A Study on Partial List Coloring

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CERTIFICATE

This is to certify that the dissertation entitled **A Study on Partial List Coloring** submitted by **Ganesh Gupta** to Indian Statistical Institute, Kolkata, in partial fulfillment for the award of the degree of **Master of Technology in Computer Science** is a bonafide record of work carried out by him under my supervision and guidance. The dissertation has fulfilled all the requirements as per the regulations of this institute and, in my opinion, has reached the standard needed for submission.

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Abstract

List coloring is a variation of coloring where instead of having a global set of colors available to choose from for each vertex, we constrain each vertex with a list of k colors, which is a k-sized subset of the global set of colors. The goal is to assign each vertex a color from its corresponding list so that no two adjacent vertices get the same color. Such a coloring is called a *list coloring* of the graph for this particular assignment of lists to vertices. If for every possible assignment of lists of size k to the vertices of a graph, there is a list coloring of the graph, then the graph is said to be *k-choosable*. The minimum value of k for which a graph is k-choosable is known as its *list chromatic number*. Therefore, for a k-choosable graph, if we assign lists of size l to its vertices, where $l < k$, it may not be possible to color all the vertices of the graph, i.e. there may not exist a list coloring for the whole graph. But can we give any lower bound on the number of vertices that can be colored using colors from their respective lists? The partial list coloring conjecture tries to answer this question. This conjecture states that for a graph G with n vertices and list chromatic number $\chi_l(G)$, and any assignment of t-sized lists to the vertices of G, where $t \leq \chi_l(G)$, at least $\frac{tn}{\chi_l(G)}$ vertices can be colored using colors from their respective lists in such a way that no two adjacent vertices get the same color. This conjecture has not yet been proven for general graphs. It has been proven for some special classes of graphs. We study the proof of this conjecture for claw-free graphs, graphs G having chromatic number at least $\frac{|V(G)|-1}{2}$, and chordless graphs.

Contents

Chapter 1

Introduction

A *proper vertex coloring* of a graph G is an assignment of a color to each vertex $v \in V(G)$ such that no two adjacent vertices are assigned the same color.

A *list vertex coloring* or *list coloring* is a variant of vertex coloring along with a constraint assigned to each vertex in the form of a list of colors.

In this report, we will study the Partial List Coloring conjecture [1] (which we shall formally state in Chapter 2) and see various kind of graphs for which this conjecture has been proved.

1.1 Definitions

Let $G(V, E)$ be a graph where V is the set of vertices of G and E is the set of edges of G.

- Degree of a vertex is the number of edges incident on it. For a vertex $v \in V(G)$, we denote the degree of v as $d(v)$.
- For $X \subseteq V$ and $U \subseteq V \setminus X$, $N_X(U)$ is defined as the set of neighbours of vertices of U in X, i.e. $N_X(U) = \{v \in X : \exists u \in U \text{ such that } (u, v) \in E\}$.
- For $S \subseteq V$, we denote by $G[S]$ the subgraph induced in G by S.
- A *proper vertex coloring* or *proper coloring* of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices are assigned the same color.
- A graph G is said to be *t-colorable* if it has a coloring using t colors.
- The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors required in any proper coloring of G. That is, it is the minimum integer t such that G is t-colorable.

- An *independent set* of a graph G is defined as a set of vertices in $V(G)$ such that no two vertices in the set share an edge between them.
- The *independence number* $\alpha(G)$ of a graph G is the maximum possible size of an independent set in G.

Chapter 2

Coloring and list coloring

Let $G(V, E)$ be a graph and let P be a set.

We say that a function $f : V \to P$ is a *coloring* of G using the set of colors P.

A *proper coloring* of G is a coloring such that $\forall (u, v) \in E$, $f(u) \neq f(v)$.

A proper coloring f of a graph which uses r colors $p_1, p_2, \ldots, p_r$, partitions the vertex set into r sets $C_1, C_2, \ldots, C_r$ such that $\forall v \in C_i$, $f(v) = p_i$. The sets $C_1, C_2, \ldots, C_r$ are also called the *color classes* of the coloring f. Since two vertices that have the same color cannot be adjacent in G, it follows that each color class of a proper coloring is an independent set in G. Conversely, if the vertex set of a graph G can be partitioned into r independent sets $I_1, I_2, \ldots, I_r$, then assigning every u the color p_i if and only if $u \in I_i$ gives a proper coloring of G using r colors. Thus, we have the following observation.

Observation 1. *A graph is r-colorable if and only if its vertex set can be partitioned into r independent sets.*

Theorem 1. *Let G be a graph with n vertices. If there is a proper coloring f of G which uses r colors, then there exists a color class of f which contains at least $\frac{n}{r}$ vertices.*

Proof:
We know that

$$\sum_{i=1}^{r} |C_i| = n$$

Suppose there does not exist a color class with at least $\frac{n}{r}$ vertices. In other words,

$$\forall i, |C_i| < \frac{n}{r}$$

Then,

$$\sum_{i=1}^{r} |C_i| < \frac{n}{r} \cdot r < n$$

Hence, a contradiction. ☐

Corollary 1. *For any graph G, $\alpha(G) \geq \frac{n}{\chi(G)}$.*

Proof:
Since $\chi(G)$ is the chromatic number of G, there exists a coloring of G which uses $\chi(G)$ colors. So by Theorem 1, there exists a color class of size at least $\frac{n}{\chi(G)}$. Since a color class is an independent set, we have $\alpha(G) \geq \frac{n}{\chi(G)}$. ☐

For a graph G and $t \leq \chi(G)$, let $\alpha_t(G)$ be the maximum cardinality of a set $S \subseteq V$ such that $G[S]$ can be properly colored using t colors.

Theorem 2. *For a graph G, if there is a proper coloring f which uses r colors, then $\alpha_t(G) \geq \frac{tn}{r}$.*

Proof:
Let $C_1, C_2, \ldots, C_r$ be the r color classes of the proper coloring f of G such that $|C_1| \geq |C_2| \geq \cdots \geq |C_r|$. Suppose $\alpha_t(G) < \frac{tn}{r}$. By Observation 1, the graph $G[C_1 \cup C_2 \cup \cdots \cup C_t]$ has a proper coloring using t colors, which implies that $|C_1| + |C_2| + \cdots + |C_t| \leq \alpha_t(G)$. Since $\alpha_t(G) < \frac{tn}{r}$, there exists $i \in \{1, 2, \ldots, t\}$ such that $|C_i| < \frac{n}{r}$. Further, $|C_{t+1}| + |C_{t+2}| + \cdots + |C_r| > n - \frac{tn}{r} > \frac{(r-t)n}{r}$. Therefore, there exists $j \in \{t+1, t+2, \ldots, r\}$ such that $|C_j| > \frac{n}{r}$, which is contradiction since we have assumed that $|C_i| \geq |C_j|$. ☐

Corollary 2. *For a graph G, $\alpha_t(G) \geq \frac{tn}{\chi(G)}$.*

Proof:
Since $\chi(G)$ is the chromatic number of G, there exists a coloring of G which uses $\chi(G)$ colors. So by Theorem 2, $\alpha_t(G) \geq \frac{tn}{\chi(G)}$. ☐

2.1 List coloring

Let $G(V, E)$ be a graph and let P be a set of colors.

A *t-assignment* of G is defined as a function $f : V \to 2^P$ such that $\forall v \in V, |f(v)| = t$. In other words, a t-assignment of G assigns to each vertex of G a list of t colors from the set of colors P.

A *proper list coloring* of a t-assignment f of G is defined as a function $l : V \to P$ such that $\forall v \in V, l(v) \in f(v)$ and $\forall (u, v) \in E, l(u) \neq l(v)$. In other words, it is a proper coloring which assigns to each vertex a color from its list of colors given by the t-assignment f.

A graph G is said to be *t-choosable* if for every possible t-assignment of G there exists a proper list coloring.

The minimum possible value of t for which a graph is t-choosable is called the *list chromatic number* of the graph. We denote the list chromatic number of a graph G by $\chi_l(G)$.

Theorem 3. *For any graph G, $\chi(G) \leq \chi_l(G)$.*

Proof:
Suppose that $\chi_l(G) < \chi(G)$. Let us consider a $\chi_l(G)$-assignment l such that $\forall v \in V, l(v) = \{p_1, p_2, ..., p_{\chi_l(G)}\}$. Since G is $\chi_l(G)$-choosable, for every $\chi_l(G)$-assignment, there exists a proper list coloring for G. Thus there exists a proper list coloring f for the $\chi_l(G)$-assignment l. Since $|\bigcup_{v \in V} l(v)| = \chi_l(G) < \chi(G)$, f is a proper coloring of G which uses less than $\chi(G)$ colors, which is a contradiction since $\chi(G)$ is the minimum number of colors that needs to be used by any proper coloring of G. □

Let G be any graph. For $t < \chi_l(G)$, let $\mathcal{L}$ be the set of all possible t-assignments. Then not all t-assignments may have a proper list coloring. For $l \in \mathcal{L}$, we define a *partial list coloring* of l as an assignment to each vertex v of some subset $V' \subseteq V$ a color in $l(v)$ such that no two adjacent vertices in V' get assigned the same color. We say that this partial list coloring "colors" the set of vertices V' (and it leaves the vertices in $V \setminus V'$ uncolored). Let $\lambda_t(l)$ be the maximum number of vertices that are colored by any partial list coloring of the t-assignment l. Let $\lambda_t(G) = \min_{l \in \mathcal{L}}(\lambda_t(l))$.

Conjecture 1 (**Partial list coloring (PLC) conjecture** [1]). *For any graph G on n vertices having list chromatic number $\chi_l(G)$, and $t \in \{1, 2, \ldots, \chi_l(G)$,*

$$\lambda_t(G) \geq \frac{tn}{\chi_l(G)}$$

This conjecture has not yet been proven for general graphs. In the next chapter, we present some classes of graphs for which this conjecture has been proven.

Chapter 3

Partial list coloring of some special classes of graphs

To understand this chapter, we first need some definitions.

3.1 Definitions

A *complete bipartite graph* $K_{m,n}$ is defined as a graph whose vertex set can be partitioned into two disjoint sets U and V, where $|U| = m$ and $|V| = n$, such that $\forall u, v$, $uv \in E(G) \Leftrightarrow u \in U, v \in V$.

3.2 Claw-free graph

A *claw* is defined to be the graph $K_{1,3}$.

A *claw-free graph* is a graph that does not contain a claw as an induced subgraph.

In a graph $G(V, E)$, we say that two edges $e_1, e_2 \in E$ are *adjacent* if they have a common vertex.

Line graph: Consider a graph $G(V, E)$. The *line graph* of G is the graph with vertex set E and edge set $\{e_1e_2 \colon e_1, e_2 \in E(G)$, e_1 and e_2 are adjacent edges in $G\}$.

Theorem 4. *The line graph of a graph is always claw-free.*

Proof:
Let us say the line graph of a graph G contains a claw as an induced subgraph. This subgraph isomorphic to $K_{1,3}$ has four vertices x, u, v, w such that u, v, w form an independent set and x is adjacent to each of u, v, w. These four vertices each represent a unique edge in the original graph G. Now the edge belonging to vertex x,

say e_x, shares a vertex with each of the edges of G represented by the vertices u, v, w, say e_u, e_v, e_w respectively. Since e_x only has 2 vertices which it can share with the 3 edges e_u, e_v, e_w, by pigeonhole principle, e_x shares the same vertex with at least two of these three edges, say e_u, e_v. Then e_u and e_v are adjacent in G, implying that uv is an edge in the line graph. This contradicts the fact that u, v, w is an independent set in the line graph. □

Theorem 5 (Janssen, Mathew and Rajendraprasad [2]). *Let G be a claw-free graph on n vertices whose list chromatic number is s. Then for every $t \in \{1, 2, \dots, s\}$, $\lambda_t(G) \geq \frac{tn}{s}$.*

Proof:

If $t = s$, then the statement of the theorem is clearly true. Suppose that $t < s$. Let l_t be a t-assignment of G. Let $T = \bigcup_{v \in V} l_t(v)$. Let $k = |T|$ and $T = \{c_1, c_2, \dots, c_k\}$. Since $t < s$, there might not exist a proper list coloring for the t-assignment l_t. Let us take another set S of $(s - t)$ colors different from the colors in T. Let $S = \{c_{k+1}, c_{k+2}, \dots, c_{k+(s-t)}\}$. We now define an s-assignment l_s of G as follows: for $v \in V$, $l_s(v) = l_t(v) \cup S$. Since s is the list chromatic number of the graph, for the s-assignment l_s, there exists a proper list coloring of G. Let us consider a proper list coloring f of l_s such that the number of vertices assigned a color in S is as small as possible. We know that f partitions the vertex set into $k + (s - t)$ color classes $C_1, C_2, \dots, C_k, C_{k+1}, C_{k+2}, \dots, C_{k+(s-t)}$ where $v \in C_i \iff f(v) = c_i$. Also $\forall i \in \{1, 2, \dots, k\}$, let $V_i = \{v \in V : c_i \in l_t(v)\}$.

We claim $|\{v \in V : f(v) \in S\}| = \sum_{i=k+1}^{k+(s-t)} |C_i| \leq \frac{(s-t)n}{s}$, which implies that $|\{v \in V : f(v) \in T\}| = \sum_{i=1}^{k} |C_i| \geq \frac{(tn)}{s}$.

Suppose for the sake of contradiction that $|\{v \in V : f(v) \in S\}| = \sum_{i=k+1}^{k+(s-t)} |C_i| > \frac{(s-t)n}{s}$. Then there exists a color class C_p where $k + 1 \leq p \leq k + (s - t)$, such that $|C_p| > \frac{n}{s}$, i.e. there exists a color class C_p corresponding to one of the colors in S whose cardinality is greater than $\frac{n}{s}$.

Then,

$$|\{v \in V : f(v) \in T\}| = \sum_{i=1}^{k} |C_i| < \frac{tn}{s}$$

Since in the summation $\sum_{i=1}^{k} |C_p \cap V_i|$, every vertex $v \in C_p$ is counted t times (corresponding to the t colors in $l_t(v)$), we have

$$\sum_{i=1}^{k} |C_p \cap V_i| = t|C_p| > \frac{tn}{s}$$

The above two inequalities imply that there exists a color class C_i, where $1 \leq i \leq k$, such that $|C_i| < |C_p \cap V_i|$.

Let Z be the smallest subset of $C_p \cap V_i$ such that $|N_G(Z) \cap C_i| < |Z|$. We say that such a Z will always exist since $Z = C_p \cap V_i$ already satisfies the condition. We claim that $\forall v \in N_G(Z) \cap C_i$, $|N_Z(v)| \geq 2$, for if $\exists v \in N_G(Z) \cap C_i$ such that $N_Z(v) \leq 1$, then we contradict the minimality of Z, since $Z \setminus N_Z(v)$ satisfies the above condition and is smaller in size than Z. Then if for some $v \in N_G(Z) \cap C_i$, we have $N_{C_p \setminus Z}(v) \neq \emptyset$, then $|N_{C_p}(v)| \geq 3$. Since C_p is an independent set in G, this implies that there is an induced subgraph of G isomorphic to a claw, which is a contradiction to the fact that G is claw-free. Thus, we can conclude that $\forall v \in N_G(Z) \cap C_i$, $N_{C_p \setminus Z}(v) = \emptyset$, i.e. $\nexists (u, v) \in E$ such that $u \in N_G(Z) \cap C_i$ and $v \in C_p \setminus Z$. This makes it possible to form another valid proper list coloring f' of l_s by taking the coloring f and then giving color c_i to Z and c_p to $N_G(Z) \cap C_i$. Since $|N_G(Z) \cap C_i| < |Z|$, f' is a proper list coloring of l_s which assigns lesser number of vertices a color from S than f, which is a contradiction to the choice of f.

□

Theorem 6 (Noel, Reed and Wu [3]). *For a graph G, $\chi(G) \geq \frac{|V(G)|-1}{2} \implies \chi(G) = \chi_l(G)$.*

Theorem 7 (Janssen, Mathew and Rajendraprasad [2]). *Let G be a graph with $\chi_l(G) = s$ and $\chi(G) \geq \frac{|V(G)|-1}{2}$. Then for every $t \in \{1, 2, \ldots, s\}$, there is an induced subgraph say H_t of G such that:*

(i) $\chi(H_t) \geq \frac{|V(H_t)|-1}{2}$

(ii) $\chi_l(H_t) = \chi(H_t) = t$

(iii) $|V(H_t)| \geq \frac{t|V(G)|}{s}$

Proof:

We shall prove this lemma by induction on $s - t$. For the base case when $s - t = 0$, or $s = t$, we can choose $H_s = G$ since $|V(G)| \geq \frac{t|V(G)|}{s}$ (as $t = s$), $\chi(G) \geq \frac{|V(G)|-1}{2}$ (as given in the statement of the lemma), and consequently by Theorem 6, $\chi(G) = \chi_l(G) = s$. Let t be an integer such that $t < s$. Suppose that the statement of the lemma is true of every r such that $s - r < s - t$, i.e. for every $r > t$. By the induction hypothesis, we have that there exists an induced subgraph H_{t+1} that satisfies the three properties in the statement of the lemma. Then $\chi(H_{t+1}) = t + 1$. Consider a proper coloring which uses $t+1$ colors and let $C_1, C_2, \ldots, C_{t+1}$ be the color classes for this coloring. Since we also have $t + 1 = \chi(H_{t+1}) \geq \frac{|V(H_{t+1})|-1}{2}$, there are two possible cases.

Case 1. $t + 1 > \frac{|V(H_{t+1})|}{2}$.

Since the number of colors is more than half of the number of vertices, there must exist one color class which contains exactly one vertex. Let C_a be that particular color class which contains exactly one vertex. We construct H_t by removing C_a from H_{t+1}, i.e. $H_t = H_{t+1} - C_a$. Since $\{C_1, C_2, \ldots, C_{t+1}\} \setminus \{C_a\}$ is a partition of the vertex set of H_t into t independent sets, we have that $\chi(H_t) = t$. Since $t+1 > \frac{|V(H_{t+1})|}{2}$, we have $t > \frac{|V(H_{t+1})|}{2} - 1 = \frac{|V(H_{t+1})|-2}{2} = \frac{|V(H_t)|-1}{2}$ (as $|V(H_{t+1})| - 1 = |V(H_t|)$. Then by Theorem 6, $\chi_l(H_t) = \chi(H_t) = t$. Further, $|V(H_t)| = |V(H_{t+1})| - 1 \geq \frac{(t+1)|V(G)|}{s} - 1 = \frac{t|V(G)|}{s} + \frac{|V(G)|}{s} - 1 \geq \frac{t|V(G)|}{s}$ (since $|V(G)| \geq \chi(G) = s$).

Case 2. $\frac{|V(H_{t+1})|-1}{2} \leq t+1 \leq \frac{|V(H_{t+1})|}{2}$.

There are two possibilities.

(a) If there exists at least one color class C_a such that $|C_a| = 2$, then we define $H_t = H_{t+1} - C_a$. As $\{C_1, C_2, \ldots, C_{t+1}\} \setminus \{C_a\}$ is a partition of $V(H_t)$ into t independent sets, we have $\chi(H_t) \leq t$. If $\chi(H_t) < t$, then adding back the vertices in C_a and giving them a new color will give us a t-coloring of H_{t+1}, which contradicts the fact that $\chi(H_{t+1}) = t+1$. Thus we have $\chi(H_t) = t$.

(b) Otherwise, every color class has size at least 3 or equal to 1. Since $\frac{|V(H_{t+1})|-1}{2} \leq t+1 \leq \frac{|V(H_{t+1})|}{2}$, we have $2t+2 \leq |V(H_{t+1})| \leq 2t+3$. Thus every color class cannot have size at least 3 since in that case $|V(H_{t+1})| \geq 3(t+1) > 2t+3$ (as $t \geq 1$), and every color class cannot have size equal to 1 since in that case $|V(H_{t+1})| = t+1 < 2t+2$. Thus there must exist color classes C_b, C_c such that $|C_b| = 1$ and $|C_c| > 2$. Consider a vertex $u \in C_c$. First, let us consider the case when $\chi(H_{t+1} - u) = t$, i.e after removal of u, the chromatic number decreases by one. In this case, we take another vertex v from C_c other than u. We can always find such a vertex v since $|C_c| > 2$. We claim that if $\chi(H_{t+1} - u) = t$, then $\chi(H_{t+1} - \{u, v\}) = t$. Otherwise, if $\chi(H_{t+1} - \{u, v\}) \leq t-1$, then adding u and v back to the graph can only increase its chromatic number by 1, since u and v do not have an edge between them, which means that $\chi(H_{t+1}) \leq t$, which is a contradiction. Therefore, $\chi(H_{t+1} - \{u, v\}) = t$. We take $H_t = H_{t+1} - \{u, v\}$. Next, let us consider the case when $\chi(H_{t+1} - \{u\}) = t+1$. In this case, we take $H_t = H_{t+1} - (C_b \cup \{u\})$. Then $\{C_1, C_2, \ldots, C_{t+1}\} \setminus \{C_b\}$ is a partition of $V(H_t)$ into t independent sets, and therefore $\chi(H_t) \leq t$. If $\chi(H_t) < t$, then adding back C_b to H_t and giving the single vertex in it a new color, we get that $\chi(H_{t+1} - \{u\}) \leq t$, which contradicts the fact that $\chi(H_{t+1} - \{u\}) = t+1$. Therefore we again have $\chi(H_t) = t$.

Thus, no matter what H_t we happen to take, $\chi(H_t) = t$. Also, $|V(H_t)| = |V(H_{t+1})| - 2$. Therefore, we have $\chi(H_t) = t = (t+1) - 1 \geq \frac{|V(H_{t+1})|-1}{2} - 1 = \frac{|V(H_t)|+1}{2} - 1 = \frac{|V(H_t)|-1}{2}$. So, by Theorem 6, $\chi_l(H_t) = \chi(H_t) = t$.

Finally, we know that $|V(H_t)| = |V(H_{t+1})| - 2 = \frac{(t+1)|V(H_{t+1})|}{t+1} - 2 = \frac{t|V(H_{t+1})|}{t+1} + \frac{|V(H_{t+1})|}{t+1} - 2 \geq \frac{t|V(H_{t+1})|}{t+1} + 2 - 2$, since we have assumed that $t+1 \leq \frac{|V(H_{t+1})|}{2}$.

Therefore, $|V(H_t)| \geq \frac{t|V(H_{t+1})|}{t+1}$. Thus $|V(H_t)| \geq \frac{t|V(G)|}{s}$, since $|V(H_{t+1})| \geq \frac{(t+1)|V(G)|}{s}$.
□

Corollary 3. *Let G be a graph on n vertices with $\chi(G) \geq \frac{n-1}{2}$. Then for every $t \in \{1, 2, \ldots, \chi_l(G)\}$, $\lambda_t(G) \geq \frac{tn}{\chi_l(G)}$.*

3.3 Minimally 2-connected graphs

A graph G is *chordless* if for each cycle in G, there does not exist any pair of nonconsecutive vertices that are adjacent in G.

A graph G is 2-vertex connected or *2-connected* if removal of any one vertex does not disconnect the graph.

A graph G is *minimally 2-connected* if it is 2-connected as well as chordless, and is not a single edge.

Lemma 1 (Plummer [4]). *A graph G is minimally 2-connected if and only if*

- *either G is a cycle*
- *or if we remove all the vertices of degree 2 from the G, we get a forest with two or more components (trees).*

Theorem 8 ([2]). *Let G be a minimally 2-connected graph. Then for every $x \in V(G)$, there exist two distinct vertices $v, w \in V(G) - \{x\}$ and a vertex $u \in V(G)$ such that $v, w \in N_G(u)$ and $d(v) = d(w) = 2$.*

Proof:
Suppose that G is a cycle. Since each vertex of a cycle has degree 2, for each $x \in V(G)$, x has two neighbours of degree 2. Let v and w be these two neighbours of x and let $u = x$. Then $v, w \in N_G(u)$ and $d(v) = d(w) = 2$, and we are done.

Now suppose that G is not a cycle. Let $T = \{x \in V(G) : d(x) = 2\}$. Then by Lemma 1, $V(G) - T$ is a forest with 2 or more components (trees). Since G is 2-connected, we know that $\forall x \in V(G), d(x) \geq 2$. Thus $\forall x \in V(G) - T$, $d(x) \geq 3$. Suppose first that $G[V - T]$ contains an isolated vertex z. Then $N_T(z) \geq 3$. We fix $u = z$. If $x = z$, then we fix any two vertices of $N_T(z)$ as v and w, since each one of them has degree 2 in G. If $x \neq z$, then since $N_T(z) \geq 3$, there exist at least two vertices in $N_T(z)$ different from x. Since these two vertices belong to T, each one of them has degree 2 in G, so we fix them as v and w. Next suppose that $G[V - T]$ does not contain any isolated vertices. Then there exist at least two trees in $G[V - T]$ each containing at least two vertices. Let T_1, T_2 be two such trees. Since T_1, T_2 are trees containing at least two vertices, each one of them contains at least two leaf vertices. Let a_1, b_1 be two leaf vertices of T_1 and a_2, b_2 be two leaf vertices of T_2. Since each of

a_1, b_1, a_2, b_2 have degree 1 in their respective trees, they have at least two neighbours each in T. If $x \notin T$, then we designate a vertex in $\{a_1, b_1, a_2, b_2\}$ that is different from x as u, and two of its neighbours in T as v and w. If $x \in T$, then since x has degree 2, there are at least two vertices in $\{a_1, b_1, a_2, b_2\}$ that are not adjacent to x. We designate one of these two vertices as u, and two of its neighbours in T as v and w. □

A graph G is said to be k-degenerate every subgraph of G contains a vertex of degree at most k. Thus, any subgraph of a k-degenerate graph is also k-degenerate.

Observation 2. *If a graph G is k-degenerate, then G is $(k+1)$-choosable.*

Proof:
We shall prove this by induction on $|V(G)|$. Let l be any $(k+1)$-assignment of G. As G is k-degenerate, G contains a vertex v of degree at most k and moreover, $G - \{v\}$ is also k-degenerate. Let l' be the $(k+1)$-assignment of $G - \{v\}$ that is obtained by setting $l'(u) = l(u)$ for every vertex $u \in V(G) \setminus \{v\}$. By the induction hypothesis, $G - \{v\}$ is $(k+1)$-choosable, and therefore there is a proper list coloring for the $(k+1)$-assignment l'. Now since in this coloring, at most k different colors appear on the neighbours of v, we can assign v a color from $l(v)$ that is different from all the colors that appear on its neighbours. It is easy to see that we now have a proper list coloring of the $(k+1)$-assignment l. Thus a proper list coloring is possible for any $(k+1)$-assignment of G, and hence G is $(k+1)$-choosable. □

For a connected graph G, we define an induced subgraph B_G of G as follows. If G contains no cut-vertices than we define $B_G = G$. For every cut-vertex v of G, we define $g(v)$ to be the size of the smallest component in $G - \{v\}$. Let x be a cut-vertex of G for which $g(x)$ is as small as possible. Let C be a component of $G - \{x\}$ with size $g(x)$. We define $B_G = G[\{x\} \cup V(C)]$. The graph B_G has the following properties:

(i) The only cut-vertex of G that is contained in B_G is x.

Suppose that B_G contains a cut-vertex y of G other than x. Then $y \in V(C) \setminus \{x\}$. Let $C_1, C_2, \ldots, C_k$ be the connected components of $G - \{x\}$ other than C. Clearly, the vertices in $V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k) \cup \{x\}$ all lie in one connected component of $G - \{y\}$. Let C' be any other connected component of $G - \{y\}$. Then it follows that every vertex in C' is from $V(C) \setminus \{y\}$, which implies that $|C'| < |C|$. Then $g(y) < g(x)$, which contradicts our choice of x.

(ii) B_G is 2-connected.

Suppose that B_G contains a cut-vertex y. Clearly $B_G - \{x\}$ results in the connected graph C, so we have $y \neq x$. Let v be a vertex from a connected component of $B_G - \{y\}$ that is different from the connected component that contains x. Clearly, every path in B_G between x and v contains y. Since x separated every vertex of C from every vertex in $G - (V(C) \cup \{x\})$, we know that every path in G between x and v lies entirely inside B_G. From our previous observation, this means that every path in G between x and v contains y. Thus x and v belong to different connected components of $G - \{y\}$, implying that y is a cut-vertex of G. But this contradicts (i).

(iii) Every vertex of B_G other than x has the same degree in B_G as well as G.

Recall that $B_G = G[\{x\} \cup V(C)]$, where C is a connected component of $G - \{x\}$. It is clear that no vertex of C is adjacent to any vertex in $V(G) \setminus (\{x\} \cup V(C)\}$. Thus every vertex of C has the same degree in B_G as well as in G.

Theorem 9. *If a graph G is chordless, then it is also 2-degenerate.*

Proof:
We can assume that G is connected as the disjoint union of 2-degenerate graphs is also a 2-degenerate graph. We shall prove this by induction on $|V(G)|$. Clearly, if $|V(G)| = 1$, then the statement is true. So we shall assume that $|V(G)| \geq 2$ and that every chordless graph containing less than $|V(G)|$ vertices is 2-degenerate. Consider the graph B_G defined as above, with x having the same meaning. By (ii) B_G is 2-connected. If B_G is a single edge, then the vertex in B_G other than x has degree 1 in B_G, and by (iii), it also has degree 1 in G. Otherwise, since B_G is an induced subgraph of G, and hence also a chordless graph, we have that B_G is a minimally 2-connected graph. By Theorem 8, in B_G, there exist two vertices v, w different from x, each having degree 2. From (iii), these vertices also have degree 2 in G. Thus in any case, G contains a vertex of degree at most 2. The graph obtained by removing this vertex is a chordless graph with lesser than $|V(G)|$ vertices and hence by the induction hypothesis is a 2-degenerate graph. It follows that G is also a 2-degenerate graph. □

Theorem 10 (Janssen, Mathew and Rajendraprasad [2]). *If G is a chordless graph with n vertices having list chromatic number s, then $\forall t \in 1, 2, \ldots, s$, $\lambda_t(G) \geq \frac{tn}{s}$.*

Proof:
We shall prove this by induction on $|V(G)|$. The base case when G contains only one vertex trivially satisfies the statement of the theorem.

First of all, note that we can assume that $t < s$ as the case when $t = s$ is trivially true. Moreover if $t = 1$, then note that the largest color class in an $\chi(G)$-coloring of G will contain at least $\frac{n}{s}$ vertices (since $\chi(G) \leq \chi_l(G) = s$). Then this color class, being an independent set, is an induced subgraph of G that is 1-choosable. Thus $\lambda_1(G) \geq \frac{n}{s}$. So we assume from here on that $1 < t < s$.

Since by Theorem 9, G is a 2-degenerate graph, we know by Observation 2 that $\chi_l(G) \leq 3$. If $s = \chi_l(G) \leq 2$, then we have already shown the theorem to be true for all possible values of t. Hence we can ssume that $s = \chi_l(G) = 3$ and $t = 2$. That is, we need to show that $\lambda_2(G) \geq \frac{2n}{3}$. If G contains a vertex v of degree at most 1, then $G - \{v\}$ is a chordless graph having lesser number of vertices than G, and hence by the induction hypothesis, we have that $\lambda_2(G-\{v\}) \geq \frac{2(n-1)}{3}$. Consider any 2-assignment l of G. Let l' be the 2-assignment of $G-\{v\}$ obtained by setting $l'(u) = l(u)$ for all $u \in V(G) \setminus \{v\}$. Since $\lambda_2(G-\{v\}) \geq \frac{2(n-1)}{3}$, there exists a partial list coloring of l' that colors $\frac{2(n-1)}{3}$ vertices of $G-\{v\}$. This is also a partial list coloring of l that colors $\frac{2(n-1)}{3}$ vertices of G. Now since the vertex v has degree at most 1, there is one color in $l(v)$ that is different from any color that has been given to a neighbour of v. We can give this color to v to obtain a partial list coloring of l that colors at least $\frac{2(n-1)}{3}+1 > \frac{2n}{3}$ vertices of G. Next, suppose that every vertex in G has degree at least 2. Let G' be any connected component of G. Consider the graph $B_{G'}$. Since $B_{G'}$ is 2-connected, is not a single edge (since we have assumed that every vertex in G has degree at least 2), and is a chordless graph (since it is an induced subgraph of the chordless graph G), we know that $B_{G'}$ is a minimally 2-connected graph. Let x be the only cut-vertex of G' in $B_{G'}$. Then by Theorem 8, there exist vertices v, w distinct from x such that both of them have degree 2 in $B_{G'}$ and they have a common neighbour u in $B_{G'}$. By (iii) v, w also have degree 2 in G', and hence also in G. Now let $H = G-\{u, v, w\}$. As H is a chordless graph on lesser number of vertices than G, we have by the induction hypothesis that $\lambda_2(H) \geq \frac{2(n-3)}{3}$. Now given any 2-assignment l of G, we can construct a 2-assignment l' of H as before (by setting $l'(z) = l(z)$ for all $z \in V(H)$), and we have that there exists a partial colouring of l' that colours $\frac{2(n-3)}{3}$ vertices of H. We now leave the vertex u uncoloured and colour the vertices v and w using colours from $l(v)$ and $l(w)$ respectively. Since u is a neighbour of both v, w, each of v, w has at most one coloured neighbour. This means v can be given a colour from $l(v)$ that is different from the colours of its neighbours, and similarly, u can be given a colour from $l(u)$ that is different from the colours of its neighbours. Now we have coloured at least $\frac{2(n-3)}{3}+2 = \frac{2n}{3}$ vertices of G. This shows that $\lambda_2(G) \geq \frac{2n}{3}$. $\square$

Bibliography

[1] Michael O. Albertson, Sara Grossman, and Ruth Haas. “Partial list colorings”. In: *Discrete Mathematics* 214.1 (2000), pp. 235–240.

[2] Jeannette Janssen, Rogers Mathew, and Deepak Rajendraprasad. “Partial list colouring of certain graphs”. In: *The Electronic Journal of Combinatorics* 22.3 (2015), P3.41.

[3] Jonathan A. Noel, Bruce A. Reed, and Hehui Wu. “A Proof of a Conjecture of Ohba”. In: *Journal of Graph Theory* 79.2 (2015), pp. 86–102.

[4] Michael D. Plummer. “On minimal blocks”. In: *Transactions of the American Mathematical Society* 134.1 (1968), pp. 85–94.