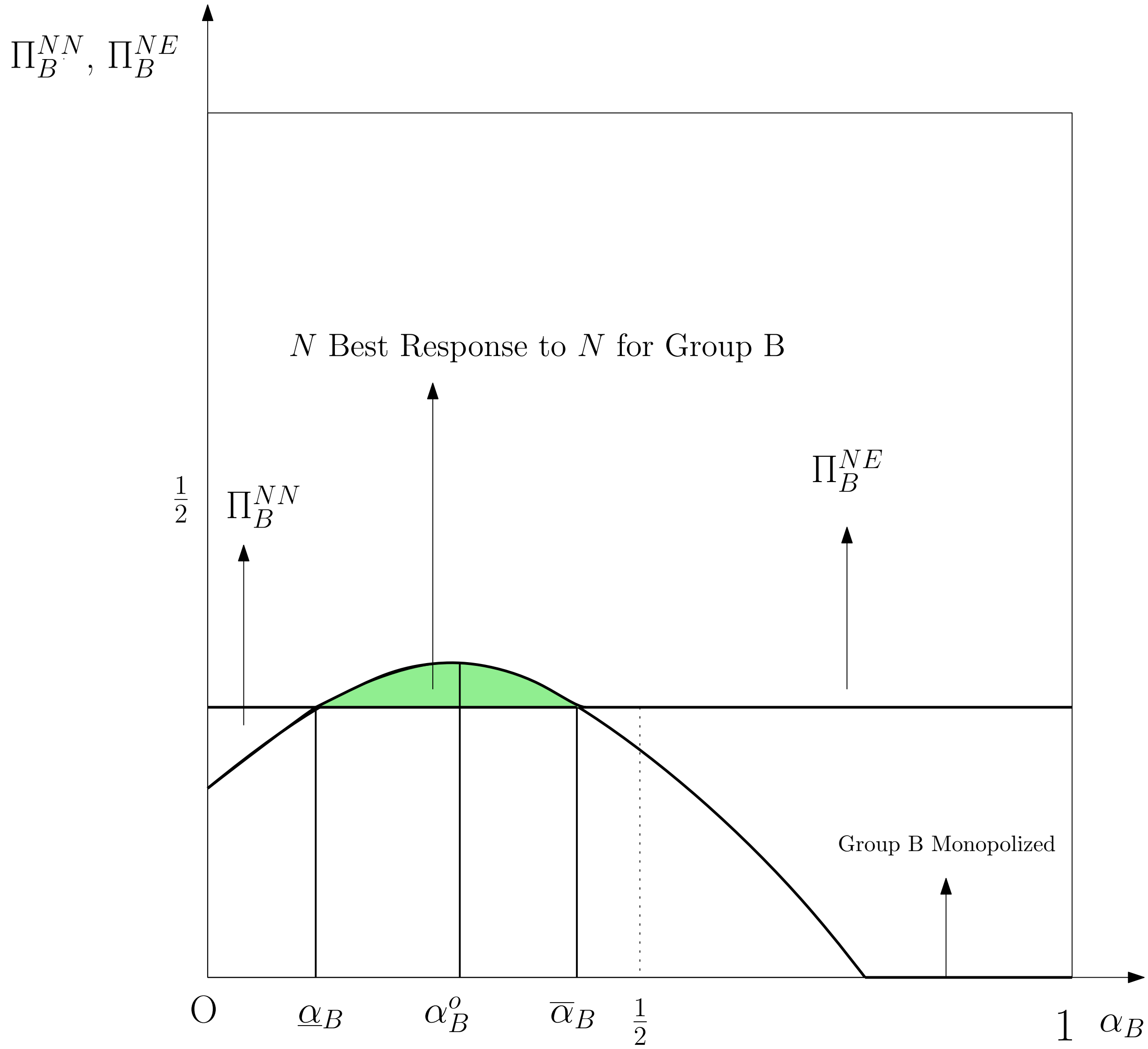

Figure 2.7: N best response to N for group B when $n_B \geqslant n_A^2$

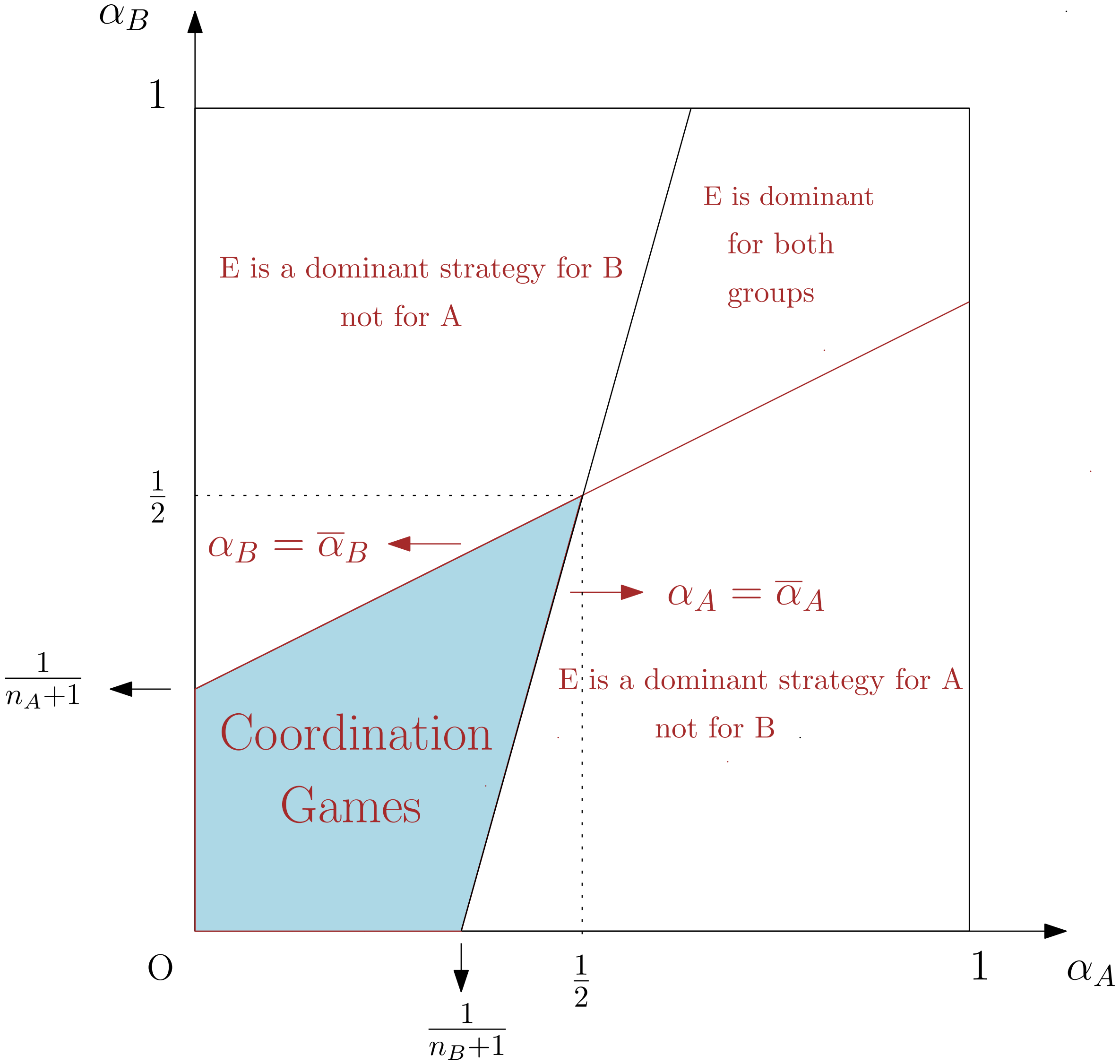

Figure 2.8: Coordination Games when $n_B < n_A^2$

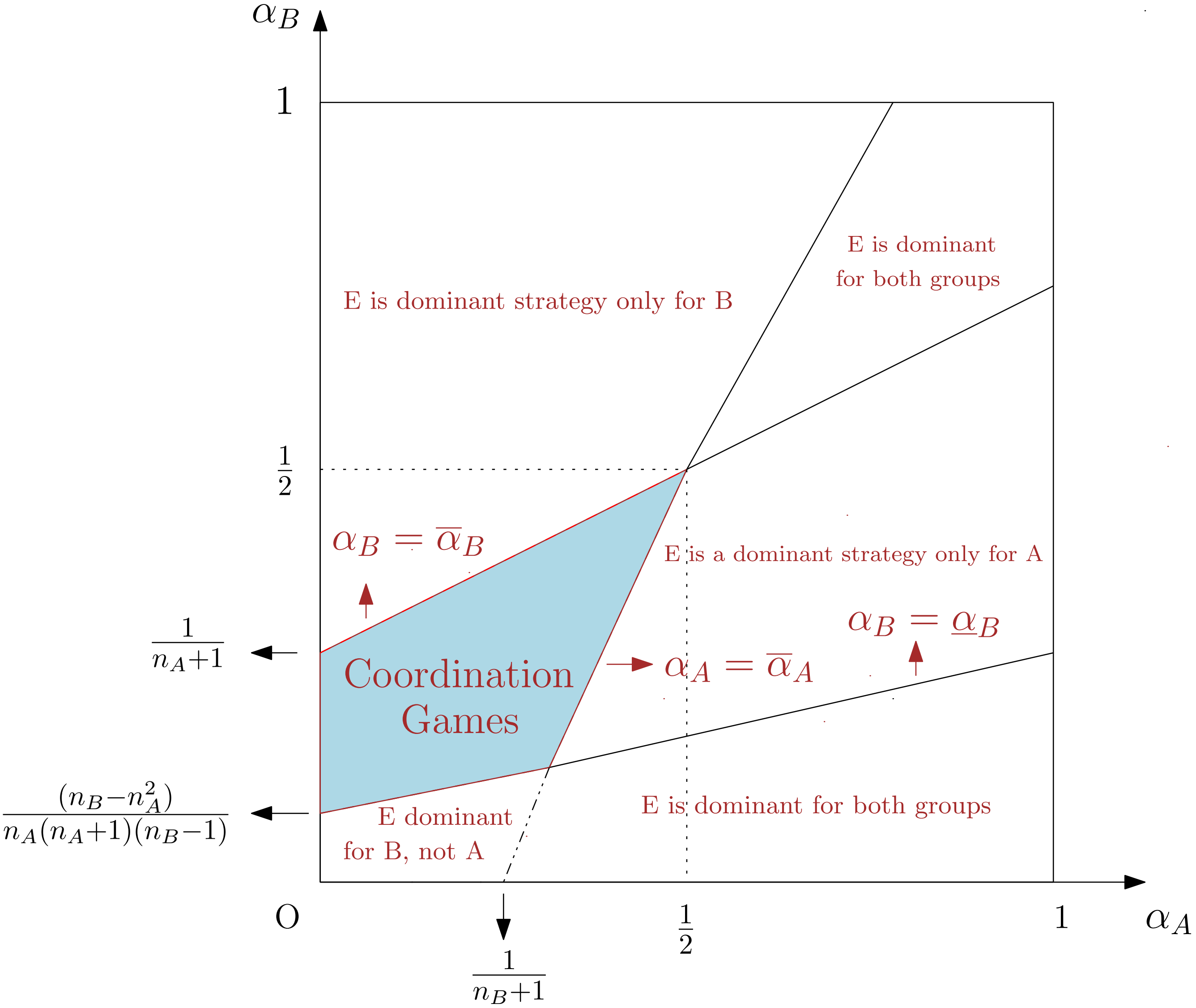

Figure 2.9: Coordination Games when $n_B \geqslant n_A^2$

2.6 Equilibrium Selection

We have been able to generate NN as a subgame perfect Nash equilibrium of an appropriately constructed two stage game, thereby providing a strategic foundations to the non-cooperative prize sharing rule N, which has been so extensively analyzed in the collective contests literature. But, given that it is an equilibrium of a Coordination game, where EE is also a Nash equilibrium, the natural next step is to consider the question of equilibrium selection, i.e., which of the equilibria are the groups likely to coordinate on? To tackle this we introduce the three refinement criteria of the Nash equilibrium solution concept, namely payoff dominance, risk dominance and the security principle[10].

If a game has multiple Nash equilibria and there is one Nash equilibrium which is Pareto superior to all other Nash equilibria then it is called payoff dominant. The notion of payoff dominance is based on the idea of collective rationality, which leads to a coordination on the Pareto superior equilibrium. The readers may refer to Harsanyi et al. (1988) for the first discussions of this refinement concept. Readers may also refer to (Schelling, 1980), who argues that efficiency based considerations may make decision makers to focus on and select a payoff dominant equilibrium point if it is unique.

A Nash equilibrium is said to be risk dominant if the losses from deviation from it is the largest among all other Nash equilibria. In the presence of high degree of uncertainty about other player's actions, this criterion seems to be more natural as players have an incentive to coordinate on it to minimize losses. A risk dominant equilibrium is defined to be one which generates the highest product of losses for the players, when there is a deviation from it. Harsanyi (1995) first made a case for risk dominance as an equilibrium selection criterion.

Interestingly, there can be a tension between the criteria of risk dominance and payoff dominance in the sense that they may make conflicting prescriptions. A Nash equilibrium can be payoff dominant but not risk dominant and vice versa. This leads to the obvious concern

[10] As an aside, readers are referred to Ray et al. (2005) to understand how existence multiple equilibria proves to be a problem for implementation of correlated equilibrium distributions.

about the relative appropriateness of the criteria? Researches have built evolutionary game theory models in an attempt to justify one or the other of the refinements, see e.g. Samuelson (1997). As it turns out, the tension between the two criterion is also a feature of our model under certain circumstances.

There also exists a substantial experimental literature, which studies how real subjects actually select between payoff dominance and risk dominance, when the two criteria make conflicting prescriptions. For a guide to that literature, the readers may look at Keser et al. (2000) and the references therein. The major takeaways from this literature is that the number of players, time horizon, pre-play communication and the structure of interactions matter. Interestingly, Keser et al. (2000) report an experiment where despite the two criteria making the same equilibrium prescription, subjects systematically deviate from playing it. Based on their conclusions, the authors claim that it is important to look for new criteria that may play an important role in equilibrium selection.

Therefore, we also consider the Security principle, see e.g. Van Huyck et al. (1990), as an additional selection criteria in this chapter. The security principle suggests players to select a course of action that maximizes their minimum payoffs over all possible actions. The idea is based on the notion of maximin introduced by Von Neumann and Morgenstern (1944). This criterion, like risk dominance, is based on the "riskiness" of the equilibrium point. Therefore, it will be salient when there is sufficient uncertainty regarding the other player's actions.

We now take up the equilibrium selection criteria one at a time. We first formally define a criterion tailored to our game Γ^C. Then we state the corresponding result.

First, we take up the equilibrium selection criteria based on "riskiness"of the equilibrium point, i.e., risk dominance and the security principle. Then, we consider the selection criterion of payoff dominance.

Definition 3 ***Risk Dominance***

Consider any game $G \in \Gamma^C$. NN is said to risk dominate EE in G iff $(\Pi_A^{NN} - \Pi_A^{EN})(\Pi_B^{NN} - \Pi_B^{NE}) \geqslant (\Pi_A^{EE} - \Pi_A^{NE})(\Pi_B^{EE} - \Pi_B^{EN})$. If the inequality holds strictly NN is said to strictly risk *dominate EE.*

For ease of stating the result we start by introducing some notations. For $i, j \in \{A, B\}$ and $j \neq i$ we define

$$\Delta_i = n_j(n_j+1)^2\Big[(\overline{\alpha}_i - \alpha_i)(\alpha_i - \underline{\alpha}_i)\Big], \tag{2.18}$$

where $\overline{\alpha}_i$ and $\underline{\alpha}_i$ are the roots of $\Pi_i^{\sigma_A\sigma_B}(\sigma_i = E, \sigma_j = N) = \Pi_i^{NN}$ as defined in (2.15) and (2.16). Δ_i is a measure of $\Pi_i^{NN} - \Pi_i^{\sigma_A\sigma_B}(\sigma_i = E, \sigma_j = N)$. As we consider only Coordination games, it is true that $\alpha_i \in (\underline{\alpha}_i, \overline{\alpha}_i)$ and therefore the right hand side of (2.18) is positive. We are now in a position to state a condition which is necessary and sufficient for equilibrium profile NN to risk dominate EE.

Proposition 5

Consider any game $G \in \Gamma^C$. NN risk dominates EE in G iff $N^4(1-2\alpha_A)^2(1-2\alpha_B)^2 \leqslant 16\Delta_A\Delta_B$.

The Proposition provides us a very easy to check condition for NN to risk dominate EE. It can be written out as follows:

$$N^4(1-2\alpha_A)^2(1-2\alpha_B)^2 \leqslant 16n_An_B(n_A+1)^2(n_B+1)^2(\overline{\alpha}_A-\alpha_A)(\alpha_A-\underline{\alpha}_A)(\overline{\alpha}_B-\alpha_B)(\alpha_B-\underline{\alpha}_B) \tag{2.19}$$

The left hand side of (2.19) is a measure of $(\Pi_A^{EE} - \Pi_A^{NE})(\Pi_B^{EE} - \Pi_B^{EN})$. It is close to zero if either α_A or α_B is close to $\frac{1}{2}$. But it is clear from Figures 2.8 and 2.9, that Γ is a Coordination game when α_A and α_B are relatively symmetric, i.e., not too far from each other. So if the left hand side has to be small when Γ is a Coordination game, we must have $\alpha_A \approx \alpha_B$ and close to $\frac{1}{2}$.

The the right hand side of (2.19) is a measure of $(\Pi_A^{NN} - \Pi_A^{EN})(\Pi_B^{NN} - \Pi_B^{NE})$. Its size depends on the product $\Delta_A\Delta_B$. The product will be close to zero if either α_A or α_B approaches any of its respective roots. But, it is clear from Figures 2.8 and 2.9, that when $\alpha_A \approx \alpha_B$ and close to $\frac{1}{2}$, both α_A and α_B are at some distance from its roots, which may make the product $\Delta_A\Delta_B$ large enough to dominate left hand side of (2.19), which is close to zero.

Therefore, Coordination games in which NN risk dominates EE, if they exist, are likely to be located around $\alpha_A = \alpha_B$. To show that the set of games in which equilibrium profile NN risk dominates EE is non-empty, we consider a subclass of Coordination games of Γ we call Symmetric Coordination games.

Definition 4 ***Symmetric Cordination games***

Consider any game $G \in \Gamma^C$. G is said to be a Symmetric Coordination game iff $n_A = n_B = n$ and $\alpha_A = \alpha_B = \alpha$. The set of all Symmetric Coordination games is denoted Γ^{SC}.

Corollary **1**

Consider any game $G \in \Gamma^{SC}$. NN risk dominates EE in G iff $\alpha \in [\frac{1}{4} - \frac{1}{4n}, \frac{1}{2})$ [11].

This result can be obtained by replacing $n_A = n_B = n$ and $\alpha_A = \alpha_B = \alpha$ in (2.19) [12] [13]. This corollary of Proposition 5 establishes that when groups participating in the collective contest are symmetric in all respects, there is a robust strategic basis of N based on the equilibrium selection criterion of risk dominance. In order to understand why these games arise, first note that at $\alpha_A = \alpha_B = \frac{1}{2}$, both the right hand side and left had side of (2.19) are zero. As we approach $\alpha_A = \alpha_B = \frac{1}{2}$ from below, along $\alpha_A = \alpha_B$, the right hand side falls at faster rate than the left hand side and therefore has to dominate it along the path, given that both have to be zero at $\alpha_A = \alpha_B = \frac{1}{2}$.

If we introduce asymmetries between groups it is unlikely that NN will pass the test of risk dominance as it becomes harder to satisfy (2.19).

Next, we consider the equilibrium selection criterion called the Security Principle, see e.g. Van Huyck et al. (1990). A secure strategy for a player is one in which the smallest payoff is

[11] These games are in fact Stag Hunt games, see e.g. Skyrms (2004)

[12] It is easiest to see if we use the form of Δ_i in (2.59).

[13] This case corresponds to Figure 2.8, with $n_A = n_B = n$.

at least as large as the smallest payoff to any other feasible strategy. Security principle selects equilibrium points implemented by secure strategies. The Security Principle, as we will see, always selects NN unlike the criterion of payoff dominance, never selects it (Proposition 7).

DEFINITION **5** ***Secure Strategy***

A strategy $\overline{\sigma}_i$ of group i is said to be secure iff $\overline{\sigma}_i = arg\Big(max_{\sigma_i \in \{E,N\}} min_{\sigma_j \in \{E,N\}} \Pi_i(\sigma_i, \sigma_j)\Big)$, $i, j \in \{A, B\}$ and $j \neq i$.

The strategy $\overline{\sigma}_i$ guarantees group i the best out of the worst of its outcomes.

DEFINITION **6** ***Security Principle***

Consider any game $G \in \Gamma^C$. NN will be said to satisfy the Security Principle in G iff N is a secure strategy for both groups A and B.

PROPOSITION **6**

Consider any game $G \in \Gamma^C$. NN satisfies the Security Principle in G.

Proof:

We do the proof assuming $i = A$.

We know that $\Pi_A^{NN} > \Pi_A^{EN}$ when Γ is a Coordination game. We also know from Proposition 7 that $\Pi_A^{EE} > \Pi_A^{NN}$ when Γ is a Coordination game. Therefore it follows that we must have $\Pi_A^{EE} > \Pi_A^{NN} > \Pi_A^{EN}$ when Γ is a Coordination game.

We can also see in proof of Proposition 3, that $\Pi_A^{EE} > \Pi_A^{NE}$ when $\alpha_A < \frac{1}{2}$. And, it can also be easily verified from Proposition 1 and 4, that $\Pi_A^{NE} > \Pi_A^{NN}$ when Γ is a Coordination game. This is true because both $P_A^{NE} > P_A^{NN}$ and $S^{NE} > S^{NN}$, i.e., not only is the net surplus higher in this case, but group A also wins the contest with a higher probability. Therefore, when Γ is a Coordination game, we have $\Pi_A^{EE} > \Pi_A^{NE} > \Pi_A^{NN} > \Pi_A^{EN}$.

As $\Pi_A^{NN} > \Pi_A^{EN}$, i.e., the minimum payoff from choosing N is strictly larger than the minimum payoff from choosing E for group A, N is a secure strategy for group A. The argument is similar for group B. ■

Finally, we consider the equilibrium selection criterion of payoff dominance.

DEFINITION **7** ***Payoff Dominance***

Consider any game $G \in \Gamma^C$. EE is said to payoff dominate NN in G iff $\Pi_A^{EE} \geqslant \Pi_A^{NN}$ and $\Pi_B^{EE} \geqslant \Pi_B^{NN}$ with one inequality holding strictly. If both inequalities hold strictly we will say that EE strictly payoff dominates NN in G.

PROPOSITION **7**

Consider any game $G \in \Gamma^C$. EE strictly payoff dominates NN in G.

To prove this result we need to show that, for $G \in \Gamma^C$ and $i = A, B$

$$P_i^{NN} S^{NN} > P_i^{EE} S^{EE} \tag{2.20}$$

We proceed by identifying games $G \in \Gamma$, such that strategy profile EE is Pareto superior to strategy profile NN, i.e., $P_i^{NN} S^{NN} \geqslant P_i^{EE} S^{EE}$, $i = A, B$. We denote such games Γ^{PS}. Then we go on to show that the set of Coordination games Γ^C is a proper subset of Γ^{PS} , i.e., $\Gamma^C \subset \Gamma^{PS}$.

The following equation represents the bigger root [14] of the quadratic equation of (2.20)

$$\alpha_j^+ = \frac{(n_j - n_i)(n_i - 1)\alpha_i + N\sqrt{((n_i - 1)\alpha_i)^2 + n_i} - 2n_i}{2n_i(n_j - 1)} \tag{2.21}$$

For instance, when $\alpha_B = \alpha_B^+$, we have $\Pi_A^{NN} = \Pi_A^{EE} = \frac{1}{4}$. If $\alpha_B < \alpha_B^+$, group B is more competitive and generates more effort, which leads to a lower S^{NN} and P_A^{NN} and hence a lower Π_A^{NN} compared to $\Pi_A^{EE} = \frac{1}{4}$. Similarly, when $\alpha_A < \alpha_A^+$, we have $\Pi_B^{NN} < \Pi_B^{EE}$. For the shapes of α_A^+ and α_B^+ look at Figure 2.10.

When both $\alpha_B \leqslant \alpha_B^+$ and $\alpha_A \leqslant \alpha_A^+$ with one inequality holding strictly, EE is Pareto superior to NN. This can be observed in Figure 2.10. It is clear from the diagram that EE Pareto superior to NN, when both α_A and α_B are substantially less than $\frac{1}{2}$.

[14] We do not report the smaller root α_j^- as it is negative and can be ignored. See proof of Proposition 7 in Appendix 1

To understand why this must be the case we refer to Figure 2.3. We start from $\alpha_A = \alpha_B = \frac{1}{2}$, where strategy profiles EE and NN are equivalent. Now, if either α_A or α_B falls then $S^{NN} < \frac{1}{2}$ and decreasing. For example, if α_A falls substantially but α_B falls infinitesimally, then P_A^{NN} rises and P_B^{NN} falls and we approach $P_A^{NN} = 1$. Here, group A captures almost the whole of the reduced net surplus, thereby getting a payoff $\Pi_A^{NN} > \Pi_A^{EE} = \frac{1}{4}$. For this case not to arise we need α_B to fall sufficiently as well.

It can also be observed in Figures 2.11 and 2.12, that $\overline{\alpha}_A$ supports α_A^+ from below and $\overline{\alpha}_B$ supports α_B^+ from above at $(\frac{1}{2}, \frac{1}{2})$ in the $\alpha_A\alpha_B$ plane. This fact helps us establish our result. If $(\alpha_A, \alpha_B) < (\frac{1}{2}, \frac{1}{2})$, then $\alpha_B^+ > \overline{\alpha}_B$ and $\alpha_A^+ > \overline{\alpha}_A$. Therefore, the set of Coordination games Γ^C, is a proper subset of the games in which EE is Pareto superior to NN.

■ **Intuition:** To understand the result, it is best to begin by noticing that the issue is only relevant in Coordination games. Further note that the Coordination games are clustered around $P_A^{NN} = P_B^{NN} = \frac{1}{2}$ (see Figures 2.3 and 2.8). The difference in probabilities of winning between the groups cannot be too large if Γ has to be a Coordination game.

We know from Proposition 1 that $P_A^{EE} = P_B^{EE} = \frac{1}{2}$. Given that the disparity in probabilities of winning between the groups cannot be large, i.e., $P_A^{NN} \approx P_A^{NN}$, and $S^{NN} < \frac{1}{2} = S^{EE}$, when Γ is Coordination game, it will be the case that each group achieves a payoff strictly less than $\frac{1}{4}$, i.e., EE Payoff dominates NN. In Coordination games, both groups essentially cancel out the gain in winning probabilities each wishes to have by choosing N. But, as both groups efforts are higher under NN, the net surplus is lower compared to EE. The net effect is that both groups lose by choosing N.

In this section we introduced several equilibrium selection criterion to check whether equilibrium NN is prescribed by any of them. When we consider the criterion of risk dominance we are able to show that there exist Coordination games in which NN risk dominates EE. We provide a necessary and sufficient conditions for NN to risk dominate EE in Proposition 5 and then go onto show existence of such games using a symmetric subclass of Coordination games in Corollary 1.

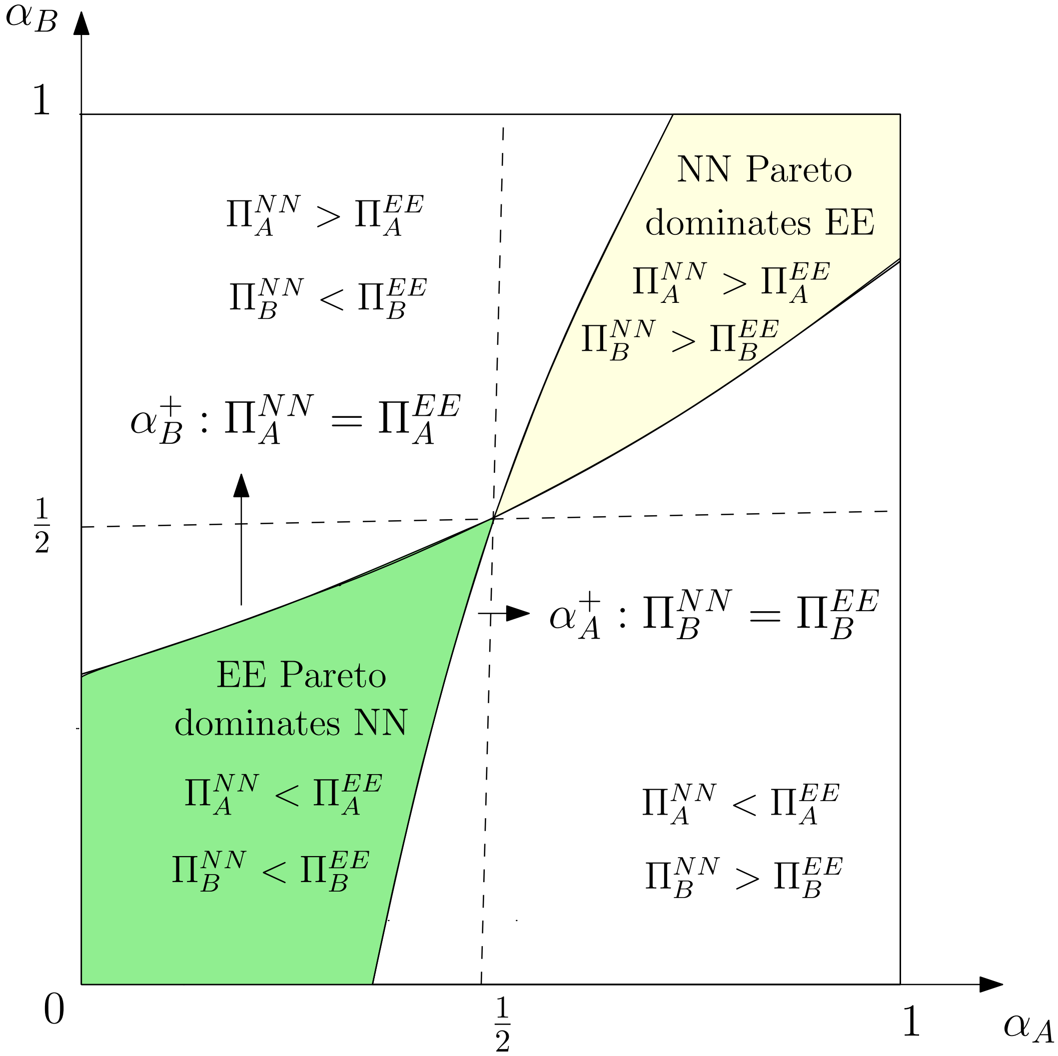

Figure 2.10: The locus of $\Pi_i^{NN} = \Pi_i^{EE}$

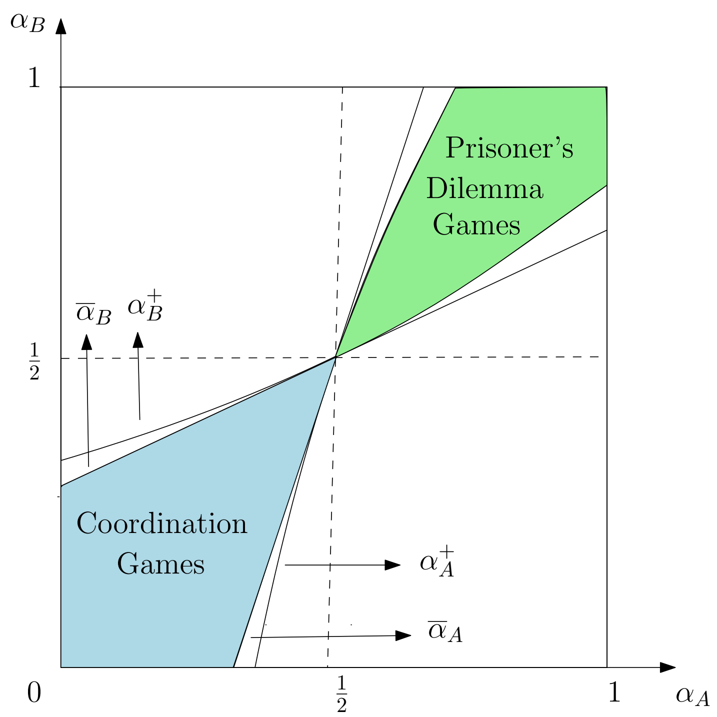

Figure 2.11: Payoff Dominance: When $n_B < n_A^2$

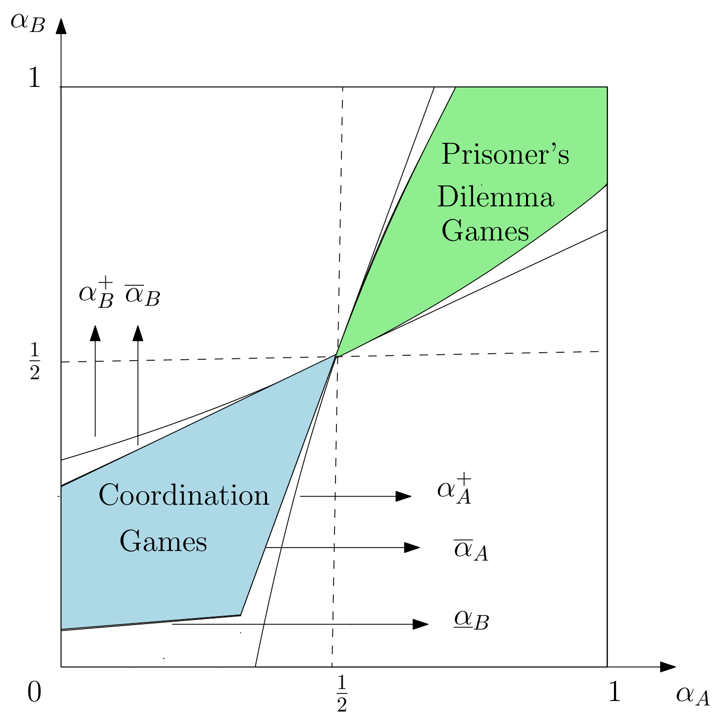

Figure 2.12: Payoff Dominance: When $n_B \geqslant n_A^2$

When we consider equilibrium selection criterion called the Security Principle, we are able to show that NN is always prescribed over EE. However, when we consider the principle of payoff dominance, equilibrium profile NN is never selected as is shown in Proposition 7. The results are therefore mixed. However, given that there exist equilibrium criteria which prescribe selection of equilibrium NN, allows us to claim that there exists a robust strategic basis of prize sharing rules N, first introduced in Nitzan (1991).

2.7 Discussion

In this section we discuss a few assumptions we made and some other properties, which we have skipped in the main body.

■ **Coordination Devices:** Given that in our model selecting equilibrium NN is essentially a failure to coordinate on a Pareto efficient equilibrium point EE, we discuss a couple of coordination devices, which may help the groups circumvent the problem.

(1) Timing of the Game: In our game we assume that in the first stage the group leaders move simultaneously to choose between E and N and having observed those choices the agents make their effort decisions simultaneously. But, it is clear that if one leader moves first, then the groups will coordinate on EE. Given the EE payoff dominates NN (Proposition 7), if one of the group leaders could choose the rule first, he would choose E and coordination failure on N will be avoided. But, the assumption of simultaneous choice of the rules is justified because in our framework of direct conflict and no communication between the groups, there is no reason to assume otherwise.

(2) Strategic Choice of Sharing Rules: In our game we have kept the α_i's fixed and provided the leader a choice between E and N. Another part of the literature considers the case, where the leaders do not have access to E. The only rule available is N but the leaders can choose $\alpha_i \in [0, 1]$. This part of the literature mostly focuses on the phenomenon of Group Size Paradox (GSP), whereby a larger group wins the contest less often due to free riding.

If we allow the leaders to choose $\alpha_i \in [0,1]$ in our model, then all equilibria will be payoff equivalent to EE. Given that the group leaders have some adjustment room with N, they will adjust N in such a manner that both groups will get fully cooperative payoffs. In fact it can shown that EE, NE ,EN will all be equilibrium profiles, with the leader of group i, choosing $\alpha_i = \frac{1}{2}$ under N. Only NN will not be an equilibrium profile. So, allowing strategic choice of sharing rules essentially gives the leaders an extra degree of freedom and help them avoid coordination failures.

■ **Prisoner's Dilemma Games:** There also exists a class of Prisoner's Dilemma games in our model. We primarily focused on the case where $(\alpha_A, \alpha_B) < (\frac{1}{2}, \frac{1}{2})$, because the focus of the chapter was on providing strategic foundations to N. But if $(\alpha_A, \alpha_B) > (\frac{1}{2}, \frac{1}{2})$, and $\alpha_A > \alpha_A^+$ and $\alpha_B > \alpha_B^+$ (α_A^+ and α_B^+ defined in (2.21)), then Γ turns out to be Prisoner's Dilemma games. Both groups have a dominant strategy E, but the strategy profile NN payoff dominates EE. So the use of grim trigger strategies, would allow us to generate NN as a subgame perfect Nash equilibrium if the first stage game is infinitely repeated [15]. The Prisoners Dilemma games can be seen in Figures 2.11 and 2.12.

When, $(\alpha_A, \alpha_B) > (\frac{1}{2}, \frac{1}{2})$, rule N makes both groups less competitive in the contest for the prize. The benefit is that a lot of net surplus gets saved and both groups benefit. But of course, given that rule N is not competitive enough, both groups have unilateral incentives of deviating to E. If the groups could write enforceable agreements they would have chosen $X = 0$. In this case mutually beneficial agreements are the ones in which $(\alpha_A, \alpha_B) > (\frac{1}{2}, \frac{1}{2})$, $\alpha_A > \alpha_A^+$ and $\alpha_B > \alpha_B^+$. But in absence of the possibility of explicit agreements between groups, one way to sustain NN as an equilibrium outcome is to repeat our stage game infinitely and use reverting to the Nash equilibrium EE forever as a punishment strategy for deviation from strategy N by any group at any stage.

[15] Ursprung (2012) recognizes that if $\alpha_A = 1$ and $\alpha_B = 1$ then Γ is a Prisoner's Dilemma game.

2.8 Extensions

In this section we provide two extensions of the main model. The first extension is to consider a general Tullock Contest Success Function and verify whether the main results of the chapter go through or not under it. The second extension is one in which we consider a situation in which the group leaders are maximizing the probabilities of winning instead of expected group payoffs. The aim again is to verify whether there are games in which rule N is chosen by the leaders in any equilibrium.

2.8.1 Generalized Tullock Contest Success Function

In this section we consider the Generalized Tullock Contest Success Function and try to replicate the main results of the paper under it. The Generalized Tullock Contest Success Function which is as follows:

$$P_i(X_i, X_j) = \begin{cases} \frac{X_i^r}{X_i^r + X_j^r}, & \text{if } X_i > 0 \text{ or } X_j > 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{2.22}$$

We will be assuming that $r \in (0, 1]$ throughout to rule out the possibility of Increasing Returns to Scale (IRS)[16].

Second Stage Choices

First, we study the second stage choice of individual efforts by group members taking as fixed the first stage choices made by the group leaders. There are four regimes to consider, .i.e., EE, EN, NE, NN.

[16]We consider the generalized Tullock form instead of the winner take all contest success function because only mixed strategy equilibria exist when we consider the the winner take all function.

Regime EE

Here we analyze the individual effort choices when both groups choose strategy E in the firsts stage of the game. The following result summarizes the essential features of this regime.

PROPOSITION **8**

In any Nash equilibrium of the effort subgame

(1) The efforts of the groups are $\left(X_A^{EE}, X_B^{EE}\right) = \left(\frac{r}{4}, \frac{r}{4}\right)$.

(2) The probabilities of winning are $\left(P_A^{EE}, P_B^{EE}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

(3) The group payoffs are $\left(\Pi_A^{EE}, \Pi_B^{EE}\right) = \left(\frac{2-r}{4}, \frac{2-r}{4}\right)$.

This result extends Proposition 1 to the generalized Tullock CSF for the case, where both leaders have chosen strategy E in the first stage of the game. One can easily make comparisons between the two results to verify that they coincide at $r = 1$. When, $r < 1$, the efforts made in the contest are lower and hence the group payoffs are higher.

Regime EN

Here we analyze the case where the leader of group A chooses E and leader of group B chooses N. The following Proposition summarizes the essential features of this regime.

As has been defined previously $\theta_B = (1 - \alpha_B)(n_B - 1)$.

PROPOSITION **9**

Along the Nash equilibrium path in the effort subgame the following equation needs to be satisfied

$$\frac{\theta_B}{r}\left[\frac{X_B}{X_A}\right]^r - n_B\frac{X_B}{X_A} + \left(1 + \frac{\theta_B}{r}\right) = 0 \tag{2.23}$$

There exists a unique $z^* = \left[\frac{X_B}{X_A}\right]^*$*, which satisfies (2.23)*

Unlike when $r = 1$, we cannot explicitly solve for individual efforts when $r < 1$. But this result proves that in any Nash equilibrium the ratios of the aggregate efforts of the two groups must be fixed. Even though we cannot explicitly solve for the ratio, we are able to prove that it exists and must be unique. Therefore, we are able to characterize all results in terms of this fixed ratio z^*.

The following result summarizes the efforts, probabilities of winning, and group payoffs in terms of z^*.

Proposition **10**

In any Nash equilibrium of the effort subgame

(1) The efforts of the groups are $\left(X_A^{EN}, X_B^{EN}\right) = \left(\frac{r(z^*)^r}{[1+(z^*)^r]^2}, \frac{r(z^*)^{r+1}}{[1+(z^*)^r]^2}\right)$.

(2) The probabilities of winning are $\left(P_A^{EN}, P_B^{EN}\right) = \left(\frac{1}{1+(z^*)^r}, \frac{(z^*)^r}{1+(z^*)^r}\right)$.

(3) The group payoffs are $\left(\Pi_A^{EN}, \Pi_B^{EN}\right) = \left(\frac{1+(1-r)(z^*)^r}{[1+(z^*)^r]^2}, \frac{(z^*)^r+(z^*)^{2r}-r(z^*)^{r+1}}{[1+(z^*)^r]^2}\right)$.

This result is an extension of Proposition 1 for the Generalized Tullock CSF, when group A chooses E and group B chooses N.

Regime NE

Here we analyze the case where group A chooses N and group B chooses E. The following Proposition summarizes the essential features of the regime.

As has been defined previously $\theta_A = (1 - \alpha_A)(n_A - 1)$.

Proposition **11**

Along the Nash equilibrium path in the effort subgame the following equation needs to be satisfied

$$\frac{\theta_A}{r}\left[\frac{X_A}{X_B}\right]^r - n_A\frac{X_A}{X_B} + \left(1 + \frac{\theta_A}{r}\right) = 0 \tag{2.24}$$

There exists a unique $y^* = \left[\frac{X_A}{X_B}\right]^*$*, which satisfies (2.24)*

This result helps pin down the unique ratio of efforts y^*, that the two groups put in when group $A$is choosing N and group B is choosing E.

PROPOSITION **12**

In any Nash equilibrium of the effort subgame

(1) The efforts of the groups are $\left(X_A^{NE}, X_B^{NE}\right) = \left(\frac{r(y^*)^{r+1}}{[1+(y^*)^r]^2}, \frac{r(y^*)^r}{[1+(y^*)^r]^2}\right)$.

(2) The probabilities of winning are $\left(P_A^{NE}, P_B^{NE}\right) = \left(\frac{(y^*)^r}{1+(y^*)^r}, \frac{1}{1+(y^*)^r}\right)$.

(3) The group payoffs are $\left(\Pi_A^{NE}, \Pi_B^{NE}\right) = \left(\frac{(y^*)^r+(y^*)^{2r}-r(y^*)^{r+1}}{[1+(y^*)^r]^2}, \frac{1+(1-r)(y^*)^r}{[1+(y^*)^r]^2}\right)$.

This result extends Proposition 1 for the Generalized Tullock CSF, where group A is choosing N and group B is choosing E.

Regime NN

In this section we consider the case where both group leaders have chosen N in the first stage of the game. We specifically restrict ourselves to the cases where there is no monopolization,.i.e. $0 < \frac{X_B}{X_A} < \infty$.

As before we define $\theta_i = (1-\alpha_i)(n_i - 1)$, $i = A, B$. The following Proposition summarizes the essential features of this regime.

PROPOSITION **13**

Along the Nash equilibrium path of the effort subgame, where neither group is monopolized, the following equation needs to be satisfied

$$n_A\theta_B\left(\frac{X_B}{X_A}\right)^r - n_B\theta_A\left(\frac{X_B}{X_A}\right)^{1-r} - n_B(r+\theta_A)\left(\frac{X_B}{X_A}\right) + n_A(r+\theta_B) = 0 \tag{2.25}$$

There exists a unique $x^* = \left[\frac{X_B}{X_A}\right]^*$, *which solves equation (2.25).*

		Group B	
		E	N
Group A	E	Π_A^{EE}, Π_B^{EE}	Π_A^{EN}, Π_B^{EN}
	N	Π_A^{NE}, Π_B^{NE}	Π_A^{NN}, Π_B^{NN}

Table 2.2: Game ψ

This result again helps pin down a unique ratio of efforts the groups must put in when both groups are choosing N. But it has to be noted that we have restricted the analyses to parameters, such that neither group is monopolized in equilibrium. Such a restriction is without any loss, given that we are only interested in verifying whether there are Coordination games like the case whre $r = 1$. Recall that groups cannot be monopolized in a Coordination game.

Proposition **14**

In any Nash equilibrium of the effort subgame, where neither group is monopolized

(1) The efforts of the groups are $\left(X_A^{NN}, X_B^{NN}\right) = \left(\frac{r(x^*)^r+(1+(x^*)^r)\theta_A}{n_A[1+(x^*)^r]^2}, \frac{(r+\theta_A)(x^*)^{r+1}+\theta_A(x^*)}{n_A[1+(x^*)^r]^2}\right)$.

(2) The probabilities of winning are $\left(P_A^{NN}, P_B^{NN}\right) = \left(\frac{1}{1+(x^*)^r}, \frac{(x^*)^r}{1+(x^*)^r}\right)$.

(3) The group payoffs are $\left(\Pi_A^{NN}, \Pi_B^{NN}\right) = \left(\frac{(n_A-\theta_A)(1+(x^*)^r)-r(x^*)^r}{n_A[1+(x^*)^r]^2}, \frac{n_A(1+(x^*)^r)(x^*)^r-(r+\theta_A)(x^*)^{r+1}-\theta_A(x^*)}{n_A[1+(x^*)^r]^2}\right)$.

This results extends Proposition 1 for the Generalized Tullock CSF for the case, where both groups choose N and neither is monopolized.

First Stage Choices

In this section we analyze the first stage choices between E and N made by the group leaders when they maximize group payoffs. The game they play is represented in Table 2.2. We denote as ψ the games in Table 2.2 . In what follows we compute the Nash equilibria of games in ψ.

Proposition **15**

Consider any game $G \in \beta$. For all $r \in (0, 1]$, EE is a Nash equilibrium of G.

This result is the counterpart of Proposition 3 for the Generalized Tullock CSF case. It clarifies that the form of contest success function is irrelevant, i.e., r does not matter. The strategy profile EE is always a Nash equilibrium. It is not surprising given rule E allows the groups to internalize all within group externalities.

Proposition **16**

Consider any game $G \in \beta$. There exists r^* such that for all $r \in (r^*, 1]$, G is a Coordination game if $\Pi_A^{NN}(r = 1) > \Pi_A^{EN}(r = 1)$ and $\Pi_B^{NN}(r = 1) > \Pi_B^{NE}(r = 1)$.

This result helps extend Proposition 4. It verifies that the Coordination problem between groups we illustrated in the main model is quite general. It says that if the parameters are such that a game is a Coordination game in the main model, i.e., $r = 1$, there exists a whole range of values of r for which it is a Coordination game. Only for low values of r do the problems of coordination between the groups get solved.

		Group B	
		E	N
Group A	E	P_A^{EE}, P_B^{EE}	P_A^{EN}, P_B^{EN}
	N	P_A^{NE}, P_B^{NE}	P_A^{NN}, P_B^{NN}

Table 2.3: Game β

2.8.2 Group Leader Maximizes Probabilities of Winning

In what follows group leaders have a choice between cooperative rule E and N. The objective is to check whether there is an equilibrium with prize sharing rules N, when group leaders maximize probabilities of winning instead of expected group payoffs. Given any configuration of parameters $(\alpha_A, \alpha_B, n_A, n_B)$, we denote the game group leaders play in the first stage as $\beta(\alpha_A, \alpha_B, n_A, n_B)$. We denote the set of all such normal form games simply as β. Games in β are bi-matrix games as represented in Table 2.3.

The following proposition characterizes the equilibria of game in β.

Proposition **17**

Consider any game $G \in \beta$.

(A) EE is a pure strategy Nash equilibrium of G iff $\alpha_A \in [\frac{1}{2}, 1]$ and $\alpha_A \in [\frac{1}{2}, 1]$.

(B) NN is a pure strategy Nash equilibrium of G iff $\alpha_A \in [0, \overline{\alpha}_A]$ and $\alpha_B \in [0, \overline{\alpha}_B]$.

(C) NE is a pure strategy Nash equilibrium of G iff $\alpha_A \in [0, \frac{1}{2}]$ and $\alpha_B \in [\overline{\alpha}_B, 1]$.

(D) EN is a pure strategy Nash equilibrium of G iff $\alpha_A \in [\overline{\alpha}_A, 1]$ and $\alpha_B \in [0, \frac{1}{2}]$.

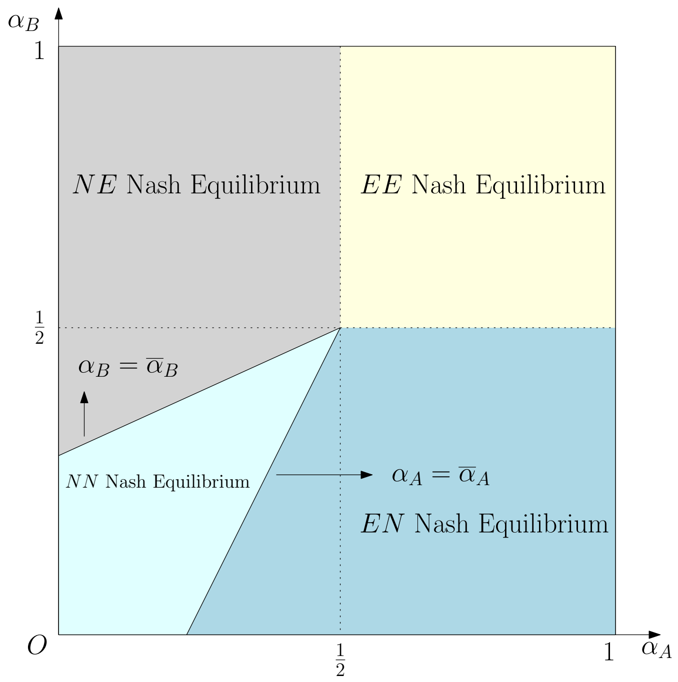

Figure 2.13: Probabilites of Winning

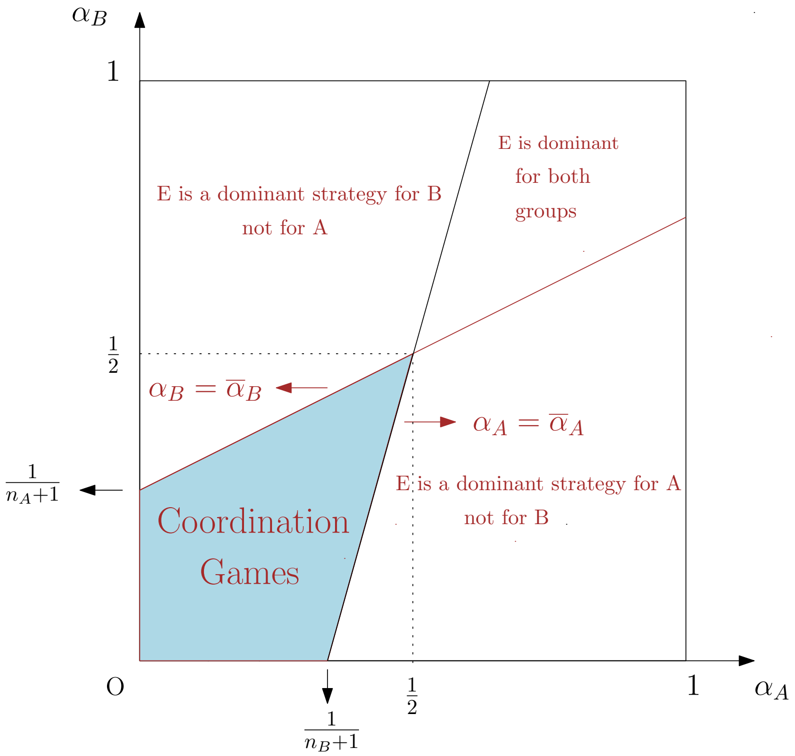

Figure 2.14: Group Payoffs

It can be seen from Figure 2.13 the games in β are no longer Coordination Games. But, we still have Nash equilibria in which N is chosen by both groups. In fact, the conditions under which NN is an equilibrium, are the same irrespective of the objective of the leader. The reason why the coordination problems arise, when leaders are maximizing group payoffs, is that even though N is useful in increasing probabilities of winning the contest, it dissipates most of the prize in unproductive effort. So, leaders would only choose N under the belief that the other group is also doing so. In fact, when leaders care about group payoffs, equilibrium NN is payoff dominated by equilibrium EE, so that selecting equilibrium NN is an instance of coordination failure between the groups.

However, when the objective of the leader is to maximize the chances of winning, the leaders are not concerned about how much prize is dissipated due to excessive efforts. In such a case, the best option for group i is to choose N when α_i is low because in such cases N is a more potent tool that E in terms of generating efforts.

2.9 Conclusion

The explicit aim of the chapter was to provide strategic foundations to the prize sharing rules introduced by Nitzan (1991), which has subsequently become the standard in the collective contests literature. To that end, we were able to uncover a class of Coordination games, where in fact the groups may end up coordinating on the Nitzan rule N, even though a cooperative option E is present. The games we study have transparency of choice and commitment to the choice by the group leaders as in Bagwell (1995). Coordinating on rule N looks like a case of coordination failure, because the equilibrium with mutual cooperation EE payoff dominates the one in which both groups choose the prize sharing rules NN.

However, when we introduce equilibrium selection criterion of risk dominance and security principle, which are based on the "riskiness" of the equilibrium point, we find that NN does indeed survive both these criterion. We provide a necessary and sufficient condition for NN

to risk dominate EE and show existence of such a class of coordination games. When we use the security principle, we find that the prescription is always to select NN. In light of these selection criterion, which prescribe selection of equilibrium profile NN, we claim that there exists a robust strategic basis to the prize sharing rules N.

We also uncover a class of Prisoner's Dilemma games where, the prize sharing rule N has a robust basis if the game is repeated infinitely and the leaders can use grim-trigger like punishment strategies.

Previously Ursprung (2012) showed in an evolutionary game theoretic model, that the extreme point $\alpha_i = 0$ of the prize sharing rule N crowds out E in the long run. We considered the whole class of rules in a 2 stage game and showed that there exist games, where the prize sharing rules may arise in equilibrium. Our analysis is complementary to theirs. It seems a worthwhile exercise to check, which parts of the rule N can actually crowd out E in the long run, given that we have been able to compute precise the conditions under which N is a Nash equilibrium in the static context.

Given, the complexity of the analysis we also did not consider what would happen if there are more than two groups. Another question which deserves attention is whether these prize sharing rules N will ever be chosen in equilibrium if efforts also had a productive component. All these issues and more, are beyond the aims and scope of the current analysis and warrant future research.

2.10 Appendix 1

2.10.1 Individual Effort Choice Problem

Proof of Proposition 1

Proposition 1 will be proved with the help of a few Lemmas, which we prove next.

Lemma **1**

If both group A and B choose Rule E, then in any Nash Equilibrium

- *Group effort levels are* $(X_i^{EE}, X_j^{EE}) = (\frac{1}{4}, \frac{1}{4})$.
- *The net surplus in the contest is* $S^{EE} = \frac{1}{2}$
- *The probabilities of winning are* $(P_i^{EE}, P_j^{EE}) = (\frac{1}{2}, \frac{1}{2})$
- *The payoffs of the groups are* $(\Pi_i^{EE}, \Pi_j^{EE}) = (\frac{1}{4}, \frac{1}{4})$.

Proof:

The payoff of member k of Group i is as follows:

$$\Pi_{ki}^{EE} = \frac{1}{n_i}(\frac{X_i}{X_i + X_j} - X_i) \tag{2.26}$$

The individual members of the groups choose efforts x_{ki} to maximize (2.26).

The following equation represents the F.O.C of any member k in group i:

$$\frac{X_j}{(X_i + X_j)^2} = 1 \tag{2.27}$$

Similarly, the following equation represents the F.O.C. of any member k in group j:

$$\frac{X_i}{(X_i + X_j)^2} = 1 \tag{2.28}$$

Adding (2.27) and (2.28) and we find that

$$X_i + X_j = \frac{1}{2} \tag{2.29}$$

Using (2.29) back in (2.27) and (2.28) we obtain that in any Nash equilibrium we must have:

$$(X_i, X_j) = (\frac{1}{4}, \frac{1}{4})$$

Again using (2.29) we get that the net surplus $S^{EE} = 1 - X_i - X_j = 1 - \frac{1}{2} = \frac{1}{2}$

The probabilities can be obtained by dividing the equilibrium efforts by (2.29) and we get $(P_i^{EE}, P_j^{EE}) = (\frac{1}{2}, \frac{1}{2})$

Using the equilibrium effort levels in (2.26) we obtain the payoffs of the groups in equilibrium are as follows:

$$(\Pi_i^{EE}, \Pi_j^{EE}) = (\frac{1}{4}, \frac{1}{4})$$

. ■

Lemma **2**

If Group i chooses E and j chooses N, then in the intra-group symmetric Nash Equilibrium

- *Group effort levels are* $(X_i, X_j) = (\frac{1+(1-\alpha_j)(n_j-1)}{(n_j+1)} - \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2}, \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2})$.
- *The net surplus in the contest is* $S^{\sigma_A \sigma_B} = 1 - \frac{1+(1-\alpha_j)(n_j-1)}{n_j+1}$.
- *The probabilities of winning are* $(P_i^{\sigma_A \sigma_B}, P_j^{\sigma_A \sigma_B}) = (\frac{1+\alpha_j(n_j-1)}{(n_j+1)}, 1 - \frac{1+\alpha_j(n_j-1)}{(n_j+1)})$.
- *The payoffs of the groups are:*
  $(\Pi_i^{\sigma_A \sigma_B}, \Pi_j^{\sigma_A \sigma_B}) = (\frac{(1+\alpha_j(n_j-1))^2}{(n_j+1)^2}, \frac{1+(1-\alpha_j)(n_j-1)}{(n_j+1)} - \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2})$

Proof:

The payoff of member k in group i is as follows:

$$\Pi_{ki} = \frac{X_i}{X_i + X_j} - X_i \tag{2.30}$$

The payoff of member k of Group j (which chooses N) is as follows:

$$\Pi_{kj} = \frac{X_j}{X_i + X_j}[(1-\alpha_j)\frac{x_{kj}}{X_j} + \frac{\alpha_j}{n_j}] - x_{kj} \tag{2.31}$$

The following equation represents the F.O.C. of member k of group i:

$$\frac{X_j}{(X_i + X_j)^2} = 1 \tag{2.32}$$

The following equation represents the F.O.C. of member k of group j:

$$\frac{X_i}{(X_i + X_j)^2}[(1-\alpha_j)\frac{x_{kj}}{X_j} + \frac{\alpha_j}{n_j}] + \frac{X_j}{X_i + X_j}[\frac{(1-\alpha_j)(X_j - x_{kj})}{X_j^2}] = 1 \tag{2.33}$$

Adding (2.33) over members in group j we reach the following condition:

$$\frac{X_i}{(X_i + X_j)^2} + \frac{(1-\alpha_j)(n_j - 1)}{X_i + X_j} = n_j \tag{2.34}$$

Adding (2.32) and (2.34) we find the total effort expended in the contest in equilibrium to be :

$$X_i + X_j = \frac{1 + (1-\alpha_j)(n_j - 1)}{n_j + 1} \tag{2.35}$$

The net surplus can obtained from (2.35) and is as follows

$$S^{\sigma_A \sigma_B} = 1 - X_i - X_j = 1 - \frac{1 + (1-\alpha_j)(n_j - 1)}{n_j + 1}$$

.

Using (2.35) in (2.32) we find that in equilibrium group j puts in

$$X_j = \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2} \tag{2.36}$$

Replacing X_j in (2.36) in (2.35) we solve for X_i in equilibrium to be

$$X_i = \frac{1+(1-\alpha_j)(n_j-1)}{(n_j+1)} - \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2} \tag{2.37}$$

To figure out the payoff of Group i we divide (2.37) by (2.35) we get the probability of group i winning the contest to be

$$P_i^{\sigma_A\sigma_B} = \frac{X_i}{X_i+X_j} = 1 - \frac{1+(1-\alpha_j)(n_j-1)}{n_j+1} \tag{2.38}$$

Subtracting X_i in (2.37) from (2.38) gives us group i's payoff in equilibrium to be

$$\Pi_i^{\sigma_A\sigma_B} = \frac{(1+\alpha_j(n_j-1))^2}{(n_j+1)^2}$$

Similarly dividing (2.36) by (2.35) we obtain the probability that group j wins the contest and subtracting X_j from the result we get the payoff of group j.

■

Lemma **3**

If both groups choose N and $\alpha_i n_j(n_i-1) - \alpha_j n_i(n_j-1) \geqslant n_i$ then group i is monopolized by group j. In the unique intra-group symmetric Nash Equilibrium

- *Group efforts are $(X_i, X_j) = (0, \frac{(1-\alpha_j)(n_j-1)}{n_j})$.*
- *The net surplus in the contest is $S^{NN} = 1 - \frac{(1-\alpha_j)(n_j-1)}{n_j}$.*
- *The probabilities of winning are $(P_i^{NN}, P_j^{NN}) = (0, 1)$.*
- *The payoffs of the groups are $(\Pi_i^{iM}, \Pi_j^{iM}) = (0, \frac{1+\alpha_j(n_j-1)}{n_j})$.*

Proof:

If both groups choose rule N, then the payoff of member k in group i is as follows

$$\Pi_{ki} = \frac{X_i}{X_i + X_j}[(1-\alpha_i)\frac{x_{ki}}{X_i} + \frac{\alpha_i}{n_i}] - x_{ki} \tag{2.39}$$

The following is the F.O.C. for member k of group i

$$\frac{X_j}{(X_i + X_j)^2}[(1-\alpha_i)\frac{x_{ki}}{X_i} + \frac{\alpha_i}{n_i}] + \frac{X_i}{X_i + X_j}[\frac{(1-\alpha_i)(X_i - x_{ki})}{X_i^2}] \leqslant 1 \tag{2.40}$$

If both groups choose rule N, then the payoff of member k in group j is as follows

$$\Pi_{kj} = \frac{X_j}{X_i + X_j}[(1-\alpha_j)\frac{x_{kj}}{X_j} + \frac{\alpha_j}{n_j}] - x_{kj} \tag{2.41}$$

The F.O.C. for member k in group j is

$$\frac{X_i}{(X_i + X_j)^2}[(1-\alpha_j)\frac{x_{kj}}{X_j} + \frac{\alpha_j}{n_j}] + \frac{X_j}{X_i + X_j}[\frac{(1-\alpha_j)(X_j - x_{kj})}{X_j^2}] \leqslant 1 \tag{2.42}$$

For all members of group i to choose $x_{ki} = 0$, the F.O.C. of group i members in (2.40) satisfied at $x_{ki} = 0$, which boils down to the following condition after we sum the F.O.C s

$$\frac{1}{X_j} + \frac{\theta_i}{X_j} \leqslant n_i \tag{2.43}$$

And summing the F.O.C.s of group j members in (2.42) ,at $X_i = 0$ we get the following condition

$$n_j X_j = \theta_j \tag{2.44}$$

For i to be monopolized in a Nash equilibrium both (2.43) and (2.44) have to be satisfied. Replacing X_j from (2.44) into (2.43) and simplifying we find that group i is monopolized if

$$\alpha_i n_j(n_i - 1) - \alpha_j n_i(n_j - 1) \geqslant n_i$$

Using (2.44) we get $X_j = \frac{\theta_j}{n_j} = \frac{(1-\alpha_j)(n_j-1)}{n_j}$

Therefore, net surplus is $S^{NN} = 1 - X_j = 1 - \frac{(1-\alpha_j)(n_j-1)}{n_j}$. Group j wins the contest with probability 1. The payoff of group i is 0, because it is monopolized. The payoff of group j is the net surplus S^{NN}, which it wins with probability 1.

■

Lemma 4

If both groups choose N and none of the groups is monopolized then in the unique intra-group symmetric Nash Equilibrium

- *Group efforts are* $(X_i, X_j) = (n_j(X^{NN})^2 - (1-\alpha_j)(n_j-1)X^{NN}, n_i(X^{NN})^2 - (1-\alpha_i)(n_i-1)X^{NN})$ *where* $X^{NN} = \frac{1+(1-\alpha_i)(n_i-1)+(1-\alpha_j)(n_j-1)}{N}$.
- *The net surplus in the contest is* $S^{NN} = 1 - \frac{1+(1-\alpha_i)(n_i-1)+(1-\alpha_j)(n_j-1)}{N}$.
- *The probabilities of winning are* $(P_i^{NN}, P_j^{NN}) = (\frac{\chi_i}{N}, 1 - \frac{\chi_i}{N})$ *where* $\chi_i = n_i + n_i(n_j - 1)\alpha_j - n_j(n_i - 1)\alpha_i$.
- *The payoffs of the groups are:* $(\Pi_i^{NN}, \Pi_j^{NN}) = ((\frac{\chi_i}{N})(1 - \frac{1+(1-\alpha_i)(n_i-1)+(1-\alpha_j)(n_j-1)}{N}), (1-\frac{\chi_i}{N})(1 - \frac{1+(1-\alpha_i)(n_i-1)+(1-\alpha_j)(n_j-1)}{N}))$.

Proof:

As none of the groups is monopolized the F.O.C. (2.40) and (2.42) hold with equality at some $x_{ki} > 0$, $\forall k \in \{2, 3..n_i\}$ and $x_{kj} > 0$, $\forall k \in \{2, 3..n_j\}$.

Using (2.40) which the F.O.C. for Group i members and summing it over all the members in i we get the following condition

$$\frac{X_j}{X^2} + \frac{\theta_i}{X} = n_i \tag{2.45}$$

Summing (2.42) over members of group j , we get the following condition

$$\frac{X_i}{X^2} + \frac{\theta_j}{X} = n_j \tag{2.46}$$

Adding (2.45) and (2.46) and simplifying we can solve for total effort X to be

$$X^{NN} = \frac{1 + \theta_i + \theta_j}{n_i + n_j} = \frac{1 + (1-\alpha_i)(n_i - 1) + (1-\alpha_j)(n_j - 1)}{N} \tag{2.47}$$

From (2.47) it follows that the net surplus is

$$S^{NN} = 1 - X^{NN} = 1 - \frac{1 + (1-\alpha_i)(n_i - 1) + (1-\alpha_j)(n_j - 1)}{N}$$

From (2.45) and (2.46) and using $\theta_r = (1-\alpha_r)(n_r - 1)$, $r = i, j$ we can deduce that

$$X_i = n_j X^2 - (1-\alpha_j)(n_j - 1)X$$

and

$$X_j = n_i X^2 - (1-\alpha_i)(n_i - 1)X$$

From these equations it is clear that the probability that group i wins the contest is

$$P_i^{NN} = \frac{X_i}{X} = n_j X - (1-\alpha_j)(n_j - 1) \tag{2.48}$$

Replacing X^{NN} from (2.47) in (2.48) and simplifying we get that $P_i^{NN} = \frac{\chi_i}{N}$ where $\chi_i = n_i + n_i(n_j - 1)\alpha_j - n_j(n_i - 1)\alpha_i$. Of course, the chances that group j wins the contest is just $P_j^{NN} = 1 - \frac{\chi_i}{N}$.

Note that $\Pi_i^{NN} = P_i^{NN} S^{NN}$. Replacing values of P_i^{NN} and S^{NN} we get our result. Similarly we can obtain the payoff of group j.

■

Proposition 1 is just sub-parts of Lemma 5, 6, 7, 8.

Proof of Proposition 2

■ Part A of the Proposition

Notice in Lemma 6 that both $\Pi_i^{\sigma_A\sigma_B}$ and $\Pi_j^{\sigma_A\sigma_B}$ are independent of α_i.

Again from Lemma 6

$$\Pi_i^{\sigma_A\sigma_B} = \frac{(1+\alpha_j(n_j-1))^2}{(n_j+1)^2}$$

This is clearly a strictly increasing function of α_j.

$$\Pi_j^{\sigma_A\sigma_B} = \frac{1+(1-\alpha_j)(n_j-1)}{(n_j+1)} - \frac{(1+(1-\alpha_j)(n_j-1))^2}{(n_j+1)^2} \tag{2.49}$$

Define $C = \frac{1+(1-\alpha_j)(n_j-1)}{(n_j+1)}$. It is easy to see that $\frac{dC}{d\alpha_j} < 0$.

Replacing value of C in (2.49) we simplify it to $\Pi_j^{\sigma_A\sigma_B} = C - C^2$

Differentiating with respect to α_j we get

$$\frac{d\Pi_j^{\sigma_A\sigma_B}}{d\alpha_j} = (1-2C)\frac{dC}{d\alpha_j}$$

As $\frac{dC}{d\alpha_j} < 0$, the sign of $\frac{d\Pi_j^{\sigma_A\sigma_B}}{d\alpha_j}$ depends on the sign of $1-2C$. If $1-2C < 0$ then $\frac{d\Pi_j^{\sigma_A\sigma_B}}{d\alpha_j} > 0$. But $1-2C < 0$ when $\alpha_j < \frac{1}{2}$. If $\alpha_j > \frac{1}{2}$, then $1-2C > 0$ and then we have $\frac{d\Pi_j^{\sigma_A\sigma_B}}{d\alpha_j} < 0$.

■ Part (B) of the Proposition

Using Lemma 8 we can write the payoff of group i as follows

$$\Pi_i^{NN} = \Big(\frac{n_i + n_i(n_j-1)\alpha_j - n_j(n_i-1)\alpha_i}{N}\Big)\Big(\frac{1+\alpha_i(n_i-1)+\alpha_j(n_j-1)}{N}\Big) \tag{2.50}$$

Notice that in both the terms within the brackets α_j enters with a positive sign. Therefore, it is the case that $\frac{d\Pi_i^{NN}}{d\alpha_j} > 0$. So we have $\frac{d\Pi_A^{NN}}{d\alpha_B} > 0$ and $\frac{d\Pi_B^{NN}}{d\alpha_A} > 0$.

Differentiating Π_i^{NN} in (2.50) with respect to α_i we get

$$\frac{d\Pi_i^{NN}}{d\alpha_i} = \frac{(n_i-1)[(n_i-n_j)(1+(n_j-1)\alpha_j) - 2n_j(n_i-1)\alpha_i]}{N^2} \tag{2.51}$$

The sign of $\frac{d\Pi_i^{NN}}{d\alpha_i}$ is the same as the sign of $(n_i - n_j)(1 + (n_j - 1)\alpha_j) - 2n_j(n_i - 1)\alpha_i$, which is the second term in brackets in the numerator.

Consider $i = A$. The term then is $(n_A - n_B)(1 + (n_B - 1)\alpha_B) - 2n_B(n_A - 1)\alpha_A$. It is negative as we have assumed $n_B \geqslant n_A$. Therefore, $\frac{d\Pi_A^{NN}}{d\alpha_A} < 0$.

Consider $i = B$. The term $(n_B - n_A)(1 + (n_A - 1)\alpha_A) - 2n_A(n_B - 1)\alpha_B > 0$ when $\alpha_B < \frac{(n_B - n_A)(1+(n_A-1)\alpha_A)}{2n_A(n_B-1)} = \alpha_B^o$. Therefore, $\frac{d\Pi_B^{NN}}{d\alpha_B} > 0$ if $\alpha_B < \alpha_B^o$. And $\frac{d\Pi_B^{NN}}{d\alpha_B} < 0$ if $\alpha_B > \alpha_B^o$.

2.10.2 Leader's Choice Problem

Proof of Proposition 3

Strategy profile EE will be a pure strategy Nash equilibrium of Γ if $\Pi_A^{EE} \geqslant \Pi_A^{NE}$ and $\Pi_B^{EE} \geqslant \Pi_B^{EN}$.

From Lemma 5 we know that $\Pi_A^{EE} = \frac{1}{4}$. And from Lemma 6 we know that

$$\Pi_A^{NE} = \frac{1 + (1 - \alpha_A)(n_A - 1)}{(n_A + 1)} - \frac{(1 + (1 - \alpha_A)(n_A - 1))^2}{(n_A + 1)^2}$$

E is a best response to E for group A if the following inequality is satisfied

$$\frac{1}{4} \geqslant \frac{1 + (1 - \alpha_A)(n_A - 1)}{(n_A + 1)} - \frac{(1 + (1 - \alpha_A)(n_A - 1))^2}{(n_A + 1)^2} \tag{2.52}$$

To see why (2.52) holds we define $x = \frac{1+(1-\alpha_A)(n_A-1)}{(n_A+1)}$. Then (2.52) can be written as

$$\frac{1}{4} \geqslant x - x^2$$

$$\Rightarrow (x - \frac{1}{2})^2 \geqslant 0$$

But this is true irrespective of the values of the parameters. Playing strategy E is a best response for group A to group B playing E.

When $\alpha_A = \frac{1}{2}$, then $x = \frac{1}{2}$ and we have

$$\Rightarrow (x - \frac{1}{2})^2 = 0$$

So in this case E is a weak best response to E for group A. In all other cases E is a strong best response for group A to E.

Similarly we can show that $\Pi_B^{EE} \geqslant \Pi_B^{EN}$ which means group B playing E is a best response to group A playing E.

Proof of Proposition 4

For strategy profile NN to be a Nash equilibrium we must have $\Pi_A^{NN} \geqslant \Pi_A^{EN}$ and $\Pi_B^{NN} \geqslant \Pi_B^{NE}$.

In general it must be true that for $i = A, B$

$$\Pi_i^{NN} \geqslant \Pi_i^{\sigma_A \sigma_B}(\sigma_i = E, \sigma_j = N) \tag{2.53}$$

Replacing the payoffs from Lemma 6 and 8 in (2.53) we get

$$\frac{\Big(n_i + n_i(n_j-1)\alpha_j - n_j(n_i-1)\alpha_i\Big)\Big(1 + (n_i-1)\alpha_i + (n_j-1)\alpha_j\Big)}{N^2} \geqslant \frac{\Big(1+\alpha_j(n_j-1)\Big)^2}{(n_j+1)^2} \tag{2.54}$$

We solve (2.54) as a quadratic equation using the Sridharacharya formula and get the following two roots:

The smaller root is

$$\underline{\alpha}_i = \frac{(n_i - n_j^2)(1+\alpha_j(n_j-1))}{n_j(n_j+1)(n_i-1)} \tag{2.55}$$

The larger root is

$$\overline{\alpha}_i = \frac{1 + \alpha_j(n_j - 1)}{(n_j + 1)} \tag{2.56}$$

It can be easily shown using Proposition 2 that $\Pi_i^{NN} \geqslant \Pi_i^{\sigma_A \sigma_B}(\sigma_i = E, \sigma_j = N)$ iff $\alpha_i \in [\underline{\alpha}_i, \overline{\alpha}_i]$. [17] In other words, if $\alpha_i \in [\underline{\alpha}_i, \overline{\alpha}_i]$ for $i = A, B$, then NN is a Nash equilibrium profile.

Now consider $i = A$. Given the assumption that $n_B \geqslant n_A$ it is clear from (2.55) that $\underline{\alpha}_A < 0$. Therefore the lower root can be ignored and the relevant range is $\alpha_A \in [0, \overline{\alpha}_A]$.

Consider $i = B$. From equation (2.55) it is clear that $\underline{\alpha}_B < 0$ iff $n_B < n_A^2$. Otherwise it is positive. If $\underline{\alpha}_B < 0$, then the relevant range for NN to be a Nash equilibrium is $\alpha_B \in [0, \overline{\alpha}_B]$. If $\underline{\alpha}_B \geqslant 0$, then the relevant range is $\alpha_B \in [\underline{\alpha}_B, \overline{\alpha}_B]$. We can write this range in a concise manner as $\alpha_B \in [max\{0, \underline{\alpha}_B\}, \overline{\alpha}_B]$.

Therefore, NN is a Nash equilibrium profile of Γ iff $\alpha_A \in [0, \overline{\alpha}_A]$ and $\alpha_B \in [max\{0, \underline{\alpha}_B\}, \overline{\alpha}_B]$. If the condition is not satisfied then in light of Proposition 3 it follows that strategy E is a strictly dominant strategy for at least one of the groups in Γ. Given that there are only two groups, Γ will be dominance solvable with the unique Nash equilibrium strategy profile EE. See Figures 2.8 and 2.9.

Proof of Proposition 5

Let us first consider the terms $\Pi_A^{NN} - \Pi_A^{EN}$ and $\Pi_B^{NN} - \Pi_B^{NE}$. In general, for $i = A, B$ we are have to consider $\Pi_i^{NN} - \Pi_i^{\sigma_A \sigma_B}$, where group i is the one which chooses E when the two group choose different strategies.

From Lemma 6 and 8 we can write the difference as follows

$$\Pi_i^{NN} - \Pi_i^{\sigma_A \sigma_B} = \left(\frac{n_i + n_i(n_j-1)\alpha_j - n_j(n_i-1)\alpha_i}{N}\right)\left(\frac{1+\alpha_i(n_i-1)+\alpha_j(n_j-1)}{N}\right) - \frac{(1+\alpha_j(n_j-1))^2}{(n_j+1)^2} \tag{2.57}$$

[17] For instance, consider group A. Starting from $\overline{\alpha}_A$ where (2.54) holds with equality, if we decrease α_A slightly, the LHS of (2.54) increases by Proposition 2 but the RHS being independent of α_A is unaffected.

Simplifying we get the following condition

$$\begin{aligned}\Pi_i^{NN} - \Pi_i^{\sigma_A\sigma_B} &= \tfrac{(n_i-1)}{N^2(n_j+1)^2}\Bigg((n_j^2 - n_i)(1+\alpha_j(n_j-1))^2 + (n_j+1)^2(n_i-n_j)(1+\alpha_j(n_j-1))\alpha_i \\ &\quad -n_j(n_j+1)^2(n_i-1)\alpha_i^2\Bigg)\end{aligned} \tag{2.58}$$

Let us define

$$\Delta_i = \frac{(n_j^2 - n_i)(1+\alpha_j(n_j-1))^2 + (n_j+1)^2(n_i-n_j)(1+\alpha_j(n_j-1))\alpha_i - n_j(n_j+1)^2(n_i-1)\alpha_i^2}{(n_i-1)} \tag{2.59}$$

Using the definition of $\overline{\alpha}_i$ and $\underline{\alpha}_i$ in (2.15) and (2.16) we can simplify and rewrite the above condition as follows

$$\Delta_i = n_j(n_j+1)^2\Big[(\overline{\alpha}_i - \alpha_i)(\alpha_i - \underline{\alpha}_i)\Big] \tag{2.60}$$

Using this definition of Δ_i in (2.59) we can write equation (2.58) as

$$\Pi_i^{NN} - \Pi_i^{\sigma_A\sigma_B} = \frac{(n_i-1)^2}{N^2(n_j+1)^2}\Delta_i \tag{2.61}$$

Now let us consider $\Pi_A^{EE} - \Pi_A^{NE}$ and $\Pi_B^{EE} - \Pi_B^{EN}$. In general for $i = A, B$ we are interested in $\Pi_i^{EE} - \Pi_i^{\sigma_A\sigma_B}$, where group i is the one which chooses N when the two groups choose different strategies.

From Lemma 5 and 6 we can write the difference as

$$\begin{aligned}\Pi_i^{EE} - \Pi_i^{\sigma_A\sigma_B} &= \tfrac{1}{4} - \tfrac{1+(1-\alpha_i)(n_i-1)}{(n_i+1)} - \tfrac{(1+(1-\alpha_i)(n_i-1))^2}{(n_i+1)^2} \\ &= (\tfrac{1}{2} - \tfrac{1+(1-\alpha_i)(n_i-1)}{(n_i+1)})^2\end{aligned} \tag{2.62}$$

This can be simplified and written as

$$\Pi_i^{EE} - \Pi_i^{\sigma_A\sigma_B} = \frac{(n_i-1)^2}{4(n_i+1)^2}(1-2\alpha_i)^2 \tag{2.63}$$

For NN to risk dominate EE we must have

$$(\Pi_A^{NN} - \Pi_A^{EN})(\Pi_B^{NN} - \Pi_B^{NE}) \geqslant (\Pi_A^{EE} - \Pi_A^{NE})(\Pi_B^{EE} - \Pi_B^{EN}) \tag{2.64}$$

Using equations (2.61) and (2.63) for groups $i = A, B$, we can immediately conclude that inequality (2.64) is satisfied iff

$$N^4(1-2\alpha_A)^2(1-2\alpha_B)^2 \leqslant 16\Delta_A\Delta_B$$

.

Proof of Proposition 7

For EE to strictly payoff dominate NN we find when is it that $\Pi_A^{NN} < \Pi_A^{EE}$ and $\Pi_B^{NN} < \Pi_B^{EE}$.

In general for $i = A, B$ we must have

$$\Pi_i^{NN} < \Pi_i^{EE} \tag{2.65}$$

Using Lemma 5 and 8 in (2.65) we get the following inequality which needs to hold

$$\frac{\Big(n_i + n_i(n_j-1)\alpha_j - n_j(n_i-1)\alpha_i\Big)(1+(n_i-1)\alpha_i+(n_j-1)\alpha_j\Big)}{N^2} < \frac{1}{4} \tag{2.66}$$

Solving (2.66) as a quadratic equation using the Sridharacharya formula we get the following two roots

The larger root is

$$\alpha_j^+ = \frac{(n_j - n_i)(n_i - 1)\alpha_i + N\sqrt{((n_i - 1)\alpha_i)^2 + n_i} - 2n_i}{2n_i(n_j - 1)} \tag{2.67}$$

The smaller root is

$$\alpha_j^- = \frac{(n_j - n_i)(n_i - 1)\alpha_i - N\sqrt{((n_i - 1)\alpha_i)^2 + n_i} - 2n_i}{2n_i(n_j - 1)} \tag{2.68}$$

Using Proposition 2 we can easily verify that EE will payoff dominate NN iff $\alpha_j \in (\alpha_j^-, \alpha_j^+)$, $j = A, B$.

We first consider group $i = A$. The roots of $\Pi_A^{NN} = \Pi_A^{EE}$ are α_B^+ and α_B^-. We now state a few important properties which these roots satisfy.

PROPERTY **1**

In the $\alpha_A\alpha_B$ plane α_B^+ lies completely above the α_A axis and α_B^- lies completely below the α_A axis and can therefore be ignored [18].

This can be verified by trying to solve either $\alpha_B^+ = 0$ or $\alpha_B^- = 0$, which gives us values of α_A at which these roots cut the α_A axis. Neither equation has a real solution as the discriminant for both these problems is $N\sqrt{1 - n_B}$, which is a complex number. Therefore, there does not exist a real α_A such that $\alpha_B^+ = 0$ or $\alpha_B^- = 0$. Therefore, neither $\alpha_B^+ = 0$ nor $\alpha_B^- = 0$ cut the α_A axis.

Replacing, $\alpha_A = 0$ in α_B^+ we find that it cuts the α_B axis at $\frac{N\sqrt{n_A} - 2n_A}{2n_A(n_B - 1)} > 0$. This combined with the observation made above helps us conclude that α_B^+ lies completely above the α_A axis

Replacing, $\alpha_A = 0$ in α_B^- we find that it cuts the α_B axis at $\frac{-N\sqrt{n_A} - 2n_A}{2n_A(n_B - 1)} < 0$. Therefore, α_B^- lies completely below the α_A axis and can be ignored.

[18] This means that the relevant zone for payoff dominance will be $\alpha_B \in [0, \alpha_B^+)$

Property **2**

α_B^+ is increasing and convex in the α_A [19].

To prove this we just look at the first and the second derivatives of α_B^+ with respect to α_A

$$\frac{d\alpha_B^+}{d\alpha_A} = \frac{(n_B - n_A)(n_A - 1) + \frac{N(n_A-1)^2\alpha_A}{\sqrt{n_A + (\alpha_A(n_A-1))^2}}}{2n_A(n_B - 1)} > 0$$

$$\frac{d^2\alpha_B^+}{d\alpha_A^2} = \frac{N(n_A - 1)^2}{2n_A(n_B - 1)}\Big(\frac{n_A}{(n_A + (\alpha_A(n_A - 1)^2)^{\frac{3}{2}}}\Big) > 0$$

Property **3**

α_B^+ passes through $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$. At $\alpha_A = \frac{1}{2}$ it is supported from below by the line $\overline{\alpha}_B$.

The first part is easily shown by replacing $\alpha_A = \frac{1}{2}$ in α_B^+. We get $\alpha_B^+ = \frac{1}{2}$

To prove the second part we note from (2.56) that the slope of $\overline{\alpha}_B$ is $\frac{d\overline{\alpha}_B}{d\alpha_A} = \frac{n_A-1}{n_A+1}$.

The slope of α_B^+ is

$$\frac{d\alpha_B^+}{d\alpha_A} = \frac{(n_B - n_A)(n_A - 1) + \frac{N(n_A-1)^2\alpha_A}{\sqrt{n_A + (\alpha_A(n_A-1))^2}}}{2n_A(n_B - 1)}$$

At $\alpha_A = \frac{1}{2}$, the slope is

$$\frac{d\alpha_B^+}{d\alpha_A} = \frac{(n_B - n_A)(n_A - 1) + \frac{N(n_A-1)^2}{n_A+1}}{2n_A(n_B - 1)} = \frac{n_A - 1}{n_A + 1}$$

Therefore, Slope of $\overline{\alpha}_B$ = Slope of α_B^+ at $\alpha_A = \frac{1}{2}$. Also at $\alpha_A = \frac{1}{2}$ we have $\overline{\alpha}_B = \frac{1}{2}$ and $\alpha_B^+ = \frac{1}{2}$. So, the curve α_B^+ and line $\overline{\alpha}_B$ have a common point and same slope at $\alpha_A = \frac{1}{2}$. Given that α_B^+ is convex and increasing and $\overline{\alpha}_B$ is increasing and linear in α_A, it follows that $\overline{\alpha}_B$ supports α_B^+ from below at $\alpha_A = \frac{1}{2}$.

Now we consider $i = B$ and state similar properties for α_A^+ and α_A^-

[19]For clear visualization note that in the $\alpha_A\alpha_B$ plane α_B^+ plots as an increasing and convex function

Property **4**

In the $\alpha_A\alpha_B$ plane α_A^+ lies completely to the right of the α_B axis and α_A^- lies completely to the left of α_A axis and can therefore be ignored.

We skip the proof as it follows exactly the same steps as Property 1.

Property **5**

α_A^+ in increasing (decreasing) in α_B if $\alpha_B > (<)\frac{n_B-n_A}{2\sqrt{n_A}(n_B-1)}$. α_A^+ is convex in α_B[20].

To prove this we just look at the first and the second derivatives of α_A^+ with respect to α_B

$$\frac{d\alpha_A^+}{d\alpha_B} = \frac{\frac{N(n_B-1)^2\alpha_B}{\sqrt{\alpha_B^2(n_B-1)^2+n_B}} - (n_B-n_A)(n_B-1)}{2n_B(n_A-1)}$$

Therefore, $\frac{d\alpha_A^+}{d\alpha_B} > 0$ iff

$$\frac{N(n_B-1)^2\alpha_B}{\sqrt{\alpha_B^2(n_B-1)^2+n_B}} > (n_B-n_A)(n_B-1)$$

Simplifying we get that this happens iff $\alpha_B > \frac{n_B-n_A}{2\sqrt{n_A}(n_B-1)}$

For convexity of α_A^+ we look at the second derivative, which is

$$\frac{d^2\alpha_A^+}{d\alpha_B^2} = \frac{N(n_B-1)^2}{2n_B(n_A-1)}\Big(\frac{n_B}{((n_B-1)^2\alpha_B^2+n_B)^{\frac{3}{2}}}\Big) > 0$$

Property **6**

α_A^+ passes through $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$. At $\alpha_B = \frac{1}{2}$ it is supported from below by the line $\overline{\alpha}_A$ [21].

We skip the proof as it follows exactly the same steps as Property 3.

Properties 1 to 6 are captured in Figure 2.10.

[20] In the $\alpha_A\alpha_B$ plane it plots as a concave function when α_A^+ is increasing and convex function when α_A^+ is decreasing. This happens because the domain of the function α_A^+. i.e., $\alpha_B \in [0, 1]$ is the vertical axis

[21] In the diagram in the $\alpha_A\alpha_B$ plane it seems that $\overline{\alpha}_A$ supports α_A^+ from above not below. But it has to be noted that that the domain $\alpha_B \in [0, 1]$ is the vertical axis and not the horizontal axis

We now proceed to show that the set of games Γ with Nash equilibria EE ad NN, i.e., $\alpha_A \in [0, \overline{\alpha}_A]$ and $\alpha_B \in [max\{0, \underline{\alpha}_B\}, \overline{\alpha}_B]$, is a proper subset of the set of games where EE payoff dominates NN, i.e., $\alpha_A \in [0, \alpha_A^+)$ and $\alpha_B \in [0, \alpha_B^+)$.

This fact directly follows from Property 3 and 6. Given for $i = A, B$, $\overline{\alpha}_i$ supports α_i^+ from below it is true that $\overline{\alpha}_i < \alpha_i^+$ except at $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$ [22], where they are equal. But we can remove $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$ as at that point all strategy profiles are Nash equilibria. In the set of games we are interested in we have $\overline{\alpha}_A < \alpha_A^+$ and $\overline{\alpha}_B < \alpha_B^+$. A look at the parametric ranges in the previous paragraph immediately confirms that the games which have Nash equilibria EE and NN are a proper subset of the games in which EE strictly payoff dominates NN. Look at Figures 2.11 and 2.12.

2.11 Appendix 2

2.11.1 Best Response Functions

Here we study the properties of the best response functions of the individual's in the two groups. To do that we start with a few notations.

We denote the best response function of the k^{th} member of group $i \in \{A, B\}$, when the group chooses $\sigma_i \in \{E, N\}$ as $R_{ik}^{\sigma_i}(X_j)$. For example, if group A chooses E, then the best response function of the k^{th} member of group A will be denoted $R_{Ak}^{E}(X_B)$, and if it chooses N, then $R_{Ak}^{N}(X_B)$.

When group i chooses E, the best response Function of member k , $R_{ik}^{E}(X_j)$ can be obtained by maximizing (2.4). It is implicitly characterized by the following first order condition:

$$\frac{X_j}{(X_i + X_j)^2} = 1 \tag{2.69}$$

Similarly, when group i chooses N, its best response function of member k, $R_{ik}^{N}(X_j)$ is obtained by maximizing (2.3). It is implicitly characterized by the following first order condi-

[22] $(\alpha_A, \alpha_B) = (\frac{1}{2}, \frac{1}{2})$ is the point at which the lines $\overline{\alpha}_i$ s support the curves

tion:

$$\frac{X_j}{(X_i+X_j)^2}[(1-\alpha_i)\frac{x_{ki}}{X_i}+\frac{\alpha_i}{n_i}]+\frac{X_i}{X_i+X_j}[\frac{(1-\alpha_i)(X_i-x_{ki})}{X_i^2}]=1 \tag{2.70}$$

Because group members are symmetric in all respects, the best response functions are the same. We can therefore apply symmetry and obtain the best response function of a representative agent of the group, which we denote $R_i^{\sigma_i}(X_j)$. This is the same notation introduced above but without the subscript k.

In the Proposition that follows, we use the following notation:

For $i \in \{A, B\}$

$$\theta_i = (1-\alpha_i)(n_i-1)$$

θ_i is a measure of competitiveness of group i weighted by group size. If α_i is low θ_i is high, so that more competitive groups will tend to have a higher θ_i. If such a group is also large, then the competitive nature of the group gets accentuated by its size. Therefore, larger groups with lower α_i's have higher θ_i's and are the most competitive ones.

Next, we state a general result about best response functions of the groups. We state the result without proof [23] but do a detailed diagrammatic analysis.

PROPOSITION **18**

For $i, j \in \{A, B\}$ and $j \neq i$

(A) If group i chooses E, then the slope of the best response function is as follows:

$$\frac{R_i^E(X_j)}{dX_j} = \frac{X_i - X_j}{2X_j}$$

Therefore, X_i is a strategic complement to X_j iff $X_i > X_j$.

[23] Available on request

(B) If group i chooses N, then the slope of the best response function is as follows:

$$\frac{R_i^N(X_j)}{dX_j} = \frac{(X_i - X_j) - \theta_i(X_i + X_j)}{2X_j + 2\theta_i(X_i + X_j)}$$

Therefore, X_i is a strategic complement to X_j iff $\frac{X_i}{X_j} > \frac{1+\theta_i}{1-\theta_i}$.

We next discuss the results summarized in Proposition 18.

■ **Both groups choose E:** The best Response Functions in this case are represented in Figure 21. Both $R_A^E(X_B)$ and $R_B^E(X_A)$ are strictly increasing when $X_A < \frac{1}{4}$ and $X_B < \frac{1}{4}$. Here, X_A and X_B are strategic complements.

The Best Response functions are well defined except at $(X_A, X_B) = (0, 0)$ and they intersect at $(X_A^{EE}, X_B^{EE}) = (\frac{1}{4}, \frac{1}{4})$, which is the unique Nash equilibrium in group efforts. At the equilibrium point, X_A and X_B are strategically independent, i.e., neither strategic complements nor strategic substitutes.

It is also important to notice that the Best Response Functions are independent of the parameters in the model.

■ **Group i chooses E, Group j chooses N:** Here, we will analyze the Best response functions of group i, which chooses E and group j, which chooses N. For ease of exposition we will assume that $i = A$ and $j = B$. The Best Response Functions in this case are represented in Figure 2.16. The Best Response function for group A, $R_A^E(X_B)$, is the same as in the previous case.

The Best Response Function of group B, $R_B^N(X_A)$, is increasing when $\frac{X_B}{X_A} > \frac{1+\theta_B}{1-\theta_B}$. The term on the right hand side is positive only when $\alpha_B \in (\frac{n_B-2}{n_B-1}, 1]$. In all other cases, the condition is trivially satisfied.

To see this clearly, in Figure 2.16, we have plotted the Best Response Function of group B for $\alpha_B = 0, \frac{1}{2}, 1$. When we increase α_B, $R_B^N(X_A)$, shifts inwards, because free riding increases within group B, which causes X_B to fall for the same group size n_B.

The Best Response Functions have a unique intersection and it is always at a point, where $R_B^N(X_A)$ is strictly decreasing. So, X_B is a strategic substitute of X_A in the neighborhood of any Nash equilibrium in group efforts .

X_A, on the other hand, is a strategic substitute to X_B as, long as $X_B > \frac{1}{4}$. So, when $X_B > \frac{1}{4}$, X_A and X_B are strategic substitutes. The Nash equilibrium in group effort levels, is stable.

However, when $X_B < \frac{1}{4}$, X_A is a strategic complement to X_B, while X_B is a strategic substitute of X_A. The Nash equilibrium in group effort levels, is unstable.

■ **Both groups choose N:** The Best Response Functions in this case are represented in Figure 2.17. As in the previous case the Best Response Function of group B, $R_B^N(X_A)$, is strictly increasing when $\frac{X_B}{X_A} > \frac{1+\theta_B}{1-\theta_B}$. However, now the Best Response function of group A, $R_A^N(X_B)$, is also increasing when $\frac{X_A}{X_B} > \frac{1+\theta_A}{1-\theta_A}$. The functions intersect uniquely to yield the Nash equilibrium in group efforts.

The functions intersect at a point, where $R_B^N(X_A)$ is decreasing. Therefore, X_B is a strategic substitute for X_A in the neighborhood of any Nash equilibrium. If, additionally at the equilibrium we have that $\frac{X_A}{X_B} < \frac{1+\theta_A}{1-\theta_A}$, so that $R_A^N(X_B)$ is also decreasing, then X_A is also a strategic substitute for X_B and the Nash equilibrium is stable.

If, however, $\alpha_A \in (\frac{n_A-2}{n_A-1}, \frac{n_A(2-N)}{(n_B-n_A)(n_A-1)} + \frac{2n_A(n_B-1)}{(n_B-n_A)(n_A-1)}\alpha_B)$, the functions intersect at a point where $R_A^N(X_B)$ is increasing. Here, X_A is a strategic complement to X_B. In this case the Nash equilibrium is unstable.

For this case to arise, we need both α_A and α_B to be sufficiently high and close to 1. One example of such a case is where $\alpha_A = 1$ and $\alpha_B = 1$. This is shown in Figure 2.17. When α_i rises, $i \in \{A, B\}$, $R_i^N(X_j)$ shifts in as free riding increases within group i but $R_j^N(X_i)$ is unaffected.

One interesting phenomenon, which arises in this case, is Monopolization of a group from the contest. If $\frac{\theta_B}{n_B} \geqslant \frac{1+\theta_A}{n_A}$, then the Best Response Function of group A is contained within the Best Response Function of group B, so that they do not intersect at any point in the interior, where both $X_A > 0$ and $X_B > 0$. Then in the Nash equilibrium in efforts, group B puts in an aggregate effort of $X_B^{NN} = \frac{1+\theta_A}{n_A}$ and group A members best respond with zero effort, so that $X_A^{NN} = 0$. So, we say that group A has been monopolized by group B. This phenomenon is captured in Figure 20. In a similar manner, group B is monopolized by group A when, $\frac{\theta_A}{n_A} \geqslant \frac{1+\theta_B}{n_B}$.

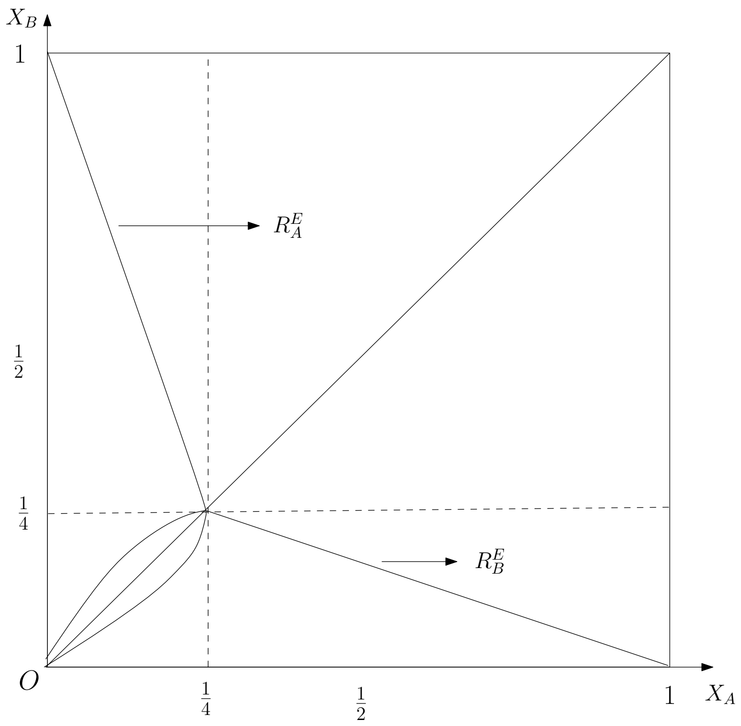

Figure 2.15: Best Responses with EE

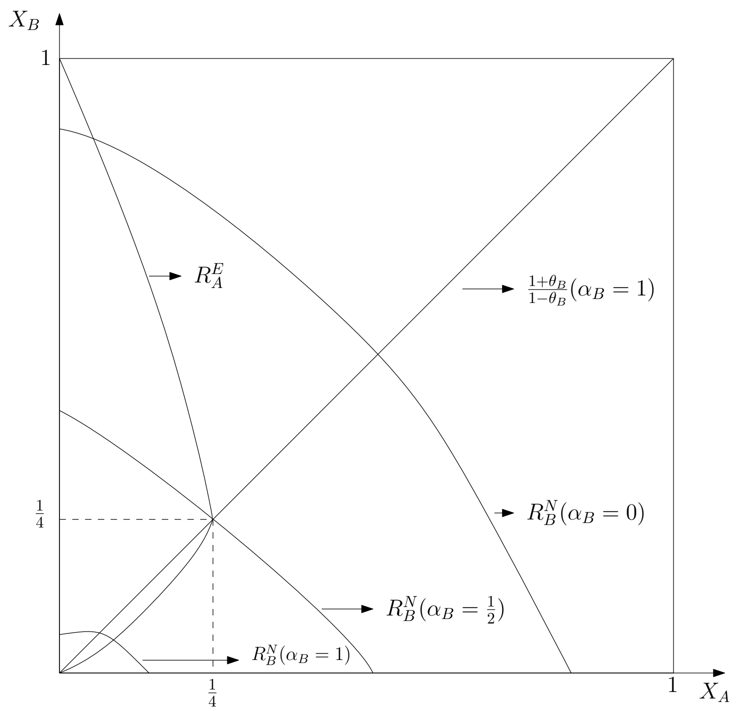

Figure 2.16: Best Responses with EN

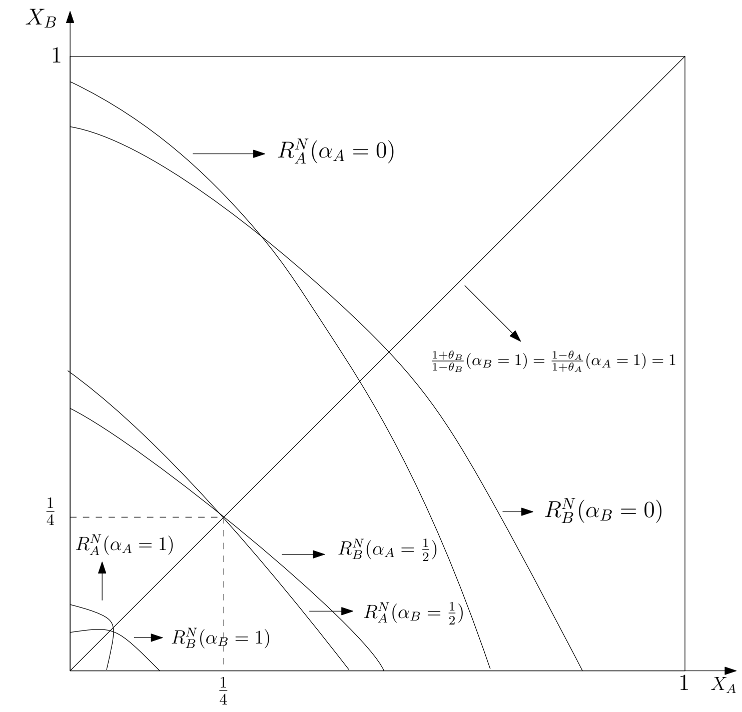

Figure 2.17: Best Responses with NN

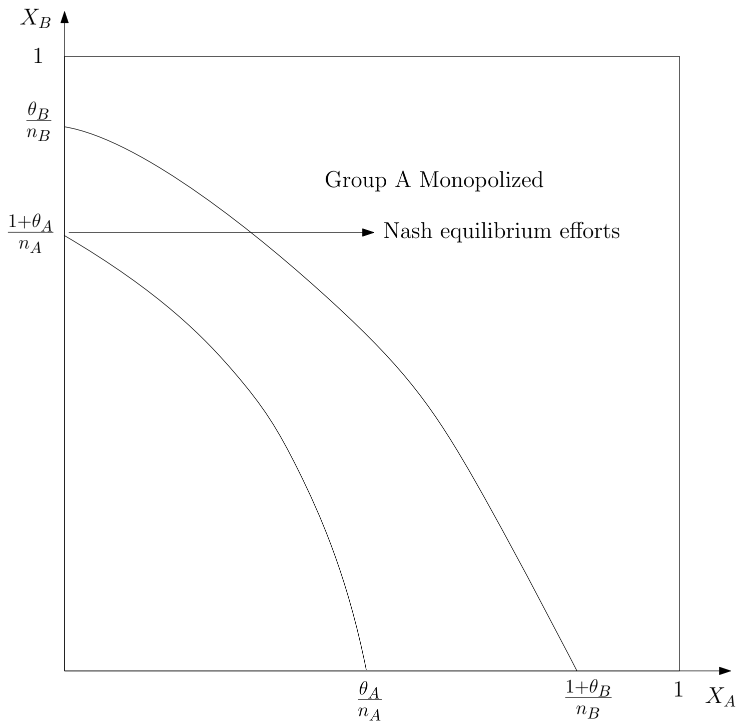

Figure 2.18: Group A Monopolized

2.12 Appendix 3

In this Appendix we provide the complete proofs of the Propositions stated in the Extensions of the chapter.

2.12.1 Generalized Tullock Contest Success Function

Proof of Proposition 8

Proof:

The expected payoff of individual k of group A is as follows

$$EU_{kA}(\text{E}) = \frac{1}{n_A}\left(\frac{X_A^r}{X_A^r + X_B^r} - X_A\right) \tag{2.71}$$

The expected payoff of individual k of group B is as follows

$$EU_{kB}(\text{E}) = \frac{1}{n_B}\left(\frac{X_A^r}{X_A^r + X_B^r} - X_A\right) \tag{2.72}$$

The following equation is the F.O.C. of members of group A

$$\frac{rX_A^{r-1}X_B^r}{(X_A^r + X_B^r)^2} = 1 \tag{2.73}$$

The following equation is the F.O.C. of members of group B

$$\frac{rX_B^{r-1}X_A^r}{(X_A^r + X_B^r)^2} = 1 \tag{2.74}$$

Equating (2.73) and (2.74) it follows that in equilibrium we muct have $X_A^{EE} = X_B^{EE} = X$. Replacing this fact in (2.73) we obtain $X = \frac{r}{4}$.

Of course, given that both groups put in equal amount of effort in equilibrium, we have $P_A^{EE} = P_B^{EE} = \frac{1}{2}$.

The payoffs can be easily obtained by replacing the efforts and probabilities of winning in the payoff function of the groups in (2.71) and (2.72). ■

Proof of Proposition 9

Proof:

The expected payoff of member k of group A is

$$EU_{kA}(\mathrm{E}) = \frac{1}{n_A}\Big(\frac{X_A^r}{X_A^r + X_B^r} - X_A\Big) \tag{2.75}$$

The expected payoff of member k of group B is

$$EU_{kB}(N) = \Big[(1-\alpha_B)\frac{x_{kB}}{X_B} + \frac{\alpha_B}{n_B}\Big]\frac{X_B^r}{X_A^r + X_B^r} - x_{kB} \tag{2.76}$$

The following equation represents the F.O.C. of the members of group A

$$\frac{rX_A^{r-1}X_B^r}{(X_A^r + X_B^r)^2} = 1 \tag{2.77}$$

The following equation represents the F.O.C. of member k of group B

$$\frac{rX_B^{r-1}X_A^r}{(X_A^r + X_B^r)^2}[(1-\alpha_B)\frac{x_{kB}}{X_B} + \frac{\alpha_B}{n_B}] + \frac{X_B^r}{X_A^r + X_B^r}[\frac{(1-\alpha_B)(X_B - x_{kB})}{X_B^2}] = 1 \tag{2.78}$$

Imposing symmetry on equation (2.78) we get

$$\frac{rX_B^{r-1}X_A^r}{(X_A^r + X_B^r)^2} + \frac{X_B^{r-1}\theta_B}{X_A^r + X_B^r} = n_B \tag{2.79}$$

Henceforth, we will work with equations (2.77) and (2.79). Define $x = X_A^r$ and $y = X_B^r$. Using this definition we can rewrite equation (2.77) as

$$\frac{rxy}{X_A(x+y)^2} = 1 \tag{2.80}$$

Similarly equation (2.79) can be rewritten as

$$\frac{rxy}{X_B(x+y)^2} + \frac{y\theta_B}{X_B(x+y)} = n_B \tag{2.81}$$

Dividing equation (2.81) by (2.80) we obtain the following equation

$$\frac{X_A}{X_B} + \frac{X_A}{X_B}\frac{\theta_B}{r}\left[1 + \frac{y}{x}\right] = n_B \tag{2.82}$$

Noticing that $\frac{y}{x} = \left(\frac{X_B}{X_A}\right)^r$ and realigning (2.82) we get the desired result.

To prove that there exists a unique solution to equation (2.23) we define $z = \frac{X_B}{X_A}$ and study the properties of the following function

$$y = \frac{\theta_B}{r}z^r - n_B z + \left(1 + \frac{\theta_B}{r}\right) \tag{2.83}$$

The function is continuous. At $z = 0$, $y = 1 + \frac{\theta_B}{r} > 0$. Also $\frac{dy}{dz} = \theta_B z^{r-1} - n_B$ and $\frac{d^2y}{dz^2} = (r-1)\theta_B z^{r-2} < 0$ as $r \in (0, 1]$. So the function is concave. The function is increasing and concave when $z < \left[\frac{\theta_B}{n_B}\right]^{\frac{1}{1-r}}$ and decreasing and concave if $z > \left[\frac{\theta_B}{n_B}\right]^{\frac{1}{1-r}}$.

To prove that $y = 0$ has a solution we study the limiting properties of y as $z \to \infty$.

$$\lim_{z \to \infty} \frac{\theta_B}{r}z^r - n_B z + \left(1 + \frac{\theta_B}{r}\right) = -\infty$$

This follows as $\left(1+\frac{\theta_B}{r}\right)$ is a positive constant and $\lim_{z\to\infty} \frac{\theta_B}{r}z^r - n_B z = \lim_{z\to\infty} z\left(\frac{\theta_B}{r}\frac{1}{z^{1-r}} - n_B\right) = -\infty$. This is true as the terms within the brackets is by bounded below by $-n_B$. The idea of the proof is captured in Figure 2.19.

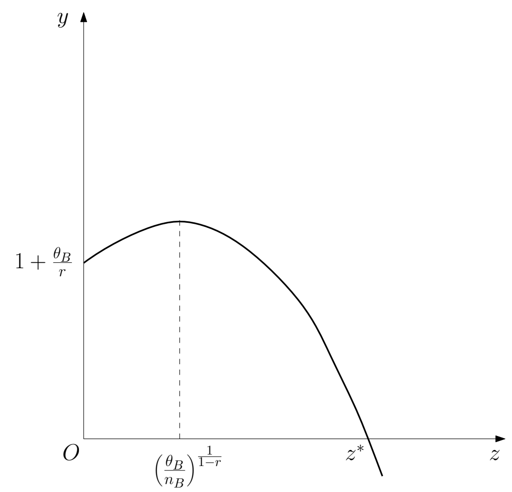

Figure 2.19: Uniqueness of z^*

■

Proof of Proposition 10

Proof:

From the previous proposition we know that $X_B = z^* X_A$ in equilibrium. Replacing this in (2.77) we get that $X_A = \frac{r(z^*)^r}{[1+(z^*)^r]^2}$ in equilibrium. And X_B is obtained from $X_B = z^* X_A$.

Replacing $X_B = z^* X_A$ in $P_A(X_A, X_B) = \frac{X_A^r}{X_A^r + X_B^r}$ we get that the equilibrium probability of winning as $P_A^{EN} = \frac{1}{1+(z^*)^r}$. And the probability of winning for group B is just $P_B^{EN} = 1 - P_A^{EN} = \frac{(z^*)^r}{1+(z^*)^r}$.

The equilibrium payoffs can be obtained from the fact that $\Pi_A^{EN} = P_A^{EN} - X_A^{EN}$ and $\Pi_B^{EN} = P_B^{EN} - X_B^{EN}$.

■

The proofs of Proposition 11 and 12 are skipped because are same as the proofs of Propositions 9 and 10.

Proof of Proposition 13

Proof:

The expected payoff of member k of group A is as follows:

$$EU_{kA}(N) = \big[(1-\alpha_A)\frac{x_{kA}}{X_A} + \frac{\alpha_A}{n_A}\big]\frac{X_A^r}{X_A^r + X_B^r} - x_{kA} \tag{2.84}$$

The expected payoff of member k of group B is as follows:

$$EU_{kB}(N) = \big[(1-\alpha_B)\frac{x_{kB}}{X_B} + \frac{\alpha_B}{n_B}\big]\frac{X_B^r}{X_A^r + X_B^r} - x_{kB} \tag{2.85}$$

The following equation represents the F.O.C. of member k of group A.

$$\frac{rX_A^{r-1}X_B^r}{(X_A^r + X_B^r)^2}[(1-\alpha_A)\frac{x_{kA}}{X_A} + \frac{\alpha_A}{n_A}] + \frac{X_A^r}{X_A^r + X_B^r}[\frac{(1-\alpha_A)(X_A - x_{kA})}{X_A^2}] = 1 \tag{2.86}$$

The following equation represents the F.O.C. of member k of group B.

$$\frac{rX_B^{r-1}X_A^r}{(X_A^r + X_B^r)^2}[(1-\alpha_B)\frac{x_{kB}}{X_B} + \frac{\alpha_B}{n_B}] + \frac{X_B^r}{X_A^r + X_B^r}[\frac{(1-\alpha_B)(X_B - x_{kB})}{X_B^2}] = 1 \tag{2.87}$$

Imposing symmetry on (2.86) we get

$$\frac{rX_A^{r-1}X_B^r}{(X_A^r + X_B^r)^2} + \frac{X_A^{r-1}\theta_A}{X_A^r + X_B^r} = n_A \tag{2.88}$$

Imposing symmetry on (2.86) we get

$$\frac{rX_B^{r-1}X_A^r}{(X_A^r + X_B^r)^2} + \frac{X_B^{r-1}\theta_B}{X_A^r + X_B^r} = n_B \tag{2.89}$$

Henceforth we will work with equations (2.88)and (2.89). Define $p = X_A^r$, $q = X_B^r$ and $z = \frac{X_B}{X_A}$. Using these definitions and dividing (2.89) by (2.88) we get

$$\frac{rpq + q(p+q)\theta_B}{rpq + p(p+q)\theta_A} = \frac{n_B}{n_A}z \tag{2.90}$$

Because neither group is monopolized $p > 0$ and $q > 0$ and we can divide the numerator and denominator of LHS of (2.90) by pq to get the following equation

$$\frac{r + \theta_B + z^r \theta_B}{r + \theta_A + \frac{\theta_A}{z^r}} = \frac{n_B}{n_A} z \tag{2.91}$$

Rearranging (2.91) we get that (2.25) must be satisfied along the Nash equilibrium path where neither group is monopolized.

Now we will show that a solution to (2.25) exists and is unique. To proceed we define $x = \frac{X_B}{X_A}$ and study the properties of the following function

$$y = n_A \theta_B x^r - n_B \theta_A x^{1-r} - n_B(r + \theta_A)x + n_A(r + \theta_B) \tag{2.92}$$

Because neither group is monopolized $x \in (0, \infty)$. It is clear that the function is continuous over its domain.

First, notice that $\lim_{x \to 0} y = n_A(r + \theta_B) > 0$. Also $\lim_{x \to \infty} y = \lim_{x \to \infty} x\big(n_A \theta_B \frac{1}{x^{1-r}} - n_B \theta_A \frac{1}{x^r} - n_B(r + \theta_A) + n_A(r + \theta_B)\frac{1}{x}\big) = -\infty$. This follows as the term within the bracket is bounded below by $-n_B(r + \theta_A)$.

These two observations immediately imply that at least one solution to $y = 0$ exists. To prove uniqueness we need some more properties of the function in (3.49).

The slope of the function (3.49) is as follows:

$$\frac{dy}{dx} = r n_A \theta_B x^{r-1} - (1 - r) n_B \theta_A x^{-r} - n_B(r + \theta_A) \tag{2.93}$$

It can be easily seen from (2.93) that $\lim_{x \to \infty} \frac{dy}{dx} = -n_B(r + \theta_A)$.

To study the properties of the slope as $x \to 0$ we write the slope as follows:

$$\frac{dy}{dx} = x^{-r}\big(r n_A \theta_B x^{2r-1} - (1 - r) n_B \theta_A - n_B(r + \theta_A) x^r\big) \tag{2.94}$$

There are three separate cases which we have to consider now. We will show that in all three cases there will exist a unique x^* which solves $y = 0$.

Case 1: $2r - 1 > 0$

In this case it can be seen from (2.94) that $\lim_{x\to 0} \frac{dy}{dx} = -\infty$. The term outside the bracket goes to ∞ and the term within the bracket is bounded below by $-(1-r)n_B\theta_A$. Also recall from (2.93) that $\lim_{x\to\infty} \frac{dy}{dx} = -n_B(r+\theta_A)$. So the function y is negatively sloped at the end points of the domain. The question to be answered is whether it can ever be positively sloped or not. We prove by contradiction that $\frac{dy}{dx} \ngeqslant 0$.

Suppose $\frac{dy}{dx} \geqslant 0$. From (2.94) it is clear that it will happen when $rn_A\theta_B x^{2r-1} \geqslant (1-r)n_B\theta_A + n_B(r+\theta_A)x^r$. The LHS is an increasing concave function with intercept 0. The RHS is an increasing concave function with positive intercept $(1-r)n_B\theta_A$. Let us assume that they intersect at z^*. Given that they cross at z^* and both are increasing and concave, it immediately follows that for all $z > z^*$ we have $\frac{dy}{dx} > 0$. But this contradicts the fact that $\lim_{x\to\infty} \frac{dy}{dx} = -n_B(r+\theta_A)$. Therefore, $\frac{dy}{dx} \ngeqslant 0$.

Therefore, $\frac{dy}{dx} < 0$ over the whole domain and $y = 0$ has a unique solution x^* in this case. Look at Figure 2.20.

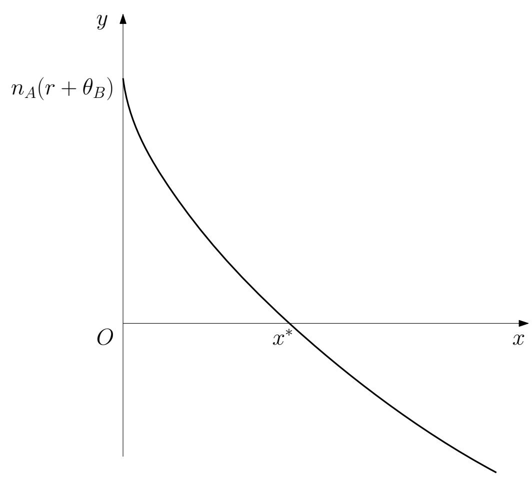

Figure 2.20: Case: $2r - 1 > 0$

Case 2: $2r - 1 < 0$

In this case note in (2.94) that $\lim_{x\to 0} \frac{dy}{dx} = \infty$. Also recall from (2.93) that $\lim_{x\to\infty} \frac{dy}{dx} = -n_B(r+\theta_A)$.

Notice that even though the slope is positive initially, i.e., $rn_A\theta_B x^{2r-1} > (1-r)n_B\theta_A + n_B(r+\theta_A)x^r$ when $x \to 0$, the LHS is a decreasing convex function and the RHS is an increasing concave function with intercept $(1-r)n_B\theta_A$. Therefore, they will intersect at some x_1 and $\frac{dy}{dx} < 0$ for all $x > x_1$. Hence, y will be a decreasing function beyond x_1 and $y = 0$ will therefore have an unique solution x^*. Look at Figure 2.21.

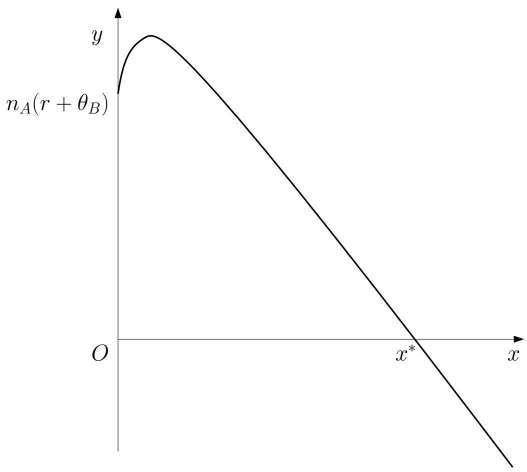

Figure 2.21: Case: $2r - 1 < 0$

Case 3: $2r - 1 = 0$

In this case (2.94) becomes

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}\big(\frac{1}{2}n_A\theta_B - \frac{1}{2}n_B\theta_A - n_B(\frac{1}{2} + \theta_A)\sqrt{x}\big) \tag{2.95}$$

Now $\lim_{x\to\ 0} \frac{dy}{dx} = \infty$ if $n_A\theta_B > n_B\theta_A$. And $\lim_{x\to\ 0} \frac{dy}{dx} = -\infty$ if $n_A\theta_B < n_B\theta_A$.

When $n_A\theta_B < n_B\theta_A$, the function y is negatively sloped throughout as $\frac{1}{2}n_A\theta_B < \frac{1}{2}n_B\theta_A + n_B(\frac{1}{2} + \theta_A)\sqrt{x}$. The LHS is a constant. The RHS is an increasing concave function which starts above $\frac{1}{2}n_A\theta_B$. Therefore $\frac{dy}{dx} < 0$ over the whole domain and $y = 0$ has unique solution. This case looks the same as Figure in 2.20.

When $n_A\theta_B > n_B\theta_A$, The function y is positively sloped initially. But, eventually becomes negatively sloped. This follows by noting that there exists some s such that $\frac{1}{2}n_A\theta_B < \frac{1}{2}n_B\theta_A + n_B(\frac{1}{2} + \theta_A)\sqrt{s}$. This is a consequence of the fact that the LHS is a constant and the RHS is an increasing and concave function, which is unbounded above. Therefore, $\frac{dy}{dx} < 0$ for all $x > s$. From this it follows that $y = 0$ will have a unique solution in this case as well. This case looks the same as in Figure 2.21.

■

Proof of Proposition 14

Proof:

From the previous Proposition we know that there exists a unique $(x^*) = \frac{X_B}{X_A}$ in equilibrium. Therefore, we have $X_B = (x^*)X_A$ in equilibrium.

Replacing this fact in the F.O.C. of group A in equation (2.88) we get the following equation

$$\frac{r(x^*)^r}{(1+(x^*)^r)^2} + \frac{\theta_A}{1+(x^*)^r} = n_A X_A \tag{2.96}$$

Solving for X_A from (2.96) we get X_A^{NN}. We get X_B^{NN}, by solving $X_B^{NN} = (x^*)X_A^{NN}$.

Replacing $X_B = (x^*)X_A$, in $P_A = \frac{X_A^r}{X_A^r + X_B^r}$ we get that $P_A^{NN} = \frac{1}{1+(x^*)^r}$. And P_B^{NN} is obtained by solving $P_B^{NN} = 1 - P_A^{NN}$.

The group payoffs can be obtained by using the computed X_A^{NN} and P_A^{NN} in $\Pi_A^{NN} = P_A^{NN} - X_A^{NN}$. Similarly, $\Pi_B^{NN} = P_B^{NN} - X_B^{NN}$.

■

Proof of Proposition 15

Proof:

To prove the results we need to show that $\Pi_A^{EE} \geqslant \Pi_A^{NE}$ and $\Pi_B^{EE} \geqslant \Pi_B^{EN}$ for all $r \in (0,1]$. We will show the second inequality $\Pi_B^{EE} \geqslant \Pi_B^{EN}$. The proof for the other will follow exactly the same steps and is skipped. There we need to show

$$\frac{2-r}{4} \geqslant \frac{(z^*)^r + (z^*)^{2r} - r(z^*)^{r+1}}{[1+(z^*)^r]^2} \tag{2.97}$$

Consider the function of the LHS of (2.97). It is a strictly decreasing linear function of r. It takes the value $\frac{1}{2}$ at $r = 0$ and the value $\frac{1}{4}$ at $r = 1$.

The function on the RHS of (2.97) takes the value $\frac{1}{2}$ at $r = 0$. At $r = 1$ and the value it takes is $\frac{(z^*)^r}{[1+(z^*)^r]^2}$. But it can be easily verified that $\frac{1}{4} \geqslant \frac{(z^*)^r}{[1+(z^*)^r]^2}$ for all z^*. Therefore at the endpoints the function on the RHS lies below the function on the LHS.

If we can show that the function on the RHS is strictly decreasing in r then we can claim that (2.97) holds for all r. That is what we do next.

Applying the Envelope Theorem we get that

$$\frac{d\Pi_B^{EN}}{dr} = -\frac{(z^*)^{r+1}}{[1+(z^*)^r]^2} < 0 \tag{2.98}$$

We know $\Pi_B^{EE} = \Pi_B^{EN}$ at $r = 0$, $\Pi_B^{EE} \geqslant \Pi_B^{EN}$ at $r = 1$. Π_B^{EE} is linearly decreasing in r and Π_B^{EN} is also strictly decreasing in r. In light of these observations we can conclude that $\forall r \in (0, 1]$, we must have $\Pi_B^{EE} \geqslant \Pi_B^{EN}$. We can show $\Pi_A^{EE} \geqslant \Pi_A^{NE}$ in a similar manner. The result is represented in Figure 2.22.

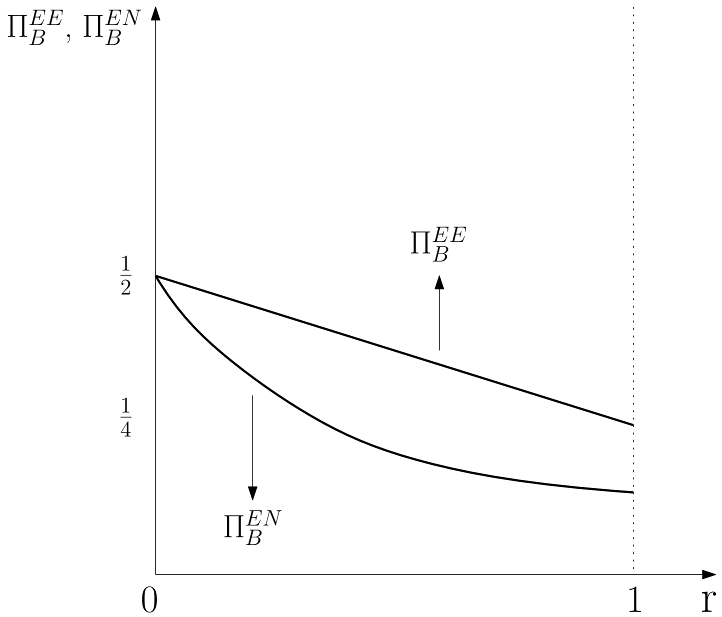

Figure 2.22: EE equilibrium

■

Proof of Proposition 16

Proof:

We provide sufficient conditions under which NN is a Nash equilibrium in G. So we will try to figure out the conditions under which $\Pi_A^{NN} > \Pi_A^{EN}$ and $\Pi_B^{NN} > \Pi_B^{NE}$. To keep the proof short we will only find the conditions under which $\Pi_A^{NN} > \Pi_A^{EN}$. We use the same method for the other inequality and therefore skip it in the proof.

To show when $\Pi_A^{NN} > \Pi_A^{EN}$ we have to show the following inequality.

$$\frac{(n_A - \theta_A)(1 + (x^*)^r) - r(x^*)^r}{n_A[1 + (x^*)^r]^2} > \frac{1 + (1 - r)(z^*)^r}{[1 + (z^*)^r]^2} \tag{2.99}$$

Notice by using Envelope Theorem we can easily show that the function on the LHS is strictly decreasing in r as

$$\frac{d\Pi_A^{NN}}{dr} = -\frac{(x^*)^r}{n_A[1 + (x^*)^r]^2} < 0 \tag{2.100}$$

Similarly, the function on the RHS is also strictly decreasing in r as

$$\frac{d\Pi_A^{EN}}{dr} = -\frac{(z^*)^r}{[1 + (z^*)^r]^2} < 0 \tag{2.101}$$

Also notice that at $r = 0$ The function on the RHS is strictly greater than LHS, i.e., $\Pi_A^{NN}(r = 0) < \Pi_A^{EN}(r = 0)$ as $\frac{2(n_A - \theta_A)}{4n_A} < \frac{1}{2}$.

Now when $\Pi_A^{NN}(r = 1) > \Pi_A^{EN}(r = 1)$ the functions will cross at some r_1 such that for all $r \in (r_1, 1]$ we have $\Pi_A^{NN} > \Pi_A^{EN}$. Similarly, we can find r_2 when considering $\Pi_B^{NN} > \Pi_B^{NE}$. Define , $r^* = max\{r_1, r_2\}$ and our result follows.

For example the sufficient condition is satisfied for group A when

$$\frac{(n_A - \theta_A) + (n_A - \theta_A - 1)(x^*)}{n_A[1 + (x^*)]^2} > \frac{1}{[1 + (z^*)]^2} \tag{2.102}$$

Similarly the sufficient condition is satisfied for group B when

$$\frac{(n_A - \theta_A)(x^*) + (n_A - 1 - \theta_A)(x^*)^2}{n_A[1 + (x^*)]^2} > \frac{1}{[1 + (y^*)]^2} \tag{2.103}$$

Recall from the analysis in the main chapter with $r = 1$, that inequalities (2.102) and

(2.103) are satisfied when $\alpha_A \in [\underline{\alpha}_A, \overline{\alpha}_A]$ and $\alpha_B \in [\underline{\alpha}_B, \overline{\alpha}_B]$. These inequalities above give us similar restrictions on α_A and α_B for the general case where $r \in (0, 1]$. Look at Figure 2.23.

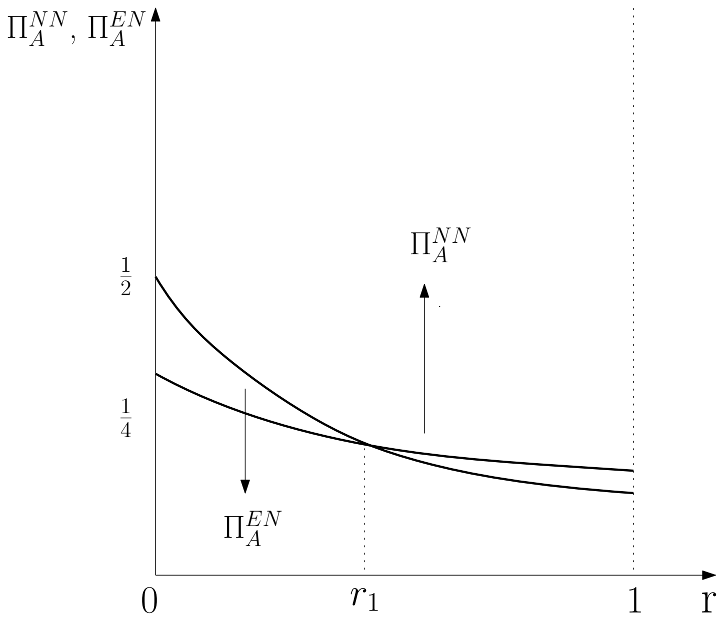

Figure 2.23: NN equilibrium

■

2.12.2 Group Leaders Maximize Probabilities of Winning

Proof of Proposition 17

Proof:

From Proposition 1 in the chapter we get the probabilities of winning for the groups.

To prove part (A) of the proposition notice that strategy E will be a best response to E for both groups if $P_A^{EE} \geqslant P_A^{NE}$ and $P_B^{EE} \geqslant P_B^{EN}$. That will be the case when the following two inequalities are satisfied

$$\frac{1}{2} \geqslant 1 - \frac{1 + \alpha_A(n_A - 1)}{n_A + 1} \tag{2.104}$$

and

$$\frac{1}{2} \geqslant 1 - \frac{1 + \alpha_B(n_B - 1)}{n_B + 1} \tag{2.105}$$

Inequality (2.104) is satisfied as long as $\alpha_A \geqslant \frac{1}{2}$. Inequality (2.105) is satisfied as long as $\alpha_B \geqslant \frac{1}{2}$.

To prove Part (B) notice that NN will be a Nash equilibrium as long as $P_A^{NN} \geqslant P_A^{EN}$ and $P_B^{NN} \geqslant P_B^{NE}$. That will be the case when the following two equations are satisfied

$$\frac{n_A + n_A(n_B - 1)\alpha_B - n_B(n_A - 1)\alpha_A}{N} \geqslant \frac{1 + \alpha_B(n_B - 1)}{n_B + 1} \tag{2.106}$$

$$\frac{n_B + n_B(n_A - 1)\alpha_A - n_A(n_B - 1)\alpha_B}{N} \geqslant \frac{1 + \alpha_A(n_A - 1)}{n_A + 1} \tag{2.107}$$

Inequality (2.106) is satisfied as long as $\alpha_A \leqslant \overline{\alpha}_A$ and inequality (2.107) is satisfied as long as $\alpha_B \leqslant \overline{\alpha}_B$.

To prove Part (C) notice that NE will be a Nash equilibrium if $P_A^{NE} \geqslant P_A^{EE}$ and $P_B^{NE} \geqslant P_B^{NN}$. The first inequality is satisfied as long as $\alpha_A \leqslant \frac{1}{2}$. It can be seen from proof of Part (A). The second inequality is satisfied as long as $\alpha_B \geqslant \overline{\alpha}_B$. This can be seen from proof of Part (B).

The proof of Part (D) is similar to proof of Part (C). ■

Chapter 3

Prize Sharing Rules in Collective Contests: When Does Group Size Matter?

3.1 Introduction

Collective contests are situations where agents organize into groups to compete over a given prize. Such situations are quite common: funds to be allocated among different departments of an organization, team sports, projects to be allocated among different divisions of a firm, regions within a country vying for shares in national grants, party members participating in pre-electoral campaigns,disputes between tribes over scarce resources.

Prizes in such contests may be purely private, e.g. money. Or the prizes may have some public characteristic like reputation or glory for the winning team. In this chapter we focus on purely private prizes. For prizes with public characteristics the reader may refer to Baik (2008), Balart et al. (2016).

One essential feature of collective contests is that a groups' performance depends on the individual contribution of its members. Departments in universities usually receive funds depending on the publication record of the department, which in turn depends on the individual publication of its members. So the group needs to coordinate and establish some rules regarding its internal organization, in particular how to share the prize in case of success. In this study we focus the prize sharing rule proposed by Nitzan (1991). The rule suggests the following way of sharing the prize within the group, if the group wins the collective contest:

$$(1-\alpha_i)\frac{x_{ki}}{X_i}+\alpha_i\frac{1}{n_i} \tag{3.1}$$

where x_{ki} is the effort put in by the k^{th} member of group i, X_i is the total effort of group i and n_i is the size of group i. α_i is weight put on egalitarian sharing of the prize within the group and $1-\alpha_i$ is the weight put on a sharing rule, which rewards higher efforts within the group, thereby inducing intra-group competition, i.e. an outlay-based incentive scheme. An increase in the weight on the egalitarian component increases free riding incentives in the group members. Whereas, an increased weight on the outlay-based component incentivizes efforts by making each members reward depend on efforts of all other members of the group.

This prize sharing rule has been extensively studied in the literature on collective contests, see e.g. Flamand et al. (2015). The popularity of this rule lies in its intuitive appeal. It combines two extreme forms of internal organization, capturing the tension between intra-group competition and the tendency to free ride on the efforts of other group members. In the situation of a collective contest, a larger weight on the outlay-based scheme helps a group generate higher efforts, thereby increasing their chances of winning the contest. But, higher efforts also eat into the surplus the groups are competing for, thereby making internal competition costly. A larger weight on the egalitarian component increases internal free riding making a group less competitive in the contest but leaves a larger surplus to be consumed in case of success. This is the trade-off, which the group leaders face when choosing its organizational form i.e., the weight he wants to put on the respective components of the prize sharing rule.

The literature on strategic choice of sharing rules see e.g. Flamand et al. (2015), allows the leader exactly this choice. A group leader can optimally choose the weight α_i for his own group. But there are two separate strands in this literature, which differ on the restrictions which are placed on that choice.

In one strand, the choice of shares α_i is restricted to the interval $[0,1]$, so that the leader can choose to reward individual efforts at most proportionally. This situation is referred to as the case of "bounded meritocracy" in Balart et al. (2016). In an alternate strand, the leader is allowed to reward efforts more than proportionally by fining members, who put in lower effort and transferring that amount to the hard working ones. In such a case the interval over which α_i is chosen is $(-\infty, 1]$ [1]. This case is called "unbounded meritocracy" in Balart et al. (2016). The literature finds that when the leaders choose the rules simultaneously, at least one of the groups chooses not to all the weight on the outlay-based component of the prize sharing rule in equilibrium i.e., the leader of at least one group chooses not to make the group maximally competitive in the contest. This is irrespective of whether the rule is "boundedly meritocratic" or "unboundedly meritocratic".

We generalize the above literature by fixing the choice of α_i to the interval $[\underline{\alpha}_i, 1]$, where $\underline{\alpha}_i \in (-\infty, 1]$ is a parameter in the model. It can be interpreted as a social norm of competitiveness within the groups. This social norm, just like group sizes, is taken as an exogenous property of the groups and denotes the maximum possible competitiveness of a group. So, we can have smaller groups with very competitive norms i.e., "small aggressive groups" or large groups with egalitarian norms i.e., "large docile groups" etc. One can imagine such group specific social norms to have developed through intra-group interactions in times of peace but which acts as constraints on the group leader in times of conflict. We assume that when competing with the other group, a leader has to respect this group specific norm while choosing how to share the prize in case of success in the contest. In our study we make necessary adjustments and call group i "boundedly meritocratic" if $\underline{\alpha}_i \geqslant 0$. Otherwise, group i is called "unboundedly meritocratic".

The above modeling innovation allows us to unify the different strands of the literature, so that both strands emerge as special cases in our model [2]. Moreover, we are able to identify situations in which both groups choose to make their groups maximally competitive in equilibrium of the contest game between the groups, i.e., both groups put maximal weight on

[1] Readers can look at Hillman and Riley (1989) for a paper where such transfers between individuals is possible.

[2] Both "bounded meritocracy" and "unbounded meritocracy" are special cases in our model

the outlay-based incentive scheme by choosing $\alpha_i = \underline{\alpha}_i$. We call a group "**hawkish**" when it chooses to put all the weight on the outlay-based scheme. Otherwise, we call a group "**dovish**".

We assume throughout that group B is at least as large as group A. We find that the smaller group A generally chooses to be hawkish. It counters the disadvantage of having smaller numbers in the contest by putting all the weight on the outlay-based component of the rule, thereby generating maximum possible efforts by its group members. In other words, the smaller group focuses exclusively on winning the contest. The larger group B, on the other hand, is usually not hawkish. In a sense, the onus of maintenance of a larger net surplus falls on the larger group, when $\underline{\alpha}_B$ is low enough. If it chooses to be hawkish, then it would win the contest more often, but most of the prize would have dissipated due to large efforts by its large numbers. It is only when $\underline{\alpha}_B$ is really high i.e., group B is sufficiently "boundedly meritocratic", that it too shifts to being hawkish in order to increase its chances of winning the contest. When $\underline{\alpha}_B$ is high, free riding becomes the overriding force in group B and larger size actually becomes a handicap. The best a larger group can do to counter the disadvantage, is take a hawkish stance. In Proposition 20 and Corollary 2, we precisely identify the conditions under which both groups choose to be hawkish in equilibrium. This is an important observation as taking a hawkish stance, which increases a group's chance of success in the contest, seems to be a natural path for a group leader to take in a collective contest.

Next, we focus on the welfare of the groups in the contest, specifically focusing on the following question: When does the larger group fare worse in the contest in terms of chances of success? The fact that larger groups may fare worse in competition with smaller ones was first identified by (Olson, 1965) and it was named The Group Size Paradox (GSP). We find that if smaller group A is "unboundedly meritocratic" then GSP cannot be avoided. This result is independent of the nature of meritocracy in the larger group B. Therefore, a necessary condition for Group B to fare better in the contest is for smaller group to be "boundedly meritocratic", i.e, the smaller group should not be in a position to undo the disadvantage of smaller numbers by being "hawkish". The situation where the larger group fares better is called Group Size Advantage (GSA) in this chapter.

A sufficient condition for group B to fare better in the contest is for group A to be "boundedly meritocratic" and group B to be "unboundedly meritocratic". In this case group A cannot undo the disadvantage of smaller numbers by using the prize sharing rule, while the rule imposes no constraint on the leader of the larger group B.

The most interesting case arises when both groups are "boundedly meritocratic". Whether group B fares better or not entirely depends on the asymmetry between the norms of comepetitiveness across groups. If the norms are too asymmetric i.e., $\underline{\alpha}_A$ very high and $\underline{\alpha}_B$ very low, or vice versa, then whichever group is less comeptitive does worse due to excessive free riding. In cases of extreme asymmetry, egalitarian groups may end up getting monopolized (Ueda (2002)).

If the norms of competitiveness are symmetric across groups, i.e.,$\underline{\alpha}_A$ and $\underline{\alpha}_B$ are very close to each other, then whether GSP arises or not depends on whether both groups egalitarian or both are competitive. If both groups are egalitarian i.e., ($\underline{\alpha}_A > \frac{1}{2}$ and $\underline{\alpha}_B > \frac{1}{2}$), then GSP occurs because free riding is the dominant force for both groups in this case and it affects the larger group more adversely. In fact, this case corresponds precisely to the type of groups (Olson, 1965) studied in *The Logic of Collective Action.* We call this class of groups **Olson's Groups.**

On the other hand, if both groups are competitive i.e., ($\underline{\alpha}_A < \frac{1}{2}$ and $\underline{\alpha}_B < \frac{1}{2}$), then intra-group competition is the dominant force for both groups. In such a case, having a larger group size is an advantage and we have GSA. This class of groups are a mirror image of the type of groups (Olson, 1965) studied [3]. We call this class of groups **Neo-Olson Groups**.

The chapter is structured as follows. In Section 2 we discuss the relevant literature. In Section 3 we describe the model. In Section 4 we analyze the second stage of the game, where

[3](Olson, 1965), however, did not study a collective contest but focused on collective action problems within a single group and related it to its size. But, his insight generalizes to a situation of collective contests.

individuals make effort choices. In Section 5 we analyze the first stage of the game where the group leaders make their choice of the sharing rule. In Section 6 we discuss when the phenomenon of Group Size Paradox arises and when it does not. In Section 7 we provide a few extensions of the basic model. Section 8 concludes. All proofs can be found in the Appendix in Section 3.10.

3.2 Literature

The literature on the prize sharing rules in collective contests owes its genesis to the influential paper by Nitzan (1991). Following its introduction the rule has become the gold standard in the field due to the simple manner it combines two extreme forms of internal organization of groups i.e. one form, which encourages intra-group competition and another which promotes egalitarinism thereby reducing internal competition. To be clear, the prize sharing rule was first analyzed in Sen (1966). But their analysis focused on the optimality of the rule in a labour cooperative (a single group of workers). Throughout this chapter we focus on collective contests, where two groups compete for a rent and the influence that has on how the groups internally organize themselves.

The literature on strategic choice of sharing rules focuses on the endogenous choice of internal organization of groups i.e. the group leaders have an option to optimally choose the weight he wants to place on the outlay based incentive scheme, which encourages higher group efforts by promoting internal competition. Two strands have emerged in the literature, which differ on the restriction placed on the leaders choice parameter. In the first strand ((Baik, 1994), (Lee, 1995), (Noh, 1999), Ueda (2002)), the leaders of the groups are allowed to choose α_i on the interval $[0, 1]$. So the outlay-based incentive can be at most proportional to efforts, i.e. the leaders cannot fine members who slack. The second strand (Baik and Shogren (1995), Baik and Lee (1997), Baik and Lee (2001), Lee and Kang (1998), Gürtler (2005)), makes the choice unrestricted , so that $\alpha_i \in (-\infty, 1]$.

In both cases the larger group chooses a less outlay-based incentive scheme than the smaller group i.e. the larger group takes a dovish stance. The reason is that there exists a trade-off between the chances of winning the contest and the size of the surplus net of efforts, which remains for *ex post* consumption. If the larger group implements maximum competition within its group, then given the advantage of size it wins the collective contest more often but the surplus that is left over is too small. As it turns out, the larger group optimally chooses a dovish stance to preserve a larger portion of the surplus.

We extend the above literature by proposing the restriction on the leaders choice of α_i to be over the $[\underline{\alpha}_i, 1]$, where $\underline{\alpha}_i \in (-\infty, 1]$ is a parameter in the model. Both strands emerge as special cases in our model. Our analysis generalizes the literature cited above and in the process allows us to analyze the conditions under which both groups choose to be hawkish , focusing just on winning the contest by putting maximal weight on the outlay-based scheme.

Additionally, we discuss conditions under which the larger group loses the contest more often, so that Group Size Paradox (GSP) applies. Even though it is not central to the main question addressed in this chapter, we still report the results given that this has been a primary focus of the literature on collective contests. For example, look at (Nitzan and Ueda, 2011), (Balart et al., 2016) and (Esteban and Ray, 2001).

3.3 Model

There are two groups A and B, of size n_i, $i = \{A, B\}$, where $n_i \in \{2, 3, ...\}$. We assume without loss of generality that group B is at least as large as A, i.e. $n_B \geqslant n_A$. We denote the total number of agents as N, so that $N = n_B + n_A$. All agents are assumed to be risk neutral.

Both groups compete for a purely private prize, the size of which we normalize to 1. The groups cannot write binding contracts among themselves regarding sharing the prize. Instead they indulge in a rent-seeking Tullock contest spending effort trying to win the contest. The outcome of this contest depends on the aggregate effort spent by the two groups. Let x_{ki}

denote the effort level of individual k belonging to group i, where effort costs are $C(x_{ki})$. For simplicity we take $C(x_{ki}) = x_{ki}$. The aggregate effort of group i is $X_i = \sum_{k=1}^{n_i} x_{ki}$.

The efforts do not add to productivity, and only determine the probability $P_i(X_i, X_j)$ that group i wins the contest. We assume that $P_i(X_i, X_j)$ takes the ratio form, i.e.

$$P_i(X_i, X_j) = \begin{cases} \frac{X_i}{X_i+X_j}, & \text{if } X_i > 0 \text{ or } X_j > 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{3.2}$$

Every group has a leader, who has the authority to enforce a sharing rule that specifies how the groups payoffs are to be shared within the group in case the groups wins the contest. Both leaders are benevolent, maximizing the expected group payoff while making their decisions.

We assume that the group leader has access to the prize sharing rules introduced by Nitzan (1991), which is described follows:

$$s_{ki}(x_{ki}, X_i; \alpha_i, n_i) = (1 - \alpha_i)\frac{x_{ki}}{X_i} + \frac{\alpha_i}{n_i}. \tag{3.3}$$

We also assume that, a group leader can choose the level of α_i for his group. Given the choice of α_i, the share of the prize the k^{th} member of group i gets is s_{ki}. It should be noted that this prize sharing rule is feasible as $\sum_{k \in n_i} s_{ki} = 1$.

The rule is a weighted average of an egalitarian component $\frac{1}{n_i}$ and a competitive component $\frac{x_{ki}}{X_i}$. The egalitarian component is an incentive scheme, which makes individual rewards independent of efforts. Therefore, a positive weight on it causes individual members of a group to free ride in effort provision. This reduces aggregate group efforts, leading to lower prize dissipation. The result is that a larger *ex post* surplus can be enjoyed by the group in case of success at the cost of lower chances of winning the contest itself.

The competitive component, on the other hand, is an outlay based incentive scheme, which rewards more those individuals, who have put in higher efforts within the group. The resultant competition within the group raises individual efforts, which in turn increases aggregate group effort. As a consequence, the chances of success in the contest increases for the group but now most of the prize gets dissipated in costly effort provision, which reduces the *ex post* surplus to be enjoyed in case of success.

In line with the literature on strategic choice of prize sharing rules, e.g. Flamand et al. (2015), this chapter explicitly focuses on how this trade-off influences the choice of α_i by the group leader.

We assume that when choosing the weights to put on the different components of the prize sharing rule, a leader is subject to group specific norms of competitiveness. In particular, the leader of group i, $i \in \{A, B\}$, is assumed to choose $\alpha_i \in [\underline{\alpha}_i, 1]$, where $\underline{\alpha}_i \in (-\infty, 1]$ [4]. In other words, the "lower bound" $\underline{\alpha}_i$ corresponds to the maximum amount of competition that a group leader can generate within his group, i.e., the maximum weight he can place on the outlay-based incentive component. This limit on the competitiveness, which is a feature specific to a group, may be imagined to have developed out of long term interactions among group members. To be clear, the restriction implies that the leader can lower competition within the group with respect to the group norm, by choosing $\alpha_i > \underline{\alpha}_i$. He, however, cannot increase internal competition beyond a certain limit given by $\underline{\alpha}_i$. In this chapter we do not go into the sources of such group specific norms and take them as fixed. For a study on the emergence of social norms in an experimental setting, readers may look at Grimalda et al. (2008).

It should also be made clear at this point that these restrictions generate an interplay of the two main forces in our model. If $\underline{\alpha}_i$ is high enough then free riding is a dominant force within group i and a larger group size is then a disadvantage as far as chances of winning the contest is concerned. On the other hand if $\underline{\alpha}_i$ is low enough then the force of competition is dominates and a larger group size would be an advantage . How these different intra-group forces play out , where two groups of different sizes and different social norms are matched in a collective contest, is the meat of this chapter.

[4]In the existing literature the cases considered are $\underline{\alpha}_i = 0$ and $\underline{\alpha}_i = -\infty$

After group i leader chooses α_i, individual k in group i chooses efforts x_{ki} to maximize his expected utility, which is as follows:

$$EU_{ki}(\mathrm{N}) = \begin{cases} s_{ki}(x_{ki}, X_i; \alpha_i, n_i)P_i(X_i, X_j) - x_{ki} & \text{if } X_i > 0, X_j \geqslant 0, \\ \frac{1}{2n_i} & \text{if } X_i = X_j = 0, \\ 0 & \text{if } X_i = 0, X_j > 0. \end{cases} \tag{3.4}$$

It should be noted that in this case only the ratio of the individual to the total group effort needs to be verifiable.

■ **Leader's Objective:** Recall that the leaders of both groups are benevolent social planners who choose $\alpha_i \in [\underline{\alpha}_i, 1]$, where $\underline{\alpha}_i \in (-\infty, 1]$, to maximize net group payoffs.

The maximization problem of leader of group i can be written as follows:

$$\max_{\alpha_i \in [\underline{\alpha}_i, 1]} P_i(X_i, X_j) - X_i \tag{3.5}$$

Given that $P_i(X_i, X_j)$ takes the ratio form it is straight forward to check that leader i's maximization problem can be re-written as follows [5]:

$$\max_{\alpha_i \in [\underline{\alpha}_i, 1]} P_i(X_i, X_j)(1 - X) \tag{3.6}$$

where $X = X_i + X_j$.

The payoff representation in (3.6) is intuitive, and captures the trade-off inherent in the group leader's maximization problem. X measures the amount of prize dissipated in the competition between the two groups. Therefore $1 - X$ is the surplus net of efforts, which remains for *ex post* consumption in case of success. The probability with which group i wins this net surplus is $P_i(X_i, X_j)$. If leader of group i wants to win the contest with a higher

[5] $P_i(X_i, X_j) - X_i = \frac{X_i}{X_i + X_j} - X_i = \frac{X_i}{X_i + X_j}(1 - X_i - X_j) = P_i(X_i, X_j)(1 - X)$

probability he has to take measures, which increase group efforts X_i. But when X_i goes up so does X, which reduces the size of the net surplus.

■ **Description of the Game:** Our game consists of two stages. In the first stage the leaders simultaneously choose their respective sharing rule $\alpha_i \in [\underline{\alpha}_i, 1]$, $i = A, B$. Having observed the choice of the sharing rules, in stage 2 all agents simultaneously decide on their own effort levels.

We denote the equilibrium of the game $\sigma^* = (\sigma_A^*, \sigma_B^*)$.

We solve for the Subgame Perfect Nash equilibrium (SPNE) of the game described above.

3.4 Choice of Individual Efforts

In this section we characterize the Nash equilibrium effort choices of individual members of the groups taking as given the sharing rules α_A and α_B, which are chosen by the group leaders in the first stage.

Before stating the results we need to state a few definitions, which we will use throughout the chapter.

First, we define the phenomenon of Monopolization of a group in the contest, which is well recognized in the collective contest literature, see e.g. Davis and Reilly (1999), Ueda (2002).

Definition 8 ***Monopolization***

A SPNE $\langle \alpha_A^*, \alpha_B^* \rangle$ is said to involve monopolization of group i, if in equilibrium group i does not put in any effort in the contest.

Equilibrium Net Surplus and Probabilities of Success

In the following proposition we report the surplus net of effort, which remains for consumption, i.e. $1-X$, which we denote S. We also report the probabilities with which each group wins the net surplus, P_i and P_j. Such a choice was made to keep the discussion in line with the basic trade-off in the model. In the Appendix we provide the relevant details. Before proceeding we introduce the following notations:

Henceforth, we denote the surplus net of efforts as S, so that $S = 1 - X$.[6]

For $i, j \in \{A, B\}$ and $i \neq j$ we define

$$\chi_i = n_i + n_i(n_j - 1)\alpha_j - n_j(n_i - 1)\alpha_i. \tag{3.7}$$

χ_i can be interpreted as a measure of the competitiveness of group i relative to group j. Note that χ_i is increasing in α_j and decreasing in α_i. When α_j is large relative to α_i, group j is relatively less competitive, which gives group i an advantage in the contest. On the other hand when α_i is large relative to α_j, group j wins the contest more often. In fact, as we see in the following Proposition, the probability with which group i wins the contest is directly proportional to χ_i.

Proposition **19**

Consider $i, j \in \{A, B\}$ and $j \neq i$.

(A) If $\chi_i \leqslant 0$ [7] then group i is monopolized by group j. In the unique intra-group symmetric Nash equilibrium of the effort subgame

(a) The net surplus in the contest is $S^{iM} = \frac{1+\alpha_j(n_j-1)}{n_j}$ [8].

(b) The probabilities of winning are $(P_i^{iM}, P_j^{iM}) = (0, 1)$.

(B) If $\chi_i > 0$ and $\chi_j > 0$ then neither group is monopolized. In the unique intra-group symmetric Nash equilibrium of the effort subgame

(a) The net surplus in the contest is $S^{NM} = \frac{1+\alpha_j(n_j-1)+\alpha_i(n_i-1)}{N}$ [9].

[6] This stands for the effective prize over which the contest takes place. See (3.6).

[7] When $\chi_i \leqslant 0$ then $\chi_j > 0$ as $\chi_i + \chi_j = N$

[8] The first component in the superscript stands for the group which is monopolized and the second stands for the word "monopolized"

[9] The first component in the superscript stands for the word "neither" and the second stands for the word "monopolized"

(b) *The probabilities of winning are* $(P_i^{NM}, P_j^{NM}) = (\frac{\chi_i}{N}, 1 - \frac{\chi_i}{N})$.

We next discuss the results summarized in Proposition 19

■ **Group i is Monopolized:** When $\chi_i \leqslant 0$ group i retires from the contest. This is exactly the same monopolization condition found by Ueda (2002). Furthermore, $\chi_i \leqslant 0$ when we have a low α_j and a high α_i. Therefore, group j members are extremely active due to individual incentives to exert effort, whereas free riding is such a dominant force in group i that individual efforts fall to zero. The effort group j exerts in this case is $X_j^{iM} = \frac{(n_j-1)(1-\alpha_j)}{n_j}$, which leaves a net surplus of $S^{iM} = \frac{1+\alpha_j(n_j-1)}{n_j}$. S^{iM} increases in α_j because the effort necessary to monopolize group i decreases with α_j, which leaves more surplus more consumption of group j.

■ **Neither group is Monopolized:** This case arises when $\chi_i > 0$ and $\chi_j > 0$, which immediately implies α_i and α_j cannot be too asymmetric. Notice that the probability that group i wins is directly proportional to χ_i. For χ_i to be high we need a α_i to be low relative to α_j, i.e., members of group i are relatively more active than members of group j.

It can be seen that the net surplus S^{NM} is increasing in both α_i and α_j. This follows from the fact that an increase in α_i or α_j exacerbates free riding within the groups, causing aggregate efforts in the contest to fall.

Proposition 19 helps us set up the optimization problems that the leaders face in the first stage. We now move to the first stage and characterize the Nash equilibrium.

3.5 Choice of Sharing Rules by Group Leaders

In this section we analyze the Nash equilibrium choice of the group leaders in the first stage. This leads us to the main result of this chapter.

First, we define the stances taken by the group leaders in equilibrium. Group i is called *hawkish* if in equilibrium its leader chooses to implement maximal competition by putting all the weight on the outlay-based component of the prize sharing rule, i.e., $\alpha_i = \underline{\alpha}_i$. A group i is called *dovish* if in equilibrium its leader puts some weight on the egalitarian component of the prize sharing rule, thereby not implementing maximum group efforts, i.e. $\alpha_i > \underline{\alpha}_i$.

It should be made clear that in this chapter the terms hawkish and dovish are not meant in the usual sense of extremes on a uni-dimensional scale. Hawkish and dovish behavior are with respect to group specific norms of competitiveness. A "hawk" focuses entirely on winning the contest by choosing $\alpha_i = \underline{\alpha}_i$. A "dove", on the other hand, does not entirely focus on winning the contest. It puts some attention on maintaining a larger net surplus by choosing $\alpha_i > \underline{\alpha}_i$.

Definition **9**

We call group i hawkish iff its leader chooses $\alpha_i = \underline{\alpha}_i$ in equilibrium. Otherwise, we call group i dovish.

3.5.1 Leader's Optimization Problem

In view of Proposition 19, we can set up the optimization problem of the group leaders noted in (3.6). We look at how the leader of group i optimally chooses α_i, given a fixed α_j.

If leader of group i wants to monopolize group j then he has to choose α_i such that $\chi_j \leqslant 0$. This observation follows from part (A) in Proposition 19. In that case we can write down his optimization problem as follows:

$$\max_{\alpha_i \in [\underline{\alpha}_i, 1]} \frac{1 + \alpha_i(n_i - 1)}{n_i} \quad \text{s.t.} \quad \chi_j \leqslant 0 \tag{3.8}$$

The solution to this problem is simple. As both the objective function and χ_j are increasing in α_i the leader will just set α_i such that $\chi_j = 0$ for given α_j. We now define a cutoff α_i^{jM} and call it the Monopolization cutoff. α_i^{jM} solves $\chi_j = 0$ at $\alpha_j = \underline{\alpha}_j$.

Definition **10** ***Monopolization Cutoff*** (α_i^{jM})

For $i, j \in \{A, B\}$ and $j \neq i$, the Monopolization Cutoff α_i^{jM} is defined as follows:

$$\alpha_i^{jM} = -\frac{1}{n_i - 1} + \frac{(n_j - 1)n_i}{(n_i - 1)n_j}\underline{\alpha}_j.$$

The cutoff α_i^{jM} is such that if group j chooses $\alpha_j = \underline{\alpha}_j$, then the best choice of group i if it wants to monopolize group j is α_i^{jM}.

Now we consider the case where neither group is monopolized, i.e., $\chi_i > 0$ and $\chi_j > 0$. In that case using part (B) of Proposition 19 and (3.6) we can write the optimization problem of the leader of group i as follows:

$$\max_{\alpha_i \in [\underline{\alpha}_i, 1]} \left(\frac{\chi_i}{N}\right)\left(\frac{1 + \alpha_j(n_j - 1) + \alpha_i(n_i - 1)}{N}\right) \quad \text{s.t.} \quad \chi_i > 0 \quad \text{and} \quad \chi_j > 0 \tag{3.9}$$

The solution to problem (3.9) is non- trivial as χ_i is decreasing in α_i but the second term in brackets, which is the net surplus S^{NM}, is increasing in α_i. So to solve it we set up the Kuhn Tucker problem. The Lagrangian of group i given $i, j \in \{A, B\}$ and $j \neq i$, can be written as follows:

$$L_i = \left(\frac{\chi_i}{N}\right)\left(\frac{1 + \alpha_j(n_j - 1) + \alpha_i(n_i - 1)}{N}\right) + \lambda_i \left(\alpha_i - \underline{\alpha}_i\right) \tag{3.10}$$

Notice that we ignore the constraints $\underline{\alpha}_i \leqslant 1$ and $\chi_i > 0$ and $\chi_j > 0$ while setting up the Lagrangian. We check later that they are satisfied. Maximizing the function in (3.10) leads to a few cutoffs we need to define. These cutoffs help us delineate the parametric space by which group's constraint binds and which group's does not in equilibrium.

Definition 11 ***Group i-Binding Cutoff*** ***(α_j^{iB})***

For $i, j \in \{A, B\}$ and $j \neq i$, Group i-the Binding Cutoff α_j^{iB} [10] *is defined as follows:*

$$\alpha_j^{iB} = \frac{n_j - n_i}{2n_i(n_j - 1)}(1 + \underline{\alpha}_i(n_i - 1)).$$

The Group i-Binding Cutoff α_j^{iB} arises from the Kuhn-Tucker conditions associated with L_i and L_j in (3.10). It arises when we assume that $\alpha_j > \underline{\alpha}_j$ and $\alpha_i = \underline{\alpha}_i$, so that $\lambda_j = 0$ and

[10] The first component of the superscript represents the group whose constraint binds and the second denotes the word binds

$\lambda_i \geqslant 0$. The cutoff helps us identify the parametric region where groups i's constraint will bind but group j's will not in equilibrium[11].

Definition **12** ***Non-Binding Cutoffs***(α_i^{NN})

For $i, j \in \{A, B\}$ and $j \neq i$ the Non-Binding cutoffs are defined as follows:

$$\alpha_i^{NN} = \frac{n_i - n_j}{N(n_i - 1)}.$$

.

The Non-Binding cutoffs, α_i^{NN} and α_j^{NN}, are obtained from the Kuhn-Tucker conditions associated with L_i and L_j in (3.10). The cutoff arises when we assume that group i chooses $\alpha_i > \underline{\alpha}_i$ and group j chooses $\alpha_j > \underline{\alpha}_j$, so that $\lambda_j = 0$ and $\lambda_i = 0$ This cutoff helps us identify the parametric zone where neither groups constraints bind in equilibrium[12].

Proposition **20**

$\forall\ i, j \in \{A, B\}$ and $j \neq i$

(a) Group i is monopolized in a Nash equilibrium iff $\underline{\alpha}_i \in [\frac{1}{n_i-1}, 1]$ and $\underline{\alpha}_j \in (-\infty, \alpha_j^{iM}]$. In this case any combination of prize sharing rules (α_i^*, α_j^*), such that $\alpha_i^* \geqslant \underline{\alpha}_i$ and $\alpha_j^* = -\frac{1}{n_j-1} + \frac{(n_i-1)n_j}{(n_j-1)n_i}\alpha_i^*$ is a Nash equilibrium.

(b) In the unique Nash equilibrium group i is hawkish and group j is dovish iff $\underline{\alpha}_i \in [\alpha_i^{NN}, \frac{1}{n_i-1})$ and $\underline{\alpha}_j \in (-\infty, \alpha_j^{iB})$. The equilibrium prize sharing rules are (α_i^*, α_j^*)=$(\underline{\alpha}_i, \alpha_j^{iB})$.

(c) In the unique Nash equilibrium both groups are dovish iff $\underline{\alpha}_i \in (-\infty, \alpha_i^{NN})$ and $\underline{\alpha}_j \in (-\infty, \alpha_j^{NN})$. The equilibrium prize sharing rules are (α_i^*, α_j^*)=$(\alpha_i^{NN}, \alpha_j^{NN})$.

(d) In all other cases in the unique Nash equilibrium both groups are hawkish. The equilibrium prize sharing rules are (α_i^*, α_j^*)=$(\underline{\alpha}_i, \underline{\alpha}_j)$.

[11]Derived in Lemma 10 and 11

[12]Derived in Lemma 9

Next we discuss the results summarized in Proposition 20

■ **Group i Monopolized:** It is clear from the bounds stated in part (a) of the result that for group i to be monopolized in equilibrium, $\underline{\alpha}_i$ has to be sufficiently high and $\underline{\alpha}_j$ sufficiently low (see Figure 3.1). Furthermore, α_j^{iB} and α_j^{iM} intersect at $\underline{\alpha}_i = \frac{1}{n_i - 1}$, so that for all $\underline{\alpha}_i < \frac{1}{n_i - 1}$ we have $\alpha_j^{iM} < \alpha_j^{iB}$. Here group j has the option to monopolize group i by choosing $\alpha_j = \alpha_j^{iM}$. But group j chooses not to do that because by choosing $\alpha_j = \alpha_j^{iB}$, which is higher, it can maintain more of the net surplus and give up only a tiny chance of winning upto group A. The choice $\alpha_j > \underline{\alpha}_j$, implies group j chooses more free riding within its group, which allows group A to survive in the contest. Of course, the benefits of a larger net surplus dominates the cost of decreased chances of winning for group j in this case.

In case $\alpha_j^{iM} > \alpha_j^{iB}$, it is again optimal for group j to choose the higher of the two in equilibrium, in order to save net surplus. But at $\alpha_j = \alpha_j^{iM}$, group i is monopolized. Given that group i will be monopolized at $\alpha_j = \alpha_j^{iM}$, any $\alpha_i \geqslant \underline{\alpha}_j$ is best response for group i, as at all such choices it gets zero payoff. For group j on the other hand, the best response is to choose a α_j, which is consistent with α_j^{iM}, given whatever choice group i makes.

■ **Group A is hawkish, Group B is dovish:** From part (b) of the proposition it is clear that this case arises when both $\underline{\alpha}_A$ and $\underline{\alpha}_B$ are low, so that both groups are potentially very competitive (see Figures 3.1 and 3.2). Because both groups are sufficiently competitive, having a larger size is an advantage in the contest. But again, because both groups are competitive, it is more difficult for group A to compete against the larger group B. So the optimal choice of group A is to be maximally competitive by choosing a hawkish stance. In other words, group A focuses entirely on its chances of winning instead of saving net surplus.

The larger group B , on the other hand, chooses to save some surplus by choosing $\alpha_B = \alpha_B^{iB} > \underline{\alpha}_B$. It has the competitive advantage of a larger size. But the larger size also means a lot of surplus will be dissipated if it focuses primarily on winning the contest by choosing a hawkish stance. So, group B leader compromises on its chances of winning by choosing to be dovish in order to save some net surplus.

Similarly, we can analyze the case, where group B is hawkish and group A is dovish. This case arises when group B is "boundedly meritocratic" but group A is "unboundedly meritocratic". This being the case, free riding is the dominant force within group B, which makes its larger size a disadvantage. On the other hand, the smaller hand has very competitive norms. Given the larger group is not much of a competition for it, group A shifts focus to saving some net surplus by taking a dovish stance.

■ **Both groups are dovish:** As can be seen in part (c) of the result, this case arises when both $\underline{\alpha}_A$ and $\underline{\alpha}_B$ are extremely low, so that both groups have extremely competitive norms. If either group focuses entirely on chances of winning by taking a hawkish stance, then a lot of surplus will be lost in costly efforts. Hence, both groups compromise on chances of winning by shifting some attention to saving net surplus.

■ **Both groups are hawkish:** This is the main result of the chapter and is succinctly summarized in Corollary 2. Look at Figure 3.2.

In this case both groups choose to be hawkish, i.e. both focus on winning the contest instead of trying to save net surplus. This case arises when group B is "boundedly meritocratic". The smaller group A could be "boundedly meritocratic" or "unboundedly meritocratic".

This case arises when social norms are such that a larger group size is a disadvantage for group B, as free riding is the dominant force within it. Group B tries to counter that disadvantage by choosing the lowest possible α_i and making its group maximally competitive in the contest.

For the smaller group on the other hand, the numbers are still a disadvantage. So, irrespective of the degree of meritocracy in its norms it tries to counter the disadvantage of smaller numbers by choosing hawkish stance.

This situation arises, when social norms of both groups are such that group sizes are a disadvantage. Hence both groups exclusively try to maximize their winning chances by choosing $\alpha_i = \underline{\alpha}_i$.

This is main observation of this chapter. We have clearly identified the circumstances under which both groups will be hawkish, which seems to be a natural stance to take in a situation of pure conflict. This has not been identified in the literature til now. It is succinctly summarized in the following corollary of Proposition 20.

Corollary **2**

In the unique Nash equilibrium both group A and group B are hawkish iff $\underline{\alpha}_B \geqslant max\{\alpha_B^{AM}, \alpha_B^{AB}\}$ *and* $\underline{\alpha}_A \geqslant max\{\alpha_A^{BM}, \alpha_A^{BB}\}$

Corollary 2 provides a lower bounds on egalitarianism, which ensure that both groups will choose to be hawkish in equilibrium. As mentioned before, it is an important observation because in the context of group conflicts, the natural path for a group leader to follow would be to try and maximize chances of winning by generating maximal efforts. In other words, it precisely captures the circumstances under which social norms have a bite for both groups. The result can be seen clearly in Figures 3.1 and 3.2.

Intuition: This result points to the fact that for the larger group B to entirely focus on winning the contest by taking a hawkish stance, it needs to have sufficiently egalitarian norms, which makes free riding the dominant force within it. In that case, having larger numbers is a disadvantage, which can only be countered by taking a hawkish stance. If it had competitive norms, larger numbers would be an advantage in terms of winning the contest but would dissipate a lot of the surplus if it tried to generate maximal efforts. So, in such a case, the larger group leader takes a dovish stance, which reduce its efforts and chances of winning below maximum but retains a larger amount of surplus, which can be had in case of success. For the smaller group, on the other hand, numbers are a disadvantage. So it generally takes a hawkish stance to counter that disadvantage by taking a hawkish stance.

Before concluding this section, let us take a closer look at Figure 3.2. In Figure 3.2 let us consider the polygon $ABCDEF$. This is the polygon of Nash equilibrium choices made by the leaders. If $(\underline{\alpha}_A, \underline{\alpha}_B)$ lies inside or on the boundary of the polygon then the Nash equilibrium is $(\alpha_A^*, \alpha_B^*) = (\underline{\alpha}_A, \underline{\alpha}_B)$. If $(\underline{\alpha}_A, \underline{\alpha}_B)$ lies outside the polygon then the Nash equilibrium is the nearest point on the boundary closest to it.

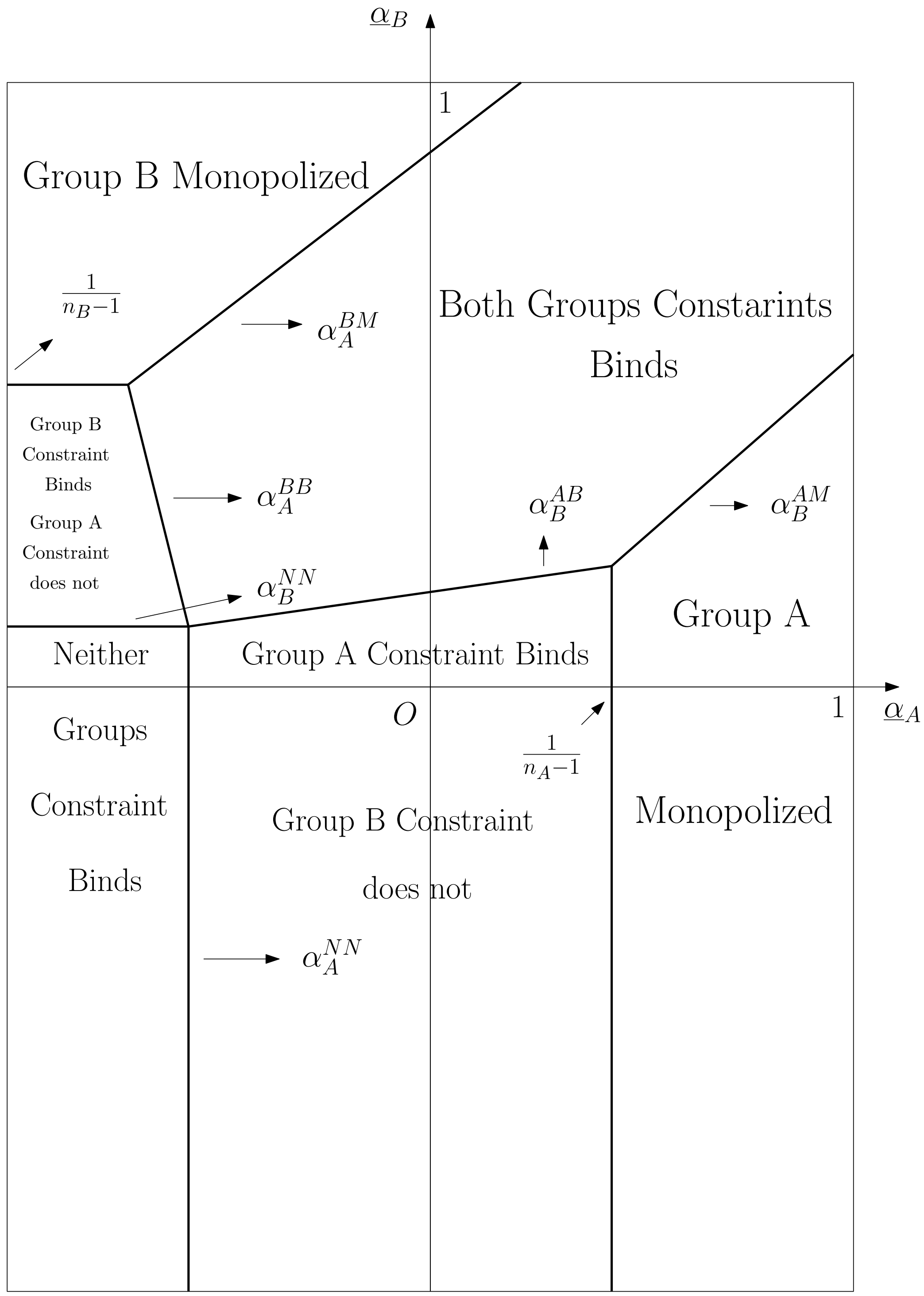

Figure 3.1: Leader's Choice in Nash Equilibrium

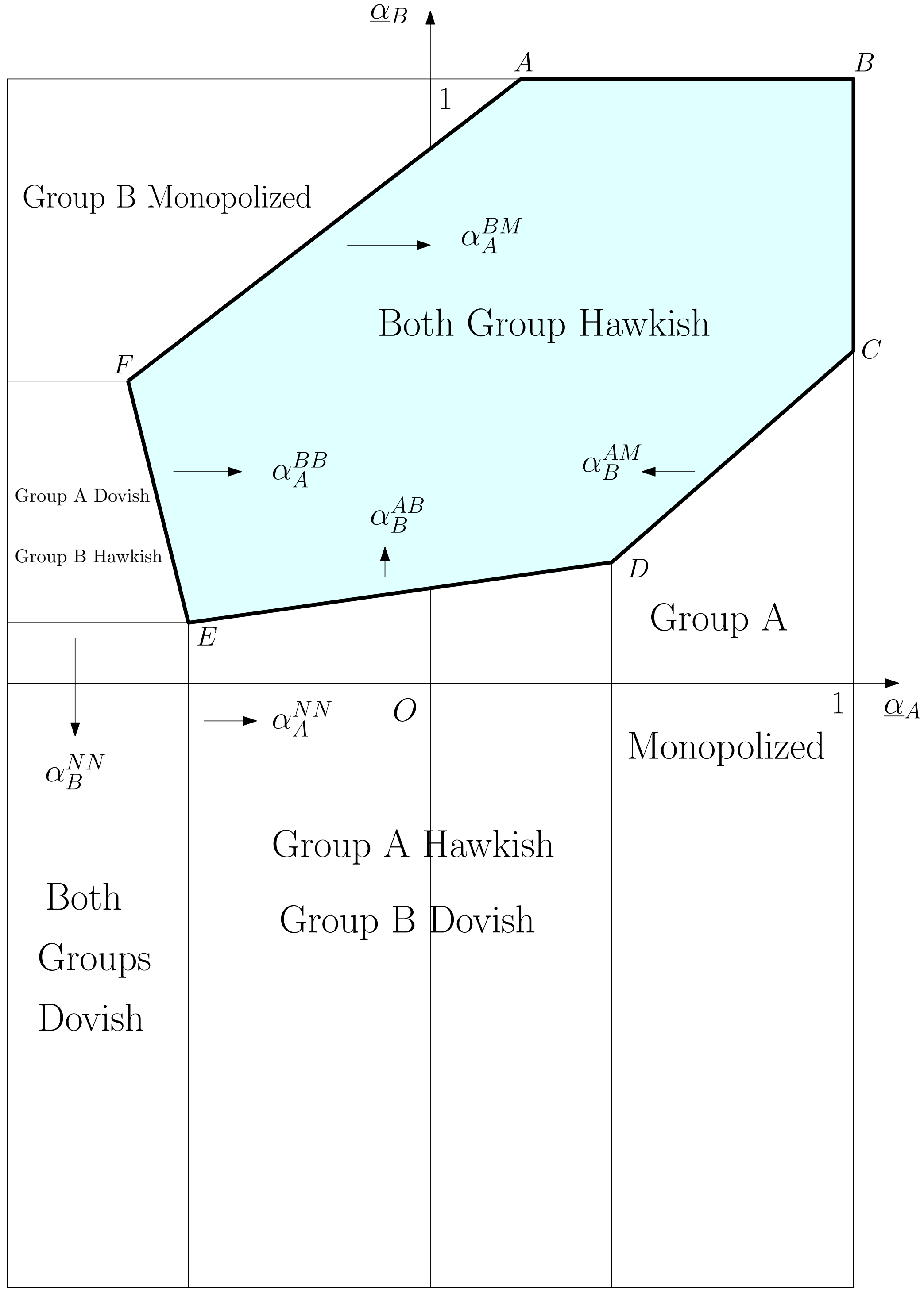

Figure 3.2: Leader's Choice in Nash Equilibrium

We conclude this section by summarizing the main takeaways. Firstly, we find that the smaller group generally takes a "hawkish" stance in the contest. The larger group, however, chooses a "hawkish" stance only in cases where it has sufficiently egalitarian norms, i.e., the incentive to free ride is so high within the group that larger numbers are actually a disadvantage. When it has sufficiently competitive internal norms, the larger group chooses a "dovish" stance to reduce its efforts and save surplus, which can be consumed *ex post* in case of success. But, the main observation is made in Corollary 2, which precisely identifies conditions under which both groups take a "hawkish" stance. Even though adoption of a "hawkish" stance by all participating groups seems to be the most natural thing to do in a purely competitive situation like ours, the conditions required for it to happen had not been identified in the previous literature.

3.6 Equilibrium Characterization

In this section we characterize the subgame perfect Nash equilibrium (SPNE) of the whole game. In Propositions 19 and 20, we characterized the Nash equilibrium of stage two and one of the game respectively. Now, we use the two propositions to characterize the (SPNE) of the game. We denote χ_i at $(\underline{\alpha}_A, \underline{\alpha}_A)$ as $\underline{\chi}_i$.

Proposition **21**

(A) If group i is monopolized, then in the SPNE

(a) The net surplus in the contest is $S^{iM} = \frac{\alpha_i(n_i-1)}{n_i}$ [13].

(b) The probabilities of winning are $(P_i^{iM}, P_j^{iM}) = (0, 1)$.

(B) If neither group is monopolized, then in the SPNE

(1) If both groups are dovish then

[13] The first component in the superscript is the group which is monopolized and the second is the the word monopolized

(a) *The net surplus in the contest is* $S^{NN} = \frac{1}{N}$.

(b) *The probabilities of winning are* $(P_i^{NN}, P_j^{NN}) = (\frac{n_j}{N}, \frac{n_i}{N})$.

(2) *If group i is hawkish but group j is dovish then*

(a) *The net surplus in the contest is* $S^{iB} = \frac{1+\underline{\alpha}_i(n_i-1)}{2n_i}$.

(b) *The probabilities of winning are* $(P_i^{iB}, P_j^{iB}) = (\frac{1-\underline{\alpha}_i(n_i-1)}{2}, \frac{1+\underline{\alpha}_i(n_i-1)}{2})$.

(3) *If both groups are hawkish then*

(a) *The net surplus in the contest is* $S^{B} = \frac{1+(n_A-1)\underline{\alpha}_A+(n_B-1)\underline{\alpha}_B}{N}$.

(b) *The probabilities of winning are* $(P_i^{B}, P_j^{B}) = (\frac{\chi_i}{N}, \frac{\chi_j}{N})$.

We next discuss the results summarized in Proposition 21.

■ **Group i is Monopolized:** This case arises when $\chi_i \leqslant 0$ as can be seen from Proposition 19. Group j's best response to any α_i is to choose α_j which solves $\chi_i = 0$. The effort is $X_j^{iM} = 1 - \frac{\alpha_i(n_i-1)}{n_i}$, which leaves a net surplus $S^{iM} = \frac{\alpha_i(n_i-1)}{n_i}$. The effort required to monopolize group i is decreasing in α_i as it easier for group j to crowd out group i, when free riding has increased within it. Therefore, the net surplus is increasing in α_i.

Given that α_i is high enough in this case, means that free riding is the dominant force in group i in this case. If now group i gets larger still, it becomes easier to monopolize group i as free riding will increase. Therefore, net surplus is rising in n_i as well.

Next, we focus on cases, where neither group is monopolized.

■ **Both groups are dovish:** Both groups are dovish means that in equilibrium $\alpha_i > \underline{\alpha}_i$ and $\alpha_i > \underline{\alpha}_i$. When both groups take a dovish stance, the total effort in equilibrium is $X^{NN} = 1 - \frac{1}{N}$, which leaves a net surplus $S^{NN} = \frac{1}{N}$. Because neither constraint binds, the probabilities of winning and net surplus are independent of $\underline{\alpha}_i$ and only depends on group sizes. In this case only groups sizes matter, i.e. social norms have no bite.

Given that both groups get to choose the globally best rules in this case, the only difference which applies between groups is one due to sizes. Increasing the size of group i decreases the effort of group i due to increased free riding. Efforts are strategic substitutes here and so the effort of group j goes up. But aggregate effort increases, thereby lowering net surplus S^{NN}. However, as the effort of group i falls, the probability of group i winning the contest goes down.

■ **Group i is hawkish, Group j is dovish:** This case arises when in equilibrium $\alpha_i = \underline{\alpha}_i$ and $\alpha_i > \underline{\alpha}_i$. In this case the aggregate effort in the Nash equilibrium is $X^{iB} = \frac{1}{2} + \frac{(n_i-1)(1-\underline{\alpha}_i)}{2n_i}$, which leaves a net surplus $S^{iB} = \frac{1}{2} - \frac{(n_i-1)(1-\underline{\alpha}_i)}{2n_i}$. When $\underline{\alpha}_i$ rises, the effort of group i decreases due to increased free riding. The effort of group j rises as efforts are strategic substitutes. Aggregate efforts decline and so the net surplus rises as $\underline{\alpha}_i$ rises. As effort of group i decreases, the probability that group i wins goes down with $\underline{\alpha}_i$.

When n_i increases, aggregate effort increases, thereby reducing the net surplus. When $\underline{\alpha}_i < 0$, effort of group i rises with n_i increasing its chances of winning. $\underline{\alpha}_i = 0$ denotes the cutoff above which the force of free riding dominates the force of competition in group i. Therefore, in terms of payoffs, larger numbers are a disadvantage for group i when $\underline{\alpha}_i > 0$ and is an advantage otherwise.

■ **Both groups are hawkish:** This case arises when in equilibrium $\alpha_i = \underline{\alpha}_i$ and $\alpha_i = \underline{\alpha}_i$. The aggregate effort level X^B is declining in $\underline{\alpha}_A$ and $\underline{\alpha}_B$ due to increased free riding. Therefore, the net surplus S^B increases in $\underline{\alpha}_A$ and $\underline{\alpha}_B$. As $\underline{\alpha}_i$ rises free riding in group i rises and so effort of group i falls. Unless both $\underline{\alpha}_A$ and $\underline{\alpha}_B$ are close to 1, efforts are strategic substitutes, so that when X_i^B rises, X_j^B falls. However, irrespective of whether X_j is a strategic complement or substitute to X_i, it can be easily verified that the aggregate efforts decline with $\underline{\alpha}_i$. Furthermore, the probability of group i winning decreases in $\underline{\alpha}_i$ and increases in $\underline{\alpha}_j$.

It should be noted that the efforts are higher when both groups are "doves" than when both groups are "hawks". This happens due to the way we have defined hawkish and dovish behavior in this chapter. A group chooses a hawkish stance in equilibrium when it has egalitarian norms and a dovish stance when it has competitive norms. If a group is egalitarian then free riding is the dominant force within it. On the other hand, if a group has competitive norms then the dominant force is that of internal competition. Even though the groups choose dovish stances under competitive norms, the reduction in efforts is not to the extent that it falls below the efforts chosen by hawkish groups, which have egalitarian norms.

3.7 When does GSP occur?

In this section we turn to the question of welfare of the groups in the collective contest. We focus on the phenomenon of Group Size Paradox (GSP), which denotes situations in which the bigger group fares worse than the smaller group in the contest. In particular we link the incidence of GSP to whether the groups are "boundedly meritocratic" or "unboundedly meritocratic". Even though GSP has been a primary focus of the literature on collective contests, e.g. (Nitzan and Ueda, 2011), (Balart et al., 2016), there is no paper we know of which analyzes how group specific social norms affect the welfare of the groups.

Definition **13**

The group size paradox (GSP) occurs in equilibrium if the bigger group wins the contest with a lower probability i.e. $P_B < P_A$. If the bigger group has at least as much chance to win the contest as the smaller group i.e., $P_B \geqslant P_A$, then we say group size advantage (GSA) occurs in equilibrium.

There is no loss in defining GSP in terms of probabilities of success. We could have alternatively defined it in terms of group efforts or payoffs, as all of them are equivalent in this framework.

Next we define a cutoff, which we will need in the next proposition.

Definition **14** ***GSP Cutoff*** *(α_B^{GSP})*

The GSP cutoff α_B^{GSP} is defined as follows:

$$\alpha_B^{GSP} = \frac{n_B - n_A}{2n_A(n_B - 1)} + \frac{(n_A - 1)n_B}{(n_B - 1)n_A}\underline{\alpha}_A.$$

This cutoff is obtained by checking when $P_B^{BB} > P_A^{BB}$ i.e. when it the case that group B wins the contest with a higher probability, where both groups are hawkish (Proposition 21).

Proposition **22**

GSP occurs iff $\underline{\alpha}_A < 0$ *or* $\underline{\alpha}_B > \alpha_B^{GSP}$.

We next discuss the result summarized in Proposition 22 by breaking it up into three different cases.

■ **Smaller group is "unboundedly meritocratic"** ($\underline{\alpha}_A < 0$):

In this case the smaller group can choose to put a larger than proportional weight on the competitive component of the rule. Allowing the smaller group this freedom allows it to counter the disadvantage of having smaller numbers in the collective contest. This is irrespective of whether the larger group is "boundedly meritocratic" or "unboundedly meritocratic".

If $\underline{\alpha}_B > 0$ then group B is "boundedly meritocratic". Being larger and "boundedly meritocratic" is doubly disadvantageous for group B. Essentially, group B contains a large number of free riders. Moreover, it does not have enough freedom to counter the force of free riding by choosing a rule, which rewards efforts more than proportionally. Therefore, the larger group always fares worse in this case.

If, on the other hand, group B is also "unboundedly meritocratic", so that $\underline{\alpha}_B < 0$, it faces the trade off between winning the contest and saving net surplus because it is larger. Group A being smaller does not face this trade off. It is optimal for group B to try and save net surplus by taking a dovish stance. In the process, group B ends up doing worse than group A, as the dovish stance increases free riding in it.

Therefore, $\underline{\alpha}_A = 0$ captures the cutoff level of competitiveness, such that below it group A is competitive enough to outdo the bigger group. In other, words group A being "boundedly meritocratic" is a sufficient condition for GSP to occur.

■ **Smaller group is "boundedly meritocratic" ($\underline{\alpha}_A > 0$) and larger group is "unboundedly meritocratic"($\underline{\alpha}_B < 0$):**

In this case the larger group has the advantage of rewarding efforts in its group more than proportionally, thereby being in a position to generate substantial efforts from its larger numbers. So it is in an advantageous position vis a vis the smaller group both with respect to size and potential level of competitiveness and hence efforts. Therefore, in equilibrium it fares better than the smaller group. We call this situation Group Size Advantage (GSA). Even though group B is dovish, the fact that group A is "boundedly meritocratic", allows it to fare better than group A in equilibrium.

■ **Both groups are "boundedly meritocratic" ($\underline{\alpha}_A \geqslant 0$ and $\underline{\alpha}_B \geqslant 0$) :**

This case, where both groups are "boundedly meritocratic" turns out to be the most interesting one. What turns out to be important is the degree of asymmetry of the norms of competitiveness across the groups. If the asymmetry is substantial, then the group with more egalitarian norms does worse unequivocally.

If the norms of competitiveness are relatively symmetric across groups, i.e. $\underline{\alpha}_A$ and $\underline{\alpha}_B$ are close to each other [14], then what determines the occurrence of GSP is whether both groups have egalitarian norms or both groups have competitive norms. Given that norms of competitiveness are symmetric across groups, what creates the difference between the groups is their relative sizes. But, the difference in sizes operate differently depending on whether both groups have competitive norms or both have egalitarian norms. Look at Figure 3.3.

If both groups are egalitarian i.e., $\underline{\alpha}_B > \frac{1}{2}$ and $\underline{\alpha}_A > \frac{1}{2}$, then the dominant force is one of free riding in both groups. Therefore, having a larger group is a disadvantage in this case. So, group B does worse than group A and GSP operates. Incidentally, this case perfectly characterizes the type of groups Olson (1965) talked about in *The Logic of Collective Action* . Olson (1965) [15] studied the case where the norms of competitiveness were symmetric across

[14] In Figure 3.3 the idea of relative symmetry is captured by drawing the 45° line and looking at clusters of $\underline{\alpha}_A$ and $\underline{\alpha}_B$ around it

[15] To be preciseOlson (1965) studied the issue of free riding in collective action with only one group. But his conclusions generalize to the collective contest scenario.

groups. Specifically, he focused on the case of full egalitarianism, i.e., $\underline{\alpha}_A = 1$ and $\underline{\alpha}_B = 1$, making the force of free riding maximal within both groups. With that situation in mind, he reached the conclusion that larger numbers are not ideal for successful collective action. We show that the force of free riding dominates as long as $\underline{\alpha}_B > \frac{1}{2}$ and $\underline{\alpha}_A > \frac{1}{2}$, thereby providing a precise characterization of the types of groups, which were the focus of Olson (1965). We call this collection of groups **Olson's Groups**.

On the other hand, if both groups are sufficiently competitive i.e., $0 \leqslant \underline{\alpha}_B < \frac{1}{2}$ and $0 \leqslant \underline{\alpha}_A < \frac{1}{2}$, then the competitive force dominates. In such a case having larger numbers is an advantage and group B fares better, so that GSA operates. Olson (1965) spoke at length about how "selective incentives" could be used to outdo the force of free riding, making collective action possible in larger groups. This case provides a perfect characterization of such a situation. The norms being symmetric across groups, only group sizes matter. Here the "selective incentives", allows the larger group to overcome the force of free riding and fare better than the smaller group. We call the collection of groups with equally competitive norms i.e., $0 \leqslant \underline{\alpha}_B < \frac{1}{2}$ and $0 \leqslant \underline{\alpha}_A < \frac{1}{2}$, the **Neo-Olson Groups**. Look at Figure 3.3.

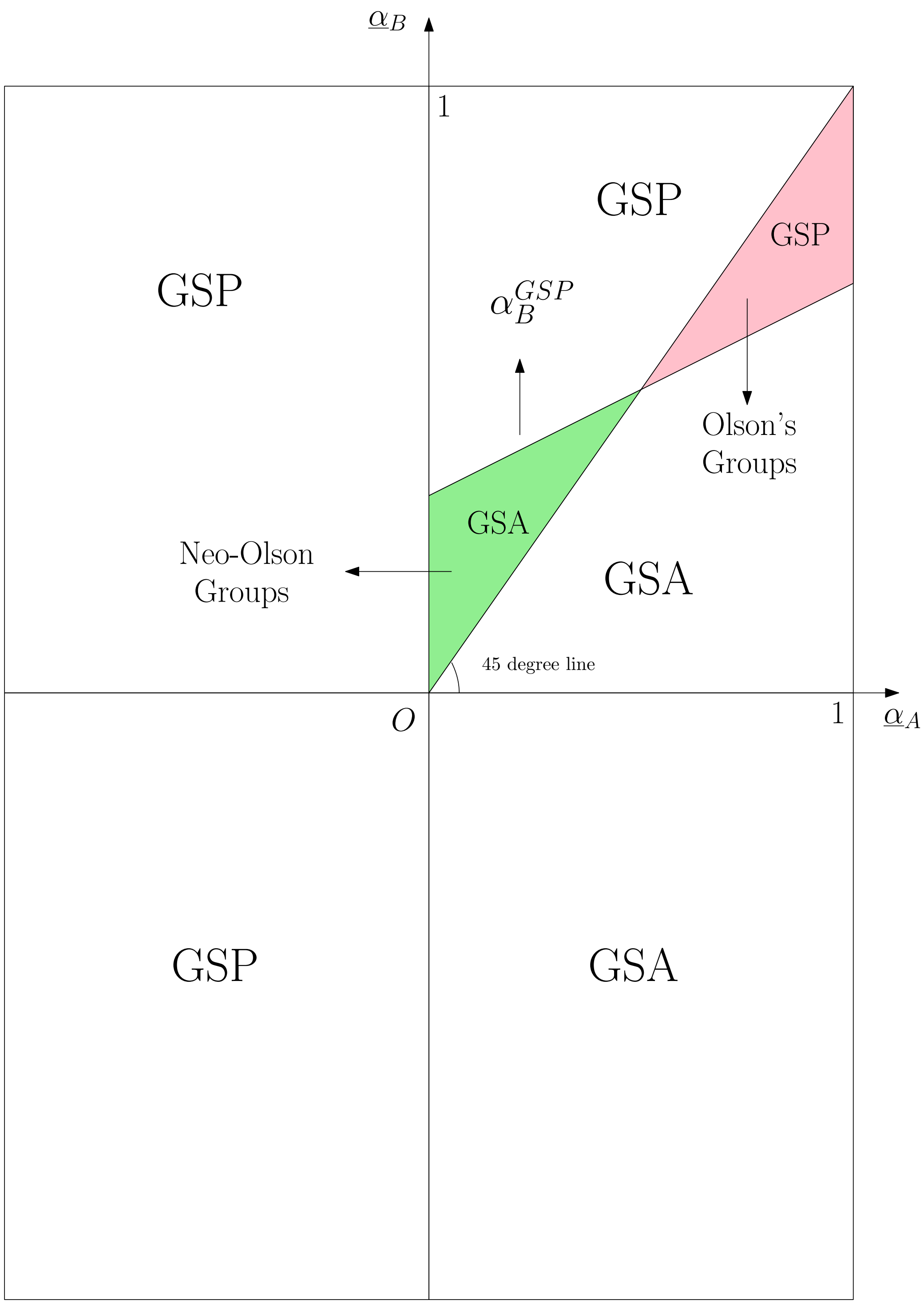

Figure 3.3: When does GSP Occur?

3.8 Extensions

In what follows we extend the basic model in two directions. First, we consider the Generalized Tullock Contest Success Function and verify whether the basic results of the main model go through or not. Secondly, we consider the case where the group leaders are actually maximizing the probability of winning the contest instead of group payoffs.

3.8.1 Generalized Tullock Contest Success Function

In this section we consider the Generalized Tullock Contest Success Function and try to replicate the main results of the paper under it. The Generalized Tullock Contest Success Function which is as follows:

$$P_i(X_i, X_j) = \begin{cases} \frac{X_i^r}{X_i^r + X_j^r}, & \text{if } X_i > 0 \text{ or } X_j > 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{3.11}$$

We will be assuming that $r \in (0, 1]$ throughout to rule out the possibility of Increasing Returns to Scale (IRS).

Second Stage Choices

The stage 2 choice of efforts is exactly the same as the **NN** regime of in Chapter 2 and the equation which needs to be satisfied in equilibrium is provided in Proposition 13. In the following result we state the efforts, probabilities and payoffs in a form that makes our subsequent calculations easier.

Proposition **23**

In any Nash equilibrium of the effort subgame, where neither group is monopolized

(1) The efforts of the groups are $\left(X_A^{NN}, X_B^{NN}\right) = \left(\frac{r(x^*)^r + (1+(x^*)^r)\theta_A}{n_A[1+(x^*)^r]^2}, \frac{(x^*)^r(r+[1+(x^*)^r]\theta_B)}{n_B[1+(x^*)^r]^2}\right)$.

(2) The probabilities of winning are $\left(P_A^{NN}, P_B^{NN}\right) = \left(\frac{1}{1+(x^*)^r}, \frac{(x^*)^r}{1+(x^*)^r}\right)$.

(3) The group payoffs are $\left(\Pi_A^{NN}, \Pi_B^{NN}\right) = \left(\frac{(n_A-\theta_A-r)P_A^{NN}+rP_A^{NN2}}{n_A}, \frac{(n_B-\theta_B-r)P_B^{NN}+rP_B^{NN2}}{n_B}\right)$.

This result extends Proposition 19 to the case, where $r < 1$.

The next result shows that the probabilities of winning P_i^{NN} decreases with α_i. Therefore if group leaders were maximizing probabilities of winning we would have $\alpha_A = \underline{\alpha}_A$ and $\alpha_B = \underline{\alpha}_B$ in a Nash equilibrium of the first stage choice by group leaders.

Proposition **24**

$\frac{dP_i^{NN}}{d\alpha_i} < 0$ *for* $i = A, B$.

Like in the main model, where $r = 1$ the probability of winning for group i is decreasing in α_i when $r < 1$. The reason is and increase in α_i only increases incentives to free ride within group i, which reduces its chances of winning the contest.

First Stage Choices

In this section we study the problem where group leaders maximize their expected group payoffs subject to the constraint $\alpha_i \geqslant \alpha_i$, $i = A, B$.

Proposition **25**

In a SPNE

(1) If $\underline{\alpha}_i \leqslant -\frac{1-r+2rP_i^{NN}}{n_i-1}$, *then group* i *will be dovish.*

(2) If $\underline{\alpha}_i > -\frac{1-r+2rP_i^{NN}}{n_i-1}$ *and* $-\frac{dP_i^{NN}}{d\alpha_i} > \frac{(n_i-1)P_i^{NN}}{(n_i-\theta_i-r)+2rP_i^{NN}}$ *at* $x = x^*$, *then group* i *will be hawkish.*

This result provides sufficient conditions for the groups to behave in a hawkish or dovish manner in equilibrium. Even though they are not necessary conditions, the result does help us put bounds on $\underline{\alpha}_i$, which will make the groups behave in a particular manner.

3.8.2 Group Leaders Maximize Probabilities of Winning

In what follows we analyze whether group leaders behave in a hawkish or dovish manner in equilibrium, when their objective is to maximize the probability of winning the contest rather than group payoffs.

We know from Proposition 9 that when neither group is monopolized the probability of winning for group i is as follows

$$P_i^{NM} = \frac{n_i + n_i(n_j - 1)\alpha_j - n_j(n_i - 1)\alpha_i}{N} \tag{3.12}$$

It can be observed in equation (3.12) that the probability of group i winning is strictly decreasing in α_i. Therefore, if the objective of group i leader is to maximize the probability of his group winning the contest then he should always choose $\alpha_i = \underline{\alpha}_i$, i.e., he should always behave in a "hawkish" manner in equilibrium. This is not surprising as " hawkish" behavior increases probability of winning at the expense of surplus. Therefore, if the primary objective is to win the contest there is no reason for any group leader to take a "dovish" stance.

In case group i is monopolized the probability of winning of group i is 0 and group j is 1. The choice of α_i is immaterial to the outcome. Group j will choose α_j such that $P_i^{NM} = 0$ in equation (3.12).

3.9 Conclusion

In this chapter we generalized the prize sharing rule proposed by Nitzan (1991) in the context of collective contests. We propose a way to model group specific norms of competitiveness and then analyze how such internal norms affect a group's chances in external conflict. The modeling innovation allowed us to characterize situations in which both groups would choose focus entirely on winning an external conflict i.e. both group take the hawkish stance. This feature despite being the most natural thing to expect in a situation of conflict, had been overlooked in literature till now.

We find that the smaller group A generally chooses to be hawkish. For group B to also behave in a hawkish manner, it has to be the case that it is sufficiently "boundedly meritocratic" i.e., $\underline{\alpha}_B \geqslant 0$ and high enough. This allows us to identify types of group conflicts, where both groups take the extremest stance possible in order to maximize the likelihood of success in the contest.

We also provide the conditions under, which GSP occurs. We find that group A being "unboundedly meritocratic" is a sufficient condition for GSP to occur. If group A is "boundedly meritocratic" and group B is "unboundedly meritocratic" then larger group size is an advantage for group B and it fares better than the smaller group. If both groups are "boundedly meritocratic", then whether GSA applies or GSP depends critically on whether the norms are symmetric across groups or not. If both group's norms are symmetric and competitive, then having a larger group is an advantage and GSA applies. If both group's norms are sufficiently egalitarian then free riding is the dominant force in both groups. In that case, being larger in size is a disadvantage and therefore GSP applies.

Even though the modeling innovation of imposing restrictions on the prize sharing rule allows us to clarify when group sizes matter and when social norms matter, what remains to be understood is where such social norms themselves come from. Given that these restrictions are interpreted as norms of competitiveness in surplus division within a group, modeling how such norms arise as a function of economic conditions a group faces in times of peace or how such norms relate to group sizes, are interesting questions that are left for future research.

3.10 Appendix 1

To prove Proposition 21 we have to first set up the individual effort choice problem of group members in stage two of the game. Then we propose and prove a set of Lemmas which help us prove the result.

3.10.1 Individual Effort Choice Problem

Taking as given (α_A, α_B) chosen by the group leaders in stage one of the game, the payoff of the k^{th} member in Group A is given as follows:

$$\pi^{kA}(X_A, X_B) = \frac{X_A}{X_A + X_B}[(1-\alpha_A)\frac{x_{kA}}{X_A} + \frac{\alpha_A}{n_A}] - x_{kA} \tag{3.13}$$

Similarly the payoff of the k^{th} member of Group B is as follows:

$$\pi^{kB}(X_A, X_B) = \frac{X_B}{X_A + X_B}[(1-\alpha_B)\frac{x_{kB}}{X_B} + \frac{\alpha_B}{n_B}] - x_{kB} \tag{3.14}$$

Both (3.13) and (3.14) are continuous except at $(X_A, X_B) = (0, 0)$. The functions are concave in x_{ki}. for $i = A, B$.

We can compute the Nash Equilibrium in individual efforts by examining the First Order Conditions of (3.13) and (3.14).

We ignore the constraint $0 \leqslant x_{ki} \leqslant 1$ while solving the problem and check later that they are indeed satisfied. We characterize within group symmetric Nash Equilibrium in our analysis.

Before proceeding we define the sets $N_i = \{1, 2...n_i\}$ for $i = A, B$.

First, we examine the First Order Conditions of the individual effort choice problem for members of both the groups. The F.O.C of (3.13) w.r.t. x_{kA} , $\forall k \in N_A$ is as follows:

$$\frac{X_B}{(X_A + X_B)^2}[(1-\alpha_A)\frac{x_{kA}}{X_A} + \frac{\alpha_A}{n_A}] + \frac{X_A}{X_A + X_B}[(1-\alpha_A)\frac{X_A - x_{kA}}{X_A^2}] \leq 1 \tag{3.15}$$

Similarly, the F.O.C of (3.14) w.r.t x_{kB} , $\forall k \in N_B$ is as follows:

$$\frac{X_A}{(X_A+X_B)^2}[(1-\alpha_B)\frac{x_{kB}}{X_B}+\frac{\alpha_B}{n_B}]+\frac{X_B}{X_A+X_B}[(1-\alpha_B)\frac{X_B-x_{kB}}{X_B^2}]\leq 1 \tag{3.16}$$

If (3.15) holds strictly then $x_{kA} = 0$, $\forall k \in N_A$. Similarly in (3.16). Both inequalities cannot hold strictly at $(x_{kA}, x_{kB}) = (0, 0)$, because it does not constitute a Nash Equilibrium. Given the Tullock Contest Success Function at $(x_{kA}, x_{kB}) = (0, 0)$, a member in one of the groups will deviate because then his group will win the contest for sure and he will get a share of the incremental group payoff. It can also be easily verified that the Second Order Conditions hold.

Therefore, there are 3 mutually exclusive cases to take care of.

▶ **CASE 1**:

Inequality (3.15) holds weakly at $x_{kA} = 0$, Inequality (3.16) holds with equality at some $x_{kB} > 0$.

Lemma **5**

If $\alpha_A n_B(n_A-1)-\alpha_B n_A(n_B-1) \geqslant n_A$ then Group A is Monopolized by Group B.In the symmetric within group Nash Equilibrium , $x^{AM}_{kA} = 0$, $\forall k \in N_A$ and $x^{AM}_{kB} = \frac{(n_B-1)(1-\alpha_B)}{n_B^2}$, $\forall k \in N_B$. The aggregate effort of group B is $X^{AM}_B = \frac{(n_B-1)(1-\alpha_B)}{n_B^2}$.

Proof: If $x^A_{kA} = 0$, $\forall k \in N_A$, then $X^A_A = n_A x^A_{kA} = 0$. But notice that at $X^A_A = 0$ the L.H.S of (3.15) is not well defined. So we will consider the limit of of L.H.S. of (3.15) as $X^A_A \to 0$.

Define $x^A_{kA} = \epsilon > 0$, $\forall k \in N_A$. Then $X^A_A = n_A x^A_{kA} = n_A\epsilon$. As n_A is finite $X^A_A \to 0$ as $\epsilon \to 0$.

We need L.H.S. of (3.15) to be well defined and (3.15) to be satisfied as a weak inequality at $x^A_{kA} = \epsilon$ and $X^A_A = n_A\epsilon$ as $\epsilon \to 0$.

We replace $x^A_{kA} = \epsilon$ and $X^A_A = n_A\epsilon$ in (3.15) and sum it over all $k \in N_A$ to arrive at the following condition:

$$\lim_{\epsilon \to 0} \frac{X_B^A}{(n_A\epsilon + X_B^A)^2} + \frac{(n_A - 1)(1 - \alpha_A)}{(n_A\epsilon + X_B^A)} \leqslant n_A \tag{3.17}$$

As this limit is well-defined we need the following condition to be satisfied if Group A is to be Monopolized.

$$n_A X_B^A \geqslant 1 + (n_A - 1)(1 - \alpha_A) \tag{3.18}$$

At $X_A^A = 0$, the L.H.S. of (3.16) is well defined. We sum (3.16) over all $k \in N_B$, to arrive at the following condition:

$$n_B X_B^A = (n_B - 1)(1 - \alpha_B) \tag{3.19}$$

For $x_{kA}^{AM} = 0$ and $x_{kB}^{AM} = \frac{(n_B-1)(1-\alpha_B)}{n_B^2}$ to be mutual best responses, both (3.18) and (3.19) need to be satisfied. Replacing X_B^A from (3.19) in (3.18) we arrive at the following condition:

$$\alpha_A n_B(n_A - 1) - \alpha_B n_A(n_B - 1) \geqslant n_A \tag{3.20}$$

Equation (3.20) needs to be satisfied if group A is to be monopolized.

■

▶ **CASE 2**:

Inequality (3.16) holds weakly at $x_{kB} = 0$, Inequality 3.15 holds with equality at some $x_{kA} > 0$.

Lemma **6**

If $\alpha_B n_A(n_B - 1) - \alpha_A n_B(n_A - 1) \geqslant n_B$ then Group B is Monopolized by Group A. In the symmetric within group Nash Equilibrium , $x_{kB}^B = 0$, $\forall k \in N_B$ and $x_{kA}^B = \frac{(n_A-1)(1-\alpha_A)}{n_A^2}$,$\forall k \in N_A$. The aggregate effort of group A is $X_A^{BM} = \frac{(n_A-1)(1-\alpha_A)}{n_A^2}$.

Proof: The proof follows exactly the same lines as Lemma 5, but with the roles of the Groups reversed. Now A Monopolizes B so $X_B^{BM} = 0$. We skip this proof. ■

▶ **CASE 3**:

Both (3.15) and (3.16) hold with equality at some $(x_{kA}, x_{kB}) > (0, 0)$

LEMMA **7**

If $-n_A > \alpha_B n_A(n_B - 1) - \alpha_A n_B(n_A - 1) < n_B$*, then neither group is Monopolized. In the symmetric within group Nash Equilibrium ,* $x_{ki}^{NM} = \frac{1}{n_i}(n_j(X^{NM})^2 - (n_j - 1)(1 - \alpha_j)X^{NM})$ *,*$\forall k \in N_i$*,* $i, j = A, B$ *and* $i \neq j$*, where the combined contest effort of the groups is* $X^{NM} = X_A^{NM} + X_B^{NM} = \frac{1+(n_A-1)(1-\alpha_A)+(n_A-1)(1-\alpha_A)}{N}$*. The probability of winning for the groups is* $(P_i^{NM}, P_j^{NM}) = (\frac{\chi_i}{N}, 1 - \frac{\chi_i}{N})$*.*

Proof: Firstly, if none of the Groups is to be Monopolized neither Lemma 5 nor Lemma 6 can apply. The antecedent of Lemma 7 follows directly by negation of Lemma 5 and Lemma 6 .

In this case all the F.O.C.'s in (3.15) and (3.16) hold with equality.

To figure out the individual efforts in the within group symmetric Nash Equilibrium we sum (3.15) over $k \in N_A$ to arrive at the following condition:

$$\frac{X_B^{NM}}{(X_A^{NM} + X_B^{NM})^2} + \frac{(1 - \alpha_A)(n_A - 1)}{X_A^{NM} + X_B^{NM}} = n_A \tag{3.21}$$

We sum (3.16) over $k \in N_B$ to arrive at the following condition:

$$\frac{X_A^{NM}}{(X_A^{NM} + X_B^{NM})^2} + \frac{(1 - \alpha_B)(n_B - 1)}{X_A^{NM} + X_B^{NM}} = n_B \tag{3.22}$$

Defining total effort in the collective contest as $X^{NM} = X_A^{NM} + X_B^{NM}$ and simplifying equations (3.21) and (3.22) we obtain:

$$x_{kA}^{NM} = \frac{1}{n_A}(n_B(X^{NM})^2 - (1 - \alpha_B)(n_B - 1)X^{NM}), \forall k \in N_A \tag{3.23}$$

and

$$x_{kB}^{NM} = \frac{1}{n_B}(n_A(X^{NM})^2 - (1-\alpha_A)(n_A-1)X^{NM}), \forall k \in N_B \tag{3.24}$$

Equations (3.23) and (3.24) are the Nash equilibrium effort levels in a within group symmetric equilibrium when both groups put in positive efforts in the collective contest.

Adding equations (3.21) and (3.22) we obtain:

$$X^{NM} = \frac{1 + (1-\alpha_A)(n_A-1) + (1-\alpha_B)(n_B-1)}{N} \tag{3.25}$$

■

Note (3.23) that $P_A^{NM} = \frac{X_A^{NM}}{X^{NM}} = n_A X^{NM} - (1-\alpha_B)(n_B-1)$

Replacing value of X^{NM} from (3.25) we get

$$P_A^{NM} = \frac{n_A + n_A(n_B-1)\alpha_B - n_B(n_A-1)\alpha_A}{N} = \frac{\chi_A}{N}$$

Similarly we can find the winning chances for group B.

Proposition 19 follows from Lemma 5, 6 and 7.

3.10.2 Leader's Choice Problem

The last sub-section dealt with the individual effort choice problem, taking as given the choices made by the respective group leaders. In this section, we focus on the choice problem of the leaders in the first stage. The leaders are assumed to choose the prize sharing rules simultaneously to maximize group payoffs. The sharing rules are subject to restrictions on competitiveness. So the problem faced by leader of group i is as follows:

$$\begin{aligned} \underset{\alpha_i}{\text{maximize}} \quad & \Pi_i(\alpha_i, \alpha_j) \\ \text{subject to} \quad & \underline{\alpha}_i \leqslant \alpha_i \leqslant 1, \ i = A, B. \end{aligned}$$

Here $\Pi_i(\alpha_i, \alpha_j)$ denotes the payoff of group i. Leader of Group i takes α_j as given. The group payoffs are also a function of the group sizes n_i and n_j, but they are suppressed for notational convenience.

To solve the above problem we set up the Kuhn-Tucker problem for the groups. To set-up the Lagrangian, however, we need to figure out the group payoffs first.

LEMMA **8**

For $i, j = A, B$,$i \neq j$

a)If Group i is Monopolized then,

$$\Pi_i^{iM} = 0 \text{ ,and } \Pi_j^{iM} = 1 - X_j^{iM}$$

b) If neither group is monopolized then

$$\Pi_i^{NM} = (1 - X^{NM})(n_j X^{NM} - (n_j - 1)(1 - \alpha_j))$$

where $1 - X^{NM}$ is the total rent ex-post and $n_j X^{NM} - (n_j - 1)(1 - \alpha_j)$ is group i's chance of winning.

Proof: The payoff function of group i can be written as follows:

$$\Pi_i(X_i, X_j) = \frac{X_i}{X_i + X_j} - X_i \tag{3.26}$$

If Group i is monopolized then from Lemma 5 and Lemma 6 we have, $X_i^{iM} = 0$ and $X_j^{iM} > 0$. Replacing in equation (3.26) we get part (a) of the Lemma.

If neither group is monopolized the from Lemma 7

$$X_i^{NM} = n_j (X^{NM})^2 - (n_j - 1)(1 - \alpha_j) X^{NM}$$

Replacing in equation (3.26) we get the expression for the group payoffs in part (b) of the Lemma.

■

Now we can set-up the Optimization Problem that the leaders of the groups face. While setting up the Lagrangian we ignore the Monopolization cases. We ignore the constraints $\alpha_i \leqslant 1$, $i = A, B$. We verify later that they are indeed satisfied in equilibrium.
The Lagrangian of the leader of Group A is as follows:

$$L_A = [1 - X^{NM}][n_B X^{NM} - (n_B - 1)(1 - \alpha_B)] + \lambda_A[\alpha_A - \underline{\alpha}_A] \tag{3.27}$$

The Lagrangian of the leader of Group B is as follows:

$$L_B = [1 - X^{NM}][n_A X^{NM} - (n_A - 1)(1 - \alpha_A)] + \lambda_B[\alpha_B - \underline{\alpha}_B] \tag{3.28}$$

λ_i is the Lagrangian multiplier of Group i. For ease of notation let us define $\theta_i = (n_i - 1)(1 - \alpha_i)$, $i \in \{A, B\}$.

The Kuhn -Tucker conditions are as follows:

$$\frac{dL_A}{d\alpha_A} = (n_B - 2n_B X^{NM} + \theta_B)\frac{dX^{NM}}{d\alpha_A} + \lambda_A = 0 \tag{3.29}$$

$$\frac{dL_B}{d\alpha_B} = (n_A - 2n_A X^{NM} + \theta_A)\frac{dX^{NM}}{d\alpha_B} + \lambda_B = 0 \tag{3.30}$$

$$\lambda_A \geqslant 0, \quad \alpha_A \geqslant \underline{\alpha}_A, \quad \lambda_A[\alpha_A - \underline{\alpha}_A] = 0 \tag{3.31}$$

$$\lambda_B \geqslant 0, \quad \alpha_B \geqslant \underline{\alpha}_B, \quad \lambda_B[\alpha_B - \underline{\alpha}_B] = 0 \tag{3.32}$$

We can use the Kuhn-Tucker conditions to break up the problem into four mutually exclusive cases. Each case is stated as Lemmas. These set of Lemmas help us prove Proposition 20

Neither Group's Constraints Bind

In this case we have $\lambda_A = 0$ and $\lambda_B = 0$.

LEMMA **9**

If neither Group's constraint binds then in Nash Equilibrium $(\alpha_A^*, \alpha_B^*) = (\alpha_A^{NN}, \alpha_B^{NN}) = (\frac{n_A - n_B}{N(n_A-1)}, \frac{n_B - n_A}{N(n_B-1)})$. *The net surplus in the contest in equilibrium is* $S^{NN} = \frac{1}{N}$. *The probabilities of winning are* $(P_A^{NN}, P_B^{NN}) = (\frac{n_B}{N}, \frac{n_A}{N})$.

Proof:

Set $\lambda_A = 0$ and $\lambda_B = 0$ in (3.29) and (3.30)

It can be easily verified that $\frac{dX^{NM}}{d\alpha_i} = \frac{-(n_i-1)}{N} < 0$ $i = A, B$. Therefore, (3.29) and (3.30) reduce to the following conditions:

$$n_B - 2n_B X^{NM} + \theta_B = 0 \tag{3.33}$$

and

$$n_A - 2n_A X^{NM} + \theta_A = 0 \tag{3.34}$$

If (3.29) and (3.30) are to hold simultaneously then the following equation must hold:

$$n_A \theta_B = n_B \theta_A \tag{3.35}$$

From Lemma 7 we know that

$$X^{NM} = \frac{1 + \theta_A + \theta_B}{N}$$

Using this fact and (3.35) in (3.33), and solving we get :

$$\theta_B = \frac{(N-2)n_B}{N} \tag{3.36}$$

Replacing θ_B from (3.36) in (3.35) we get:

$$\theta_A = \frac{(N-2)n_A}{N} \tag{3.37}$$

Using the definition of θ_i in (3.36) and (3.37), we get that in a Nash Equilibrium

$$(\alpha_A^{NN}, \alpha_B^{NN}) = (\frac{n_A - n_B}{N(n_A-1)}, \frac{n_B - n_A}{N(n_B-1)})$$

The net surplus and probabilities of winning can be obtained by replacing the Nash equilibrium values of (α_A, α_B) in part (B) of Proposition 19

■

Group A's Constraint Binds, Group B's Constraint does not Bind

This is the case which corresponds to $\lambda_A \geqslant 0$ and $\lambda_B = 0$

Lemma **10**

If Group A's constraint binds but Groups B's does not then in Nash Equilibrium $(\alpha_A^*, \alpha_B^*) = (\alpha_A^{AB}, \alpha_B^{AB}) = (\underline{\alpha}_A, \frac{(n_B-n_A)(1+(n_A-1)\underline{\alpha}_A)}{2n_A(n_B-1)})$. *The net surplus in the contest in equilibrium is* $S^{AB} = \frac{1+\underline{\alpha}_A(n_A-1)}{2n_A}$. *The probabilities of winning are* $(P_A^{AB}, P_B^{AB}) = (\frac{1-\underline{\alpha}_A(n_A-1)}{2}, \frac{1+\underline{\alpha}_A(n_A-1)}{2})$.

Proof:

Set $\lambda_B = 0$ in (3.30) and noting that $\frac{dX^{NM}}{d\alpha_B} = \frac{-(n_B-1)}{N} < 0$, the following condition is the relevant one

$$n_A - 2n_A X^{NM} + \theta_A = 0 \tag{3.38}$$

Replacing X^{NM} from Lemma 7 in (3.38) simplifying we get

$$Nn_A + N\theta_A = 2n_A(1 + \theta_A + \theta_B) \tag{3.39}$$

Solving for θ_B from (3.39)

$$\theta_B = \frac{n_A(N-2) + (n_B - n_A)\theta_A}{2n_A} \tag{3.40}$$

By definition $\theta_B = (n_B - 1)(1 - \alpha_B)$. Applying this definition and the fact that $\alpha_A = \underline{\alpha}_A$ and simplifying the above equation we get

$$\alpha_B^{AB} = \frac{(n_B - n_A)(1 + (n_A - 1)\underline{\alpha}_A)}{2n_A(n_B - 1)} \tag{3.41}$$

Therefore in this case the in a Nash equilibrium we have

$$(\alpha_A^{AB}, \alpha_B^{AB}) = (\underline{\alpha}_A, \frac{(n_B - n_A)(1 + (n_A - 1)\underline{\alpha}_A)}{2n_A(n_B - 1)})$$

We, however, need to verify that $\lambda_A \geqslant 0$. To do that we use (3.29). We know that $\frac{dX^{NM}}{d\alpha_A} = \frac{-(n_A-1)}{N} < 0$. Therefore to show that $\lambda_A \geqslant 0$, we need to show that $(n_B - 2n_B X^{NM} + \theta_B) \geqslant 0$. This is satisfied as long as

$$\underline{\alpha}_A \geqslant \frac{n_A - n_B}{N(n_A - 1)} = \alpha_A^{NB}$$

This is the choice made by group A when none of the constraints bind in Lemma 9. This condition delineates the zone where groups A's constraint binds and where it does not in equilibrium.

The net surplus and probabilities of winning can be obtained by replacing the Nash equilibrium values of (α_A, α_B) in part (B) of Proposition 19

■

Group B's Constraint Binds, Group A's Constraint does not Bind

This is the case where we have $\lambda_A = 0$ and $\lambda_B \geqslant 0$

Lemma **11**

If Group B's constraint binds but Groups A's does not then in Nash Equilibrium $(\alpha_A^*, \alpha_B^*) = (\alpha_A^{BB}, \alpha_B^{BB}) = (\frac{(n_A-n_B)(1+(n_B-1)\underline{\alpha}_B)}{2n_B(n_A-1)}, \underline{\alpha}_B)$. *The net surplus in the contest in equilibrium is* $S^{BB} = \frac{1+\underline{\alpha}_B(n_B-1)}{2n_B}$. *The probabilities of winning are* $(P_A^{BB}, P_B^{BB}) = (\frac{1+\underline{\alpha}_B(n_B-1)}{2}, \frac{1-\underline{\alpha}_B(n_B-1)}{2})$.

Proof:

The proof follows exactly the same line as the proof of Lemma 10, but now the relevant first order condition being (3.29). Therefore, we skip the proof.

■

Both Groups Constraint Binds

This is the case where we must have $\lambda_A \geqslant 0$ and $\lambda_B \geqslant 0$

Lemma **12**

If both Group A and Group B's constraint binds then in Nash Equilibrium $(\alpha_A^*, \alpha_B^*) = (\underline{\alpha}_A, \underline{\alpha}_B)$. *The net surplus in the contest in equilibrium is* $S^B = \frac{1+(n_A-1)\underline{\alpha}_A+(n_B-1)\underline{\alpha}_B}{N}$. *The probabilities of winning are* $(P_A^B, P_B^B) = (\frac{\chi_A}{N}, \frac{\chi_A}{N})$.

Proof:

In this case $\alpha_A^* = \underline{\alpha}_A$ and $\alpha_B^* = \underline{\alpha}_B$.

However, for this case to be valid we need to verify that $\lambda_A \geqslant 0$ and $\lambda_B \geqslant 0$. In light of the fact that $\frac{dX^{NN}}{d\alpha_i} = \frac{-(n_i-1)}{N} < 0$, we can immediately conclude from (3.29) and (3.30) that $\lambda_i \geqslant 0$ as long as $(n_j - 2n_jX^T + \underline{\theta}_j) \geqslant 0$, $i, j = A, B$ and $i \neq j$, where X^T is given in (**??**).

We work with the expression $(n_j - 2n_jX^{NM} + \underline{\theta}_j)$ to find conditions under which it is non-negative. Replacing X^T in the expression we get

$$n_j - 2n_j\frac{1+\underline{\theta}_i+\underline{\theta}_j}{N} + \underline{\theta}_j \geqslant 0$$

$$\Rightarrow Nn_j - 2n_j - 2n_j\underline{\theta}_i - 2n_j\underline{\theta}_j + N\underline{\theta}_j \geqslant 0$$

$$\Rightarrow (N-2)n_j + (n_i - n_j)(1-\underline{\alpha}_j)(n_j - 1) - 2n_j(1-\underline{\alpha}_i)(n_i - 1) \geqslant 0$$

$$\Rightarrow 2n_j(n_i - 1)\underline{\alpha}_i - (n_i - n_j)(n_j - 1)\underline{\alpha}_j \geqslant n_i - n_j$$

Simplifying we get that $\lambda_A \geqslant 0$ and $\lambda_B \geqslant 0$ as long as

$$\underline{\alpha}_B \geqslant \frac{(n_B - n_A)(1 + (n_A - 1)\underline{\alpha}_A)}{2n_A(n_B - 1)} = \alpha_B^{AB} \tag{3.42}$$

and

$$\underline{\alpha}_A \geqslant \frac{(n_A - n_B)(1 + (n_B - 1)\underline{\alpha}_B)}{2n_B(n_A - 1)} = \alpha_A^{BB} \tag{3.43}$$

where α_B^{AB} is the equilibrium choice of group B in the case where the constraint of group A binds but group B does not (Lemma 9) and α_A^{BB} is the equilibrium choice of group A in the case where group B's constraint binds but group A's does not (Lemma 10). So, these conditions cleanly delineate the zones of equilibria characterized in Lemma 9, 10 and 11.

The net surplus and probabilities of winning can be obtained by replacing the Nash equilibrium values of (α_A, α_B) in part (B) of Proposition 19

■

Having exhaustively analyzed the cases where neither group is monopolized, now we bring in monopolization to check when a group is monopolized in equilibrium.

Lemma **13**

Group i is monopolized in a Nash equilibrium iff $\underline{\alpha}_i \in [\frac{1}{n_i-1}, 1]$ and $\underline{\alpha}_j \in (-\infty, \alpha_j^M]$. In this case any combination of prize sharing rules (α_i^*, α_j^*), such that $\alpha_i^* \geqslant \underline{\alpha}_i$ and $\alpha_j^* = -\frac{1}{n_j-1} + \frac{(n_i-1)n_j}{(n_j-1)n_i}\alpha_i^*$ is a Nash equilibrium. The net surplus in the contest in equilibrium is $S^{iM} = \frac{\alpha_i(n_i-1)}{n_i}$. The probabilities of winning are $(P_i^{iM}, P_j^{iM}) = (0, 1)$.

Proof:

Let us consider the case where $i = A$. The proof for $i = B$ will be analogous and is skipped.

Now for group A to be monopolized we know from Lemma 5 that the following condition needs to be satisfied

$$\alpha_A n_B(n_A - 1) - \alpha_B n_A(n_B - 1) \geqslant n_A \tag{3.44}$$

So if group B were to monopolize group A, then given any choice of α_A, group B's best response is to choose

$$\alpha_B = -\frac{1}{n_B - 1} + \frac{(n_A - 1)n_B}{(n_B - 1)n_A}\alpha_A \tag{3.45}$$

This is so because it is the most egalitarian and hence the least costly way in which group B could monopolize group A. This is obtained by solving for α_B from (3.44) with an equality. As for choice of of group A we have the following two cases

Case 1:

Suppose in an equilibrium, group A behaves in a hawkish manner, so that $\alpha_A = \underline{\alpha}_A$. To monopolize A, group B will choose from (3.45).

$$\alpha_B^M = -\frac{1}{n_B - 1} + \frac{(n_A - 1)n_B}{(n_B - 1)n_A}\underline{\alpha}_A \tag{3.46}$$

Now in this case, group B in equilibrium obtains a payoff of $\Pi_B^{AM} = \frac{\underline{\alpha}_A(n_A-1)}{n_A})$ (see Proposition 21).

If instead it were to deviate to α_B^{AB} it would get $\Pi_B^{AB} = \frac{(1+(n_A-1)\underline{\alpha}_A)^2}{4n_A}$ (see Lemma 10).

But notice that $\Pi_B^{AB} \geqslant \Pi_B^{AM}$. Therefore group B always wants to deviate to α_B^{AB}. This deviation is not possible if $\underline{\alpha}_A > \frac{1}{n_A-1}$ [16], because then $\Pi_A^{AB} < 0$, so that group A is drops out. Given that group A will drop out group B's best response is to choose α_B^M, because $\alpha_B^M > \alpha_B^{AB}$ in this case and choosing α_B^M is the less costly way to monopolize A.

As group A gets zero payoff when monopolized, $\underline{\alpha}_A$ is a best response to α_B^M. Therefore, when $\underline{\alpha}_A > \frac{1}{n_A-1}$, $(\underline{\alpha}_A, \alpha_B^M)$ constitute a Nash equilibrium in which group A is monopolized.

[16]This is where α_B^{AB} and α_B^M intersect

If, however, $\underline{\alpha}_A < \frac{1}{n_A-1}$, then $\Pi_A^{AB} > 0$. Given that it is always optimal for group B to deviate to α_B^{AB}, it will do so and group A will not be monopolized. Therefore, there does not exist a Nash equilibrium in which A is monopolized when $\underline{\alpha}_A < \frac{1}{n_A-1}$.

Case 2:

Group A acts in a dovish manner $\alpha_A > \underline{\alpha}_A$ in equilibrium.

When $\underline{\alpha}_A < \frac{1}{n_A-1}$, the best response for group B is to choose α_B such that group A is not monopolized. Given that group B will not monopolize group A, the best response for group A do deviate to a hawkish stance, as its payoff is decreasing in α_A. Therefore, there does not exist a Nash equilibrium in which group A is dovish when $\underline{\alpha}_A < \frac{1}{n_A-1}$.

If $\underline{\alpha}_A \geqslant \frac{1}{n_A-1}$, nothing which group A does can guarantee it a positive payoff. So group A is indifferent and can choose any $\alpha_A \geqslant \underline{\alpha}_A$. In this case the best group B can do is to choose the least costly way to monopolize A by choosing α_B given in (3.45).

The fact that group A is indifferent between choices of α_A when it is monopolized in equilibrium, gives rise to multiple Nash equilibria. But, we can get around this issue by assuming that when indifferent group A chooses $\alpha_A = \underline{\alpha}_A$, because this choice is immune to trembles in strategies of group B. ■

Proposition 20 follows directly from Lemma 9, 10, 11, 12, 13. Also look at Figures 3.1 and 3.2.

The net surplus and probabilities of winning can be obtained by replacing the Nash equilibrium values of (α_A, α_B) in part (B) of Proposition 19.

■ Proof of Proposition 21

The proof directly follows from Proposition 20 noting that $\Pi_i = P_i S$.

■ Proof of Proposition 22.

Proof: To prove this Proposition we use Figures 3.1 and break up the proposition into four mutually exclusive cases. [17]

[17] Even though GSP has been defined in terms of winning probabilities in the chapter, we proceed by comparing payoffs of the groups, as these are equivalent in our framework.

▶ **Case 1:** $\underline{\alpha}_A \geqslant 0$ and $\underline{\alpha}_B < 0$

From Figure 3.1 it is clear that in this case either group A is Monopolized or we are in the case where Group A's constraint binds but Group B's does not.

If group A is monopolized then of course the larger group B wins the contest with probability 1 and GSA applies.

If group A is not monopolized then Lemma 10 applies. We can immediately verify that $\Pi_B^{AB} \geqslant \Pi_A^{AB}$. This inequality holds as long as $\underline{\alpha}_A \geqslant 0$. So again GSA applies.

▶ **Case 2:** $\underline{\alpha}_A \geqslant 0$ and $\underline{\alpha}_B \geqslant 0$

From Figure 3.1 it is clear that in this case we have many subcases, i.e., group A can be monopolized, group B can be monopolized, both groups constraints may bind and we may also be in situation where Group A's constraint binds but Group B's does not.

But just considering the case where both group's constraint binds helps us to cleanly delineate the parametric zone into zones where GSP or GSA applies. When both group's constraints bind then Lemma 12 applies. It can be easily verified that $\Pi_A^B > \Pi_B^B$ if and only if $\underline{\alpha}_B > \alpha_B^{GSP} = \frac{n_B - n_A}{2n_A(n_B-1)} + \frac{(n_A-1)n_B}{(n_B-1)n_A}\underline{\alpha}_A$.

α_B^{GSP} intersects α_B^{AB} at $\underline{\alpha}_A = 0$ and lies above it at any $\underline{\alpha}_A > 0$. Also, α_B^{GSP} lies entirely above α_B^{AM} at any $\underline{\alpha}_A \geqslant 0$. So, these cases belong where $\underline{\alpha}_B \leqslant \alpha_B^{GSP}$, and therefore GSA should apply in these cases. It can be easily verified from Lemma 10 and Lemma 13, that it is indeed the case. Look at Figure 3.4.

Also, α_B^{GSP} lies completely below α_A^{BM}. So, the cases in which group B is monopolized belong where $\underline{\alpha}_B > \alpha_B^{GSP}$, and therefore GSP applies.

α_B^{GSP} provides a clear delineation of this parametric zone, i.e., $\underline{\alpha}_A \geqslant 0$ and $\underline{\alpha}_B \geqslant 0$, as far as occurrence of GSP or GSA is concerned.

▶ **Case 3:** $\underline{\alpha}_A < 0$ and $\underline{\alpha}_B < 0$

From Figure 3.1 it is clear that either we are in the case where Group A's constraint binds but Group B's does not or we are in the case where neither groups constraint binds.

In the case where neither groups constraint binds Lemma 9 applies. It can be immediately verified from the Lemma that $\Pi_A^{NN} > \Pi_B^{NN}$. Therefore, GSP applies in such cases.

In the case where Group A's constraint binds but Group B's does not, Lemma 10 applies. And again it is straightforward to check from the Lemma that $\Pi_B^{AB} < \Pi_A^{AB}$ when $\underline{\alpha}_A < 0$. So, again GSP applies.

► **Case 4:** $\underline{\alpha}_A < 0$ and $\underline{\alpha}_B \geqslant 0$

From Figure 3.1 it is clear that this case has many subcases, i.e., neither group's constraints bind, group B is monopolized, Group A's constraint binds but Group B's does not and also Group B's constraint binds but Group A's does not. In what follows we consider each case one by one.

If we are in the case where group B is monopolized, then group A wins with probability 1 and GSP applies.

If neither groups constraint binds then Lemma 9 applies. It can be immediately verified from the Lemma 9 that $\Pi_A^{NN} > \Pi_B^{NN}$. Therefore, GSP applies in such cases.

If group A's constraint binds but group B's does not then, Lemma 10 applies. It can be easily verified from Lemma 10 that $\Pi_B^{AB} < \Pi_A^{AB}$ when $\underline{\alpha}_A < 0$. Therefore, GSP applies in this case.

If group B's constraint binds but group A's does not then, Lemma 11 applies. It can be easily verified from Lemma 11 that $\Pi_B^{BB} < \Pi_A^{BB}$ when $\underline{\alpha}_B \geqslant 0$. Therefore, GSP applies in this case too.

The last case to consider is the one where both groups constraint binds. We saw that in **Case 2** that GSP applies when $\underline{\alpha}_B > \alpha_B^{GSP}$. When both groups constraints bind in this case, the condition for GSP to occur is still $\underline{\alpha}_B > \alpha_B^{GSP}$ as Lemma 12 still applies. But, we also noted in the proof of **Case 2** that α_B^{GSP} intersects α_B^{AB} at $\underline{\alpha}_A = 0$. In this particular case, α_B^{GSP} lies entirely below α_B^{AB}. For both groups constraints to bind it must be the case that $\underline{\alpha}_B > \alpha_B^{AB}$. But because $\alpha_B^{AB} > \alpha_B^{GSP}$ in this case, it follows that $\underline{\alpha}_B > \alpha_B^{GSP}$. Therefore, GSP applies in this case as well. For visual clarity consider the dotted section of α_B^{GSP} in Figure

3.4.

Proposition 22 directly follows from the above four cases and can be visualized in Figure 3.4

■

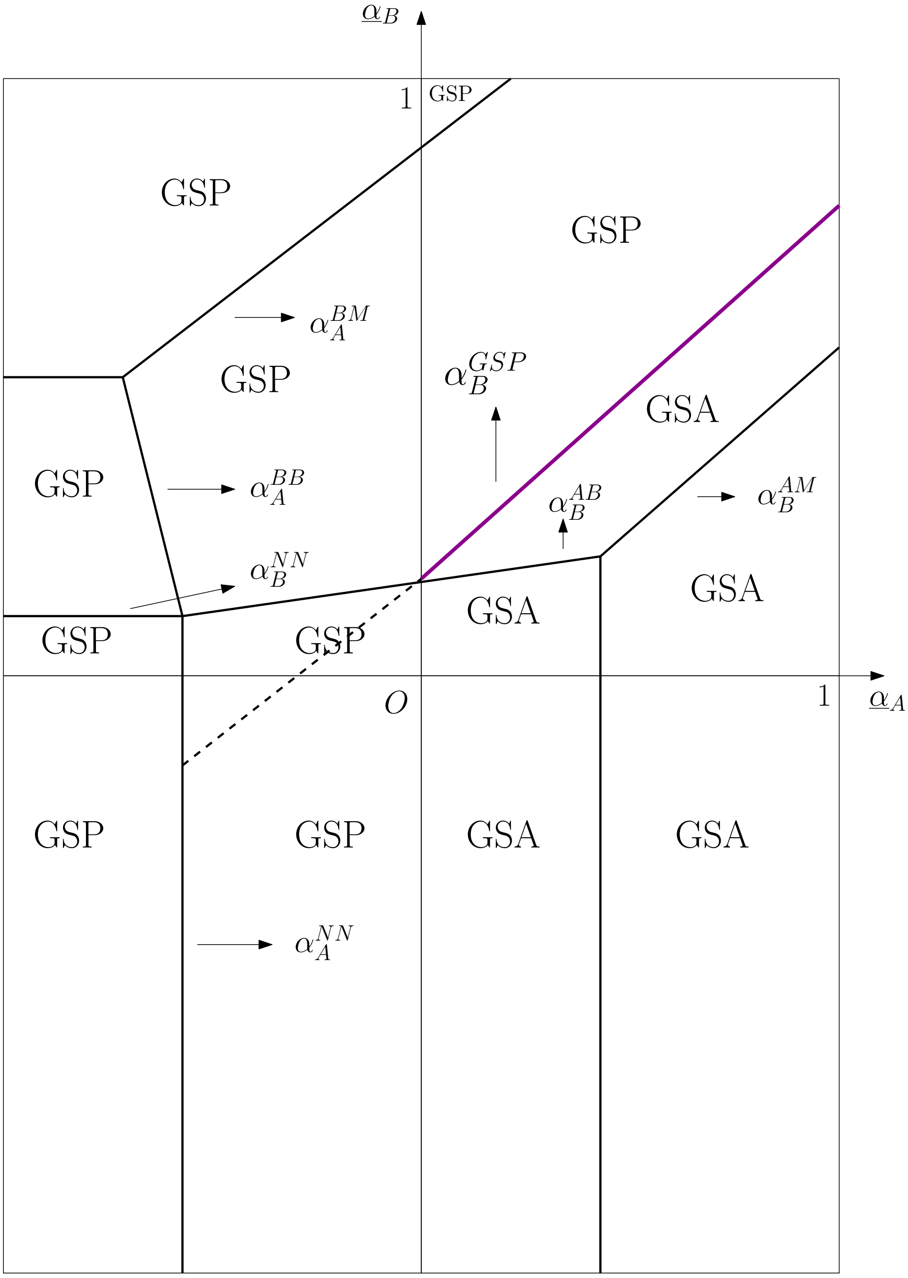

Figure 3.4: GSP-GSA

3.11 Appendix 2

Proof of Proposition 23

Proof:

Using the fact that $X_B = x^* X_A$ in (2.88) we get X_A^{NN}. We get X_B^{NN}, by replacing $X_B^{NN} = (x^*) X_A^{NN}$ in (2.89).

Replacing $X_B = (x^*) X_A$, in $P_A = \frac{X_A^r}{X_A^r + X_B^r}$ we get that $P_A^{NN} = \frac{1}{1+(x^*)^r}$. And P_B^{NN} is obtained by solving $P_B^{NN} = 1 - P_A^{NN}$.

The group payoffs can be obtained by using the computed X_A^{NN} and P_A^{NN} in $\Pi_A^{NN} = P_A^{NN} - X_A^{NN} = P_A^{NN} - \frac{r(x^*)^r + (1+(x^*)^r)\theta_A}{n_A[1+(x^*)^r]^2} = P_A^{NN} - \frac{1}{1+(x^*)^r}\left(\frac{r(x^*)^r + (1+(x^*)^r)\theta_A}{n_A[1+(x^*)^r]}\right) = P_A^{NN}(1 - \frac{\theta_A}{n_A} - \frac{r(x^*)^r}{n_A[1+(x^*)^r]}) = \frac{P_A^{NN}}{n_A}\left(n_A - \theta_A - r(1 - \frac{1}{1+(x^*)^r})\right) = \frac{(n_A - \theta_A - r)P_A^{NN} + rP_A^{NN\,2}}{n_A}$.

Similarly we can find Π_B^{NN}. ■

Proof of Proposition 24

Proof:

To prove this we use equation (2.25).

$$n_A \theta_B (x^*)^r - n_B \theta_A (x^*)^{1-r} - n_B(r + \theta_A)(x^*) + n_A(r + \theta_B) = 0 \tag{3.47}$$

Differentiating the equation with respect to α_A and rearranging we get

$$\frac{dx^*}{d\alpha_A} = \frac{-n_B(n_A - 1)[x^* + (x^*)^{1-r}]}{rn_A\theta_B(x^*)^{r-1} - (1-r)n_B\theta_A(x^*)^{-r} - n_B(r + \theta_A)} \tag{3.48}$$

The numerator is clearly negative. As for the the denominator, we have to consider the following function

$$y = n_A\theta_B x^r - n_B\theta_A x^{1-r} - n_B(r + \theta_A)x + n_A(r + \theta_B) \tag{3.49}$$

Therefore slope of the function evaluated at x^* is as follows:

$$\left.\frac{dy}{dx}\right|_{x=x^*} = rn_A\theta_B x^{*r-1} - (1-r)n_B\theta_A x^{*-r} - n_B(r+\theta_A) \tag{3.50}$$

Notice that the right hand side of (3.50) is the denominator of (3.48). We can also see in Figure 2.20 and 2.21 that $\left.\frac{dy}{dx}\right|_{x=x^*} < 0$. Therefore, we can conclude that $\frac{dx^*}{d\alpha_A} > 0$.

Now we know that $P_A^{NN} = \frac{1}{1+(x^*)^r}$.

Differentiating with respect to α_A we get $\frac{dP_A^{NN}}{d\alpha_A} = -\frac{r(x^*)^{r-1}}{(1+(x^*)^r)^2}\frac{dx^*}{d\alpha_A} < 0$.

Similarly we can show that $\frac{dP_B^{NN}}{d\alpha_B} < 0$.

■

Proof of Proposition 25

Proof:

To understand when group i constraint will be binding, i.e., $\alpha_i = \underline{\alpha}_i$ in equilibrium we need to find conditions when $\frac{d\Pi_i^{NN}}{d\alpha_i} < 0$. Note from Proposition 23 that the payoff of group i in equilibrium is

$$\Pi_i^{NN} = \frac{(n_i-\theta_i-r)P_i^{NN} + rP_i^{NN^2}}{n_i} \tag{3.51}$$

Taking the derivative with respect to α_i we get

$$\frac{d\Pi_i^{NN}}{d\alpha_i} = \frac{(n_i-1)P_i^{NN} + (n_i-\theta_i-r+2rP_i^{NN})\frac{dP_i^{NN}}{d\alpha_i}}{n_i} \tag{3.52}$$

We know that $\frac{dP_i^{NN}}{d\alpha_i} < 0$ from Proposition 24. So to determine the sign of $\frac{d\Pi_i^{NN}}{d\alpha_i}$ we need to focus on the sign of $n_i-\theta_i-r+2rP_i^{NN}$. This will be non-positive as long as $\underline{\alpha}_i \leqslant -\frac{1-r+2rP_i^{NN}}{n_i-1}$. But in that case $\frac{d\Pi_i^{NN}}{d\alpha_i} > 0$ and the group leader will choose $\alpha_i > \underline{\alpha}_i$ in equilibrium, which is defined to be dovish behavior.

If on the other hand $\underline{\alpha}_i > -\frac{1-r+2rP_i^{NN}}{n_i-1}$ then to have $\frac{d\Pi_i^{NN}}{d\alpha_i} < 0$ we also require $-\frac{dP_i^{NN}}{d\alpha_i} > \frac{(n_i-1)P_i^{NN}}{(n_i-\theta_i-r)+2rP_i^{NN}}$ to hold. In this case, we will have group i leader behaving in a hawkish manner.

■

Chapter 4

Bargaining for Assembly

4.1 Introduction

In many real life situations a buyer needs to acquire multiple inputs to implement a project. Examples include acquiring multiple plots of land to set up a factory, hiring faculty to set up an academic department, acquiring different molecules to make a new drug, among others. In most of these situations the inputs are owned by different individuals. Consequently, the buyer needs to bargain successfully with multiple sellers owning distinct sets of inputs. We refer to such situations as assembly problems.

For illustration, let us consider a buyer, who owns a production process, which can be made operational with inputs viz., capital and labor. The buyer owns none of the inputs and needs to acquire them from respective owners. The degree of complementarity between different inputs turns out to be an important determinant of how bargaining between the buyer and sellers unfolds. Roy Chowdhury and Sengupta (2012) show that when inputs are perfectly complementary the buyer is subject to holdout by the sellers and therefore, realizes little share of surplus in absence of an outside option.

We are skeptical about the assumption of perfect complementarity of inputs and want to analyze whether holdout persists when we allow different degrees of complementarity. To do that we introduce production processes modeled as graphs. Each node on the graph represents an input and an edge between a pair of nodes represents the complementarity of these inputs in the production process. The buyer wants to purchase a path of a desired length, called a feasible path. Inputs can be substitutes because we assume that the buyer may not need to acquire inputs from all sellers to implement the project — this is the situation when there are more than one feasible path in the graph representing the production process. Examples of such production processes is provided in subsection 4.3.1.

Most of the existing literature on multilateral bargaining assumes perfect complementarity of inputs and focuses on the effect of different bargaining protocols on the incidence of hold out. In contrast, we focus on the importance of the features of the underlying production process for holdout to occur using a standard extension of the Rubinstein protocol to multilateral bargaining.

We are able to show that full surplus extraction by the buyer within two periods is a subgame perfect equilibrium of our bargaining model if and only if (a) there are no critical sellers and (b) there exist at least two feasible paths with minimum sum of seller valuations.

The equilibria we characterize have the following features:

- Suppose the underlying graph has no critical sellers and seller valuations are identical.
  - If the graph contains a cycle of a minimal length, the buyer can extract full surplus in the first period itself regardless of whether she or seller makes the first offer. This represents the case of perfect substitutability between all individual inputs.
  - If the graph contains two disjoint paths, the buyer can extract full surplus in the second period if she is making the first offer. She extracts full surplus in the first period if sellers are making the first offer. This is the case of path-path substitutability.
  - For any other graph without critical sellers, which do not fall into the previous two classes, the buyer can extract full surplus in the second period if she is making the first offer. She extracts full surplus in the first period if sellers are making the first offer. This is the case of node-path substitutability.

- Suppose the underlying graph has no critical sellers and seller valuations are unequal. The buyer cannot extract full surplus except in a special case.

- Suppose there is exactly one critical seller in the graph. We show that the buyer cannot earn more than $\frac{1}{1+\delta}$ of the maximum surplus in any equilibrium. If there are more than one critical sellers, buyer cannot earn more than $\frac{1-\delta}{1+\delta}$ of the maximum surplus in any equilibrium.

Our results highlight the importance of indispensable inputs (critical sellers) to the incidence of strategic delays and the buyer getting minimal share of the surplus. In the process, we underscore the importance of studying underlying production processes in models of multilateral bargaining, as it affects how bargaining unfolds in critical ways. Even though the production process we propose is not the most general one, it has wide applications. We conjecture that our basic insight about the importance critical sellers in multilateral bargaining will carry over to more general production processes if a critical seller is appropriately defined in such a context.

We structure our chapter as follows. In the next section we discuss the relevant literature. Subsequently, we lay down the preliminary structure of our model and present two important results from the literature. Then we present our main results for different cases of our model. All proofs are presented in the Appendix. The next section offers detailed discussion of the main results. The final sections offer concluding remarks.

4.2 Literature

Situations where a buyer needs to buy complementary inputs from different sellers is quite common. For example, the famous railroad example by Coase (1960) presents a situation where the railroad has to acquire plots of land from several farmers. Given the complementarity inherent in such activities, the outcome is likely to exhibit hold out, allowing sellers to extract a greater share of the surplus. In such scenarios hold out is expected to cause inefficiencies viz. strategic delays, implementation of sub optimal projects or complete breakdown of negotiations. This problem has been studied in the land assembly context (Asami, 1985; O'Flaherty, 1994; Cai, 2000b, 2003; Menezes and Pitchford; Miceli and Segerson, 2012; Roy Chowdhury and Sengupta, 2012; Göller and Hewer, 2015).

In one of the earliest papers on the topic, Asami (1985) models a land market with multiple buyers and multiple sellers as a cooperative game. He finds that in a core allocation, competition prevents agents from claiming surplus, but some agents, e.g. a critical seller or a lonely buyer are able to extract positive surplus. In contrast, our approach is non-cooperative and allows for general contiguity structures and valuations. However, it retains all the features of Asami (1985) pertaining to the single buyer problem.

Strategic exchange is usually modeled in economics using bargaining games, where agents on one side of the market propose prices (or, equivalently, shares of the surplus), and those on the other side accept or reject. The legitimate range of price offers, the sequencing of the offers and the possible length of the negotiation process are given by the bargaining protocol, which is common knowledge (see Osborne and Rubinstein (1990) for a survey). The bargaining protocol we follow is a natural extension of Rubinstein (1982) and is the same one used in Roy Chowdhury and Sengupta (2012). We also assume complete information, i.e., all relevant information pertaining to the game is common knowledge among players. So our model belongs to the wider class of strategic bargaining models with complete information (e.g., Fernandez and Glazer (1991)).

Closer to our setting, Menezes and Pitchford study a non-cooperative game of entry into an efficient bargaining process. They show that there is inefficiency in the entry decision and relate it to the degree of complementarity in production. Cai (2000b, 2003), shows how inefficiency due to hold-out may arise by using a circular bargaining protocol, where the buyer follows a fixed order of bargaining with sellers; sellers who reject an offer are pushed to the end of the queue. In contrast, we do not study entry and we assume a simultaneous offers game. Thus we are closest to Roy Chowdhury and Sengupta (2012). Also like most of the above papers (except Cai (2003)), we analyze the cash offers model, where payment is made immediately upon agreement.

Roy Chowdhury and Sengupta (2012) have studied the problem of a buyer bargaining with multiple sellers holding an item each, where all items are complementary. Either side of the market can open the negotiations. Suppose the buyer begins by making simultaneous offers to active sellers. A seller can accept or reject the offer he receives. On acceptance, the seller surrenders his plot in lieu of the cash offer and leaves the market. Sellers rejecting buyer's offer make counteroffers in the next period that the buyer can accept or reject. Bargaining continues till either the buyer quits to avail an outside option or realize an agreement with all sellers. They focus on the role of transparent protocols and outside options: buyer can extract higher surplus with transparent protocols if he has an outside option; holdout may be unavoidable with less transparent protocols even in presence of an outside option.

We use the same bargaining protocol as Roy Chowdhury and Sengupta (2012) but assume it to be transparent; we also assume that the buyer has no outside option, similar to their benchmark model. We introduce competition among sellers in the model by allowing for more sellers than the number of items required. Such competition has the familiar flavor of Bertrand games covered in the applied game theoretic literature. Our graph theoretic approach allows us to explore different degrees complementarity among inputs and relates it to the phenomenon of holdout in an intuitive way. For a paper, which analyzes bilateral bargaining between agents on a graph with complementary pieces of information, the reader is referred to Jiménez Martínez and Dam (2011).

A number of contributions in the literature have applied protocols where the buyer engages in a sequence of bilateral negotiations with sellers (Cai, 2000b; Suh and Wen, 2006, 2009; Li, 2010a). Delay is embedded in such protocols in the sense that at least k periods are required for successful assembly if the buyer needs to assemble k units. Unless the buyer's budget per period is limited, or the application in question involves bargaining with agents in different levels of supply chain (e.g., wholesaler, retailer etc.), a rational buyer would minimize such delay. Notice that our protocol effectively allows for bilateral negotiations: for example, if the buyer has to assemble two items, she can choose to make a positive offer to seller 1 and a negative offer to seller 2. This allows a bilateral negotiation with seller 1 in period 1 and that with seller 2 in period 2. Alternatively, she can make simultaneous offers to both sellers. Consequently, whether she chooses to bargain simultaneously or in a sequence is a choice to be made as an equilibrium response. Readers interested in the equilibria in sequential protocols may refer to the papers cited above.

We want to distinguish our contribution from two seemingly related strands of literature. First, it is distinct from the literature on bilateral trade on networks (see the survey by Manea (2016)): in this literature, bilateral trade takes place in each period between a random pair of nodes on a network. In contrast, we use a network to model the complementarity of inputs owned by sellers. The buyer is isolated from this network, but wants to purchase all nodes on a feasible path. She can make an offer to any seller and vice versa, but no seller can make any offer to another seller. Secondly, it is distinct from the literature on spectrum and package auctions (see the survey by Bichler and Goeree (2017)): in such auctions, multiple buyers have possibly different valuations over different "packages" of radio spectrum. In contrast, our single-minded buyer has the same valuation over every feasible path.

Sarkar (2017) investigated the existence of direct mechanisms that are "successful" in the sense of Myerson and Satterthwaite (1983)[1] when agents have private and independent valuations and seller valuations are drawn from the same prior. Although a successful direct mechanism may exist for certain priors, it is not easy to interpret the form of such a mechanism[2]. In contrast, bargaining has a natural interpretation in a complete information framework. It also enables us to study the equilibrium strategies of the agents in depth. It remains to be investigated whether the generalized Rubinstein bargaining protocol that we use can lead to efficient outcomes under asymmetric information.

[1] A mechanism is "successful" in this sense if it is ex-post efficient, interim incentive compatible, interim individually rational and ex post budget balanced.

[2] See Krishna and Perry (2000) for the construction of a successful mechanism

There is also a literature in multiagent contract theory that maybe of secondary interest to our readership. Segal (1999) analyzes the problem of contracting with externalities. With public commitment, inefficiency arises because of externalities in agents' reservation utilities. Genicot and Ray (2006) analyses a game where a principal offers contracts to a set of agents whose outside option depends on the number of agents not contracted. In this game, competition among agents is exploited to force agents to inferior contractual terms.

A natural follow-up of our exercise is to investigate the impact of formation of seller coalitions on equilibrium payoffs (see Ray (2007) for a survey of coalition formation). A complete analysis of this question is beyond the scope of this chapter. In our discussion section, we provide an example to show that if the sellers are allowed to form coalitions, the buyer may not be able to extract full surplus even when sellers have identical valuations.

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4.3 Preliminaries

4.3.1 Graphs and assembly problems

Sellers of inputs are located on nodes of a graph. Two sellers are connected by an edge if the corresponding inputs are complementary in buyer's production process. In an application like land acquisition, adjacency can be interpreted in the usual physical sense. A path is a sequence of connected nodes. The buyer wants to purchase a path of desired length[3], say k. This implies that the buyer can combine any k complementary inputs to produce output. We will denote a path by $\mathcal{P}$ and the corresponding sum of seller valuations by $\mathcal{S}$. An assembly problem with complete information is a tuple: $\langle \Gamma, k, v, \delta \rangle$. Here Γ is a graph of order n; positive integer k is the desired minimum length of the path buyer is interested in purchasing; if the buyer cannot acquire a path of size k or more, the project is not feasible and the value he gets is normalized to 0; the first component of $v \equiv (v_0, v_1, \ldots, v_n)$ denotes the valuation of the

[3]This can be relaxed to include any special graph of a fixed size. Rights of passage directly motivates the desire to purchase a path in our case.

buyer for a path of length k or more, and other components denote the valuation of the sellers for their respective items; the real number $\delta \in [0,1]$ denotes the common rate at which the $n+1$ agents discount future payoffs. Note that efficiency would require the buyer to purchase only paths of length k, unless some of the sellers have zero valuation. We assume that there exists a path $\mathcal{P} \in \Gamma$, such that it results in a positive surplus: $v_0 - \sum_{i \in \mathcal{P}} v_i > 0$. Given such a graph Γ, the expression $\max_{\mathcal{P} \in \Gamma} \left(v_0 - \sum_{i \in \mathcal{P}} v_i\right)$ is referred to as "full surplus" or "efficient surplus".

The variety of possible graph structures can be large since a graph with n nodes can have up to $\binom{n}{2}$ edges. We group possible graphs into four mutually exclusive and exhaustive classes and analyze them independently to reach our main result. The four classes are as follows:

Graphs with Critical Sellers: A seller is critical if he lies on every feasible path (see Figure 4.1). This implies that the corresponding input is complementary with respect to every feasible production plan. If there is only one feasible path in Γ, all sellers in that path are critical. But if there are multiple feasible paths, a seller must lie in their intersection in order to qualify as critical. If there are multiple feasible paths, the number of critical sellers cannot exceed $k-1$: not all sellers on a single path can be critical. Paths of length less than k that do not have an intersection with any feasible path can be excluded from the analysis, because the buyer's valuation over such paths is zero.

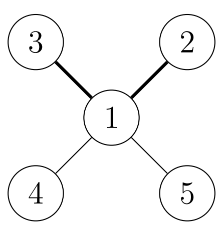

Figure 4.1: A feasible path in the star graph when $k = 3$; seller 1 is critical.

Consider graphs with critical sellers, referred to as Γ^* (see Figure 4.1 and Figure 4.2). In this class, inputs belonging to critical sellers are not substitutable but those belonging to non-critical sellers are substitutable in a limited sense.

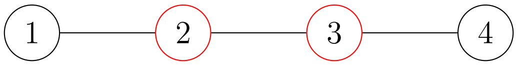

Figure 4.2: A line graph with two critical sellers marked red;k=3

Graphs with a $k+1$ Cycle: Consider graphs with cycles of order $k+1$, referred to as $\Gamma^{\triangle}$ (see Figure 4.3). Here, every input on a feasible path can be substituted by another input on the graph. Note that when Γ is a complete graph of order n, which we denote as an assembly

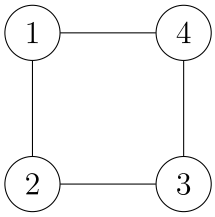

Figure 4.3: A cycle of length 4.

problem by $\langle n, k, v, \delta \rangle$, it belongs the the class $\Gamma^{\triangle}$.

Graphs with Disjoint Paths: Next, we consider graphs with two disjoint paths, referred to as Γ^{D} (see Figure 4.4). Here, no individual input is completely substitutable, but a feasible path can be substituted by another feasible path.

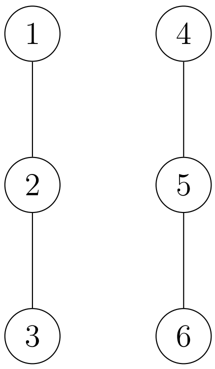

Figure 4.4: Graph with disjoint feasible paths; k=3

Oddball Graphs: Finally, consider graphs where (i) there is no cycle of length $k+1$, (ii) no two paths are disjoint and (iii) the intersection of all feasible paths is empty, referred to as

Γ^O (see Figure 4.5). For convenience, we will refer this class as oddball. In this class, inputs in the intersection of two or more feasible paths cannot be substituted with respect to these feasible paths, but they are substitutable with respect to inputs on other feasible paths.

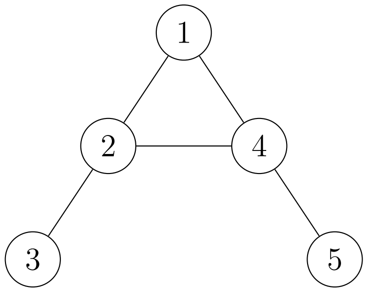

Figure 4.5: An oddball graph, $n = 5$, $k = 3$

Facts 1-5 below imply that single component graphs with (a) critical sellers, (b) $k+1$-cycle, (c) disjoint paths, and (d) oddball are four mutually exclusive and exhaustive categories. A graph may have multiple components from different classes.

- **Fact 1:** All sellers on a single path of length k are critical, regardless of whether this path is a cycle.
- **Fact 2:** The number of critical sellers on a single path reduces with its length.
- **Fact 3:** No cycle of length more than k has a critical seller.
- **Fact 4:** Cycles of length $2k$ or more have at least two disjoint feasible paths and hence, no critical seller.
- **Fact 5:** The oddball class covers all cycles of length larger than $k + 1$ but smaller than $2k$. Further, since every pair of feasible paths on an oddball graph intersect at least once, it also covers graphs containing cycles of length less than or equal to k.

Applications

The following examples illustrate the natural appeal of using graph structures for modeling assembly.

The government wants to build a flyover or a gas pipeline by combining three ($k = 3$) plots out of 6 available plots ($n = 6$) numbered 1 through 6. Every combination of three plots may not be feasible because of practical reasons, say, the route between some pair of plots may have protected forest cover. Each plot can be represented by a node on a graph; nodes that do not have any forest cover between them are connected by an edge. The government then needs to pick a path of length 3 on the resulting graph. Let us now interpret alternative graph structures in this context.

Consider Figure 4.1. There is no forest cover between plot 1 and any other node, but the path between every pair of other nodes have forest cover. Plot 1 must be a part of every path — it is perfectly complementary to other plots in any production plan to construct the bridge. We refer to such a plot as a critical plot and the corresponding seller as a critical seller. Note that there can be more than one critical seller (Figure 4.2). If there were only three plots without forest cover between any pair of plots, then all three of them would be critical.

Consider any pair of plots in Figure 4.3: either there is no forest cover between them, or there exist an uninterrupted access from one plot to the other via some other plot. This case represents perfect substitutability between any pair of nodes. If a particular plot in a combination is replaced with some other plot, the resulting combination remains feasible.

Consider plots 1, 2 and 3 in Figure 4.4: there is no forest cover between 1 and 2 or 2 and 3, and consequently, there is an uninterrupted access between 1 and 3 via 2. Similarly for the set consisting plots 4, 5 and 6. The government can substitute path 456 with path 123 but cannot substitute a node within a path with a node outside a path. In this sense, nodes on a particular path are perfect complements but the paths are perfect substitutes.

Consider Figure 4.5 containing a graph we call *oddball*. There is no critical seller, no pair of paths is disjoint and not every node on a feasible path can be replaced by another node. There is a cycle but it is unlike Figure 4.3. This is a case where a plot on a path can be substituted by a set of plots to maintain feasibility. For instance, consider the path 123 — 1 or 3 can be replaced by 4 but 2 can be replaced by 4 and 5, while plot 3 is going to be wasted.

Notice that the congruence of inputs matter in our model. This idea of congruence has natural appeal in land assembly because the buyer needs rights of passage from one plot to the other to implement the project. Another natural example is that of a mobile service provider trying to purchase spectrum in multiple districts. The buyer values contiguity of districts where spectrum is acquired because it ensures seamless connectivity across the coverage area. A rather unconventional example in this context is a situation where a music composer wants to assemble different parts from other songs to compile a new score. But these songs are owned by different copyright holders. Portions of songs have to be harmonically close enough to each other to be combined to yield a meaningful score.

Certain production processes may use a different idea of congruence that may not immediately be amenable to our graph-theoretic treatment. Consider a pharmaceutical company which wants to create a drug by assembling molecules owned by different companies. Let us use Figure 4.1 to illustrate this case, renaming plots as molecules. Let us say that the only combinations of molecules that work are 1, 2 and 3 or 1, 4 and 5. But now 2, 1 and 4 may not always make a feasible combination because of their chemical properties. In a similar vein, suppose 1 represents a factory, 2 and 4 are wholesalers while 3 and 5 are retailers; a brand wants to create a complete supply chain by signing agreements with the factory, a wholesaler and a retailer. Notice that now 214 no longer remains feasible as it contains two wholesalers and no retailer. In this case, it makes more sense to use graphs like Figure 4.4, repeating 1 in both paths, or keep 1 out of the situation completely and consider disjoint paths with a retailer and a wholesaler each.

4.3.2 Bargaining protocol

We apply a variant of the Rubinstein bargaining protocol which is due to Roy Chowdhury and Sengupta (2012). In each period, either the sellers propose individual offers of surplus shares to the buyer or the buyer proposes a vector of offers of surplus shares to active sellers. Suppose the buyer proposes surplus shares. The sellers can individually accept or reject the offer. The sellers who reject buyer's offer propose individual shares to the buyer in the next period that the buyer may accept or reject. If the buyer reaches an agreement with any of the sellers in some period, she immediately purchases his plot and the seller leaves the market. The buyer and the remaining seller then resumes bargaining. The game continues till the buyer is able to purchase at least one feasible path.

We allow the buyer to utilize negative surplus offers to exclude particular sellers from the bargaining process — such offers translate into prices that are less than seller's valuation and therefore, rejected. This facilitates the buyer avoid the commitment involved in a cash offers bargaining protocol. Also, such negative offers enable the buyer to choose sequences of sellers to bargain with in each period as discussed in our introduction. Notice that a seller cannot possibly make a negative offer to the buyer in our setting, since it delays trade with the buyer or eliminates the prospect of trade[4]. Bilateral bargaining models, like that by Rubinstein (1982) do not use this feature, while multilateral models like Roy Chowdhury and Sengupta (2012) do.

4.3.3 Existing results

Two special cases of bargaining for assembly in our sense have been studied in the literature and these are given below for completeness.

The bilateral game studied by Rubinstein (1982) is a special assembly problem with $n = k = 1$. Here the only seller present is critical. The Subgame Perfect Nash Equilibrium of this game, which is now a standard result, is presented below without a proof.

Theorem **1** (**Rubinstein** (**1982**)) *Consider the model where the buyer bargains with one seller for one input: $\langle n = 1, k = 1, v_0 > v_1, \delta \rangle$. There is a unique SPNE of the model described as follows:*

> *Agent i proposes a share $\frac{\delta}{1+\delta}$ of the surplus to j whenever she has to propose, and accept any share at least equal to $\frac{\delta}{1+\delta}$ whenever j has to propose.*

[4]Allowing negative offers to sellers makes more sense when there are multiple buyers.

The game ends in the first period itself, with buyer proposing $\frac{\delta}{1+\delta}$ to the seller and the seller accepting it.

The model studied by Roy Chowdhury and Sengupta (2012) is a special assembly problem with $n = k \geq 2$ and all seller valuations are identical. Since the buyer wants all n plots, all sellers are critical here.

THEOREM **2** (**Roy Chowdhury and Sengupta** (**2012**)) *Consider the model $\langle n \geq 2, k = n, v_1 \leq \cdots \leq v_n, v_0 > \sum_{i=1}^{n} v_i, \delta \rangle$. The buyer's equilibrium payoff cannot be more than $\frac{1-\delta}{1+\delta}$ of the full surplus for any $\delta > 0$.*

In terms of our model this situation pertains to the case with only one feasible path in a graph. Two types of equilibrium outcomes are identified: if $1 > \frac{n\delta}{1+\delta}$, the buyer offers $\frac{\delta}{1+\delta}$ to every seller in the first period and all of them accept. Otherwise, the buyer would offer zero in the first period, all but $r \geq 2$ sellers would accept, and in the second period these r sellers would demand P such that $1 - rP = \delta\left(1 - \frac{r\delta}{1+\delta}\right)$. Here, r is the highest positive integer such that $1 > \frac{r\delta}{1+\delta}$.

4.3.4 Two examples

We are interested in assembly problems where not all sellers are critical — this corresponds to the cases where Γ contains at least two feasible paths. Also, we allow for arbitrary seller valuations. Subsection 4.4.1 discusses the case where no seller is critical, while Subsection 4.4.2 discusses the case with 1 or more critical sellers.

The essential arguments of our main results presented in the next section are illustrated below using the simplest such cases: Example 1 presents the case of one buyer bargaining for one item from two sellers holding an item each and having the same valuation; Example 2 deals with the case of one buyer bargaining for one item from two sellers holding an item each and having different valuations.

Example **1** Consider the model $\langle n = 2, k = 1, v_0 > v_1 = v_2, \delta\rangle$. Suppose the buyer makes offers of zero surplus to seller 1 and negative surplus to seller 2. If seller 1 rejects the buyer's offer, he would compete with seller 2 in the next period and offer the entire surplus to the buyer. If sellers 1 and 2 are making offers in the first period, they cannot make equal positive claims: one of the sellers have the incentive to reduce her claim and increase payoff. On the other hand, if their claims are unequal, the seller with the lowest claim has the incentive to increase her claim slightly and increase his payoff. Consequently, none of the sellers 1 and 2 can extract any surplus. The game ends immediately with the buyer extracting full surplus. The equilibrium outcome is identical even when the sellers are proposing first.

The situation described in Example 1 is identical to the well-known Bertrand model of price competition between firms producing the same product at identical marginal costs. In this model, competition between the sellers drives prices down to the marginal cost. The buyer is able to extract full surplus. Note that in our model the competition is among feasible paths. Consequently, the richness of the underlying graph structure allows for results that are richer than simple Bertrand competition. However, the spirit of the argument applied for richer graph structures is in the nature of Bertrand competition.

The simple example below illustrates that the buyer may not be able to extract efficient surplus when seller valuations are not identical. This example is in the lines of Blume (2003) who characterizes a class of equilibria in the Bertrand model of price competition when firms have asymmetric marginal costs.

Example **2** Consider the land acquisition problem $\langle n, k, v, \delta\rangle$ such that $n = 2; k = 1, v_1 < v_2 < v_0$. We claim that the buyer cannot extract the efficient surplus in equilibrium. Consider the following strategies of the sellers: in any continuation game where the two sellers are making offers, seller 1 offers to sell at a price of v_2 and seller 2 mixes prices in $(v_2, v_2 + \gamma)$, $\gamma > 0$, with uniform probability. In any continuation game where the buyer is making an offer, seller 1 accepts a surplus of at least $\delta(v_2 - v_1)$ and seller 2 accepts any positive surplus. Given these strategies, following is a best response for the buyer: in any continuation game where the buyer is making an offer, she offers a surplus of $\delta(v_2 - v_1)$ to seller 1 and a negative surplus to seller 2. In any continuation game where the sellers are making an offer, she accepts any surplus offer which is less than or equal to $v_2 - v_1$. If the buyer proposes first, trade takes place in the first period itself with seller 1; seller 1 is able to extract a surplus of $\delta(v_2 - v_1)$. If the sellers propose first, trade takes place in the first period, where seller 1 is able to extract a surplus of $(v_2 - v_1)$. To check that this is an equilibrium, note that when making an offer, buyer cannot offer any higher surplus to seller 1 as it would be accepted. The buyer cannot offer positive surplus to seller 2, since he would accept it. Any lower surplus offer would be rejected by seller 1. When sellers are making offers, the buyer cannot reject the offer of seller 1 either because that would reduce her share of surplus. Seller 1 cannot reduce his offer because it would be accepted. Any higher offer by seller 1 would be rejected, thus leading to a lower surplus for him. If $v_1 < v_0 < v_2$, only the trade with seller 1 is feasible. In this circumstance, we are back to the equilibrium outcome of the familiar bilateral bargaining model by Rubinstein (1982).

4.4 Results

In this section, we consider subgame perfect equilibrium outcomes of our simultaneous-offer protocol in assembly problems $\langle \Gamma, k, v, \delta \rangle$ where Γ has at least two different feasible paths and v is any arbitrary valuation profile. The results given in this section characterize buyer's prospects of full surplus extraction in such equilibria. In the next subsection, we show that the buyer extracts full surplus within two periods if the underlying graph does not contain a critical seller and at least two feasible paths have the minimum sum of seller valuations. The following subsection characterizes the highest share of the surplus the buyer can achieve in the presence of critical sellers.

4.4.1 Possibility of full surplus extraction

The following result characterizes when it is possible for the buyer to extract full surplus in an equilibrium of our protocol.

THEOREM **3** *There exists* $\bar{\bar{\delta}} \in [0, 1)$ *such that for all* $\delta > \bar{\bar{\delta}}$ *the buyer extracts full surplus in at most two periods in an equilibrium if and only if*

- $\Gamma \neq \Gamma^*$*, i.e., there does not exist a critical seller in the underlying graph, and*
- $\mathcal{S}_1 = \mathcal{S}_2$*, i.e., there exist at least two paths with the minimum sum of valuations.*

REMARK **1** The two conditions for full surplus extraction are independent of each other. For illustration, consider Figure 4.1: this graph contains a critical seller and multiple feasible paths when $k = 3$. The existence of the critical seller on the graph is independent of the sum of seller valuations on feasible paths and vice-versa.

In what follows, we break down this result into several Propositions — Propositions 1-3 correspond to the "if part" of this result, while Propositions 4 and 5 correspond to the "only if" part of the result. Propositions 1-3 apply to problems where the valuations of sellers are equal and the underlying graph does not contain a critical seller, i.e., either the graph has a $k+1$-cycle, or it has at least a pair of disjoint paths, or it is an oddball graph. We show that there exist equilibria where the buyer extracts full surplus in the first period for any $\delta \in [0, 1]$ if either the underlying graph is a $k+1$-cycle, or sellers are making first offers. If the underlying graph is not a $k+1$-cycle, the buyer can extract full surplus only in the second period while making the first offers provided δ is sufficiently large. Proposition 4 shows that there is no equilibrium with full surplus extraction when there is a critical seller in the assembly problem. Proposition 5 shows the impossibility of full surplus extraction when seller valuations are unequal. We present a set of examples after each result to illustrate the essential argument. Detailed proofs are presented in the appendix.

Equal seller valuations and no critical sellers

Here we consider assembly problems without critical sellers where seller valuations are equal. Propositions 26–28 claim existence of equilibria with full surplus extraction in these problems within at most two periods.

PROPOSITION **26** *Consider an assembly problem $\langle \Gamma^{\triangle}, k, v, \delta \rangle$ such that $v_1 = \cdots = v_n, v_0 > kv_1$. The buyer extracting full surplus is an equilibrium outcome.*

REMARK **2** Notice that in this case the existence of equilibrium with full surplus extraction is not dependent on the magnitude of the discount factor δ.

EXAMPLE **3 (A 4-cycle)** Consider a cycle of length 4 (see Figure 4.3) . Note that there are 4 feasible paths of length 3. Every pair of feasible paths has a non-empty intersection. But the intersection of all 4 feasible paths is empty. We argue that there exists an equilibrium where the buyer extracts full surplus. First note that bargaining continues if and only if there are at least two active sellers. Consider the following strategy of the buyer: She picks a feasible path. Whenever she is proposing, she offers sellers from the picked path their valuations (equivalently, zero surplus), and the remaining seller strictly less than his valuation (equivalently, negative surplus). Whenever the sellers are proposing, she accepts the required number of offers from the lowest seller claims, provided she can afford. Consequently, all active sellers claiming zero surplus whenever they are required to make an offer is a best response. To check this, note that no active seller can gain by deviating for one stage when all of them claim zero surplus. If active sellers make identical positive surplus claims, one of them can reduce his claim by a small amount and make a gain. If active sellers make unequal claims, then a seller with lower claim can increase his claim by a small amount and make a gain. Now consider a stage where the buyer is making an offer. Active sellers who are made zero surplus offers would immediately accept: if any such seller rejects, he reaches a continuation game where the maximum he can gain is zero. Hence this is an equilibrium. Agreement takes place in the first period itself with 3 sellers who are made zero surplus offers.The equilibrium outcome does not change whether the buyer moves first, or the sellers.

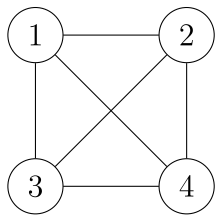

Figure 4.6: A complete graph of order 4; a cycle of order 4 is a subgraph.

Note that any complete graph of order $n > k$ contains a cycle of length $k + 1$. This results in the following Corollary.

COROLLARY **3** *Consider an assembly problem* $\langle n, k, v, \delta \rangle$ *such that* $v_1 = \cdots = v_n, v_0 > kv_1$. *The buyer extracting full surplus is an equilibrium outcome.*

REMARK **3** In the special case when $k = 2$, then the above result is also true for any graph containing a cycle of length more than 3. But it is not true for $k > 2$. For instance, consider the cycle of length 5 when $k = 3$ (see Figure 4.7). Suppose the buyer wants to make offers that are acceptable to sellers 1,2 and 3 in the first period itself. Sellers 1 and 3 will accept a zero surplus offer since if they reject, they have to compete with sellers 5 or 4. Seller 3, on the other hand, will not accept a surplus of less than δv_1, since if he rejects an offer, he has to compete with sellers 4 and 5 together. Therefore, the buyer has two ways to complete the transaction in the first period: either (i) she makes zero surplus offers to 4 sellers on the graph and makes a negative offer to the remaining seller, or (ii) she makes zero surplus offers to sellers 1 and 3, make a surplus offer of δv_1 to seller 2, and negative offers to the remaining sellers. In this particular case, she would prefer (ii) over (i).

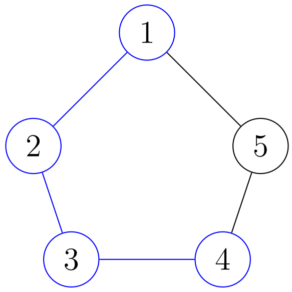

Figure 4.7: A cycle of length 5; Γ^{SO} in blue

Proposition 27 *Consider an assembly problem* $\langle \Gamma^D, k, v, \delta \rangle$ *such that* $v_1 = \cdots = v_n, v_0 > kv_1$*. (a) If the sellers move first, the buyer extracts full surplus in the first period. (b) If the buyer moves first, there exists* $\bar{\delta}$ *such that* $\forall \delta > \bar{\delta}$ *there is an equilibrium where the buyer extracts full surplus in the second period*

Remark 4 If $\delta < \bar{\delta}$ then full surplus extraction is not possible in the equilibrium: either buyer purchases items from all sellers on a single path by paying positive surplus shares; or, she purchases $2k$ items by offering zero surplus shares to all sellers on two disjoint paths.

Example 4 (Two disjoint feasible paths) Consider a graph with two disjoint paths of length 3 (see Figure 4.4). We will show that if the sellers move first, the buyer achieves full surplus in the first period itself.If the buyer moves first and δ is large, there is an equilibrium where buyer extracts full surplus in the second period.Consider the following strategy of the buyer: whenever the buyer is proposing, she makes negative offers to all sellers. Whenever the sellers are proposing, the buyer accepts the claims of sellers on a path with the lowest sum of claims provided her share of surplus is non-negative and rejects all other claims. In case the sum of claims on two feasible paths are same, she accepts claims from one of the paths chosen with equal probability. We claim that, given the above strategy, sellers in the two disjoint feasible paths claiming zero surplus whenever they are required to make an offer is a best response. No seller can gain by deviating for one stage when the sum of surplus claims on either path is zero. If the sum of surplus claims on both paths are equal and positive, a seller on either path can reduce his claim by a small amount and make a gain. If the sum of surplus claims on two paths are unequal, then any seller on the path corresponding to the lower sum can increase his claim by a small amount and make a gain. Hence these are not equilibrium claims. To rule out other possible deviations, note that buyer can make zero surplus offers to sellers on both paths; sellers on both paths would accept these offers. To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0 - 6v_1}{v_0 - 3v_1}$. The buyer can also make acceptable offers of surplus shares, $2\delta v_1$, to each seller on one path and negative offers to all other sellers, provided $v_0 - 3v_1 - 6\delta v_1 > 0$. This is because, by rejecting a first period offer from the buyer, a seller on the chosen path competes with sellers on the other path; the highest surplus he can claim in a continuation game where he and the other sellers are making offers is $3v_1 - v_1 = 2v_1$. To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0 - 3v_1 - 6\delta v_1}{v_0 - 3v_1}$. Thus, provided $\delta > \max\{\frac{v_0 - 6v_1}{v_0 - 3v_1}, \frac{v_0 - 3v_1 - 6\delta v_1}{v_0 - 3v_1}\}$, the buyer extracting full surplus in the second period is an equilibrium outcome in the strategies described above for large δ.

PROPOSITION **28** *Consider an assembly problem $\langle \Gamma^O, k, v, \delta \rangle$ such that $v_1 = \cdots = v_n, v_0 > kv_1$. (a) If the sellers move first, the buyer extracts full surplus in the first period. (b) If the buyer moves first, there exists $\overline{\delta}$ such that $\forall \delta > \overline{\delta}$ there is an equilibrium where the buyer extracts full surplus in the second period.*

REMARK **5** If $\delta < \bar{\delta}$ then full surplus extraction is not possible in the equilibrium: either buyer purchases items from all sellers on a single path by paying positive surplus shares; or, she purchases more than k items by offering zero surplus shares to corresponding sellers.

EXAMPLE **5 (An oddball graph)** Consider the graph in Figure 4.5 below with $k = 3$. We will show that if the sellers move first, the buyer extracts full surplus in the first period itself.If the buyer moves first and δ is large, there is an equilibrium where buyer extracts full surplus in the second period. Consider feasible path $\{125\}$.For each node x on this feasible path there exists another node outside this path and a corresponding edge such that exclusion of x from the path leaves at least one feasible path of length 3. For instance, exclusion of node 2, would leave the graph with feasible paths $\{134\}$. For each node x on a feasible path, let $s(x)$ be the order of the smallest subgraph such that the union of the graph excluding x and the subgraph contains a feasible path. For example, in Figure 4.5, $s(1) = s(5) = 1$ and $s(2) = 2$.

Consider the following strategy of the buyer: In any continuation game where the buyer has the first move , the buyer makes negative offers to all sellers. In any continuation game where sellers have the first move, the buyer accepts the claims of sellers on a path with the lowest sum of claims provided her share of surplus is non-negative, and reject all other claims. In case the sum of claims on the two feasible paths are same, she accepts claims from one of the paths chosen with equal probability. We claim that given the above strategy, sellers claiming zero surplus at any subgame they are required to make an offer is a best response. To check this, note that no seller can gain by deviating for one stage when the sum of seller claims across paths is zero. This is because, for each node, there is always a feasible path in the graph that excludes it. If sums across feasible paths are positive, a seller on one of the paths can reduce his claim by a small amount and make a gain. If sums across paths are unequal, then a seller on a path with lower sum of claims can increase his claim by a small amount and make a gain. Hence these are not best responses. To disallow possible deviations, note that buyer can make zero surplus offers to all sellers on the path picked, and negative surplus offers to all other sellers; sellers on the path picked would accept these offers. To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0 - 4v_1}{v_0 - 3v_1}$. The buyer can also make acceptable offers of surplus shares to sellers on a path and negative offers to all other sellers. Seller corresponding to node x_i on the path picked accepts any surplus share at least equal to $\delta(s(x_i) - 1)v_1$. This is because, by rejecting a first period offer from the buyer, a seller on the chosen path competes with sellers on the other path; the highest surplus he can claim in a continuation game where he and the other sellers are making offers is $(s(x_i) - 1)v$.To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{\sum_{i \in \mathcal{P}} (s(x_i) - 1) v_1}{v_0 - 3v_1}$. Thus, provided $\delta > \max\left\{\frac{v_0 - 4v_1}{v_0 - 3v_1}, \frac{\sum_{i \in \mathcal{P}} (s(x_i) - 1) v_1}{v_0 - 3v_1}\right\}$, the buyer extracting full surplus in the second period is an equilibrium outcome in the strategies described above.

Remark **6** Observe that in Propositions 1-3 the underlying graph does not contain a critical seller. If the graph has only one component, then the corresponding bargaining game has an equilibrium where the buyer extracts full surplus in at most two periods. Suppose the graph has multiple components. When seller valuations are identical, there exists an equilibrium where the buyer extracts full surplus in the first period itself if and only if the graph contains a $k+1$-cycle. Otherwise, (a)competing paths lie in different components or (b) form an oddball graph. In these cases, there exist an equilibrium where the buyer extracts full surplus in the second period if she is making the first set of offers, or in the first period itself if sellers are making the first set of offers.

Critical sellers

Here we consider assembly problems where the underlying graph contains at least one critical seller. This result is obtained without any assumption on valuations.

Proposition **29** *Suppose* $\Gamma = \Gamma^*$*. The buyer cannot extract full surplus in an equilibrium.*

Remark **7** The above result implies that in any equilibrium of $\Gamma = \Gamma^*$ a critical seller always earns a positive surplus share. In fact, if there is more than one critical seller, each will take away some positive share of the surplus. The buyers share of the surplus is decreasing in the number of critical sellers till it reaches zero. The non-critical sellers may or may not get positive shares.

Example **6 (Graph with critical sellers)** Consider the situation in Figure 4.2 where the numbers marking the nodes represent valuations. Suppose the buyer makes zero surplus offers in the first period. By Theorems 1 and 2, at least one critical seller , say, seller 2, rejects this offer and claims $\frac{1}{1+\delta}$ of the efficient surplus in the next period which buyer must accept. Consequently, the buyer cannot extract full surplus within the first two periods in an equilibrium. Suppose sellers make the first offers. As argued, critical seller 2 can claim strictly positive surplus. So full surplus extraction cannot take place in the first period. But if the buyer rejects all seller offers in the first period, then she is on a continuation game where she is proposing to all sellers. By the previous argument cannot extract full surplus in such a continuation game.

Unequal seller valuations and no critical sellers

In this subsection, we consider the case where seller valuations are not equal. In this case, the sum of seller valuations may differ over paths. The path corresponding to the least sum of seller valuations is efficient in the sense that it corresponds to highest potential surplus. It follows that if possible, the buyer would prefer to purchase the efficient path.

Let $\mathcal{P}_i$ denote the path corresponding to the i-th smallest sum of valuations on a path in Γ. We will refer to a set of assembly problems as rich if there does not exist two disjoint paths $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{S}_1 = \mathcal{S}_2$. Suppose the richness condition is not satisfied. The buyer, if offering first, can offer negative surplus shares to all sellers who reject such offers. In the next period, sellers on $\mathcal{P}_1$ and $\mathcal{P}_2$ cannot claim any surplus: the buyer extracts full surplus in the second period. If the sellers are making offers first, sellers on these two paths cannot claim any surplus share.

Proposition **30** *Consider the rich class of assembly problems $\langle \Gamma, k, v, \delta \rangle$ such that $v_1 \leq \cdots \leq v_n$ with at least one strict inequality. There does not exist any equilibrium where the buyer extracts full surplus.*

Note that extracting full surplus implies trade taking place with only the k sellers on $\mathcal{P}_1$. There may exist equilibria where the buyer offers zero surplus to more than k sellers who accept. But this reduces the buyer's surplus strictly below $v_0 - \mathcal{S}_1$.

For the formal proof of Proposition 30, see Appendix 4.9.We apply the method of contradiction showing that at least one seller getting zero surplus share has a profitable deviation. Thus, full surplus extraction is not an equilibrium outcome. Here we present four examples pertaining to the varieties of graph structures discussed in Section 4.3. In all these examples, we have $k = 3$.

Example **7 (A 4-cycle)** Consider the cycle in Figure 4.3 where the numbers marking the nodes represent valuations. Suppose the buyer makes zero surplus offers to sellers on the efficient path {123} in the first period and negative surplus offers to seller 4. At least one seller, say, seller 1, would reject this offer and claim a price of 4, the valuation of the seller outside this path, in the next period. The buyer must accept, provided the surplus on the path excluding seller 1, $v_0 - 7 > 0$. If this inequality does not hold, the buyer must offer $\frac{\delta}{1+\delta}$ times the efficient surplus, i.e., $v_0 - 6$ to this seller. So, suppose buyer makes negative offers to all sellers in the first period. Note that the sum of valuations on the four paths {123}, {234}, {341} and {412} are 6, 9, 8 and 7 respectively. Seller 1, being in the intersection of {123} and {412} can raise his price claim to 2: thus the sum of claims over the four paths become 7, 9, 9 and 8. Either the buyer accepts this claim, or she rejects and offers $\frac{\delta}{1+\delta}$ times the efficient surplus to this seller. Not all sellers would claim zero surplus when proposing first: for example, seller 1 can claim a price of 4, or if $v_0 - 7 < 0$, she can claim $\frac{1}{1+\delta}$ times the efficient surplus. If the buyer rejects all seller offers in the first period, then she is on a continuation game where she is proposing to all sellers. We have already argued that she cannot extract full surplus in such a continuation game.

Example **8 (Two disjoint feasible paths)** Consider Figure 4.4 where the numbers marking the nodes represent valuations. Suppose the buyer makes zero surplus offers to all sellers on {123} in the first period and negative surplus offers to the remaining sellers. Seller 1 can reject this offer and claim a price equivalent to the sum of valuations on {456}, i.e., 15 in the next period which buyer must accept, provided the corresponding surplus is positive, i.e., $v_0 - 20 > 0$. If this inequality does not hold, the buyer must offer $\frac{\delta}{1+\delta}$ times the efficient surplus to this seller. So, suppose buyer makes negative offers to all sellers in the first period. If $v_0 > 15$, sellers on {123} can make claims summing up to 15 in the second period. Either the buyer accepts this claim, or she rejects and offers $\frac{\delta}{1+\delta}$ times the efficient surplus, $v_0 - 6$ to sellers on $\{123\}$. Not all sellers would claim zero surplus when proposing first: at least one seller can claim a price of 15, or, if $v_0 - 20 < 0$, she can claim $\frac{1}{1+\delta}$ times the efficient surplus $v_0 - 6$. If the buyer rejects all seller offers in the first period, then she is on a continuation game where she is proposing to all sellers. We have already argued that she cannot extract full surplus in such a continuation game.

Example **9 (An oddball graph)** Consider the assembly problem in Figure 4.5 where the numbers marking the nodes represent valuations. Suppose the buyer makes zero surplus offers to sellers on the efficient path $\{123\}$ in the first period. Seller 1 would reject this offer and claim a price of 4 in the next period which buyer must accept, provided $v_0 - 9 > 0$. If this inequality does not hold, the buyer must offer $\frac{\delta}{1+\delta}$ times the efficient surplus $v_0 - 6$ to this seller. So, suppose buyer makes negative offers to all sellers in the first period. Note that seller 1 lies in the intersection of multiple paths. He can raise his claim by at least 1. Either the buyer accepts this claim, or she rejects and offers $\frac{\delta}{1+\delta}$ times the efficient surplus $v_0 - 6$. Not all sellers would claim zero surplus when proposing first: as argued before, seller 1 can claim a price of 4, or if $v_0 - 9 < 0$, she can claim $\frac{1}{1+\delta}$ times the efficient surplus $v_0 - 6$. If the buyer rejects all seller offers in the first period, then she is on a continuation game where she is proposing to all sellers. We have already argued that she cannot extract full surplus in such a continuation game.

4.4.2 Buyer's surplus share in presence of critical sellers

Consider an assembly problem with critical sellers. We provide bounds on buyer's surplus share in such a problem when she purchases an efficient path in the equilibrium.

Remark **8** In equilibria where the buyer purchases an inefficient path or purchases more than k items, she cannot realize the efficient surplus or the full surplus. We cannot characterize bounds on the buyer's realized surplus share in these equilibria since these are dependent on valuation profiles and the value of δ.

By Proposition 4 the buyer cannot extract full surplus in the equilibria of the class of problems considered here. Hence her maximum surplus share is bounded above by 1, but it is not the least upper bound. We will show that buyer's equilibrium surplus share cannot exceed $\frac{1}{1+\delta}$ with one critical seller, and $\frac{1-\delta}{1+\delta}$ with more than one critical sellers.

Note that there exist assembly problems where these bounds are exactly achieved: for example, when $n = k = 1$, the bound given by Theorem 4 is exactly achieved if the buyer is making the first offer (recall Theorem 1). It is also exactly achieved when Γ is a single line graph with three nodes, $k = 2$ and the buyer is making the first offer. When $n = k = 2$, the bound given by Theorem 5 is exactly achieved if the buyer is making the first offer (recall Theorem 2). It is also exactly achieved when Γ is a single line graph with four nodes, $k = 3$ and the buyer is making the first offer.

Theorem 4 *Consider an assembly problem $\langle \Gamma, k, v, \delta \rangle$ with exactly one critical seller. In any equilibrium where the buyer purchases an efficient path, her share of surplus cannot exceed* $\frac{1}{1+\delta}$.

Theorem 5 *Consider an assembly problem $\langle \Gamma, k, v, \delta \rangle$ with m critical sellers, where $2 \leq m \leq k$. In any equilibrium where the buyer purchases an efficient path, her share of surplus cannot exceed* $\frac{1-\delta}{1+\delta}$.

4.5 Discussion

Our first result claims that if valuations of the sellers are identical and the underlying graph structure does not have a critical seller, there exist equilibria where the buyer extracts full surplus within two periods. Here we have considered the simple advantages of position that certain sellers exact in a graph, and abstracted from advantages due to differences in seller valuations.

We have considered four mutually exclusive and exhaustive categories of graphs, viz., (a) graphs containing cycles of order $k + 1$, (b) graphs with two disjoint paths, (c) graphs with critical sellers, and (d) oddball graphs where (i) there is no cycle of length $k+1$, (ii) no two paths are disjoint and (iii) the intersection of all feasible paths is empty. These categories can be easily interpreted in terms of complementarity and substitutability as we have done in Section 4.3. Of particular interest is the $k+1$ cycle, where every item on a feasible path can be completely substituted by another item on the graph: only in this case, the buyer is able to extract full surplus in the first period, regardless of whether the buyer makes the first offer or the sellers. In other words, in this case, no seller has any positional advantage. Thus, it is comparable to the pure Bertrand competition visible in Example 1. At the other extreme is the graph with critical sellers: such critical sellers exhibit full positional advantage and prevent the buyer from extracting surplus beyond a point, regardless of whoever makes the first offer. Such sellers exhibit full complementarity with respect to any feasible path on the graph.

The cases of graphs with disjoint feasible paths and oddball graphs lie between these two extremes. If the buyer picks a feasible path on any of these graphs, its nodes have limited substitutability. Note that our bargaining protocol only permits cash offers with full commitment. Consequently, once the buyer commits to a seller on a feasible path, she tends to commit to all sellers on that feasible path. Thus, the buyer has to cough up positive shares of the surplus if she is making the first offer. However, the buyer can avoid this commitment problem by making negative offers to all sellers and to push the outcome towards Bertrand competition in the second period. For a patient buyer, the loss of surplus by shifting the onus of bargaining to the sellers is not very significant.

The interpretation of these graphs in the context of anti-commons applications like land acquisition is immediate. The notions of complements and substitutes also arise naturally in contexts like acquiring patent rights for drug manufacturing or obtaining rights for musical scores for a documentary.

Proposition 30 shows that full surplus extraction is not robust with respect to changes in the valuation structure. In fact, the buyer cannot extract full surplus whenever the valuation profile of the sellers shows slightest degree of asymmetry. The positional advantages that certain sellers hold become more pronounced when their valuations are asymmetric. In this sense, asymmetric valuations enable sellers in the efficient feasible path exercise monopoly power of a nature we had seen in Example 2.

It must be noted that earlier inefficiency results in the literature, like Theorem 2, focused on the extreme case where all sellers are critical. Our generalized model, in contrast, shows that the inefficiency result pertains to the rather extreme case of graphs with critical sellers or when seller valuations are asymmetric.

In Propositions 4 and 5, we show that the bounds provided by Roy Chowdhury and Sengupta on the surplus share the buyer can extract in an assembly with critical sellers carries over to our generalized structure. The bounds are tight because the non-critical sellers can be made to compete by buyers making unacceptable offers when it is their turn to offer. This strategy however does not work in case of critical sellers, some of whom must be given a positive share.The importance of the timing of the offers in the bargaining protocol is also highlighted in the proofs of the results.

Recall that we are using subgame perfection in an infinite horizon sequential game of complete information. In each period the surplus remains the same because payments made by buyers in previous periods are sunk. An agreement in a period is followed by a continuation game that has a smaller number of sellers and a reduced demand from the buyer. These continuation games include as subgames (a) Rubinstein games with one critical seller, b) games with multiple critical sellers studied in Roy Chowdhury and Sengupta (2012) and (c) games which have only non-critical sellers which correspond to Theorem 1. If the game has only non-critical sellers the buyer can walk away with the entire surplus. But if the game has critical sellers we show that in any equilibrium where the buyer gets more than the bound provided, some critical seller has incentive to deviate and force more out of the buyer. The minimum the buyer has to give up to get agreements from all critical sellers provides us the bound proposed in the results.

An obvious extension of this exercise is to investigate the impact of coalition formation among sellers on the surplus shares. For example, consider the problem where one item is required and two sellers are present. First, notice that by making alternate offers to one of the sellers according to the equilibrium strategy specified by Rubinstein (1982) and by excluding the other seller using negative offers, the buyer can assure herself $\frac{\delta}{1+\delta}$ share of full surplus. If we allow sellers to use trigger strategies, there exists an equilibrium where both sellers collude to claim $\frac{1}{1+\delta}$ of the full surplus and the buyer picks one of them with equal probability provided $\delta > \frac{1}{\sqrt{2}}$. This equilibrium is sustained by the following trigger strategy: if any seller deviates by charging less than $\frac{1}{1+\delta}$, the other seller charges zero surplus share in the subsequent period. The buyer then rejects the deviating seller's offer and chooses to purchase from the other seller. The collusive payoff $\frac{1}{2(1+\delta)}$ is greater than the non-collusive payoff $1-\delta$ if $\delta > \frac{1}{\sqrt{2}}$. In this equilibrium, both sellers gets positive expected payoff. If $\delta < \frac{1}{\sqrt{2}}$, sellers compete and earn zero surplus shares in the equilibrium. A complete investigation of seller coalitions is beyond the scope of current chapter[5].

4.6 Public Policy Implications

The main policy prescription of our paper would relate to examination of the contiguity structure of plots before entering bargaining with landowners to implement a project. It is a direct implication of our results that bargaining is likely to enter delays and hold-out if there are critical sellers. So in order to implement a project without costly delays it would make sense to first study the plot structure in the area. If the number of critical sellers is reasonably high it makes sense to implement the project at an alternative location. The basic trade-off, that needs to be solved is balancing the benefits of implementing a project at a productive location versus the costs, which arise due to delays and critical sellers holding out.

Additionally, this paper also makes a case for application of eminent domain in particular cases. If a location is particularly productive for a project but there are a few critical sellers

[5] The interested reader may refer to Gupta and Sarkar (2019).

who hold out, it makes sense for the government to apply eminent domain and shift the ownership of such land to more productive users.

4.7 Conclusion

In this chapter, we modeled the assembly problem as a bargaining game between one buyer and multiple sellers located on the nodes of a graph. In our simple bargaining problem without transaction costs, the buyer, using competition between sellers, is able to implement an efficient project without significant delay when valuations are identical. Positional advantages, or equivalently complementarities, can be exercised only under cases, when sellers are critical. The second result states that asymmetry of seller valuations is an additional source of inefficiency: such asymmetry provides additional monopoly power to efficient sellers even if they are not critical and prevent the buyer from efficient assembly. Thus, our results qualify the claim by Coase (1960) when sellers are not "monopolistic" in terms of positional advantage or due to valuations.

4.8 Appendix

4.8.1 Proof of Proposition 26

We will first prove the case of a graph which is a cycle of length $k+1$.

Lemma **14** *Consider an assembly problem $\langle \Gamma^{\Delta}, k, v, \delta \rangle$ such that $v_1 = \cdots = v_{k+1}, v_0 > kv_1$. The buyer extracting full surplus is an equilibrium outcome.*

Proof: Consider the following strategy of the buyer: She picks a feasible path. In any continuation game where $m < k$ plots have already been acquired and the buyer has the first move , the buyer offers $k - m$ sellers zero surplus and make negative offers to the remaining seller. In any continuation game where $m < k$ plots have already been acquired and sellers make the first, the buyer accepts the lowest $k - m$ claims provided her share of the surplus is non-negative and reject all other claims. If more than $k - m$ sellers are making identical lowest offers, she accepts $k - m$ offers with equal probability.

We claim that given the above strategy, all active sellers claiming zero surplus at any continuation game they are required to make an offer is a best response. Let x_i be the surplus claim of active seller i. No seller can gain by deviating for one stage when $x_i = x_j = 0, i \neq j$. Hence it is an equilibrium. If $x_i = x_j > 0, i \neq j$, either seller i or j can reduce his claim by a small amount and make a gain. If $x_i > x_j \geq 0, i \neq j$, then seller j can increase his claim by a small amount and make a gain. Hence these are not equilibrium claims.

At any subgame where the buyer is making an offer and m plots have already been acquired, the active seller who is made a negative offer rejects it. Simultaneously, $k - m$ would immediately accept corresponding zero offers, since if any of these sellers reject such offers, they reach a continuation game where the maximum he can gain by rejecting buyer's offers is zero. Hence this is an equilibrium. Trade takes place in the first period itself when $m = 0$, with k sellers who are made zero surplus offers. ■

Note that by Lemma 14 equilibrium outcome does not change whether the buyer moves first, or the sellers.

It follows immediately that such an equilibrium can be obtained for any graph containing a cycle of length $k + 1$ as a subgraph.

4.8.2 Proof of Proposition 27

Consider the following strategy of the buyer: In any continuation game where the buyer makes the first , she makes negative offers to all sellers. In any continuation game where sellers have the first move, the buyer accepts the claims of sellers on a path with the lowest sum of claims provided her share of surplus is non-negative and reject all other claims. In case the sum of claims on the two feasible paths are same, she accepts claims from one of the paths chosen with equal probability.

We claim that, given the above strategy, sellers in the two disjoint feasible paths claiming zero surplus at any subgame they are required to make an offer is a best response. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two feasible paths in Γ^D. Let x_i be the surplus claim of active seller i. No seller can gain by deviating for one stage when $\sum_{i\in\mathcal{P}_1} x_i = \sum_{i\in\mathcal{P}_2} x_i = 0$. Hence it is an equilibrium. If $\sum_{i\in\mathcal{P}_1} x_i = \sum_{i\in\mathcal{P}_2} x_i > 0$, a seller on one of the paths can reduce his claim by a small amount and make a gain. If $\sum_{i\in\mathcal{P}_1} x_i > \sum_{i\in\mathcal{P}_2} x_i$, then any seller on $\mathcal{P}_2$ can increase his claim by a small amount and make a gain. Hence these are not equilibrium claims.

Part (a) of the claim follows immediately. For part (b), note that buyer can make zero surplus offers to sellers on both paths, and negative surplus offers to all other sellers; sellers on both paths would accept these offers. To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0-2kv_1}{v_0-kv_1}$. The buyer can also make acceptable offers of surplus shares, $\delta(k-1)v_1$, to each seller on one path and negative offers to all other sellers, provided $v_0 - kv_1 - \delta k(k-1)v_1 > 0$. This is because, by rejecting a first period offer from the buyer, a seller on the chosen path competes with sellers on the other path; the highest surplus he can claim in a continuation game where he and the other sellers are making offers is $(k-1)v_1$.To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0-kv_1-\delta k(k-1)v_1}{v_0-kv_1}$. Thus, provided $\delta > \max\{\frac{v_0-2kv_1}{v_0-kv_1}, \frac{v_0-kv_1-\delta k(k-1)v_1}{v_0-kv_1}\}$, the buyer extracting full surplus in the second period is an equilibrium outcome in the strategies described above.

4.8.3 Proof of Proposition 28

We introduce some notation in the next two paragraphs that would be useful in proving the next result.

We note that each graph Γ^O has a subgraph Γ^{SO} such that (i) it contains a feasible path $\mathcal{P}$, (ii) for each node $x \in \mathcal{P}$ there exists a node $y \in \Gamma^O - \Gamma^{SO}$ and an edge $e(y,z), z \in \Gamma^{SO}$ such that $\Gamma^{SO} - x + z$ contains a feasible path of length k. For instance, in Figure 4.7, the path

{1234} qualifies as Γ^{SO}. Figure 4.5 shows one more example. Observe that the order of any Γ^{SO} would vary from k to $n-1$. For any given Γ^{O}, let Γ^{SO*} be the smallest of all $\Gamma^{SO} \subset \Gamma^{O}$ with order m^*.

Further, pick any feasible path $\mathcal{P}$ of length k on Γ^{O}. For each x on $\mathcal{P}$, let $s(x)$ be the order of the smallest subgraph Γ_S of Γ^O such that $(\mathcal{P} - x) \cup \Gamma^S$ is a feasible path of length k. For example, in Figure 4.5, $s(1) = s(5) = 1$ and $s(2) = 2$.

Consider the following strategy of the buyer: In any continuation game where the buyer has the first move , the buyer makes negative offers to all sellers. In any continuation game where sellers have the first move, the buyer accepts the claims of sellers on a path with the lowest sum of claims provided her share of surplus is non-negative and reject all other claims. In case the sum of claims on the two feasible paths are same, she accepts claims from one of the paths chosen with equal probability.

We claim that given the above strategy, sellers claiming zero surplus at any subgame they are required to make an offer is a best response. Let $\mathcal{P}_1, \dots \mathcal{P}_m$ be the feasible paths in Γ^O. Let x_i be the surplus claim of active seller i. No seller can gain by deviating for one stage when $\sum_{i \in \mathcal{P}_1} x_i = \cdots = \sum_{i \in \mathcal{P}_m} x_i = 0$. This is because, for each x_i, there is always a feasible path in Γ^O that does not contain x_i. Hence it is an equilibrium. If $\sum_{i \in \mathcal{P}_1} x_i = \cdots = \sum_{i \in \mathcal{P}_m} x_i > 0$, a seller on either path can reduce his claim by a small amount and make a gain. If $\sum_{i \in \mathcal{P}_1} x_i > \sum_{i \in \mathcal{P}_2} x_i$, then any seller on $\mathcal{P}_2$ can increase his claim by a small amount and make a gain. Hence these are not equilibrium claims.

Part (a) of the claim follows immediately. For part (b), note that buyer can make zero surplus offers to all sellers on Γ^{SO*}, and negative surplus offers to all other sellers; sellers on Γ^{SO*} would accept these offers. To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0 - m^* v_1}{v_0 - k v_1}$. The buyer can also make acceptable offers of surplus shares to sellers on a path and negative offers to all other sellers. If $\mathcal{P}$ is the picked path and x_i is the node corresponding to seller i, he accepts any surplus share at least equal to $\delta(s(x_i) - 1)v_1$. This is possible when $v_0 - kv_1 - \delta \sum_{i \in \mathcal{P}} (s(x_i) - 1)v_1 > 0$. This is because, by rejecting a first period offer from the buyer, a seller on the chosen path competes with sellers on the other path; the highest surplus he can claim in a continuation game where he and the other sellers are making offers is $(s(x_i) - 1)v_1$.To ensure that this deviation in the first stage is not profitable for the buyer, we require $\delta > \frac{v_0 - kv_1 - \delta \sum_{i \in \mathcal{P}} (s(x_i)-1)v_1}{v_0 - kv_1}$. Thus, provided $\delta > \max\{\frac{v_0 - m^* v_1}{v_0 - kv_1}, \frac{v_0 - kv_1 - \delta \sum_{i \in \mathcal{P}} (s(x_i)-1)v_1}{v_0 - kv_1}\}$, the buyer extracting full surplus in the second period is an equilibrium outcome in the strategies described above.

4.8.4 Proof of Proposition 29

Suppose there is an equilibrium where the buyer obtains full surplus. Such an equilibrium entails a strategy profile, where the buyer always make zero surplus offers and the sellers accept such offers at some period. Alternatively, if the sellers are to make an offer they ask for zero surplus at some period and the buyer accepts it. We will show by contradiction, that in any such equilibrium a critical seller has profitable deviation.

Suppose, if possible, that the buyer obtains full surplus in an equilibrium at period t. This implies that all sellers on an efficient path are selling their items at period t or some period before t. Consider a critical seller with whom trade takes place at period $\hat{t}$. Now let us consider the following deviation by the critical seller at $\hat{t}$: If he is offering at $\hat{t}$ he asks for something positive and if the buyer offers according to the equilibrium strategy he rejects it. In either case, the critical seller moves to a continuation game in period $t+1$ where he is the only active seller. By Theorem 1 the critical seller obtains a positive surplus share in the continuation game. This constitutes a profitable deviation for the critical seller.

4.9 Proof of Proposition 30

Again the idea of the proof is to use the method of contradiction. Let us suppose that there is an equilibrium where the buyer extracts full surplus. This entails a a strategy profile, where the buyer always make zero surplus offers and the sellers accept such offers at some period.

Alternatively, if the sellers are to make an offer they ask for zero surplus at some period and the buyer accepts it. We find a contradiction to it.

Case 1 ($\Gamma = \Gamma^{\Delta}$): Suppose the buyer obtains full surplus in an equilibrium at period t. This implies that all sellers on the efficient path sell their items at t or prior to t. Let us pick a seller i on the efficient path and suppose he is last active at period $\hat{t} \leqslant t$. Since buyer extracts full surplus in the proposed equilibrium either i proposes zero surplus share at $\hat{t}$ or accepts a zero surplus share offer at $\hat{t}$. Now pick the seller j who is on $\mathcal{P}_2$ but not on $\mathcal{P}_1$. Since by assumption $\mathcal{S}_1 < \mathcal{S}_2$, $v_i < v_j$. Consider the following deviation for seller i at $\hat{t}$: i makes a surplus offer of $\frac{1}{1+\delta}(v_0 - \mathcal{S}_1)$ if $v_0 < \mathcal{S}_2$ and $v_j - v_i - \epsilon$ if $v_0 > \mathcal{S}_2$ and accepts offer greater than $\frac{\delta}{1+\delta}(v_0 - \mathcal{S}_1)$ if $v_0 < \mathcal{S}_2$ and $\delta(v_j - v_i - \epsilon)$ if $v_0 > \mathcal{S}_2$. Here ϵ is a small positive quantity. If the buyer keeps rejecting i offer and keeps offering less than the claim of i then we reach a continuation game where i is the only active seller on the efficient path. Note that for small ϵ, buyer would never agree to trade with seller j. In this continuation game i can ensure a positive surplus. This leads us to a contradiction.

Case 2 ($\Gamma = \Gamma^{D}$): Suppose the buyer obtains full surplus in an equilibrium at period t. This implies that all sellers on the efficient path sell their items at t or prior to t. Let us pick a seller i on the efficient path and suppose he is last active at period $\hat{t} \leqslant t$. Since buyer extracts full surplus in the proposed equilibrium either i proposes zero surplus share at $\hat{t}$ or accepts a zero surplus share offer at $\hat{t}$.

Now consider the deviation strategy of i where he makes an offer of $\frac{1}{1+\delta}(v_0 - \mathcal{S}_1)$ if $v_0 < \mathcal{S}_2$ and $\mathcal{S}_2 - v_i - \epsilon$ if $v_0 > \mathcal{S}_2$ and accept offers of at least $\frac{\delta}{1+\delta}(v_0 - \mathcal{S}_1)$ if $v_0 < \mathcal{S}_2$ and $\delta(\mathcal{S}_2 - v_i - \epsilon)$ if $v_0 > \mathcal{S}_2$. Then there exists a continuation game at period $t+1$ where i is the only remaining active seller on the efficient path and can guarantee himself a positive payoff.

Case 3 ($\Gamma = \Gamma^{O}$): Suppose the buyer obtains full surplus in an equilibrium at period t. This implies that all sellers on the efficient path sell their items at t or prior to t. Let us pick a seller i at the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$ and suppose he is last active at period $\hat{t} \leqslant t$.Since buyer extracts full surplus in the proposed equilibrium either i proposes zero surplus share at

$\hat{t}$ or accepts a zero surplus share offer at $\hat{t}$.

Now consider a deviation strategy for seller i: suppose the cheapest path on the subgraph excluding i is $\mathcal{P}_R$. If $v_0 > \mathcal{S}_R$, i claims $\mathcal{S}_R - \mathcal{S}_1 - \epsilon$ and accepts no less than $\delta(\mathcal{S}_R - \mathcal{S}_1 - \epsilon)$. If $v_0 \leq \mathcal{S}_R$, i claims $\frac{1}{1+\delta}(v_0 - \mathcal{S}_1)$ and accepts no less than $\frac{\delta}{1+\delta}(v_0 - \mathcal{S}_1)$. In this case there is a continuation game at $t+1$ where i is the only active seller on the efficient path and can guarantee himself a positive surplus.

Remark 9 See Example 2 for an equilibrium in the simple model where the buyer agrees to trade with sellers on the efficient path.

4.10 Proof of Theorem 4

By contradiction: Suppose there exists an equilibrium where the buyer purchases the efficient feasible path and gets a surplus share strictly higher than $\frac{1}{1+\delta}$. This implies that in equilibrium the critical seller gets a surplus share strictly less than $\frac{\delta}{1+\delta}$. Since in equilibrium the buyer purchases an efficient feasible path and realizes a strictly positive surplus, the game terminates at some finite period t. There are three mutually exclusive and exhaustive cases with respect to the sellers who agree in period t.

Case (a): Suppose only the critical seller agrees to trade at t. By Theorem 1, in the continuation game beginning at t, he gets a surplus share equal to $\frac{\delta}{1+\delta}$ if the buyer is making an offer and $\frac{1}{1+\delta}$ if himself making an offer. Consequently, buyer's surplus share cannot exceed $\frac{1}{1+\delta}$.

Case (b): Suppose the critical seller and at least one non-critical seller agree to trade in period t. If the buyer is making an offer in period t, she must offer the critical seller $\frac{\delta}{1+\delta}$ of full surplus, otherwise this seller can reject the offer and move to period $t+1$, where he can earn a surplus share of $\frac{1}{1+\delta}$.

Suppose the sellers are making offers, the critical seller claims $x < \frac{\delta}{1+\delta}$ and the buyer gets the maximum possible surplus share $\bar{X} > \frac{1}{1+\delta}$. Suppose the critical seller offers to sell at $x + \epsilon$

surplus share instead. If the buyer accepts the offer, her surplus share is $\bar{X} - \epsilon$. If she rejects only this offer, her surplus share in the continuation game is $\frac{1}{1+\delta}$. If she rejects some other offers too, her surplus share in the continuation game cannot exceed $\bar{X}$. Consequently, she would accept the increment claimed by the critical seller in period $t-1$ if $\bar{X} - \epsilon > \delta \max\{\frac{1}{1+\delta}, \bar{X}\} = \delta\bar{X}$. Hence the critical seller claiming strictly less than $\frac{\delta}{1+\delta}$ at t cannot be an equilibrium.

Case (c): Suppose only non-critical sellers agree to trade in period t and agreement takes place with the critical seller in period $t-1$. Suppose the buyer gets the maximum possible surplus share $\bar{X} > \frac{1}{1+\delta}$. Suppose sellers are making offers at $t-1$. The critical seller claiming $x < \frac{\delta}{1+\delta}$ can successfully claim a small increment as shown above. Suppose the buyer is making offers in period $t-1$ and realizes the highest possible surplus share $\bar{X} > \frac{1}{1+\delta}$. Buyer's maximum surplus share in the continuation game beginning at period $t-1$ is $\bar{X} = \delta - x$ which is strictly positive since it is greater than $\frac{1}{1+\delta}$ by the contradiction hypothesis; therefore, $x < \delta$. Now suppose the critical seller rejects buyer's offer of x and charges a surplus share of $\frac{x}{\delta} + \epsilon$ in the next period. Since sellers are offering in this period, all non-critical sellers agree to trade at their valuations. It follows that if the buyer accepts the critical seller's new offer, she realizes a surplus of $1 - \frac{x}{\delta} - \epsilon$ in the continuation game beginning period t, whereas if she rejects, the surplus share she gets is at most $\delta\bar{X} = \delta^2 - \delta x$. For $\epsilon < (1-\delta^2)\left(1 - \frac{x}{\delta}\right)$, the buyer would accept the critical seller's new offer. Note that the buyer cannot be making offers in any period prior to $t-1$ where the critical seller agrees to trade: the non-critical sellers would agree to trade in the very next period ending the game. The critical seller is always able to claim a small increment if he is proposing anything below $\frac{\delta}{1+\delta}$ in any period prior to $t-1$.

4.11 Proof of Theorem 5

We will apply an induction argument on the number of critical sellers in the problem. Consider an equilibrium in an assembly problem with two critical sellers. Suppose the final agreement takes place at period t. There are three mutually exclusive and exhaustive cases with respect to the sellers who agree at t.

Case (a): If only two critical sellers and no non-critical sellers agree to trade at t, then by Roy Chowdhury and Sengupta (2012), buyer's surplus share cannot exceed $\frac{1-\delta}{1+\delta}$. Suppose there are some non-critical sellers who also agree to trade at t. If buyer is making an offer at t, she must offer $\frac{\delta}{1+\delta}$ of full surplus to both critical sellers, otherwise one of them can reject the offer and claim $\frac{1}{1+\delta}$ of the full surplus at $t+1$. If the sellers are making an offer, any critical seller receiving less than $\frac{\delta}{1+\delta}$ of full surplus can reject such an offer and claim $\frac{1}{1+\delta}$ at $t+1$.

Case (b): If only one critical seller agrees to trade at t, then this critical seller earns at least $\frac{\delta}{1+\delta}$. Suppose not. If the buyer is making offers at t, then this seller can reject offers less than $\frac{\delta}{1+\delta}$ and successfully claim $\frac{1}{1+\delta}$ at $t+1$. If sellers are making offers, this critical seller is claiming $x < \frac{\delta}{1+\delta}$. Suppose the buyer gets a surplus share of X in the continuation game beginning at t. If this critical seller claims an increment ϵ over x, the buyer would accept as long as $X - \epsilon > \delta X$. Consider the other critical seller and suppose he agrees to trade at $t-1$. Suppose sellers are making offers in period $t-1$, this critical seller claims a surplus share of $x < \frac{\delta}{1+\delta}$ and the buyer's maximum surplus share in the continuation game beginning at period $t-1$ is $X > \frac{1-\delta}{1+\delta}$. If this seller claims ϵ increment on his claim, the buyer would accept as long as $X - \epsilon > \delta X$ because if she rejects this offer, the surplus share she can earn at t where both critical sellers are active is X. Suppose the buyer is making offers at $t-1$, offers $x < \frac{\delta}{1+\delta}$ to the critical seller and earns the maximum possible surplus share $X > \frac{1-\delta}{1+\delta}$. Note that in the next period only one critical seller would claim $\frac{1}{1+\delta}$. Consequently, $X = \frac{\delta^2}{1+\delta} - x > \frac{1-\delta}{1+\delta} > 0$. Therefore, $x < \frac{\delta^2}{1+\delta}$. Suppose the critical seller rejects this offer and makes a counteroffer of $\frac{x}{\delta} + \epsilon$ next period: the buyer can guarantee herself a surplus share $\frac{\delta}{1+\delta} - \frac{x}{\delta} - \epsilon$ by accepting the deviating seller's offer and making a fresh offer to the other critical seller next period. If she rejects this offer, in the continuation game beginning period t, both critical sellers are present, and the buyer's surplus share can be at most $\delta X = \frac{\delta^3}{1+\delta} - \delta x$. For small ϵ, the buyer would accept the deviating critical seller's offer. Similar arguments work if a critical seller is agreeing to trade in a period prior to $t-1$.

Case (c): Suppose only non-critical sellers agree to trade at t. If agreement takes place with both critical sellers at $t-1$, then it cannot be that sellers are making offers at $t-1$ and the buyer's equilibrium surplus share is more than $\frac{1-\delta}{1+\delta}$: like above, it implies that at least one of the critical sellers is claiming $x < \frac{\delta}{1+\delta}$. He can claim a sufficiently small increment ϵ over x which the buyer cannot reject. Suppose the buyer is making an offer at $t-1$ and realizes the maximum possible surplus share of $X > \frac{1-\delta}{1+\delta}$. Then she must be making an offer of $x < \frac{\delta}{1+\delta}$ to some critical seller. Notice in this case, $X = \delta - x_1 - x_2$, where x_i is the surplus share of critical seller i. Since $X > \frac{1-\delta}{1+\delta} > 0$, $x_1, x_2 < \delta$. Now a critical seller who is getting less than $\frac{\delta}{1+\delta}$, say 1, can reject this offer and claim $\frac{x_1}{\delta} + \epsilon$. If the buyer accepts this offer, she gets $1 - x_2 - \frac{x_1}{\delta} + \epsilon$ at t and at most δX if she rejects. For small ϵ, the buyer would accept this offer. Recall that it cannot be that some critical seller agrees to trade prior to $t-1$ since non-critical sellers agree to trade in at most two periods. Similar arguments work if a critical seller is agreeing to trade in a period prior to $t-1$.

Suppose the claim is true for $c = 2, \ldots, m-1$ critical sellers. We will show that it is true for $c = m$. Note that by the induction hypothesis, the claim holds whenever the number of sellers agreeing to trade at t is between 2 and $m-1$. Suppose m critical sellers agree to trade in period t. If the buyer is making offers at t, she cannot offer any critical seller less than $\frac{\delta}{1+\delta}$ because such a seller can reject and claim $\frac{1}{1+\delta}$ in period $t+1$. If sellers are making offers at t and the buyer is getting a surplus share of $X > \frac{1-\delta}{1+\delta}$, one of the critical sellers must be claiming $x < \frac{\delta}{1+\delta}$. Such a seller can successfully claim a small increment ϵ over x because if the buyer rejects this offer, her maximum possible surplus share is δX. If only one critical seller agrees to trade at t, he cannot be receiving less than $\frac{\delta}{1+\delta}$. If buyer is making offers at t and he is getting less than $\frac{\delta}{1+\delta}$, he can reject and claim $\frac{1}{1+\delta}$ in period $t+1$. If sellers are making offers at t and the buyer is getting a surplus share of $X > \frac{1-\delta}{1+\delta}$, this critical seller must be claiming $x < \frac{\delta}{1+\delta}$. He can successfully claim a small increment ϵ over x because if the buyer rejects this offer, her maximum possible surplus share is δX. Now consider $m-1$ critical sellers agreeing to trade at period $t-1$. Note that it implies all m sellers were present at period $t-1$, otherwise the induction hypothesis would apply. If the sellers are making offers at period $t-1$, it cannot be that a critical seller claims $x < \frac{\delta}{1+\delta}$ and the buyer gets a surplus of $X > \frac{1-\delta}{1+\delta}$: such a seller can always claim a small increment that the buyer cannot reject. Suppose the buyer is making offers $x_1, \ldots, x_m$ at $t-1$ to m sellers and if possible, realizing the highest possible surplus share $X > \frac{1-\delta}{1+\delta}$. Since one of these sellers, say 1, would reject this offer and make a counteroffer in period t, buyer's equilibrium surplus share in the continuation game beginning period $t-1$ is $X = \frac{\delta^2}{1+\delta} - \sum_{i\neq 1} x_i$ which is greater than $\frac{1-\delta}{1+\delta}$ by the contradiction hypothesis. Therefore, for each seller $i \neq 1$, $x_i < \frac{\delta^2}{1+\delta} - \frac{1-\delta}{1+\delta}$, since $x_i \geq 0$ for all i. By refusing an offer any seller $i \neq 1$ can make a counteroffer $\frac{1}{1+\delta}$ in period t. Note that then the continuation game beginning at period t has exactly two active sellers, and the buyer's equilibrium surplus share cannot be more than $\frac{1-\delta}{1+\delta}$ in this continuation game and the deviating critical seller has a guaranteed surplus share of $\frac{\delta}{1+\delta}$ in period $t+1$. This deviation therefore earns the critical seller $\frac{\delta^3}{1+\delta} > \frac{\delta^2}{1+\delta} - \frac{1-\delta}{1+\delta}$. Note that it cannot be the case that critical sellers agree to trade in three or more different periods and the buyer gets a surplus share more than $\frac{1-\delta}{1+\delta}$, because then there are at least three critical sellers and the induction hypothesis applies.

Finally, consider the case where only non-critical sellers agree to trade at t. This implies all m critical sellers must have agreed to trade at period $t-1$. If the sellers are making offers at $t-1$, it cannot be that any seller is claiming $x < \frac{\delta}{1+\delta}$ and the buyer is getting a surplus share of $X > \frac{1-\delta}{1+\delta}$. Such a seller can successfully claim an increment in surplus share over x. Suppose the buyer is making offers $x_1, \ldots, x_n$ at $t-1$ and gets the highest possible surplus share $X > \frac{1-\delta}{1+\delta}$. Since only non-critical sellers make offers in period t, $X = \delta - \sum_i x_i > \frac{1-\delta}{1+\delta} > 0$. Therefore, $x_i < \delta$ for all i. A critical seller, say 1, can reject the buyer's offer at $t-1$ and claim $\frac{x_1}{\delta} + \epsilon$ at t. If the buyer accepts this claim, buyer's equilibrium surplus share is $1 - \sum_{i\neq 1} x_i - \frac{x_1}{\delta} - \epsilon$ at t; if she rejects, her equilibrium surplus share is at most $\delta X = \delta^2 - \delta \sum_i x_i - \frac{x_1}{\delta} - \epsilon$ at t. Therefore, for small ϵ, the buyer would accept the claim $\frac{x_1}{\delta} + \epsilon$ at t. Similar arguments work if a critical seller is agreeing to trade in a period prior to $t-1$.

Chapter 5

Conclusion

In this chapter, I take up the questions addressed in the thesis, discuss in brief the main findings and suggest tentative ways forward. Some of the ideas discussed are projects currently underway.

In the first chapter, we considered the situation of a collective contest between two differently sized groups over a private prize. The aim of the chapter was to provide strategic foundations to certain prize sharing rules, which may be used by groups in such a situation. In particular, we considered the prize sharing rule introduced in Nitzan (1991). Even though this rule is considered a standard in the literature, it was ad hoc in the sense that there was nothing but an intuitive basis to it.

In order to provide the rules microfoundations, we proposed another rule which can be considered to be the first best rule in our set up. We asked whether the prize sharing rule will ever be chosen by any group when the first best rule is also present. We found that under certain circumstances, the prize sharing rule being chosen by both groups may indeed be an equilibrium of an appropriately constructed two stage game. But, it is an equilibrium of a Coordination game where both groups choosing the first best rule is also a Nash equilibrium. It essentially captures a situation of failure by the groups to coordinate on the Pareto superior equilibrium involving the first best rules. But, the equilibrium with prize sharing rules survives the selection criterion of risk dominance under certain circumstances. And the equilibrium always survives the criterion called the security principle. Given that there exist

Nash equilibrium refinement criteria which select the equilibrium with the prize sharing rules, we claim that there indeed exists a strategic basis to the prize sharing rules.

Besides the obvious extensions of considering larger number of groups, or testing the prize sharing rules against other intuitive rules, an intriguing possibility is to test the model using an appropriately constructed experiment. It is not hard to imagine group leaders choosing inefficient institutions within his group, given it is in conflict with some other group. The best way to construct an experiment would be to create two differently sized groups in a laboratory setting and choose a leader from each group. Then just follow the exact approach taken in the chapter and play our game (maybe a simplified version), with the constructed groups. If the group leaders systematically choose the inefficient prize sharing rule in equilibrium at a rate significantly greater than zero, then it would support the theoretical results of our model.

In the second chapter we tried to model social norms of competitiveness within groups and how that affected their performance in a situation of conflict with other groups. We showed that large egalitarian groups are the worst performing ones. It will be very difficult to test the predictions of our model empirically or experimentally in a laboratory. But, it would be a worthwhile exercise to find anecdotal evidence which either supports or refutes the predictions of the model.

For instance, our model has an application to the theory of organizations. Given a world where organizations of different sizes are always in conflict over some scarce resource, our model predicts that large dispersed organizations will always fare worse than smaller, close knit and more competitive organizations. For example, consider the wide, dispersed and the larger identity, we call the Hindu identity. Our model would predict that such an identity will more often than not lose out in competition to the much smaller, more coherent and competitive identities, we can call the caste identities. Looking for evidence along such lines to test our model is something we wish to do in future.

Moving away from the theme of collective contests, our final chapter considers a situation of multilateral bargaining between a buyer and several sellers. Each seller owns an input each and the buyer needs to bargain successfully with a subset of sellers to implement a grand project. The common theme in this literature is the problem of hold-out, whereby sellers delay agreeing to accept offers in order to hold the buyer hostage, once the buyer has already made agreements with some of them. The problem of hold out may lead to delay or non-implementation of efficient projects.

What lies behind the phenomenon of hold out, is the extreme complemetaririties assumed between inputs. We contend that such extreme degrees of complementarity between inputs is not realistic. We try to model different degrees of complementarities between inputs using a graph theoretic model, where each seller is a node on the graph and an edge exists between two sellers in case they own complementary inputs. The buyer needs to pick a path of a particular size to implement his project. The possibility of multiple such paths nicely builds in the idea of substitutability between inputs.

We go on to show that the problem of hold-out more or less vanishes unless there exist sellers, who belong to every path, i.e., a seller who is perfectly complementary to the production process. This helps us show that the phenomenon of hold out critically depends on the assumption of perfect complementarity and its incidence may thus been overstated in the literature. Currently we are working on a project, the aim of which is to compute the coalition proof Nash equilibria of the model in the chapter. Also, given the graph theoretic approach does not give rise to the most general production processes, an interesting idea is to generalize our model using the coalitional bargaining approach and verify whether our results hold in that general model or not.

All the extensions and possible experiments suggested here are worthwhile future projects but are much beyond the scope of this thesis.

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