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A RANDOM FUNCTIONAL CENTRAL LIMIT THEOREM FOR MARTINGALES

By

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1. Introduction. The last decade has witnessed great developments in the area of martingale central limit theorems (CLT). A recent paper by B. M. BROWN [3] may be referred to for a brief outline of the historical development, and a good bibliography. It may be mentioned that the most general types of results in this direction provide not only a proof of the classical Lindeberg—Feller CLT for martingales, but also guarantee the weak convergence of all finite dimensional distributions of an a.e. sample continuous stochastic process to those of a Wiener process. A *functional CLT* (also known as an *invariance principle*) is proved which says that the distributions of the said process converge weakly to a Wiener measure on $C[0, 1]$.

Functional CLT's were proved for martingales under the stationarity and ergodicity assumptions by BILLINGSLEY [1], [2] and IBRAGIMOV [6]. These conditions were relaxed and replaced by a Lindeberg-type condition by BROWN [3]. The present paper extends Brown's results to a martingale sequence with random indices, proving a functional CLT. The main results are given in section 2. Classical random CLT's for martingales are proved by CSORGO [4] and PRAKASA RAO [7]. We shall see at the end of section 2 that conditions imposed by them for proving the CLT imply ours, and are, in fact, much more restrictive.

2. The main results. We adopt the same notations as BROWN's [3]. Let $\{S_n, \mathcal{F}_n, n \geqq 1\}$ be a martingale sequence on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with $S_0 = 0$. Define $X_n = S_n - S_{n-1}, n \geqq 1$. $\mathcal{F}_0$ need not be the trivial σ-field $\{\emptyset, \Omega\}$. Let $\mathbf{E}_{j-1}(Y) = \mathbf{E}(Y/\mathcal{F}_{j-1})$. Define

$$\sigma_j^2 = \mathbf{E}_{j-1}(X_j^2), \qquad j \geqq 1, \tag{2.1}$$

$$V_n^2 = \sum_{j=1}^{n} \sigma_j^2, \qquad n \geqq 1, \tag{2.2}$$

$$s_n^2 = \mathbf{E}(V_n^2) = \mathbf{E}(S_n^2), \quad n \geqq 1. \tag{2.3}$$

We assume the following two conditions are satisfied:

$$V_n^2 s_n^{-2} \xrightarrow{\mathbf{P}} 1 \quad \text{as} \quad n \to \infty \tag{2.4}$$

$$s_n^{-2} \sum_{j=1}^{n} \mathbf{E}[X_j^2 I(|X_j| \geqq \varepsilon s_n)] \to 0, \tag{2.5}$$

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as $n\to\infty$ for all $\varepsilon>0$, where $I(B)$ denotes the indicator function of B. Martingale functional CLT's were proved by BROWN [3] under (2.4) and (2.5). Under the same conditions, we shall prove here the following two theorems, the second one giving in fact a random functional CLT for martingales. With this end, first define the process

$$\xi_n(t) = s_n^{-1}(S_r + X_{r+1}(ts_n^2 - s_r^2)/(s_{r+1}^2 - s_r^2)), \tag{2.6}$$

for $0\le t\le 1$, and $s_n^{-2}s_r^2 \le t \le s_n^{-2}s_{r+1}^2$, $r=0,1,\dots,n-1$, $s_0=0$. Then we have the following two theorems.

THEOREM 1. *Let $B\in\mathcal{F}_k$ and $\mathbf{P}(B)>0$. Then under (2.4) and (2.5) one has*

$$\lim_{n\to\infty} \mathbf{P}(s_n^{-1}S_n \le x|B) = \Phi(x) = (2\pi)^{-\frac12}\int_{-\infty}^{x} \exp(-\tfrac12 y^2)\,dy,$$

for all x. Further, all the finite dimensional distributions of $\xi_n(t)$ converge weakly under the measure $\mathbf{P}_B$ to the finite dimensional distributions of the Wiener measure, where

$$\mathbf{P}_B(A) = \mathbf{P}(A|B) \quad \textit{for any} \quad A\in\mathcal{F}. \tag{2.7}$$

THEOREM 2. *Let $\{\nu_n\}$ be a sequence of positive integer valued random variables (rv's) defined on $(\Omega,\mathcal{F})$. Also, let there exist a sequence $\{a_n\}$ of positive integers such that $a_n\to\infty$ as $n\to\infty$, and*

$$s_{\nu_n}^2/s_{a_n}^2 \xrightarrow{\mathbf{P}} \lambda, \tag{2.8}$$

for some positive rv λ. Then under (2.4) and (2.5) the process $\{\xi_{\nu_n}(t):0\le t\le 1\}$ converges weakly to the Wiener measure.

Theorem 1 ensures a Rényi type mixing condition (see RENYI [8]). This is the major tool used in proving theorem 2 along the lines of BILLINGSLEY [2].

PROOF OF THEOREM 1. Let $B\in\mathcal{F}_k$ and $\mathbf{P}(B)>0$. Define

$$S_n^B = S_n \quad \text{if} \quad n\ge k+1, \quad S_k^B = 0, \quad X_{n,B} = S_n^B - S_{n-1}^B \quad \text{for} \quad n\ge k+1, \tag{2.9}$$

so that $X_{n,B}=X_n$ if $n\ge k+2$, $X_{k+1,B}=S_{k+1}$. To prove the theorem, first observe that $\{S_n^B, \mathcal{F}_n, n\ge k+1\}$ is a martingale on $(\Omega,\mathcal{F},\mathbf{P}_B)$. To see this, note that with the use of the notation $\mathbf{E}_B(f)$ for $\int f\,d\mathbf{P}_B$, one has for any $A\in\mathcal{F}$, and for any $n\ge k+2$,

$$\begin{aligned}\int_A \mathbf{E}_B(X_n^B|\mathcal{F}_{n-1})\,d\mathbf{P}_B &= \int_A \mathbf{E}_B(X_n|\mathcal{F}_{n-1})\,d\mathbf{P}_B = \int_A X_n\,d\mathbf{P}_B = \\ &= (\mathbf{P}(B))^{-1}\int_{A\cap B} X_n\,d\mathbf{P} = (\mathbf{P}(B))^{-1}\int_{A\cap B}\mathbf{E}(X_n|\mathcal{F}_{n-1})\,d\mathbf{P} = \\ &= (\mathbf{P}(B))^{-1}\int_A I_B\mathbf{E}(X_n|\mathcal{F}_{n-1})\,d\mathbf{P}.\end{aligned} \tag{2.10}$$

But $B\in\mathcal{F}_k\subset\mathcal{F}_{n-1}(n\ge k+2)\Rightarrow I_B\mathbf{E}(X_n|\mathcal{F}_{n-1})$ is $\mathcal{F}_{n-1}$ measurable. Hence

$$\mathbf{E}_B(X_n|\mathcal{F}_{n-1}) = I(B)\mathbf{E}(X_n|\mathcal{F}_{n-1}) = 0 \quad \text{a.e.} \quad [\mathbf{P}_B].$$

Define now $\sigma^2_{n,B} = \mathbf{E}_B(X^2_{n,B}|\mathcal{F}_{n-1})$, $n \geqq k+1$. Proceeding as in the earlier paragraph one gets

$$\sigma^2_{n,B} = I(B)\mathbf{E}(X^2_n|\mathcal{F}_{n-1}) \quad \text{a.e} \quad [\mathbf{P}_B]. \tag{2.11}$$

Let $V^2_{n,B} = \sum_{j=k+1}^{n} \sigma^2_{j,B}$ for $n \geqq k+1$. Then one gets

$$\begin{aligned} V^2_{n,B} &= I(B)\sum_{j=k+2}^{n} \mathbf{E}(X^2_j|\mathcal{F}_{j-1}) + \sigma^2_{k+1,B} \quad \text{a.e.} \quad [\mathbf{P}_B] \\ &= I(B)V^2_n - I(B)\sum_{j=1}^{k+1} \mathbf{E}(X^2_j|\mathcal{F}_{j-1}) + \sigma^2_{k+1,B} \quad \text{a.e.} \quad [\mathbf{P}_B[. \end{aligned} \tag{2.12}$$

Hence, $V^2_{n,B}/s^2_n \xrightarrow{\mathbf{P}_B} 1$ as $n \to \infty$ and proceeding as in lemma 1 of BROWN [3]

$$\mathbf{E}_B(V^2_{n,B}/s^2_n) \to 1 \quad \text{as} \quad n \to \infty. \tag{2.13}$$

This leads to

$$V^2_{n,B}(\mathbf{E}_B V^2_{n,B})^{-1} \xrightarrow{\mathbf{P}_B} 1 \quad \text{as} \quad n \to \infty. \tag{2.14}$$

Again the Lindeberg condition for the sequence $\{X_{n,B}, n \geqq k+1\}$ of rv's namely

$$(\mathbf{E}_B V^2_{n,B})^{-1} \sum_{j=k+1}^{n} \mathbf{E}_B[X^2_{j,B} I(|X_{j,B}| > \varepsilon \mathbf{E}_B^{\frac{1}{2}}(V^2_{n,B}))] \to 0 \tag{2.15}$$

as $n \to \infty$ for all $\varepsilon > 0$, follows from the definition of $X_{j,B}$'s ($j \geqq k+1$), (2.5) and (2.13). Hence, by theorem 2 of BROWN [3] one has

$$\lim_{n\to\infty} \mathbf{P}_B((\mathbf{E}_B V^2_{n,B})^{-\frac{1}{2}} S_{n,B} \leqq x) = \Phi(x). \tag{2.16}$$

(2.9), (2.13) and (2.16) now lead to

$$\lim_{n\to\infty} \mathbf{P}(s_n^{-1} S_n \leqq x|B) = \Phi(x). \tag{2.17}$$

From the same theorem of Brown, it follows that the finite dimensional distributions of $\xi_n(t)$ converge weakly under the measure $\mathbf{P}_B$ to the corresponding finite dimensional distributions of the Wiener measure. This completes the proof of Theorem 1.

REMARK 1. It is also possible to have a result similar as Brown's theorem 3. This essentially says that if $\{D, \mathcal{D}, \mathbf{W}\}$ is the probability space, where $D = D[0, 1]$, $\mathcal{D}$ is the Borel σ-field generated by open sets in D, and $\mathbf{W}$ is Wiener measure on $D[0, 1]$, then $\mathbf{P}(\xi_n \in A|B) \to \mathbf{W}(A)$, for all $\mathbf{W}$-continuity sets A in D and $B \in \mathbf{U}$, where $\mathbf{U} = \{B : B \in \mathcal{F}_k$ for some $k \geqq 1$ and $\mathbf{P}(B) > 0\}$. Note that Brown's result is for all $\mathbf{W}'$-continuity sets A in $C = C[0, 1]$, where $\mathbf{W}'$ is Wiener measure on C, but the extension is trivial since $\mathbf{W}(D - C) = 0$.

PROOF OF THEOREM 2. We proceed analogously as theorem 10.3 and 17.2 of BILLINGSLEY [3]. First, let p_n be a sequence of real numbers such that

$$s_{p_n} \to \infty \text{ and } s_{p_n}/s_n \to 0 \quad \text{as} \quad n \to \infty.$$

Defining now $Y_n(t)=0$ if $ts_n^2<s_{p_n}^2$ and $=s_n^{-1}(S_r-S_{p_n})$ if $s_{p_n}^2\leqq s_r^2\leqq ts_n^2\leqq s_{r+1}^2$ for all $r\geqq p_n$, one gets from (2.6)

$$|\xi_n(t)-Y_n(t)| = \begin{cases}|\xi_n(t)| & \text{if } ts_n^2 < s_{p_n}^2 \\ s_n^{-1}|S_{p_n}+X_{r+1}(ts_r^2-s_n^2)/(s_{r+1}^2-s_r^2)| & \\ \text{if } s_{p_n}^2 \leqq s_r^2 \leqq ts_n^2 \leqq s_{r+1}^2 \text{ for all } r \geqq p_n. & \end{cases} \tag{2.18}$$

Now from (2.6)

$$\sup_{0\leqq t\leqq s_{p_n}^2} |\xi_n(t)| \leqq s_n^{-1} \max_{1\leqq i\leqq p_n} (|S_i|+|X_i|) \leqq 3s_n^{-1} \max_{1\leqq i\leqq p_n} |S_i|, \tag{2.19}$$

using $X_i=S_i-S_{i-1}$ $(1\leqq i\leqq p_n)$. Thus, from (2.18),

$$\sup_t |\xi_n(t)-Y_n(t)| \leqq 3s_n^{-1} \max_{1\leqq i\leqq p_n} |S_i|+s_n^{-1} \max_{1\leqq r\leqq n} |X_r|. \tag{2.20}$$

The Kolmogorov inequality for martingales gives

$$\mathbf{P}\left\{\max_{1\leqq i\leqq p_n} |S_i| > \varepsilon s_n\right\} \leqq \varepsilon^{-2}s_n^{-2}s_{p_n}^2 \to 0 \tag{2.21}$$

as $n\to\infty$. Also, from (2.5), $s_n^{-1} \max_{1\leqq r\leqq n} |X_r| \xrightarrow{\mathbf{P}} 0$ as $n\to\infty$. (2.20) and (2.21) now give

$$\sup_t |\xi_n(t)-Y_n(t)| \xrightarrow{\mathbf{P}} 0 \text{ as } n \to \infty. \tag{2.22}$$

By virtue of theorem 1 (the remarks following it), (2.22) and theorem 3 of Brown, it follows that

$$|\mathbf{P}\{(Y_n\in A)\cap B\}-\mathbf{P}(Y_n\in A)\mathbf{P}(B)| \to 0 \tag{2.23}$$

as $n\to\infty$ for all $B\in\mathrm{U}$ and for all W-continuity sets A in D.

Next we proceed as theorem 17.2 of Billingsley, changing the definitions of $\Phi_n(t, w)$ in his (17.16) by

$$\Phi_n(t, w) = ts_{v_n(w)}^2/s_{a_n}^2 \quad \text{if} \quad s_{v_n(w)}^2/s_{a_n}^2 \leqq 1$$

and $t\theta$ otherwise. This leads to the result.

REMARK 2. Condition (2.8) seems to be more involved than the usual condition

$$v_n/a_n \xrightarrow{\mathbf{P}} \lambda, \quad \text{as} \quad n \to \infty, \tag{2.24}$$

where λ is a positive rv. It is easy to check that for a stationary sequence $\{X_i\}$ of rv's with $\mathbf{E}(X_1)=0$, $\mathbf{E}(X_1^2)=\sigma^2$, (2.8) in fact reduces to (2.24). However, (2.24) along with (2.4) and (2.5) will not lead to theorem 2 in general. The following example illustrates this.

Let $X_1, X_2, \ldots$ be independent normal variables with zero means, and $\mathbf{V}(X_1)=$ $=\mathbf{V}(X_2)=1$, $\mathbf{V}(X_i)=\exp(i/\log i)-\exp((i-1)/\log(i-1))$ for $i\geqq 3$. Then,

$$s_n^2 = \mathbf{V}(S_n) = \exp(n/\log n)-\exp(2/\log 2)+2, \tag{2.25}$$

for $n\geqq 3$. Define $\xi_n(t)$ $(0\leqq t\leqq 1)$ as in (2.6). Using $n/\log n-(n-1)/\log(n-1)\to 0$ as $n\to\infty$, it is easy to check that $\mathbf{V}(X_n)/s_n^2\to 0$ as $n\to\infty$. In this case $V_n^2s_n^{-2}=1$ for all

$n \geqq 1$ so that (2.4) is satisfied. Also, appealing to theorem 2, p. 492 of FELLER [5] we find that (2.5) is satisfied. Hence, from BROWN's [3] result, ξ_n converges weakly in $C[0, 1]$ to W, the standard Brownian motion process.

Define now

$$v_n = \max\{j \leqq n : S_j \geqq 0\}, \quad n \geqq 1; \tag{2.26}$$

$$m_n = [n - (\log n)^2], \quad n \geqq 1, \tag{2.27}$$

$[u]$ denoting the integer part of u. Then, since, $\log(s_{m_n}^2/s_n^2) \sim -\log n$ (by $a_n \sim b_n$ we mean $a_n/b_n \to 1$ as $n \to \infty$), it follows that

$$s_{m_n}^2/s_n^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{2.28}$$

Note now that using (2.28) and the weak convergence of ξ_n to W in $C[0, 1]$,

$$\mathbf{P}(v_n < m_n) = \mathbf{P}\left\{\sup_{s_{m_n}^2 s_n^{-2} \leqq t \leqq 1} W_n(t) < 0\right\} \to \mathbf{P}\left\{\sup_{0 \leqq t \leqq 1} W(t) < 0\right\} = 0 \quad \text{as} \quad n \to \infty. \tag{2.29}$$

From the definition of m_n in (2.27), it follows now that

$$v_n/n \xrightarrow{\mathbf{P}} 1. \tag{2.30}$$

However, $\xi_{v_n}(1) = s_{v_n}^{-1} S_{v_n} \geqq 0$ for all n, so that ξ_{v_n} does not converge weakly in $C[0, 1]$ to W.

REMARK 3. CSORGO [4] proved a random CLT for a sequence of martingales with $\mathbf{E}(X_1^2) = \sigma_1^2$, $\mathbf{E}(X_k^2 | \mathcal{F}_{k-1}) = \sigma_1^2$ for all $k \geqq 2$. It is easy to see then that $V_n^2/s_n^2 = 1$ for all n so that (2.4) is automatically satisfied. Also, $s_n^{-2} \max_{1 \leqq j \leqq n} \sigma_j^2 = n^{-1} \to 0$ as $n \to \infty$. Further, defining

$$\varphi_j(t) = \mathbf{E}[\exp(itX_j) | \mathcal{F}_{j-1} [= \mathbf{E}_{j-1}[\exp(itX_j)]], \quad j \geqq 1, \tag{2.24}$$

$$f_n(t) = \prod_{j=1}^{n} \varphi_j(t/s_n), \quad n \geqq 1, \tag{2.25}$$

one gets $f_n(t) = \left(1 - \dfrac{t^2}{2n} + o(n^{-1})\right)^n$, so that $\log f_n(t) \to -\frac{1}{2}t^2$ as $n \to \infty$. It follows now from theorem 1 of BROWN [3] that (2.5) holds. Thus Csorgo's assumptions imply ours.

REMARK 4. PRAKASA RAO [7] proved a random CLT under stationarity and ergodicity conditions of BILLINGSLEY [1], [2] along with the strong mixing condition

$$|\mathbf{P}(B|A) - \mathbf{P}(B)| \leqq \psi(n), \tag{2.26}$$

if $A \in \mathcal{F}_l$, $B \in$ the σ-algebra generated by $(X_{l+n}, X_{l+n+1}, \ldots)$ for a fixed l, where $1 \geqq \psi(1) \geqq \psi(2) \geqq \ldots$, $\lim_{n \to \infty} \psi(n) = 0$. One can see easily that stationarity and ergodicity imply (2.5) and (2.4), where (2.26) is redundant.

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