Essays in Multidimensional Mechanism Design

Kolagani Paramahamsa

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Supervisor: Debasis Mishra,
Professor, EPU, ISI Delhi.

Abstract

This thesis analyzes three problems where a monopolistic seller is selling to an agent with multidimensional private information. While our understanding of such problems is comprehensive if the agent's private information is one-dimensional, problems with multidimensional private information are known to be ubiquitous but analytically notorious. The three chapters in this thesis make progress in understanding optimal mechanism design in such multidimensional screening problems.

In the first problem, the seller is selling an object to an agent who exhibits behavioral preferences, in a departure from the standard rational models. Behavioral preferences arise because the agent is budget constrained and needs approval from a manager for outcomes beyond the budget. Such lexicographic decision-making and different preferences of the agent and the manager make this a two-dimensional mechanism design problem where the agent's aggregate choice shows intransitivity. We characterize the expected revenue-maximizing mechanism of the seller in this problem.

In the second problem, the seller is selling a pair of goods to the agent. The agent demands two goods in a particular ratio; the bundle's valuation and the ratio are the buyer's private information. We characterize the expected revenue-maximizing mechanism of the seller in this problem.

In the third model, a seller is selling an object with an inherent value and an attribute value to the agent. The value of the attribute depends on the level of the attribute, which is commonly observed by the seller and the agent, and is contractible. Since the attribute's level is not known when signing the contract, the seller can offer contingent contracts. We characterize the expected revenue-maximizing contingent contract. In particular, we show that the expected revenue-maximizing contract is deterministic under reasonable restrictions on priors.

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Contents

Chapter 1

Introduction

Mechanism design primarily deals with designing incentives for agents to achieve the designer's objectives in various settings in an economy. This approach has been successful in deepening our understanding of the modern economy. One of the principal objectives of the theory is to characterize the revenue-maximizing mechanisms. Towards achieving this objective, the revelation principle states that it is without loss of generality to focus on mechanisms that induce agents to report truthfully. Therefore, an essential aspect of this exercise is incentivizing the agents to report their private information truthfully.

The theory is extensive in models with one-dimensional private information. In these models, the agents' private information consists of just one number. Although these models explain many scenarios quite elegantly, the economy often exhibits instances of multidimensional private information. However, the nature of incentive constraints in a multidimensional model makes it tougher to design mechanisms. For instance, a complete solution to a simple two good revenue-maximizing selling mechanism still eludes us.

Another critical feature of mechanism design is the simplicity of the mechanisms and their implementation. Deterministic mechanisms are appealing due to their simplicity of description and that their implementation does not involve the designer to use a randomization device. While one-dimensional models typically result in a deterministic mechanism, randomization is necessary in many multidimensional mechanism design cases.

In this thesis, we consider three natural multidimensional models and solve for the revenue-maximizing mechanisms. In all the three models, we consider scenarios in which one seller is interested in maximizing revenue in the sale of the object(s) she possesses. We consider an object's sale to a behavioral agent in the second chapter, in a departure from the standard rational agent models in the literature. In the third chapter, we consider a two object sale where the agent demands the objects in a particular ratio. In the fourth chapter, we model a scenario in which an object's sale is contingent on a future outcome dependent on its attribute. In all the chapters, we describe practical and simple revenue-maximizing mechanisms. Either the mechanisms are deterministic, or their menu is finite and easier to describe.

1.1 Selling to a naive (agent, manager) pair

In this chapter, we consider a model in which a seller is selling a good to an (agent, manager) pair. The agent is budget constrained, but the manager is not. Both value the good differently and want to jointly acquire it, but they make decisions in a lexicographic manner. In particular, for any pair of outcomes, the agent first compares using her valuation. If she cannot compare them (due to budget constraint), then the manager compares. Sales to such an (agent, manager) pair who take decisions lexicographically, where the agent is budget-constrained, is not uncommon: (child, parent) pair deciding to buy some product; (management, board) pair of a company making decisions to acquire another company; (department, dean) pair deciding to recruit a faculty candidate.

In the first part of the chapter, we model this assuming that budget constraint is public information while how the agent and manager value the object is their private information. In the second part of the chapter, the budget constraint is also private information, making it a three-dimensional private information model.

We are interested in the optimal (expected revenue-maximizing) mechanism under incentive and individual rationality constraints. We show that the optimal mechanism is either a posted price mechanism or a mechanism involving a pair of posted prices (a menu of three outcomes). In the latter case, the optimal mechanism involves randomization and *pools* types in the middle.

1.2 Selling two complementary goods

In this chapter, we model a scenario where a seller is selling a pair of complementary goods to an agent. The agent consumes the goods only in a certain *ratio* and freely disposes of excess in either of the goods. In other words, the agent has a Leontief utility function. The value of the bundle and the ratio are private information of the agent; that is, the type-space is two-dimensional. For instance, a firm needs two inputs in a particular ratio to produce a final product. A consumer treats a pair of goods, hardware, and software, for instance, as perfect complements.

We characterize the incentive constraints and show that the optimal mechanism (expected revenue maximizing) is a *ratio-dependent posted price* mechanism for a class of distributions; that is, it has a different posted price for each ratio report. We identify additional sufficient conditions on the joint distribution for a *posted price* to be an optimal mechanism. We also show that the optimal mechanism is a posted price mechanism when the value and the ratio types are independently distributed.

1.3 Selling an object with an uncertain attribute

In this chapter, a seller is selling an object to a buyer who has an inherent value and an additional value from an attribute to the object. Both these values are drawn from a distribution and are private information of the buyer. The seller can potentially offer contracts contingent on the attribute realization. The payoff from the attribute is contingent on its level. This level is unknown (both to the seller and the buyer) when signing the contract but is contractible as it is revealed publicly later.

For example, in the transfer of a football player between two clubs, the buying club derives a privately known inherent value from signing the player through advertising and jersey sale rights. It also derives value from the player's future performance, number of goals scored, for example. While the per-goal value derived is private information of the buying club, the number of goals scored is public knowledge. Contingent contracts are commonplace in such settings.

We show that the optimal (revenue-maximizing) mechanism is deterministic under some regularity conditions over the distribution of the values. The optimal allocation takes a simple threshold (function of attribute realization) structure. Further, we provide an ex-post individually rational implementation of the mechanism.

Chapter 2

Selling to a naive (agent, manager) pair

2.1 Introduction

An (agent, manager) pair needs to buy a good. The agent (she) is budget constrained, but the manager (he) is not budget constrained. A seller offers a menu of (quantity, price) bundles to the them in a mechanism. If the agent's best bundle is within her budget, she buys it. Else, she contacts the manager. The manager is not budget constrained and can give any amount of funding as long as she respects his preference. Implicitly, the manager's payoff is linked to the agent's payoff in a monotone way and hence, the manager is willing to fund (without any side payments). This may be because both the manager and the agent need to acquire the good for the firm, and their payoff depends on the payoff of the firm. They have subjective valuation of the good for the firm. The valuations of the agent and the manager may be different because either there is inherent uncertainty about the valuation of the good and the agent and the manager may be differently informed about it or they use different attributes of the good to determine its valuation.

Our objective here is to capture a setting where an agent's behavior contradicts standard notions of rationality - ideally, the agent and the manager should get together and choose the best option according their joint estimate of the good's valuation. However, they are *naive*: (a) the agent only contacts the manager when she cannot choose the best bundle due to budget constraint; (b) whenever she contacts the manager, she respects his decision; and (c) the manager can impose his preference only when contacted by the agent. This makes the problem different from standard monopoly pricing problems. Sales to such an (agent, manager) pair who take decisions lexicographically, where the agent is budget constrained, is not uncommon: (child, parent) pair making decision to buy some product; (management, board) pair of a company making decisions to acquire another company; (department, dean) pair making decision to recruit a faculty candidate. A department (or, child or management) only contacts the dean (or, parent or board respectively) when it cannot take a decision about a new faculty candidate due to budget constraint. But once it contacts the dean, it has to respect the dean's preference. [1] We are interested in finding the optimal mechanism for selling to such an (agent, manager) pair.

The private information or *type* in our model is a pair of valuations: agent's own valuation and manager's valuation. Later, we discuss an extension where the budget is also a private information. There is no information transmission story here - even though the agent does not know the valuation of the manager, she can readily access the preference of the manager, but does so only when she cannot make a decision due to budget constraint. Hence, her decisions depend on her valuation *and* the manager's valuation. The incentive constraints in our model are quite different from a standard model of mechanism design. This is because the sequential nature of decision-making generates cyclic preference of the (agent, manager) pair. Hence, no utility representation is possible for such preferences, and the incentive constraints are *ordinal* in nature. In particular, if a mechanism assigns bundle (q, p) to a type, where q is quantity and p is price, then a manipulation to get another (quantity, price) pair (q', p') is possible if (a) the agent finds (q', p') more attractive than (q, p) and p' is less than the budget or (b) she cannot compare these two pairs (because the preferred pair is beyond budget) but the manager finds (q', p') more attractive than (q, p). An incentive compatible mechanism guards against all such manipulations.

Contributions. We fully characterize the optimal (expected revenue maximizing incentive compatible and individually rational) mechanism for the seller in our model. The optimal mechanism is either a posted-price mechanism (the no-haggling solution of Mussa and Rosen (1978); Riley and Zeckhauser (1983)) or a mechanism involving two posted-prices - we call it the POST-2 mechanism. The POST-2 mechanism has a pair of posted prices P_1 and P_2, both greater than the budget B. If the agent's valuation of the good is less than P_1, then the

[1]The dean and the department cannot jointly evaluate a faculty candidate because the dean is time constrained, and may be involved with a number of other such responsibilities. Similarly, the company board has delegated responsibility to the management with a budget constraint. Burkett (2015) shows that such arrangements can come out of an equilibrium contracting agreement between a *(principal, agent) pair* participating in a mechanism.

object is not sold (and no payments are made). If the agent's valuation of the good is more than P_1, then the object is sold with probability $\frac{B}{P_1}$ at per unit price P_1 (i.e., total payment is B). The remaining probability $(1 - \frac{B}{P_1})$ is sold at per unit price P_2 if the valuation of both the agent and the manager exceeds P_2. Hence, a POST-2 mechanism involves an extra layer of *pooling* of types in the middle and involves randomization. [2]

We provide a simple condition on the budget when a POST-2 mechanism is optimal. There are three special cases, where our problem reduces to a standard revenue maximization problem of a monopolist: (1) when budget of the agent is sufficiently high (then the agent can make all the decisions); (2) when budget of the agent is zero (then the manager makes all the decisions); and (3) when the preferences of the agent and the manager are identical. In all these cases, a posted-price mechanism is optimal (Mussa and Rosen, 1978; Riley and Zeckhauser, 1983) - call the optimal posted-price in such settings a *monopoly reserve price*. We show that if the budget of the agent is below the monopoly reserve price, a POST-2 mechanism is optimal.

Our optimal mechanism is simple since it can be described by a single parameter or a pair of parameters, and involves a menu of size two or three. Further, our result works for a rich class of priors (over values of the two rationales), which allows for correlation. The nature of incentive constraints in our problem implies that there is no revenue equivalence theorem to work with. Compared to a standard multi-object monopolist, where one runs into difficulty even in the two-object case (Manelli and Vincent, 2007; Hart and Nisan, 2017), we still have tractability in our multidimensional model because of the nature of decision-making and the incentive constraints.

We also consider an extension of our model where the budget information (along with values of the agent and the manager) is private. By restricting our attention to a reasonable class of mechanisms, we derive an optimal mechanism over this class of mechanisms - the projection of this optimal mechanism on the valuations space for each budget is (i) a POST-2 mechanism if the budget is low and (ii) a POST-1 mechanism if the budget is high. This

[2] Randomization is often seen in practice: same product is sold with different quality levels; limited shares of a company are possible to acquire instead of complete acquisition; a faculty candidate considers different levels of teaching in the contract when being hired etc. However, our optimal mechanism design recommends a particular kind of randomization. We do not know if such particular randomization is seen in practical problems. Our results suggest that whenever a designer believes he is confronted with an (agent, manager) pair described in our model, it is optimal to offer such randomization in the menu.

shows some robustness of our main result.

2.2 An illustration

We explain using a simple example why a posted price mechanism need not be optimal in our model. For simplicity, consider a setting where valuations of the agent and the manager, $v \equiv (v_1, v_2)$, are distributed in $[0,1] \times [0,1]$. We assume that both the agent and the manager have quasilinear preferences. So, the agent evaluates options using v_1 and the manager evaluates options using v_2. Consider a budget $B > 0$. Suppose the seller uses a posted price mechanism with price $p > B$. We argue that such a posted price mechanism cannot be optimal. To see this, consider the menu in a posted price mechanism: $\{(1,p),(0,0)\}$, i.e., take the object with probability 1 at price p or get nothing at zero price. If $v \equiv (v_1, v_2)$ is such that $v_1 \leq p$ the agent will prefer $(0,0)$ to $(1,p)$ and she will take this decision without consulting the manager. If $v \equiv (v_1, v_2)$ is such that $v_2 \leq p$ and $v_1 \geq p$, then the agent prefers $(1,p)$ to $(0,0)$ but she cannot take this decision since $p > B$. Hence, she consults the manager who prefers $(0,0)$ to $(1,p)$. Hence, $(0,0)$ will be preferred over $(1,p)$ at such profiles. So, the only region where $(1,p)$ is preferred to $(0,0)$ is when $\min(v_1, v_2) \geq p$ - this is when both the agent and the manager prefers $(1,p)$ to $(0,0)$. This is shown in the left graph of Figure 2.1.

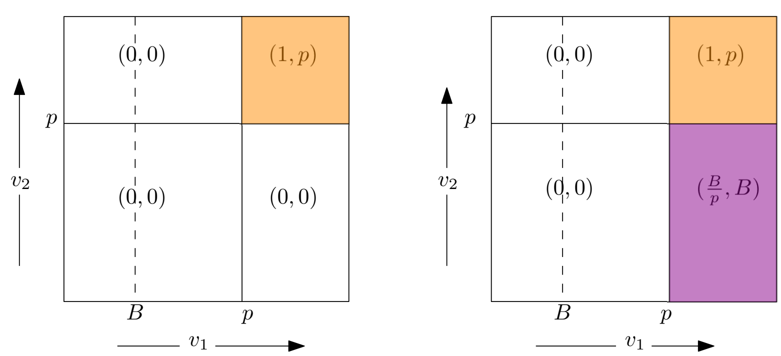

Figure 2.1: Non-optimality of posted prices

Now, consider another mechanism with a menu of three outcomes: $\{(1,p),(\frac{B}{p}, B),(0,0)\}$. So, the new menu contains an outcome that involves randomization and a payment of B. Consider the profile of values $v \equiv (v_1, v_2)$. Using the same argument as before, we see that if $\min(v_1, v_2) \geq p$, then the (agent,manager) pair prefers $(1,p)$ to the other two outcomes in the menu. Similarly, if $v_1 \leq p$, then the $(0,0)$ is preferred to the other two outcomes in the menu. However, if $v_1 \geq p$ but $v_2 \leq p$, then $v_1 - p \geq \frac{B}{p}(v_1 - p)$. But $p > B$ implies that the agent cannot compare $(1, p)$ and $(\frac{B}{p}, B)$ - i.e., the preferred outcome $(1, p)$ is beyond beyond the budget. However, since $v_2 \leq p$, we see that $\frac{B}{p}(v_2 - p) \geq v_2 - p$. So, the manager prefers $(\frac{B}{p}, B)$ to $(1, p)$. The agent prefers $(\frac{B}{p}, B)$ to $(0, 0)$ because $\frac{B}{p}(v_1 - p) \geq 0$ and she can compare these outcomes (within budget). Hence, the $(\frac{B}{p}, B)$ is preferred to the other outcomes in the menu by the (agent, manager) pair when $v_1 \geq p$ but $v_2 \leq p$. This is shown the right graph of Figure 2.1. This graph has an extra positive measure region where revenue of B can be earned by the seller at every profile in this region. Hence, this mechanism generates strictly larger revenue than the posted price mechanism. As is apparent, the seller is able to exploit the lexicographic nature of decision-making of the (agent, manager) pair to extract more revenue than in a posted price mechanism. Our main result will show that it cannot exploit any more than this, i.e., such a mechanism will be optimal.

The above discussion shows that a posted price mechanism which posts a price above the budget cannot be optimal. Our main result will formalize this intuition - for low enough budgets, we will show that the optimal mechanism will involve randomization but we can be precise about the nature of the randomization. The optimal mechanism will be a posted price mechanism for "high enough" budgets. But for budgets below a certain threshold, it will be a mechanism involving an extra layer of pooling in the middle.

The rest of the paper is structured as follows. In the next section, we introduce our model formally. In Section 2.4, we introduce our notion of incentive compatibility and state our main results. The proofs of our main results are quite long. So, we have put them in Appendix 4.5. We give a brief overview of the proofs in Section 2.4.5. Section 2.5 discusses a different notion of incentive compatibility and compares it with the notion we use for our results. Section 2.6 contains an extension where budget is also considered private information of the agent. The proofs of Section 2.6 is given in Appendix 2.9. Supplementary Appendix 2.10 contains some missing proofs and discussions.

2.3 The model

A seller is selling a single object to an agent who evaluates options along with her manager. She has a publicly observable budget $B \in (0, \beta)$, where $\beta > 0$ - Section 2.6 deals with the private budget case. A consumption bundle is a pair (a, t), where $a \in [0, 1]$ is the allocation probability and $t \in \mathbb{R}$ is the transfer - amount *paid* by the agent. The set of all consumption bundles is denoted by $Z \equiv [0,1] \times \mathbb{R}$. The agent and the manager evaluate the outcomes in Z using **quasilinearity**. Hence, their individual preference can be captured by valuations: a generic valuation of the agent is denoted as v_1 and a generic valuation of the manager is denoted by v_2. We assume that $v_1, v_2 \in V \equiv [0, \beta]$ - all our results extend even if we allow for the fact $v_i \in [0, \beta_i]$ for each $i \in \{1, 2\}$ and $\beta_1 \neq \beta_2$. Since the budget is publicly observable in this section, the only private information in the model are the two valuations (v_1, v_2).

Preference (rationale) of the agent with valuation v_1 is denoted by $\succeq_{v_1}$. Formally, $\succeq_{v_1}$ is a binary relation (incomplete): $\forall\ (a,t), (a', t') \in Z$,

$$\big[(a,t) \succeq_{v_1} (a', t')\big] \quad \Leftrightarrow \quad \big[av_1 - t \geq a'v_1 - t' \text{ and } t \leq B\big].$$

Notice that t' need not be below B in the above definition. This is consistent with our story that the agent makes a decision whenever she can.

Preference of the manager with valuation v_2 is denoted by $\succeq_{v_2}$. Formally, $\forall\ (a,t), (a', t') \in Z$,

$$\big[(a,t) \succeq_{v_2} (a', t')\big] \quad \Leftrightarrow \quad \big[av_2 - t \geq a'v_2 - t'\big].$$

Hence, $\succeq_{v_2}$ is complete. Notice that both $\succeq_{v_1}$ and $\succeq_{v_2}$ are transitive.

We denote the **aggregate preference** of the (agent, manager) pair with type $v \equiv (v_1, v_2)$ as $\succeq_v$. The preference $\succeq_v$ is a complete binary relation derived from $\succeq_{v_1}$ and $\succeq_{v_2}$ as follows. For every $(a,t), (a', t') \in Z$,

$$\big[(a,t) \succeq_v (a', t')\big] \Leftrightarrow$$

$$\text{either } \big[(a,t) \succeq_{v_1} (a', t')\big] \text{ or } \big[(a,t) \not\succeq_{v_1} (a', t'), (a', t') \not\succeq_{v_1} (a,t), (a,t) \succeq_{v_2} (a', t')\big].$$

As is expected, $\succeq_v$ is intransitive for some $v \equiv (v_1, v_2)$ - a formal lemma is given in Supplementary Appendix 2.10.1 at the end. An important consequence of this lemma is that there is *no utility representation* of the preference of our (agent, manager) pair. As discussed earlier, the aggregate preference captures the decision making process of the (agent, manager) pair. For every pair of outcomes, first the agent tries to compare. The manager compares only if the agent fails to compare due to budget constraint. Potentially, the agent can strategize when approaching the manager but we rule this out due to naivete. We interpret this decision-making process further after defining the incentive constraints.

We assume that the random variable $v \equiv (v_1, v_2)$ over $V \times V$ follows a distribution G with G_1 being the marginal for agent's valuation and G_2 being the marginal for manager's valuation. Both G_1 and G_2 are assumed to be differentiable functions with positive densities g_1 and g_2 respectively. Notice that we allow for values of the agent and the manager to be correlated. Our results will require some restrictions in G_1, which we will state later.

2.4 The optimal mechanism

2.4.1 Incentive compatibility

Since the preference of the (agent, manager) pair is completely captured by $v \equiv (v_1, v_2)$, we will refer to v as the **type** in our model - Section 2.6 discusses the private budget case, where the type will be (v_1, v_2, B). A (direct) **mechanism** is a pair of maps: an allocation rule $f : V^2 \to [0, 1]$ and a payment rule $p : V^2 \to \mathbb{R}$. For every $v \in V^2$, $f(v)$ denotes the allocation probability and $p(v)$ denotes the payment of this type.

The restriction to such direct mechanisms is without loss of generality as a version of the revelation principle holds in our setting - see Section 2.5. [3] Hence, we can discuss about incentive compatibility of direct mechanisms.

Definition **2.1** *A mechanism (f, p) is* **incentive compatible** *if for all $u, v \in V^2$,*

$$(f(u), p(u)) \ \succeq_u \ (f(v), p(v)).$$

Fix a mechanism (f, p) and let the range of the mechanism be

$$R^{f,p} := \{(a, t) : (f(v), p(v)) = (a, t) \text{ for some } v \in V^2\}.$$

Consider a type $u \equiv (u_1, u_2)$. The designer has assigned the bundle $(f(u), p(u))$ to this type. For every $(a, t) \in R^{f,p}$, there are two possibilities of manipulation. First, the agent can manipulate - this is possible if $au_1 - t > f(u)u_1 - p(u)$ with $t \leq B$. Second, the manager can manipulate and this is possible if the agent could not take a decision, contacted the manager, and $au_2 - t > f(u)u_2 - p(u)$. Our notion of incentive compatibility thus guards against two kinds of manipulations: one where the agent can take her own decision and manipulates, and the other where the agent cannot decide due to budget constraint and the manager manipulates.

[3] Though direct reporting of valuations of the agent and the manager may seem unrealistic in this setting, we can think of the direct mechanism as announcing a menu of outcomes and the agent choosing the best outcome from this menu (with the help of her manager).

In general, preferences over outcomes in $R^{f,p}$ may violate transitivity. However, our notion of incentive compatibility requires that at every type u, the outcome $(f(u), p(u))$ is preferred to any other outcome in $R^{f,p}$. This implies that if the designer wants type u to choose $(f(u), p(u))$ from the menu $R^{f,p}$, then it must be the case that for any other outcome (a, t) in $R^{f,p}$, the agent does not prefer (a, t) to $(f(u), p(u))$ or the agent cannot compare (a, t) and $(f(u), p(u))$, but the manager does not prefer (a, t) to $(f(u), p(u))$. Our notion of incentive compatibility implies that the outcome chosen for every type is not involved in a cycle. This allows us to rule out Dutch book arguments (or money pump) using our notion of incentive compatibility. We discuss another notion of incentive compatibility and its relation to our notion later in Section 2.5.

Thus, our notion of incentive compatibility can be broken down into two distinct cases. Fix $u, v \in V^2$. Then, there are two ways in which bundle $(f(u), p(u))$ can be (weakly) preferred over $(f(v), p(v))$ by a type u.

1. First, the agent prefers $(f(u), p(u))$ over $(f(v), p(v))$. This is possible if $p(u) \leq B$ and

$$u_1 f(u) - p(u) \geq u_1 f(v) - p(v).$$

2. Second, the agent cannot compare $(f(u), p(u))$ and $(f(v), p(v))$, but the manager prefers $(f(u), p(u))$ over $(f(v), p(v))$. This means $u_2 f(u) - p(u) \geq u_2 f(v) - p(v)$. Further, since the agent cannot compare these two outcomes, one of the following conditions must hold.

   (a) $u_1 f(u) - p(u) > u_1 f(v) - p(v)$ but $p(u) > B$.

   (b) $u_1 f(v) - p(v) > u_1 f(u) - p(u)$ but $p(v) > B$.

   (c) $u_1 f(v) - p(v) = u_1 f(u) - p(u)$ but $\min(p(u), p(v)) > B$.

Besides, incentive compatibility, we will impose a natural participation constraint. For this, we will assume that outside option of the (agent, manager) pair is the outcome $(0, 0)$, where she receives nothing and pays nothing.

Definition **2.2** *A mechanism (f, p) is* **individually rational** *if for all $v \in V^2$,*

$$(f(v), p(v)) \ \succeq_v \ (0, 0).$$

It is useful to note that the above individual rationality condition can be equivalently stated as follows. A mechanism (f, p) is individually rational if for all $v \in V^2$ (a) when $p(v) \leq$

B, we have $v_1 f(v) - p(v) \geq 0$ and (b) when $p(v) > B$, we have $v_1 f(v) - p(v) \geq 0$ and $v_2 f(v) - p(v) \geq 0$. This leads us to the following characterization of individual rationality. Such characterizations are well known in standard settings and the result below shows that it extends to our model too.

LEMMA **2.1** *Consider any incentive compatible mechanism (f, p). Then, (f, p) is individually rational if and only if $p(0, 0) \leq 0$.*

Proof: Suppose that $p(0, 0) \leq 0$. Consider any $u \in V^2$ with $p(u) \leq B$. Incentive compatibility and the fact that $p(u) \leq B$ and $p(0, 0) \leq 0 < B$ imply that $(f(u), p(u)) \succeq_u (f(0, 0), p(0, 0))$, which further implies that $u_1 f(u) - p(u) \geq u_1 f(0, 0) - p(0, 0)$. This combined with the fact that $u_1 f(0, 0) - p(0, 0) \geq 0$ (since $-p(0, 0), f(0, 0) \geq 0$), we conclude $(f(u), p(u)) \succeq_u (0, 0)$.

Similarly, consider any $v = (v_1, v_2) \in V^2$ with $p(v) > B$. Incentive compatibility and the fact that $p(0, 0) \leq 0 < B$, $p(v) > B$ imply that the agent cannot compare $(f(v), p(v))$ and $(f(0, 0), p(0, 0))$ but the manager prefers $(f(v), p(v))$ to $(f(0, 0), p(0, 0))$. This implies that $v_1 f(v) - p(v) \geq v_1 f(0, 0) - p(0, 0)$ and $v_2 f(v) - p(v) \geq v_2 f(0, 0) - p(0, 0)$. These inequalities imply that $v_1 f(v) - p(v) \geq 0$ and $v_2 f(v) - p(v) \geq 0$ as $-p(0, 0), f(0, 0) \geq 0$. From this we conclude $(f(v), p(v)) \succeq_v (0, 0)$.

For the other direction, consider the type $(0, 0) \in V$. Individual rationality implies that $(f(0, 0), p(0, 0)) \succeq_{(0,0)} (0, 0)$. This implies that $-p(0, 0) \geq 0$. ■

2.4.2 New mechanisms

Incentive compatibility has different implications in our model because of the sequential nature of decision-making. There are some simple mechanisms that are incentive compatible and resemble similar mechanisms in standard settings where decisions are taken using a single preference relation.

DEFINITION **2.3** *A mechanism (f, p) is a* POST-1 *mechanism if there exists a $K_1 \in [0, B]$ such that*

$$(f(v), p(v)) = \begin{cases} (0, 0) & \textit{if } v_1 \leq K_1 \\ (1, K_1) & \textit{otherwise.} \end{cases}$$

A POST-1 mechanism is a mechanism where the object is allocated by only considering the value of the agent. So, it can be thought of as a posted price mechanism *for* the agent. This is because it posts a price K_1 which is less than the budget B, and hence, the agent can make a decision using her preference. So, if her value is less than K_1, then the object is not allocated. Else, the object is allocated with probability 1. It is easy to see that such a mechanism is incentive compatible and individually rational.

We now introduce a new class of mechanisms that we call the POST-2 mechanisms. Unlike the POST-1 mechanism, the POST-2 mechanism considers the values of both the agent and the manager.

DEFINITION **2.4** *A mechanism* (f, p) *is a* POST-2 *mechanism if there exists a* $K_1, K_2 \in [B, \beta]$ *with* $K_1 \leq K_2$*, such that*

$$(f(v), p(v)) = \begin{cases} (0,0) & \text{if } v_1 \leq K_1 \\ (1, B + K_2(1 - \frac{B}{K_1})) & \text{if } \min(v_1, v_2) > K_2 \\ (\frac{B}{K_1}, B) & \text{otherwise} \end{cases}$$

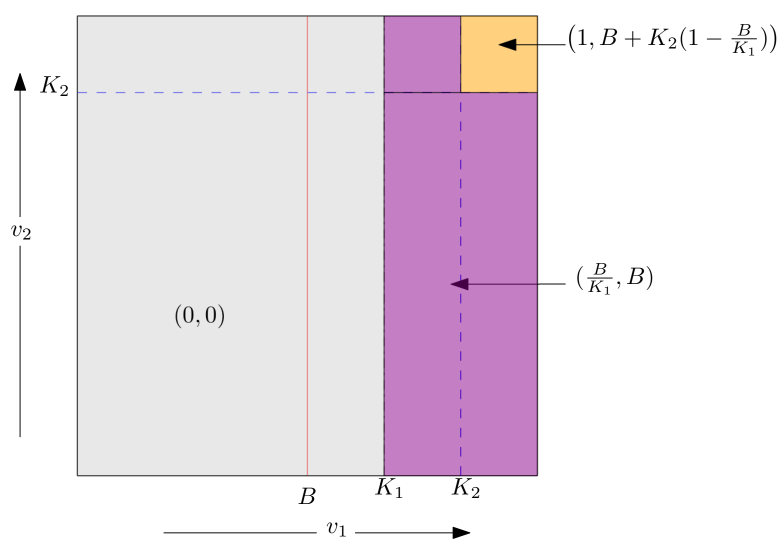

Figure 2.2: POST-2 mechanism

The POST-2 mechanism has a pair of posted prices. The first posted price K_1 is for the agent. If the value of the agent is below K_1, then the object is not sold. Else, the the object is sold with probability $\frac{B}{K_1}$ at per unit price of K_1, i.e., the total price paid equals K_1 times the probability of winning, which is $K_1 \times \frac{B}{K_1} = B$. The remaining probability $(1 - \frac{B}{K_1})$ is sold at per unit price K_2 if the values of both the agent and the manager exceed K_2. Figure

2.2 gives a graphical illustration of a POST-2 mechanism. We show below that a POST-2 mechanism is incentive compatible and individually rational.

PROPOSITION **2.1** *Every* POST-2 *mechanism is incentive compatible and individually rational.*

Though, we provide a formal proof of this result (and all subsequent omitted proofs) in the Appendix, we explain how the notion of incentive compatibility and the lexicographic decision-making make the result possible. There are three outcomes in the "menu" (range) of a POST-2 mechanism. The outcomes $(0, 0)$ and $(\frac{B}{K_1}, B)$ are outcomes which can be compared using preference of the agent. On the other hand, outcome $(1, B + K_2(1 - \frac{B}{K_1}))$ has payment more than B. So, if a type $v \equiv (v_1, v_2)$ is assigned this outcome, incentive compatibility requires that $(1, B + K_2(1 - \frac{B}{K_1}))$ is preferred to $(0, 0)$ and $(\frac{B}{K_1}, B)$ by *both* the agent and the manager. It is easy to verify that this is possible if $v_1, v_2 \geq K_2$ and $K_2 \geq K_1$. Similarly, the other incentive constraints can be shown to hold.

A POST-2 mechanism uses the naivety of the (agent, manager) pair by posting a pair of prices. There are other kinds of mechanisms that can be incentive compatible. Our main result below shows that the optimal mechanism can be either a POST-1 or a POST-2 mechanism.

2.4.3 Main results

The expected (ex-ante) revenue of a mechanism (f, p) is given by

$$\text{REV}(f, p) = \int_{V^2} p(v) dG(v)$$

We say that a mechanism (f, p) is **optimal** if (a) (f, p) is incentive compatible and individually rational, and (b) $\text{REV}(f, p) \geq \text{REV}(f', p')$ for any other incentive compatible and individually rational mechanism (f', p').

For the optimality of our mechanisms, we will need a condition on the marginal distribution of the agent. Define the function H_1 as follows:

$$H_1(x) = xG_1(x) \; \forall \; x \in [0, \beta].$$

THEOREM **2.1** *Suppose H_1 is a strictly convex function. Then, either a* POST-1 *or a* POST-2 *mechanism is an optimal mechanism.*

Our results are slightly stronger than what Theorem 2.1 suggests. We prove that among all mechanisms which has a positive measure of types where the payment is more than the budget, a POST-2 mechanism is optimal. In the remaining class of mechanisms, a POST-1 mechanism is optimal. The strict convexity assumption of H_1 is satisfied by a variety of distributions, including the uniform distribution. [4]

We can be more precise about the optimization programs that need to be solved to get the optimal mechanism in Theorem 2.1. In particular, we either need to solve a one-variable or a two-variable optimization program.

PROPOSITION **2.2** *Suppose H_1 is strictly convex. Then, the expected revenue from the optimal mechanism is* $\max(R_1, R_2)$*, where*

$$R_1 = \max_{K_1 \in [0,B]} K_1(1 - G_1(K_1))$$
$$R_2 = \max_{K_2 \in [B,\beta], K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

The maximization expressions for R_1 and R_2 reflect the expected revenue from a POST-1 and POST-2 mechanism respectively.

If the budget B is high enough, then the POST-1 mechanism becomes optimal - intuitively, the agent makes more decisions and screening along her valuation becomes optimal. It is more interesting to see how much restriction on budget we need to get POST-2 mechanism to be optimal. Below, we derive such a sufficient condition on the budget.

Define the optimal monopoly reserve price as $\bar{K}$

$$\bar{K} := \arg\max_{r \in [0,\beta]} r(1 - G_1(r)).$$

If H_1 is a strictly convex function, $\bar{K}$ is uniquely defined since $x - xG_1(x)$ is a strictly concave function. The interpretation of $\bar{K}$ is that if the agent was *not* budget-constrained,

[4]Such a distributional assumption has appeared in the context of mechanism design before (Che and Gale, 2000). The strict convexity of H_1 requires that the function $G_1(x) + xg_1(x)$ is strictly increasing. This is equivalent to requiring $g_1(x)\big(x - \frac{1-G_1(x)}{g_1(x)}\big)$ being strictly increasing. The standard regularity condition in mechanism design requires increasingness of the bracketed term only.

then the optimal mechanism would have involved a posted-price of $\bar{K}$. Our other main result shows that if the budget constraint is less than $\bar{K}$, then the optimal mechanism is a POST-2 mechanism.

PROPOSITION **2.3** *Suppose H_1 is strictly convex and $B \leq \bar{K}$. Then, the optimal mechanism is a* POST-2 *mechanism. In particular, it is a solution to the following program.*

$$\max_{K_2 \in [B,\beta],\ K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

Proof: Since H_1 is strictly convex, $r(1 - G_1(r))$ is strictly increasing for all $r \leq \bar{K}$. Using $B \leq \bar{K}$, we get that $B(1 - G_1(B)) \geq r(1 - G_1(r))$ for all $r \leq B$. Hence, R_1 defined as the maximum possible revenue in a posted-price mechanism in our problem (Proposition 2.2) is

$$R_1 = \max_{K_1 \in [0,B]} K_1(1 - G_1(K_1)) = B(1 - G_1(B)).$$

But the POST-2 mechanism with $K_1 = K_2 = B$ generates a revenue of $B(1 - G_1(B))$. This proves the theorem. ■

The optimality of POST-2 mechanism is possible even for $B > \bar{K}$. Proposition 2.3 only gives a sufficient condition on the budget for optimality of a POST-2 mechanism. The exact optimal mechanism is difficult to describe in general. Section 2.4.6 works out the exact optimal mechanism for the uniform distribution prior.

2.4.4 Limiting cases

It is interesting to see what our result says in three extreme cases. First, as $B \to \beta$, then the expected revenue from any POST-2 mechanism tends to 0 (since $K_1, K_2 \geq B$). As a result, a POST-1 mechanism becomes optimal.

Second, as $B \to 0$, the expected revenue from a POST-1 mechanism is zero (since posted price is not more than B in a POS-1 mechanism), but using the expression of revenue for optimal POST-2 mechanism given by Proposition 2.2, we see that it is independent of K_1:

$$\max_{K_2 \in [0,\beta]} K_2\Big(1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big)$$

Hence, the optimal POST-2 mechanism can have $K_1 = K_2$ and chooses K_2 that maximizes the product of K_2 and the probability measure of the square on the north-east corner of Figure 2.2 (where $v_1 \geq K_2$ and $v_2 \geq K_2$). Note that since $\frac{B}{K_1} \to 0$, there are only two outcomes in the menu such a mechanism: $(0,0)$ and $(1, K_2)$. Thus the optimal mechanism converges to the optimal posted-price mechanism for the *manager* - just as we described in Section 2.2, only types in the north-east square will choose outcome $(1, K_2)$ in a posted-price mechanism with a posted-price K_2. Note that such a posted price mechanism is *not* a POST-1 mechanism because a POST-1 mechanism has a posted price less than or equal to the budget.

Finally, though our results require that we *do not* have perfect correlation, it is interesting to see what happens as we approach the perfect correlation case. As we approach perfect correlation, we have for all x, $G(x,x) \to G_i(x)$ for each $i \in \{1,2\}$. Hence, using Proposition 2.2, we conclude that the optimal POST-2 mechanism revenue is given by

$$\max_{K_2 \in [B,\beta], K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big]$$
$$= \max_{K_2 \in [B,\beta], K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2)\Big].$$

The above expression is just maximizing the expected revenue of the following class of mechanisms. Pick any $K_2 \in [B, \beta]$ and $K_1 \in [B, K_2]$ and define a mechanism (f, p) as follows:

$$(f(v), p(v)) = \begin{cases} (0,0) & \text{if } v_1 \leq K_1 \\ (1, B + K_2(1 - \frac{B}{K_1})) & \text{if } v_1 > K_2 \\ (\frac{B}{K_1}, B) & \text{otherwise} \end{cases}$$

A straightforward calculation reveals that the revenue from this mechanism is exactly the expression in the maximization term above. Of course, this mechanism is an incentive compatible mechanism in a standard model where there is just the agent with type v_1. But, we know that the optimal mechanism in such a model is a posted-price mechanism with some posted-price p^* and revenue $p^*(1 - G_1(p^*))$. Hence, the revenue R_2 from the optimal POST-2 mechanism must satisfy $R_2 \leq p^*(1 - G_1(p^*))$. If R_2 is strictly higher than the revenue from the optimal POST-1 mechanism, then $p^* \leq B$ will imply that a POST-1 mechanism is also optimal, a contradiction. Hence, $p^* > B$ must hold when a POST-2 mechanism is the optimal mechanism. But a POST-2 mechanism generating a revenue of $p^*(1 - G_1(p^*))$ with $p^* > B$ is a POST-2 mechanism with $K_1 = K_2 = p^*$. Thus $R_2 = p^*(1 - G_1(p^*))$, where $K_1 = K_2 = p^*$. Finally, note that as $G(x,x) \to G_i(x)$ for each x and for each i, the probability measure of the rectangle $\{v : v_1 > K_2, v_2 < K_2\}$ tends to zero. Hence, this POST-2 mechanism approaches a standard posted-price mechanism with two outcomes in the menu.

2.4.5 Sketch of the proofs

We give an overview of the proof of Theorem 2.1 in this section. Fix a mechanism (f, p), and define the following partitioning of the type space:

$$V^+(f,p) := \{v : p(v) > B\}$$
$$V^-(f,p) = \{u : p(u) \le B\}.$$

The proof considers two classes of mechanisms, those (f, p) where $V^+(f, p)$ has non-zero Lebesgue measure and those where $V^+(f, p)$ has zero Lebesgue measure. Define the following partitioning of the class of mechanisms:

$$M^+ := \{(f,p) : V^+(f,p) \text{ has positive Lebesgue measure}\}$$
$$M^- := \{(f,p) : V^+(f,p) \text{ has zero Lebesgue measure}\}.$$

The proof of Theorem 2.1 is completed by proving the following proposition.

PROPOSITION **2.4** *Suppose H_1 is strictly convex. Then, the following are true.*

1. *There exists a* POST-1 *mechanism $(f, p) \in M^-$ which is incentive compatible and individually rational such that for every incentive compatible and individually rational mechanism $(f', p') \in M^-$, we have*

$$\textsc{Rev}(f,p) \ge \textsc{Rev}(f',p').$$

2. *There exists a* POST-2 *mechanism $(f, p) \in M^+$ which is incentive compatible and individually rational such that for every incentive compatible and individually rational mechanism $(f', p') \in M^+$, we have*

$$\textsc{Rev}(f,p) \ge \textsc{Rev}(f',p').$$

The proof of (1) in Proposition 2.4 uses somewhat familiar ironing arguments. However, proof of (2) in Proposition 2.4 is quite different, and requires a lot of work to get to a simpler class of mechanisms where ironing can be applied. The proof proceeds by deriving some necessary conditions for incentive compatibility and reducing the space of mechanisms. It can be broken down into three steps.

1. STEP 1. The first step of the proof uses just incentive constraints to show that every incentive compatible mechanism has a simple form. In particular, there is a cutoff $K \ge B$ such that for all types v with $\min(v_1, v_2) > K$, the outcome of the mechanism is constant (with payment greater than the budget). This implication comes purely from the incentive constraints in the mechanism.

2. STEP 2. In the next step, we show that the *optimal* mechanism must belong to a class of simple mechanisms. In this class of mechanisms, there is a cutoff K (identified in Step 1), such that the outcome of the mechanism for types v with $\min(v_1, v_2) > K$ is one constant (where payment is greater than the budget) and for types v with $v_1 \geq K$ but $\min(v_1, v_2) \leq K$, it is another constant (where payment is equal to budget). For types v with $v_1 < K$, payment is not more than the budget.

3. STEP 3. In this step, we further relax the class of above mechanisms. We show that it is without loss of generality to consider only those mechanisms where for all types u, v with $u_1 = v_1 < K$, the outcomes at u and v are the same. These steps allow us to apply standard ironing arguments and get to a POST-2 mechanism.

In summary, though the proof does not introduce new tools to deal with multidimensional mechanism design problems, it illustrates that multidimensional mechanism design problems may be tractable under certain behavioral assumptions.

2.4.6 Uniform distribution

In this section, we work out the exact optimal mechanism for the uniform distribution case. All the proofs of this section are given in Supplementary Appendix 2.10.2.

We assume that $\beta = 1$ and G is the uniform distribution over $[0,1] \times [0,1]$. Call a POST-2 mechanism defined by posted prices (K_1^*, K_2^*) optimal POST-2 mechanism if it solves the optimization program in Proposition 2.2. Our result shows that for uniform distribution $K_1^* = K_2^*$.

LEMMA **2.2** *Suppose $\beta = 1$ and G is the uniform distribution over $[0,1] \times [0,1]$. If (K_1^*, K_2^*) are values of (K_1, K_2) in the optimal* POST-2 *mechanism, then $K_1^* = K_2^*$.*

Further, the optimal POST-2 *mechanism must satisfy:*

1. *if $B \geq \frac{1}{2}(3 - \sqrt{5})$, then $K_1^* = K_2^* = B$,*

2. *if $B < \frac{1}{2}(3 - \sqrt{5})$, then $K_1^* = K_2^* = \frac{1}{3}\big(B + 2 - \sqrt{(B^2 + B + 1)}\big)$.*

Using this lemma, we can provide a complete description of the optimal mechanism for the uniform distribution case.

PROPOSITION **2.5** *Suppose $\beta = 1$ and G is the uniform distribution over $[0,1] \times [0,1]$. Then, the optimal mechanism is the following.*

1. *If $B > \frac{1}{2}$, then a* POST-1 *mechanism with $K_1 = \frac{1}{2}$ is optimal.*

2. *If $B \in [\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}]$, then a* POST-1 *mechanism with $K_1 = B$ is optimal. In this case, a* POST-2 *mechanism with $K_1 = K_2 = B$ is also optimal.*

3. *If $B \in (0, \frac{1}{2}(3-\sqrt{5}))$, then a* POST-2 *mechanism with*

$$K_1 = K_2 = \frac{1}{3}\big(B + 2 - \sqrt{(B^2 + B + 1)}\big)$$

is optimal.

Notice that as $B \to 0$, the optimal mechanism is a posted price mechanism with price $\frac{1}{3}$. So, in the limiting case when the agent has zero budget to make decisions, the optimal mechanism is *not* a posted price mechanism with posted price $\frac{1}{2}$ - which is the optimal posted price in the standard model. To see why, consider the limiting case $B = 0$. Suppose the seller uses a posted price mechanism with price p. Who are the types who will accept this price? This is shown in the left graph in Figure 2.1. All the types (v_1, v_2) such that $v_1 < p$ will choose outcome $(0, 0)$. All types (v_1, v_2) with $v_1 > p$ but $v_2 < p$ will also choose outcome $(0, 0)$ - this is because even though the agent prefers $(1, p)$ over $(0, 0)$, it cannot make a decision because of budget constraint. Thus, the only types (v_1, v_2) which will prefer $(1, p)$ to $(0, 0)$ are those with $v_1 > p, v_2 > p$. Hence, the expected revenue from a posted price mechanism is $p(1 - p)^2$, which is maximized at $\frac{1}{3}$. This argument establishes the optimal posted price mechanism. Proposition 2.5 shows that it is the optimal mechanism.

On the other extreme, when $B \to \beta$, the optimal mechanism is a posted price mechanism with price $\frac{1}{2}$. This is because the agent makes all the decisions now and for any price p, the types that accept this price are just the types with $v_1 > p$. An optimal solution thus gives a posted price of $\frac{1}{2}$ as in a standard model.

2.5 Notion of incentive compatibility

In this section, we discuss some issues related to the revelation principle and our notion of incentive compatibility. We show here a version of the revelation principle holds in our setting. To define an arbitrary mechanism, let M be a message space and $\mu : M \to Z$ be a mechanism. A strategy of the (agent, manager) pair is a map $s : V \to M$. We say that mechanism μ **implements** the direct revelation mechanism (f, p) if there exists a strategy $s : V \to M$ such that

- EQUILIBRIUM. $\mu(s(v)) \succeq_v \mu(m)\ \forall\ v \in V,\ \forall\ m \in M.$

- OUTCOME. $\mu(s(v)) = (f(v), p(v))\ \forall\ v \in V.$

Suppose μ implements (f, p). Then, fix some $v, v' \in V$ and note that $(f(v), p(v)) = \mu(s(v)) \succeq_v \mu(s(v')) = (f(v'), p(v'))$, which proves incentive compatibility of (f, p). Hence, the revelation principle holds in this setting. It is well known that with behavioral agents, the revelation principle may not hold in general (de Clippel, 2014). There are at least two assumptions in our model which allows the revelation principle to work. The first is the completeness of our relation $\succeq_v$ (even though it may be intransitive). The second, and more important one, is the notion of incentive compatibility we use. We discuss this issue in detail next.

The primitives of our model involves how the (agent, manager) pair chooses from pairs of outcomes. We are silent about how it chooses from a subset of alternatives. This is consistent with Tversky (1969) and most of the literature which works on binary choice models (Rubinstein, 1988; Tadenuma, 2002; Houy and Tadenuma, 2009). Our incentive constraints are appropriate for this binary choice model.

In Supplementary Appendix 2.10.3, we consider a model where we extend our framework to allow for choice from any subset of outcomes. We adapt a model of Manzini and Mariotti (2012) for this purpose. We then propose a notion of incentive compatibility which is appropriate for choice correspondences - we call it *choice-incentive compatibility*. We argue that both the notions of incentive compatibility are independent. However, there are two main reasons why we use our existing notions of incentive compatibility instead of choice-incentive compatibility. First, to be able to use choice-incentive compatibility, we have to *assume* that the (agent, manager) pair chooses from subsets of outcomes using some choice procedure. The current primitives of our model are much simpler - it just makes assumptions on how we choose between pairs of outcomes. Importantly, our notion of incentive compatibility allows us tractability using minimal assumptions about deviations from rationality. Second, if the primitives of the model are choice correspondences, then a revelation principle need not hold - see de Clippel (2014). This implies that the space of mechanisms are more complex than the set of direct revelation mechanisms. In summary, it is not clear how an optimal mechanism will look like if we considered a model assuming certain choice behavior of agents over subsets of outcomes and choice-incentive compatibility as the notion of our incentive compatibility. We leave this issue for future research.

2.6 Private budgets: a partial result

In this section, we consider the scenario when budget is private information. This may be the case in various examples that we considered - the budget of the agent may not be observable to the seller. In such cases, the type space is three-dimensional. We only have a partial description of an optimal mechanism in this case.

We will assume that both the values and the budget lie in $[0, \beta]$. Thus, the type space is $W \equiv [0, \beta]^3$. A type will be denoted by $(v, B) \equiv (v_1, v_2, B)$, where v_1 and v_2 are the values of the agent and the manager respectively and B is the budget. For any type $(v, B) \in W$, the preferences over the outcome space is same as the preferences of the type $v \in V$ with budget B in the public budget case. Since the outcome space is the same, this is well defined as before. For any type (v, B), we denote the corresponding preference as $\succeq_{(v,B)}$.

The seller has a prior Φ over the type space W. A (direct) **mechanism** is a pair of maps: an allocation rule $f : W \to [0, 1]$ and a payment rule $p : W \to \mathbb{R}$. The incentive compatibility and individual rationality constraints are as before.

Definition **2.5** *A mechanism (f, p) is* **incentive compatible** *if for all $(u, B), (v, B') \in W$,*

$$\big(f(u, B), p(u, B)\big) \succeq_{(u,B)} \big(f(v, B'), p(v, B')\big).$$

A mechanism (f, p) is **individually rational** *if for all $(v, B) \in W$,*

$$\big(f(v, B), p(v, B)\big) \succeq_{(v,B)} \big(0, 0\big).$$

We will only consider the following class of mechanisms in this section for our main result.

Definition **2.6** *A mechanism (f, p) is* **manager non-trivial** *if there exists some budget $B \in [0, \beta]$ and $V' \subseteq [0, \beta]^2$ such that V' has non-zero Lebesgue measure in $[0, \beta]^2$ and*

$$p(v, B) > B \ \forall \ v \in V'.$$

A manager non-trivial mechanism rules out the possibility that at every budget B, the payment is not more than B at almost every valuation profile (given B). We only consider optimality in the class of manager non-trivial mechanisms. We believe that manager non-triviality is a reasonable restriction to impose on the class of mechanisms in this setting - in the absence of this, the agent will take all the decisions in a mechanism. As before, the expected revenue of a mechanism (f, p) is

$$\textsc{Rev}(f, p) := \int_W p(v, B) d\Phi(v, B).$$

A manager non-trivial mechanism (f, p) is **partially optimal** if it is incentive compatible and individually rational and there is no other manager non-trivial mechanism (f', p') which is incentive compatible and individually rational and $\textsc{Rev}(f', p') > \textsc{Rev}(f, p)$. Even though manager non-trvial mechanisms are a natural class of mechanisms to consider, our reason for restricting attention to this class is tractability. However, we give sufficient conditions on the distributions under which a partially optimal mechanism is optimal.

We now introduce an analogue of the POST-2 mechanism in the private budget case.

DEFINITION **2.7** *A mechanism (f, p) is a* POST* *mechanism if there exists $K \in [0, \beta]$ such that*

$$(f(v,B), p(v,B)) = \begin{cases} \Big(1, K\Big) & \textit{if } \Big(\min(v_1, v_2) > K \textit{ and } B < K\Big) \\ & \textit{or } \Big(v_1 > K \textit{ and } B \geq K\Big) \\ (0,0) & \textit{if } v_1 \leq K \\ (\frac{B}{K}, B) & \textit{if } v_1 > K,\ v_2 \leq K \textit{ and } B < K \end{cases}$$

A pictorial description of a POST* mechanism is given in Figure 2.3. The similarity

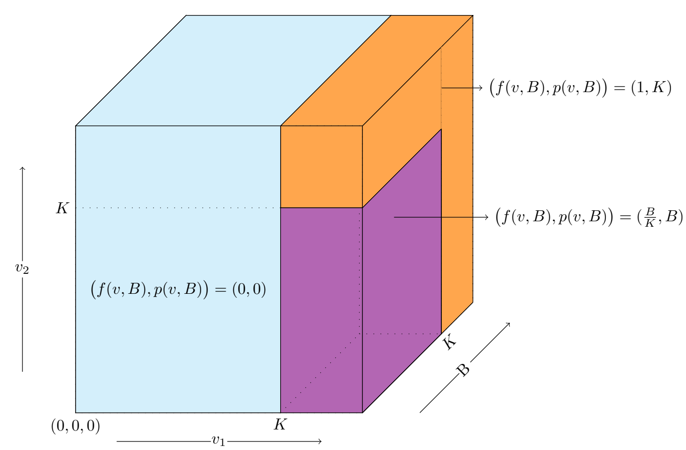

Figure 2.3: Illustration of a POST* mechanism

between POST-2 and POST* is deceiving since POST-2 is defined for a fixed budget B but POST* is defined for all values of budget. As a result, the menu size of POST* is infinite - a separate outcome is chosen for every budget in the third case of the definition of POST* mechanism. Notice that choice of $K \in [0, \beta]$ pins down a POST* mechanism. So, a POST* mechanism is defined by a single parameter. On the other hand, a POST-2 mechanism requires specification of two parameters. However, if we fix a POST* mechanism, defined by choosing K, and consider a budget $B < K$, then the projection of this POST* mechanism at B is a POST-2 mechanism with the two parameters of the POST-2 mechanism equal to K. Similarly, if we take $B > K$, then the projection of a POST* mechanism at B is a posted price mechanism.

We show below that every POST* mechanism is incentive compatible and individually rational.

PROPOSITION **2.6** *Every* POST* *mechanism is manager non-trivial, incentive compatible, and individually rational.*

The main result of this section establishes the partial optimality of POST* mechanism.

THEOREM **2.2** *A partially optimal mechanism is a* POST* *mechanism.*

We emphasize here that unlike Theorem 2.1, Theorem 2.2 does not require any distributional assumption. This is a consequence of the ironing required to arrive at the optimal mechanism in Theorem 2.1, and the absence of any ironing in the proof of Theorem 2.2 - see the respective proofs in Appendix. Intuitively, with private budgets, the set of incentive constraints become larger and the need for ironing reduces. We should also note here that if the lower support of budget is positive (for simplicity, we have assumed it to be zero), Theorem 2.2 goes through with some minor changes, but it requires the distribution to satisfy the same condition as in Theorem 2.1. This is because, in that case, we need ironing to arrive at an optimal mechanism (very similar to Theorem 2.1). We skip these details for the interest of space but it is available upon request.

The derivation of an optimal mechanism without the manager non-triviality assumption for the private budget case seems difficult - even in the standard model, the private budget case is significantly complicated (Che and Gale, 2000). In Supplementary Appendix 2.10.4, we state a sufficient condition on distributions (satisfied if values and budget are independently and uniformly distributed) that guarantee the optimality of a POST* mechanism.

2.7 Related literature

Our paper is related to a couple of strands of literature in mechanism design. We go over them in some detail. Before doing so, we relate our work to two papers which seem directly related to our work. The first is the work of Burkett (2016), who studies a principal-agent model where the agent is participating in an auction mechanism with a third-party. In his model, there is a third-party which has proposed a mechanism for selling a single good. After the third-party announces a mechanism, the principal in his model announces another mechanism, which he terms as a *contract*, to the agent. The sole purpose of the contract is to determine the amount the agent will bid in the third-party mechanism. In his model, the value of the good to the agent is the *only* private information - the value of the good to the principal can be determined from the value of the agent. The main result in this paper is that the optimal contract for the principal is a "budget-constraint" contract, which specifies a cap on the report of each type of the agent to the third-party mechanism and involves no side-payments between the principal and the agent. [5]

Though related, our model is quite different. In our model, the values of the agent and the manager can be completely different (at a technical level, Burkett (2016) has a one-dimensional mechanism design problem, whereas ours is a two-dimensional mechanism design problem). Further, we do not model decision-making by our (agent, manager) pair via a contract. In other words, the naive decision-making in our model makes it quite different from Burkett (2015, 2016).

Another closely related paper is Malenko and Tsoy (Forthcoming), who study a model where a single good is sold to a set of buyers. Each buyer is advised by a unique advisor. Each buyer does not know her value but the advisor knows. However, the advisor has some bias, which is commonly known. Before the start of the auction, there is communication from the advisor to the buyer, which influences how much the buyer bids in the auction. The aim of Malenko and Tsoy (Forthcoming) is to compare standard auction formats in the presence of such uncertain buyers being advised by biased consultants. They find that standard sealed-bid auctions are revenue equivalent, but ascending-price auction generates more expected revenue than sealed-bid auctions. While their focus is on the effect of communication on equilibrium of standard form auctions, ours is a mechanism design problem where the (agent, manager) pair do not engage in any communication. Our novelty is to solve for the optimal contract of a seller in the presence of a naive (agent, manager) pair.

Behavioral mechanism design. We discuss some literature in mechanism design which looks at specific models of behavioral agents and designing optimal contracts for selling to

[5] In a related paper, Burkett (2015) considers first-price and second-price auctions and compares their revenue and efficiency properties when a seller is faced with such principal-agent pairs.

such agents. A very detailed survey with excellent examples can be found in Koszegi (2014). Our literature survey is limited in nature as we focus on models which are closer to ours.

A stream of papers investigate the optimal contract for a firm to a consumer in a two-period model, where the consumer has time inconsistent preferences. These papers differ in the way it treats inconsistent preferences and non-common priors between firm and the consumer.

Eliaz and Spiegler (2006) consider a model where the type of the agent is his "cognitive" ability. In their model, there are two periods and the agent enjoys a valuation for an action in each period. In period 2, the agent's valuation may change to another value. Agents differ in their subjective assessment of the probability of that transition. So, in their model, the type is the subjective probability of the agent. They show how the optimal contract treats sophisticated and naive agents. While this paper allows agents to be time-inconsistent, in another paper, Eliaz and Spiegler (2008) study a similar model but do not allow time inconsistency. There, they allow the monopolist to have a separate belief about the change of state. They characterize the optimal contract and show the implications of non-common priors on the menu of optimal contract and ex-post efficiency. Grubb (2009) considers a two period model where a firm is selling a divisible good to consumers. The private type of the consumer is his demand in period 2. In period 1, the firm offers them a tariff which is accepted or rejected. If accepted, the consumers buy the quantity in period 2 once they realize their demand. The key innovation in his paper is again the lack of common prior between consumers and the firm - in particular, he shows that if the prior of the consumers is such that it *underestimates* the variance of the actual prior (for instance, if the consumer prior has the same mean as the firm, then consumer prior is a mean-preserving spread of the firm prior), then the optimal tariff of the firm must have three parts (with quantities offered at zero marginal cost).

de Clippel (2014) studies complete information implementation with behavioral agents - his main results extend Maskin's characterization (Maskin, 1999) to environments with behavioral agents. Esteban et al. (2007) consider a model where agents have temptation and self control preferences as in Gul and Pesendorfer (2001), and characterize the optimal contract - also see related work on self control preferences in DellaVigna and Malmendier (2004). There are several other papers who consider time inconsistent preferences and analyze the optimal contracting problem. Carbajal and Ely (2016) consider a model of optimal price discrimination when buyers have loss averse preferences with state dependent reference points. They characterize the optimal contract in their model.

Multidimensional mechanism design. The type space of our agent is two-dimensional. It is well known that the problem of finding an optimal mechanism for selling multiple goods (even to a single buyer) is notorious. A long list of papers have shown the difficulties involved in extending the one-dimensional results in Mussa and Rosen (1978); Myerson (1981); Riley and Zeckhauser (1983) to multidimensional framework - see Armstrong (2000); Manelli and Vincent (2007) as examples. Even when the seller has just *two* objects and there is just one buyer with additive valuations (i.e., value for both the objects is sum of values of both the objects), the optimal mechanism is difficult to describe (Manelli and Vincent, 2007; Daskalakis et al., 2017; Hart and Nisan, 2017). This has inspired researchers to consider *approximately* optimal mechanisms (Chawla et al., 2007, 2010; Hart and Nisan, 2017) or additional robustness criteria for design (Carroll, 2017). Compared to these problems, our two-dimensional mechanism design problem becomes tractable because of the nature of incentive constraints, which in turn is a consequence of the preference of the agent.

Mechanism design with budget constraints. In our model, the agent is budget constrained but the manager is not. We compare this with the literature in the standard model when there is a single object and the buyer(s) is budget constrained. The space of mechanisms is restricted to be such that payment is no more than the budget. This feasibility requirement on the mechanisms essentially translates to a violation of quasilinearity assumption of the buyer's preference for prices above the budget (utility assumed to be $-\infty$) but below the budget the utility is assumed to be quasilinear. This introduces additional complications for finding the optimal mechanism. Laffont and Robert (1996) show that an all-pay-auction with a suitable reserve price is an optimal mechanism for selling an object to multiple buyers who have publicly known budget constraints. When the budget is private information, the problem becomes even more complicated - see Che and Gale (2000) for a description of the optimal mechanism for the single buyer case and Pai and Vohra (2014) for a description of the optimal mechanism for the multiple buyers case. All these mechanisms involve randomization but the nature of randomization is quite different from ours. This is because the source of randomization in all these papers is either due to budget being private information (hence, part of the type, as in Che and Gale (2000); Pai and Vohra (2014)) or because of multiple agents with budget being common knowledge (as in Laffont and Robert (1996); Pai and Vohra (2014)). Indeed, with a single agent and public budget, the optimal mechanism in a standard single object allocation model is a posted price mechanism. This can be contrasted with our result where we get randomized optimal mechanism even with one (agent, manager) pair and budget being common knowledge. This shows that the lexicographic decision making using two rationales plays an important role in making a POST-2 mechanism optimal. Also, the set of menus in the optimal mechanism in the standard single object auction with budget constraint may have more than three outcomes. Further, the outcomes in the menu of these optimal mechanisms are not as simple as our POST-2 mechanism. Finally, like us, these papers assume that budget is exogenously determined by the agent. If the buyer can choose his budget constraint, then Baisa and Rabinovich (2016) shows that the optimal mechanism in a multiple buyers setting allocates the object efficiently whenever it is allocated - this is in contrast to the exogenous budget case (Laffont and Robert, 1996; Pai and Vohra, 2014).

2.8 Appendix: Omitted Proofs of Section 2.4

This section contains all omitted proofs of Section 2.4 - except for proofs of Section 2.4.6, which are given in the Supplementary Appendix 2.10.2.

2.8.1 Proof of Proposition 2.1

Proof: Consider a POST-2 mechanism (f, p) defined by parameters K_1 and K_2 with $B \leq K_1 \leq K_2$. Since $p(0,0) = 0$, Lemma 2.1 implies that (f, p) is individually rational if it is incentive compatible. We show incentive compatibility of (f, p). We will denote by $\bar{u} \to \tilde{u}$ the incentive constraint associated with type $\bar{u}$ when it cannot misreport $\tilde{u}$.

Consider types u, v, s taken from three different regions in Figure 2.2 with three different outcomes. In particular, u, v, s satisfy: $u_1 \leq K_1$, $\min(v_1, v_2) \leq K_2$ but $v_1 > K_1$, and $\min(s_1, s_2) > K_2$. Note that

$$(f(u), p(u)) = (0, 0),\ (f(v), p(v)) = (\frac{B}{K_1}, B),\ \text{and } (f(s), p(s)) = (1, B + K_2(1 - \frac{B}{K_1})).$$

We consider incentive compatibility of each of these types.

1. $u \to v, u \to s$. Note that since $u_1 \leq K_1$, we have $u_1 \frac{B}{K_1} - B \leq 0$. Hence, type u weakly prefers $(0,0)$ to $(\frac{B}{K_1}, B)$. Similarly,
$$\begin{aligned} u_1 - B - K_2\big(1 - \frac{B}{K_1}\big) &\leq K_1 - B - K_2 + \frac{K_2}{K_1}B \\ &= (K_2 - K_1)\Big(\frac{B}{K_1} - 1\Big) \leq 0, \end{aligned}$$
where first inequality is due to $u_1 \leq K_1$ and the second is due to $K_2 \geq K_1$ and $B \leq K_1$. Hence, u prefers $(0,0)$ to $(f(s), p(s))$.

2. $v \to u, v \to s$. For $v \to u$, we note that
$$v_1 \frac{B}{K_1} - B \geq 0$$
This follows from the fact that $v_1 > K_1$. Hence, incentive constraint $v \to u$ holds as $p(v) = B$.
For $v \to s$, we note that
$$\begin{aligned} \min(v_1, v_2) - B - K_2\Big(1 - \frac{B}{K_1}\Big) &\leq \min(v_1, v_2) - B - \min(v_1, v_2)\Big(1 - \frac{B}{K_1}\Big) \\ &= \frac{B}{K_1}\min(v_1, v_2) - B. \end{aligned}$$

If $\min(v_1, v_2) = v_1$, then we see that $(f(v), p(v))$ is preferred to $(f(s), p(s))$. Else, $\min(v_1, v_2) = v_2$. In that case since $p(s) > B$, even if the agent prefers $(f(s), p(s))$ to $(f(v), p(v))$, she cannot compare. But the manager prefers $(f(v), p(v))$ to $(f(s), p(s))$. Hence, incentive constraint $v \to s$ holds.

3. $s \to u, s \to v$. Note that for $x \in \{s_1, s_2\}$, we have

$$\begin{aligned} 0 \le \frac{K_2}{K_1}B - B \le \frac{B}{K_1}x - B \\ &= x - B - x\Big(1 - \frac{B}{K_1}\Big) \\ &\le x - B - K_2\Big(1 - \frac{B}{K_1}\Big), \end{aligned}$$

where the inequalities follow from the fact that $\min(s_1, s_2) > K_2 \ge K_1 \ge B$. This shows that *both* the dimensions at s prefer $(f(s), p(s))$ to $(f(v), p(v))$ and $(f(u), p(u))$. Because $p(s) > B$, the incentive constraints $s \to v$ and $s \to u$ hold.

■

2.8.2 Proofs of Theorem 2.1 and Propositions 2.2 and 2.4

In this section, we provide the proof of the main results - Theorem 2.1 and Propositions 2.2 and 2.4. It is clear that Proposition 2.4 immediately implies Theorem 2.1. So, we first provide a proof of Proposition 2.4, followed by a proof of Proposition 2.2.

Preliminary Lemmas

We start off by proving a series of necessary conditions for incentive compatibility. The first lemma is a monotonicity condition of allocation rule: for every incentive compatible mechanism, type with higher payment implies higher allocation probability. Hence, the outcomes in the range of an incentive compatible mechanism are ordered in a natural sense.

LEMMA **2.3** *For any incentive compatible mechanism (f, p), if $p(u) < p(v)$ for any u, v, then $f(u) < f(v)$.*

Proof: Take any u, v such that $p(u) < p(v)$. Incentive compatibility implies that

$$(f(v), p(v)) \succeq_v (f(u), p(u)).$$

If $p(v) \leq B$, then we must use the incentive constraints in $\succeq v_1$, which gives us

$$v_1 f(v) - p(v) \geq v_1 f(u) - p(u) > v_1 f(u) - p(v),$$

where the last inequality uses $p(v) > p(u)$. This implies $f(u) < f(v)$. If $p(v) > B$, then using the incentive constraint in $\succeq_{v_2}$, we have

$$v_2 f(v) - p(v) \geq v_2 f(u) - p(u) > v_2 f(u) - p(v),$$

where the last inequality uses $p(v) > p(u)$. This implies $f(u) < f(v)$. ■

Lemma **2.4** *For any incentive compatible mechanism (f, p), for all u, v*

1. *if $p(u), p(v) \leq B$ and $u_1 > v_1$, then $f(u) \geq f(v)$,*
2. *if $p(u), p(v) > B$ and $u_2 > v_2$, then $f(u) \geq f(v)$.*

Proof: Take any u, v. If $p(u), p(v) \leq B$, then adding the incentive constraints using $\succeq_{v_1}$ and $\succeq_{u_1}$gives us the desired result and if $p(u), p(v) > B$, then adding the incentive constraints using $\succeq_{v_2}$ and $\succeq_{u_2}$ gives us the desired result. ■

Lemma **2.5** *For any incentive compatible mechanism (f, p), for all u, v the following holds:*

$$\Big[p(u) \leq B < p(v)\Big] \Rightarrow \Big[\min(v_1, v_2) \geq \min(u_1, u_2)\Big].$$

Proof: Since $p(u) \leq B < p(v)$, by Lemma 2.3, $f(v) > f(u)$. We consider the incentive constraint from v to u first. This gives us

$$v_2 f(v) - p(v) \geq v_2 f(u) - p(u). \tag{2.1}$$

$$v_1 f(v) - p(v) > v_1 f(u) - p(u). \tag{2.2}$$

Using $f(v) > f(u)$, and aggregating Inequalities 2.1 and 2.2 gives us

$$\min(v_1, v_2)\big(f(v) - f(u)\big) \geq p(v) - p(u). \tag{2.3}$$

Incentive compatibility from u to v implies one of the two conditions to holds:

Case 1. $\succeq_{u_1}$ prefers $(f(u), p(u))$ to $(f(v), p(v))$: this gives

$$u_1 f(u) - p(u) \geq u_1 f(v) - p(v) \text{ or } p(v) - p(u) \geq u_1(f(v) - f(u)).$$

Adding with Inequality 2.3, we get,

$$(\min(v_1, v_2) - u_1)(f(v) - f(u)) \geq 0.$$

Then, $f(v) > f(u)$ implies that $\min(v_1, v_2) \geq u_1$.

CASE 2. $\succeq_{u_1}$ does not prefer $(f(u), p(u))$ to $(f(v), p(v))$ but budget has a bite - so, $\succeq_{u_2}$ prefers $(f(u), p(u))$ to $(f(v), p(v))$: this gives

$$u_2 f(u) - p(u) \geq u_2 f(v) - p(v). \tag{2.4}$$

Adding Inequalities (2.4) and (2.3), we get $(\min(v_1, v_2) - u_2)(f(v) - f(u)) \geq 0$. Since $f(v) > f(u)$, we get $\min(v_1, v_2) \geq u_2$.

Combining both the cases, $\min(v_1, v_2) \geq \min(u_1, u_2)$. ■

Now, fix a mechanism (f, p), and define

$$\begin{aligned} V^+(f,p) &:= \{v : p(v) > B\} \\ V^-(f,p) &= \{u : p(u) \leq B\}. \end{aligned}$$

LEMMA **2.6** *Fix an incentive compatible mechanism* (f, p). *If* $V^+(f,p)$ *and* $V^-(f,p)$ *are non-empty, then the following holds:*

$$\inf_{v \in V^+(f,p)} \min(v_1, v_2) = \sup_{u \in V^-(f,p)} \min(u_1, u_2).$$

Proof: Since $V^+(f,p)$ is non-empty and $\min(v_1, v_2) \geq 0$, we have that $\inf_{v \in V^+(f,p)} \min(v_1, v_2)$ is a non-negative real number - we denote it as $\underline{v}$. By Lemma 2.5, $\sup_{u \in V^-(f,p)} \min(u_1, u_2)$ is also a non-negative real number as it is bounded above - we denote this as $\bar{v}$.

First, we show that $\underline{v} \geq \bar{v}$. If not, then $\underline{v} < \bar{v}$. Then, there is some v such that $\underline{v} < \min(v_1, v_2) < \bar{v}$. By definition of $\underline{v}$, there is a v' such that $\min(v'_1, v'_2)$ is arbitrarily close to $\underline{v}$ and $p(v') > B$. Since $\min(v'_1, v'_2) < \min(v_1, v_2)$, Lemma 2.5 gives us $p(v) > B$. Similarly, by definition of $\bar{v}$, there is a u' such that $\min(u'_1, u'_2)$ is arbitrarily close to $\bar{v}$ and $p(u') \leq B$. Since $\min(u'_1, u'_2) > \min(v_1, v_2)$, Lemma 2.5 gives us $p(v) \leq B$, giving us the desired contradiction.

Next, we show that $\underline{v} = \bar{v}$. If not, $\underline{v} > \bar{v}$. But this is not possible since for any v with $\underline{v} > \min(v_1, v_2) > \bar{v}$, we will have both $p(v) \leq B$ and $p(v) > B$, giving us a contradiction. ■

For any mechanism (f, p), we will denote by $K_{(f,p)}$ the following:

$$K_{(f,p)} := \inf_{v \in V^+(f,p)} \min(v_1, v_2) = \sup_{u \in V^-(f,p)} \min(u_1, u_2). \tag{2.5}$$

By Lemma 2.6, this is well-defined if $V^+(f, p)$ and $V^-(f, p)$ is non-empty.

Lemma **2.7** *If (f, p) is an incentive compatible and individual rational mechanism, then $V^-(f, p)$ is non-empty.*

Proof: Lemma 2.1 ensures that $(0, 0) \in V^-(f, p)$ if (f, p) is incentive compatible and individually rational. ■

Define the following partitioning of the class of mechanisms:

$$M^+ := \{(f, p) : V^+(f, p) \text{ has positive Lebesgue measure}\}$$
$$M^- := \{(f, p) : V^+(f, p) \text{ has zero Lebesgue measure}\}.$$

We now prove a series of Lemmas for M^+ class of mechanisms.

Lemmas for M^+

The following lemma shows that $K_{(f,p)}$ is well defined if $(f, p) \in M^+$.

Lemma **2.8** *Suppose (f, p) is an incentive compatible and individually rational mechanism.*

1. *If $V^+(f, p)$ is non-empty, then $K_{(f,p)}$ defined in Equation (2.5) exists and satisfies: for all $v \in V$,*

$$\Big[\min(v_1, v_2) > K_{(f,p)}\Big] \Rightarrow \Big[p(v) > B\Big],$$
$$\Big[\min(v_1, v_2) < K_{(f,p)}\Big] \Rightarrow \Big[p(v) \le B\Big].$$

2. *If $(f, p) \in M^+$, then $\beta > K_{(f,p)} > B$.*

Proof: The first part follows from Lemma 2.6, Lemma 2.7, and the definition of M^+.

For the second part, we first argue that $K_{(f,p)} \ge B$. Suppose $K_{(f,p)} < B$. Then, for some v with $K_{(f,p)} < \min(v_1, v_2) \le B$, we have $p(v) > B$. But this violates individual rationality.

Now, assume for contradiction $K_{(f,p)} = B$. In that case, fix some $\epsilon \in (0, 1)$ and positive integer k, and consider the type $v^{k,\epsilon} \equiv (B + \epsilon^k, B + \epsilon^k)$. By (1), we know that $p(v^{k,\epsilon}) > B$. By individual rationality,

$$(B + \epsilon^k) f(v^{k,\epsilon}) \ge p(v^{k,\epsilon}) > B.$$

This gives us $f(v^{k,\epsilon}) > \frac{B}{B+\epsilon^k}$. Since $B + \epsilon > B + \epsilon^k$ for all $k > 1$, by (1) of Lemma 2.4, we have $f(v^{1,\epsilon}) \geq f(v^{k,\epsilon}) > \frac{B}{B+\epsilon^k}$. As $\frac{B}{B+\epsilon^k}$ can be made arbitrarily close to 1, we conclude that $f(v^{1,\epsilon}) = 1$ - notice that $v^{1,\epsilon} \equiv (B+\epsilon, B+\epsilon)$ and the claim holds for *all* $\epsilon \in (0,1)$. By Lemma 2.3, for all $\epsilon, \epsilon' \in (0,1)$, since $f(v^{1,\epsilon}) = f(v^{1,\epsilon'}) = 1$, we get that $p(v^{1,\epsilon}) = p(v^{1,\epsilon'})$. Denote $p(v^{1,\epsilon}) = B + \delta$, where $\epsilon \in (0,1)$. By definition, $\delta > 0$. Now, individual rationality requires that for *every* $\epsilon \in (0,1)$,

$$(B+\epsilon)f(v^{1,\epsilon}) - p(v^{1,\epsilon}) = (B+\epsilon) - (B+\delta) \geq 0.$$

But this will mean $\epsilon > \delta$ for all $\epsilon \in (0,1)$. Since $\delta > 0$ is fixed, this is a contradiction.

Finally, we know that $(f,p) \in M^+$ implies $V^+(f,p)$ has positive Lebesgue measure. If $\beta = K_{(f,p)}$, then by (1), we know that $V^+(f,p)$ has zero Lebesgue measure, which is a contradiction. ■

Next, we show a useful inequality involving $K_{(f,p)}$ for any $(f,p) \in M^+$.

Lemma **2.9** *Suppose (f,p) is an incentive compatible and individually rational mechanism. If $(f,p) \in M^+$, then for all types $u \in V$ with $B < p(u)$, we must have*

$$K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) \geq K_{(f,p)} f(u) - p(u).$$

Proof: First, consider two types $v \equiv (K_{(f,p)}, 0)$ and $v' \equiv (K_{(f,p)}, K_{(f,p)} - \epsilon)$, where $\epsilon > 0$ such that $K_{(f,p)} - \epsilon > 0$. Notice that $\min(v_1, v_2) < K_{(f,p)}$ and $\min(v'_1, v'_2) < K_{(f,p)}$. Hence, by Lemma 2.8, $p(v) \leq B$ and $p(v') \leq B$. As a result incentive constraints $v \to v'$ and $v' \to v$ imply that

$$\begin{aligned} K_{(f,p)} f(v) - p(v) &\geq K_{(f,p)} f(v') - p(v') \\ K_{(f,p)} f(v') - p(v') &\geq K_{(f,p)} f(v) - p(v). \end{aligned}$$

This gives us

$$K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) = K_{(f,p)} f(K_{(f,p)}, K_{(f,p)} - \epsilon) - p(K_{(f,p)}, K_{(f,p)} - \epsilon). \tag{2.6}$$

Now, assume for contradiction that for some u with $p(u) > B$ we have

$$K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) < K_{(f,p)} f(u) - p(u).$$

We can choose an $\epsilon > 0$ but arbitrarily close to zero such that

$$K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) < \big(K_{(f,p)} - \epsilon\big) f(u) - p(u).$$

Using Equation 2.6, we get,

$$K_{(f,p)}f(K_{(f,p)},K_{(f,p)}-\epsilon)-p(K_{(f,p)},K_{(f,p)}-\epsilon)<\big(K_{(f,p)}-\epsilon\big)f(u)-p(u).$$

But then

$$\begin{aligned}(K_{(f,p)}-\epsilon)f(K_{(f,p)},K_{(f,p)}-\epsilon)&-p(K_{(f,p)},K_{(f,p)}-\epsilon)\\ &<K_{(f,p)}f(K_{(f,p)},K_{(f,p)}-\epsilon)-p(K_{(f,p)},K_{(f,p)}-\epsilon)\\ &<\big(K_{(f,p)}-\epsilon\big)f(u)-p(u)<K_{(f,p)}f(u)-p(u).\end{aligned}$$

Hence, the incentive constraint $(K_{(f,p)},K_{(f,p)}-\epsilon)\to u$ does not hold - a contradiction. ■

LEMMA **2.10** *Suppose $(f,p)\in M^+$ is an incentive compatible and individually rational mechanism. Then, for any $\gamma\in(K_{(f,p)},\beta]$, the following limits exist:*

$$\begin{aligned}\lim_{\delta\to0^+}f(K_{(f,p)}+\delta,\gamma)&=A_{(f,p),\gamma}\\ \lim_{\delta\to0^+}p(K_{(f,p)}+\delta,\gamma)&=P_{(f,p),\gamma}.\end{aligned}$$

Further, the following equations hold:

$$K_{(f,p)}A_{(f,p),\gamma}-P_{(f,p),\gamma}=K_{(f,p)}f(K_{(f,p)},0)-p(K_{(f,p)},0)\tag{2.7}$$

$$\gamma A_{(f,p),\gamma}-P_{(f,p),\gamma}=\gamma f(\beta,\gamma)-p(\beta,\gamma).\tag{2.8}$$

Proof: Fix any $\gamma\in(K_{(f,p)},\beta]$ and any $\delta>0$ such that $K_{(f,p)}+\delta\leq\beta$ - by Lemma 2.8, such $\delta>0$ exists. Consider two types $v\equiv(K_{(f,p)}+\delta,\gamma)$ and $v'\equiv(\beta,\gamma)$. By Lemma 2.8, $p(v),p(v')>B$. The pair of incentive constraints between v and v' gives us

$$\begin{aligned}\gamma f(v)-p(v)&\geq\gamma f(v')-p(v')\\ \gamma f(v')-p(v')&\geq\gamma f(v)-p(v).\end{aligned}$$

Combining these and using the definition of v', we get

$$\gamma f(v)-p(v)=\gamma f(\beta,\gamma)-p(\beta,\gamma).\tag{2.9}$$

Now, consider $v''\equiv(K_{(f,p)},0)$. By Lemma 2.8, $p(v'')\leq B$. But $p(v)>B$ implies that incentive constraint $v\to v''$ must imply

$$\begin{aligned}(K_{(f,p)}+\delta)f(v)-p(v)&\geq(K_{(f,p)}+\delta)f(v'')-p(v'')\\ &\geq K_{(f,p)}f(v)-p(v)+\delta f(v''),\end{aligned}$$

where the second inequality comes from Lemma 2.9 and the fact that $p(v) > B$. Using Equation 2.9, we replace $p(v)$ in the previous equation to get,

$$\begin{aligned}(K_{(f,p)} + \delta)f(v) &\geq (K_{(f,p)} + \delta)f(v'') - p(v'') + \gamma f(v) - \gamma f(\beta,\gamma) + p(\beta,\gamma)\\ &\geq K_{(f,p)}f(v) + \delta f(v'')\end{aligned}$$

Rearranging terms, we get

$$\begin{aligned}\Big[\gamma - K_{(f,p)}\Big]f(v) &\leq \Big[\gamma f(\beta,\gamma) - p(\beta,\gamma)\Big] - \Big[K_{(f,p)}f(v'') - p(v'')\Big]\\ &\leq \Big[\gamma - K_{(f,p)}\Big]f(v) + \delta\Big[f(v'') - f(v)\Big]\end{aligned}$$

Since $v'' \equiv (K_{(f,p)}, 0)$ is independent of δ and $v \equiv (K_{(f,p)} + \delta, \gamma)$, we get that

$$\Big[\gamma - K_{(f,p)}\Big]\lim_{\delta\to 0^+} f(K_{(f,p)} + \delta, \gamma) = \Big[\gamma f(\beta,\gamma) - p(\beta,\gamma)\Big] - \Big[K_{(f,p)}f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0)\Big].$$

This gives us the desired expression for $A_{(f,p),\gamma}$. Using Equation 2.9, we also get the desired expression for $P_{(f,p),\gamma}$.

Then, it is routine to check that Equations (2.7) and (2.8) hold. ■

LEMMA **2.11** *Suppose $(f,p) \in M^+$ is an incentive compatible and individually rational mechanism. For every $\delta \in (0, \beta - K_{(f,p)}]$ and $\gamma \in (K_{(f,p)}, \beta]$, the following is true:*

1. $f(K_{(f,p)} + \delta, \gamma) \geq A_{(f,p),\gamma}$,
2. $p(K_{(f,p)} + \delta, \gamma) \geq P_{(f,p),\gamma}$.

Proof: Fix any $\delta \in (0, \beta - K_{(f,p)}]$ and $\gamma \in (K_{(f,p)}, \beta]$ and let $v \equiv (K_{(f,p)} + \delta, \gamma)$. By Lemma 2.8, we know that $p(v) > B$. Then Lemma 2.9 applies and we must have,

$$K_{(f,p)}f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) \geq K_{(f,p)}f(v) - p(v).$$

Equation 2.7 then directly implies,

$$K_{(f,p)}A_{(f,p),\gamma} - P_{(f,p),\gamma} \geq K_{(f,p)}f(v) - p(v).$$

Combining Equations (2.8) and (2.9) yields,

$$\gamma A_{(f,p),\gamma} - P_{(f,p),\gamma} = \gamma f(v) - p(v).$$

Combining the above two expressions gives us

$$K_{(f,p)}\Big(A_{(f,p),\gamma} - f(v)\Big) \geq P_{(f,p),\gamma} - p(v) = \gamma\Big(A_{(f,p),\gamma} - f(v)\Big).$$

Since $\gamma > K_{(f,p)}$, we get $A_{(f,p),\gamma} \leq f(v)$, which further implies $P_{(f,p),\gamma} \leq p(v)$. This gives us the desired results. ■

Lemma **2.12** *Suppose $(f,p) \in M^+$ is an incentive compatible and individually rational mechanism. For every $\gamma_1, \gamma_2 \in (K_{(f,p)}, \beta]$,*

$$A_{(f,p),\gamma_1} = A_{(f,p),\gamma_2}$$
$$P_{(f,p),\gamma_1} = P_{(f,p),\gamma_2}.$$

Proof: Fix any $\gamma_1, \gamma_2 \in (K_{(f,p)}, \beta]$. First, we note that Equation 2.7 implies

$$K_{(f,p)}A_{(f,p),\gamma_1} - P_{(f,p),\gamma_1} = K_{(f,p)}A_{(f,p),\gamma_2} - P_{(f,p),\gamma_2}. \tag{2.10}$$

Assume for contradiction that $A_{(f,p),\gamma_1} < A_{(f,p),\gamma_2}$, which implies that $P_{(f,p),\gamma_1} < P_{(f,p),\gamma_2}$. Then Equation 2.10 combined with the fact that $K_{(f,p)} < \gamma_1$ implies

$$\gamma_1 A_{(f,p),\gamma_1} - P_{(f,p),\gamma_1} < \gamma_1 A_{(f,p),\gamma_2} - P_{(f,p),\gamma_2}.$$

Let $\Delta > 0$ be defined by the equation

$$\Delta = \big[\gamma_1\big(A_{(f,p),\gamma_2} - A_{(f,p),\gamma_1}\big)\big] - \big[P_{(f,p),\gamma_2} - P_{(f,p),\gamma_1}\big]. \tag{2.11}$$

Fix some $\delta > 0$ be such that the following inequality holds

$$p(K_{(f,p)} + \delta, \gamma_2) - P_{(f,p),\gamma_2} < \Delta.$$

Existence of such a δ is guaranteed by the definition of $P_{(f,p),\gamma_2}$. Lemma 2.11 implies that

$$0 \le \gamma_1\big(f(K_{(f,p)} + \delta, \gamma_2) - A_{(f,p),\gamma_2}\big).$$

Adding above two inequalities we arrive at

$$\gamma_1 A_{(f,p),\gamma_2} - P_{(f,p),\gamma_2} < \Delta + \gamma_1 f(K_{(f,p)} + \delta, \gamma_2) - p(K_{(f,p)} + \delta, \gamma_2).$$

Substituting Δ from Equation 2.11 we get

$$\gamma_1 A_{(f,p),\gamma_1} - P_{(f,p),\gamma_1} < \gamma_1 f(K_{(f,p)} + \delta, \gamma_2) - p(K_{(f,p)} + \delta, \gamma_2).$$

Combining this with Equation 2.8 we get

$$\gamma_1 f(\beta, \gamma_1) - p(\beta, \gamma_1) < \gamma_1 f(K_{(f,p)} + \delta, \gamma_2) - p(K_{(f,p)} + \delta, \gamma_2).$$

By Lemma 2.8, we know that $p(\beta, \gamma_1) > B$ and $p(K_{(f,p)} + \delta, \gamma_2) > B$. Then, the above inequality implies that the incentive constraint $(\beta, \gamma_1) \to (K_{(f,p)} + \delta, \gamma_2)$ does not hold, which is a contradiction. ∎

In light of Lemma 2.12, for every incentive compatible and individually rational mechanism (f,p) in M^+ we denote $A_{(f,p),\gamma}$ and $P_{(f,p),\gamma}$ defined in the Lemma 2.10 by $A_{(f,p)}$ and $P_{(f,p)}$, i.e., we drop the subscript γ.

A structure lemma for M^+ mechanisms

The following lemma identifies an important structure of incentive compatible and individually rational mechanisms in M^+.

LEMMA **2.13** *Suppose $(f,p) \in M^+$ is an incentive compatible and individually rational mechanism. Then the following are true.*

1. *$p(u) = P_{(f,p)}$ and $f(u) = A_{(f,p)}$, for all u with $u_2 \in (K_{(f,p)}, \beta)$ and $u_1 > K_{(f,p)}$.*
2. *$P_{(f,p)} > B$.*
3. *$A_{(f,p)} > f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\Big[B - p(K_{(f,p)}, 0)\Big]$.*

Proof: PROOF OF (1). Consider a type $(K_{(f,p)} + \delta, \beta)$ for some $\delta > 0$ but close to zero. By Lemma 2.8, we know that $p(K_{(f,p)} + \delta, \beta) > B$. Now, choose any u with $u_2 \in (K_{(f,p)}, \beta)$ and $u_1 > K_{(f,p)}$. By Lemma 2.8, we have $p(u) > B$. By Lemma 2.4, we get $f(K_{(f,p)}+\delta, \beta) \geq f(u)$. Now, the incentive constraint $u \to (K_{(f,p)} + \delta, \beta)$ implies

$$\begin{aligned} u_2 f(u) - p(u) &\geq u_2 f(K_{(f,p)} + \delta, \beta) - p(K_{(f,p)} + \delta, \beta) \\ \Rightarrow p(K_{(f,p)} + \delta, \beta) - p(u) &\geq u_2\Big[f(K_{(f,p)} + \delta, \beta) - f(u)\Big] \geq 0. \end{aligned}$$

Since this holds for all $\delta > 0$ but arbitrarily close to zero,

$$P_{(f,p)} = \lim_{\delta \to 0^+} p(K_{(f,p)} + \delta, \beta) \geq p(u).$$

Now, applying Lemmas 2.11 and 2.12, we have

$$P_{(f,p)} \leq p(u).$$

The above two inequalities give us $p(u) = P_{(f,p)}$. Then, using Equations (2.8) and (2.9) give us $f(u) = A_{(f,p)}$.

PROOF OF (2). By Lemma 2.8, for all u with $u_2 \in (K_{(f,p)}, \beta)$ and $u_1 > K_{(f,p)}$, we have $p(u) > B$. By (1), the result then follows.

Proof of (3). Assume for contradiction that

$$A_{(f,p)} \leq f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\Big[B - p(K_{(f,p)}, 0)\Big].$$
$$\Leftrightarrow K_{(f,p)}A_{(f,p)} - B \leq K_{(f,p)}f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0).$$

Using the expression of $A_{(f,p)}$ and $P_{(f,p)}$ in Lemma 2.10, we get that

$$K_{(f,p)}f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) = K_{(f,p)}A_{(f,p)} - P_{(f,p)}.$$

Substituting this above, we get $P_{(f,p)} \leq B$. This contradicts (2) above. ∎

Lemma 2.13 shows that how certain regions in the type space look like for any incentive compatible and individually rational mechanism (f, p). This is shown in Figure 2.4.

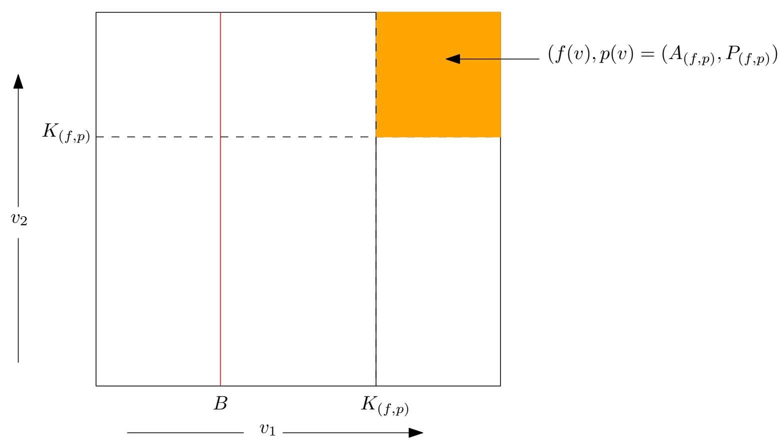

Figure 2.4: Implication of Lemma 2.13

Notice that Lemma 2.13 is silent about the outcome of the mechanism for types v with $v_1 > K_{(f,p)}$ and $v_2 = \beta$.

Reduction of space of M^+ mechanisms: implications of optimality

The next lemma shows that it is without loss of generality to make the outcomes for those types also $(A_{(f,p)}, P_{(f,p)})$.

Lemma **2.14** *Suppose $(f, p) \in M^+$ is an incentive compatible and individually rational mechanism. Then, there is another incentive compatible and individually rational mechanism (f', p') such that*

$$(f'(v), p'(v)) = \begin{cases} (A_{(f,p)}, P_{(f,p)}) & \text{if } v_1 > K_{(f,p)} \text{ and } v_2 = \beta \\ (f(v), p(v)) & \text{otherwise.} \end{cases}$$

and

$$p'(v) \geq p(v) \textit{ for almost all } v.$$

Proof: By Lemma 2.13, the only difference between the mechanisms (f', p') and (f, p) is at v with $v_1 > K_{(f,p)}$ and $v_2 = \beta$ with $\beta > K_{(f,p)}$ (see (2) in Lemma 2.8). Also, such a modification changes the outcome at these types to $(A_{(f,p)}, P_{(f,p)})$ which is already in the menu of outcomes in the original mechanism (f, p). Hence, the only possibility of a manipulation in (f', p') is for type (v_1, β) with $v_1 > K_{(f,p)}$ to report another type v' to get $(f(v'), p(v')) \neq (A_{(f,p)}, P_{(f,p)})$. This manipulation is possible if $p(v') \leq B$ and

$$v_1 f(v') - p(v') > v_1 A_{(f,p)} - P_{(f,p)}$$

or $p(v') > B$ and

$$\beta f(v') - p(v') > \beta A_{(f,p)} - P_{(f,p)}.$$

Now, consider a type u such that $u_1 = v_1$ and $u_2 = \beta - \epsilon$ for small enough $\epsilon > 0$. Note that $(f(u), p(u)) = (f'(u), p'(u)) = (A_{(f,p)}, P_{(f,p)})$ by Lemma 2.13. Since $\epsilon > 0$ is small enough, this implies that one of the above constraints must hold for type u too, which further implies that type u can manipulate the mechanism (f, p). This is a contradiction.

Since $p'(0, 0) = p(0, 0) = 0$, individual rationality follows from Lemma 2.1. Since (f', p') is a modification of (f, p) at measure zero profiles, $p'(v) \geq p(v)$ for almost all v. ■

Lemma 2.14 has a straightforward implication - we can assume without loss of generality that the top (and right) boundary of the upper rectangle in Figure 2.4 is assigned outcome $(A_{(f,p)}, P_{(f,p)})$. This simplifies our analysis. Using Lemmas 2.13 and 2.14, we assume that every incentive compatible and individually rational mechanism $(f, p) \in M^+$ has the feature that for all v with $\min(v_1, v_2) > K_{(f,p)}$, we have $((f(v), p(v)) = (A_{(f,p)}, K_{(f,p)})$.

Next, we will look at a subclass of mechanisms which fixes some other regions of the type space. Further, we will show that such a restriction is also without loss of generality for optimal mechanisms. To show this property, we consider an arbitrary incentive compatible and individually rational mechanism $(f, p) \in M^+$. We then construct a new incentive compatible and individually rational mechanism which generates more expected revenue and has the property we require. The new mechanism, which we denote as (f', p') is defined as follows.

$$(f'(v), p'(v)) = \begin{cases} (f(v), p(v)) & \text{if } v_1 < K_{(f,p)} \text{ or } \min(v_1, v_2) > K_{(f,p)} \\ \Big(f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big), B\Big) & \text{otherwise} \end{cases}$$

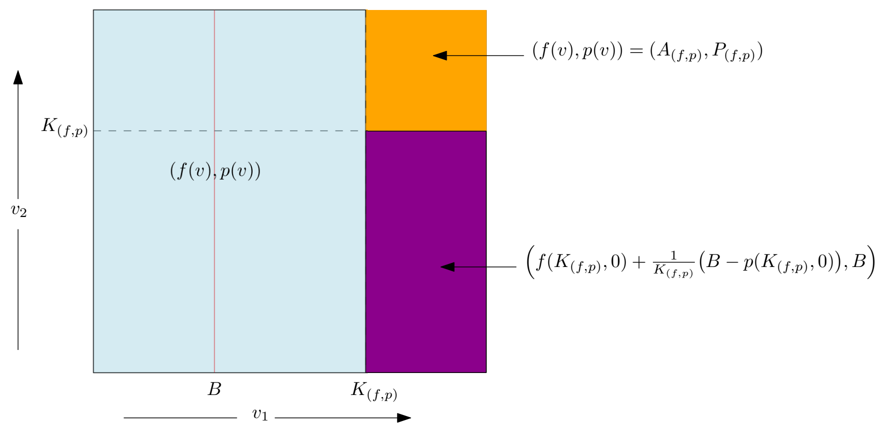

Figure 2.5: New mechanism

The new mechanism is shown in Figure 2.5. The rectangle at the top-right corner of the type space (excluding the lower boundaries) continues to have the outcome $(A_{(f,p)}, P_{(f,p)})$ - by Lemma 2.13, this is the same outcome as in the original mechanism (f, p). The outcomes in the big white rectangle to the left (but excluding the right boundary) is left unchanged. Note that $v_1 < K_{(f,p)}$ implies $p'(v) = p(v) \leq B$ by Lemma 2.8 in this region. The outcomes along the vertical line corresponding to $K_{(f,p)}$ value of the agent and the outcomes for all types such that $v_1 > K_{(f,p)}$ and $v_2 \leq K_{(f,p)}$ is assigned value

$$\left(f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big), B\right)$$

We prove the following.

Lemma **2.15** *If $(f, p) \in M^+$ is an incentive compatible and individually rational mechanism, then the mechanism (f', p') is incentive compatible, individually rational, and*

$$p'(v) \geq p(v) \text{ for almost all } v.$$

Proof: As stated earlier, we assume $(f, p) \in M^+$ is an incentive compatible and individually rational mechanism such that $(f(v), p(v)) = (A_{(f,p)}, P_{(f,p)})$ for all v with $\min(v_1, v_2) > K_{(f,p)}$. Since $p(0, 0) = p'(0, 0)$ and (f, p) is individually rational, Lemma 2.1 implies that (f', p') is also individually rational if we can show that (f', p') is incentive compatible. First, we establish that $p'(v) \geq p(v)$ for **almost** all $v \in V$. To see this, first observe that $p(v)$ and $p'(v)$ may be unequal *only* when v belongs to the following set of types:

$$\tilde{V} := \{v : v_1 \geq K_{(f,p)} \text{ and } \min(v_1, v_2) \leq K_{(f,p)}\}.$$

Now, consider the set of types $\bar{V} := \{v : \big(v_1 > K_{(f,p)}, v_2 \leq K_{(f,p)}\big) \text{ or } v_1 = K_{(f,p)}\}$. For each $v \in \bar{V}$, we have $p'(v) = B$ and $p(v) \leq B$ (due to Lemma 2.8). The set of types $\tilde{V} \setminus \bar{V}$ forms a set of measure zero. So, for almost all v, we have $p'(v) \geq p(v)$.

For incentive compatibility, we consider a partition of the type space as follows:

$$\begin{aligned}
V^1 &:= \{v : \min(v_1, v_2) > K_{(f,p)}\} \\
V^2 &:= \{v : v_1 < K_{(f,p)}\} \\
V^3 &:= (V \times V) \setminus (V^1 \cup V^2).
\end{aligned}$$

For any $v, v' \in V^1 \cup V^2$, we have $(f'(v), p'(v)) = (f(v), p(v))$ and $(f'(v'), p'(v')) = (f(v'), p(v'))$. Since (f, p) is incentive compatible, the incentive constraints $v \to v'$ and $v' \to v$ hold. For any $v, v' \in V^3$, we have $(f'(v), p'(v)) = (f'(v'), p'(v'))$. Hence, the incentive constraints $v \to v'$ and $v' \to v$ hold.

Hence, we pick $u \in V^1, s \in V^2, t \in V^3$, and verify the incentive constraints

$$s \to t, t \to s, t \to u, u \to t.$$

1. $s \to t$. Note that $p(K_{(f,p)}, 0) \leq B$ and since $p(s) \leq B$, incentive constraint $s \to (K_{(f,p)}, 0)$ in (f, p) implies that

$$\begin{aligned}
s_1 f(s) - p(s) &\geq s_1 f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) \\
&\geq s_1 f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) - \Big[B - p(K_{(f,p)}, 0)\Big]\Big(1 - \frac{s_1}{K_{(f,p)}}\Big),
\end{aligned}$$

   where the inequality follows because $p(K_{(f,p)}, 0) \leq B$ and $s_1 < K_{(f,p)}$. Using $f(s) = f'(s)$, $p(s) = p'(s)$, and a slight rearrangement of RHS of the above inequality gives us

$$\begin{aligned}
s_1 f'(s) - p'(s) &\geq s_1 \Big[f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big)\Big] - B \\
&= s_1 f'(t) - p'(t).
\end{aligned}$$

   Hence, the incentive constraint $s \to t$ holds for (f', p').

2. $t \to s$. Since $p(s) \leq B$, incentive constraint $(K_{(f,p)}, 0) \to s$ in (f, p) implies that

$$\begin{aligned}
&K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) \geq K_{(f,p)} f(s) - p(s) \\
\Rightarrow\ &K_{(f,p)}\Big[f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big)\Big] - B \geq K_{(f,p)} f(s) - p(s) \\
\Rightarrow\ &K_{(f,p)} f'(t) - p'(t) \geq K_{(f,p)} f'(s) - p'(s).
\end{aligned}$$

This implies that

$$K_{(f,p)}\Big[f'(t) - f'(s)\Big] \geq p'(t) - p'(s).$$

But $p'(t) = B \geq p'(s) = p(s)$ implies that $f'(t) \geq f'(s)$. Using the fact that $t_1 \geq K_{(f,p)}$, we get

$$t_1\Big[f'(t) - f'(s)\Big] \geq p'(t) - p'(s),$$

Since $p'(t) = B$ and $p'(s) \leq B$, this is the desired incentive constraint $t \to s$ in (f', p').

3. $t \to u$, $u \to t$. By Lemma 2.10, we know that

$$\begin{aligned} K_{(f,p)}f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) &= K_{(f,p)}A_{(f,p)} - P_{(f,p)} \\ \Leftrightarrow K_{(f,p)}\Big[f(K_{(f,p)}, 0) - \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big)\Big] - B &= K_{(f,p)}A_{(f,p)} - P_{(f,p)}. \end{aligned}$$

Hence, we get

$$K_{(f,p)}\Big[f'(u) - f'(t)\Big] = p'(u) - p'(t). \tag{2.12}$$

Using Lemma 2.13, $p'(u) = p(u) = P_{(f,p)} > p'(t) = B$. Hence, Equation 2.12 implies that $f'(u) > f'(t)$. Using $\min(u_1, u_2) > K_{(f,p)}$, we get

$$\begin{aligned} u_1 f'(u) - p'(u) &\geq u_1 f'(t) - p'(t) \\ u_2 f'(u) - p'(u) &\geq u_2 f'(t) - p'(t). \end{aligned}$$

Hence, the incentive constraint $u \to t$ holds in (f', p').

Similarly, we now use the fact that $\min(t_1, t_2) \leq K_{(f,p)}$. If $\min(t_1, t_2) = t_1$, then using Equation 2.12, we get

$$t_1 f'(t) - p'(t) \geq t_1 f'(u) - p'(u).$$

Else, $\min(t_1, t_2) = t_2$, in which case again, we get

$$t_2 f'(t) - p'(t) \geq t_2 f'(u) - p'(u).$$

So, one of the above constraints must hold. Since $p'(t) = B$ and $p'(u) > B$, this ensures that the incentive constraint $t \to u$ holds in (f', p').

∎

Ironing Lemmas

The final Lemma before we start ironing, further simplifies the class of mechanisms that we need to consider for optimal mechanism design.

Lemma **2.16** *Suppose $(f, p) \in M^+$ is an incentive compatible and individually rational mechanism. Then, there exists another mechanism $(\hat{f}, \hat{p})$ such that*

1. $(\hat{f}(v), \hat{p}(v)) = (f(v), p(v))$ *for all v with $v_1 \geq K_{(f,p)}$,*
2. $(\hat{f}(v), \hat{p}(v)) = (\hat{f}(u), \hat{p}(u))$ *for all u, v with $u_1 = v_1 < K_{(f,p)}$,*
3. $\hat{p}(u) \geq p(u)$ *for all u,*
4. $\hat{p}(0, 0) = p(0, 0)$,
5. *incentive constraints $u \to v$ for every u, v with $\hat{p}(u), \hat{p}(v) \leq B$ hold in $(\hat{f}, \hat{p})$.*

Proof: Consider an incentive compatible and individually rational mechanism (f, p), and let $K_{(f,p)}$ be as defined in Lemma 2.8. We complete the proof in two steps.

Step 1. In this step, we show some implications of incentive constraints $u \to v$, where $u_1, v_1 < K_{(f,p)}$. Consider any $(u_1, u_2), (u_1, u_2')$ such that $u_1 < K_{(f,p)}$. Then, by Lemma 2.8, we have $p(u_1, u_2) \leq B$ and $p(u_1, u_2') \leq B$. Hence, the relevant pair of incentive constraints give us:

$$u_1 f(u_1, u_2) - p(u_1, u_2) \geq u_1 f(u_1, u_2') - p(u_1, u_2')$$
$$u_1 f(u_1, u_2') - p(u_1, u_2') \geq u_1 f(u_1, u_2) - p(u_1, u_2).$$

This gives us

$$u_1 f(u_1, u_2) - p(u_1, u_2) = u_1 f(u_1, u_2') - p(u_1, u_2'). \tag{2.13}$$

Also, notice that Equation 2.13 implies that for all $u_2 \in [0, \beta]$,

$$p(0, u_2) = p(0, 0) \tag{2.14}$$

Finally, since only incentive constraints corresponding to agent's value are relevant in this region, revenue equivalence formula implies that for every $u_1 < K_{(f,p)}$ and $u_2, u_2' \in [0, \beta]$, we have

$$u_1 f(u_1, u_2) - p(u_1, u_2) = \int_0^{u_1} f(x, u_2)dx - p(0, u_1) = \int_0^{u_1} f(x, u_2)dx - p(0, 0)$$

$$u_1 f(u_1,u_2') - p(u_1,u_2') = \int_0^{u_1} f(x,u_2')dx - p(0,u_1) = \int_0^{u_1} f(x,u_2')dx - p(0,0)$$

Using Equation 2.13, we get

$$\int_0^{u_1} f(x,u_2)dx = \int_0^{u_1} f(x,u_2')dx.$$

Hence, we can write for every $u_1 < K_{(f,p)}$ and every $u_2 \in [0,\beta]$,

$$u_1 f(u_1,u_2) - p(u_1,u_2) = \int_0^{u_1} f(x,0)dx - p(0,0). \tag{2.15}$$

Notice that the RHS of the above equation is independent of u_2. Denoting the RHS of the above equation as $\mathcal{U}^{(f,p)}(u_1)$, we see that

$$u_1 \sup_{u_2\in[0,\beta]} f(u_1,u_2) = \sup_{u_2\in[0,\beta]} p(u_1,u_2) + \mathcal{U}^{(f,p)}(u_1). \tag{2.16}$$

Notice that f and p are bounded from above (p is bounded from above because $p(u_1,u_2) \le B$ for each $u_2 \in [0,\beta]$ due to Lemma 2.8). As a result, the supremums in the above equation exist. We denote this supremums as follows:

$$\alpha(u_1) := \sup_{u_2\in[0,\beta]} f(u_1,u_2)\ \forall\ u_1 < K_{(f,p)} \tag{2.17}$$

$$\pi(u_1) := \sup_{u_2\in[0,\beta]} p(u_1,u_2)\ \forall\ u_1 < K_{(f,p)}. \tag{2.18}$$

We use these to define our new mechanism in the next step.

Step 2. Now, we define the following mechanism $(\hat{f},\hat{p})$. For every v with $v_1 \ge K_{(f,p)}$, we have $(\hat{f}(v),\hat{p}(v)) = (f(v),p(v))$. For all v with $v_1 < K_{(f,p)}$, we define

$$\hat{f}(v) := \alpha(v_1); \hat{p}(v) := \pi(v_1).$$

By definition of $\hat{p}$, it is clear that $\hat{p}(v) \ge p(v)$ for all v. Also, Equation 2.14 ensures that $\hat{p}(0,0) = \pi(0) = p(0,0)$. Hence, (1), (2), (3), (4) hold for $(\hat{f},\hat{p})$.

For (5), assume for contradiction that the incentive constraint $u \to v$ in $(\hat{f},\hat{p})$ does not hold for some u, v with $\hat{p}(u), \hat{p}(v) \le B$. So, the violation of incentive constraint must happen for value of the agent. Note that by definition of $\hat{p}$, we must have $p(u) \le B$ and $p(v) \le B$. Also, incentive constraints cannot be violated if $u_1, v_1 \ge K_{(f,p)}$ since (f,p) is incentive compatible and $(\hat{f}(u),\hat{p}(u)) = (f(u),p(u))$ and $(\hat{f}(v),\hat{p}(v)) = (f(v),p(v))$. The other possibilities are analyzed below.

CASE 1. $u_1, v_1 < K_{(f,p)}$. In that case, we must have

$$u_1\alpha(u_1) - \pi(u_1) < u_1\alpha(v_1) - \pi(v_1) = (u_1 - v_1)\alpha(v_1) + v_1\alpha(v_1) - \pi(v_1).$$

Using Equation (2.16), we get that

$$\mathcal{U}^f(u_1) < \mathcal{U}^f(v_1) + (u_1 - v_1)\alpha(v_1).$$

By definition, there exists, $y \in [0, \beta]$ such that $\alpha(v_1)$ is arbitrarily close to $f(v_1, y)$. Using Equation (2.15) gives us

$$u_1 f(u_1, y) - p(u_1, y) < v_1 f(v_1, y) - p(v_1, y) + (u_1 - v_1)f(v_1, y) = u_1 f(v_1, y) - p(v_1, y).$$

This contradicts incentive compatibility of (f, p).

CASE 2. $u_1 < K_{(f,p)}$ and $v_1 \geq K_{(f,p)}$. In that case, we must have

$$u_1\alpha(u_1) - \pi(u_1) < u_1 f(v) - p(v).$$

But using Equations (2.15) and (2.16), we see that there is some y such that

$$u_1 f(u_1, y) - p(u_1, y) < u_1 f(v) - p(v)$$

which contradicts incentive compatibility of (f, p).

CASE 3. $u_1 \geq K_{(f,p)}$ and $v_1 < K_{(f,p)}$. In that case, we must have

$$u_1 f(u) - p(u) < u_1\alpha(v_1) - \pi(v_1) = (u_1 - v_1)\alpha(v_1) + \mathcal{U}^f(v_1).$$

Now, pick y such that $\alpha(v_1)$ is arbitrarily close to $f(v_1, y)$. By Equations (2.15) and (2.16), we get

$$u_1 f(u) - p(u) < (u_1 - v_1)f(v_1, y) + v_1 f(v_1, y) - p(v_1, y) = u_1 f(v_1, y) - p(v_1, y).$$

This contradicts incentive compatibility of (f, p) and completes the proof. ■

DEFINITION **2.8** *We call a mechanism (f, p)* **simple** *if there exists $K, A, \hat{A}, P$ with $K \in (0, B)$, $P \in (B, \beta]$, $A, \hat{A} \in [0, 1]$, $A > \hat{A}$ such that*

1. $p(0, 0) \leq 0$.

2. $K(A - \hat{A}) = P - B$ with $KA - P \geq 0$.

3. $(f(v), p(v)) = (A, P)$ for all v with $\min(v_1, v_2) > K$,

4. $p(v) \leq B$ for all v with $v_1 < K$.

5. $(f(v), p(v)) = (\hat{A}, B)$ for all v with $\min(v_1, v_2) \leq K$ and $v_1 \geq K$.

6. $(f(v), p(v)) = (f(v'), p(v'))$ for all v, v' with $v_1 = v'_1 < K$.

7. incentive constraints $v \to v'$ hold for all types with $p(v), p(v') \leq B$.

Based on Lemmas 2.15 and 2.16, the following is a simple corollary.

Corollary **2.1** *If (f, p) is an optimal mechanism in M^+, then there is a simple mechanism $(\hat{f}, \hat{p})$ such that*

$$\textsc{Rev}(f, p) \leq \textsc{Rev}(\hat{f}, \hat{p}).$$

Proof: Suppose (f, p) is an optimal mechanism in M^+, then Lemma 2.15 says that there is another incentive compatible and individually rational mechanism (f', p') such that $\textsc{Rev}(f', p') \geq \textsc{Rev}(f, p)$. Using $K = K_{(f,p)}$, Lemma 2.16 shows that (f', p') satisfies all the properties of a simple mechanism. ■

Because of property (6), for any simple mechanism (f, p), we denote the allocation probability at any type v with $v_1 < K$ as simply $\alpha^f(v_1)$ and the payment as $\pi^p(v_1)$. We also denote by $\alpha^f(K) \equiv \hat{A}$ and $\pi^p(K) \equiv B$, where $\hat{A}$ is the parameter specified in the simple mechanism (f, p).

Lemma **2.17** *Suppose (f, p) is a simple mechanism with parameters $(K, A, \hat{A}, P)$. Then, the revenue from (f, p) is*

$$\textsc{Rev}(f, p) = G_1(K)\Big[B - K\alpha^f(K)\Big] + \int_0^K h(x)\alpha^f(x)dx$$
$$+ B(1 - G_1(K)) + K(A - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K, K)),$$

where $h(x) = xg_1(x) + G_1(x)$ for all $x \in [0, K]$.

Proof: Fix a simple mechanism with parameters $(K, A, \hat{A}, P)$. We divide the proof into two parts, where we compute revenue from two disjoint regions of the type space.

Region 1. Here, we consider all v such that $v_1 \leq K$. By properties (4) and (5) of the simple mechanism, payments in this region of type space is not more than B and by property (7), all the incentive constraints in this region hold. Using standard Myersonian techniques, it is easy to see that

$$\alpha^f(v_1) \geq \alpha^f(v_1') \qquad \forall\ v_1' < v_1 \leq K \tag{2.19}$$

$$\pi^p(v_1) = \pi^p(0) + v_1\alpha^f(v_1) - \int_0^{v_1} \alpha^f(x)dx \qquad \forall\ v_1 \leq K \tag{2.20}$$

Hence, the expected payment from this region is

$$\begin{aligned}
\int_0^K \pi^p(v_1)g_1(v_1)dv_1 &= \int_0^K \pi^p(0)g_1(v_1)dv_1 + \int_0^K v_1\alpha^f(v_1)g_1(v_1)dv_1 - \int_0^K \big(\int_0^{v_1} \alpha^f(x)dx\big)g_1(v_1)dv_1 \\
&= G_1(K)\pi^p(0) + \int_0^K v_1\alpha^f(v_1)g_1(v_1)dv_1 - \int_0^K \big((G_1(K) - G_1(v_1)\big)\alpha^f(v_1)dv_1 \\
&= G_1(K)\big[\pi^p(0) - \int_0^K \alpha^f(x)dx\big] + \int_0^K h(x)\alpha^f(x)dx \\
&= G_1(K)\big[\pi^p(K) - K\alpha^f(K)\big] + \int_0^K h(x)\alpha^f(x)dx \\
&= G_1(K)\big[B - K\alpha^f(K)\big] + \int_0^K h(x)\alpha^f(x)dx,
\end{aligned}$$

where the last but one equality follows from Equation 2.20 at $v_1 = K$ and the last equality follows from the fact $\pi^p(K) = B$.

Region 2. Finally, we consider all v such that $v_1 > K$. By definition, the expected revenue from this region is

$$\begin{aligned}
&B(1 - G_1(K)) + (P - B)(1 - G_1(K) - G_2(K) + G(K, K)) = \\
&\qquad B(1 - G_1(K)) + K(A - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K, K)),
\end{aligned}$$

where the equality follows from property (2) of simple mechanism.

Putting together the revenues from both the regions, we get the desired expression of the expected revenue from the simple mechanism. ■

We now prove that for every simple mechanism, there is a post-2 mechanism that generates as much expected revenue.

Lemma **2.18** *For every simple mechanism (f, p), there is a* post-2 *mechanism $(\bar{f}, \bar{p})$ such that*

$$\text{Rev}(\bar{f}, \bar{p}) \geq \text{Rev}(f, p).$$

Proof: Suppose (f, p) is a simple mechanism with parameters $(K, A, \hat{A}, P)$. Now, by property (5) of the simple mechanism, Equation 2.20 along with property (1) imply that

$$\pi^f(K) = B \leq K\alpha^f(K) - \int_0^K \alpha^f(x)dx. \tag{2.21}$$

Now, define a POST-2 mechanism by parameters:

$$K_1 := \frac{B}{\hat{A}} = \frac{B}{\alpha^f(K)}, \quad K_2 := K.$$

By property (1) of simple mechanism, we get that $K_1 = \frac{B}{\alpha^f(K)} \leq K_2 = K$. Also, $K_1 > B$. This means that the new mechanism is a well-defined POST-2 mechanism. Denote this mechanism as (f', p').

It is also easily verified that it is a simple mechanism: the parameters are

$$K' := K_2 = K; A' = 1; \hat{A}' := \hat{A} = \alpha^f(K); P' := B + K_2(1 - \frac{B}{K_1}) = B + K(1 - \alpha^f(K)),$$

and also note that every POST-2 mechanism is incentive compatible (Proposition 2.1). Note here that $\alpha^{f'}(K) = \alpha^f(K)$. Also, $\alpha^{f'}(x) = 0$ for all $x \leq K_1$ and $\alpha^{f'}(x) = \frac{B}{K_1} = \alpha^f(K)$ for all $x \in (K_1, K]$. Using these observations and Lemma 2.17,

$$\begin{aligned}
&\textsc{Rev}(f', p') - \textsc{Rev}(f, p) \\
&= \left(G_1(K)\Big[B - K\alpha^f(K)\Big] + \int_0^K h(x)\alpha^{f'}(x)dx + B(1 - G_1(K)) + \right. \\
&\left. K(1 - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K, K)) \right) \\
&- \left(G_1(K)\Big[B - K\alpha^f(K)\Big] + \int_0^K h(x)\alpha^f(x)dx + B(1 - G_1(K)) + \right. \\
&\left. K(A - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K, K)) \right) \\
&\geq \int_0^K h(x)\alpha^{f'}(x)dx - \int_0^K h(x)\alpha^f(x)dx \\
&\geq \int_{K_1}^K h(x)\big(\alpha^f(K) - \alpha^f(x)\big)dx - \int_0^{K_1} h(x)\alpha^f(x)dx. \\
&\geq (K - K_1)h(K_1)\alpha^f(K) - h(K_1)\int_{K_1}^K \alpha^f(x)dx - h(K_1)\int_0^{K_1} \alpha^f(x)dx \\
&\text{(using } h \text{ and } \alpha \text{ to be increasing functions)} \\
&= (K - K_1)h(K_1)\alpha^f(K) - h(K_1)\int_0^K \alpha^f(x)dx \\
&\geq h(K_1)(K - K_1)\alpha^f(K) - h(K_1)(K - K_1)\alpha^f(K) \\
&\text{(using Equation (2.21) and definition of } K_1\text{)} \\
&= 0.
\end{aligned}$$

■

Proof of Proposition 2.4

The proof of (2) in Proposition 2.4 now follows from Corollary 2.1 and Lemma 2.18. Proof of (1) in Proposition 2.4 is given below.

This requires to show that the optimal mechanism in M^- is a POST-1 mechanism. Every mechanism $(f, p) \in M^-$ satisfies the property that types satisfying $p(v) > B$ have zero measure. We first argue that it is without loss of generality to assume that $p(v) \leq B$ for *all* v. To see this, note that by (1) in Lemma 2.8 and the fact that $V^+(f, p)$ has zero measure, it must be that $K_{(f,p)} = \beta$. Let $\pi^p(\beta) := \sup_{v_2<\beta} p(\beta, v_2)$ and $\alpha^f(\beta) := \sup_{v_2<\beta} f(\beta, v_2)$. Observe that

$\alpha^p(\beta) \leq B$. Hence, we consider the following mechanism (f', p'): $(f'(v), p'(v)) = (f(v), p(v))$ if $v \notin V^+(f,p)$ and $(f'(v), p'(v)) = (\alpha^f(\beta), \pi^p(\beta))$ otherwise. By construction, the expected revenue of (f', p') is the same as (f, p) and $p'(v) \leq B$ for all v. Further, (f', p') is incentive compatible (we only need to worry about incentive constraints of types $v \in V^+(f,p)$, and they hold because for all v, $p'(v) \leq B$ implies we only need to check incentive constraints for value of agent, which holds due to an argument similar to that in Lemma 2.16(5)). Individual rationality of (f', p') follows from Lemma 2.1.

Now, we state an analogue of Lemma 2.16 for M^- class of mechanisms - the proof of this lemma is identical to that of Lemma 2.16, and is skipped.

Lemma **2.19** *Suppose $(f,p) \in M^-$ is an incentive compatible and individually rational mechanism. Then, there exists another mechanism $(\hat{f}, \hat{p})$ such that*

1. *$(\hat{f}(v), \hat{p}(v)) = (\hat{f}(u), \hat{p}(u))$ for all u, v with $u_1 = v_1$,*
2. *$\hat{p}(u) \geq p(u)$ for all u,*
3. *$\hat{p}(0,0) = p(0,0)$,*
4. *$(\hat{f}, \hat{p})$ is incentive compatible and individually rational.*

Using Lemma 2.19, we only focus on mechanisms satisfying the properties stated in Lemma 2.19. Let (f,p) be such a mechanism and define α^f and π^p as before, i.e., $\alpha^f(v_1) = f(v_1, v_2)$ and $\pi^p(v_1) = p(v_1, v_2)$ for all v with $v_1 < \beta$.

Hence, the expected revenue from a mechanism (f,p) given in Lemma 2.19 is given by

$$\begin{aligned}
\textsc{Rev}(f,p) &= p(0,0) + \int_0^\beta u_1 \alpha^f(u_1) g_1(u_1) du_1 - \int_0^\beta \Big(\int_0^{u_1} \alpha^f(x) dx \Big) g_1(u_1) du_1 \\
&= p(0,0) + \int_0^\beta x \alpha^f(x) g_1(x) dx - \int_0^\beta (1 - G_1(x)) \alpha^f(x) dx \\
&= p(0,0) + \int_0^\beta \Big[h(x) - 1 \Big] \alpha^f(x) dx.
\end{aligned}$$

We now construct another posted-price mechanism (f', p') that generates no less revenue than (f,p). The posted-price mechanism (f', p') is defined as follows. Let $K_1 := \frac{\pi^f(\beta)}{\alpha^f(\beta)}$. For all v with $v_1 \leq K_1$, we set

$$f'(v) = 0, p'(v) = 0$$

and for all v with $v_1 > K_1$, we set

$$f'(v) = \alpha^f(\beta), \quad p'(v) = K_1 \alpha^f(\beta) = \pi^p(\beta).$$

It is not difficult to see that (f', p') is individually rational and incentive compatible. The expected revenue from (f', p') is given by

$$\textsc{Rev}(f', p') = K_1 \alpha^f(\beta)(1 - G_1(K_1))$$

Now, note that

$$\alpha^f(\beta) \int_{K_1}^{\beta} \Big[h(x) - 1\Big] dx = \alpha^f(\beta)\Big(K_1 - K_1 G_1(K_1)\Big) = \textsc{Rev}(f', p').$$

So, we get

$$\begin{aligned}
\textsc{Rev}(f', p') - \textsc{Rev}(f, p) &= \left(\alpha^f(\beta) \int_{K_1}^{\beta} \Big[h(x) - 1\Big] dx\right) - \left(p(0,0) + \int_0^{\beta} \Big[h(x) - 1\Big] \alpha^f(x) dx\right) \\
&= \alpha^f(\beta) \int_{K_1}^{\beta} h(x) dx - \int_0^{\beta} h(x) \alpha^f(x) dx + \int_0^{\beta} \alpha^f(x) dx - (\beta - K_1)\alpha^f(\beta) - p(0,0) \\
&= \alpha^f(\beta) \int_{K_1}^{\beta} h(x) dx - \int_0^{\beta} h(x) \alpha^f(x) dx + \int_0^{\beta} \alpha^f(x) dx - \beta\alpha^f(\beta) - \pi^p(\beta) - p(0,0) \\
&\text{(Using definition of } K_1\text{)} \\
&= \alpha^f(\beta) \int_{K_1}^{\beta} h(x) dx - \int_0^{\beta} h(x) \alpha^f(x) dx \\
&\text{(Using revenue equivalence formula (Equation 2.20) at } \beta\text{)} \\
&= \int_{K_1}^{\beta} \Big[\alpha^f(\beta) - \alpha^f(x)\Big] h(x) dx - \int_0^{K_1} \alpha^f(x) h(x) dx \\
&\geq h(K_1) \int_{K_1}^{\beta} \Big[\alpha^f(\beta) - \alpha^f(x)\Big] dx - h(K_1) \int_0^{K_1} \alpha^f(x) dx \\
&\text{(since } h \text{ is increasing and } \alpha \text{ is non-decreasing)} \\
&= h(K_1)(\beta - K_1)\alpha^f(\beta) - h(K_1) \int_0^{\beta} \alpha^f(x) dx \\
&\geq h(K_1)(\beta - K_1)\alpha^f(\beta) - h(K_1)(\beta - K_1)\alpha^f(\beta) \\
&\text{(Using revenue equivalence formula (Equation 2.20) at } \beta \text{ and } p(0,0) \leq 0\text{)} \\
&= 0.
\end{aligned}$$

Hence, every optimal mechanism in M^- is a posted-price mechanism described in (f', p'). It is characterized by a posted-price K_1 and an allocation probability α if the value of the agent is above the posted price. The optimization program can be written as follows.

$$\max_{K_1,\alpha} K_1\alpha(1 - G_1(K_1))$$
$$\text{subject to}$$
$$K_1\alpha \leq B$$
$$\alpha \in [0, 1].$$

We argue that the optimal solution to this program must have $\alpha = 1$. To see this, let K^* be the unique solution to the following optimization

$$\max_{K_1\in[0,B]} K_1(1 - G_1(K_1)).$$

The fact that this optimization program has a unique solution follows from the fact that $x - xG_1(x)$ is strictly concave (since $xG_1(x)$ is strictly convex). Hence, the revenue from the solution when $\alpha = 1$ is $K^*(1 - G_1(K^*))$. Now, suppose the optimal solution has $\hat{K}$ and $\hat{\alpha}$. Note that the $\hat{K}\hat{\alpha} \leq B$. So, define $\tilde{K} = \hat{K}\hat{\alpha} \leq B$. By definition,

$$\begin{aligned} K^*(1 - G_1(K^*)) &\geq \tilde{K}(1 - G_1(\tilde{K})) \\ &= \hat{K}\hat{\alpha}(1 - G_1(\hat{K}\hat{\alpha})) \\ &\geq \hat{K}\hat{\alpha}(1 - G_1(\hat{K})), \end{aligned}$$

where the final inequality used the fact that $G_1(\hat{K}\hat{\alpha}) \leq G_1(\hat{K})$. This implies that the optimal solution must have $\alpha = 1$ and K_1 must be the unique solution to $K_1(1 - G_1(K_1))$ with the constraint $K_1 \in [0, B]$. Hence, the optimal solution in M^- must be a posted price mechanism, where the posted price is a unique solution to

$$\max_{K_1\in[0,B]} K_1(1 - G_1(K_1)).$$

Proof of Proposition 2.2

We now combine the optimal solutions in M^+ and M^- as follows. The optimal in M^- is a solution to

$$\max_{K_1\in[0,B]} K_1(1 - G_1(K_1)).$$

The optimal in M^+ is a solution to

$$\max_{K_2\in(B,\beta),K_1\in[B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

Notice that the optimization for M^+ does not admit $K_2 = B$. But if $K_2 = B$ and $K_1 \in [B, K]$, we must have $K_1 = B$ and then the objective function value reduces to $B(1 - G_1(B))$. This is the same objective function value of the program for M^- when $K_1 = B$. Similarly, if $K_2 = \beta$ is allowed in the optimization for M^+, we see that the objective function is maximized at $K_1 = B$ giving a value of $B(1 - G_1(B))$ to the objective function. Again, this is the same objective function value of the program for M^- when $K_1 = B$.

Summarizing these findings, we get that the expected revenue from the optimal mechanism is $\max(R_1, R_2)$, where

$$R_1 = \max_{K_1 \in [0,B]} K_1(1 - G_1(K_1))$$

$$R_2 = \max_{K_2 \in [B,\beta], K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

This proves Proposition 2.2.

2.9 Appendix: Proofs of Section 2.6

This appendix contains all omitted proofs of Section 2.6.

2.9.1 Proof of Proposition 2.6

We establish a stronger result. We show that a larger class mechanisms, which includes the POST* mechanism, is incentive compatible.

DEFINITION **2.9** *A mechanism (f, p) is a generalized* POST* *(*G-POST**) mechanism if there exists $K, P \in (0, \beta]$ and $A \in [0, 1]$ such that*

$$0 \leq A - \frac{P}{K} \leq 1 - \frac{B}{K}$$

and for all $(v, B) \in W$

$$(f(v,B), p(v,B)) = \begin{cases} (A - \frac{P}{K}, 0) & \text{if } v_1 \leq K \\ \Big(A, P\Big) & \text{if } \{\min(v_1, v_2) > K \text{ and } B < P\} \\ & \text{or } \{v_1 > K \text{ and } B \geq P\} \\ (A - \frac{P-B}{K}, B) & \text{if } v_1 > K,\ v_2 \leq K \text{ and } B < P \end{cases}$$

Note that if we put $A = 1, P = K$, we get a POST* mechanism. We prove the following proposition, which implies Proposition 2.6.

PROPOSITION **2.7** *Every* G-POST* *mechanism is manager non-trivial, incentive compatible, and individually rational.*

Proof: It is clear that a G-POST* mechanism is manager non-trivial. Individual rationality will follow from Lemma 2.1 once we show incentive compatibility. So, we show incentive compatibility below.

Fix a G-POST* mechanism (f, p) defined by parameters K, P, A. Partition the type space W into three regions:

$$\begin{aligned} W^1 &:= \{(u, B) : u_1 \leq K\}, \\ W^2 &:= \{(u, B) : \min(u_1, u_2) > K, B < P\} \cup \{(u, B) : u_1 > K, B \geq P\}, \\ W^3 &:= \{(u, B) : u_1 > K, u_2 \leq K, B < P\}. \end{aligned}$$

By definition, we have $(f(u, B), p(u, B)) = (f(u', B'), p(u', B'))$ if $(u, B), (u', B') \in W^1$ or $(u, B), (u', B') \in W^2$. Now, pick $(u, B), (u', B') \in W^3$ with $B < B'$. Notice that

$$K\Big[f(u, B) - f(u', B')\Big] = p(u, B) - p(u', B') = B - B' < 0.$$

This gives us $f(u,B) < f(u',B')$. Since, $u'_1 > K$, we get

$$u'_1\Big[f(u,B) - f(u',B')\Big] < p(u,B) - p(u',B'),$$

which implies that incentive constraint $(u',B') \to (u,B)$ holds for (f,p). Similarly, using $u_2 \le K$, we notice that

$$u_2\Big[f(u,B) - f(u',B')\Big] \ge p(u,B) - p(u',B').$$

Using $p(u',B') = B' > B$, the above inequality implies that incentive constraint $(u,B) \to (u',B')$ also holds for (f,p).

We now show incentive constraints hold across each pair of types in W^1, W^2, W^3. For this, pick $(u,B) \in W^1, (u',B') \in W^2, (u'',B'') \in W^3$. By definition, we have

$$Kf(u,B) - p(u,B) = Kf(u',B') - p(u',B') = Kf(u'',B'') - p(u'',B'') = KA - P. \quad (2.22)$$

Now, we consider three cases.

Case 1. $(u,B) \to (u',B')$ and $(u',B') \to (u,B)$. Using Equation (2.22), we get

$$K\Big[f(u,B) - f(u',B')\Big] = p(u,B) - p(u',B') = -P < 0.$$

Using $u_1 < K$, we get

$$u_1 f(u,B) - p(u,B) \ge u_1 f(u',B') - p(u',B').$$

This is enough for incentive constraint $(u,B) \to (u',B')$ since $p(u,B) = 0$.

Similarly, using $u'_1 > K$ implies

$$u'_1 f(u',B') - p(u',B') \ge u'_1 f(u,B) - p(u,B). \quad (2.23)$$

This is enough for incentive constraint $(u',B') \to (u,B)$ if $p(u',B') = P \le B'$. Else, $p(u',B') = P > B'$, which also means $\min(u'_1, u'_2) > K$. But this means, we also have

$$u'_2 f(u',B') - p(u',B') \ge u'_2 f(u,B) - p(u,B). \quad (2.24)$$

Inequalities (2.23) and (2.24) ensure that incentive constraint $(u',B') \to (u,B)$ holds.

Case 2. $(u',B') \to (u'',B'')$ and $(u'',B'') \to (u',B')$. Using Equation (2.22) and $B'' < P$, we get

$$K\Big[f(u',B') - f(u'',B'')\Big] = p(u',B') - p(u'',B'') = P - B'' > 0.$$

Since $u''_2 \leq K$, we get

$$u''_2 f(u'', B'') - p(u'', B'') \geq u''_2 f(u', B') - p(u', B').$$

This is enough for incentive constraint $(u'', B'') \to (u', B')$ to hold since $p'(u', B') = P > B''$.

Similarly, using $u'_1 > K$ implies

$$u'_1 f(u', B') - p(u', B') > u'_1 f(u'', B'') - p(u'', B''). \tag{2.25}$$

This is enough for incentive constraint $(u', B') \to (u'', B'')$ if $p(u', B') = P \leq B'$. Else, $p(u', B') = K > B'$, which also means $\min(u'_1, u'_2) > K$. But this means, we also have

$$u'_2 f(u', B') - p(u', B') > u'_2 f(u'', B'') - p(u'', B''). \tag{2.26}$$

Inequalities (2.25) and (2.26) ensure that incentive constraint $(u', B') \to (u'', B'')$ holds.

Case 3. $(u, B) \to (u'', B'')$ and $(u'', B'') \to (u, B)$. Using Equation (2.22), we get

$$K\Big[f(u, B) - f(u'', B'')\Big] = p(u, B) - p(u'', B'') = 0 - B'' \leq 0.$$

Using $u_1 \leq K$, we get

$$u_1 f(u, B) - p(u, B) \geq u_1 f(u'', B'') - p(u'', B'').$$

This is enough for incentive constraint $(u, B) \to (u'', B'')$ since $p(u, B) = 0$. Also, since $u''_1 > K$, we get

$$u''_1 f(u'', B'') - p(u'', B'') \geq u''_1 f(u, B) - p(u, B).$$

This is enough for incentive constraint $(u'', B'') \to (u, B)$ since $p(u'', B'') = B''$. ■

2.9.2 Proof of Theorem 2.2

We give the proof of Theorem 2.2. We start by giving some preparatory lemmas.

Preparatory Lemmas

Fix a manager non-trivial mechanism (f, p). Let

$$B^+_{(f,p)} := \{B : \{v \in V : p(v, B) > B\} \text{ has non-zero measure}\}.$$

By manager non-triviality $B^+_{(f,p)}$ is non-empty. This means for any $B \in B^+_{(f,p)}$, we observe that $V^+(f, p)$ defined in the public budget case has non-zero measure and hence (f, p) restricted to B belongs to M^+. We can then directly state equivalent of lemmas from the public budget case for any $B \in B^+_{(f,p)}$.

Lemma **2.20** *Suppose (f,p) is an incentive compatible and individually rational mechanism satisfying manager non-triviality. Then, for any $B \in B^+_{(f,p)}$, there exists $P_{(f,p),B}$, $A_{(f,p),B}$ and $K_{(f,p),B}$ such that the following are true.*

1. $p(u,B) = P_{(f,p),B}$ *and* $f(u,B) = A_{(f,p),B}$, *for all* u *with* $u_2 \in (K_{(f,p),B}, \beta)$ *and* $u_1 > K_{(f,p),B}$.

2. $A_{(f,p),B} > f(K_{(f,p),B}, 0, B) + \frac{1}{K_{(f,p),B}}\Big[B - p(K_{(f,p),B}, 0, B)\Big]$.

3. $\beta A_{(f,p),B} - P_{(f,p),B} = \beta f(u,B) - p(u,B)$ *for all* u *with* $u_2 = \beta$ *and* $u_1 > K_{(f,p),B}$.

4. $K_{(f,p),B}A_{(f,p),B} - P_{(f,p),B} = K_{(f,p),B} f(K_{(f,p),B}, 0, B) - p(K_{(f,p),B}, 0, B)$.

Proof: Fix any $B \in B^+_{(f,p)}$. Define $K_{(f,p),B}$ as in Lemma 2.6 and $P_{(f,p),B}$, $A_{(f,p),B}$ as in Lemma 2.10. Then it is easy to see that the first two statements are direct equivalent statements from Lemma 2.13. (3) follows by combining Lemma 2.12 with Equations 2.8 and 2.9. Combining Equation 2.7 with Lemma 2.12 we get (4). ■

Lemma **2.21** *Suppose (f,p) is an incentive compatible and individually rational mechanism satisfying manager non-triviality. Then, there exists $P_{(f,p)}$, $A_{(f,p)}$ and $K_{(f,p)}$ such that the following hold.*

1. $p(u,B) = P_{(f,p)}$, $f(u,B) = A_{(f,p)}$ *for all* $(u,B) \in W$ *with*
$$u_1 > K_{(f,p)},\ u_2 \in (K_{(f,p)}, \beta) \text{ and } B < P_{(f,p)}.$$

2. *If* $B < P_{(f,p)}$, *then* $B \in B^+_{(f,p)}$.

3. $p(u,B) \le B$ *for all* $(u,B) \in W$ *with* $(u_1,u_2) \neq (\beta,\beta)$ *and* $B \ge P_{(f,p)}$.

4. $p(u,B) = P_{(f,p)}$ *and* $f(u,B) = A_{(f,p)}$ *for all* $(u,B) \in W$ *with* $B \ge P_{(f,p)}$, $u_1 \in (K_{(f,p)}, \beta)$, *and* $u_2 < \beta$

5. $K_{(f,p)}A_{(f,p)} - P_{(f,p)} = K_{(f,p)} f(K_{(f,p)}, 0, B) - p(K_{(f,p)}, 0, B)$ *for all* $B < P_{(f,p)}$.

6. $p(u,B) \le p(K_{(f,p)}, 0, B')$ *for all* $(u,B) \in W$ *with* $u_1 < K_{(f,p)}$ *and for all* $B' < P_{(f,p)}$.

7. $p(u,B) \le 0$ *for all* $(u,B) \in W$ *with* $u_1 < K_{(f,p)}$.

Proof: Proofs of (1) and (2). Fix an incentive compatible and individually rational mechanism (f, p) and pick any $\acute{B} \in B^+_{(f,p)}$. From Lemma 2.20, we know that there exist $K_{(f,p),\acute{B}}$, $P_{(f,p),\acute{B}}$, and $A_{(f,p),\acute{B}}$ such that $p(u, \acute{B}) = P_{(f,p),\acute{B}} > \acute{B}$ and $f(u, \acute{B}) = A_{(f,p),\acute{B}}$, for all $u \in V$ with $u_2 \in (K_{(f,p),\acute{B}}, \beta)$ and $u_1 > K_{(f,p),\acute{B}}$. We do the proof in two steps.

Step 1. Consider an outcome (a, t) in the range of the mechanism. First, consider the case when $t < P_{(f,p)}$. Analogous to Lemma 2.3, it can be shown that incentive compatibility of (f, p) implies that $a < A_{(f,p),\acute{B}}$. Now, consider any type of the form $(v, \acute{B})$ where $v_1 = v_2 = x \in (K_{(f,p),\acute{B}}, \beta)$. Such a v exists since $K_{(f,p),\acute{B}} < \beta$. Lemma 2.20 implies that $(f(v, \acute{B})), p(v, \acute{B})) = (A_{(f,p),\acute{B}}, P_{(f,p),\acute{B}})$. Incentive compatibility from $(v, \acute{B})$ to any type with the outcome (a, t) gives us:

$$xA_{(f,p),\acute{B}} - P_{(f,p),\acute{B}} \geq xa - t.$$

Since this is true for *all* $x \in (K_{(f,p),\acute{B}}, \beta)$ and noting that $t < P_{(f,p),\acute{B}}$ and $a < A_{(f,p),\acute{B}}$ we conclude that

$$xA_{(f,p),\acute{B}} - P_{(f,p),\acute{B}} > xa - t \text{ for all } x \in (K_{(f,p),\acute{B}}, \beta). \tag{2.27}$$

If $t > P_{(f,p)}$, a similar reasoning establishes that Inequality (2.27) continues to hold (the only adjustment we need to do is that a will be strictly greater than $A_{(f,p)}$).

Step 2. Pick any budget B' with $B' \neq \acute{B}$ but $B' < P_{(f,p),\acute{B}}$. Further, pick any type (u, B') with $u_1 > K_{(f,p),\acute{B}}$ and $u_2 \in (K_{(f,p),\acute{B}}, \beta)$. We will argue that $(f(u, B'), p(u, B')) = (A_{(f,p),\acute{B}}, P_{(f,p),\acute{B}})$. Assume for contradiction, $(f(u, B'), p(u, B')) = (a, t)$ for some $(a, t) \neq (A_{(f,p),\acute{B}}, P_{(f,p),\acute{B}})$. Since Inequality (2.27) holds for $x = u_2$, incentive compatibility implies that $t \leq B'$ and

$$u_1 a - t \geq u_1 A_{(f,p),\acute{B}} - P_{(f,p),\acute{B}}.$$

But $B' < P_{(f,p),\acute{B}}$ implies that $t < P_{(f,p),\acute{B}}$, and hence, $a < A_{(f,p),\acute{B}}$. So, for any $x \in (K_{(f,p),\acute{B}}, \beta)$ with $x < u_1$, we must have

$$xa - t > xA_{(f,p),\acute{B}} - P_{(f,p),\acute{B}},$$

which is a contradiction to Inequality (2.27).

So, we conclude that for all $u_1 > K_{(f,p),\acute{B}}$ and $u_2 \in (K_{(f,p),\acute{B}}, \beta)$, we have $(f(u, B'), p(u, B')) = (A_{(f,p),\acute{B}}, P_{(f,p),\acute{B}})$. Further, this ensures that $B' \in B^+_{(f,p)}$. Hence, we have shown that for any

$\acute{B} \in B^+_{(f,p)}$ and any $B' < P_{(f,p),\acute{B}}$, we have

$$B' \in B^+_{(f,p)}. \tag{2.28}$$

Now, Lemma 2.20 implies that for every (u, B') with $u_1 > K_{(f,p),B'}$ and $u_2 \in (K_{(f,p),B'}, \beta)$, we have $p(u, B') = P_{(f,p),B'}$, we get that $P_{(f,p),B'} = P_{(f,p),\acute{B}}$. Consequently, $A_{(f,p),B'} = A_{(f,p),\acute{B}}$. Clearly, $K_{(f,p),B'} \leq K_{(f,p),\acute{B}}$. But since $P_{(f,p),B'} = P_{(f,p),\acute{B}}$ and the choice of $B', \acute{B}$ is arbitrary, we could swap their positions to conclude $K_{(f,p),\acute{B}} = K_{(f,p),B'}$.

We can now define $P_{(f,p)} := P_{(f,p),\acute{B}}$, $A_{(f,p)} := A_{(f,p),\acute{B}}$, and $K_{(f,p)} := K_{(f,p),\acute{B}}$. This concludes proof of (1).

For (2), by manager non-triviality, $B^+_{(f,p)}$ is non-empty, and using the conclusion in (1) along with the set inclusion in (2.28), we get that for all $B < P_{(f,p)}$, we have $B \in B^+_{(f,p)}$.

From this step, using Inequality (2.27), we can write that for all outcomes $(a, t) \neq (A_{(f,p)}, P_{(f,p)})$ in the mechanism, we must have

$$xA_{(f,p)} - P_{(f,p)} > xa - t \qquad \forall\ x \in (K_{(f,p)}, \beta). \tag{2.29}$$

This obviously implies that if $a > A_{(f,p)}$, then

$$xA_{(f,p)} - P_{(f,p)} > xa - t \qquad \forall\ x < \beta. \tag{2.30}$$

Proof of (3) and (4). Fix any type (u, B) such that $B > P_{(f,p)}$, and $(u_1, u_2) \neq (\beta, \beta)$. Assume for contradiction that $p(u, B) > B$ - this implies that $f(u, B) > A_{(f,p)}$. Since $p(u, B) > B > P_{(f,p)}$ and $f(u, B) > A_{(f,p)}$, the following inequalities must hold for incentive compatibility

$$\begin{aligned} u_1 f(u, B) - p(u, B) &\geq u_1 A_{(f,p)} - P_{(f,p)} \\ u_2 f(u, B) - p(u, B) &\geq u_2 A_{(f,p)} - P_{(f,p)} \end{aligned}$$

This contradicts Inequality (2.30) for $x = u_1$ or $x = u_2$ (note that $f(u, B) > A_{(f,p)}$). This proves (2).

Fix any (u, B) such that $B \geq P_{(f,p)}$, $u_1 \in (K_{(f,p)}, \beta)$, and $u_2 < \beta$. From (2) above, we have $p(u, B) \leq B$. Substituting $x = u_1$ in Inequality (2.29), we notice that for every other outcome (a, t) in the range of the mechanism, we have

$$u_1 A_{(f,p)} - P_{(f,p)} > u_1 a - t.$$

Hence, the agent prefers $(A_{(f,p)}, P_{(f,p)})$ to any other outcome (a,t) in the range of the mechanism. By incentive compatibility $(f(u,B), p(u,B)) = (A_{(f,p)}, P_{(f,p)})$. This proves (3).

Proof of (5). By (1), we know that every $B < P_{(f,p)}$ belongs to $B^+_{(f,p)}$. Then, (4) in Lemma 2.20 gives the result.

Proof of (6). Fix any $(u,B) \in W$ such that $u_1 < K_{(f,p)}$. Since $u_1 < K_{(f,p)}$, Lemma 2.8 implies that $p(u,B) \le B$.

Substituting $x = K_{(f,p)}$ and $(a,t) = \big(f(u,B), p(u,B)\big)$, Inequality (2.29) implies

$$K_{(f,p)}A_{(f,p)} - P_{(f,p)} \ge K_{(f,p)}f(u,B) - p(u,B)$$

Now pick $B' < P_{(f,p)}$ and use (4) above to get

$$K_{(f,p)}f(K_{(f,p)},0,B') - p(K_{(f,p)},0,B') \ge K_{(f,p)}f(u,B) - p(u,B). \tag{2.31}$$

Now, assume for contradiction that $p(u,B) > p(K_{(f,p)},0,B')$. Since, $p(u,B) \le B$ we have $p(K_{(f,p)},0,B') < B$. Then incentive constraint $(u,B) \to (K_{(f,p)},0,B')$ implies that

$$u_1 f(u,B) - p(u,B) \ge u_1 f(K_{(f,p)},0,B') - p(K_{(f,p)},0,B'). \tag{2.32}$$

Adding Inequalities (2.31) and (2.32), and using $u_1 < K_{(f,p)}$, we get $f(u,B) \le f(K_{(f,p)},0,B')$. But this implies that $p(u,B) \le p(K_{(f,p)},0,B')$, which is contradiction.

Proof of (7). This is a corollary to (5) above. Set $B' = 0$ and the result follows since $p(K_{(f,p)},0,0) \le 0$ from Lemma 2.8. ■

Figure 2.6 gives a pictorial description of an incentive compatible and individually rational mechanism as implied by Lemma 2.21.

Optimality of POST*

We now complete the proof of Theorem 2.2 by using the preparatory lemmas. For every incentive compatible, individually rational, and manager non-trivial mechanism (f,p), we first construct a new G-POST* mechanism (f',p') in the following way.

$$(f'(v,B), p'(v,B)) = \begin{cases} (A_{(f,p)}, P_{(f,p)}) & \text{if } \Big(\min(v_1,v_2) > K_{(f,p)} \text{ and } B < P_{(f,p)}\Big) \\ & \text{or } \Big(v_1 > K_{(f,p)} \text{ and } B \ge P_{(f,p)}\Big) \\ \big(A_{(f,p)} - \frac{1}{K_{(f,p)}}P_{(f,p)}, 0\big) & \text{if } v_1 \le K_{(f,p)} \\ \big(A_{(f,p)} - \frac{1}{K_{(f,p)}}(P_{(f,p)} - B), B\big) & \text{if } v_1 > K_{(f,p)},\ v_2 \le K_{(f,p)} \text{ and } B < P_{(f,p)} \end{cases}$$

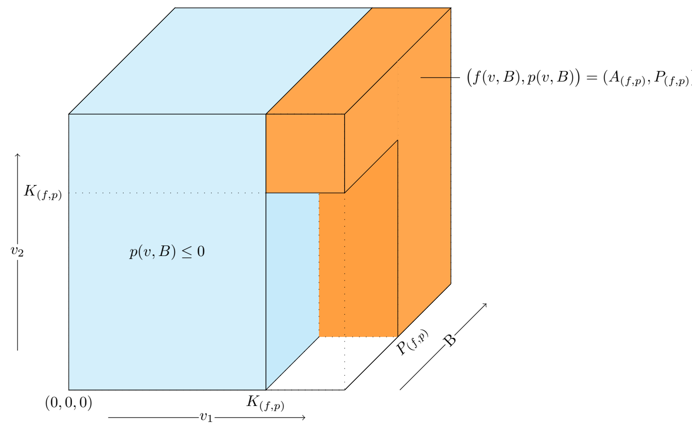

Figure 2.6: Structure of incentive compatible and individually rational mechanism

The new mechanism (f', p') is shown in Figure 2.7. It is easy to verify that $f'(v, B) \in [0, 1]$ for all $(v, B) \in W$. To see this, assume for contradiction that $A_{(f,p)} - \frac{1}{K_{(f,p)}}(P_{(f,p)} - B) > 1$ when $B < P_{(f,p)}$. Then, we get $K_{(f,p)}A_{(f,p)} - P_{(f,p)} > K_{(f,p)} - B$, which is a contradiction since $A_{(f,p)} \in [0, 1]$ and $B < P_{(f,p)}$. This shows that $A_{(f,p)} - \frac{1}{K_{(f,p)}}(P_{(f,p)} - B) \leq 1$, which also implies that $A_{(f,p)} - \frac{1}{K_{(f,p)}}P_{(f,p)} \leq 1$. Finally, $A_{(f,p)} - \frac{1}{K_{(f,p)}}P_{(f,p)} \geq 0$ follows from (5) in Lemma 2.21 and individual rationality of (f, p).

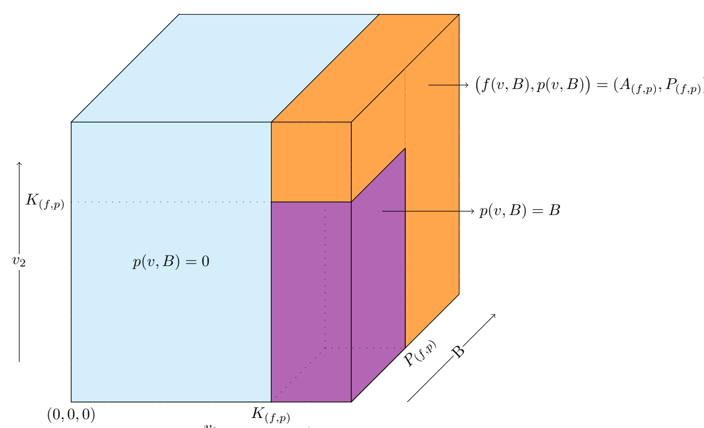

Figure 2.7: Mechanism (f', p')

Lemma **2.22** *If (f,p) is an incentive compatible, individually rational, manager non-trivial mechanism, then the* G-POST* *mechanism (f',p') is a manager non-trivial, incentive compatible, individually rational, and*

$$p'(v,B) \geq p(v,B) \text{ for almost all } (v,B) \in W.$$

Proof: Since (f',p') is a G-POST* mechanism, Proposition 2.7 implies that (f',p') is a manager non-trivial, incentive compatible, individually rational. We establish that $p'(v,B) \geq p(v,B)$ for **almost** all $(v,B) \in W$. To see this, consider the following three cases.

- Case 1. Consider $(v,B) \in W$ such that $\{\min(v_1,v_2) > K_{(f,p)}$ and $B < P_{(f,p)},\ v_2 \neq \beta\}$ or $\{v_1 \in (K_{(f,p)}, \beta)$ and $B \geq P_{(f,p)},\ v_2 \neq \beta\}$. By (1) and (4) in Lemma 2.21,

$$p'(v) = P_{(f,p)} = p(v).$$

- Case 2. Consider $(v,B) \in W$ such that $v_1 < K_{(f,p)}$. By (7) in Lemma 2.21, we have $p'(v,B) = 0 \geq p(v,B)$.

- Case 3. Finally, consider $(v,B) \in W$ such that $v_2 < K_{(f,p)},\ v_1 > K_{(f,p)}$ and $B < P_{(f,p)}$. By (2) in Lemma 2.21, we get that $B \in B^+_{(f,p)}$. Then, since $\min(v_1,v_2) < K_{(f,p)}$, by the definition of $K_{(f,p)}$, we get $p(v,B) \leq B = p'(v,B)$, which concludes this case.

Denote by W' the set of type profiles covered in the above three cases. It is easy to see (for instance, refer to Figure 2.7) that $W \setminus W'$ has zero Lebesgue measure. So, for almost all (v,B), we have $p'(v,B) \geq p(v,B)$. ■

The proof of Theorem 2.2 is completed by the following lemma.

Lemma **2.23** *For every* G-POST* *mechanism (f,p), there is a* POST* *mechanism (f',p') such that*

$$p'(v,B) \geq p(v,B)\ \forall\ (v,B) \in W.$$

Proof: Take any G-POST* mechanism (f,p) defined by parameters A, P, K. Consider the POST* mechanism (f',p') defined by parameter K. By definition of G-POST* mechanism (f,p), we know that $K \geq P$. Now, consider the following cases:

- $p'(v,B) = p(v,B) = 0$ for all (v,B) if $v_1 \leq K$.

- $p'(v,B) = p(v,B) = B$ for all (v,B) if $v_1 > K$, $v_2 \leq K$ and $B < P$.

- $p'(v,B) = K \geq P = p(v,B)$ for all (v,B) if $\{\min(v_1,v_2) > K$ and $B < K\}$ or $\{v_1 > K$ and $B \geq K\}$

- $p'(v, B) = K \geq P = p(v, B)$ for all (v, B) if $v_1 > K$, $v_2 \leq K$ and $P \leq B < K$.

This concludes the proof. ■

Lemma 2.23 thus establishes that a POST* mechanism is a partially optimal mechanism, which concludes the proof of Theorem 2.2.

2.10 Supplementary Appendix

2.10.1 Intransitive preferences

Lemma **2.24 (Intransitive preference)** *For any type $v = (v_1, v_2)$ with $v_1, v_2 > 0$ and $v_1 \neq v_2$ there exist three outcomes $(a,t),(b,t'),(c,t'') \in Z$ such that*

$$(a,t) \succ_v (b,t') \succ_v (c,t'') \succ_v (a,t),$$

where $\succ_v$ is the strict part of the relation $\succeq_v$.

Proof: We consider two cases where $v_1 < v_2$ and then $v_1 > v_2$. The proof is by construction of three outcomes as stated above.

Case 1. Fix any $v = (v_1, v_2)$ such that $0 < v_1 < v_2$. Consider three outcomes

$$(a,t) := (\frac{1}{2}, B),\ (b,t') := (1, B + \frac{3v_1}{8} + \frac{v_2}{8}),\ \text{and}\ (c,t'') = (\frac{3}{4} - \frac{v_1}{8v_2}, B + \frac{v_1}{8}).$$

First,

$$v_1 a - t = \frac{1}{2}v_1 - B = v_1 - B - \frac{v_1}{2} > v_1 - B - \big(\frac{3v_1}{8} + \frac{v_2}{8}\big) = v_1 b - t',$$

where the inequality is true because $v_1 < v_2$. Combining this with $t \leq B$ gives us

$$(a,t) \succ_v (b,t').$$

Second,

$$\begin{aligned} v_2 b - t' &= v_2 - B - \big(\frac{3v_1}{8} + \frac{v_2}{8}\big) = v_2 - B - \big(\frac{v_1}{4} + \frac{v_1 + v_2}{8}\big) \\ &> v_2 - B - \big(\frac{v_1}{4} + \frac{v_2}{4}\big) = v_2\big(\frac{3}{4} - \frac{v_1}{8v_2}\big) - B - \frac{v_1}{8} \\ &= v_2 c - t''. \end{aligned}$$

where the inequality is true because $v_1 < v_2$. Combining this with the fact that $t', t'' > B$, we have

$$(b,t') \succ_v (c,t'').$$

Third,

$$v_1 c - t'' = v_1\big(\frac{3}{4} - \frac{v_1}{8v_2}\big) - B - \frac{v_1}{8} > \frac{3}{4}v_1 - B - \frac{v_1}{4} = \frac{1}{2}v_1 - B = v_1 a - t,$$

where the inequality is true because $v_1 < v_2$. Hence, $(a,t) \not\succeq_{v_1} (c,t'')$.

But since $t'' > B$, we need to compare the outcomes with respect to v_2. For that, notice

$$v_2 c - t'' = v_2\left(\frac{3}{4} - \frac{v_1}{8v_2}\right) - B - \frac{v_1}{8} = v_2\left(\frac{3}{4} - \frac{v_1}{4v_2}\right) - B > \frac{1}{2}v_2 - B,$$

where the inequality is due to $v_1 < v_2$. This implies that $(c,t'') \succ_v (a,t)$.

Case 2. Fix any $v = (v_1, v_2)$ such that $v_1 > v_2$. Set $K = \max(2, \left\lceil \frac{v_2}{B} \right\rceil)$, where we use the notation that $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Consider three outcomes

$$(a,t) := (1 - \frac{2}{K}, B - \frac{v_2}{K}),\ (b,t') := (1, B + \frac{v_2(3 - \frac{v_2}{v_1})}{2K}),\ \text{and}\ (c,t'') := (1 - \frac{7 - 3(\frac{v_2}{v_1})}{4K}, B).$$

The value of K set above ensures that all the consumption bundles are feasible.
First,

$$(v_1 b - t') - (v_1 a - t) = \frac{1}{K}(2v_1 - v_2) - \frac{1}{2K}\frac{v_2}{v_1}(3v_1 - v_2) \geq \frac{1}{K}(2v_1 - v_2) - \frac{1}{2K}(3v_1 - v_2) > 0,$$

where the inequalities are true because $v_1 > v_2$. Since $t' > B$ we have $(b,t') \not\succeq_{v_1} (a,t)$. We need to check the outcomes with respect to v_2. For that, notice

$$(v_2 a - t) - (v_2 b - t') = \frac{v_2}{v_1}\left(\frac{3v_1 - v_2}{2K}\right) - \frac{v_2}{K} > 0.$$

The inequality is true because $v_1 > v_2$. From above discussions, we have

$$(a,t) \succ_v (b,t').$$

Second,

$$(v_2 b - t') - (v_2 c - t'') = \frac{1}{4K}\Big(\big(7 - 3\frac{v_2}{v_1}\big) - \big(6 - 2\frac{v_2}{v_1}\big)\Big) = \frac{1}{4K}(1 - \frac{v_2}{v_1}) > 0,$$

where the inequality is due to $v_1 > v_2$. Also, notice that from above we derive $t' - t'' < v_2(b-c) < v_1(b-c)$ which implies $v_1 b - t' > v_1 c - t''$. Combining the above two results with the fact that $t' > B$, we conclude that

$$(b,t') \succ_v (c,t'').$$

Third,

$$(v_1 c - t'') - (v_1 a - t) = \frac{1}{K}(2v_1 - v_2) - \frac{1}{4K}\big(7v_1 - 3v_2\big) = \frac{1}{4K}(v_1 - v_2) > 0.$$

The inequality is because $v_1 > v_2$. Noticing that $t'' \leq B$, we have $(c,t'') \succ_v (a,t)$. ■

2.10.2 Proofs for the uniform distribution case

In this section, we give the proofs of Lemma 2.2 and Proposition 2.5.

Proof of Lemma 2.2

Proof: Suppose (K_1^*, K_2^*) are values of (K_1, K_2) in the optimal POST-2 mechanism. By definition $K_1^* \leq K_2^*$. Using the uniform distribution of G, we see that (K_1^*, K_2^*) are optimal solutions to the following optimization problem:

$$\max_{K_2 \in [B,1],\ K_1 \in [B,K_2]} B\Big[1 - K_1\Big] + \big(1 - \frac{B}{K_1}\big) K_2 (1 - K_2)^2. \tag{2.33}$$

We consider the following optimization problem, where we fix the value of K_1^* and maximize over all K_2:

$$\max_{K_2 \in [0,1]} B\Big[1 - K_1^*\Big] + \big(1 - \frac{B}{K_1^*}\big) K_2 (1 - K_2)^2.$$

Notice that the objective function is strictly concave in K_2, and the unique maximum occurs when $K_2 = \frac{1}{3}$.

Now, assume for contradiction $K_1^* < K_2^*$. We consider two cases and reach a contradiction in both the cases.

CASE 1. Suppose $K_1^* \geq \frac{1}{3}$. Then, $K_2^* > \frac{1}{3}$. But $K_2 = K_1^*$ and K_1^* defines a feasible POST-2 mechanism, and generates more revenue. This is a contradiction.

CASE 2. Suppose $K_1^* < \frac{1}{3}$. Since $K_2^* \geq K_1^*$, we see that $K_2 = \frac{1}{3}$ and K_1^* defines a feasible POST-2 mechanism and generates more revenue. Hence, K_2^* must be equal to $\frac{1}{3}$. Now, fixing the value of K_2 at $\frac{1}{3}$, we optimize the Expression (2.33) with relaxed constraints on K_1:

$$\max_{K_1 \in [0,1]} B\Big[1 - K_1\Big] + \big(1 - \frac{B}{K_1}\big) \frac{4}{27}.$$

This objective function is strictly concave with a unique maxima at $K_1 = \frac{2}{3\sqrt{3}} > \frac{1}{3}$. Hence, the objective function of the Expression in (2.33) is higher at $K_1 = \frac{1}{3} = K_2^*$ than at (K_1^*, K_2^*) with $K_1^* < \frac{1}{3}$. Further, $K_1 = K_2 = \frac{1}{3}$ is a POST-2 mechanism since (K_1^*, K_2^*) with $K_2^* = \frac{1}{3}$ is a POST-2 mechanism. This is a contradiction.

Using this, we can conclude that the optimal POST-2 mechanism is a solution to the following single-variable constrained optimization problem.

$$\max_{K \in [B,1]} B\big(1-K\big) + \big(K-B\big)(1-K)^2. \tag{2.34}$$

We denote $J(K) := B\big(1-K\big) + \big(K-B\big)(1-K)^2$ for all K. Notice that

$$\begin{aligned} J'(K) &= 3K^2 - K(2B+4) + (B+1) \\ J''(K) &= 6K - (2B+4). \end{aligned}$$

Note that

$$J'(B) = B^2 - 3B + 1 = \Big(B - \frac{3-\sqrt{5}}{2}\Big)\Big(B - \frac{3+\sqrt{5}}{2}\Big).$$

Hence, $J'(B) \leq 0$ if and only if $B \geq \frac{1}{2}\big(3-\sqrt{5}\big)$.

Notice that $J''(K) = 0$ for $K = \frac{1}{3}(B+2)$. Hence, $J'(K)$ is decreasing in $[B, \frac{1}{3}(B+2)]$ and increasing in $[\frac{1}{3}(B+2), 1]$. Also, $J'(1) = -B < 0$. Hence, if $J'(B) \leq 0$, we must have $J'(K) < 0$ for all $K \in (B, 1]$.

PROOF OF (1). This implies that for $B \geq \frac{1}{2}\big(3-\sqrt{5}\big)$, we have $J'(K) < 0$ for all $K \in (B, 1]$. This implies that J is decreasing in $[B, 1]$, and hence, the optimal solution of Optimization (2.34) must have $K = B$. Then, the first part implies that the optimal POST-2 mechanism must have $K_1^* = K_2^* = B$.

PROOF OF (2). If $B < \frac{1}{2}\big(3-\sqrt{5}\big)$, then $J'(B) > 0$ and $J'(K) = 0$ at a unique point

$$K = \frac{1}{3}\big(B + 2 - \sqrt{(B^2+B+1)}\big).$$

Denote this point of inflection as $\tilde{K}$. Notice that $J'(K) < 0$ for all $K > \tilde{K}$, and, hence, J is decreasing after $\tilde{K}$. Further, $\tilde{K} < \frac{1}{3}(B+2)$ and $J''(K) < 0$ for all $K < \tilde{K}$. This means J is strictly concave from B to $\frac{1}{3}(B+2)$. Combining these observations, we conclude that $K = \tilde{K}$ solves the Optimization in (2.34). The first part implies that the optimal POST-2 mechanism must have

$$K_1^* = K_2^* = \frac{1}{3}\big(B + 2 - \sqrt{(B^2+B+1)}\big),$$

if $B < \frac{1}{2}\big(3-\sqrt{5}\big)$. ■

Proof of Proposition 2.5

Proof: To do the proof, we first compute the optimal POST-1 mechanism, which is the solution to the following optimization program:

$$\max_{K_1 \in [0,B]} K_1(1 - K_1). \tag{2.35}$$

It is clear the optimal POST-1 mechanism is $K_1 = \frac{1}{2}$ if $B > \frac{1}{2}$ and $K_1 = B$ if $B \leq \frac{1}{2}$. Now, we consider the three cases separately.

CASE 1 - $B > \frac{1}{2}$. Optimal POST-1 mechanism generates a revenue of $\frac{1}{4}$. By Lemma 2.2, optimal POST-2 mechanism generates a revenue of $B(1 - B)$, which is less than $\frac{1}{4}$. Hence, the optimal mechanism is a POST-1 mechanism with $K_1 = \frac{1}{2}$.

CASE 2 - $B \in [\frac{1}{2}(3 - \sqrt{5}), \frac{1}{2}]$. In this case, both the optimal POST-1 mechanism and the optimal POST-2 mechanism (due to Lemma 2.2) generates a revenue of $B(1 - B)$. Hence, the optimal POST-1 mechanism with $K_1 = B$ is optimal.

CASE 3 - $B \in (0, \frac{1}{2}(3-\sqrt{5}))$. In this case, the optimal POST-1 mechanism generates a revenue of $B(1-B)$, which is also the revenue generates by a POST-2 mechanism with $K_1 = K_2 = B$. But the optimal POST-2 is unique and has $K_1 = K_2 = \frac{1}{3}\big(B + 2 - \sqrt{(B^2 + B + 1)}\big)$ due to Lemma 2.2. Hence, the result follows. ■

2.10.3 An alternate notion of incentive compatibility

In this section, we adapt the choice correspondence procedure defined in Manzini and Mariotti (2012) to propose an extension of our binary choice model. We then propose an appropriate notion of incentive compatibility for this model and show its relation to our notion of incentive compatibility.

Consider a type $v \equiv (v_1, v_2)$. For any subset of outcomes $S \subseteq Z$, define

$$M^1(S; v_1) := \{(a,t) \in S : av_1 - t \geq a'v_1 - t' \ \forall \ (a',t') \in S \text{ and } t \leq B\}$$

and define

$$M^2(S; v_2) := \{(a,t) \in S : av_2 - t \geq a'v_2 - t' \ \forall \ (a',t') \in S\}.$$

Using $M^1(S; v_1)$ and $M^2(S; v_2)$, we can now define a choice correspondence $C^v : 2^Z \to 2^Z$ with $\emptyset \neq C^v(S) \subseteq S$ for each $S \subseteq Z$ as follows:

$$C^v(S) = \begin{cases} M^1(S; v_1) & \text{if } M^1(S; v_1) \neq \emptyset \\ M^2(S; v_2) & \text{otherwise} \end{cases}$$

Intuitively, first, the agent tries to choose from S using v_1, and if the maximal elements according to her preference satisfy budget constraint, then they are chosen. Otherwise, the maximal elements according to the manager are chosen. This is a plausible extension of our binary choice model to accommodate choice from arbitrary subsets.

If we *assume* that our (agent, manager) pair makes choices using such choice correspondences (or some other choice correspondence "consistent" with type v), then a familiar notion of incentive compatibility for choice correspondences can be applied. In particular, we say that (f, p) is **choice-incentive compatible** if for every v,

$$(f(v), p(v)) \in C^v(R^{f,p}),$$

where $R^{f,p}$ is the range of the mechanism (f, p). This definition can be extended to arbitrary mechanisms $\mu : M \to Z$ defined on message space M. Notice that our definition requires that

$$(f(v), p(v)) \succeq_v (a, t) \; \forall \; (a, t) \in R^{f,p}.$$

If the (agent, manager) pair makes choices using C^v for each type v, we show that choice-incentive compatibility and incentive compatibility are independent conditions. We give two examples below to illustrate this.

Example **2.1**

To see this, consider a type space with three types $V := \{v, v', v''\}$, where

$$v = (1, 1.2), v' = (0, 0), v'' = (1, 1).$$

Assume $B = 0.5$ and consider the following mechanism (f, p) defined on this type space.

$$(f(v), p(v)) := (1, 0.6), \quad (f(v'), p(v')) := (0.81, 0.4), \quad (f(v''), p(v'')) = (0.924, 0.51).$$

We can check that

$M^1(R^{f,p}; v_1) = \emptyset, M^1(R^{f,p}; v'_1) = \{(f(v'), p(v'))\}, M^1(R^{f,p}; v''_1) = \emptyset$
$M^2(R^{f,p}; v_2) = \{(f(v), p(v))\}, M^2(R^{f,p}; v'_2) = \{(f(v'), p(v'))\}, M^2(R^{f,p}; v''_2) = \{(f(v''), p(v''))\}.$

Hence, we get

$$C^v(R^{f,p}) = \{(f(v), p(v))\}, C^{v'}(R^{f,p}) = \{(f(v'), p(v'))\}, C^{v''}(R^{f,p}) = \{(f(v''), p(v''))\}.$$

Hence, (f, p) is choice-incentive compatible. But it can also be checked that

$$(f(v), p(v)) = (1, 0.6) \not\succeq_v (0.81, 0.4).$$

Hence, (f, p) is not incentive compatible.

Example **2.2**

Now, consider another type space $V' = \{u, u', u''\}$, where

$$u = (3, 2),\ u' = (0, 0),\ \text{and } u'' = (2.5, 2.5).$$

As before, assume $B = 0.5$. Now, consider the following mechanism (f', p') defined on the type space V'.

$$(f'(u), p'(u)) := (0.99, 0.49),\quad (f'(u'), p'(u')) := (0.989, 0.487),\quad (f'(u''), p'(u'')) = (1, 0.51).$$

Now, the following binary relations can be verified.

$$\begin{aligned}(0.99, 0.49) \succeq_u (0.989, 0.487), (0.99, 0.49) \succeq_u (1, 0.51).\\ (0.989, 0.487) \succeq_{u'} (0.99, 0.49), (0.989, 0.487) \succeq_{u'} (1, 0.51).\\ (1, 0.51) \succeq_{u''} (0.99, 0.49), (1, 0.51) \succeq_{u''} (0.989, 0.487).\end{aligned}$$

This shows that (f', p') is incentive compatible. But notice that

$$M^1(R^{f',p'}; u_1) = \emptyset, M^2(R^{f',p'}; u_2) = \{(0.989, 0.487)\}.$$

Hence, $(f'(u), p'(u)) = (0.99, 0.49) \notin C^u(R^{f',p'})$. This shows that (f', p') is not choice-incentive compatible.

2.10.4 A sufficient condition for optimality of POST*

In this section, we will identify some restrictions on the distribution that ensures that POST* is an *optimal* mechanism for the private budgets case. We summarize our assumptions below.

Definition **2.10** *We say distribution* Φ *satisfies* **Assumption A** *if*

- *Values and budget are distributed independently, i.e., there exists a prior G over $V \equiv [0,\beta] \times [0,\beta]$ and a prior Π over $[0,\beta]$ such that $\Phi(v,B) = G(v)\Pi(B)$ for all (v,B).*

- *Marginal G_1 satisfies the property that $H_1(x) := xG_1(x)\ \forall\ x$ is strictly convex.*

- *Finally, define $\bar{K}$ as before: $\bar{K} := \arg\max_{r\in[0,\beta]} r(1 - G_1(r))$ - this is well defined because H_1 is strictly convex. Then, the following must hold:*

$$[1-G(\bar{K},\beta)-G(\beta,\bar{K})+G(\bar{K},\bar{K})]\int_0^{\bar{K}}(\bar{K}-B)d\Pi(B) \geq \int_0^{\bar{K}} B[G_1(\bar{K})-G_1(B)]d\Pi(B)$$

If G is the uniform distribution over $[0,1]\times[0,1]$ and Π is uniform over $[0,1]$, then the resulting distribution satisfies Assumption A.

Proposition **2.8** *If Φ satisfies Assumption A, then a* post* *mechanism is optimal.*

Proof: Fix any B in $(0,\beta)$ and consider the optimal post-1 mechanism in M^- derived in Proposition 2.4. We use this mechanism for each B (using the expression in Proposition 2.2) to define a new mechanism (f', v') for the private budget case - for $B \in \{0,\beta\}$, we use the limiting mechanisms of the post-1 mechanism suggested in Proposition 2.2.

$$(f'(v),p'(v)) = \begin{cases} (1,B) & \text{if } v_1 > B \text{ and } B < \bar{K} \\ (1,\bar{K}) & \text{if } v_1 > \bar{K} \text{ and } B \geq \bar{K} \\ (0,0) & \text{otherwise.} \end{cases}$$

Of course, this mechanism is not incentive compatible in the private budget case - when $v_1 > B > 0$, the (agent, manager) pair has an incentive to report a budget equal to zero get the outcome $(1,0)$. But notice that the expected revenue of the optimal mechanism in the class of incentive compatible and individually rational mechanisms that are not manager non-trivial cannot exceed the expected revenue of (f',p').

Now, consider the post* mechanism by setting $K = \bar{K}$:

$$(f^*(v),p^*(v)) = \begin{cases} (1,\bar{K}) & \text{if } \{v_1 > \bar{K} \text{ and } B \geq \bar{K}\} \text{ or } \{v_1, v_2 > \bar{K} \text{ and } B < \bar{K}\} \\ (\frac{B}{\bar{K}}, B) & \text{if } v_1 > \bar{K},\ v_2 \leq \bar{K}, \text{ and } B < \bar{K} \\ (0,0) & \text{otherwise} \end{cases}$$

The two mechanisms are shown in Figures 2.8 and 2.9 below.

We argue that post* generates weakly greater expected revenue that (f',p') under Assumption A. Hence, the optimal mechanism must be a post* mechanism by Theorem 2.2.

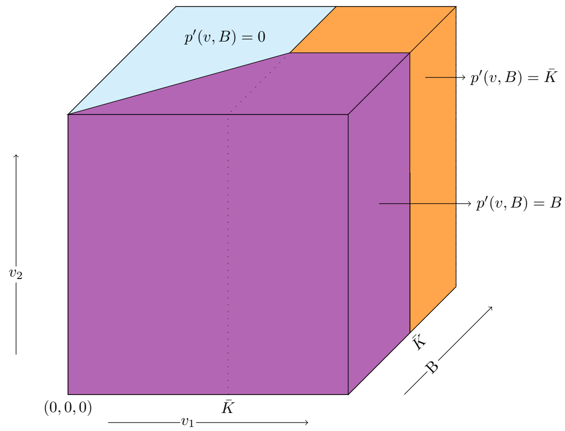

Figure 2.8: Upper bound

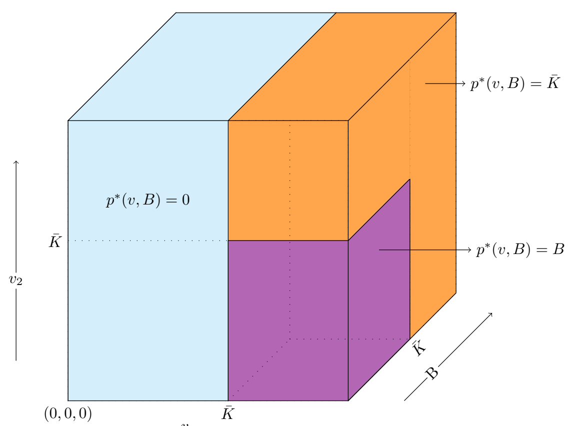

Figure 2.9: Lower bound

Note that (f', p') and (f^*, p^*) yield the same revenue for the following types:

$$(v, B) \text{ such that } B \geq \bar{K}$$

$$(v, B) \text{ such that } v_1 > \bar{K},\ v_2 \leq \bar{K}, \text{ and } B < \bar{K}$$

$$(v, B) \text{ such that } v_1 \leq B, \text{ and } B < \bar{K}$$

So, we ignore these types and focus on rest of the types.

- for any type (v, B) such that $v_1, v_2 > \bar{K}$ and $B < \bar{K}$, revenue from (f^*, p^*) is $\bar{K}$ whereas revenue from (f', p') is B; so the difference in revenue is $\bar{K} - B$.

- for any type (v, B) such that $v_1 \in (B, \bar{K}]$ and $B < \bar{K}$, revenue from (f^*, p^*) is 0 whereas revenue from (f', p') is B; so the difference in revenue is B.

Then the condition for revenue from (f^*, p^*) to be more than that of (f', p') is:

$$[1 - G(\bar{K}, \beta) - G(\beta, \bar{K}) + G(\bar{K}, \bar{K})] \int_0^{\bar{K}} (\bar{K} - B) d\Pi(B) \geq \int_0^{\bar{K}} B[G_1(\bar{K}) - G_1(B)] d\Pi(B)$$

This holds because of Assumption A. ■

Chapter 3

Selling two complementary goods

3.1 Introduction

An agent consumes a pair of goods only in a specific ratio of quantities. For instance, a firm needs two inputs in a particular ratio to produce a final product; that is, the firm has a Leontief production function. A consumer treats a pair of goods, coffee and sugar, for example, as perfect complements. That is, the consumer has Leontief preferences, and hence discards excess in either of the goods (after consuming them in the ratio of quantities). A seller who owns one unit each of the two *divisible* goods is selling to such an agent; what is the revenue-maximizing optimal mechanism in this setting?

We set up this as a mechanism design problem. For any allocation of the bundle of goods, the agent evaluates it using the ratio in which she consumes. The agent has quasilinear preferences across such bundles of goods whose quantities are in the desired ratio. The agent's payoff is determined by a *value*, the ratio, and quantity of the bundle he consumes. The value is interpreted as the payoff from consuming one unit of one of the goods combined with the other good in the desired ratio. Both per unit value from the consumption of the bundle of goods and the ratio itself are *private* information of the agent. The central theme of the paper is in finding the revenue-maximizing mechanism in such an environment.

For each report, a mechanism assigns quantities of both the goods and payment to be made by the agent. Due to the revelation principle, we focus, without loss of generality, on direct mechanisms that are incentive compatible. An agent could potentially misreport both on value and ratio dimensions. Dealing with incentive constraints in multi-dimensional mechanism design problems is difficult (Manelli and Vincent, 2007; Carroll, 2017). We present this natural two-dimensional mechanism design model and show that a simple class of *non-wasteful* mechanisms are optimal under some conditions over seller's beliefs on agent's type.

Note that we consider a *divisible* goods model, while the optimal mechanism in the indivisible model may involve randomization. (Hart and Reny (2015); Thanassoulis (2004)).

We show that a POSTED PRICE mechanism or a RATIO-DEPENDENT POSTED PRICE mechanism is optimal. The former is a mechanism in which the seller offers one unit of one of the goods and the other good in the desired ratio at some fixed price. In the latter mechanism, each type gets the same bundle as in the former mechanism, but the price depends on the reported ratio. We first show that it is without loss of generality to focus on mechanisms in which allocations to any type are in the desired ratio; that is, the agent, after a truthful report, does not dispose of either of the goods that the mechanism allocates. This result allows us to use Myersonian techniques. We then characterize incentive compatible mechanisms and provide sufficient conditions over the seller's belief on the type-space for simple non-wasteful mechanisms to be optimal. We fully describe these mechanisms over the parameters of the problem.

3.1.1 Related Literature

Armstrong (1996); Rochet and Chone (1998) analyze the standard model with divisible goods while Mcafee and Mcmillan (1988); Manelli and Vincent (2006) among others analyze the problem of indivisible goods. The optimal mechanism is known to be stochastic (that is, allocation of the objects is randomized) for many distributions (Hart and Reny (2015); Thanassoulis (2004)) in the case of the indivisible goods. Manelli and Vincent (2006); Devanur et al. (2020); Bikchandani and Mishra (2020) are among the papers that find sufficient conditions (on type distribution) under which deterministic mechanisms are optimal. Our model considers divisible complementary goods and finds sufficient conditions under which one of the goods is allocated to the maximum quantity. This maximum quantity allocation is interpreted as a deterministic mechanism in the standard indivisible goods model (see Pavlov (2011)).

Devanur et al. (2020) consider a model in which there are multiple copies of a good for sale. The agent derives a constant marginal 'value' up to a 'quantity' of the goods and no value beyond the desired quantity. The value and quantity are private information of the agent. They find conditions for deterministic mechanisms to be optimal and focus on the computational complexity of the problem. Our paper differs from theirs in that we consider a pair of heterogeneous goods with a privately known ratio of consumption while they consider heterogeneous goods with privately known demand. The two optimization exercises are similar after we prove our first result of 'non-wastefulness.' While they use the 'utility' approach to show that there exists a deterministic mechanism under some conditions, we use the Myersonian approach to characterize the optimal mechanism under a different set of conditions.

Fiat et al. (2016)'s model has two-dimensional private type for a single object. One dimension is for 'value,' which is constant up to a 'deadline' and suddenly goes to zero beyond the deadline. The paper characterizes optimal mechanisms, not just focussing on deterministic mechanisms. The agent's utility in their model changes sharply beyond the deadline, while in our model (and in Devanur et al. (2020)'s model), the utility is continuous, as a function of the allocations.

3.2 The model

A seller is selling a pair of divisible goods to an agent. The seller has one unit each of the goods, denoted by GOOD_1 and GOOD_2, and has no value for them. A consumption bundle for the agent is a tuple (a_1, a_2, t), where $a_1, a_2 \in [0, 1]$ is the allocation quantities of GOOD_1 and GOOD_2, respectively, and $t \in \mathbb{R}$ is the transfer - the amount *paid* by the agent.

The agent treats the goods as perfect complements, that is any two allocations (a_1, a_2) and (a_1', a_2') with $\min\{\frac{a_1}{k}, a_2\} = \min\{\frac{a_1'}{k}, a_2'\}$ are payoff equivalent, where $k \in K \equiv (0, 1]$ is the ratio of quantities of GOOD_1 and GOOD_2 that the agent demands. If the agent gets (a_1, a_2) and her desired ratio is k, then she can produce $\min\{\frac{a_1}{k}, a_2\}$ of the final good, which she values at v per unit, where $v \in V \equiv [0, 1]$. Both v and k are private information of the agent, therefore the agent has a "type" $(v, k) \in V \times K$.

The utility derived by agent of type (v, k) from an outcome (a_1, a_2, t) is given by,

$$U_{(v,k)}(a_1, a_2, t) := v \min\{\frac{a_1}{k}, a_2\} - t.$$

GOOD_2 is the *primary* good, whereas GOOD_1 is its *complement* which is always consumed lesser in quantity than the former as $k \in (0, 1]$. For instance, consider an agent with type $(v, k) = (\frac{1}{2}, \frac{1}{3})$. From an outcome $(a_1, a_2, t) = (\frac{1}{4}, 1, t)$, the agent derives a utility of

$$\frac{1}{2}. \min\{\frac{\frac{1}{4}}{\frac{1}{3}}, 1\} = \frac{1}{2}.\frac{3}{4} - t.$$

We assume that the random variables v, k follow a joint distribution function G with strictly positive density function g. We use g_v, g_k to denote marginal density functions of V and K, respectively. $g(v|k)$ denotes the conditional density of v given k.

3.3 Optimal Mechanism

An allocation function $f : V \times K \to [0,1]^2$ and a payment function $p : V \times K \to \mathbb{R}$ define a direct mechanism (f,p). For any allocation function f, we use subscript notations f_1 and f_2 to denote allocations corresponding to $\textsc{good}_1$ and $\textsc{good}_2$, respectively. Standard revelation principle argument implies that we can focus, without loss of generality, on incentive compatible direct mechanisms.

Definition **3.1** *A mechanism (f,p) is* **incentive compatible (IC)** *if for all $(v,k),(v',k') \in V \times K$,*

$$U_{(v,k)}(f(v,k),p(v,k)) \geq U_{(v,k)}(f(v',k'),p(v',k'))$$

IC condition ensures that the agent has the incentive to report his type - both value and ratio - truthfully. We also impose a participation constraint; that is, the utility for every type of the agent is at least zero from participating in the mechanism.

Definition **3.2** *A mechanism (f,p) is* **individually rational (IR)** *if for all $(v,k) \in V \times K$,*

$$U_{(v,k)}(f(v,k),p(v,k)) \geq 0.$$

Notation. We use $(v,k) \to (v',k')$ to denote the incentive constraint for the type (v,k) to not misreport as type (v',k').

3.3.1 Non-wasteful Mechanisms

We state two simple classes of non-wasteful mechanisms and show that they are IC and IR.

Definition **3.3** *A mechanism (f,p) is* posted price *mechanism if there exists a $\rho^* \in [0,1]$ such that*

$$(f(v,k),p(v,k)) = \begin{cases} (0,0,0) & \text{if } v \leq \rho^* \\ (k,1,\rho^*) & \text{otherwise.} \end{cases}$$

In a posted price mechanism there exists a price ρ such that all the types whose value is less than ρ get no good and pay nothing. A type (v,k) with $v > \rho$ gets k units of $\textsc{good}_1$, 1 unit of $\textsc{good}_2$, and pays ρ to the seller.

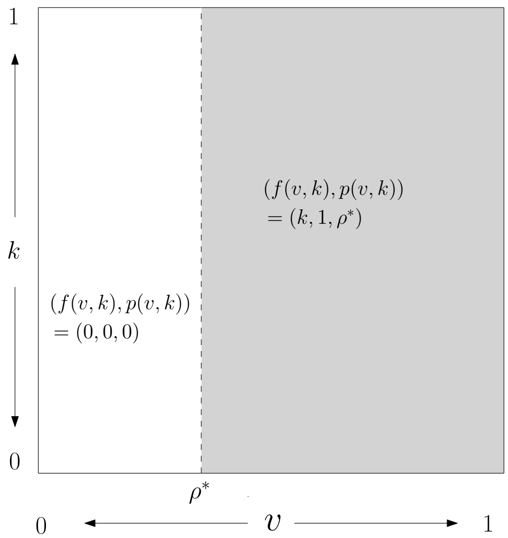

Figure 3.1: POSTED PRICE

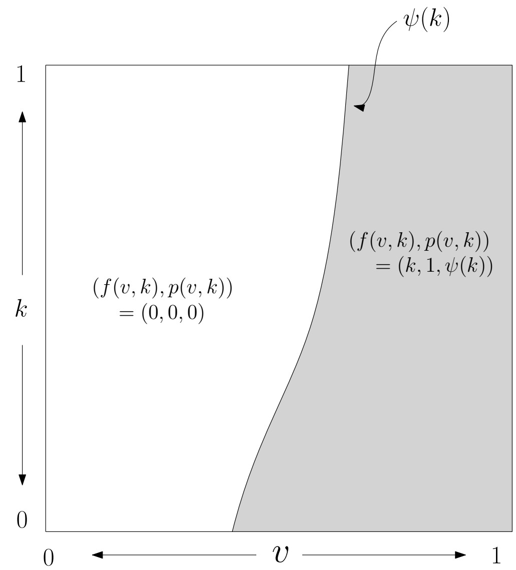

Figure 3.2: RATIO-DEPENDENT POSTED PRICE

DEFINITION **3.4** *A mechanism* (f,p) *is* RATIO-DEPENDENT POSTED PRICE *mechanism if there exists a function* $\psi : K \to V$ *such that for all* $k' > k$,

$$\psi(k) \leq \psi(k'),$$
$$\frac{k}{k'}\psi(k') \leq \psi(k), \textit{and}$$
$$(f(v,k),p(v,k)) = \begin{cases} (0,0,0) & \textit{if } v \leq \psi(k) \\ (k,1,\psi(k)) & \textit{otherwise.} \end{cases}$$

For instance $\psi(k) = (\frac{1}{k+2})^{\frac{1}{k+1}}$ satisfies the conditions that defines a RATIO-DEPENDENT POSTED PRICE mechanism, and this is not a POSTED PRICE mechanism. Observe that the POSTED PRICE mechanism is a special case of the RATIO-DEPENDENT POSTED PRICE mechanism by setting $\psi(k) = \rho$ for all k. Our next proposition shows that the RATIO-DEPENDENT POSTED PRICE mechanism is IC and IR. Therefore, this also proves that the POSTED PRICE mechanism is IC and IR. While the POSTED PRICE mechanism has a unique price in the menu, the RATIO-DEPENDENT POSTED PRICE mechanism has a potentially infinite menu. We call these mechanisms *non-wasteful* as they allocate GOOD_2 (primary good) and GOOD_1 (complement good) in the desired ratio. They are simple to describe since the primary good is allocated fully or not allocated at all.

PROPOSITION **3.1** *A* RATIO-DEPENDENT POSTED PRICE *mechanism is IC and IR.*

Proof: Consider a RATIO-DEPENDENT POSTED PRICE mechanism (f, p) defined by a function ψ. We first show that (f, p) is IR. For any type (v, k),

$$U_{(v.k)}(f(v,k), p(v,k)) = \begin{cases} 0 & \text{if } v \leq \psi(k) \\ v - \psi(k) & \text{otherwise.} \end{cases}$$

Clearly, $U_{(v.k)}(f(v,k), p(v,k)) \geq 0$ and hence (f, p) is IR. We now show that (f, p) is IC. Without loss of generality, consider any two representative types $(v, k), (v', k')$ such that $k' \geq k$. Note that $\frac{k}{k'}\psi(k') \leq \psi(k) \leq \psi(k')$.

$\underline{(v, k) \to (v', k')}$. Note that $\big(f(v', k'), p(v', k')\big)$ is either $(0, 0, 0)$ or $(k', 1, \psi(k'))$, we only need to check deviation to the latter outcome because IR implies (v, k) does not deviate to a type with outcome $(0, 0, 0)$. We check deviation to the outcome $(k', 1, \psi(k'))$ in two cases.
Case 1: $v \leq \psi(k)$. (v, k) has no incentive to deviate to (v', k') because

$$U_{(v,k)}(0,0,0) = 0 \geq v - \psi(k) \geq v - \psi(k') = v \min\{\frac{k'}{k}, 1\} - \psi(k') = U_{(v,k)}(k', 1, \psi(k')).$$

Case 2: $v > \psi(k)$.

$$U_{(v,k)}(k, 1, \psi(k)) = v - \psi(k) \geq v - \psi(k') = v \min\{\frac{k'}{k}, 1\} - \psi(k') = U_{(v,k)}(k', 1, \psi(k')),$$

The second inequality in the first case and the inequality in the second case come from the fact that $\psi(k) \leq \psi(k')$. This means (v, k) has no incentive to deviate to (v', k').

$\underline{(v', k') \to (v, k)}$. Again we only need to check (v', k') deviating to the outcome $(k, 1, \psi(k))$.
Case 1: $v' \leq \psi(k')$.

$$U_{(v',k')}(0,0,0) = 0 \geq \psi(k')\frac{k}{k'} - \psi(k) \geq v'\frac{k}{k'} - \psi(k) = v' \min\{\frac{k}{k'}, 1\} - \psi(k) = U_{(v',k')}(k, 1, \psi(k)).$$

Case 2: $v' > \psi(k')$.

$$U_{(v',k')}(k', 1, \psi(k')) = v' - \psi(k') \geq v'\frac{k}{k'} - \psi(k) = v' \min\{\frac{k}{k'}, 1\} - \psi(k) = U_{(v',k')}(k, 1, \psi(k)).$$

The first inequality in the first case comes from the condition that $\psi(k) \geq \frac{k}{k'}\psi(k')$. The inequality in the second case comes from the following argument. $\psi(k) \geq \frac{k}{k'}\psi(k')$ implies $\psi(k')(1 - \frac{k}{k'}) \geq \psi(k') - \psi(k)$. Since $v' > \psi(k')$, we have $v'(1 - \frac{k}{k'}) \geq \psi(k') - \psi(k)$. Rearranging the terms we get the inequality. ■

3.3.2 Optimal Mechanism

We now describe our optimal mechanism. The expected (ex-ante) revenue of a mechanism (f, p) is given by

$$\Pi(f, p) = \int_{V \times K} p(v, k) dG(v, k).$$

We say that a mechanism (f, p) is **optimal** if

- (f, p) is IC and IR,
- and $\Pi(f, p) \geq \Pi(f', p')$ for any other IC and IR mechanism (f', p').

We can restrict the class of mechanisms to optimize over due to the following result.

Proposition **3.2** *For every IC and IR mechanism (f, p) there exists another IC and IR mechanism (f', p') such that*

1. $\Pi(f', p') = \Pi(f, p)$, *and*
2. $f_1'(v, k) = k f_2'(v, k)$ *for all* (v, k). - **non-wasteful allocation**

Omitted proofs are relegated to the Appendix 3.5. Proposition 3.2 implies that, to find the optimal mechanism it is without loss of generality to focus on the class of mechanisms with the property that allocation of GOOD_1 is k times allocation of GOOD_2. To prove this, we start with an arbitrary IC and IR mechanism (f, p) and construct the desired form mechanism (f', p') while keeping the revenue constant. (f', p') is derived from (f, p) by reducing the allocation of one of the goods so that the allocation ratio is as reported. The payments remain the same. For an insight into why (f', p') is IC, observe that the utility of any type in (f', p') is the same as that in (f, p). There is no incentive to misreport in (f', p') as the utility from misreporting to the same type is weakly lower than that in (f, p), which is IC. Weakly lower utility in (f', p') from misreporting is since only one of the good's allocation is lower while keeping the payment and the other good's allocation unchanged.

These **non-wasteful** mechanisms are denoted by,

$$\mathcal{M} := \{(f, p) : f_1(v, k) = k f_2(v, k) \text{ for all } (v, k)\}.$$

Note: If $(f, p) \in \mathcal{M}$ then $\min\{\frac{f_1(v,k)}{k}, f_2(v, k)\} = f_2(v, k)$ for all (v, k). In the next Proposition and rest of the paper we use the following fact without explicitly stating: for any k,

$$U_{(v,k)}(f(v', k), p(v', k)) = v f_2(v', k) - p(v', k) \text{ for all } v, v'.$$

This result allows us to focus only on one of the allocation function components f_2, and deduce f_1 from it in the final step. However, this does not reduce the problem to a one-dimensional exercise as incentive constraints across the ratio dimension are crucial to the optimal program. The following result makes this clear.

Characterization IC Mechanisms

We characterize the IC mechanisms in the class $\mathcal{M}$.

PROPOSITION **3.3** *$(f,p) \in \mathcal{M}$ is IC if and only if the following are true for any (v,k),*

(1) $f_2(v,k) \leq f_2(v',k)$ *for all* $v' > v$,

(2) $p(v,k) = p(0,1) + vf_2(v,k) - \int_0^v f_2(t,k)dt$,

(3) $\int_0^v f_2(t,k')dt \leq \int_0^v f_2(t,k)dt$ *for all* $k' > k$,

(4) $\int_0^{v\frac{k}{k'}} f_2(t,k)dt \leq \int_0^v f_2(t,k')dt$ *for all* $k' > k$.

The conditions (1) and (2) in Proposition 3.3 correspond to IC constraints between two types on a horizontal line in the type-space (see Figure 3.3). Mechanisms in $\mathcal{M}$ have the property of reducing the IC constraints on any horizontal line equivalent to that of a one-dimensional problem. This is the same as Myerson (1981)'s IC characterization when restricted to any k. However, Proposition 3.3 shows that some 'vertical' and 'diagonal' constraints are enough to guarantee the incentive compatibility of the mechanism. Condition (3) corresponds to the vertical constraints, while (4) corresponds to the diagonal constraints. The arrows in Figure 3.3 indicate the direction in which the incentive constraints need to be satisfied. To see why condition (3) and (4) are necessary for IC, observe that applying conditions (1) and (2) for the types indicated in the figure and then simplifying the IC expression yields the expressions. Interestingly, these 'local' constraints are enough to guarantee global incentive compatibility. Describing optimal mechanisms in multidimensional models is difficult partly because we cannot pin down the binding constraints (Rochet and Chone (1998)). However, due to the incentive constraints characterization, we can do so in this model.

We state two lemmas which we use in our analysis.

LEMMA **3.1** *If a mechanism (f,p) is IC, then $p(0,k) = p(0,1)$ for all k.*

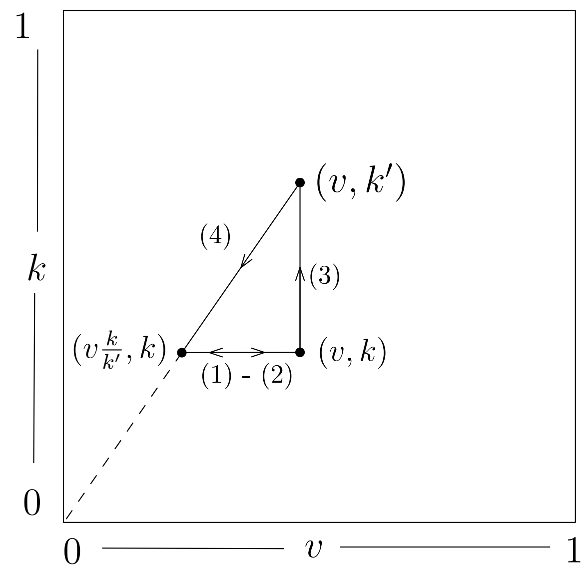

Figure 3.3: IC Constraints

Proof: For any k, $(0,1) \to (0,k)$ implies that $-p(0,1) \geq -p(0,k)$ while $(0,k) \to (0,1)$ implies that $-p(0,k) \geq -p(0,1)$. ∎

In line with the other models in mechanism design, the following standard result holds in this setting too.

LEMMA **3.2** *An IC mechanism (f,p) is individually rational if and only if,*

$$p(0,1) \leq 0.$$

Proof: Fix an IC mechanism (f,p). Suppose that (f,p) is IR. Consider the type $(0,1)$. IR implies that $U_{(0,1)}(f(0,1), p(0,1)) \geq 0$, this simplifies to $p(0,1) \leq 0$. To show the other way, fix any (v,k) and observe that $U_{(v,k)}(f(v,k),(v,k)) = v\min\{\frac{f_1(v,k)}{k}, f_2(v,k)\} - p(v,k) \geq v\min\{\frac{f_1(0,k)}{k}, f_2(0,k)\} - p(0,k) \geq -p(0,k) \geq 0$. The first inequality is from the incentive constraint $(v,k) \to (0,k)$, the second from the fact that allocation functions are non-negative. The third is true since $p(0,1) \leq 0$ implies that $p(0,k) \leq 0$ for all k due to Lemma 3.1. ∎

Using Lemma 3.1, Lemma 3.2, and Proposition 3.3, the equivalent optimal program can now be written as follows,

$$\max_{(f,p)\in\mathcal{M}} \int_0^1 \int_0^1 \big[p(0,1) + v f_2(v,k) - \int_0^v f_2(t,k)dt\big] g(v,k)dvdk \tag{O}$$

$$f_2(v,k) \le f_2(v',k) \text{ for all } v < v', k, \tag{C1}$$

$$\int_0^v f_2(t,k')dt \le \int_0^v f_2(t,k)dt \text{ for all } v, k' > k, \tag{C2}$$

$$\int_0^{v\frac{k}{k'}} f_2(t,k)dt \le \int_0^v f_2(t,k')dt \text{ for all } v, k' > k, \tag{C3}$$

$$p(0,1) \le 0. \tag{C4}$$

Notice that the $p(0,1)$ appears only in the (C4) constraint and to maximize (O) we set $p(0,1) = 0$ without changing any other constraints. We rewrite the objective function (O) by changing the order of integration. Also, since f_2 uniquely determines f_1 and p by Propositions 3.2 and 3.3, respectively, we suppress these decision variables in the optimal program and rewrite it as follows:

Optimal Program

$$\max_{f_2:V\times K\to[0,1]} \int_0^1 \left[\int_0^1 \big(v - \frac{1-G(v|k)}{g(v|k)}\big) f_2(v,k)g(v|k)dv\right] g_k(k)dk \tag{O}$$

$$f_2(v,k) \le f_2(v',k) \text{ for all } v < v', k, \tag{C1}$$

$$\int_0^v f_2(t,k')dt \le \int_0^v f_2(t,k)dt \text{ for all } v, k' > k, \tag{C2}$$

$$\int_0^{v\frac{k}{k'}} f_2(t,k)dt \le \int_0^v f_2(t,k')dt \text{ for all } v, k' > k. \tag{C3}$$

3.3.3 Main Results

We impose the following restrictions on G for our next results.

DEFINITION **3.5** *A distribution G is satisfies* CONDITION A *if for any k, $v(1-G(v|k))$ is strictly concave in v.*

This condition has been used in the literature before (Che and Gale, 2000; Devanur et al., 2020; Mishra and Paramahamsa, 2018). Let,

$$\phi(v,k) := v - \frac{1-G(v|k)}{g(v|k)}.$$

For a k, this is the standard virtual valuation expression. Condition A is equivalent to strictly increasing $\phi(v,k)g(v|k)$ for every k. Notice that $\phi(0,k) < 0$ and $\phi(1,k) > 1$ for all k and that continuity of G ensures continuity of $\phi(v,k)g(v|k)$. Since $g(v|k) > 0$, the solution to $\phi(v,k)g(v|k) = 0$ and $\phi(v,k) = 0$ is the same and unique, for any k. Therefore, whenever Condition A is satisfied, for any k, there exists a unique $v \in (0,1)$ such that $\phi(v,k) = 0$. We denote the value satisfying this equation by $\phi_k^{-1}(0)$.

Definition **3.6** *A distribution G is said to satisfy* Condition B *if it satisfies* Condition A *and for all $k < k'$ the following is true,*

$$\frac{k}{k'}\phi_{k'}^{-1}(0) \leq \phi_k^{-1}(0) \leq \phi_{k'}^{-1}(0).$$

The uniform distribution satisfies this condition as $\phi_k^{-1}(0) = \phi_{k'}^{-1}(0)$ for all k, k'. For a given ratio k, $\phi_k^{-1}(0)$ represents the price at which the seller extracts maximum surplus. The condition says that as k increases, this is increasing but relative to k it is decreasing. We derive the optimal mechanism for a distribution that satisfies this condition after stating the next result.

Theorem **3.1** *If G satisfies Condition B, then the following* ratio-dependent posted price *mechanism is optimal,*

$$\big(f(v,k), p(v,k)\big) = \begin{cases} (0,0,0) & v \leq \phi_k^{-1}(0) \\ (k,1,\phi_k^{-1}(0)) & \textit{otherwise} \end{cases}$$

Proof: Ignoring the constraints (C1), (C2), and (C3), a point-wise maximization (for each k) of the objective function (O) implies that the optimal allocation function f_2 is as in the statement of the theorem, since $\phi(v,k) \leq 0$ for all (v,k) with $v \leq \phi_k^{-1}(0)$ and $\phi(v,k) > 0$ for all (v,k) with $v > \phi_k^{-1}(0)$, due to Condition A. Condition B implies that this mechanism is indeed ratio-dependent posted price mechanism. We have already shown this mechanism to be IC (Proposition 3.1). Hence, the ignored constraints hold. ■

Example 1. Consider a density function $g(v,k) = \frac{v^k}{\ln 2}$. We evaluate the conditional density to $g(v|k) = v^k(k+1)$. From this we derive the virtual valuation to,

$$\phi(v,k) = v - \frac{1 - v^{k+1}}{v^k(k+1)}.$$

We can show that $\phi(v,k)$ is strictly increasing by first order condition, and $\phi_k^{-1}(0) = (\frac{1}{k+2})^{\frac{1}{k+1}}$ satisfies Condition A. Therefore, the optimal mechanism for this distribution evaluates as,

$$\big(f(v,k), p(v,k)\big) = \begin{cases} (0,0,0) & v \leq (\frac{1}{k+2})^{\frac{1}{k+1}} \\ (k, 1, (\frac{1}{k+2})^{\frac{1}{k+1}}) & \text{otherwise} \end{cases}$$

Observe that, for this result we can replace Condition A with a more standard regularity condition, that $\phi(v,k)$ is strictly increasing in v. For a detailed comparison of the regularity condition with Condition A see Devanur et al. (2020), Section 6.1. We use Condition A as we require it for our next result.

Definition **3.7** *A distribution G is satisfies* Condition B′ *if it satisfies* Condition A *and for all $k < k'$ the following is true,*

$$\phi_k^{-1}(0) > \phi_{k'}^{-1}(0).$$

Notice that in Condition B we have $\phi_k^{-1}(0)$ to be strictly decreasing in k whereas the opposite is true in Condition B′.

Theorem **3.2** *If G satisfies* Condition B′*, then* posted price *mechanism is optimal.*

Following is an example of a distribution that satisfies Condition B′ and the optimal mechanism.

Example 2. Consider a density function $g(v,k) = \frac{2}{3}(v+2k)$. The cdf of this distribution is $\frac{vk}{3}(v+2k)$. We evaluate the conditional density to $g(v|k) = \frac{v+2k}{0.5+2k}$. From this we derive the virtual valuation to,

$$\phi(v,k) = \frac{1.5v^2 + 4kv - 2k - 0.5}{v+2k}.$$

We can show that $\phi(v,k)$ is strictly increasing by first order condition, and that

$$\phi_k^{-1}(0) = \frac{-4k + \sqrt{16k^2 + 12k + 3}}{3}$$

is decreasing in k. Therefore, the optimal mechanism for this distribution evaluates to,

$$\big(f(v,k),p(v,k)\big)=\begin{cases}(0,0,0) & v\leq\rho^*\\ (k,1,\rho^*) & \text{otherwise}\end{cases}$$

$\rho^*=\text{argmax}_p\, p\big(1-G_v(p)\big)$, this evaluates to $\rho^*=\frac{\sqrt{13}-2}{3}$.

The following proposition describes the optimal mechanism when the value and ratio random variables are independent. The following result does not require any other condition on the type distribution.

PROPOSITION **3.4** *If $g(v,k)=g_v(v)g_k(k)$, then following* POSTED PRICE *mechanism is optimal,*

$$\big(f(v,k),p(v,k)\big)=\begin{cases}(k,1,p^*) & v\geq p^*\\ (0,0,0) & \textit{otherwise}\end{cases}$$

where p^* is any p that maximizes $p\big(1-G_v(p)\big)$

Proof: We solve the reduced problem by ignoring constraints (C2) and (C3); this can be written as:

Using $g(v|k)=g_v(v)$ we rewrite (O) as,

$$\max_{f_2:V\times K\to[0,1]}\int_0^1\left[\int_0^1\big[v-\frac{1-G_v(v)}{g_v(v)}\big]g_v(v)f_2(v,k)dv\right]g_k(k)dk. \tag{O}$$

$$f_2(v,k)\leq f_2(v',k)\text{ for all }v<v',k. \tag{C1}$$

We first maximize the objective function point-wise for each k, while satisfying the constraint for that k. To that end, fix some k, and observe that maximizing the term inside large bracket along with the monotonocity constraint is the same as in the standard Myerson's problem for a general distribution. Therefore, the solution of f_2, as described in Myerson (1981), is a step function as follows,

$$f_2(v,k)=\begin{cases}1 & v\geq p^*\\ 0 & \text{otherwise}\end{cases}$$

p^* is any p that maximizes $p\big(1-G_v(p)\big)$

Since we have picked an arbitrary k, and this allocation function is independent of k, the point-wise maximization must yield a POSTED PRICE mechanism. We need to verify that the constraints (C2) and (C3) are also satisfied. But since we have shown in Proposition 3.1 that a POSTED PRICE mechanism is IC mechanism; this fact together with Proposition 3.3 implies constraints (C2) and (C3) are satisfied. ■

3.4 Concluding Remarks

Often, models in multidimensional are intractable, even in the two-dimensional case. Even if some of the models are tractable, it is hard to derive a reduced-form solution for the optimal mechanism. In this paper, we consider a two-dimensional private information model with a 'separation' in between the dimensions. This feature helps us solve the problem and provide a reduced-form solution that is simple and intuitive. The POSTED PRICE mechanism can be described by one parameter and involves a finite menu of outcomes. While the RATIO-DEPENDENT POSTED PRICE mechanism involves a potentially infinite size of the menu, it has a simple feature of allocating the *primary* good fully and the *secondary* good in the desired ratio.

There are three main directions we intend to extend this work. First, to explore results in a broader class of distributions, and identifying non-wasteful mechanisms beyond RATIO-DEPENDENT POSTED PRICE mechanism. Second, consider a multi-good perfect complements model. Third, consider a scenario in which multiple agents compete for the same pair of complementary goods.

3.5 Appendix: Omitted Proofs

3.5.1 Proof of the Proposition 3.2.

Proof: Fix an IC and IR mechanism (f, p) and define (f', p') as follows,

$$(f'(v,k), p'(v,k)) := \begin{cases} \big(f_1(v,k), \frac{f_1(v,k)}{k}, p(v,k)\big) & \text{if } \frac{f_1(v,k)}{k} \le f_2(v,k) \\ \big(kf_2(v,k), f_2(v,k), p(v,k)\big) & \text{if } \frac{f_1(v,k)}{k} > f_2(v,k). \end{cases}$$

The new mechanism generates as much revenue as the original mechanism and satisfies the *non-wasteful* allocation condition. Showing that it satisfies IC and IR conditions will prove the proposition. Fix any type (v, k) and to show that this type does not deviate to some other type (u, j), we do this in two cases.

Case 1 - $\frac{f_1(u,j)}{j} \le f_2(u,j)$.

$$\begin{aligned} U_{(v,k)}(f'(v,k), p'(v,k)) &= U_{(v,k)}(f(v,k), p(v,k)) \\ &\ge U_{(v,k)}(f(u,j), p(u,j)) \\ &= v \min\{\frac{f_1(u,j)}{k}, f_2(u,j)\} - p(u,j) \\ &\ge v \min\{\frac{f_1(u,j)}{k}, \frac{f_1(u,j)}{j}\} - p(u,j) \\ &= U_{(v,k)}(f_1(u,j), \frac{f_1(u,j)}{j}, p(u,j)) \\ &= U_{(v,k)}(f'(u,j), p'(u,j)). \end{aligned}$$

Case 2 - $\frac{f_1(u,j)}{j} > f_2(u,j)$.

$$\begin{aligned} U_{(v,k)}(f'(v,k), p'(v,k)) &= U_{(v,k)}(f(v,k), p(v,k)) \\ &\ge U_{(v,k)}(f(u,j), p(u,j)) \\ &= v \min\{\frac{f_1(u,j)}{k}, f_2(u,j)\} - p(u,j) \\ &\ge v \min\{\frac{jf_2(u,j)}{k}, f_2(u,j)\} - p(u,j) \\ &= U_{(v,k)}(jf_2(u,j), f_2(u,j), p(u,j)) \\ &= U_{(v,k)}(f'(u,j), p'(u,j)). \end{aligned}$$

In both the cases, first inequality is by incentive compatibility of (f, p), second inequality by the condition that defines the particular case, the first and last equations by construction of (f', p'), and the rest by definitions. Using first equations and the fact that (f, p) is IR implies that (f', p') is IR. ■

3.5.2 Proof of the Proposition 3.3.

Proof: Let a mechanism $(f, p) \in \mathcal{M}$ be IC, then to show (1) and (2) fix some k. For any $v' > v$, consider the following IC constraints,

$$\begin{aligned}(v, k) \to (v', k) &\equiv v f_2(v, k) - p(v, k) \geq v f_2(v', k) - p(v', k)\\ (v', k) \to (v, k) &\equiv v' f_2(v', k) - p(v', k) \geq v' f_2(v, k) - p(v, k).\end{aligned}$$

After suppressing k in the above inequalities notice that these are the standard one-dimensional IC constraints between two types v, v'. Therefore, in similar fashion to the one-dimensional problem we get (1) by adding the inequalities. For any k applying Myerson (1981)'s revenue equivalence formula we get

$$p(v, k) = p(0, k) + v f_2(v, k) - \int_0^v f_2(t, k) dt \text{ for all } v$$

Applying Lemma 3.1 to this expression we get (2).

To show (3) and (4) consider any $v, k' > k$. IC constraint $(v, k) \to (v, k')$ implies that,

$$\begin{aligned}U_{(v,k)}(f(v, k), p(v, k)) &\geq U_{(v,k)}(f(v, k'), p(v, k'))\\ \implies v f_2(v, k) - p(v, k) &\geq v \min\{\frac{f_1(v, k')}{k}, f_2(v, k')\} - p(v, k')\\ &= v \min\{\frac{k' f_2(v, k')}{k}, f_2(v, k')\} - p(v, k')\\ &= v f_2(v, k') - p(v, k')\\ \implies \int_0^v f_2(t, k) dt &\geq \int_0^v f_2(t, k') dt\end{aligned}$$

IC constraint $(v,k') \to (v\frac{k}{k'},k)$ implies that,

$$
\begin{aligned}
U_{(v,k')}(f(v,k'),p(v,k')) &\geq U_{(v,k')}(f(v\frac{k}{k'},k),p(v\frac{k}{k'},k)) \\
\implies v f_2(v,k') - p(v,k') &\geq v\min\{\frac{f_1(v\frac{k}{k'},k)}{k'}, f_2(v\frac{k}{k'},k)\} - p(v\frac{k}{k'},k) \\
&= v\min\{\frac{k}{k'}f_2(v\frac{k}{k'},k), f_2(v\frac{k}{k'},k)\} - p(v\frac{k}{k'},k) \\
&= v\frac{k}{k'}f_2(v\frac{k}{k'},k) - p(v\frac{k}{k'},k) \\
&= U_{(v\frac{k}{k'},k)}(f(v\frac{k}{k'},k),p(v\frac{k}{k'},k)) \\
\implies \int_0^v f_2(t,k')dt &\geq \int_0^{v\frac{k}{k'}} f_2(t,k)dt
\end{aligned}
$$

The first equality in both the constraints uses the fact that $(f,p) \in \mathcal{M}$. The second implication uses the necessary condition (2) of this Proposition.

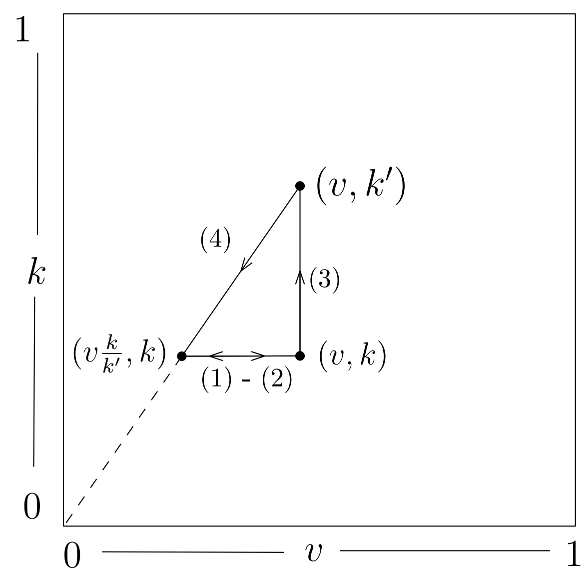

Figure 3.4: IC Constraints

For the only if part, fix any k and notice that IC constraints of the type $(v,k) \to (v',k')$ when $k = k'$ are satisfied by conditions (1) and (2) as this is equivalent to standard one-dimensional one agent model. Therefore, it is enough to show that any type (v,k) does not deviate to a (v',k') in the following two cases.

Case 1: $k' > k$**.**

$$\begin{aligned}
U_{(v,k)}(f(v,k),p(v,k)) &= vf_2(v,k) - p(v,k) \\
&= \int_0^v f_2(t,k)dt - p(0,1) \\
&\geq \int_0^v f_2(t,k')dt - p(0,1) \\
&= vf_2(v,k') - p(v,k') \\
&\geq vf_2(v',k') - p(v',k') \\
&= v\min\{\frac{k'}{k}f_2(v',k'), f_2(v',k')\} - p(v',k') \\
&= U_{(v,k)}(f(v',k'),p(v',k'))
\end{aligned}$$

The second and third equation uses condition (2). The second inequality is from IC constraint $(v,k') \to (v',k')$ which in turn come from conditions (1) and (2) as argued already. The first inequality is from condition (3).

Case 2: $k' < k$**.**

$$\begin{aligned}
U_{(v,k)}(f(v,k),p(v,k)) &= vf_2(v,k) - p(v,k) \\
&= \int_0^v f_2(t,k)dt - p(0,1) \\
&\geq \int_0^{v\frac{k'}{k}} f_2(t,k')dt - p(0,1) \\
&= v\frac{k'}{k}f_2(v\frac{k'}{k},k') - p(v\frac{k'}{k},k') \\
&\geq v\frac{k'}{k}f_2(v',k') - p(v',k') \\
&= v\min\{\frac{k'}{k}f_2(v',k'), f_2(v',k')\} - p(v',k') \\
&= U_{(v,k)}(f(v',k'),p(v',k'))
\end{aligned}$$

The second and third equation uses condition (2), the second inequality is from IC constraint $(v\frac{k'}{k},k') \to (v',k')$ which in turn come from conditions (1) and (2) as argued already. The first inequality is from condition (4). ■

3.5.3 Proof of the Theorem 3.2.

We solve for the optimal mechanism by ignoring the constraint (C3). We show that the optimal in this reduced problem is a POSTED PRICE mechanism. We first prove the following

Lemma towards this.

Lemma **3.3** *If G satisfies* Condition A *then for every mechanism $(f,p)\in\mathcal{M}$ that satisfies constraints (C1), (C2), then the mechanism $(f',p')\in\mathcal{M}$ defined by,*

$$f_2'(v,k)=\begin{cases}0 & \text{if } v\le 1-\int_0^1 f_2(t,k)dt\\ 1 & \text{otherwise.}\end{cases}$$

satisfies constraints (C1), (C2) and generates more(weakly) expected revenue than (f,p).

Proof: It is straightforward to see that constraint (C1) is satisfied. For (C2), observe that, for any (v,k),

$$\int_0^v f_2'(t,k)dt=\begin{cases}0 & \text{if } v\le 1-\int_0^1 f_2(t,k)dt\\ v-1+\int_0^1 f(t,k)dt & \text{otherwise.}\end{cases}\tag{3.1}$$

Fix any $k'>k$, and since (f,p) satisfies constraint (C2) we have,

$$\int_0^1 f_2(t,k')dt\le\int_0^1 f_2(t,k)dt.\tag{3.2}$$

If $v\le 1-\int_0^1 f_2(t,k')dt$, then $\int_0^v f_2'(t,k')dt=0\le\int_0^v f_2'(t,k)dt$, as $f_2'(v,k)\ge 0\ \forall(v,k)$.

Else if $v>1-\int_0^1 f_2(t,k')dt$, then $v>1-\int_0^1 f_2(t,k)dt$ by equation 3.2. Therefore, $\int_0^v f_2'(t,k')dt=v-1+\int_0^1 f_2(t,k')\le v-1+\int_0^1 f_2(t,k)=\int_0^v f_2'(t,k)dt$. The inequality is by equation 3.2. The equations are by expression 3.1.

Now we show that (f',p') generates weakly more expected revenue than (f,p). Fix any k. Denote $\beta_{(f,p,k)}:=1-\int_0^1 f_2(t,k)dt$ and consider the difference in expected revenue of the two mechanisms,

$$\begin{aligned}\int_0^1\phi(v,k)g(v|k)\big(f_2'(v,k)-f_2(v,k)\big)dv &= \int_{\beta_{(f,p,k)}}^1\phi(v,k)g(v|k)\big(f_2'(v,k)-f_2(v,k)\big)dv\\ &\quad-\int_0^{\beta_{(f,p,k)}}\phi(v,k)g(v|k)f_2(v,k)dv\\ &\ge\phi(\beta_{(f,p,k)},k)g(\beta_{(f,p,k)}|k)\int_{\beta_{(f,p,k)}}^1\big(f_2'(v,k)-f_2(v)\big)dv\\ &\quad-\phi(\beta_{(f,p,k)},k)g(\beta_{(f,p,k)}|k)\int_0^{\beta_{(f,p,k)}}f_2(v,k)dv\\ &=\phi(\beta_{(f,p,k)},k)g(\beta_{(f,p,k)}|k)\Big(\int_0^1(f_2'(v,k)-f_2(v,k))dv\Big)\\ &=0\end{aligned}$$

The equations use the definition of (f', p') and rearranging of terms, the inequality is from the fact that $\phi(v,k)g(v|k)$ is increasing. Since we have shown this for an arbitrary k therefore expected revenue from (f', p') is greater(weakly) than (f, p). ■

Proof of Theorem 3.2.

Lemma 3.3 implies that, without loss of generality, we can focus on mechanisms $(f, p) \in \mathcal{M}$ such that there exists $\rho(k)$ **increasing** in k and,

$$f_2(v,k) = \begin{cases} 0 & \text{if } v \le \rho(k) \\ 1 & \text{otherwise.} \end{cases}$$

ρ is increasing because (C2) is satisfied in Lemma 3.3, and due to the definition of f' in Lemma 3.3. We will show that we can improve such a mechanism to a POSTED PRICE mechanism. Fix any such mechanism (f, p) and note that CONDITION B$'$ implies $\phi_k^{-1}(0) > \phi_{k'}^{-1}(0)\ \forall k' > k$. Consider the following three mutually exclusive and exhaustive cases:

1. $\rho(1) \le \phi_1^{-1}(0)$. Consider the following mechanism (f', p') defined by,

$$f_2'(v,k) = \begin{cases} 0 & \text{if } v \le \rho(1) \\ 1 & \text{otherwise.} \end{cases}$$

   Fix any k, note that $\rho(k) \le \rho(1)$. If $v \le \rho(k)$ or $v > \rho(1)$ then $f_2'(v,k) = f_2(v,k)$. $\rho(1) \le \phi_1^{-1}(0) \le \phi_k^{-1}(0)$ implies $\phi(\rho(1),k)g(\rho(1)|k) \le \phi(\phi_k^{-1}(0),k)g(\phi_k^{-1}(0)|k) = 0$ since $\phi(v,k)\rho(v|k)$ is increasing in v. This also implies $\int_{\rho(k)}^{\rho(1)} f_2(v,k)\phi(v,k)g(v|k)dv \le 0$. Noticing $\int_{\rho(k)}^{\rho(1)} f_2'(v,k)\phi(v,k)g(v|k)dv = 0$ by construction implies revenue in (f', p') is more than that of (f, p).

   With some abuse of notation , we use $\rho(0^+)$ to denote $\lim_{k\to 0^+} \rho(k)$, and $\phi_{0^+}^{-1}(0)$ to denote $\lim_{k\to 0^+} \phi_k^{-1}(0)$.

2. $\rho(1) > \phi_1^{-1}(0)$ and $\rho(0^+) < \phi_{0^+}^{-1}(0)$. ρ is increasing in k, and $\phi_k^{-1}(0)$ is strictly decreasing in k and continuous, hence the function $\rho(k) - \phi_k^{-1}(0)$ is strictly increasing (and continuous a.e.). Therefore, there exists a unique k^* such that $\rho(k) > \phi_k^{-1}(0)\ \forall k > k^*$ and $\rho(k) < \phi_k^{-1}(0)\ \forall k < k^*$. Let $v^* := \rho(k^*)$. Define a POSTED PRICE mechanism (f', p') as follows,

$$f_2'(v,k) = \begin{cases} 0 & \text{if } v \le v^* \\ 1 & \text{otherwise.} \end{cases}$$

   We show that (f', p') generates more expected revenue than (f, p) for every k in two following cases.

(a) Fix any $k > k^*$. Note that $v^* \leq \rho(k)$. If $v \leq v^*$ or $v > \rho(k)$ then $f_2'(v,k) = f_2(v,k)$. Since $\phi_k^{-1}(0) \leq \phi_{k^*}^{-1}(0) = v^*$ and $\phi(v,k)g(v|k)$ increasing in v we have $\phi(v,k)g(v|k) > 0$ for all $v > v^*$. Therefore, $\int_{v^*}^{\rho(k)} \big(f_2'(v,k) - f_2(v,k)\big)\phi(v,k)g(v|k)dv \geq 0$ since $f_2'(v,k) = 1$ in this range.

(b) Fix any $k < k^*$. Note that $v^* \geq \rho(k)$. If $v > v^*$ or $v \leq \rho(k)$ then $f_2'(v,k) = f_2(v,k)$. Since $\phi_k^{-1}(0) \geq \phi_{k^*}^{-1}(0) = v^*$ and $\phi(v,k)g(v|k)$ increasing in v we have $\phi(v,k)g(v|k) < 0$ for all $v < v^*$. Therefore, $\int_{\rho(k)}^{v^*} \big(f_2'(v,k) - f_2(v,k)\big)\phi(v,k)g(v|k)dv \geq 0$ since $f_2'(v,k) = 0$ in this range.

3. $\rho(0^+) \geq \phi_{0^+}^{-1}(0)$. Consider the following mechanism (f', p') defined by,

$$f_2'(v,k) = \begin{cases} 0 & \text{if } v \leq \rho(0^+) \\ 1 & \text{otherwise.} \end{cases}$$

Fix any k and note that $\rho(k) \geq \rho(0^+)$. If $v \leq \rho(0^+)$ then $f_2'(v,k) = f_2(v,k)$. Since $\phi_k^{-1}(0) \leq \phi_{0^+}^{-1}(0) \leq \rho(0^+)$ and $\phi(v,k)g(v|k)$ increasing in v we have $\phi(v,k)g(v|k) > 0$ for all $v > \rho(0^+)$. Therefore, $\int_{\rho(0^+)}^{1} \big(f_2'(v,k) - f_2(v,k)\big)\phi(v,k)g(v|k)dv \geq 0$ since $f_2'(v,k) = 1$ in this range.

In each of the above three cases, we have shown that the revenue is higher in a POSTED PRICE mechanism for an arbitrary k. Therefore, a POSTED PRICE mechanism is optimal in the reduced problem we considered. This also implies it is the optimal mechanism since we have shown that a POSTED PRICE mechanism satisfies all the constraints, including the ignored constraint (C3).

Chapter 4

Selling an object with an attribute

4.1 Introduction

In many single object sale settings, along with inherent value, the buyer also derives additional value from an attribute of the object. Consider, for instance, an actor interested in selling rights to her book to a publisher. The publisher derives a reputational value from signing the author, which is privately known to him. Besides, there is a value generated by the book's sale. The per-unit value generated by the book's sale, which depends on the publisher's investments to print and promote the book, is also his private information. The number of books sold is unknown when signing the contract but is public information later. The buyer and the seller hold a common belief over this uncertainty. Typically, in such settings, the seller offers a contract contingent upon such a *publicly* observable quantity. The central theme of the paper is in finding the revenue-maximizing mechanism in such an environment.[1]

Other examples include the transfer of a football player between two clubs or of public-private partnerships in provision of infrastructure. The buying club derives a privately known inherent value from signing the player, through advertising and jersey sale rights of the player, for example. It also derives value from the player's future performance, number of goals scored, for example. While the per-goal value derived is private information of the buying club, the number of goals scored is public knowledge. Such contingent clauses are standard in contracts between FIFA clubs.[2] In highway infrastructure projects, there is

[1] As opposed to a standard model in which all the attributes of an object are collapsed to one-dimensional 'value', we model this in two dimensions. Ability to write contingent contract on one of the dimensions does not allow us to collapse two dimensions into one.

[2] https://resources.fifa.com/image/upload/global-transfer-market-report-2019-men.pdf?

inherent value from the project as well as value from the tolls collected which is uncertain at the time of contracting.

We model this as a mechanism design problem in which the buyer has two-dimensional private information when signing the contract. If the type-space is one-dimensional (in a single object sale, for instance), the optimal mechanism is deterministic (Myerson (1981); Riley and Zeckhauser (1983)), but the standard multi-object optimal mechanism maybe stochastic (that is, allocation of the objects is randomized) for many distributions (Hart and Reny (2015); Thanassoulis (2004)). Although there is one object for sale in our model, multi-dimensional private information combined with a contingent contract may imply a stochastic optimal mechanism.

Contribution. We provide sufficient conditions (on the distribution of the buyer's type) under which there exists an optimal mechanism that is deterministic. The class of these deterministic mechanisms takes a simple threshold structure over the attribute realization. They have the following feature: for each report of the agent, the mechanism offers outcomes contingent on the attribute's realized level. Each of these outcomes has an allocation probability of either 0 or 1. For some reports, the allocation probability is 1 if and only if the realized attribute level is *below* some level, whereas, for some reports, the allocation probability is 1 if and only if the realized attribute level is *above* some level. The types mapped to the former class of outcomes have higher inherent value relative to that of the attribute while the opposite is true for the types mapped to the latter class of outcomes. This disincentivizes misreporting. For the rest of the reports, the agent pays nothing and gets nothing irrespective of the attribute realization.

We propose two implementations for the optimal mechanism. In the first implementation, the buyer pays an upfront fee to choose from a menu of allocations that depend on the attribute realization. In the second, we propose an ex-post individually rational mechanism. The seller does not use any randomization device to implement the mechanism, although she uses the implicit randomization caused by the attribute's uncertainty. The example discussed below illustrates that for some distributions, the seller *has* to use randomization to achieve optimal revenue. We rule out such a possibility for the distributions we consider. The condition we impose on the buyer's distribution is standard in the literature; this goes back to Mcafee and Mcmillan (1988) and has been more recently used by Pavlov (2011); Bikchandani and Mishra (2020) in the multi-dimensional case. Che and Gale (2000); Devanur et al. (2020) have used a one-dimensional version of the condition.

`cloudid=x2wrqjstwjoailnncnod` - refer to page 6 for the definition of Conditional transfer fee.

While we impose conditions on the buyer's type distribution to arrive at our results, we do not impose restrictions on the attribute distribution. Our approach follows Rochet (1987); Mcafee and Mcmillan (1988) in writing the optimal program in terms of the utility of the agent. Then we adapt Pavlov (2011)'s approach of improving upon an arbitrary mechanism to our model.

4.1.1 Related literature

The model of attributes analyzed in this paper is standard in the literature. Eliaz and Frug (2018) do an equilibrium analysis of two-sided private information where attributes of an object are realized over time. Smolin (2020) studies an information disclosure problem in the attribute model in which the seller has access to statistical experiments over the attributes. Although their model has multi-dimensional private information similar to the current paper, the paper fixes the mechanism of sale and focuses on the optimal experiment that reveals information to the buyer. The attribute model can also be found in several other papers, for example - Lancaster (1966); Gabaix and Laibson (2006); Neeman (1996).

Another strand of literature this paper is related to is contingent auctions (see Skrzypacz (2013) for a survey). The critical difference between this literature and our paper is that the buyer's payoff/profit is fully observable and contractible in the literature. Cremer (1987) argued that this leads to full surplus extraction by the seller. Consequently, the literature has focused on the revenue ranking of auctions (Demarzo et al. (2005), for example) rather than finding the optimal mechanism. Explanations for contingent auctions without full surplus extraction in practice include - one agent not having all the bargaining power, moral hazard, budget constraints (Skrzypacz (2013)). Payoff observability is a reasonable assumption in the examples they consider, such as an oil field lease where a government can observe the revenues or merger of two firms where one firm's stakeholders can learn about the joint firm's profits. However, the buyer's payoff is neither fully observable nor contractible for the examples we discussed in the introduction. Only a *factor* of the payoff is observable and contractible. Therefore, we add to this literature by showing the existence of *optimal contingent contracts* where full surplus extraction is not possible due to the payoff's partial contractibility.

This paper is also related to dynamic mechanism design literature. Courty and Li (2000) is among the first papers to study the sequential screening framework. The main difference between sequential screening papers and the current paper is that in the former, there is private information to be learned by the agent after the contract is signed. In the latter, the information revelation is public after the contract is signed. Future realization being public here implies that we have to deal with fewer constraints, although the dynamic nature of the constraints is still crucial. Another contrast is the presence of multi-dimensional private information when signing the contract. To our knowledge, all the dynamic mechanism design papers in the literature consider models that have one-dimensional private information when signing the contract, even though the private information is multi-dimensional during the play of the game.

On a technical level, our paper is close to the two-dimensional mechanism design problem and, more generally, multi-dimensional mechanism design. A complete solution to this problem is hard to solve, even for the two-dimensional case (Manelli and Vincent (2007); Daskalakis et al. (2017)).

The following example illustrates the fact that the optimal mechanism is stochastic for some distributions.

Example. The buyer's type is either $u = (u_1, u_2) = (1, 10)$ or $w = (w_1, w_2) = (3, 0)$ with the seller's belief over these types being 0.6 and 0.4, respectively. The ex-post utility of a buyer with type $v = (v_1, v_2)$ from consuming an outcome (a, t) when x is the level of attribute realized is given by,

$$(v_1 + v_2 x)a - t,$$

where a is the allocation probability of the object, and t is the transfer made by the buyer to the seller. v_1 is the inherent value derived from the object, whereas v_2 is the additional payoff derived from the object when one unit of the attribute is provided. In this example, the seller and the buyer believe x is drawn from $\{0, 1\}$ with equal probability. The contract is implemented after x is realized[3]. A direct mechanism maps the buyer's report and realization of x to an allocation probability and payment pair. In the mechanism shown in Table 4.1, for example, the type w is mapped to $(1, 3)$ and $(0, 0)$ when realized x is 0 and 1, respectively. That is, if the buyer reports w, then he gets the object with probability 1 and pays 3 when realized x is 0, whereas she does not get the object and pays nothing when realized x is 1.

The optimal deterministic mechanism (shown in Table 4.1) generates an ex-ante expected revenue of 4.2, this is beaten by a stochastic mechanism (shown in Table 4.2) generating a revenue of 4.35. The proof of this can be found in the Appendix 4.5.1[4]. Entries in the tables are (allocation probability, payment) pairs.

To illustrate the incentive constraints, consider the mechanism in Table 4.2 and the type $u = (1, 10)$, truthful report yields interim expected utility of $\frac{1}{2}[(1)(1)-6]+\frac{1}{2}[(1+10)(1)-6] =$

[3] One may refer to Timing in Section 4.3 for a clearer understanding of the play of the game.

[4] All the omitted proofs can be found in the Appendix 4.5.

Table 4.1: Optimal deterministic mechanism.

Report	$x = 0$	$x = 1$
u	$(1, 6)$	$(1, 6)$
w	$(1, 3)$	$(0, 0)$

Table 4.2: A stochastic mechanism.

Report	$x = 0$	$x = 1$
u	$(1, 6)$	$(1, 6)$
w	$(1, 3)$	$(0.25, 0.75)$

0. Misreporting to w gives an expected utility of $\frac{1}{2}[(1)(1) - 3] + \frac{1}{2}[(1 + 10)(0.25) - 0.75] = 0$. Therefore, there is no incentive for agent with type u to misreport as w.

4.2 The model

A monopolist (seller) is selling a single unit of an indivisible good to an agent. The agent derives two values from the good; first, there is an inherent value to the agent from consumption of the good; second, there is an additional value from an attribute. The 'level' of the attribute is unknown to the seller and the agent when signing the contract but revealed publicly only over time. Therefore, this is contractible in the mechanism, but the payoff derived from the attribute is private information of the agent. The seller and the agent are assumed to be risk-neutral. The ex-post payoff of the agent with type $v = (v_1, v_2)$ from consumption of an outcome (a, t) when realized attribute 'level' is x is given by,

$$(v_1 + v_2 x)a - t. \tag{4.1}$$

For any generic type v, we use the following subscript notation: v_1 is interpreted as the inherent value of consuming the good. v_2 is per-unit value of the attribute, which is the additional payoff from consuming the good when the attribute is fully provided. When signing the contract, the pair $(v_1, v_2) \in V \equiv [0, 1]^2$ is private information of the agent while $x \in X \equiv [0, 1]$ is a random variable on which both the seller and the agent have a common prior with strictly positive density function ψ. Our results hold even if the support of V is set to $[0, \bar{v}_1] \times [0, \bar{v}_2]$ for some $\bar{v}_1, \bar{v}_2 > 0$, we stick to the present form for ease of exposition.

The seller has a belief G over the type-space V. The corresponding joint density function g is continuous and strictly positive. Densities g and ψ are assumed to be independent.

4.3 The Optimal Mechanism

The seller can potentially offer a contract contingent on the attribute's realization[5]. Therefore, a **direct mechanism** is a pair of functions, an allocation function, $f : V \times X \to [0,1]$, and a payment function $p : V \times X \to \mathbb{R}$.

Timing of the game:

1. Seller announces and *commits* to a mechanism,
2. Agent observes his type v,
3. Agent signs the contract,
4. x is revealed *publicly*,
5. Mechanism is implemented and payoffs are realized.

Given a payment function p we define $\bar{p} : V \to \mathbb{R}$ as the interim payment of the agent:[6],

$$\bar{p}(v) := \int_X p(v,x)\psi(x)dx.$$

An agent with type v and reporting v' to the mechanism (f,p) derives a (interim expected) utility given by,

$$\begin{aligned} U_{(f,p)}(v,v') &:= \int_X \big[(v_1 + v_2 x) f(v',x) - p(v',x)\big]\psi(x)dx \\ &= v_1 \int_X f(v',x)\psi(x)dx + v_2 \int_X x f(v',x)\psi(x)dx - \bar{p}(v'). \end{aligned} \tag{4.2}$$

The equation is derived by using the definition of $\bar{p}$ and rearranging the terms. We will use this form of utility expression in our analysis. We simply use $U_{(f,p)}(v)$ to denote $U_{(f,p)}(v,v)$, that is, the utility by truth-telling.

Definition **4.1** *A mechanism (f,p) is said to be incentive compatible (IC) if for every v, v',*

$$U_{(f,p)}(v) \geq U_{(f,p)}(v,v')$$

[5]Note that this is more general than a contract that is not dependent on the attribute realization.

[6]This notation will be useful for our analysis.

The condition implies that in an incentive compatible mechanism, an agent with type v derives higher expected utility from truth-telling than misreporting to any other type v'. Note that the agent can potentially misreport in one or both the dimensions. Due to the revelation principle, it is without loss of generality to focus on incentive compatible mechanisms. We also impose a participation constraint.

DEFINITION **4.2** *A mechanism (f, p) is said to be individually rational (IR) if for every v,*

$$U_{(f,p)}(v) \geq 0$$

The expected (ex-ante) revenue of a mechanism (f, p) is given by

$$\begin{aligned}\Pi(f,p) &= \int_V \Big(\int_X p(v,x)\psi(x)dx \Big) dG(v), \\ &= \int_V \bar{p}(v) dG(v). \end{aligned} \tag{4.3}$$

We say that a mechanism (f, p) is **optimal** if

- (f, p) is IC and IR,
- and $\Pi(f, p) \geq \Pi(f', p')$ for any other IC and IR mechanism (f', p').

Since the objective and the constraints only depend on $\bar{p}$, the optimal program is written as:

$$\max_{(f,\bar{p})} \int_V \bar{p}(v) dG(v) \tag{O}$$

$$U_{(f,\bar{p})}(v) \geq U_{(f,\bar{p})}(v, v') \text{ for all } v, v' \tag{IC}$$

$$U_{(f,\bar{p})}(v) \geq 0 \text{ for all } v, \tag{IR}$$

$$0 \leq f(v, x) \leq 1 \text{ for all } (v, x). \tag{Feasibility}$$

The payment function p can later be derived from $\bar{p}$; this can be interpreted as an upfront payment to be made to the seller when signing the contract. We discuss this and other implementations later in detail, in the Implementation section (4.3.2).

The following lemma is due to Rochet (1987).[7]

[7] We skip the proof as it is derived by applying Rochet (1987)'s Proposition 2 to our setting. We use the fact that utility function in equation 4.2 is linear in v.

LEMMA **4.1** *A mechanism (f, p) is IC iff,*

$$U_{(f,p)} \text{ is convex, and,}$$
$$\Big[\int_X f(v,x)\psi(x)dx \quad , \quad \int_X xf(v,x)\psi(x)dx\Big] \in \partial U(v) \text{ for almost all } v.$$

We suppress the subscript of the utility function whenever there is no confusion but note that there is an underlying mechanism when we mention utility functions. Since U is convex, $\nabla U(v)$ is defined almost everywhere and equals $[\int_X f(v,x)\psi(x)dx \quad \int_X xf(v,x)\psi(x)dx]$. Therefore, using Lemma 4.1 the optimal program can be reformulated as,

$$\max_{(f,\bar{p})} \int_V [\nabla U(v).v - U(v)]dG(v) \tag{O}$$
$$U \text{ is convex,} \tag{IC}$$
$$U(v) \geq 0 \text{ for all } v, \tag{IR}$$
$$0 \leq f(v,x) \leq 1 \text{ for all } (v,x). \tag{Feasibility}$$

LEMMA **4.2** *The seller's expected revenue from an IC mechanism $(f, \bar{p})$ is*

$$\int_0^1 U(1,v_2)g(1,v_2)dv_2 + \int_0^1 U(v_1,1)g(v_1,1)dv_1 - \int_V \big[3g(v) + v.\nabla g(v)\big]U(v)dv$$

This lemma is derived using integration by parts. Mcafee and Mcmillan (1988) is the first to use this approach in solving a multi-dimensional optimization problem. We skip the proof as it is a direct application of their analysis.

4.3.1 Results

To prove our results, we require the following condition on the seller's belief over the agent's type. This condition is standard in the literature and satisfied by a large class of distributions.

DEFINITION **4.3** *We say that belief g satisfies Condition A iff, for all v,*

$$3g(v) + v.\nabla g(v) \geq 0. \tag{4.4}$$

Before we state our main result, we partition the type-space into three parts,

$$V^1 = \{v \in V : v_1 = 1\},$$
$$V^2 = \{v \in V : v_2 = 1, v_1 < 1\},$$
$$V^0 = V/\{V^1 \cup V^2\}.$$

Theorem **4.1** *If g satisfies Condition A, then there is an optimal mechanism $(f, \bar{p})$ which is deterministic. Moreover, $(f, \bar{p})$ satisfies the following properties,*

1. *For every $v \in V^1$, there exists some κ_v such that,*

$$f(v, x) = \begin{cases} 1 & \text{if } x < \kappa_v, \\ 0 & \text{if } x \geq \kappa_v. \end{cases}$$

2. *For every $v \in V^2$, there exists some γ_v such that,*

$$f(v, x) = \begin{cases} 0 & \text{if } x \leq \gamma_v, \\ 1 & \text{if } x > \gamma_v. \end{cases}$$

3. *For every $v \in V^0$, either*

   - $(f(v, x), \bar{p}(v)) = (0, 0)\ \forall x$, *or*
   - *there exists some $u \in V^1 \cup V^2$ such that $(f(v, x), \bar{p}(v)) = (f(u, x), \bar{p}(u))\ \forall x$.*

The theorem shows that there exists an optimal mechanism, which is **deterministic**. The first point states that for every type in V^1 (the right line in Figure 4.1), if the attribute level is *below* a certain threshold (that corresponds to the type), the object is allocated with probability one; otherwise, the seller retains the object. In contrast, for every type in V^2 (the top line in Figure 4.1), the object is given with probability one if the realized level of the attribute is *above* a certain threshold that corresponds to the type. Otherwise, the seller retains the object. The third point states that every other type not included in the above two cases (all the interior points, left and bottom boundaries in Figure 4.1) is mapped to the outcome $(0, 0)$ or to one of the outcomes that are mapped to types in V^1 or V^2.

For an insight into the optimal mechanism, note that the types in V^1 and V^2 differ in terms of the inherent value of the object relative to that of the attribute. While types in V^1 have a higher inherent value relative to the attribute, the opposite is true for the types in V^2. The mechanism assigns the good to types in V^2 for a higher realization of the attribute. Whereas the types in V^1 get the good for lower attribute realizations, thereby incentivizing from misreporting. More on this in Remark 4.1 after the next Proposition.

Proof sketch.[8] To solve for the optimal mechanism, we fix an IC and IR mechanism $(f, \bar{p})$ and construct another indirect mechanism that is an improvement over this in terms

[8]The proof is in Appendix 4.5.2.

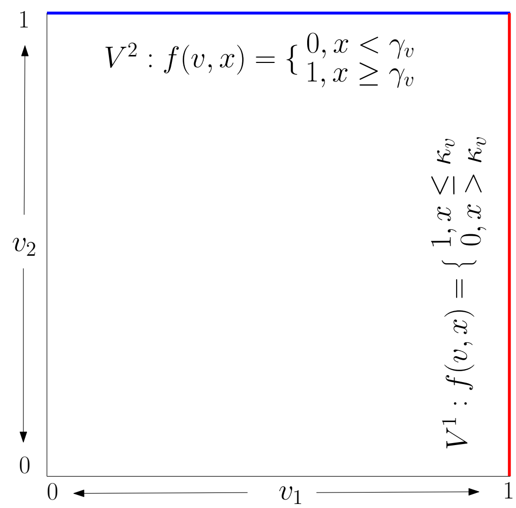

Figure 4.1: Optimal mechanism

of revenue. Given Condition A, Lemma 4.2 implies weakly increasing the revenue by keeping the utility of types in V^1 and V^2 constant while weakly reducing the utilities of types in the rest of the type-space. This exercise is done by manipulating the mechanism $(f, \bar{p})$, first by deleting the outcomes mapped to types in V^0, and second, by adjusting the outcomes mapped to V^1 and V^2 such that their utilities remain the same. Third, we add a null message to the mechanism and map it to the outcome $(0, 0)$, irrespective of the attribute realization. Fourth we make sure that the utility of any type in V^0 is weakly lower in the new mechanism by reporting as any type in $V^1 \cup V^2$ or the null message. Notice that the new mechanism is IR since the outcome $(0, 0)$ is present. Since we started with an arbitrary mechanism and reached a mechanism with the desired structure, we claim that the theorem is true. Note that this exercise does not imply that every optimal mechanism takes this form; we only claim that there exists at least one such optimal mechanism.

In the theorem, we did not discuss the payment function or the nature of the thresholds defining the mechanism. The following proposition states these two properties of the optimal mechanism.

Proposition **4.1** *For the optimal mechanism $(f, \bar{p})$ defined in Theorem 4.1, the following statements are true.*

1. *For all $v, v' \in V^1$ with $v_2 < v_2'$,*

$$\kappa_v \leq \kappa_{v'},$$
$$\bar{p}(v) = \bar{p}(1,0) + \int_{\kappa_{(1,0)}}^{\kappa_v} \psi(x)dx + v_2 \int_0^{\kappa_v} x\psi(x)dx - \int_0^{v_2} \Big(\int_0^{\kappa_{(1,t)}} x\psi(x)dx \Big) dt.$$

2. *For all $v, v' \in V^2$ with $v_1 < v_1'$,*

$$\gamma_v \geq \gamma_{v'},$$
$$\bar{p}(v) = \bar{p}(0,1) + \int_{\gamma_v}^{\gamma_{(0,1)}} x\psi(x)dx + v_1 \int_{\gamma_v}^{1} \psi(x)dx - \int_0^{v_1} \Big(\int_{\gamma_{(t,1)}}^{1} \psi(x)dx \Big) dt.$$

The Proposition states for types in V^1, V^2, there is monotonicity of the thresholds described in Theorem 4.1. The payment of any type in V^1, V^2 is pinned down by the thresholds, and payment by the respective lowest type - $(1,0)$ for V^1, $(0,1)$ for V^2. The result is similar to Myerson (1981)'s revenue equivalence result applied over V^1 and V^2 separately. This characterization of the payments will help determine the optimal thresholds, which will lead to a full description of the mechanism. The proof of the Proposition is in Appendix 4.5.2.

Remark **4.1** *The proposition shows that the utilities (and contingent allocations) are monotonic for types in V^1 or V^2. To see why, observe that for any $v, v' \in V^1$ with $v_2 < v_2'$ we have $\kappa_v < \kappa_{v'}$. From the theorem and the equation 4.2, the utility of the type $v = (1, v_2)$ is equals*

$$\int_0^{\kappa_v} \psi(x)dx + v_2 \int_0^{\kappa_v} x\psi(x)dx.$$

This is clearly less than the utility of the type $v' = (1, v_2')$ which equals to

$$\int_0^{\kappa_{v'}} \psi(x)dx + v_2' \int_0^{\kappa_{v'}} x\psi(x)dx,$$

since $\kappa_v < \kappa_{v'}$ and $v_2 < v_2'$. A similar argument can be given for types in V^2, that the utility of types in increasing as the v_1 component is increasing, using the theorem and the proposition together. This monotonicity of utilities with respect to the agent's private information is due to incentive constraints and is true in other multidimensional models (for example see Rochet (1987)).

However, in the theorem we state that for types in V^1, the allocation function $f(v, x)$ is decreasing with respect to attribute realization x, which is public information. This is due to separation of types in V^1 from V^2 in the optimal mechanism. In the theorem, for the types *in V^2, the allocation function is an increasing function with respect to attribute realization. The example discussed in the Introduction section illustrates this, note that both mechanisms in Table 4.1 and 4.2 are incentive compatible. For the type $w = (3, 0)$ the allocation is decreasing in x, whereas for the type $u = (1, 10)$ the allocation is (weakly) increasing in x. This is due to difference in relative weights attached inherent value and the attribute value across these types.*

4.3.2 Ex-post IR Implementation

From the optimal program, it is clear that seller's (ex-ante) expected revenue depends on 'average' (over attribute realization) payment $\bar{p}$ and there is freedom on how the payment function p (of the direct mechanism (f, p)) is chosen. However, $\bar{p}$ can readily be interpreted as a one-time upfront payment when signing the contract, while the allocation of the object is dependent on the attribute realization. Given the optimal mechanism derived above, this upfront payment has an interpretation as payment made to secure the object conditional on certain levels of the attribute. This implementation is interim IR, but not ex-post IR for all realizations of the attribute. The following Proposition addresses this issue.

PROPOSITION **4.2** *For every IR mechanism $(f, \bar{p})$, there exists a payment function p such that the mechanism (f, p) is ex-post IR and implements $(f, \bar{p})$.*

Proof: Fix some mechanism $(f, \bar{p})$ that is (interim) IR, then,

$$\int_X (v_1 + v_2 x) f(v, x) \psi(x) dx - \bar{p}(v) \geq 0 \ \forall v.$$

Let ϵ_v be defined by:

$$\epsilon_v := \int_X (v_1 + v_2 x) f(v, x) \psi(x) dx - \bar{p}(v).$$

Define payment function p by the following equation:

$$p(v, x) := (v_1 + v_2 x) f(v, x) - \epsilon_v \ \forall (v, x). \tag{4.5}$$

The ex-post utility of any type (v, x) in the mechanism (f, p) is $(v_1 + v_2 x) f(v, x) - p(v, x)$, but this equals ϵ_v by equation 4.5. However, $\epsilon_v \geq 0 \ \forall v$ by construction (since $(f, \bar{p})$ is IR). Therefore, the mechanism (f, p) is ex-post IR. Also, (f, p) implements $(f, \bar{p})$ as $\int_X p(v, x) \psi(x) dx = \bar{p}(v) \ \forall v$ by using equation 4.5 and the defintion of ϵ_v. ■

The direct mechanism can be implemented by a menu in which each item is a mapped to allocation and payment pairs contingent on attribute realization, as in the example discussed in the Introduction section.

4.4 Concluding Remarks

We show that when Condition A is satisfied, the seller need not use any randomization device. A cursory look at the incentive constraints would reveal that the optimization exercise is technically close to the standard two-dimensional optimization problem (Pavlov (2011); Bikchandani and Mishra (2020)). In the standard problem, though, Condition A does not guarantee a deterministic optimal mechanism. One reason it is true in the current model is that the seller implicitly uses the randomness of the attribute level (that is revealed publicly at implementation). Nevertheless, this implicit randomization is not sufficient to guarantee a deterministic optimal in the model, as the example in the Introduction section illustrated.

The main difference is that in the standard problem there is more freedom on how allocation functions corresponding to each of the dimensions can be mapped. In the current model, there is a precise relationship across dimensions, determined by the belief over the attribute. This in the form of an extra constraint to the optimal program can make the problem potentially tractable. More precisely, given the optimal mechanism in Theorem 4.1, we can write the equation 4.2 in one of the following forms,

$$\begin{aligned} U(v,v') &= v_1 \int_0^{\kappa_{v'}} \psi(x)dx + v_2 \int_0^{\kappa_{v'}} x\psi(x)dx - p(v'), \text{ or} \\ &= v_1 \int_{\gamma_{v'}}^1 \psi(x)dx + v_2 \int_{\gamma_{v'}}^1 x\psi(x)dx - p(v') \end{aligned}$$

depending on whether $v' \in V^1$ or V^2. Technically, it amounts to determining thresholds for each type in V^1 and V^2 to pin down allocations. This is just one number as compared to a 2-object sale in which allocation of both the objects need to be determined simultaneously.

This is a non-trivial problem, since even in a simple example in which V is uniformly distributed and X is a binary random variable, the number of potential mechanisms is large. We suggest a possible direction to this in the next subsection.

Other interpretations of the model. - One interpretation is the sale of online advertisement slots. A firm buying a slot might derive payoff from two sources: first, by showing the ad to a consumer, and second, when a consumer clicks on the ad. While whether the ad was clicked or not is public information, the value from showing the ad and click on it is private information. Therefore, the seller can offer a mechanism contingent only on the clicks.

Another interpretation of x is the durability of a good. Suppose the good has multiple uses where v_1 represents immediate consumption value while v_2 represents use-value over time. The seller offers a contract that depends on the good's durability but cannot contract upon the utility the buyer derives from the good over time.

4.4.1 Characterizing the optimal mechanism

The precise nature of thresholds in Theorem 4.1 needs to be solved. We leave this for future work in this paper, but in this section, we discuss an observation that could potentially describe the solution in more detail. The arguments in this section are not rigorous but instead, illustrate a potential approach for complete characterization.

Fix some optimal mechanism $(f, \bar{p})$ as described in the Theorem 4.1 and we partition the type-space as follows:

$$
\begin{aligned}
V^{1*} &= \{v \in V : (f(v), \bar{p}(v)) = (f(u), \bar{p}(u)) \text{ for some } u \in V^1\} \\
V^{2*} &= \{v \in V : (f(v), \bar{p}(v)) = (f(u), \bar{p}(u)) \text{ for some } u \in V^2\} \\
V^{0*} &= V / V^{1*} \cup V^{2*}
\end{aligned}
$$

The objective function (O) can be written as,

$$
\int_{V^{0*}} \bar{p}(v) d(G(v)) + \int_{V^{1*}} \bar{p}(v) d(G(v)) + \int_{V^{2*}} \bar{p}(v) d(G(v)).
$$

The first term in the above expression is equal to zero as $\bar{p}(v) = 0 \ \forall v \in V^{0*}$. Let $g^1(v)$ be the density function of the types in V^{1*} that report v. Using Proposition 4.1 and rearranging the terms we can write the second term as,

$$
\int_0^1 \left(v_2 - \frac{\int_{v_2}^1 g^1(v) dv}{g(v)} \right) \left(\int_0^{\kappa_v} x \psi(x) dx \right) g^1(v) dv + \int_0^1 \left(\bar{p}(1,0) + \int_{\kappa_{(1,0)}}^{\kappa_v} \psi(x) dx \right) g^1(v) dv
$$

Note that the second term in the above expression is positive, and the first term is positive when $v_2 = 1$. If we can show that g^1 is continuous then for some v_2* we can set $\kappa_v = 1$ for $v_2 \geq v_2*$ without violating constraints. We can make a similar argument for V^2. The observations lead to the following conjecture.

Conjecture **1** *In the optimal mechanism $(f, \bar{p})$, there exists a type v with $v_1, v_2 < 1$ such that,*

$$
f(v', x) = 1 \ \forall (v', x) \ \textit{with} \ v'_1 \geq v_1, v'_2 \geq v_2.
$$

The Conjecture states that the object is sold with probability 1 irrespective of attribute realization if the reported type is in the 'north-east' corner of the type-space. A similar result may be expected for types in the 'south-west' - that the object is not assigned irrespective of the attribute realization.

4.5 Appendix: Omitted Proofs

4.5.1 Proof of the example.

For optimal *deterministic* mechanism, the seller solves the following optimal program (the IR, IC constraints are interim while the objective is ex-ante). Let a_{ij} and t_{ij} denote allocation probability and payment, respectively when the buyer reports i, and the attribute realization is j.

$$\max_{a_{ij},t_{ij}:i\in\{u,w\},j\in\{0,1\}} 0.6\big[\frac{1}{2}t_{u0}+\frac{1}{2}t_{u1}\big]+0.4\big[\frac{1}{2}t_{w0}+\frac{1}{2}t_{w1}\big] \tag{O}$$

$$\frac{1}{2}(u_1a_{u0}-t_{u0})+\frac{1}{2}((u_1+u_2)a_{u1}-t_{u1})\geq 0 \tag{IR$_u$}$$

$$\frac{1}{2}(w_1a_{w0}-t_{w0})+\frac{1}{2}((w_1+w_2)a_{w1}-t_{w1})\geq 0 \tag{IR$_w$}$$

$$\frac{1}{2}(u_1a_{u0}-t_{u0})+\frac{1}{2}((u_1+u_2)a_{u1}-t_{u1})\geq \frac{1}{2}(u_1a_{w0}-t_{w0})+\frac{1}{2}((u_1+u_2)a_{w1}-t_{w1}) \tag{IC$_{u\to w}$}$$

$$\frac{1}{2}(w_1a_{w0}-t_{w0})+\frac{1}{2}((w_1+w_2)a_{w1}-t_{w1})\geq \frac{1}{2}(w_1a_{u0}-t_{u0})+\frac{1}{2}((w_1+w_2)a_{u1}-t_{u1}) \tag{IC$_{w\to u}$}$$

$$a_{ij}\in\{0,1\} \text{ for all } i\in\{u,w\}, j\in\{0,1\}. \tag{Feasibility}$$

Denoting $t_i := t_{i0}+t_{i1}$ for $i\in\{u,w\}$, substituting $u=(1,10), w=(3,0)$, and simplifying the expressions, we re-write the optimal program as below,

$$\max_{a_{ij},t_i:i\in\{u,w\},j\in\{0,1\}} 0.3t_u+0.2t_w \tag{O}$$

$$a_{u0}+11a_{u1}-t_u\geq 0 \tag{IR$_u$}$$

$$3a_{w0}+3a_{w1}-t_w\geq 0 \tag{IR$_w$}$$

$$a_{u0}+11a_{u1}-t_u\geq a_{w0}+11a_{w1}-t_w \tag{IC$_{u\to w}$}$$

$$3a_{w0}+3a_{w1}-t_w\geq 3a_{u0}+3a_{u1}-t_u \tag{IC$_{w\to u}$}$$

$$a_{ij}\in\{0,1\} \text{ for all } i\in\{u,w\}, j\in\{0,1\}. \tag{Feasibility}$$

In a feasibled solution, if $a_{u0} > a_{u1}$, we can swap them and slacken constraints IR$_u$ and IC$_{u\to w}$ while holding remaining constraints the same, thereby improving the objective. Therefore, $a_{u0}\leq a_{u1}$ in the optimal. By a similar argument we can show that $a_{w0}\geq a_{w1}$ in the optimal. If $(a_{u0},a_{u1})=(0,0)$, we can set $(a_{u0},a_{u1})=(1,1)$ and $t_u=12$ and improve the revenue without violating any constraints. If $(a_{w0},a_{w1})=(0,0)$, we can set $(a_{w0},a_{w1})=(1,0)$

and $t_w = 1$ and improve the revenue without violating any constraints. That leaves us with following four cases:

Case 1. $(a_{u0}, a_{u1}) = (0, 1), (a_{w0}, a_{w1}) = (1, 0)$. IR_u, IR_w imply $t_u \leq 11$ and $t_w \leq 3$, respectively. Setting $t_u = 11$ and $t_w = 3$ does not violate the other constraints, hence generating revenue of 3.9.

Case 2. $(a_{u0}, a_{u1}) = (0, 1), (a_{w0}, a_{w1}) = (1, 1)$. IR_w implies $t_w \leq 6$ and $\mathrm{IC}_{u\to w}$ implies $t_u \leq t_w - 1$. Setting $t_u = 5$ and $t_w = 6$ does not violate the other constraints, hence generating revenue of 2.7.

Case 3. $(a_{u0}, a_{u1}) = (1, 1), (a_{w0}, a_{w1}) = (1, 0)$. IR_u, IR_w imply $t_u \leq 12$ and $t_w \leq 3$, respectively. Setting $t_u = 12$ and $t_w = 3$ does not violate the other constraints, hence generating revenue of 4.2.

Case 4. $(a_{u0}, a_{u1}) = (1, 1), (a_{w0}, a_{w1}) = (1, 1)$. IR_w implies $t_w \leq 6$. Adding $\mathrm{IC}_{u\to w}$ and $\mathrm{IC}_{w\to u}$ yields $t_u = t_w$. Setting $t_u = t_w = 6$ does not violate the other constraints, hence generating revenue of 3.

Therefore, an optimal deterministic mechanism is from Case 3, as described in Table 4.1. To solve for optimal (not restricting deterministic) mechanism one needs to solve the same optimal program except replacing the feasibility constraint by new feasibility constraint - $0 \leq a_{ij} \leq 1$ for all $i \in \{u, w\}, j \in \{0, 1\}$. One can easily verify that the mechanism in Table 4.2 satisfies the constraints and generates more revenue than the optimal deterministic mechanism; therefore, the optimal mechanism is stochastic.

4.5.2 Proof of Theorem 4.1 and Proposition 4.1.

Fix an IC and IR mechanism $(f, \bar{p})$. We construct an indirect mechanism that is an improvement over $(f, \bar{p})$ in terms of revenue. This indirect mechanism takes the form stated in the Theorem. Since we started with an arbitrary mechanism, the Theorem is then proven. Towards proof, we define two terms based on the mechanism $(f, \bar{p})$.

For any $v \in [0, 1]$, κ_v solves:

$$\int_0^{\kappa} x\psi(x)dx = \int_0^1 xf(v, x)\psi(x)dx.$$

For any $v \in [0,1]$, γ_v solves:

$$\int_{\gamma}^{1} \psi(x)dx = \int_{0}^{1} f(v,x)\psi(x)dx.$$

The above two terms are well-defined due to the following argument. In the first expression, note that the LHS is continuous and strictly monotonic (since $\psi(x) > 0$), ranging from 0 to $\int_0^1 x\psi(x)dx$. RHS is a number in the same range since $0 \leq f(v,x) \leq 1 \ \forall (v,x)$. This implies existence and uniqueness of κ_v, an identical argument suffices to show the same for γ_v.

We will use the following preparatory lemma in the proof of the theorem.

LEMMA **4.3** *For any $(f, \bar{p})$, and any v, the following statements are true,*

(a) $\int_0^{\kappa_v} \psi(x)dx \geq \int_0^1 f(v,x)\psi(x)dx$,

(b) $\int_{\gamma_v}^1 x\psi(x)dx \geq \int_0^1 xf(v,x)\psi(x)dx$.

Proof: Fix some v and observe the following statements,

(a)

$$\int_0^{\kappa_v} x\psi(x)dx - \int_0^1 xf(v,x)\psi(x)dx = 0$$

(by the definition of κ_v)

$$\implies \int_0^{\kappa_v} x(1-f(v,x))\psi(x)dx - \int_{\kappa_v}^1 xf(v,x)\psi(x)dx = 0$$

(by splitting the integral and rearranging the terms)

$$\implies \int_0^{\kappa_v} \kappa_v(1-f(v,x))\psi(x)dx - \int_{\kappa_v}^1 \kappa_v f(v,x)\psi(x)dx \geq 0$$

(by a property of definite integrals)

$$\implies \int_0^{\kappa_v} (1-f(v,x))\psi(x)dx - \int_{\kappa_v}^1 f(v,x)\psi(x)dx \geq 0$$

(since $\kappa_v \geq 0$)

$$\implies \int_0^{\kappa_v} \psi(x)dx - \int_0^1 f(v,x)\psi(x)dx \geq 0$$

(by rearranging the terms).

(b)

$$\int_{\gamma_v}^{1} \psi(x)dx - \int_{0}^{1} f(v,x)\psi(x)dx = 0$$

(by the definition of γ_v)

$$\implies \int_{\gamma_v}^{1} (1 - f(v,x))\psi(x)dx - \int_{0}^{\gamma_v} f(v,x)\psi(x)dx = 0$$

(by splitting the integral and rearranging the terms)

$$\implies \int_{\gamma_v}^{1} \gamma_v(1 - f(v,x))\psi(x)dx - \int_{0}^{\gamma_v} \gamma_v f(v,x)\psi(x)dx = 0$$

(multiplying both sides by γ_v)

$$\implies \int_{\gamma_v}^{1} x(1 - f(v,x))\psi(x)dx - \int_{0}^{\gamma_v} x f(v,x)\psi(x)dx \geq 0$$

(by a property of definite integrals)

$$\implies \int_{\gamma_v}^{1} x\psi(x)dx - \int_{0}^{1} x f(v,x)\psi(x)dx \geq 0$$

(by rearranging the terms).

■

Now, we will construct the desired indirect mechanism. Let $M = V^1 \cup V^2 \cup \{\emptyset\}$. Define a mechanism $\hat{f} : M \times X \to [0,1]$, $\hat{\bar{p}} : M \to \mathbb{R}$ as follows,

If $m \in V^1$ then,

$$\hat{f}(m,x) = \begin{cases} 1 & \text{if } x < \kappa_m, \\ 0 & \text{if } x \geq \kappa_m. \end{cases}$$

$$\hat{\bar{p}}(m) = \bar{p}(m) + (\int_{0}^{\kappa_m} \psi(x)dx - \int_{0}^{1} f(m,x)\psi(x)dx).$$

Else if $m \in V^2$ then,

$$\hat{f}(m,x) = \begin{cases} 0 & \text{if } x < \gamma_m, \\ 1 & \text{if } x \geq \gamma_m. \end{cases}$$

$$\hat{\bar{p}}(m) = \bar{p}(m) + (\int_{\gamma_m}^{1} x\psi(x)dx - \int_{0}^{1} x f(m,x)\psi(x)dx).$$

Else if $m = \emptyset$ then,

$$\hat{f}(m, x) = 0, \hat{\bar{p}}(m) = 0 \ \forall x.$$

The utility of any type v in the mechanism $(\hat{f}, \hat{\bar{p}})$ denoted by $\hat{U}(v)$ is given by,

$$\hat{U}(v) = \max_{m \in M} v_1 \int_0^1 \hat{f}(m, x)\psi(x)dx + v_2 \int_0^1 x\hat{f}(m, x)\psi(x)dx - \hat{\bar{p}}(m).$$

We fix a generic type v and evaluate its utility when choosing a message in each subset of the partition of M, viz., V^1, V^2, and $\{\emptyset\}$ in three points below.

1. For any message $m = (1, m_2) \in V^1$ the utility of a generic type v is

$$\begin{aligned}
&v_1 \int_0^1 \hat{f}(m, x)\psi(x)dx + v_2 \int_0^1 x\hat{f}(m, x)\psi(x)dx - \hat{\bar{p}}(m) \\
&= v_1 \int_0^{\kappa_m} \psi(x)dx + v_2 \int_0^{\kappa_m} x\psi(x)dx - \bar{p}(m) - \Big(\int_0^{\kappa_m} \psi(x)dx - \int_0^1 f(m, x)\psi(x)dx\Big) \\
&= v_1 \int_0^{\kappa_m} \psi(x)dx + v_2 \int_0^1 xf(m, x)\psi(x)dx - \bar{p}(m) - \Big(\int_0^{\kappa_m} \psi(x)dx - \int_0^1 f(m, x)\psi(x)dx\Big) \\
&= v_1 \int_0^1 f(m, x)\psi(x)dx + v_2 \int_0^1 xf(m, x)\psi(x)dx - \bar{p}(m) - (1 - v_1)\big(\int_0^{\kappa_m} \psi(x)dx - \int_0^1 f(m, x)\psi(x)dx\big) \\
&\leq v_1 \int_0^1 f(m, x)\psi(x)dx + v_2 \int_0^1 xf(m, x)\psi(x)dx - \bar{p}(m) \\
&= U(v, m) \leq U(v).
\end{aligned}$$

The first equality is from the construction of the mechanism $(\hat{f}, \hat{\bar{p}})$ while the second uses the definition of κ_m in the second term of the expression. To arrive at the third equality, we first add and subtract $v_1 \int_0^1 f(m, x)\psi(x)dx$ and then rearrange the terms. The first inequality uses the fact that $v_1 \leq 1$ and $\int_0^{\kappa_m} \psi(x)dx \geq \int_0^1 f(m, x)\psi(x)dx$ (this is proved in Lemma 4.3(a)). The second inequality is due to the fact that $(f, \bar{p})$ is IC.

Note that both the inequalities hold with equality when $v \in V^1$. The first since $v_1 = 1$, and the second since (f, p) is IC. That is, a type $v \in V^1$ does not gain by misreporting to any other type in V^1, and indeed gets the same utility in $(\hat{f}, \hat{\bar{p}})$ as in $(f, \bar{p})$ by reporting its own type.

2. For any message $m = (m_1, 1) \in V^2$ the utility of a generic type v is

$$
\begin{aligned}
& v_1 \int_0^1 \hat{f}(m,x)\psi(x)dx + v_2 \int_0^1 x\hat{f}(m,x)\psi(x)dx - \hat{\bar{p}}(m) \\
&= v_1 \int_{\gamma_m}^1 \psi(x)dx + v_2 \int_{\gamma_m}^1 x\psi(x)dx - \bar{p}(m) - \Big(\int_{\gamma_m}^1 x\psi(x)dx - \int_0^1 xf(m,x)\psi(x)dx \Big) \\
&= v_1 \int_0^1 f(m,x)\psi(x)dx + v_2 \int_{\gamma_m}^1 x\psi(x)dx - \bar{p}(m) - \Big(\int_{\gamma_m}^1 x\psi(x)dx - \int_0^1 xf(m,x)\psi(x)dx \Big) \\
&= v_1 \int_0^1 f(m,x)\psi(x)dx + v_2 \int_0^1 xf(m,x)\psi(x)dx - \bar{p}(m) - (1-v_2)\big(\int_{\gamma_m}^1 x\psi(x)dx - \int_0^1 xf(m,x)\psi(x)dx \big) \\
&\leq v_1 \int_0^1 f(m,x)\psi(x)dx + v_2 \int_0^1 xf(m,x)\psi(x)dx - \bar{p}(m) \\
&= U(v,m) \leq U(v).
\end{aligned}
$$

The first equality is from the construction of the mechanism $(\hat{f}, \hat{\bar{p}})$ while the second uses the definition of γ_m. To arrive at the third equality, we first add and subtract $v_2 \int_0^1 xf(m,x)\psi(x)dx$ and then rearrange the terms. The first inequality uses the fact that $v_2 \leq 1$ and $\int_{\gamma_m}^1 x\psi(x)dx \geq \int_0^1 xf(m,x)\psi(x)dx$ (this is proved in Lemma 4.3(b)).The second inequality is due to the fact that $(f, \bar{p})$ is IC.

Note that both the inequalities hold with equality when $v \in V^2$. The first since $v_2 = 1$, and the second since (f, p) is IC. That is, a type $v \in V^2$ does not gain by misreporting to any other type in V^2, and indeed gets the same utility in $(\hat{f}, \hat{\bar{p}})$ as in $(f, \bar{p})$ by reporting its own type.

3. For the message $m = \emptyset$ the utility of a generic type v is 0. Since $(f, \bar{p})$ is IR we have $0 \leq U(v)$.

Any type gets weakly less utility in the new mechanism since the utility is weakly less by choosing a message in any of the partition's three subsets. Also, the utility of types in V^1 or V^2 remains unchanged by choosing a message that is their type. We summarize these observations as,

$$
\begin{aligned}
\hat{U}(v) &\leq U(v) \text{ for all } v, \\
\hat{U}(v) &= U(v) \text{ for all } v \in V^1 \cup V^2.
\end{aligned}
\tag{4.6}
$$

Using equations 4.6 and Lemma 4.2 we conclude that the difference in revenue of the two mechanisms

$$\Pi(\hat{f}, \hat{\bar{p}}) - \Pi(f, \bar{p}) = \int_V [3g(v) + v.\nabla g(v)]\big(U(v) - \hat{U}(v)\big)dv \geq 0.$$

The inequality is due to Condition A. The mechanism $(\hat{f}, \hat{\bar{p}})$ is IR because of the existence of the outcome $(0, 0)$ for $\emptyset$ message in the mechanism. Clearly, this mechanism satisfies feasibility constraint by construction. This completes proof of Theorem 4.1.

Proof of Proposition 4.1. We have argued in the proof of the Theorem 4.1 that the mechanism $(\hat{f}, \hat{\bar{p}})$ is IC when restricted to types in V^1 and V^2. Since these are one-dimensional subset of the type-space, we can apply Myerson (1981)'s IC characterization on V^1 and V^2 separately. From this we apply results from Theorem 4.1 to arrive at the results in the proposition. This requires changing the order of integration, we skip the details of the derivation.

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