Around Fatou Theorem and Its Converse on Certain Lie Groups

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Dedicated to the memory of

my parents

Hemanta Kumar Sarkar and Anita Sarkar

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Contents

Chapter 1

Introduction

In this thesis, by a measure μ on some locally compact Hausdorff space we will always mean a complex Borel measure or a signed Borel measure such that the total variation $|\mu|$ is locally finite, that is, $|\mu|(K)$ is finite for all compact sets K. It is well-known that if μ is such a measure on some second countable, locally compact Hausdorff space then $|\mu|$ is regular (see [Rud87, Theorem 2.18]). As we will always work with measures on some second countable, locally compact Hausdorff space, we will always assume, without loss of generality, that the total variation of a measure is regular. If $\mu(E)$ is nonnegative for all Borel measurable sets E then μ will be called a positive measure. Our motivation is certain classical results of Fatou and their converse which relates various differentiability properties of μ at a boundary point $x_0 \in \mathbb{R}^n$, with various types of boundary behavior of the Poisson integral $P[\mu]$ of the measure μ at x_0.

Definition 1.0.1. Given a measure μ on $\mathbb{R}^n$, the symmetric derivative $D_{sym}\mu(x_0)$ of μ at a point $x_0 \in \mathbb{R}^n$, is given by the limit

$$D_{sym}\mu(x_0) = \lim_{r \to 0} \frac{\mu(B(x_0, r))}{m(B(x_0, r))}, \tag{1.0.1}$$

provided the limit exists. Here,

$$B(x, r) = \{\xi \in \mathbb{R}^n \mid \|x - \xi\| < r\},$$

is the open ball of radius $r > 0$, with center at $x \in \mathbb{R}^n$, with respect to the Euclidean metric and m denotes the Lebesgue measure on $\mathbb{R}^n$.

Given a measure μ on $\mathbb{R}^n$, its Poisson integral $P[\mu]$ is given by the convolution

$$P[\mu](x, y) = \int_{\mathbb{R}^n} P(x - \xi, y)\, d\mu(\xi), \tag{1.0.2}$$

whenever the integral above converges absolutely for $(x, y) \in \mathbb{R}^{n+1}_+$. Here, the kernel $P(x, y)$ is the standard Poisson kernel of $\mathbb{R}^{n+1}_+$ and is given by the formula

$$P(x, y) = c_n \frac{y}{(y^2 + \|x\|^2)^{\frac{n+1}{2}}}, \quad (x, y) \in \mathbb{R}^{n+1}_+, \tag{1.0.3}$$

where $c_n = \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)$. It is known that if the integral above converges absolutely for some $(x_0, y_0) \in \mathbb{R}^{n+1}_+$, then it converges absolutely for all other points in $\mathbb{R}^{n+1}_+$. This observation follows from the following limiting behavior (see (2.2.11)).

$$\lim_{\|\xi\|\to\infty} \left(\frac{y_0^2 + \|x_0 - \xi\|^2}{y_1^2 + \|x_1 - \xi\|^2}\right)^{\frac{n+1}{2}} = 1, \quad \text{for any given} \quad (x_1, y_1) \in \mathbb{R}^{n+1}_+.$$

Moreover, $P[\mu]$ defines a harmonic function in $\mathbb{R}^{n+1}_+$. For a function $f \in L^r(\mathbb{R}^n)$, $r \in [1, \infty]$, its Poisson integral is defined analogously and is denoted by Pf. So, Pf is the Poisson integral of the measure μ, where $d\mu = f\, dm$.

For a complex-valued function ϕ on $\mathbb{R}^n$, we define

$$\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right), \quad x \in \mathbb{R}^n,\ t \in (0, \infty). \tag{1.0.4}$$

It then follows that

$$P(x, y) = \mathrm{P}_y(x), \quad \mathrm{P}(x) = \frac{c_n}{(1 + \|x\|^2)^{\frac{n+1}{2}}}, \quad x \in \mathbb{R}^n,\ y \in (0, \infty), \tag{1.0.5}$$

with $\|\mathrm{P}\|_{L^1(\mathbb{R}^n)} = 1$. We say that the Poisson integral $P[\mu]$ of a measure μ is well-defined if

$$\int_{\mathbb{R}^n} \frac{1}{(1 + \|\xi\|^2)^{\frac{n+1}{2}}}\, d|\mu|(\xi) < \infty.$$

Our starting point is the following well-known result which was proved by Fatou [Fat06] in the case $n = 1$.

Theorem 1.0.2. *Suppose μ is a measure with well-defined Poisson integral $P[\mu]$. If there exists $x_0 \in \mathbb{R}^n$, $L \in \mathbb{C}$, such that*

$$D_{sym}\mu(x_0) = L,$$

then

$$\lim_{y \to 0} P[\mu](x_0, y) = L.$$

It is important to note here that the theorem above concerns existence of limits at a single point $x_0 \in \mathbb{R}^n$, and has nothing to do with almost everywhere existence of the above limits. Fatou's theorem were later generalized in various directions. One such generalization was obtained by Saeki [Sae96], which generalizes the result of Fatou for more general approximate identities like $\{\phi_t\}$. The detailed version of Saeki's result is as follows. We recall that a function $\phi : \mathbb{R}^n \to \mathbb{C}$, is a radial function if

$$\phi(x) = \phi(\xi), \quad \text{whenever } \|x\| = \|\xi\|.$$

For a radial function $\phi : \mathbb{R}^n \to \mathbb{C}$, we will occasionally interpret ϕ as a function on $[0, \infty)$, in the following way.

$$\phi(r) = \phi(x), \quad \text{whenever} \quad r = \|x\|, \ x \in \mathbb{R}^n.$$

Also, a function $\psi : \mathbb{R}^n \to \mathbb{R}$, is said to be radially decreasing if

$$\psi(x) \geq \psi(\xi), \quad \text{whenever} \quad \|x\| \leq \|\xi\|.$$

In this thesis, we will always assume that if $\psi : \mathbb{R}^n \to [0, \infty)$, is a nonzero radially decreasing function, then ψ is bounded by $\psi(0) \in (0, \infty)$. More precisely,

$$\psi(x) \leq \psi(0), \quad \text{for all} \quad x \in \mathbb{R}^n.$$

We now define a notion called comparison condition which will be used several times in the latter part of this thesis.

Definition 1.0.3. We say that a function $\phi : \mathbb{R}^n \to (0, \infty)$, satisfies the comparison condition if

$$\sup \left\{ \frac{\phi_t(x)}{\phi(x)} \mid t \in (0,1),\ \|x\| > 1 \right\} < \infty. \tag{1.0.6}$$

For examples of functions satisfying the comparison condition (1.0.6), we refer the reader to Example 2.1.2.

Given a measure μ and a complex-valued function ϕ on $\mathbb{R}^n$, we define the convolution integral $\phi[\mu](x,t)$ by

$$\phi[\mu](x,t) = \mu * \phi_t(x) = \int_{\mathbb{R}^n} \phi_t(x - \xi)\, d\mu(\xi), \tag{1.0.7}$$

whenever the integral converges absolutely for $(x,t) \in \mathbb{R}^{n+1}_+$.

Remark 1.0.4. It was proved in [Sae96, P. 137] that if μ is a measure on $\mathbb{R}^n$, and ϕ is a nonnegative, radially decreasing function on $\mathbb{R}^n$, then finiteness of $|\mu| * \phi_{t_0}(x_0)$, for some $(x_0, t_0) \in \mathbb{R}^n$, implies the finiteness of $|\mu| * \phi_t(x)$ for all $(x,t) \in \mathbb{R}^n \times (0, t_0)$. In this case, we say that $\phi[\mu]$ is well-defined in $\mathbb{R}^n \times (0, t_0)$. We note that if $|\mu|(\mathbb{R}^n)$ is finite then $\phi[\mu]$ is well-defined in $\mathbb{R}^{n+1}_+$.

The importance of the condition (1.0.6) stems from the following theorem (see [Sae96, Theorem 1.1]) which generalizes Theorem 1.0.2 for more general kernels.

Theorem 1.0.5. *Suppose that $\phi : \mathbb{R}^n \to (0,\infty)$, satisfies the following conditions:*

1) ϕ is radial, radially decreasing function with $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$.

2) ϕ satisfies the comparison condition (1.0.6).

Suppose μ is a measure on $\mathbb{R}^n$ such that $|\mu| * \phi_{t_0}(x_1)$ is finite for some $t_0 \in (0,\infty)$, and $x_1 \in \mathbb{R}^n$. If for some $x_0 \in \mathbb{R}^n$, and $L \in \mathbb{C}$, we have $D_{sym}\mu(x_0) = L$, then

$$\lim_{t \to 0} \mu * \phi_t(x_0) = L. \tag{1.0.8}$$

It was also shown in [Sae96, Remark 1.6] that the theorem above fails in the absence of the comparison condition (1.0.6).

For us, the main concern of Chapter 2 is the following classical question regarding the converse implication of Theorem 1.0.2.

Question: Given a measure μ on $\mathbb{R}^n$, with well-defined Poisson integral $P[\mu]$, and $x_0 \in \mathbb{R}^n$, $L \in \mathbb{C}$, is it true that

$$\lim_{y \to 0} P[\mu](x_0, y) = L,$$

implies that

$$D_{sym}\mu(x_0) = L\,?$$

In [Loo43, P.246], for $n = 1$, Loomis had shown by an example that in general, the answer to the question above is negative. However, Loomis also proved in the same paper that the converse of Theorem 1.0.2 does hold true for $n = 1$, if the measure μ is assumed to be positive (see [Loo43, P.239-240]). Rudin generalized the result of Loomis for dimension $n > 1$.

Theorem 1.0.6 ([Rud78, Theorem A]). *Suppose μ is a positive measure on $\mathbb{R}^n$ with well-defined Poisson integral $P[\mu]$. If there exists $x_0 \in \mathbb{R}^n$, and $L \in [0, \infty)$, such that*

$$\lim_{y \to 0} P[\mu](x_0, y) = L,$$

then $D_{sym}\mu(x_0) = L$.

According to Rudin, the theorem above is a Tauberian theorem with positivity of μ being the Tauberain condition. The proof of this theorem, given in [Rud78], depends heavily on Wiener's Tauberian theorem for the multiplicative group $(0, \infty)$. In [RU88, Theorem 2.3], a different proof of this theorem without using Wiener's Tauberian theorem was given. But this proof crucially uses the fact that $P[\mu]$ is a harmonic function in $\mathbb{R}^{n+1}_+$. However, it will be too simplistic to think that Theorem 1.0.6 is valid only for harmonic functions. In fact, Gehring [Geh60, Theorem 4] proves a similar result for positive solution of the heat equation in one dimension. Later, Watson [Wat77, Theorem 4] generalized the result of Gehring for higher dimensions. In [Khe94, Theorem 3], it has been shown that results analogous to Theorem 1.0.6 hold true for positive eigenfunctions of the Laplacian in $\mathbb{R}^{n+1}_+$. We refer the reader to [CD99, Dub04, Geh57, Log15] for related results. This motivated us to ask the following question:

Question: Can one find a necessary and sufficient condition on $\phi : \mathbb{R}^n \to (0, \infty)$, satisfying the conditions $1)$ and $2)$ of Theorem 1.0.5, which ensures that

$$\lim_{t\to 0} \mu * \phi_t(x_0) = L,$$

implies $D_{sym}\mu(x_0) = L$, whenever μ is a positive measure?

This question is being answered in Chapter 2. The main result proved in Chapter 2 is Theorem 2.1.3 which provides a necessary and sufficient condition on the function ϕ (as in Theorem 1.0.5), under which a result analogous to Theorem 1.0.6 holds. We will then use this theorem (Theorem 2.1.3) to prove a result analogous to Theorem 1.0.6 for certain positive eigenfunctions of the Laplace-Beltrami operator on real Hyperbolic spaces.

So far, we have restricted our discussion only to the existence of the vertical limit of the Poisson integral $P[\mu]$ and the symmetric derivative of the measure μ. We will now shift our focus to the nontangential convergence of Poisson integral. We need the following definitions to proceed further.

Definition 1.0.7 ([SW71, P.62]). i) For $x_0 \in \mathbb{R}^n$, and $\alpha \in (0, \infty)$, we define the conical region $S(x_0, \alpha)$ with vertex at x_0 and aperture α by

$$S(x_0, \alpha) = \{(x, y) \in \mathbb{R}^{n+1}_+ \mid \|x - x_0\| < \alpha y\}.$$

ii) A complex-valued function u defined on $\mathbb{R}^{n+1}_+$, is said to have nontangential limit $L \in \mathbb{C}$, at $x_0 \in \mathbb{R}^n$, if, for every $\alpha \in (0, \infty)$,

$$\lim_{\substack{(x,y)\to(x_0,0)\\(x,y)\in S(x_0,\alpha)}} u(x, y) = L.$$

Let $\beta : \mathbb{R} \to \mathbb{C}$, be such that $\beta = \sum_{j=1}^{4} \epsilon_j \beta_j$, where each β_j is increasing and right-continuous on $\mathbb{R}$, and ϵ_j are ± 1 or $\pm i$ and let μ_β be the Lebesgue-Stieltjes measure on $\mathbb{R}$, induced by β. In other words,

$$\mu_\beta((a, b]) = \beta(b) - \beta(a), \quad a,\ b \in \mathbb{R},\ \ a < b. \tag{1.0.9}$$

We refer to [SS05, P.281-284] for detailed discussion on Lebesgue-Stieltjes measure. In his paper [Fat06], Fatou also considered the nontangential convergence of Poisson integrals of Lebesgue-Stieltjes measures and proved the following.

Theorem 1.0.8. *Suppose that $P[\mu_\beta]$ is well-defined in $\mathbb{R}^2_+$, where β and μ_β are as above. If β is differentiable at some $x_0 \in \mathbb{R}$, then $P[\mu_\beta]$ has nontangential limit $\beta'(x_0)$ at x_0.*

As shown by Loomis [Loo43, P.246], converse of Theorem 1.0.8 fails in general. However, Loomis also proved in the same paper that the converse does hold true if β is a real-valued increasing function on $\mathbb{R}$ (see [Loo43, Theorem 1]). It is well-known that for a positive measure μ on $\mathbb{R}$, its distribution function $F : \mathbb{R} \to \mathbb{R}$, given by

$$F(x) = \mu\left((0,x]\right), \; x > 0; \quad F(x) = -\mu\left((x,0]\right), \; x < 0; \quad F(0) = 0,$$

is right-continuous and increasing. Moreover, $\mu_F = \mu$, where μ_F is defined according to (1.0.9). The reason for discussing Lebesgue-Stieltjes measures is that they are related to characterization of a large class of harmonic functions in $\mathbb{R}^2_+$. In fact, characterization of positive harmonic functions in $\mathbb{R}^2_+$, due to Loomis and Widder says the following:

Theorem 1.0.9 ([LW42, Theorem 4, P.645])**.** *If u is a positive harmonic function in $\mathbb{R}^2_+$, then there exists a right-continuous and increasing function β defined on $\mathbb{R}$ (unique up to an additive constant) and a nonnegative constant C such that*

$$u(x,y) = Cy + P[\mu_\beta](x,y), \quad (x,y) \in \mathbb{R}^2_+, \tag{1.0.10}$$

where μ_β is the Lebesgue-Stieltjes measure induced by β.

We are now ready to present Loomis's result.

Theorem 1.0.10 ([Loo43, Theorem 1])**.** *Suppose that u is a positive harmonic function in $\mathbb{R}^2_+$, $x_0 \in \mathbb{R}$, $L \in [0,\infty)$, and that β as in (1.0.10). If for two distinct real numbers α_1, α_2*

$$\lim_{y\to 0} u(x_0 + \alpha_1 y, y) = L = \lim_{y\to 0} u(x_0 + \alpha_2 y, y),$$

then $\beta'(x_0) = L$.

In order to extend Theorem 1.0.8 and Theorem 1.0.10 in higher dimensions, one needs to generalize the notion of derivative of the distribution function of a measure defined on $\mathbb{R}$. It is not hard to see that symmetric derivative of measure is not a right candidate for this purpose. Indeed, we consider the measure $d\mu = \chi_{[0,1]}\, dm$. Then for all $r \in (0,1)$,

$$\frac{\mu((-r,r))}{m((-r,r))} = \frac{1}{2r}\int_{-r}^{r} \chi_{[0,1]}\, dm = \frac{1}{2r}\int_{0}^{r} dm = \frac{1}{2}.$$

This shows that the symmetric derivative of μ at 0, $D_{sym}\mu(0)$, equals to $1/2$. On the other hand, the distribution function F of μ is given by

$$F(x) = x\chi_{[0,1]}(x) + \chi_{(1,\infty)}(x), \quad x \in \mathbb{R},$$

which is not differentiable at zero. The correct generalization of derivative, in this context, turns out to be the notion of strong derivative of a measure introduced by Ramey and Ullrich.

Definition 1.0.11 ([RU88, P.208])**.** Given a measure μ on $\mathbb{R}^n$, we say that μ has strong derivative $L \in \mathbb{C}$, at $x_0 \in \mathbb{R}^n$, if

$$\lim_{r\to 0} \frac{\mu(x_0 + rB)}{m(rB)} = L,$$

holds for every open ball $B \subset \mathbb{R}^n$, where $rB = \{rx \mid x \in B\}$, $r > 0$. The strong derivative of μ at x_0, if it exists, is denoted by $D\mu(x_0)$.

We note that for a measure μ, if $D\mu(x_0) = L$, then $D_{sym}\mu(x_0)$ is also equal to L. However, the converse is not true. To see this, we refer the reader to Remark 3.1.3, where we have given an example of a measure μ on $\mathbb{R}$ such that $D_{sym}\mu(0) = 1/2$, but the strong derivative of μ fails to exist at 0. We also refer the reader to Theorem 3.1.4, where the relation between strong derivative and derivative has been explained. In fact, we have shown in Theorem 3.1.4 that a measure μ on $\mathbb{R}$ has strong derivative L at some $x_0 \in \mathbb{R}$ if and only if its distribution function has derivative L at x_0.

The relation between the strong derivative and the nontangential convergence of the Poisson integral was first established by Ramey and Ullrich [RU88]. In the following we present the result of Ramey and Ullrich. Much of our work in this thesis will revolve around this theorem. Ramey and Ullrich proved their results for positive harmonic functions. Positive harmonic functions in the upper half-space $\mathbb{R}^{n+1}_+$, are essentially characterized by Poisson integral of positive measures defined on $\mathbb{R}^n$. In fact, the following higher dimensional analogue of the aforementioned result of Loomis and Widder (Theorem 1.0.9) is known.

Theorem 1.0.12 ([ABR01, Theorem 7.26]). *If u is a positive harmonic function in $\mathbb{R}^{n+1}_+$, then there exists a unique positive measure μ (known as the boundary measure of u) on $\mathbb{R}^n$, and a nonnegative constant C such that*

$$u(x,y) = Cy + P[\mu](x,y), \quad (x,y) \in \mathbb{R}^{n+1}_+.$$

We are now ready to present the result of Ramey and Ullrich.

Theorem 1.0.13 ([RU88, Theorem 2.2]). *Suppose that u is positive and harmonic in $\mathbb{R}^{n+1}_+$, with boundary measure μ and that $L \in [0,\infty)$, $x_0 \in \mathbb{R}^n$. Then the following statements are equivalent:*

i) u *has nontangential limit* L *at* $x_0 \in \mathbb{R}^n$.

ii) There exists a positive number α *such that*

$$\lim_{\substack{(x,y)\to(x_0,0)\\(x,y)\in S(x_0,\alpha)}} u(x,y) = L.$$

iii) μ *has strong derivative* L *at* x_0.

A generalization of this theorem has been proved by Logunov [Log15, Theorem 10] for positive solutions of a more general class of second order uniformly elliptic operators on $\mathbb{R}^{n+1}_+$ containing the Laplacian. Theorem 1.0.13 has also been extended for a more general classes of measures in [BC90, RU88]. However, in this thesis we will restrict ourselves only to positive measures.

Inspired by the works of Fatou and Loomis, Gehring [Geh60], initiated the study of Fatou-type theorems and their converse for solutions of the heat equation in $\mathbb{R}^2_+$. The heat equation in $\mathbb{R}^{n+1}_+$ is given by

$$\Delta u(x,t) = \frac{\partial}{\partial t} u(x,t), \quad (x,t) \in \mathbb{R}^{n+1}_+,$$

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$, is the Laplacian of $\mathbb{R}^n$. We recall that the fundamental solution of the heat equation in the Euclidean upper half-space $\mathbb{R}^{n+1}_+$, is known as the Gauss-Weierstrass kernel or the heat kernel of $\mathbb{R}^n$, and is given by

$$W(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4t}}, \quad (x,t) \in \mathbb{R}^{n+1}_+. \tag{1.0.11}$$

We observe that

$$W(x,t) = w_{\sqrt{t}}(x), \quad w(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4}}, \quad x \in \mathbb{R}^n,\ t > 0. \tag{1.0.12}$$

In the literature, $w_{\sqrt{t}}$ is also denoted by h_t. The Gauss-Weierstrass integral of a measure μ on $\mathbb{R}^n$, is given by the convolution

$$W[\mu](x,t) = \mu * w_{\sqrt{t}}(x) = \int_{\mathbb{R}^n} W(x-y,t)\, d\mu(y), \tag{1.0.13}$$

whenever the integral above converges absolutely for $(x,t) \in \mathbb{R}^{n+1}_+$. As in the case of Poisson integral of measures, it is known that [Wat12, Theorem 4.4], if $W[|\mu|](x_0,t_0)$ is finite at some point $(x_0,t_0) \in \mathbb{R}^{n+1}_+$, then $W[|\mu|](x,t)$ is also finite for all $(x,t) \in \mathbb{R}^n \times (0,t_0)$. In this case, we say that $W[\mu]$ is well-defined in $\mathbb{R}^n \times (0,t_0)$. Moreover, $W[\mu]$ is a solution of the heat equation in the strip $\mathbb{R}^n \times (0,t_0)$. Widder proved that positive solutions of the heat equation in $\mathbb{R}^2_+$, have the similar characterization as positive harmonic functions in $\mathbb{R}^2_+$.

Theorem 1.0.14 ([Wid44, Theorem 6])**.** *If u is a positive solution of the heat equation in $\mathbb{R}^2_+$, then there exists a right-continuous, increasing function β defined on $\mathbb{R}$ (unique up to an additive constant) such that*

$$u(x,t) = W[\mu_\beta](x,t) = \int_{\mathbb{R}} W(x-\xi,t)\, d\mu_\beta(\xi), \quad x \in \mathbb{R},\ \ t \in (0,\infty), \tag{1.0.14}$$

where μ_β is the Lebesgue-Stieltjes measure induced by β.

The natural approach regions to consider for studying boundary behavior of solutions of heat equation in $\mathbb{R}^{n+1}_+$, are the parabolic regions.

Definition 1.0.15. i) For $\alpha \in (0, \infty)$, we define the parabolic region $\mathtt{P}(x_0, \alpha)$ with vertex at $x_0 \in \mathbb{R}^n$ and aperture α by

$$\mathtt{P}(x_0, \alpha) = \{(x, t) \in \mathbb{R}^{n+1}_+ \mid \|x - x_0\|^2 < \alpha t\}. \tag{1.0.15}$$

ii) A complex-valued function u defined on $\mathbb{R}^{n+1}_+$, is said to have parabolic limit $L \in \mathbb{C}$, at $x_0 \in \mathbb{R}^n$, if, for every $\alpha \in (0, \infty)$,

$$\lim_{\substack{(x,t)\to(x_0,0)\\(x,t)\in \mathtt{P}(x_0,\alpha)}} u(x, t) = L.$$

The following are the analogues of results of Fatou and Loomis for solutions of the heat equations in $\mathbb{R}^2_+$ due to Gehring.

Theorem 1.0.16 ([Geh60, Theorem 2, Theorem 5]). *i) Suppose that $W[\mu_\beta]$ is well-defined in $\mathbb{R}^2_+$, where β and μ_β are as in Theorem 1.0.8. If β is differentiable at some $x_0 \in \mathbb{R}$, then $W[\mu_\beta]$ has parabolic limit $\beta'(x_0)$ at x_0.*

ii) Suppose that u is a positive solution of the heat equation in $\mathbb{R}^2_+$, with β as in Theorem 1.0.14 and that $x_0 \in \mathbb{R}$, $L \in [0, \infty)$. If for two distinct real numbers α_1, α_2

$$\lim_{t\to 0} u(x_0 + \alpha_1\sqrt{t}, t) = L = \lim_{t\to 0} u(x_0 + \alpha_2\sqrt{t}, t),$$

then $\beta'(x_0) = L$.

There are two kinds of results related to the result of Ramey and Ullrich (Theorem 1.0.13) which will be proved in this thesis. Results of the first kind proved in Chapter 3, and 4, revolve around the parabolic convergence of the positive solutions of the heat equation in $\mathbb{R}^n \times (0, \infty)$, and in $G \times (0, \infty)$, where G stands for certain nilpotent Lie groups known as stratified Lie groups. In chapter 3, we will use the idea of Theorem 1.0.13 to prove higher dimensional analogue of the results of Gehring (Theorem 1.0.16) regarding parabolic convergence of positive solution of the heat equation in $\mathbb{R}^{n+1}_+$ (see Theorem 3.3.2). In Chapter 4, Theorem 3.3.2 will be further generalized for positive solutions of the heat equation on stratified Lie groups.

Result of the second kind deals with certain Riemannian manifolds called Harmonic NA groups (also known as Damek-Ricci spaces) which generalizes the Euclidean upper half-space. Our result will deal with the relationship between admissible convergence (in the sense of Korányi [KP76, P.158]) of certain positive eigenfunctions of the Laplace-Beltrami operator (including positive harmonic functions), on these spaces and the strong derivative of a measure on the boundary, which is a group of Heisenberg type. This result extends the theorem of Ramey and Ullrich (Theorem 1.0.13) to a more general class of spaces which includes Riemannian symmetric spaces of noncompact type with real rank one. We will discuss this result in Chapter 6.

To proceed further, we need some more definitions.

Definition 1.0.17. Given a measure μ on $\mathbb{R}^n$, we consider the following sets.

i) A point $x_0 \in \mathbb{R}^n$, is called a Lebesgue point of a measure μ on $\mathbb{R}^n$, if there exists $L \in \mathbb{C}$, such that
$$\lim_{r\to 0} \frac{|\mu - Lm|(B(x_0,r))}{m(B(0,r))} = 0.$$
The set of all of Lebesgue points of a measure μ is called the Lebesgue set of μ and is denoted by $\mathrm{L}_n(\mu)$.

ii) A point $x_0 \in \mathbb{R}^n$, is called a σ-point of μ if there exists $L \in \mathbb{C}$ satisfying the following: for each $\epsilon > 0$, there exists $\delta > 0$ such that
$$|(\mu - Lm)(B(x,r))| < \epsilon(\|x - x_0\| + r)^n,$$
whenever $\|x - x_0\| < \delta$, and $r \in (0,\delta)$. In this case, we will denote the complex number L by $D_\sigma\mu(x_0)$. The set of all σ-points is called the σ-set of μ and is denoted by $\Sigma_n(\mu)$.

iii) $\mathsf{S}_n(\mu) = \{x_0 \in \mathbb{R}^n \mid$ the strong derivative of μ exists at $x_0\}$.

It can be shown that $\mathrm{L}_n(\mu)$ includes almost all points of $\mathbb{R}^n$, and that
$$\mathrm{L}_n(\mu) \subset \Sigma_n(\mu),$$
for a measure μ on $\mathbb{R}^n$. The set $\mathrm{L}_n(\mu)$ was introduced by Saeki [Sae96, P.135] generalizing the notion of Lebesgue point of a locally integrable function. As has been mentioned already,

the set $\mathsf{S}_n(\mu)$ was defined by Ramey and Ullrich [RU88, P.208]. The set $\Sigma_n(\mu)$ was defined by Shapiro [Sha06, P.3182] for locally integrable functions and for measures it was defined in [Sar21b, Definition 2.1].

We again look back at the theorem of Fatou (Theorem 1.0.8) regarding nontangential convergence of Poisson integral of measures. According to the result of Ramey and Ullrich (Theorem 1.0.13), if μ is a positive measure then the nontangential convergence of its Poisson integral $P[\mu]$ takes place precisely at the points of $\mathsf{S}_n(\mu)$. However, the same is not known about measures (not necessarily positive) with well-defined Poisson integrals. In [Sha06, Theorem 1], Shapiro proved that if $d\mu = f\,dm$, for some $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $P[\mu]$ has nontangential limit at all points of $\Sigma_n(\mu)$. Shapiro also gave an example of a function $f \in L^p(\mathbb{R}^2)$, for all $1 \leq p \leq \infty$, such that the origin is a σ-point of f but not a Lebesgue point. In Chapter 5, we will define the notion of Lebesgue points and σ-points of a measure on a stratified Lie group and prove an analogue of Shapiro's result. When we specialize to the Euclidean space, our results include the following.

i) For a measure μ on $\mathbb{R}^n$, the following set containment holds.

$$\mathrm{L}_n(\mu) \subseteq \Sigma_n(\mu) \subseteq \mathsf{S}_n(\mu).$$

In one dimension, the following equality holds.

$$\Sigma_1(\mu) = \mathsf{S}_1(\mu).$$

ii) Nontangential convergence of the convolution integral $\phi[\mu]$ takes place on the set $\Sigma_n(\mu)$ for a fairly general class of radial kernels containing the Poisson kernel. This extends the result of Shapiro [Sha06, Theorem 1].

In the same chapter, we shall construct a measure on the Heisenberg group such that the set of all Lebesgue points of the measure is strictly contained in that of all σ-points. A result analogous to ii) above was proved in [EH06, Theorem 3.4]. However, by means of an example, we shall show that our result does not follow from that of [EH06, Theorem 3.4].

The last result we will be discussing in this thesis was proved by Repnikov and Eidelman [RE66, RE67] regarding large time behavior of certain solutions of the heat equation. They proved, among other things, the following result.

Theorem 1.0.18 ([RE66, Theorem 1]). *Let* $f \in L^\infty(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, $L \in \mathbb{C}$. *Then*

$$\lim_{r\to\infty} \frac{1}{m(B(x_0,r))} \int_{B(x_0,r)} f(x)\, dm(x) = L, \tag{1.0.16}$$

if and only if

$$\lim_{t\to\infty} f * w_{\sqrt{t}}(x_0) = L, \tag{1.0.17}$$

where $w_{\sqrt{t}}(x)$ *is the heat kernel of* $\mathbb{R}^n$ *(see 1.0.12).*

We will extend the above theorem for two different approximate identities $\{\phi_t\}$ and $\{\psi_t\}$ (see Theorem 2.1.14). Precisely, we will find a set of sufficient conditions on the function ϕ such that for $f \in L^\infty(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ and $L \in \mathbb{C}$

$$\lim_{t\to\infty} f * \phi_t(x_0) = L,$$

implies that

$$\lim_{t\to\infty} f * \psi_t(x_0) = L,$$

for all radial function $\psi \in L^1(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} \psi(x)\, dm(x) = 1.$$

We will further show that one of these conditions is also necessary. We will then use this result to prove a theorem analogous to Theorem 1.0.18 regarding asymptotic behavior of certain eigenfunctions of the Laplace-Beltrami operator on real hyperbolic spaces (see Corollary 2.2.6 in Chapter 2). As such there does not exist any connection between Theorem 1.0.6 and Theorem 1.0.18, at least in this generality, apart from the fact that they seem complementary to each other in some sense. However, from our viewpoint, the main reason for including both these results in the same chapter (Chapter 2) is the fact that proof of both these results depend crucially on the Wiener Tauberian theorem for the multiplicative group $(0, \infty)$.

We now describe the chapterwise content of the thesis.

Chapter 2: Theorem 2.1.3 extends the result of Rudin (Theorem 1.0.6) for a fairly general class of radial kernels containing the Poisson kernel. Necessity of these conditions has been discussed in Example 2.1.9. Our next result, Theorem 2.1.14, extends the result of Repnikov and Eidelman (Theorem 1.0.18) for a suitable class of approximate identities satisfying certain conditions. Example 2.1.16 discusses necessity of these conditions. We have then used Theorem 2.1.3 to prove a result analogous to Theorem 1.0.6 for certain positive eigenfunctions of the Laplace-Beltrami operator on real hyperbolic spaces $\mathbb{H}^n$, $n \geq 2$ (see Theorem 2.2.4). Analogue of Theorem 1.0.18 for certain eigenfunctions of the Laplace-Beltrami operator on real hyperbolic spaces $\mathbb{H}^n$ have also been proved in this chapter (see Theorem 2.2.6). Results of these chapter have appeared in [Sar].

Chapter 3: In this chapter, our main result is Theorem 3.3.2, which extends the result of Gehring (Theorem 1.0.16) in Euclidean upper half-spaces $\mathbb{R}^{n+1}_+$, regarding parabolic convergence of positive solutions of the heat equation. This result has appeared in [Sar21c].

Chapter 4: Our aim, in this chapter, is to prove a variant of Theorem 1.0.16, for positive solutions of the heat equation on stratified Lie groups. This result can also be viewed as a generalization of Theorem 3.3.2. After discussing necessary prerequisite regarding the analysis on these groups, we prove our main theorem which is Theorem 4.4.2.

Chapter 5: The main result of this chapter is Theorem 5.2.12, which generalizes Shapiro's theorem (Theorem 5.1.2) for a fairly general class of radial kernels as well as for measures. One of our result of this chapter, in particular, shows the relationship between the sets $\mathrm{L}_n(\mu)$, $\Sigma_n(\mu)$, $\mathsf{S}_n(\mu)$ (see Definition 1.0.17), for a measure μ on $\mathbb{R}^n$.

Chapter 6: In this chapter, we prove an analogue of Theorem 1.0.13 for certain positive eigenfunctions of the Laplace-Beltrami operator on Harmonic NA groups. Our main result in this chapter is Theorem 6.4.2.

Chapter 2

Generalization of a theorem of Loomis and Rudin

In this chapter, we generalize the result of Loomis and Rudin (Theorem 1.0.6), to show that analogous result remain valid for more general convolution integrals other than the Poisson integral. We will then apply this result to prove a result regarding boundary behavior of certain positive eigenfunctions of the Laplace-Beltrami operator on real hyperbolic spaces $\mathbb{H}^n$, $n \geq 2$. Our other aim, in this chapter, is to prove a generalization of the result of Repnikov and Eidelman (Theorem 1.0.18) regarding large time behavior of bounded solution of the heat equation. We will then use this result to prove a result regarding asymptotic behavior of certain eigenfunctions of the Laplace-Beltrami operator on real hyperbolic spaces $\mathbb{H}^n$. The main results of this chapter are Theorem 2.1.3, Theorem 2.1.14, Theorem 2.2.4, and Theorem 2.2.6.

2.1 The Euclidean spaces

We recall that for a measure μ on $\mathbb{R}^n$ and a complex-valued function ϕ on $\mathbb{R}^n$, we have defined the convolution integral $\phi[\mu]$ by

$$\phi[\mu](x,t) = \mu * \phi_t(x) = \int_{\mathbb{R}^n} \phi_t(x-\xi)\, d\mu(\xi),$$

whenever the integral converges absolutely for $(x,t) \in \mathbb{R}^{n+1}_+$. Here,

$$\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right), \quad x \in \mathbb{R}^n,\ t \in (0,\infty).$$

We say that the convolution integral $\phi[\mu]$ is well-defined in $E \subseteq \mathbb{R}^{n+1}_+$, if the integral above converges absolutely for all $(x,t) \in E$. It is well-known that if $\phi \in L^1(\mathbb{R}^n)$, with

$$\int_{\mathbb{R}^n} \phi(x)\, dm(x) = 1,$$

then $\{\phi_t \mid t \in (0,\infty)\}$ is an approximate identity [SW71, Theorem 1.18], where m is the Lebesgue measure on $\mathbb{R}^n$. Throughout this chapter, whenever an integral is involved, we will write dx instead of $dm(x)$ and hope that it will not create any confusion.

Remark 2.1.1. As we have already mentioned, it was proved in [Sae96, P. 137] that if ϕ is a nonnegative, radially decreasing function on $\mathbb{R}^n$ then finiteness of $|\mu| * \phi_{t_0}(x_0)$, for some $(x_0,t_0) \in \mathbb{R}^{n+1}_+$, implies the finiteness of $|\mu| * \phi_t(x)$ for all $(x,t) \in \mathbb{R}^n \times (0,t_0)$, that is, $\phi[\mu]$ is well-defined in $\mathbb{R}^n \times (0,t_0)$. Note that if $|\mu|(\mathbb{R}^n)$ is finite, then $\phi[\mu]$ is well-defined in $\mathbb{R}^{n+1}_+$.

In Saeki's theorem (Theorem 1.0.5), which generalize Fatou's theorem (Theorem 1.0.2) for an approximate identity $\{\phi_t \mid t \in (0,\infty)\}$, it is necessary for ϕ to satisfy the comparison condition (1.0.6).

Example 2.1.2. The following are some simple examples of functions which satisfy the condition (1.0.6).

i) For $\alpha \in [n/2,\infty)$, and $\kappa \in [0,\infty)$, we define

$$K(x) = \frac{1}{(1+\|x\|^2)^\alpha \log(2+\|x\|^\kappa)}, \quad x \in \mathbb{R}^n.$$

We have for $t \in (0,1)$, and $\|x\| > 1$,

$$\begin{aligned}
\frac{K_t(x)}{K(x)} &= t^{-n}\frac{t^{2\alpha}(1+\|x\|^2)^\alpha \log(2+\|x\|^\kappa)}{(t^2+\|x\|^2)^\alpha \log\left(2+\frac{\|x\|^\kappa}{t^\kappa}\right)} \\
&\leq t^{2\alpha-n}\frac{(1+\|x\|^2)^\alpha \log(2+\|x\|^\kappa)}{\|x\|^{2\alpha}\log\left(2+\|x\|^\kappa\right)} \\
&\leq t^{2\alpha-n}\left(1+\frac{1}{\|x\|^2}\right)^\alpha.
\end{aligned}$$

This shows that K satisfies the comparison condition (1.0.6). In fact, in case of $\alpha > n/2$, we have the following stronger result.

$$\lim_{t\to 0} \frac{K_t(x)}{K(x)} = 0,$$

uniformly for $x \in B(0,\epsilon)^c$, for any $\epsilon > 0$. In particular, taking $\alpha = \frac{n+1}{2}$, $\kappa = 0$, we see that

$$\mathrm{P}(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}(1+\|x\|^2)^{-\frac{n+1}{2}}, \quad x \in \mathbb{R}^n$$

satisfies the comparison condition (1.0.6).

ii) For positive real numbers α and β, we define

$$G(x) = e^{-\alpha\|x\|^\beta}, \quad x \in \mathbb{R}^n.$$

For $t \in (0,1)$, and $\|x\| > 1$,

$$\frac{G_t(x)}{G(x)} = t^{-n}e^{-\alpha\left(\frac{1}{t^\beta}-1\right)\|x\|^\beta} \leq t^{-n}e^{-\alpha\left(\frac{1}{t^\beta}-1\right)} = e^\alpha t^{-n} e^{-\frac{\alpha}{t^\beta}}.$$

Taking limit as $t \to 0$, we get

$$\lim_{t\to 0} \frac{G_t(x)}{G(x)} = 0,$$

uniformly for $x \in \mathbb{R}^n \setminus \overline{B(0,1)}$. Thus, G satisfies the comparison condition (1.0.6). In particular, the Gaussian

$$w(x) = (4\pi)^{-\frac{n}{2}}e^{-\frac{\|x\|^2}{4}}, \quad x \in \mathbb{R}^n$$

satisfies the comparison condition (1.0.6).

We recall that the symmetric derivative of a measure μ on $\mathbb{R}^n$, at a point $x_0 \in \mathbb{R}^n$, is defined as

$$D_{sym}\mu(x_0) = \lim_{r\to 0} \frac{\mu(B(x_0,r))}{m(B(x_0,r))},$$

whenever the limit exists. Here, $B(x_0,r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ is the open ball of radius r with center at x_0 with respect to the Euclidean metric. We are now in a position to present the our first result of this chapter, which generalizes the result of Loomis and Rudin (Theorem 1.0.6).

Theorem 2.1.3. *Suppose $\phi : \mathbb{R}^n \to (0, \infty)$, satisfies the following conditions:*

1) ϕ is radial, radially decreasing measurable function with $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$.

2) ϕ satisfies the comparison condition (1.0.6).

3) For all $y \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} \phi(x)\|x\|^{iy}\, dx \neq 0. \tag{2.1.1}$$

Suppose μ is a positive measure on $\mathbb{R}^n$, such that $\mu * \phi_{t_0}(0)$ is finite for some $t_0 \in (0, \infty)$. If for some $x_0 \in \mathbb{R}^n$, and $L \in [0, \infty)$

$$\lim_{t\to 0} \mu * \phi_t(x_0) = L,$$

then $D_{sym}\mu(x_0) = L$.

This theorem will be proved after we prove the necessary lemmas. Our first lemma shows that the comparison condition (1.0.6) can be used to reduce matters to the case of a finite positive measure μ.

Lemma 2.1.4. *Suppose $\phi : \mathbb{R}^n \to (0, \infty)$, satisfies the conditions 1) and 2) of Theorem 2.1.3. If μ is a positive measure such that $\mu * \phi_{t_0}(0)$ is finite for some $t_0 \in (0, \infty)$, then*

$$\lim_{t\to 0} \mu * \phi_t(0) = \lim_{t\to 0} \tilde{\mu} * \phi_t(0), \tag{2.1.2}$$

where $\tilde{\mu}$ is the restriction of μ on the closed ball $\overline{B(0, t_0)}$. Moreover,

$$D_{sym}\mu(0) = D_{sym}\tilde{\mu}(0). \tag{2.1.3}$$

Proof. We write for $t \in (0, t_0)$,

$$\begin{aligned} \mu * \phi_t(0) &= \int_{\{x\in\mathbb{R}^n \mid \|x\|\le t_0\}} \phi_t(x)\, d\mu(x) + \int_{\{x\in\mathbb{R}^n \mid \|x\|> t_0\}} \phi_t(x)\, d\mu(x) \\ &= \tilde{\mu} * \phi_t(0) + \int_{\{x\in\mathbb{R}^n \mid \|x\|> t_0\}} \phi_t(x)\, d\mu(x). \end{aligned} \tag{2.1.4}$$

Since ϕ is a radial, radially decreasing function, we have for any $r \in (0, \infty)$,

$$\int_{r/2 \leq \|x\| \leq r} \phi(x)\, dx \geq \omega_{n-1} \phi(r) \int_{r/2}^{r} s^{n-1}\, ds = C_n r^n \phi(r),$$

where ω_{n-1} is the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$, and C_n is a positive constant which depends only on the dimension. Since ϕ is an integrable function, the integral on the left hand side converges to zero as r goes to zero and infinity. Hence, it follows that

$$\lim_{\|x\| \to 0} \|x\|^n \phi(x) = \lim_{\|x\| \to \infty} \|x\|^n \phi(x) = 0. \tag{2.1.5}$$

We denote the integral appearing on the right-hand side of (2.1.4) by $I(t)$. Then, for $t \in (0, 1)$

$$\begin{aligned} I(tt_0) &= (tt_0)^{-n} \int_{\{x \in \mathbb{R}^n \mid \|x\| > t_0\}} \phi\left(\frac{x}{tt_0}\right) d\mu(x) \\ &= \int_{\{x \in \mathbb{R}^n \mid \|x\| > t_0\}} \frac{\left(\frac{\|x\|}{tt_0}\right)^n \phi\left(\frac{x}{tt_0}\right)}{\|x\|^n \phi_{t_0}(x)} \phi_{t_0}(x)\, d\mu(x). \end{aligned} \tag{2.1.6}$$

From (2.1.5) we get that

$$\lim_{t \to 0} \left(\frac{\|x\|}{tt_0}\right)^n \phi\left(\frac{x}{tt_0}\right) = 0,$$

for each fixed $x \in \mathbb{R}^n$. Also, by the comparison condition (1.0.6), we have

$$\frac{\left(\frac{\|x\|}{tt_0}\right)^n \phi\left(\frac{x}{tt_0}\right)}{\|x\|^n \phi_{t_0}(x)} = \frac{\phi_t\left(\frac{x}{t_0}\right)}{\phi\left(\frac{x}{t_0}\right)} \leq C, \quad \text{whenever} \quad \|x\| > t_0,\ 0 < t < 1,$$

for some positive constant C. Since $\phi_{t_0} \in L^1(\mathbb{R}^n, d\mu)$, it follows from (2.1.6), by the dominated convergence theorem that

$$\lim_{t \to 0} I(tt_0) = 0.$$

Consequently,

$$\lim_{t \to 0} I(t) = \lim_{t \to 0} I(tt_0^{-1} t_0) = \lim_{t \to 0} \int_{\{x \in \mathbb{R}^n \mid \|x\| > t_0\}} \phi_t(x)\, d\mu(x) = 0.$$

This proves (2.1.2). Proof of (2.1.3) follows easily as for all $r \in (0, t_0)$,

$$\tilde{\mu}\left(B(0, r)\right) = \mu\left(B(0, r)\right).$$

□

Next, we are going to prove two simple lemmas which will be used in the proof of the main theorem.

Lemma 2.1.5. *Let $\phi : \mathbb{R}^n \to (0, \infty)$, be a radial and radially decreasing measurable function. Let μ be a finite positive measure on $\mathbb{R}^n$ and*

$$v(t) = \mu * \phi_t(0), \quad t \in (0, \infty).$$

If

$$\lim_{t \to 0} v(t) = L < \infty,$$

then

i) v is a bounded function on $(0, \infty)$.

ii) The function

$$M(r) = \frac{\mu(B(0,r))}{m(B(0,r))}, \quad r \in (0, \infty), \tag{2.1.7}$$

is a bounded function on $(0, \infty)$.

Proof. The proof of $i)$ is simple. Indeed, as v has finite limit L at zero, there exists some $\delta \in (0, \infty)$, such that

$$0 \leq v(t) \leq L + 1,$$

for all $t \in (0, \delta)$. On the other hand, for all $t \geq \delta$

$$0 \leq v(t) = \int_{\mathbb{R}^n} \phi_t(x)\, d\mu(x) = t^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x}{t}\right) d\mu(x) \leq \delta^{-n} \phi(0) \mu(\mathbb{R}^n).$$

As $\phi(0)$ and $\mu(\mathbb{R}^n)$ are finite quantities, v is bounded on $(0, \infty)$.

To prove $ii)$, it suffices to show that

$$M(r) \leq C_{n,\phi} v(r), \quad \text{for all} \quad r \in (0, \infty). \tag{2.1.8}$$

Using the fact that ϕ is radial and radially decreasing, we observe that

$$v(r) \;=\; r^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x}{r}\right)\, d\mu(x)$$

$$\begin{aligned} &\geq r^{-n} \int_{B(0,r)} \phi\left(\frac{x}{r}\right) d\mu(x) \\ &\geq r^{-n} \int_{B(0,r)} \phi(1)\, d\mu(x) \\ &= m(B(0,1))\phi(1)M(r), \end{aligned}$$

for all $r \in (0,\infty)$. As $\phi(1) \in (0,\infty)$, the inequality (2.1.8) follows by setting $C_{n,\phi} = (m(B(0,1))\phi(1))^{-1}$. This completes the proof of ii). □

Remark 2.1.6. We observe that the inequality (2.1.8) remains valid even if μ is an infinite positive measure. This observation will be used in the proof of Theorem 2.1.10.

To prove our next lemma we will have to use the convolution on the multiplicative group $(0,\infty)$, with Haar measure ds/s. To differentiate with the convolution on $\mathbb{R}^n$, we write

$$f *_{(0,\infty)} g(t) = \int_0^\infty f(s) g\left(\frac{t}{s}\right) \frac{ds}{s},$$

where f and g are integrable on $(0,\infty)$, with respect to the Haar measure ds/s.

Lemma 2.1.7. *Suppose $k \in L^\infty((0,\infty))$, is such that*

$$\lim_{t\to 0} k(t) = L,$$

for some $L \in \mathbb{C}$. Then for all $f \in L^1((0,\infty), ds/s)$, with

$$\int_0^\infty f(s)\frac{ds}{s} = 1,$$

we have

$$\lim_{t\to 0} f *_{(0,\infty)} k(t) = L.$$

Proof. Let f be as above. We note that for each $t \in (0,\infty)$,

$$\begin{aligned} \left| f *_{(0,\infty)} k(t) - k(t) \right| &= \left| \int_0^\infty f(s) k\left(\frac{t}{s}\right) \frac{ds}{s} - \int_0^\infty f(s) k(t) \frac{ds}{s} \right| \\ &\leq \int_0^\infty |f(s)| \left| k\left(\frac{t}{s}\right) - k(t) \right| \frac{ds}{s}. \end{aligned} \tag{2.1.9}$$

Since $k(t)$ has limit L as t goes to zero, it follows that for each fixed $s \in (0, \infty)$,

$$\lim_{t \to 0} \left| k\left(\frac{t}{s}\right) - k(t) \right| = 0.$$

The integrand on the right-hand side of (2.1.9) is bounded by $2\|k\|_{L^\infty((0,\infty))}|f|$, an integrable function on $(0, \infty)$. Using dominated convergence theorem we conclude from (2.1.9) that

$$\lim_{t \to 0} \left| f *_{(0,\infty)} k(t) - k(t) \right| = 0,$$

which in turn, implies that

$$\lim_{t \to 0} f *_{(0,\infty)} k(t) = \lim_{t \to 0} k(t) = L. \qquad \square$$

The following versions of Wiener's Tauberian theorem [Rud91, Theorem 9.7] for the multiplicative group $(0, \infty)$, will be used multiple times in the remaining part of this chapter.

Theorem 2.1.8. *Suppose $\psi \in L^\infty((0, \infty))$, and $K \in L^1((0, \infty), ds/s)$ with the Fourier transform $\hat{K}$ everywhere nonvanishing on $\mathbb{R}$.*

1. *If for some $a \in \mathbb{C}$*

$$\lim_{t \to \infty} K *_{(0,\infty)} \psi(t) = a\hat{K}(0),$$

   then for all $f \in L^1((0, \infty), dt/t)$,

$$\lim_{t \to \infty} f *_{(0,\infty)} \psi(t) = a\hat{f}(0).$$

2. *If for some $a \in \mathbb{C}$*

$$\lim_{t \to 0} K *_{(0,\infty)} \psi(t) = a\hat{K}(0),$$

   then for all $f \in L^1((0, \infty), dt/t)$,

$$\lim_{t \to 0} f *_{(0,\infty)} \psi(t) = a\hat{f}(0).$$

We now prove our main result of this chapter.

Proof of Theorem 2.1.3. Without loss of generality, we can assume that x_0 is the origin. Indeed, we consider the translated measure $\mu_0 = \tau_{-x_0}\mu$, where

$$(\tau_\xi \mu)(E) = \mu(E - \xi), \tag{2.1.10}$$

for all Borel subsets $E \subset \mathbb{R}^n$. Using translation invariance of the Lebesgue measure it follows from the definition of symmetric derivative that $D_{sym}\mu_0(0)$ and $D_{sym}\mu(x_0)$ are equal. On the other hand, for a function $f : \mathbb{R}^n \to \mathbb{C}$, its left-translate by $\xi \in \mathbb{R}^n$, is defined as

$$\tau_\xi f(x) = f(x - \xi), \quad x \in \mathbb{R}^n. \tag{2.1.11}$$

As translation commutes with convolution, it also follows that

$$\mu_0 * \phi_t(0) = (\tau_{-x_0}\mu * \phi_t)(0) = \tau_{-x_0}(\mu * \phi_t)(0) = \mu * \phi_t(x_0),$$

for all $t \in (0, \infty)$. This shows that we can assume that x_0 is the origin. In view of Lemma 2.1.4, we can restrict μ on $\overline{B(0, t_0)}$, if necessary, to assume that μ is a finite positive measure. As before, we define

$$v(t) = \mu * \phi_t(0),\ \ 0 < t < \infty; \qquad M(r) = \frac{\mu(B(0, r)}{m(B(0, r))}, \quad 0 < r < \infty.$$

From Lemma 2.1.5, we know that both v and M are bounded functions on $(0, \infty)$. Following [Rud78], we consider the following function on $(0, \infty)$.

$$H(t) = \begin{cases} 0, & t \in (0, 1) \\ nt^{-n}, & t \geq 1. \end{cases}$$

Clearly, $H \in L^1((0, \infty), dt/t)$ with $\|H\|_{L^1((0,\infty),dt/t)} = 1$. We observe that for $r \in (0, \infty)$,

$$\begin{aligned} H *_{(0,\infty)} v(r) &= \int_0^\infty H\left(\frac{r}{s}\right) v(s)\, \frac{ds}{s} \\ &= n \int_0^r \left(\frac{s}{r}\right)^n \mu * \phi_s(0)\, \frac{ds}{s} \\ &= nr^{-n} \int_0^r s^n \int_{\mathbb{R}^n} s^{-n}\phi\left(\frac{x}{s}\right)\, d\mu(x)\, \frac{ds}{s} \\ &= nr^{-n} \int_{\mathbb{R}^n} \int_0^r \phi\left(\frac{x}{s}\right)\, \frac{ds}{s}\, d\mu(x), \end{aligned} \tag{2.1.12}$$

where we have used Fubini's theorem and the fact that ϕ is positive to obtain the last equality. Since M is a bounded function, it follows from the definition of M (2.1.7) that

$$\lim_{r\to 0}\mu(B(0,r)) = \lim_{r\to 0} m(B(0,r))M(r) = 0,$$

that is, μ has no point mass at the origin. We can now write, from (2.1.12), the convolution $H *_{(0,\infty)} v$ in a different way.

$$\begin{aligned}
H *_{(0,\infty)} v(r) &= nr^{-n}\int_{\mathbb{R}^n\setminus\{0\}}\int_0^r \phi\left(\frac{x}{s}\right)\frac{ds}{s}\,d\mu(x)\\
&= nr^{-n}\int_{\mathbb{R}^n\setminus\{0\}}\int_0^{\frac{r}{\|x\|}} \phi\left(\frac{x}{t\|x\|}\right)\frac{dt}{t}\,d\mu(x)\\
&\qquad \text{(change of variables, } s = t\|x\|\text{)}\\
&= nr^{-n}\int_{\mathbb{R}^n\setminus\{0\}}\int_0^{\frac{r}{\|x\|}} \phi\left(\frac{1}{t}\right)\frac{dt}{t}\,d\mu(x)\\
&= n\int_0^\infty\int_{B(0,\frac{r}{t})\setminus\{0\}} d\mu(x)\phi\left(\frac{1}{t}\right)t^{-n}\left(\frac{t}{r}\right)^n\frac{dt}{t}\\
&\qquad \text{(by Fubini's theorem)}\\
&= n\,m(B(0,1))\int_0^\infty \frac{\mu\left(B\left(0,\frac{r}{t}\right)\right)}{m\left(B\left(0,\frac{r}{t}\right)\right)}t^{-n}\phi\left(\frac{1}{t}\right)\frac{dt}{t}\\
&= n\,m(B(0,1))\int_0^\infty M\left(\frac{r}{t}\right)t^{-n}\phi\left(\frac{1}{t}\right)\frac{dt}{t}. \qquad (2.1.13)
\end{aligned}$$

By defining

$$g(s) = n\,m(B(0,1))s^{-n}\phi\left(\frac{1}{s}\right), \quad s\in(0,\infty),$$

equation (2.1.13) can be rewritten as

$$H *_{(0,\infty)} v(r) = M *_{(0,\infty)} g(r), \quad r\in(0,\infty).$$

By hypothesis, $v(r)$ converges to L as r goes to zero and hence by Lemma 2.1.7, so does $H *_{(0,\infty)} v(r)$. It now follows from the equation above that

$$\lim_{r\to 0} M *_{(0,\infty)} g(r) = L. \qquad (2.1.14)$$

We want to use the Wiener Tauberian theorem (Theorem 2.1.8) to deduce, from (2.1.14), that for all $f \in L^1\Big((0,\infty), \frac{ds}{s}\Big)$ with integral one,

$$\lim_{r\to 0} M *_{(0,\infty)} f(r) = L. \tag{2.1.15}$$

In order to do so, we need to show that g is of integral one with everywhere nonvanishing Fourier transform on the multiplicative group $(0,\infty)$. This can be deduced from the assumption (2.1.1). Precisely, for all $y \in \mathbb{R}$ we have

$$\begin{aligned}
\widehat{g}(y) &= \int_0^\infty g(s) s^{-iy}\,\frac{ds}{s} \\
&= n\, m(B(0,1)) \int_0^\infty s^{-n}\phi\left(\frac{1}{s}\right) s^{-iy}\,\frac{ds}{s} \\
&= n\, m(B(0,1)) \int_0^\infty \phi(t) t^{iy} t^n\,\frac{dt}{t} \\
&\qquad \text{(change of variables, } t = 1/s) \\
&= \int_{\mathbb{R}^n} \phi(x)\|x\|^{iy}\,dx.
\end{aligned}$$

We observe that by considering $y = 0$, it also follows that $g \in L^1((0,\infty), dt/t)$, with $\hat{g}(0) = 1$. Validity of the limit (2.1.15) now follows from the Wiener's Tauberian theorem (Theorem 2.1.8). In the final part of the proof we shall use (2.1.15) to deduce that

$$D_{sym}\mu(0) = \lim_{r\to 0} M(r) = L. \tag{2.1.16}$$

We fix an arbitrary $\gamma \in (1,\infty)$. We choose two positive functions $f_1 \in C_c((0,\infty))$, $f_2 \in C_c((0,\infty))$, such that $\|f_i\|_{L^1((0,\infty),dt/t)} = 1$, for $i = 1, 2$, and

$$\text{supp } f_1 \subset [1,\gamma], \quad \text{supp } f_2 \subset \left[\frac{1}{\gamma}, 1\right].$$

By monotonicity of μ we have for $t \in [1,\gamma]$ and $r \in (0,\infty)$,

$$m\left(B\left(0,\frac{r}{t}\right)\right) M\left(\frac{r}{t}\right) = \mu\left(B\left(0,\frac{r}{t}\right)\right) \leq \mu(B(0,r)),$$

and hence

$$M\left(\frac{r}{t}\right) \leq \frac{\mu(B(0,r))}{m\left(B\left(0,\frac{r}{t}\right)\right)} = t^n M(r) \leq \gamma^n M(r). \tag{2.1.17}$$

By a similar argument it follows that for $t \in [1/\gamma, 1]$ and $r \in (0,\infty)$,

$$M\left(\frac{r}{t}\right) \geq \gamma^{-n} M(r). \tag{2.1.18}$$

Now, for $r \in (0,\infty)$

$$M *_{(0,\infty)} f_1(r) = \int_1^\gamma f_1(t) M\left(\frac{r}{t}\right) \frac{dt}{t} \leq \int_1^\gamma \gamma^n M(r) f_1(t) \frac{dt}{t} = \gamma^n M(r), \tag{2.1.19}$$

where the inequality follows from (2.1.17). Similarly, using (2.1.18) we get

$$M *_{(0,\infty)} f_2(r) \geq \gamma^{-n} M(r) \quad r \in (0,\infty). \tag{2.1.20}$$

Combining (2.1.19) and (2.1.20) we get

$$\gamma^{-n} M *_{(0,\infty)} f_1(r) \leq M(r) \leq \gamma^n M *_{(0,\infty)} f_2(r), \quad r \in (0,\infty).$$

Allowing r tending to zero in the inequality above and using (2.1.15) we get

$$\gamma^{-n} L \leq \liminf_{r\to 0} M(r) \leq \limsup_{r \to 0} M(r) \leq \gamma^n L.$$

This implies (2.1.16) as $\gamma > 1$ is arbitrary. This completes the proof. □

We now show by an example that Theorem 2.1.3 fails in the absence of condition (2.1.1).

Example 2.1.9. Suppose $\phi : \mathbb{R}^n \to (0,\infty)$, is such that it satisfies the first two conditions of Theorem 2.1.3 but does not satisfy the third condition. That is, there exists $y_0 \in \mathbb{R}$, such that

$$\int_{\mathbb{R}^n} \phi(x) \cos(y_0 \log \|x\|)\, dx = \int_{\mathbb{R}^n} \phi(x) \sin(y_0 \log \|x\|)\, dx = 0. \tag{2.1.21}$$

As $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$, and ϕ is strictly positive, we have $y_0 \neq 0$. We consider the function

$$\begin{aligned} f(x) &= 2 + \cos(y_0 \log \|x\|), \quad x \in \mathbb{R}^n \setminus \{0\}, \\ &= 1, \qquad x = 0, \end{aligned}$$

and define a positive measure, $d\mu(x) = f(x)\, dm(x)$.

We will show that

$$\lim_{t \to 0} \mu * \phi_t(0) = 2, \tag{2.1.22}$$

but the symmetric derivative of μ does not exist at zero. Now, for all $t \in (0, \infty)$

$$\begin{aligned}
\mu * \phi_t(0) &= t^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x}{t}\right) f(x)\, dx \\
&= \int_{\mathbb{R}^n} \phi(x) f(tx)\, dx \\
&= 2\int_{\mathbb{R}^n} \phi(x) dx + \int_{\mathbb{R}^n} \phi(x) \cos(y_0 \log t + y_0 \log \|x\|)\, dx \\
&= 2 + \cos(y_0 \log t) \int_{\mathbb{R}^n} \phi(x) \cos(y_0 \log \|x\|)\, dx \\
&\qquad - \sin(y_0 \log t) \int_{\mathbb{R}^n} \phi(x) \sin(y_0 \log \|x\|)\, dx \\
&= 2,
\end{aligned}$$

where the last equality follows from (2.1.21). This implies the limit (2.1.22). On the other hand, for any $r \in (0, \infty)$

$$\begin{aligned}
\frac{1}{m(B(0,r))} \int_{B(0,r)} f(x)\, dx &= \frac{1}{m(B(0,r))} \int_{B(0,r)} (2 + \cos(y_0 \log \|x\|))\, dx \\
&= 2 + \frac{1}{m(B(0,r))} \mathrm{Re} \left[\int_{B(0,r)} \|x\|^{iy_0}\, dx \right] \\
&= 2 + \frac{\omega_{n-1}}{m(B(0,1))} \mathrm{Re} \left(\frac{r^{iy_0}}{n + iy_0} \right) \\
&= 2 + \frac{n \cos(y_0 \log r) + y_0 \sin(y_0 \log r)}{n^2 + y_0^2}.
\end{aligned} \tag{2.1.23}$$

As y_0 is nonzero, taking $r_j = e^{-\frac{j\pi}{|y_0|}}$, for $j \in \mathbb{N}$, we get from the equation above that

$$\frac{\mu(B(0, r_j))}{m(B(0, r_j))} = \frac{1}{m(B(0, r_j))} \int_{B(0, r_j)} f(x)\, dx = 2 + \frac{n(-1)^j}{n^2 + y_0^2}, \quad j \in \mathbb{N}.$$

Since $r_j \to 0$, as $j \to \infty$, it follows from the relation above that $D_{sym}\mu(0)$ does not exist.

It now remains to construct a function ϕ on $\mathbb{R}^n$, as above. To do this, we first consider the following functions defined on $(0, \infty)$:

$$g(r) = \chi_{[e^{-1}, e]}(r), \qquad f(r) = \frac{r^n}{(1 + r)^{2n}}.$$

Clearly, f and g both are in $L^1((0,\infty), dr/r)$, and the function $r \mapsto r^{-n} f(r)$ is decreasing in $(0,\infty)$. Moreover, for all $y \in \mathbb{R} \setminus \{0\}$

$$\int_0^\infty g(r) r^{iy} \, \frac{dr}{r} = \frac{2 \sin y}{y},$$

which vanishes for $y = \pi$. We now define

$$\psi(s) = f *_{(0,\infty)} g(s) = \int_0^\infty f\left(\frac{s}{r}\right) g(r) \, \frac{dr}{r}, \quad s \in (0,\infty).$$

Then $\psi \in L^1((0,\infty), ds/s)$, with $\widehat{\psi}(\pi) = 0$, and ψ is strictly positive. Hence,

$$c_\psi := \int_0^\infty \psi(s) \, \frac{ds}{s} > 0.$$

As the function $r \mapsto r^{-n} f(r)$ is decreasing in $(0,\infty)$ and g is nonnegative, it follows that

$$\frac{\psi(s)}{s^n} = \int_0^\infty \frac{f(s/r)}{(s/r)^n} g(r) r^{-(n+1)} \, dr, \tag{2.1.24}$$

is also decreasing. We observe that for each $r \in (0,\infty)$

$$\lim_{s\to 0} \frac{f(s/r)}{s^n} = \lim_{s\to 0} \frac{r^n}{(r+s)^{2n}} = r^{-n},$$

and

$$\frac{f(s/r)}{s^n} g(r) \le r^{-n} g(r), \quad s \in (0,\infty).$$

From the expression (2.1.24), it follows by applying dominated convergence theorem that

$$\lim_{s\to 0} \frac{\psi(s)}{s^n} = \int_0^\infty g(r) r^{-(n+1)} dr > 0.$$

Finally, we define $\phi : \mathbb{R}^n \to (0,\infty)$, by

$$\phi(x) = \begin{cases} \dfrac{1}{c_\psi \omega_{n-1}} \dfrac{\psi(\|x\|)}{\|x\|^n}, & x \neq 0, \\ \dfrac{1}{c_\psi \omega_{n-1}} \lim\limits_{s\to 0} \dfrac{\psi(s)}{s^n}, & x = 0. \end{cases}$$

By construction, ϕ is strictly positive, radial and radially decreasing on $\mathbb{R}^n$.

For all $y \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} \phi(x)\|x\|^{iy}\, dx = \omega_{n-1}\frac{1}{c_\psi \omega_{n-1}}\int_0^\infty \frac{\psi(s)}{s^n}s^{n-1+iy}\, ds = \frac{1}{c_\psi}\int_0^\infty \psi(s)s^{iy}\,\frac{ds}{s}.$$

This shows that $\phi \in L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \phi(x)\, dx = 1,$$

and

$$\int_{\mathbb{R}^n} \phi(x)\|x\|^{i\pi} dx = 0.$$

It remains to show that ϕ satisfies the comparison condition (1.0.6). The function ψ has the following explicit expression

$$\psi(s) = \int_{e^{-1}}^{e} \frac{r^n s^n}{(r+s)^{2n}}\,\frac{dr}{r}. \tag{2.1.25}$$

Differentiation under the integral sign yields

$$\psi'(s) = \int_{e^{-1}}^{e} \frac{n(r+s)^{2n-1}s^{n-1}(r-s)}{(r+s)^{4n}} r^{n-1}\, dr,$$

which is negative if $s \in (e,\infty)$. Hence, ψ is decreasing in (e,∞). Now, for $\|x\| \in (e,\infty)$, and $t \in (0,1)$ we have from the definition of ϕ

$$\frac{\phi_t(x)}{\phi(x)} = \frac{\psi(\|x\|/t)}{\psi(\|x\|)} \le \frac{\psi(\|x\|)}{\psi(\|x\|)} = 1. \tag{2.1.26}$$

We will now deal with the case $\|x\| \in (1,e]$, and $t \in (0,1)$. Using the expression (2.1.25) of ψ, it follows that for $s \in (1,e]$

$$\psi(s) \ge \int_{e^{-1}}^{e} \frac{e^{-(n-1)}}{(2e)^{2n}}\, dr = a_n > 0. \tag{2.1.27}$$

Also, for $t \in (0,1)$, and $s \in (1,e]$

$$\psi\left(\frac{s}{t}\right) = \int_{e^{-1}}^{e} \frac{t^n s^n r^{n-1}}{(tr+s)^{2n}}\, dr \le \int_{e^{-1}}^{e} s^{-n} r^{n-1}\, dr \le \int_{e^{-1}}^{e} r^{n-1}\, dr = b_n. \tag{2.1.28}$$

It now follows from (2.1.27) and (2.1.28) that for $\|x\| \in (1,e]$, $t \in (0,1)$

$$\frac{\phi_t(x)}{\phi(x)} = \frac{\psi(\|x\|/t)}{\psi(\|x\|)} \le a_n^{-1} b_n. \tag{2.1.29}$$

Inequalities (2.1.26) and (2.1.29) together imply that ϕ satisfies the comparison condition (1.0.6).

In the following theorem we show that Theorem 2.1.3 remains valid for a restricted class of measures in the absence of condition (2), that is, the comparison condition (1.0.6).

Theorem 2.1.10. *Suppose $\phi : \mathbb{R}^n \to (0, \infty)$, satisfies the following conditions:*

1) ϕ is radial, radially decreasing measurable function with $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$.

2) ϕ satisfies the condition (2.1.1).

Suppose that μ is a positive measure on $\mathbb{R}^n$ such that

$$\mu(B(0,r)) = O(r^n), \;\; as \;\; r \to \infty, \tag{2.1.30}$$

and that $\mu * \phi_{t_0}(0)$ is finite for some $t_0 \in (0, \infty)$. If for some $x_0 \in \mathbb{R}^n$, and $L \in [0, \infty)$

$$\lim_{t \to 0} \mu * \phi_t(x_0) = L,$$

then $D_{sym}\mu(x_0) = L$.

Proof. Without loss of generality, as before, we assume that x_0 is the origin. We will use the same notation as in the proof of Theorem 2.1.3. From the proof of Theorem 2.1.3 we observe that it suffices to prove the boundedness of the functions v and M and then the rest of the arguments remains same. Using Remark 2.1.1, we note that v is well-defined in $(0, t_0]$ and the fact that v is well-defined in (t_0, ∞) will be shown to be a consequence of the condition (2.1.30) in the hypothesis. According to Remark 2.1.6, the boundedness of v implies boundedness of M. Therefore, it suffices to prove that under the hypothesis of the theorem the function v is bounded. Since v has limit L at the origin, there exists $\delta \in (0, t_0)$, such that v is bounded on $(0, \delta]$. We observe that for $t > \delta$,

$$\begin{aligned} \mu * \phi_t(0) &= t^{-n} \int_{B(0,t)} \phi\left(\frac{x}{t}\right) \, d\mu(x) + t^{-n} \sum_{k=1}^{\infty} \int_{2^{k-1}t \le \|x\| < 2^k t} \phi\left(\frac{x}{t}\right) \, d\mu(x) \\ &\le \phi(0) m(B(0,1)) \frac{\mu(B(0,t))}{m(B(0,t))} + \sum_{k=1}^{\infty} \phi(2^{k-1}) t^{-n} \mu(B(0, 2^k t)) \end{aligned}$$

$$\begin{aligned}
&= \phi(0)m(B(0,1))\frac{\mu(B(0,t))}{m(B(0,t))} + m(B(0,1))\sum_{k=1}^{\infty}2^{nk}\phi(2^{k-1})\frac{\mu(B(0,2^k t))}{m(B(0,2^k t))}\\
&\leq m(B(0,1))\left(\phi(0)+\sum_{k=1}^{\infty}2^{nk}\phi(2^{k-1})\right)\sup_{r>\delta}\frac{\mu(B(0,r))}{m(B(0,r))} \qquad (2.1.31)
\end{aligned}$$

As ϕ is radial and radially decreasing, we have

$$\begin{aligned}
\int_1^{\infty}\phi(r)r^{n-1}dr &= \sum_{k=1}^{\infty}\int_{2^{k-1}}^{2^k}\phi(r)r^{n-1}dr\\
&\geq \sum_{k=1}^{\infty}\phi(2^k)\int_{2^{k-1}}^{2^k}r^{n-1}\,dr\\
&= \sum_{k=1}^{\infty}\phi(2^k)\frac{2^{nk}-2^{n(k-1)}}{n}\\
&= \frac{2^{-n}-2^{-2n}}{n}\sum_{k=1}^{\infty}\phi(2^k)2^{n(k+1)}.
\end{aligned}$$

Integrability of ϕ now implies that

$$\sum_{k=1}^{\infty}2^{nk}\phi(2^{k-1})<\infty.$$

Hence, we conclude from inequality (2.1.31) that for all $t>\delta$,

$$\mu*\phi_t(0)\leq C_{n,\phi}\sup_{t>\delta}\frac{\mu(B(0,t))}{m(B(0,t))}=C_{n,\phi}\sup_{t>\delta}M(t), \qquad (2.1.32)$$

where

$$C_{n,\phi}=m(B(0,1))\left(\phi(0)+\sum_{k=1}^{\infty}2^{nk}\phi(2^{k-1})\right)<\infty.$$

As μ satisfies (2.1.30), there exist positive constants C and r_0 such that for all $r\geq r_0$,

$$\frac{\mu(B(0,r))}{m(B(0,r))}\leq C.$$

On the other hand, for all $r\in(\delta,r_0)$,

$$\frac{\mu(B(0,r))}{m(B(0,r))}\leq\frac{\mu(B(0,r))}{m(B(0,\delta))}\leq\frac{\mu(B(0,r_0))}{m(B(0,\delta))}.$$

Therefore,

$$\sup_{r>\delta}M(r)=\sup_{r>\delta}\frac{\mu(B(0,r))}{m(B(0,r))}<\infty.$$

Existence of v on (t_0, ∞), and boundedness of v on (δ, ∞), now follow from boundedness of M on (δ, ∞), and (2.1.32). This completes the proof. $\square$

Remark 2.1.11. i) If μ is an absolutely continuous measure with L^p density, then μ satisfies the growth condition (2.1.30). Indeed, if $d\mu = f\, dm$, with $f \in L^p(\mathbb{R}^n)$, $p \in (1, \infty]$, then by the Hölder's inequality we have

$$\begin{aligned} |\mu(B(0,r))| \leq \int_{B(0,r)} |f|\, dm &\leq \|f\|_{L^p(\mathbb{R}^n)} \left(m(B(0,1))\right)^{\frac{1}{p'}} r^{\frac{n}{p'}} \\ &\leq \|f\|_{L^p(\mathbb{R}^n)} \left(m(B(0,1))\right)^{\frac{1}{p'}} r^{n}, \end{aligned}$$

for all $r > 1$, where p' is the conjugate exponent of p. If $d\mu = f\, dm$, with $f \in L^1(G)$, then $|\mu|(G)$ is finite. Hence, in this case the comparison condition (1.0.6) in Theorem 2.1.3 is not necessary.

ii) The condition (2.1.1) on ϕ is necessary in this case as well and can be seen from Example 2.1.9 by noting that the positive measure $d\mu(x) = f(x)\, dm(x)$ described there satisfies (2.1.30), in fact,

$$\mu(B(0,r)) \leq 3\, m(B(0,r)), \quad \text{for all} \quad r \in (0, \infty).$$

As an application of Theorem 2.1.3, we now suggest an alternative proof of a result of Watson [Wat77, Theorem 4] (see also [Geh60, Theorem 4] for the case $n = 1$, due to Gehring) regarding boundary behavior of positive solutions of the heat equation in $\mathbb{R}^{n+1}_+$ along the normal. Like positive harmonic functions in $\mathbb{R}^{n+1}_+$, we have similar characterization of the positive solutions of the heat equation.

Lemma 2.1.12 ([Wat12, Theorem 4.18])**.** *Let u be a positive solution of the heat equation in $\mathbb{R}^n \times (0, t_0)$, for some $t_0 \in (0, \infty]$. Then there exists a unique positive measure μ (known as the boundary measure of u) on $\mathbb{R}^n$, such that*

$$u(x,t) = W[\mu](x,t) = \int_{\mathbb{R}^n} W(x - \xi, t)\, d\mu(\xi), \quad x \in \mathbb{R}^n, \ t \in (0, t_0).$$

We recall that $W[\mu]$ is the Gauss-Weierstrass integral of μ (see (1.0.13)).

Corollary 2.1.13. *Let u be a positive solution of the heat equation with boundary measure μ in $\mathbb{R}^n \times (0, t_0)$, for some $t_0 \in (0, \infty]$. If for some $x_0 \in \mathbb{R}^n$, and $L \in [0, \infty)$*

$$\lim_{t \to 0} u(x_0, t) = L,$$

then

$$D_{sym}\mu(x_0) = L.$$

Proof. We consider the Gaussian $w = W(\cdot, 1)$ given in (1.0.12). Clearly, w is a strictly positive, radial, radially decreasing function on $\mathbb{R}^n$. Moreover, $\|w\|_{L^1(\mathbb{R}^n)} = 1$ [SW71, P.9], and w satisfies the comparison condition (1.0.6) (see Example 2.1.2, ii)). To apply Theorem 2.1.3, all we need is to show that w satisfies (2.1.1). Now, for all $y \in \mathbb{R}$,

$$\begin{aligned}
\int_{\mathbb{R}^n} w(x)\|x\|^{iy} dx &= (4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{\|x\|^2}{4}} \|x\|^{iy} \, dx \\
&= (4\pi)^{-\frac{n}{2}} \omega_{n-1} \int_0^\infty e^{-\frac{r^2}{4}} r^{iy} r^{n-1} \, dr \\
&= (4\pi)^{-\frac{n}{2}} \omega_{n-1} 2^{n-1+iy} \int_0^\infty e^{-t} t^{\frac{iy+n}{2}-1} \, dt \\
&\qquad \text{(by the substitution } r = 2\sqrt{t}) \\
&= (4\pi)^{-\frac{n}{2}} \omega_{n-1} 2^{n-1+iy} \Gamma\left(\frac{n+iy}{2}\right),
\end{aligned}$$

which is nonzero. As

$$\lim_{t \to 0} u(x_0, t) = \lim_{t \to 0} W[\mu](x_0, t) = \lim_{t \to 0} \mu * w_{\sqrt{t}}(x_0) = \lim_{t \to 0} \mu * w_t(x_0) = L,$$

the proof follows by Theorem 2.1.3. □

Our next result is a generalization of the result of Repnikov-Eidelman (Theorem 1.0.18) alluded to in the introduction.

Theorem 2.1.14. *Suppose $\phi \in L^1(\mathbb{R}^n)$, and $\psi \in L^1(\mathbb{R}^n)$, are radial functions such that*

$$\int_{\mathbb{R}^n} \phi(x) \, dx = \int_{\mathbb{R}^n} \psi(x) \, dx = 1. \tag{2.1.33}$$

Further assume that ϕ satisfies the condition (2.1.1).

If $f \in L^\infty(\mathbb{R}^n)$, is such that for some $x_0 \in \mathbb{R}^n$, and $L \in \mathbb{C}$,

$$\lim_{t\to\infty} f * \phi_t(x_0) = L,$$

then

$$\lim_{t\to\infty} f * \psi_t(x_0) = L.$$

Proof. Using polar coordinates we write

$$\begin{aligned} f * \phi_t(x_0) &= t^{-n} \int_{\mathbb{R}^n} f(x_0 - x)\phi\left(\frac{x}{t}\right) dx \\ &= t^{-n} \int_0^\infty \int_{S^{n-1}} f(x_0 - r\omega)\, d\sigma(\omega)\phi\left(\frac{r}{t}\right) r^{n-1}\, dr \\ &= \int_0^\infty f_0(r)\phi\left(\frac{r}{t}\right)\left(\frac{r}{t}\right)^n \frac{dr}{r}, \end{aligned} \tag{2.1.34}$$

where

$$f_0(r) = \int_{S^{n-1}} f(x_0 - r\omega)\, d\sigma(\omega), \quad r > 0,$$

with σ being the rotation invariant measure on the unit sphere S^{n-1}. Clearly, f_0 is a bounded function on $(0, \infty)$. We set

$$g_\phi(s) = s^{-n}\phi(s^{-1}), \quad s > 0.$$

From (2.1.34) we get the relation

$$f * \phi_t(x_0) = f_0 *_{(0,\infty)} g_\phi(t), \quad t > 0. \tag{2.1.35}$$

A similar computation shows that

$$f * \psi_t(x_0) = f_0 *_{(0,\infty)} g_\psi(t), \quad t > 0, \tag{2.1.36}$$

where

$$g_\psi(s) = s^{-n}\psi(s^{-1}), \quad s > 0.$$

Since ϕ, ψ are radial and integrable functions on $\mathbb{R}^n$, it follows that g_ϕ and g_ψ belong to the space $L^1((0,\infty), \frac{ds}{s})$.

Moreover, by (2.1.33)

$$\int_0^\infty g_\phi(s)\,\frac{ds}{s} = \int_0^\infty g_\psi(s)\,\frac{ds}{s} = \frac{1}{\omega_{n-1}}. \tag{2.1.37}$$

A simple calculation as in the proof of Theorem 2.1.3, shows that the Fourier transform of g_ϕ on the multiplicative group $(0,\infty)$ satisfies

$$\widehat{g_\phi}(y) = \int_0^\infty g_\phi(s)s^{-iy}\,\frac{ds}{s} = \frac{1}{\omega_{n-1}}\int_{\mathbb{R}^n}\phi(x)\|x\|^{iy}\,dx \neq 0,$$

for all $y\in\mathbb{R}$, as ϕ satisfies (2.1.1). Using the equations (2.1.35) and (2.1.37), it follows from the hypothesis that

$$\lim_{t\to\infty} f_0 *_{(0,\infty)} g_\phi(t) = \lim_{t\to\infty} f*\phi_t(x_0) = L = L\omega_{n-1}\widehat{g_\phi}(0).$$

From Wiener's tauberian theorem (Theorem 2.1.8) and (2.1.37) it follows that

$$\lim_{t\to\infty} f_0 *_{(0,\infty)} g_\psi(t) = L\omega_{n-1}\widehat{g_\psi}(0) = L.$$

An application of the relation (2.1.36) completes the proof. □

Remark 2.1.15. We will now show that the result of Repnikov and Eidelman (Theorem 1.0.18) can be proved using the last theorem. We choose $\phi = w$, and $\psi = m(B(0,1))^{-1}\chi_{B(0,1)}$. We have already observed in the proof of Corollary 2.1.13 that ϕ satisfies all the conditions of Theorem 2.1.14. We observe that ψ is a radial and integrable function on $\mathbb{R}^n$ with

$$\int_{\mathbb{R}^n}\psi(x)\,dx = 1.$$

Hence, to deduce Theorem 1.0.18, it suffices for us to show that ψ also satisfies (2.1.1). Now, for all $y\in\mathbb{R}$

$$\begin{aligned}\int_{\mathbb{R}^n}\psi(x)\|x\|^{iy}\,dx &= m(B(0,1))^{-1}\int_{B(0,1)}\|x\|^{iy}\,dx\\ &= m(B(0,1))^{-1}\omega_{n-1}\int_0^1 r^{iy+n-1}\,dr\\ &= m(B(0,1))^{-1}\omega_{n-1}\frac{1}{n+iy},\end{aligned}$$

which is nonzero. Now, suppose $f \in L^\infty(\mathbb{R}^n)$, and $x_0 \in \mathbb{R}^n$, $L \in \mathbb{C}$. Applying Theorem 2.1.14 twice, it follows that

$$\lim_{t\to\infty} f * w_{\sqrt{t}}(x_0) = \lim_{t\to\infty} f * w_t(x_0) = L,$$

if and only if

$$\lim_{t\to\infty} f * \psi_t(x_0) = \lim_{t\to\infty} \frac{1}{m(B(x_0,t))} \int_{B(x_0,t)} f(x)\, dx = L.$$

This proves Theorem 1.0.18.

We show by an example that condition (2.1.1) is necessary for the validity of Theorem 2.1.14 as well.

Example 2.1.16. Suppose $\phi \in L^1(\mathbb{R}^n)$, is a radial function such that

$$\int_{\mathbb{R}^n} \phi(x)\, dx = 1. \tag{2.1.38}$$

Assume that there exists $y_0 \in \mathbb{R}$, such that

$$\int_{\mathbb{R}^n} \phi(x) \|x\|^{iy_0} dx = 0, \tag{2.1.39}$$

From (2.1.38), it is clear that y_0 is nonzero. Consider the function

$$\begin{aligned} f(x) &= \|x\|^{iy_0}, \quad x \in \mathbb{R}^n \setminus \{0\}, \\ &= 1, \quad x = 0. \end{aligned}$$

Then $f \in L^\infty(\mathbb{R}^n)$. We have that for all $t \in (0,\infty)$,

$$f * \phi_t(0) = t^{-n} \int_{\mathbb{R}^n} \|x\|^{iy_0} \phi\left(\frac{x}{t}\right) dx = t^{-n} \int_{\mathbb{R}^n} \|t\xi\|^{iy_0} \phi(\xi) t^n\, d\xi = t^{iy_0} \int_{\mathbb{R}^n} \phi(\xi) \|\xi\|^{iy_0}\, d\xi,$$

and hence by (2.1.39)

$$\lim_{t\to\infty} f * \phi_t(0) = 0.$$

As in Remark 2.1.15, we again consider the function $\psi = m(B(0,1))^{-1} \chi_{B(0,1)}$.

Then ψ is nonnegative, radial with $\|\psi\|_{L^1(\mathbb{R}^n)} = 1$. We observe that

$$\begin{aligned} f * \psi_t(0) &= m(B(0,1))^{-1}t^{-n}\int_{B(0,t)} f(x)\,dx \\ &= m(B(0,1))^{-1}t^{-n}\int_{B(0,t)} \|x\|^{iy_0}\,dx \\ &= \omega_{n-1}m(B(0,1))^{-1}t^{-n}\int_0^t r^{iy_0}r^{n-1}\,dr \\ &= \omega_{n-1}m(B(0,1))^{-1}\frac{t^{iy_0}}{n+iy_0}. \end{aligned}$$

As y_0 is nonzero, taking $t_j = e^{\frac{j\pi}{|y_0|}}$, for $j \in \mathbb{N}$, we get from the equation above that

$$f * \psi_{t_j}(0) = \omega_{n-1}m(B(0,1))^{-1}\frac{(-1)^j}{n+iy_0}, \quad j \in \mathbb{N}.$$

Since $t_j \to \infty$, as $j \to \infty$, it follows that $f * \psi_t(0)$ does not converge to any limit as t goes to infinity. This establishes the necessity of condition (2.1.1).

Since a bounded harmonic function u on $\mathbb{R}^{n+1}_+$ is the Poisson integral of a unique boundary function $f \in L^\infty(\mathbb{R}^n)$ (see [SW71, Theorem 2.5]), the following result is a simple consequence of Theorem 2.1.14.

Corollary 2.1.17. *Suppose u is a bounded harmonic function on $\mathbb{R}^{n+1}_+$, with boundary function f. Then for $x_0 \in \mathbb{R}^n$, and $L \in \mathbb{C}$,*

$$\lim_{y\to\infty} u(x_0, y) = L,$$

if and only if

$$\lim_{r\to\infty} \frac{1}{m(B(x_0,r))}\int_{B(x_0,r)} f(x)dx = L.$$

Proof. We have (see 1.0.2)

$$u(x,y) = f * \mathrm{P}_y(x), \quad x \in \mathbb{R}^n, \ y \in (0,\infty).$$

The function P is radial and positive with $\|\mathrm{P}\|_{L^1(\mathbb{R}^n)} = 1$ (see 1.0.5).

We have for any $s \in \mathbb{R}$ (see [Rud78, Equation (24)]),

$$\int_{\mathbb{R}^n} (1 + \|x\|^2)^{-\frac{n+1}{2}} \|x\|^{is}\, dx = c_n \frac{\Gamma\left(\frac{n+is}{2}\right)\Gamma\left(\frac{1-is}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)} \neq 0.$$

This shows that P satisfies (2.1.1). We have already shown in Remark 2.1.15, that the function $m(B(0,1))^{-1}\chi_{B(0,1)}$ also satisfies (2.1.1). Applying Theorem 2.1.14 twice, first with $\phi = \mathrm{P}$ and then with $\phi = m(B(0,1))^{-1}\chi_{B(0,1)}$, we get the result. □

Remark 2.1.18. i) If $f \in L^q(\mathbb{R}^n)$, for some $q \in [1, \infty)$, then the result is trivially true, as can be seen by using Hölder's inequality. Indeed,

$$|f * \mathrm{P}_y(x)| \leq \|f\|_{L^q(\mathbb{R}^n)} \|\mathrm{P}_y\|_{L^{q'}(\mathbb{R}^n)} = c_{n,q} y^{-\frac{n}{q}} \|f\|_{L^q(\mathbb{R}^n)},$$

for all $(x, y) \in \mathbb{R}^{n+1}_+$, where q' is the conjugate exponent of q. On the other hand, for any $x \in \mathbb{R}^n$, $r > 0$

$$\frac{1}{m(B(x_0, r))} \int_{B(x_0,r)} f(x)dx \leq C_{n,q} r^{-\frac{n}{q}} \|f\|_{L^q(\mathbb{R}^n)},$$

where $c_{n,q}$, $C_{n,q}$ are two positive constants. Thus, $L = 0$ in both cases.

ii) For $f \in L^\infty(\mathbb{R}^n)$, the complex number L can be nonzero, as can be seen by taking f to be the constant function 1.

2.2 Real hyperbolic spaces

In this section, we will apply Theorem 2.1.3 and Theorem 2.1.14 in the context of real hyperbolic spaces and prove some analogous results for certain eigenfunctions of the Laplace-Beltrami operator. We start with a brief review of some basic facts about real hyperbolic spaces (see [Dav90, Sto16]).

We consider the Poincaré upper half space model of the n-dimensional real hyperbolic space

$$\mathbb{H}^n = \{(x, y) \mid x \in \mathbb{R}^{n-1}, y \in (0, \infty)\}, \quad n \geq 2,$$

equipped with the standard hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The boundary of $\mathbb{H}^n$ is identified with $\mathbb{R}^{n-1}$. The Laplace-Beltrami operator for $\mathbb{H}^n$ is given by the formula [Dav90, P. 176]

$$\Delta_{\mathbb{H}^n} = y^2 \left(\Delta_x + \frac{\partial^2}{\partial y^2} \right) - (n-2) y \frac{\partial}{\partial y}.$$

The expression for the corresponding Poisson kernel $\mathcal{P}$ is given by [Sto16, P. 76]

$$\mathcal{P}(x, y) = c_n \frac{y^{n-1}}{(y^2 + \|x\|^2)^{n-1}}, \quad (x, y) \in \mathbb{H}^n, \tag{2.2.1}$$

where c_n is a positive constant so that

$$\int_{\mathbb{R}^{n-1}} \mathcal{P}(x, 1)\, dx = 1.$$

We note that if

$$\phi(x) = \frac{c_n}{(1 + \|x\|^2)^{n-1}}, \quad x \in \mathbb{R}^{n-1},$$

then for $y \in (0, \infty)$

$$\phi_y(x) = y^{-(n-1)} \phi\left(\frac{x}{y}\right) = \mathcal{P}(x, y).$$

It is known that various classes of harmonic functions on $\mathbb{H}^n$, are Poisson integral of functions or measures defined on the boundary $\mathbb{R}^{n-1}$ [BSOS03, Sto16]. One such class is the collection of positive harmonic functions. More precisely, we have the following (see [Sto16, P. 113]): *given any positive harmonic function u on $\mathbb{H}^n$, there exists a unique positive measure μ on $\mathbb{R}^{n-1}$ and a nonnegative constant C such that*

$$u(x, y) = C y^{n-1} + \int_{\mathbb{R}^{n-1}} \mathcal{P}(x - \xi, y)\, d\mu(\xi), \quad (x, y) \in \mathbb{H}^n.$$

It turns out that more general eigenfunctions of $\Delta_{\mathbb{H}^n}$ can be obtained by considering the complex power of the Poisson kernel.

For $\lambda \in \mathbb{C}$, the λ-Poisson kernel is given by the formula

$$\mathcal{P}_\lambda(x,y) = \left(\frac{\mathcal{P}(x,y)}{\mathcal{P}(0,1)}\right)^{\frac{1}{2}-\frac{i\lambda}{n-1}} = \left[\frac{y^{n-1}}{(y^2+\|x\|^2)^{n-1}}\right]^{\frac{1}{2}-\frac{i\lambda}{n-1}}, \quad (x,y) \in \mathbb{H}^n. \tag{2.2.2}$$

It is known that for $\lambda \in \mathbb{C}$, the function $\mathcal{P}_\lambda$ is an eigenfunctions of $\Delta_{\mathbb{H}^n}$ and satisfies the following eigenvalue equation (see [ADY96, P. 654, Equation (2.35)]).

$$\Delta \mathcal{P}_\lambda = -(\lambda^2+\rho^2)\mathcal{P}_\lambda, \quad \text{where } \rho = \frac{n-1}{2}.$$

From the explicit expression (2.2.2), we observe that

$$|\mathcal{P}_\lambda(x,y)| = y^{\rho+\mathrm{Im}(\lambda)} \frac{1}{(y^2+\|x\|^2)^{\rho+\mathrm{Im}(\lambda)}}, \quad (x,y) \in \mathbb{H}^n.$$

Here and hereafter, $\mathrm{Im}(z)$ will denote the imaginary part of the complex number z. This shows that for $\mathrm{Im}(\lambda) \in (0,\infty)$, and $1 \le p \le \infty$, we have $\mathcal{P}_\lambda(\cdot,y) \in L^p(\mathbb{R}^{n-1})$, for all $y \in (0,\infty)$. Moreover, the following formula is known [Kum16, Lemma 2.3].

$$\int_{\mathbb{R}^{n-1}} \mathcal{P}_\lambda(x,1)\, dx = \frac{\mathbf{c}(-\lambda)}{c_n}, \tag{2.2.3}$$

where $\mathbf{c}(\lambda)$ is the Harish-Chandra $\mathbf{c}$-function for $\mathbb{H}^n$, and is given by

$$\mathbf{c}(\lambda) = 2^{n-1-2i\lambda} \frac{\Gamma(2i\lambda)\Gamma(n/2)}{\Gamma(\frac{n-1}{2})\Gamma(1/2+i\lambda)}, \quad \mathrm{Im}(\lambda) < 0. \tag{2.2.4}$$

It is clear from the formula above that the $\mathbf{c}$-function has no pole or zero in $\{\lambda \in \mathbb{C} \mid \mathrm{Im}(\lambda) < 0\}$. Therefore, for $\mathrm{Im}(\lambda) > 0$, we can normalize $\mathcal{P}_\lambda$, to define

$$P_\lambda(x,y) = d_\lambda \mathcal{P}_\lambda(x,y), \quad (x,y) \in \mathbb{H}^n, \tag{2.2.5}$$

where $d_\lambda = c_n \mathbf{c}(-\lambda)^{-1}$, so that

$$\int_{\mathbb{R}^{n-1}} P_\lambda(x,1)\, dx = 1. \tag{2.2.6}$$

Using the expression (2.2.2) we have the following important observation

$$\begin{aligned}
P_\lambda(x,y) &= d_\lambda \left[\frac{y^{n-1}}{(y^2+\|x\|^2)^{n-1}}\right]^{\frac{1}{2}-\frac{i\lambda}{n-1}} \\
&= y^{\frac{n-1}{2}+i\lambda} y^{-(n-1)} \frac{d_\lambda}{(1+\|y^{-1}x\|^2)^{\frac{n-1}{2}-i\lambda}} \\
&= y^{\rho+i\lambda} y^{-(n-1)} P_\lambda\left(\frac{x}{y},1\right) \\
&= y^{\rho+i\lambda}\left(\psi^\lambda\right)_y(x), \qquad (2.2.7)
\end{aligned}$$

where

$$\psi^\lambda(x) = P_\lambda(x,1) = d_\lambda \frac{1}{(1+\|x\|^2)^{\rho-i\lambda}}, \quad \left(\psi^\lambda\right)_y(x) = y^{-(n-1)}\psi^\lambda\left(\frac{x}{y}\right). \qquad (2.2.8)$$

It follows from (2.2.6) that for $\mathrm{Im}(\lambda) \in (0,\infty)$,

$$\int_{\mathbb{R}^{n-1}} \psi^\lambda(x)\, dx = 1. \qquad (2.2.9)$$

Hence, we get from (2.2.7) that for all $y \in (0,\infty)$,

$$\int_{\mathbb{R}^{n-1}} P_\lambda(x,y)\, dx = y^{\rho+i\lambda},$$

whenever $\mathrm{Im}(\lambda) \in (0,\infty)$. Using the fact that $\mathbf{c}(-i\rho) = 1$, it follows from (2.2.5) that $P_{i\rho} = \mathcal{P}$. For $\mathrm{Im}(\lambda) > 0$, we define the λ-Poisson transform of a measure μ on $\mathbb{R}^{n-1}$, as the convolution

$$P_\lambda[\mu](x,y) = \int_{\mathbb{R}^{n-1}} P_\lambda(x-\xi,y)\, d\mu(\xi), \quad (x,y) \in \mathbb{H}^n, \qquad (2.2.10)$$

whenever the integral converges absolutely.

Lemma 2.2.1. *Suppose that μ is a measure on $\mathbb{R}^{n-1}$ and that Im$(\lambda) > 0$. If the integral in (2.2.10) converges absolutely for some $(x_0,y_0) \in \mathbb{H}^n$, then it converges absolutely for all $(x,y) \in \mathbb{H}^n$.*

Proof. We fix a point (x_1,y_1) in $\mathbb{H}^n$. We observe from (2.2.5) and (2.2.2) that

$$|P_\lambda(x,y)| = |d_\lambda| y^{\rho+\mathrm{Im}(\lambda)} \frac{1}{(y^2+\|x\|^2)^{\rho+\mathrm{Im}(\lambda)}}, \quad (x,y) \in \mathbb{H}^n.$$

As $P_\lambda(\cdot, y_1)$ is continuous, it is enough to show that for some large $R \in (0,\infty)$,

$$\int_{\|\xi\|>R} |P_\lambda(x_1-\xi, y_1)| \; d|\mu|(\xi) < \infty.$$

We observe that for large $\|\xi\|$

$$\begin{aligned} \frac{y_0^2 + \|x_0-\xi\|^2}{y_1^2+\|x_1-\xi\|^2} &= \frac{y_0^2+\|x_0\|^2+\|\xi\|^2-2\langle x_0,\xi\rangle}{y_1^2+\|x_1\|^2+\|\xi\|^2-2\langle x_1,\xi\rangle} \\ &\le \frac{y_0^2+\|x_0\|^2+\|\xi\|^2+2\|x_0\|\|\xi\|}{y_1^2+\|x_1\|^2+\|\xi\|^2-2\|x_1\|\|\xi\|} \\ &= \frac{\frac{y_0^2}{\|\xi\|^2}+\frac{\|x_0\|^2}{\|\xi\|^2}+1+2\frac{\|x_0\|}{\|\xi\|}}{\frac{y_1^2}{\|\xi\|^2}+\frac{\|x_1\|^2}{\|\xi\|^2}+1-2\frac{\|x_1\|}{\|\xi\|}} \end{aligned}$$

This shows that

$$\limsup_{\|\xi\|\to\infty} \frac{y_0^2 + \|x_0-\xi\|^2}{y_1^2+\|x_1-\xi\|^2} \le 1.$$

Similarly, one can prove that

$$\liminf_{\|\xi\|\to\infty} \frac{y_0^2 + \|x_0-\xi\|^2}{y_1^2+\|x_1-\xi\|^2} \ge 1.$$

Hence,

$$\lim_{\|\xi\|\to\infty} \frac{y_0^2 + \|x_0-\xi\|^2}{y_1^2+\|x_1-\xi\|^2} = 1. \tag{2.2.11}$$

We obtain from the equation above that

$$\lim_{\|\xi\|\to\infty} \left|\frac{P_\lambda(x_1-\xi,y_1)}{P_\lambda(x_0-\xi,y_0)}\right| = \left(\frac{y_1}{y_0}\right)^{\rho+\mathrm{Im}(\lambda)} \lim_{\|\xi\|\to\infty} \left(\frac{y_0^2 + \|x_0-\xi\|^2}{y_1^2+\|x_1-\xi\|^2}\right)^{\rho+\mathrm{Im}(\lambda)} = \left(\frac{y_1}{y_0}\right)^{\rho+\mathrm{Im}(\lambda)}.$$

Thus, there exists some $R \in (0,\infty)$, such that

$$\left|\frac{P_\lambda(x_1-\xi,y_1)}{P_\lambda(x_0-\xi,y_0)}\right| < 1 + \left(\frac{y_1}{y_0}\right)^{\rho+\mathrm{Im}(\lambda)}, \quad \text{for all } \|\xi\| > R.$$

Hence,

$$\int_{\|\xi\|>R} |P_\lambda(x_1-\xi,y_1)| \; d|\mu|(\xi) \le 1 + \left(\frac{y_1}{y_0}\right)^{\rho+\mathrm{Im}(\lambda)} \int_{\|\xi\|>R} |P_\lambda(x_0-\xi,y_0)| \; d|\mu|(\xi).$$

This completes the proof as the right-hand side of the inequality above is finite by the hypothesis. $\square$

In view of the previous lemma, we say that $P_\lambda[\mu]$ is well-defined in $\mathbb{H}^n$, whenever the integral in the right-hand side of (2.2.10) converges absolutely for some $(x, y) \in \mathbb{H}^n$. As P_λ is an eigenfunction of $\Delta_{\mathbb{H}^n}$ with eigenvalue $-(\lambda^2 + \rho^2)$, it follows that $P_\lambda[\mu]$, provided it is well-defined, also satisfies the eigenvalue equation

$$\Delta_{\mathbb{H}^n} P_\lambda[\mu] = -(\lambda^2 + \rho^2) P_\lambda[\mu].$$

The relation (2.2.7) implies that $P_\lambda[\mu]$ can be rewritten as

$$P_\lambda[\mu](x, y) = y^{\rho + i\lambda} \left(\mu * \left(\psi^\lambda \right)_y \right)(x), \quad x \in \mathbb{R}^{n-1},\ y > 0. \tag{2.2.12}$$

From (2.2.8) and (2.2.4), we note that $\psi^\lambda(x)$ is positive for all $x \in \mathbb{R}^{n-1}$, if λ is equal to $i\beta$, for some $\beta \in (0, \infty)$. In this case, $P_\lambda[\mu]$ is a positive eigenfunction, provided it is well-defined, with eigenvalue $(\beta^2 - \rho^2)$, whenever μ is a positive measure. In fact, we have the following characterization of positive eigenfunctions of $\Delta_{\mathbb{H}^n}$.

Lemma 2.2.2 ([DR92, Theorem 7.11])**.** *If u is a positive eigenfunction of $\Delta_{\mathbb{H}^n}$ with eigenvalue $\beta^2 - \rho^2$, for some $\beta \in (0, \infty)$, then there exists a unique positive measure μ (known as the boundary measure of u) on $\mathbb{R}^{n-1}$ and a nonnegative constant C, such that*

$$u(x, y) = C y^{\beta + \rho} + P_{i\beta}[\mu](x, y), \quad \textit{for all} \quad (x, y) \in \mathbb{H}^n. \tag{2.2.13}$$

Remark 2.2.3. i) From the characterization of the positive eigenfunctions of $\Delta_{\mathbb{H}^n}$ given in [Hel77, Theorem 4.1], it follows that if a positive function u satisfies the equation

$$\Delta_{\mathbb{H}^n} u + \lambda u = 0,$$

in $\mathbb{H}^n$, then one must have $\lambda \in (-\infty, \rho^2]$.

ii) The result of Damek and Ricci [DR92, Theorem 7.11] also characterizes all positive eigenfunction of $\Delta_{\mathbb{H}^n}$ with eigenvalue $\beta^2 - \rho^2$, where $\beta \in (-\infty, 0]$. But our next theorem, Theorem 2.2.4, does not apply to these eigenfunctions.

We are now ready to prove an analogue of the result of Loomis and Rudin (Theorem 1.0.6) for positive eigenfunctions of $\Delta_{\mathbb{H}^n}$.

Theorem 2.2.4. *Suppose u is a positive eigenfunction of $\Delta_{\mathbb{H}^n}$, with boundary measure μ and eigenvalue $\beta^2 - \rho^2$, for some $\beta > 0$. If there exists $x_0 \in \mathbb{R}^{n-1}$, and $L \in [0, \infty)$, such that*

$$\lim_{y \to 0} y^{\beta-\rho} u(x_0, y) = L, \tag{2.2.14}$$

then $D_{sym}\mu(x_0) = L$.

Proof. As μ is the boundary measure of u, the expressions (2.2.13) and (2.2.12) imply that

$$u(x, y) = Cy^{\beta+\rho} + y^{\rho-\beta} \left(\mu * \left(\psi^{i\beta}\right)_y\right)(x), \quad x \in \mathbb{R}^{n-1},\ y > 0.$$

Hence,

$$y^{\beta-\rho} u(x, y) = Cy^{2\beta} + \left(\mu * \left(\psi^{i\beta}\right)_y\right)(x), \quad x \in \mathbb{R}^{n-1},\ y > 0.$$

By the hypothesis (2.2.14) and the fact that $\beta \in (0, \infty)$, it follows from the equation above that

$$\lim_{y \to 0} \left(\mu * \left(\psi^{i\beta}\right)_y\right)(x_0) = \lim_{y \to 0} y^{\beta-\rho} u(x_0, y) = L.$$

We recall from (2.2.8) that

$$\psi^{i\beta}(x) = d_{i\beta} \left(\frac{1}{1 + \|x\|^2}\right)^{\rho+\beta}, \quad x \in \mathbb{R}^{n-1}.$$

It is clear from the expression above that $\psi^{i\beta}$ is a strictly positive, radial and radially decreasing function on $\mathbb{R}^{n-1}$. Moreover, taking $\alpha = \rho + \beta$, and $\kappa = 0$, in Example 2.1.2, i), we see that $\psi^{i\beta}$ satisfies the comparison condition (1.0.6). In order to apply Theorem 2.1.3, we need to show that $\psi^{i\beta}$ satisfies the condition (2.1.1). To do this, we will need the following well-known formula [Gra14, P.420].

$$\int_0^{\pi/2} (\sin\theta)^z (\cos\theta)^w \, d\theta = \frac{\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{w+1}{2}\right)}{2\Gamma\left(\frac{z+w+2}{2}\right)}, \quad \mathrm{Re}(z) > -1,\ \mathrm{Re}(w) > -1. \tag{2.2.15}$$

Now, for any $\lambda \in \mathbb{C}$ with $\text{Im}(\lambda) > 0$, and $s \in \mathbb{R}$ we have

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \psi^{\lambda}(x)\|x\|^{is}\,dx &= d_{\lambda}\int_{\mathbb{R}^{n-1}}\left(\frac{1}{1+\|x\|^2}\right)^{\rho-i\lambda}\|x\|^{is}\,dx \\
&= d_{\lambda}\omega_{n-2}\int_0^{\infty}\left(\frac{1}{1+r^2}\right)^{\rho-i\lambda} r^{is}r^{n-2}\,dr \\
&= d_{\lambda}\omega_{n-2}\int_0^{\pi/2}\left(\frac{1}{\sec^2\theta}\right)^{\rho-i\lambda}(\tan\theta)^{n-2+is}(\sec\theta)^2\,d\theta \\
&\qquad \text{(using the substitution } r=\tan\theta) \\
&= d_{\lambda}\omega_{n-2}\int_0^{\pi/2}(\cos\theta)^{(2\rho-2i\lambda-n-is)}(\sin\theta)^{n-2+is}\,d\theta \\
&= d_{\lambda}\omega_{n-2}\frac{\Gamma\left(\frac{2\rho-2i\lambda-n-is+1}{2}\right)\Gamma\left(\frac{n-1+is}{2}\right)}{2\Gamma(\rho-i\lambda)}, \qquad (2.2.16)
\end{aligned}$$

where the last equality follows from (2.2.15), as $\text{Im}(\lambda) \in (0,\infty)$. As, the expression on the right-hand side of (2.2.16) is nonzero, it follows that $\psi^{i\beta}$ satisfies (2.1.1). In view of (2.2.9), the proof now follows simply by applying Theorem 2.1.3. □

The last topic we are going to discuss in this chapter is related to the result of Repnikov and Eidelman (Theorem 1.0.18). It is known that the exact analogue of the result of Repnikov and Eidelman (Theorem 1.0.18) is false on $\mathbb{H}^n$ (see [NRS21, Rep02]). However, an analogue of Corollary 2.1.17 (which we view as a variant of Theorem 1.0.18) can be proved for $\mathbb{H}^n$.

We define for $\lambda \in \mathbb{C}$ and f a measurable function on $\mathbb{R}^{n-1}$, the λ-Poisson transform $P_\lambda f$, by a convolution analogous to (2.2.10)

$$P_\lambda f(x,y) = \int_{\mathbb{R}^{n-1}} P_\lambda(x-\xi,y)f(\xi)\,d\xi, \quad (x,y)\in\mathbb{H}^n,$$

whenever the integral converges absolutely. Since the kernel $P_\lambda(\cdot,y)$ is integrable for every $y \in (0,\infty)$, for $\text{Im}(\lambda) \in (0,\infty)$, the λ-Poisson transform $P_\lambda f$ is well-defined in $\mathbb{H}^n$, for $f \in L^\infty(\mathbb{R}^{n-1})$. We will now prove an analogue of Corollary 2.1.17 for certain eigenfunctions of $\Delta_{\mathbb{H}^n}$. In order to do this, we will need the following characterization of eigenfunctions of $\Delta_{\mathbb{H}^n}$.

Lemma 2.2.5 ([BSOS03, Theorem 3.6])**.** *Suppose u is an eigenfunction of $\Delta_{\mathbb{H}^n}$ with eigenvalue $-(\lambda^2+\rho^2)$, where* $\text{Im}(\lambda) \in (0,\infty)$*. Then $u = P_\lambda f$, for some $f \in L^\infty(\mathbb{R}^{n-1})$, if and* *only if*

$$\sup_{y>0} y^{\mathrm{Im}(\lambda)-\rho}\|u(.,y)\|_{L^\infty(\mathbb{R}^{n-1})} < \infty. \tag{2.2.17}$$

The function f will be called the boundary function of u. The following result, for $\lambda = i\rho$, can be thought of as an exact analogue of Corollary 2.1.17.

Theorem 2.2.6. *Suppose u is an eigenfunction of $\Delta_{\mathbb{H}^n}$ with eigenvalue $-(\lambda^2+\rho^2)$, where Im$(\lambda) \in (0,\infty)$. Further suppose that u satisfies (2.2.17) and f is the boundary function of u. Then for $x_0 \in \mathbb{R}^{n-1}$, and $L \in \mathbb{C}$,*

$$\lim_{y\to\infty} y^{-(\rho+i\lambda)}u(x_0,y) = L,$$

if and only if

$$\lim_{r\to\infty} \frac{1}{m(B(x_0,r))}\int_{B(x_0,r)} f(x)dx = L.$$

Proof. Since f is the boundary function of u, we have

$$u(x,y) = P_\lambda f(x) = \int_{\mathbb{R}^{n-1}} P_\lambda(x-\xi,y)f(\xi)\,d\xi,\ \ (x,y)\in\mathbb{H}^n.$$

The equation (2.2.7) now implies that

$$y^{-(\rho+i\lambda)}u(x_0,y) = \left(f * \left(\psi^\lambda\right)_y\right)(x_0).$$

We also have

$$\frac{1}{m(B(x_0,r))}\int_{B(x_0,r)} f(x)dx = f * \left(m(B(0,1))^{-1}\chi_{B(0,1)}\right)_r (x_0).$$

Now, (2.2.16) shows that ψ^λ satisfies (2.1.1). Also, from (2.2.8) and (2.2.9), we have that ψ^λ is radial and is of integral one. We have already observed in Remark 2.1.15 that $m(B(0,1))^{-1}\chi_{B(0,1)}$ also obeys (2.1.1). Application of Theorem 2.1.14 twice, first with $\phi{=}\psi^\lambda$ and then with $\phi = m(B(0,1))^{-1}\chi_{B(0,1)}$ finishes the proof. □

Remark 2.2.7. It is known that the analogue of Theorem 2.2.4 (for $\beta=\rho$), is false for complex hyperbolic spaces (see [Rud08, P.78]). However, it is not known to us, at the moment, whether the exact analogue of Theorem 2.2.6 holds for complex hyperbolic spaces.

Chapter 3

Parabolic convergence of positive solutions of the heat equation in $\mathbb{R}^{n+1}_+$

In this chapter, we study the parabolic convergence of positive solutions of the heat equation in the Euclidean upper half-space $\mathbb{R}^{n+1}_+$. We prove that the existence of the parabolic limit of a positive solution of the heat equation at a point in the boundary is equivalent to the existence of the strong derivative of the boundary measure of the solution at that point. Moreover, the parabolic limit and the strong derivative are equal. This extends the result of Gehring (Theorem 1.0.16) in higher dimensions. The main result of this chapter is Theorem 3.3.2.

3.1 Introduction

We recall that the heat equation in $\mathbb{R}^{n+1}_+$ is given by

$$\Delta u(x,t) = \frac{\partial}{\partial t} u(x,t), \quad (x,t) \in \mathbb{R}^{n+1}_+,$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator on $\mathbb{R}^n$. The fundamental solution of the heat equation is the Gauss-Weierstrass kernel or the heat kernel of $\mathbb{R}^n$, and is given by

$$W(x,t) = w_{\sqrt{t}}(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4t}}, \quad (x,t) \in \mathbb{R}^{n+1}_+$$
$$w(\xi) = (4\pi)^{-\frac{n}{2}} e^{-\frac{\|\xi\|^2}{4}}, \quad \xi \in \mathbb{R}^n.$$

The Gauss-Weierstrass integral of a measure μ on $\mathbb{R}^n$, is given by the convolution

$$W[\mu](x,t) = \mu * w_{\sqrt{t}}(x) = \int_{\mathbb{R}^n} W(x-y,t)\, d\mu(y),$$

whenever the integral above converges absolutely for $(x,t) \in \mathbb{R}^{n+1}_+$.

Remark 3.1.1. ([Wat12, Theorem 4.4]). As we have already mentioned, it is known that if $W[|\mu|](x_0,t_0)$ is finite at some point $(x_0,t_0) \in \mathbb{R}^{n+1}_+$, then $W[|\mu|](x,t)$ is also finite for all $(x,t) \in \mathbb{R}^n \times (0,t_0)$. Moreover, $W[\mu]$ is a solution of the heat equation in the strip $\mathbb{R}^n \times (0,t_0)$. In this case, we say that $W[\mu]$ is well-defined in $\mathbb{R}^n \times (0,t_0)$.

For the sake of completeness, we would like to state the following theorem, which describes the relationship of the boundary behavior of Gauss-Weierstrass integral of a measures on $\mathbb{R}^n$ along the normal with the symmetric derivative of the measure. We recall that the symmetric derivative $D_{sym}\mu(x_0)$ of a measure defined on $\mathbb{R}^n$, at point $x_0 \in \mathbb{R}^n$, is given by the limit

$$D_{sym}\mu(x_0) = \lim_{r\to 0} \frac{\mu(B(x_0,r))}{m(B(x_0,r))},$$

provided the limit exists.

Theorem 3.1.2. *Suppose that μ is a measure on $\mathbb{R}^n$, with well-defined Gauss-Weierstrass integral $W[\mu]$ in $\mathbb{R}^n \times (0,t_0)$, for some $t_0 \in (0,\infty)$. If $x_0 \in \mathbb{R}^n$, $L \in \mathbb{C}$, then the following statements holds.*

i) If $D_{sym}\mu(x_0) = L$, then

$$\lim_{t\to 0} W[\mu](x_0) = L.$$

ii) If, additionally, we assume μ to be positive, then

$$\lim_{t\to 0} W[\mu](x_0) = L$$

implies that $D_{sym}\mu(x_0)$ is also equal to L.

The proof of the first part of the theorem above follows from the result of Saeki (Theorem 1.0.5) (see also [Wat77, Theorem 3] for an alternative proof) and the second part has already been discussed in Chapter 2 (Corollary 2.1.13). It was Gehring [Geh60, Theorem 3], who had shown, for $n = 1$, that the conclusion of the second part of the theorem above does not hold true for signed measures.

We now focus on the main topic of this chapter, namely the extension of Gehring's result (Theorem 1.0.16) in higher dimensions. The main tool used in Gehring's proof is Wiener's Tauberian theorem. But it is not at all clear to us at the moment whether the same approach can be adapted to prove analogue of Theorem 1.0.16, for $n > 1$. As we have pointed out in the introduction of this thesis, correct interpretation of the derivative of the distribution function of a measures on $\mathbb{R}$, is crucial in order to prove any higher dimensional analogue of this theorem. It turns out that the strong derivative of a measure is a right candidate for this purpose.

We recall that a measure μ on $\mathbb{R}^n$, has strong derivative $D\mu(x_0) = L \in \mathbb{C}$, at $x_0 \in \mathbb{R}^n$, if

$$\lim_{r\to 0} \frac{\mu(x_0 + rB)}{m(rB)} = L,$$

holds for every open ball $B \subset \mathbb{R}^n$.

Remark 3.1.3.

i) It is clear from the definition above that if $D\mu(x_0) = L$, then $D_{sym}\mu(x_0) = L$. However, the converse is not true and can be seen from the following elementary example. Consider the measure $d\mu = \chi_{[0,1]}\, dm$ on $\mathbb{R}$. Then

$$\frac{\mu\left((-r, r)\right)}{m\left((-r, r)\right)} = \frac{1}{2r}\int_{-r}^{r} \chi_{[0,1]}\, dm = \frac{1}{2r}\int_{0}^{r} dm = \frac{1}{2}, \quad r \in (0, 1).$$

Thus, $D_{sym}\mu(0) = 1/2$. However, the strong derivative of μ at zero does not exist. To see this, consider an interval of the form $I_1 = (x - t, x + t)$, with $x \in (0, \infty)$, and $t \in (0, x)$. Then for all positive number r smaller than $1/(x + t)$, we see that rI_1 is a subset of $[0, 1]$ and hence

$$\lim_{r\to 0} \frac{\mu(rI_1)}{m(rI_1)} = \lim_{r\to 0} \frac{m(rI_1)}{m(rI_1)} = 1.$$

On the other hand, if we choose $I_2 = (x-t, x+t)$, with $x \in (-\infty, 0)$, and $t \in (0, -x)$, then for all $r \in (0, \infty)$, rI_2 and $[0,1]$ are disjoint as $r(x+t)$ is negative. Hence

$$\lim_{r\to 0} \frac{\mu(rI_2)}{m(rI_2)} = 0.$$

It follows that the strong derivative of μ at zero does not exist.

ii) From an interesting example constructed by Shapiro in [Sha06, Section 3], it can be seen that there exists an absolutely continuous measure μ on $\mathbb{R}^n$, $n > 1$ (which can also be choosen to be positive), and a point $x_0 \in \mathbb{R}^n$, such that x_0 is not a Lebesgue point of μ but the strong derivative of μ exists at x_0. More details on these can be found in Chapter 5.

The following theorem shows that the strong derivative of a measure is a natural generalization of the derivative of the distribution function of a measure defined on the real line.

Theorem 3.1.4. *Let μ be a measure on $\mathbb{R}$, with distribution function F. Then F is differentiable at $x_0 \in \mathbb{R}$, if and only if the strong derivative of μ at x_0 exists. In either case,*

$$F'(x_0) = D\mu(x_0).$$

Proof. Suppose that F is differentiable at $x_0 \in \mathbb{R}$, with $F'(x_0) = L \in \mathbb{C}$. Then for any interval of the form $I = (x-s, x+s)$, where $x \in \mathbb{R}$, $s \in (0, \infty)$, we have

$$\begin{aligned}
\lim_{r\to 0} \frac{\mu(x_0 + rI)}{m(rI)} &= \lim_{r\to 0+} \frac{F(x_0 + rx + rs) - F(x_0 + rx - rs)}{2rs} \\
&= \lim_{r\to 0+} \left(\frac{F(x_0 + r(x+s)) - F(x_0)}{r(x+s)} \times \frac{(x+s)}{2s} \right. \\
&\qquad \left. - \frac{F(x_0 + r(x-s)) - F(x_0)}{r(x-s)} \times \frac{(x-s)}{2s} \right) \\
&= \left(L \times \frac{(x+s)}{2s} - L \times \frac{(x-s)}{2s} \right) = L.
\end{aligned}$$

Thus, $D\mu(x_0)$ equals L.

Conversely, suppose that $D\mu(x_0)$ equals L, for some $L \in \mathbb{C}$. Then for every interval of the form $I = (x - s, x + s)$, with $x \in \mathbb{R}$, $s \in (0, \infty)$, we have

$$\begin{aligned} L &= \lim_{r \to 0+} \frac{\mu(x_0 + rI)}{m(rI)} \\ &= \lim_{r \to 0+} \frac{\mu_F\left((x_0 + rx - rs, x_0 + rx + rs)\right)}{2rs} \\ &= \lim_{r \to 0+} \frac{F(x_0 + rx + rs) - F(x_0 + rx - rs)}{2rs}. \end{aligned}$$

Now, by choosing $x = s = 1/2$ (a one dimensional specialty), in the equation above, we get

$$L = \lim_{r \to 0+} \frac{F(x_0 + r) - F(x_0)}{r},$$

that is, the right-hand derivative of F at x_0 is L. Similarly, by choosing $x = -1/2$, and $s = 1/2$, we get that the left-hand derivative of F at x_0 is also L. □

Our main result (Theorem 3.3.2) can be succinctly stated as follows: for a positive measure μ on $\mathbb{R}^n$, $W[\mu]$ has parabolic limit $L \in [0, \infty)$, at $x_0 \in \mathbb{R}^n$, if and only if $D\mu(x_0) = L$. We refer to Definition 1.0.15, for the definition of parabolic limit. The proof of this result is based on the proof of Theorem 1.0.13, due to Ramey and Ullrich. However, two relatively recent results on the qualitative properties of the solution of heat equation plays an important role in the proof of the theorem. One of them is an analogue of Montel's theorem valid for solutions of the heat equation (see Lemma 3.2.7), due to Bär [Bär13]. The other one is a result of Poon [Poo96] on the unique continuation of the solutions of the heat equation. In the next section, we state and prove all the preliminary results needed to prove Theorem 3.3.2. The statement and proof of this theorem is given in the last section.

3.2 Auxilary results

Let M denote the set of all measure μ on $\mathbb{R}^n$, such that the Gauss-Weierstrass integral $W[\mu]$ is well-defined in $\mathbb{R}^{n+1}_+$. In view of the Remark 3.1.1, we have

$$M = \{\mu \text{ is a measure on } \mathbb{R}^n \mid W[|\mu|](0, t) \text{ is finite for all } t \in (0, \infty)\}. \tag{3.2.1}$$

As $w \in L^p(\mathbb{R}^n)$, for all $p \in [1,\infty]$, it follows from the definition of $W[\mu]$ (see 1.0.13) that $L^p(\mathbb{R}^n) \subset M$. For $f \in L^p(\mathbb{R}^n)$, $p \in [1,\infty]$, we denote

$$Wf(x,t) = f * w_{\sqrt{t}}(x) = \int_{\mathbb{R}^n} W(x-\xi,t) f(\xi)\, dm(\xi), \qquad x \in \mathbb{R}^n,\ t \in (0,\infty).$$

As $\{w_{\sqrt{t}} \mid t \in (0,\infty)\}$ is an approximate identity, we see that if $f \in C_c(\mathbb{R}^n)$, then $Wf(\cdot,t)$ converges to f uniformly as t goes to zero. However, a stronger result is true.

Lemma 3.2.1. *If $f \in C_c(\mathbb{R}^n)$, then*

$$\lim_{t\to 0} \frac{Wf(\cdot,t)}{w} = \frac{f}{w},$$

uniformly on $\mathbb{R}^n$.

Proof. We assume that supp$f \subset B(0,R)$, for some $R \in (0,\infty)$. Since w is bounded below on $B(0,2R)$ by a positive number,

$$\lim_{t\to 0} \frac{Wf(x,t)}{w(x)} = \frac{f(x)}{w(x)},$$

uniformly for $x \in B(0,2R)$. Hence, it suffices to prove that

$$\lim_{t\to 0} \frac{Wf(\cdot,t)}{w} = 0,$$

uniformly for $x \in B(0,2R)^c$. We have

$$\frac{Wf(x,t)}{w(x)} = \frac{1}{w(x)t^{\frac{n}{2}}} \int_{B(0,R)} f(\xi) w\left(\frac{x-\xi}{\sqrt{t}}\right) dm(\xi), \quad x \in \mathbb{R}^n. \tag{3.2.2}$$

For $x \in B(0,2R)^c$, and $\xi \in B(0,R)$, we have

$$\|\xi\| < R \le \frac{\|x\|}{2},$$

and hence it follows from the triangle inequality that

$$\|x-\xi\| \ge \|x\| - \|\xi\| > \|x\| - \frac{\|x\|}{2} = \frac{\|x\|}{2}. \tag{3.2.3}$$

Since w is radially decreasing, we get from (3.2.2) and (3.2.3) that for $x \in B(0,2R)^c$,

$$\begin{aligned}\left|\frac{Wf(x,t)}{w(x)}\right| &\leq \frac{1}{w(x)t^{\frac{n}{2}}}\int_{B(0,R)}|f(\xi)|w\left(\frac{x-\xi}{\sqrt{t}}\right)\,dm(\xi)\\ &\leq \frac{1}{w(x)t^{\frac{n}{2}}}\int_{B(0,R)}|f(\xi)|w\left(\frac{x}{2\sqrt{t}}\right)\,dm(\xi)\\ &= \frac{w\left(\frac{x}{2\sqrt{t}}\right)}{w(x)t^{\frac{n}{2}}}\|f\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{3.2.4}$$

Using the expression of w (see (1.0.12)), it follows that for $x \in B(0,2R)^c$, and $t \in (0,1/4)$,

$$\frac{w\left(\frac{x}{2\sqrt{t}}\right)}{w(x)t^{\frac{n}{2}}} = t^{-\frac{n}{2}}e^{-\frac{\|x\|^2}{4}\left(\frac{1}{4t}-1\right)} \leq t^{-\frac{n}{2}}e^{-R^2\left(\frac{1}{4t}-1\right)} \leq \frac{n!4^n}{R^{2n}}\frac{t^{\frac{n}{2}}}{(1-4t)^n}.$$

Using the inequality above in (3.2.4), and then taking limit as t goes to zero on both sides of (3.2.4), we complete the proof. □

Lemma 3.2.2. *If $\nu \in M$, and $f \in C_c(\mathbb{R}^n)$, then for each fixed $t \in (0,\infty)$,*

$$\int_{\mathbb{R}^n} Wf(x,t)\,d\nu(x) = \int_{\mathbb{R}^n} W[\nu](x,t)f(x)\,dm(x).$$

Proof. The result will follow by interchanging integrals using Fubini's theorem. Assuming supp$f \subset B(0,R)$, it suffices to show that

$$\int_{\mathbb{R}^n}\int_{B(0,R)}|f(\xi)|W(x-\xi,t)\,dm(\xi)\,d|\nu|(x) < \infty.$$

Now,

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{B(0,R)}|f(\xi)|W(x-\xi,t)\,dm(\xi)\,d|\nu|(x)\\ = \;& (4\pi t)^{-\frac{n}{2}}\int_{B(0,2R)}\int_{B(0,R)}e^{-\frac{\|x-\xi\|^2}{4t}}|f(\xi)|\,dm(\xi)\,d|\nu|(x)\\ &+(4\pi t)^{-\frac{n}{2}}\int_{B(0,2R)^c}\int_{B(0,R)}e^{-\frac{\|x-\xi\|^2}{4t}}|f(\xi)|\,dm(\xi)\,d|\nu|(x)\\ \leq\;& (4\pi t)^{-\frac{n}{2}}\int_{B(0,2R)}\int_{B(0,R)}|f(\xi)|\,dm(\xi)\,d|\nu|(x)\\ &+(4\pi t)^{-\frac{n}{2}}\int_{B(0,2R)^c}\int_{B(0,R)}e^{-\frac{\|x\|^2}{16t}}|f(\xi)|\,dm(\xi)\,d|\nu|(x)\\ &\qquad\qquad\text{(using the inequality (3.2.3), in the second integral)}\\ \leq\;& (4\pi t)^{-\frac{n}{2}}|\nu|(B(0,2R))\|f\|_{L^1(\mathbb{R}^n)} + 4^{\frac{n}{2}}\|f\|_{L^1(\mathbb{R}^n)}W[|\nu|](0,4t).\end{aligned}$$

The last quantity is finite as $\nu \in M$. This completes the proof. $\square$

The following notions will be used throughout the chapter.

Definition 3.2.3. i) A sequence of complex-valued functions $\{u_j \mid j \in \mathbb{N}\}$, defined on $\mathbb{R}^{n+1}_+$, is said to converge normally to a function u if $\{u_j\}$ converges to u uniformly on all compact subsets of $\mathbb{R}^{n+1}_+$.

ii) A sequence of complex-valued functions $\{u_j \mid j \in \mathbb{N}\}$, defined on $\mathbb{R}^{n+1}_+$, is said to be locally bounded if given any compact set $K \subset \mathbb{R}^{n+1}_+$, there exists a positive constant C_K (depending only on K) such that for all j and all $x \in K$,

$$|u_j(x)| \leq C_K.$$

v) A sequence $\{\mu_j\}$ of positive measures on $\mathbb{R}^n$, is said to converge to a positive measure μ on $\mathbb{R}^n$, in weak* if

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} f(x)\, d\mu_j(x) = \int_{\mathbb{R}^n} f(x)\, d\mu(x),$$

for all $f \in C_c(\mathbb{R}^n)$.

Lemma 3.2.4. *Suppose that $\{\mu_j \mid j \in \mathbb{N}\} \subset M$, and $\mu \in M$, are positive measures. If $\{W[\mu_j]\}$ converges normally to $W[\mu]$, then $\{\mu_j\}$ converges to μ in weak*.*

Proof. Let $f \in C_c(\mathbb{R}^n)$, with supp$f \subset B(0,R)$, for some $R \in (0,\infty)$. We need to show that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} f(x)\, d\mu_j(x) = \int_{\mathbb{R}^n} f(x)\, d\mu(x).$$

For any $t \in (0,\infty)$, we write

$$\begin{aligned} &\int_{\mathbb{R}^n} f(x)\, d\mu_j(x) - \int_{\mathbb{R}^n} f(x)\, d\mu(x) \\ = \;& \int_{\mathbb{R}^n} (f(x) - Wf(x,t))\, d\mu_j(x) + \int_{\mathbb{R}^n} Wf(x,t)\, d\mu_j(x) - \int_{\mathbb{R}^n} Wf(x,t)\, d\mu(x) \\ &\quad + \int_{\mathbb{R}^n} (Wf(x,t) - f(x))\, d\mu(x). \end{aligned} \tag{3.2.5}$$

Given $\epsilon > 0$, by Lemma 3.2.1, we get some $t_0 \in (0, \infty)$, such that for all $x \in \mathbb{R}^n$,

$$\frac{|Wf(x,t_0) - f(x)|}{w(x)} < \epsilon. \tag{3.2.6}$$

Using Lemma 3.2.2 for $t = t_0$, in the second and third integral on the right-hand side of (3.2.5), it follows that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} f(x)\, d\mu_j(x) - \int_{\mathbb{R}^n} f(x)\, d\mu(x) \right| \\
\leq\ & \int_{\mathbb{R}^n} |f(x) - Wf(x,t_0)|\, d\mu_j(x) + \int_{B(0,R)} |W[\mu_j](x,t_0) - W[\mu](x,t_0)||f(x)|\, dm(x) \\
&\quad + \int_{\mathbb{R}^n} |Wf(x,t_0) - f(x)|\, d\mu(x) \\
=\ & I_1(j) + I_2(j) + I_3.
\end{aligned} \tag{3.2.7}$$

Applying (3.2.6), we obtain

$$I_1(j) = \int_{\mathbb{R}^n} \frac{|Wf(x,t_0) - f(x)|}{w(x)} w(x)\, d\mu_j(x) \leq \epsilon \int_{\mathbb{R}^n} w(x)\, d\mu_j(x) = \epsilon W[\mu_j](0,1),$$

for all $j \in \mathbb{N}$. By the same argument, we also have

$$I_3 \leq \epsilon W[\mu](0,1).$$

Since $\{W[\mu_j]\}$ converges to $W[\mu]$ normally, the sequence $\{W[\mu_j](0,1)\}$, in particular, is bounded. Hence, by setting

$$C = \sup_{j\in\mathbb{N}} W[\mu_j](0,1) + W[\mu](0,1),$$

we get that

$$I_1(j) + I_3 \leq 2C\epsilon.$$

Again using the normal convergence of $\{W[\mu_j]\}$ to $W[\mu]$, we get some $j_0 \in \mathbb{N}$, such that for all $j \geq j_0$,

$$\|W[\mu_j] - W[\mu]\|_{L^\infty(\overline{B(0,R)} \times \{t_0\})} < \epsilon.$$

This shows that for all $j \geq j_0$,

$$I_2(j) \leq \epsilon \|f\|_{L^1(\mathbb{R}^n)}.$$

Hence, for all $j \geq j_0$

$$\left| \int_{\mathbb{R}^n} f(x)\, d\mu_j(x) - \int_{\mathbb{R}^n} f(x)\, d\mu(x) \right| \leq \epsilon (2C + \|f\|_{L^1(\mathbb{R}^n)}). \qquad \square$$

We will also need the following measure theoretic result.

Lemma 3.2.5 ([RU88, Proposition 2.6])**.** *Suppose that $\{\mu_j \mid j \in \mathbb{N}\}$, μ are positive measures on $\mathbb{R}^n$, and that $\{\mu_j\}$ converges to μ in weak*. Then for some $L \in [0, \infty)$, $\mu = Lm$ if and only if $\{\mu_j(B)\}$ converges to $Lm(B)$ for every open ball $B \subset \mathbb{R}^n$.*

We shall next prove a result regarding comparison of the Hardy-Littlewood maximal function and the heat maximal function. This result, perhaps, is well-known to the experts, but since we could not find any reference of this result in the form in which it will be needed, we include a proof of it in the following. We recall that for a positive measure μ, its Hardy-Littlewood maximal function $M_{HL}(\mu)$ is defined by

$$M_{HL}(\mu)(x_0) = \sup_{r>0} \frac{\mu(B(x_0, r))}{m(B(x_0, r))}, \qquad x_0 \in \mathbb{R}^n.$$

Lemma 3.2.6. *If $\mu \in M$ (see (3.2.1)), is a positive measure and $\alpha \in (0, \infty)$, then there exist positive constants c_α and c_n such that*

$$c_n M_{HL}(\mu)(x_0) \leq \sup_{t>0} W[\mu](x_0, t^2) \leq \sup_{(x,t) \in P(x_0, \alpha)} W[\mu](x, t) \leq c_\alpha M_{HL}(\mu)(x_0), \tag{3.2.8}$$

for all $x_0 \in \mathbb{R}^n$. The constants c_n and c_α are independent of x_0.

Proof. We recall that

$$\mathtt{P}(x_0, \alpha) = \{(x, t) \in \mathbb{R}^{n+1}_+ \mid \|x - x_0\|^2 < \alpha t\}.$$

The second inequality follows easily as the set $\{(x_0, t^2) \mid t \in (0, \infty)\}$ is contained in $\mathtt{P}(x_0, \alpha)$ for any $\alpha \in (0, \infty)$. The proof of the third inequality can be found in [Sae96, P.137, inequality (b)]. The proof in [Sae96] proves the case $x_0 = 0$. The general case then follows simply by considering the translated measure $\tau_{-x_0}\mu$. Proof of the first inequality is easy, as for any

$t \in (0, \infty)$,

$$
\begin{aligned}
W[\mu](x_0, t^2) &= (4\pi t^2)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{\|x_0-\xi\|^2}{4t^2}}\, d\mu(\xi) \\
&\geq (4\pi)^{-\frac{n}{2}} t^{-n} \int_{B(x_0,2t)} e^{-\frac{\|x_0-\xi\|^2}{4t^2}}\, d\mu(\xi) \\
&\geq (4\pi)^{-\frac{n}{2}} t^{-n} \int_{B(x_0,2t)} e^{-1}\, d\mu(y) \\
&= e^{-1} m(B(0,1)) 2^n (4\pi)^{-\frac{n}{2}} \frac{\mu(B(x_0, 2t))}{m(B(x_0, 2t))}.
\end{aligned}
$$

Taking supremum over t on both sides we get

$$
c_n M_{HL}(\mu)(x_0) \leq \sup_{t>0} W[\mu](x_0, t^2), \tag{3.2.9}
$$

where $c_n = e^{-1} m(B(0,1)) 2^n (4\pi)^{-\frac{n}{2}}$. □

As we have mentioned earlier, to prove our main result we will need an analogue of Montel's theorem for solutions of the heat equation. Using the fact that the heat operator, $\frac{\partial}{\partial t} - \Delta$, is hypoelliptic on $\mathbb{R}^{n+1}_+$, one can get a following Montel-type result for solutions of the heat equation from a very general theorem due to Bär.

Lemma 3.2.7 ([Bär13, Theorem 4])**.** *Let $\{u_j\}$ be sequence of solutions of the heat equation in $\mathbb{R}^{n+1}_+$. If $\{u_j\}$ is locally bounded, then it has a subsequence which converges normally to a solution v of the heat equation in $\mathbb{R}^{n+1}_+$.*

Given a function F on $\mathbb{R}^{n+1}_+$, and $r \in (0, \infty)$, we define the nonisotropic dilation F_r of F as follows

$$
F_r(x,t) = F(rx, r^2 t), \quad (x,t) \in \mathbb{R}^{n+1}_+. \tag{3.2.10}
$$

Remark 3.2.8. This notion of nonisotropic dilation is crucial for us primarily because of the following reasons.

i) If F is a solution of the heat equation then so is F_r, for every $r \in (0, \infty)$. This follows easily by standard differentiation rules.

ii) $(x,t) \in \mathrm{P}(0, \alpha)$, if and only if $(rx, r^2 t) \in \mathrm{P}(0, \alpha)$, for every $r \in (0, \infty)$.

Given $\nu \in M$, and $r \in (0,\infty)$, we also define the dilate ν_r of ν by

$$\nu_r(E) = r^{-n}\nu(rE), \tag{3.2.11}$$

for every Borel set $E \subseteq \mathbb{R}^n$, where $rE = \{rx \mid x \in E\}$. We now prove a simple lemma involving the above notions of dilates.

Lemma 3.2.9. *If $\nu \in M$, then for every $r \in (0,\infty)$,*

$$W[\nu_r](x,t) = W[\nu](rx, r^2t), \quad \textit{for all } (x,t) \in \mathbb{R}^{n+1}_+.$$

Proof. For a Borel set $E \subseteq \mathbb{R}^n$, it follows from the definition of ν_r (3.2.11), that

$$\int_{\mathbb{R}^n} \chi_E \, d\nu_r = r^{-n}\nu(rE) = r^{-n}\int_{\mathbb{R}^n} \chi_{rE}(x)\, d\nu(x) = r^{-n}\int_{\mathbb{R}^n} \chi_E\left(\frac{x}{r}\right) d\nu(x).$$

Hence, for all nonnegative measurable functions f on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} f(x)\, d\nu_r(x) = r^{-n}\int_{\mathbb{R}^n} f\left(\frac{x}{r}\right) d\nu(x).$$

Hence, for all $(x,t) \in \mathbb{R}^{n+1}_+$,

$$\begin{aligned} W[\nu_r](x,t) &= \int_{\mathbb{R}^n} W(x-\xi,t)\, d\nu_r(\xi) \\ &= r^{-n}\int_{\mathbb{R}^n} W\left(x - \frac{\xi}{r}, t\right) d\nu(\xi) \\ &= r^{-n}(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{\|x - r^{-1}\xi\|^2}{4t}}\, d\nu(\xi) \\ &= (4\pi t r^2)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{\|rx-\xi\|^2}{4tr^2}}\, d\nu(\xi) \\ &= W[\nu](rx, r^2t). \quad \square \end{aligned}$$

We end this section with an uniqueness theorem for solutions of the heat equation in $\mathbb{R}^{n+1}_+$. If u is a solution of the heat equation in $\mathbb{R}^{n+1}_+$, then the question of joint real analyticity of u in (x,t) variable seems to be an involved one. Nevertheless, the following uniqueness result, due to Poon, holds.

Lemma 3.2.10 ([Poo96, Theorem 1.2]). *Let u be a solution of the heat equation in $\mathbb{R}^{n+1}_+$. If u vanishes of infinite order in space-time at a point $(x_0, t_0) \in \mathbb{R}^{n+1}_+$, then u is identically zero.*

Here, vanishing of infinite order in space-time at a point $(x_0, t_0) \in \mathbb{R}^{n+1}_+$, means that there exists a positive constant C and an open neighbourhood V of (x_0, t_0) such that

$$|u(x,t)| \leq C(\|x - x_0\| + |t - t_0|)^k,$$

for all $k \in \mathbb{N}$, and for all $(x, t) \in V$. As an immediate corollary of Poon's result, we have the following.

Corollary 3.2.11. *Let u be a solution of the heat equation in $\mathbb{R}^{n+1}_+$. Then u cannot vanish on a nonempty open set in $\mathbb{R}^{n+1}_+$ without being identically zero.*

3.3 The main result

We shall first prove a special case of our main result. The proof of the main result will follow by reducing matters to this special case.

Theorem 3.3.1. *Suppose that u is a positive solution of the heat equation in $\mathbb{R}^{n+1}_+$, and that $L \in [0, \infty)$. If the boundary measure μ (see Theorem 2.1.12) of u is finite, then the following statements hold.*

i) If there exists $\eta \in (0, \infty)$, such that

$$\lim_{\substack{(x,t)\to(0,0)\\(x,t)\in P(0,\eta)}} u(x,t) = L, \tag{3.3.1}$$

then the strong derivative of μ at the origin is also equal to L.

ii) If the strong derivative of μ at the origin is equal to L, then u has parabolic limit L at the origin.

Proof. We first prove $i)$. We choose an open ball $B_0 \subset \mathbb{R}^n$, and a sequence of positive numbers $\{r_j\}$ converging to zero and then consider the quotient

$$L_j = \frac{\mu(r_jB_0)}{m(r_jB_0)}, \quad j \in \mathbb{N}. \tag{3.3.2}$$

Assuming (3.3.1), we will prove that $\{L_j\}$ is a bounded sequence and every convergent subsequence of $\{L_j\}$ converges to L. We first choose a positive number s such that B_0 is contained in $B(0,s)$. Then

$$L_j \leq \frac{\mu(r_jB(0,s))}{m(r_jB_0)} = \frac{\mu(r_jB(0,s))}{m(r_jB(0,s))} \times \frac{m(B(0,s))}{m(B_0)} \leq C_s M_{HL}(\mu)(0), \tag{3.3.3}$$

where $C_s = \frac{m(B(0,s))}{m(B_0)}$.

Since μ is the boundary measure for u, we have

$$u(x,t) = W[\mu](x,t), \quad \text{for all } (x,t) \in \mathbb{R}^{n+1}_+.$$

Thus, (3.3.1) shows that $W[\mu](0,t^2)$ converges to L as t tends to zero, which implies, in particular, that there exists a positive number δ such that

$$\sup_{t<\delta} W[\mu](0,t^2) < \infty.$$

Since μ is a finite measure we also have that for all $t \geq \delta$,

$$W[\mu](0,t^2) \leq (4\pi t^2)^{-\frac{n}{2}} \int_{\mathbb{R}^n} d\mu \leq (4\pi\delta^2)^{-\frac{n}{2}} \mu(\mathbb{R}^n).$$

Combining the above two inequalities, we obtain

$$\sup_{t>0} W[\mu](0,t^2) < \infty.$$

Lemma 3.2.6 now implies that $M_{HL}(\mu)(0)$ is finite. Boundedness of the sequence $\{L_j\}$ is now a consequence of the inequality (3.3.3). It remains to prove that every convergent subsequence of $\{L_j\}$ converges to L. We choose a convergent subsequence of $\{L_j\}$, for the sake of simplicity, again denoted by $\{L_j\}$. For $j \in \mathbb{N}$, we define

$$u_j(x,t) = u(r_j x, r_j^2 t), \quad (x,t) \in \mathbb{R}^{n+1}_+.$$

Then by Remark 3.2.8, i), $\{u_j\}$ is a sequence of positive solutions of the heat equation in $\mathbb{R}^{n+1}_+$. We claim that $\{u_j\}$ is locally bounded. To prove this claim, we choose any compact set $K \subset \mathbb{R}^{n+1}_+$. Then there exists a positive number α such that K is contained in the parabolic region $\mathtt{P}(0,\alpha)$. Indeed, we consider the map

$$(x,t) \mapsto \frac{\|x\|^2}{t}, \quad (x,t) \in \mathbb{R}^{n+1}_+.$$

Clearly, this map is continuous. As K is compact, image of K under this map is bounded and hence

$$\frac{\|x\|^2}{t} < \alpha t, \quad \text{for all } (x,t) \in K,$$

for some positive number α. Using the invariance of $\mathtt{P}(0,\alpha)$ under nonisotropic dilation (see Remark 3.2.8, ii)) and Lemma 3.2.6, it follows that for all $j \in \mathbb{N}$

$$\sup_{(x,t)\in \mathtt{P}(0,\alpha)} u_j(x,t) = \sup_{(x,t)\in \mathtt{P}(0,\alpha)} u(x,t) = \sup_{(x,t)\in \mathtt{P}(0,\alpha)} W[\mu](x,t) \leq c_\alpha M_{HL}(\mu)(0).$$

Hence, $\{u_j\}$ is locally bounded. Applying Lemma 3.2.7, we extract a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ which converges normally to a positive solution v of the heat equation in $\mathbb{R}^{n+1}_+$. We now show that v is identically equals to L in $\mathtt{P}(0,\eta)$. To show this, we take $(x_0,t_0) \in \mathtt{P}(0,\eta)$. Since $\{r_{j_k}\}$ converges to zero as k goes to infinity and $u(x,t)$ has limit L, as (x,t) tends to $(0,0)$ within $\mathtt{P}(0,\eta)$, we must have

$$v(x_0,t_0) = \lim_{k\to\infty} u_{j_k}(x_0,t_0) = \lim_{k\to\infty} u(r_{j_k}x_0, r_{j_k}^2 t_0) = L,$$

as $(r_{j_k}x_0, r_{j_k}^2 t_0) \in \mathtt{P}(0,\eta)$, for all $k \in \mathbb{N}$. Therefore, v is identically equals to L in the open set $\mathtt{P}(0,\eta)$. It is now immediate from the Corollary 3.2.11 that v is identically equal to L in $\mathbb{R}^{n+1}_+$, that is,

$$v \equiv L = W[Lm]. \tag{3.3.4}$$

On the other hand, by Lemma 3.2.9 we have for all $(x,t) \in \mathbb{R}^{n+1}_+$,

$$u_{j_k}(x,t) = u(r_{j_k}x, r_{j_k}^2 t) = W[\mu](r_{j_k}x, r_{j_k}^2 t) = W[\mu_{r_{j_k}}](x,t), \tag{3.3.5}$$

where $\mu_{r_{j_k}}$ is the dilate of μ by r_{j_k} according to (3.2.11). It follows from (3.3.4) and (3.3.5) that $\{W[\mu_{r_{j_k}}]\}$ converges normally to $W[Lm]$. Therefore, by Lemma 3.2.4, the sequence of measures $\{\mu_{r_{j_k}}\}$ converges to Lm in weak* and hence by Lemma 3.2.5, $\{\mu_{r_{j_k}}(B)\}$ converges to $Lm(B)$ for every open ball $B \subset \mathbb{R}^n$. In particular,

$$Lm(B_0) = \lim_{k\to\infty} \mu_{r_{j_k}}(B_0) = \lim_{k\to\infty} {r_{j_k}}^{-n}\mu(r_{j_k}B_0) = m(B_0) \lim_{k\to\infty} \frac{\mu(r_{j_k}B_0)}{m(r_{j_k}B_0)}.$$

The equality above, together with (3.3.2), implies that the sequence $\{L_{j_k}\}$ converges to L and hence, so does $\{L_j\}$. Thus, every convergent subsequence of the bounded sequence $\{L_j\}$ converges to L. This implies that $\{L_j\}$ itself converges to L. Since B_0 and $\{r_j\}$ are arbitrary, μ has strong derivative L at zero. This completes the proof of $i)$.

Now, we prove $ii)$. We assume that the strong derivative of μ at zero is equal to L. We fix a positive number α and a sequence $\{(x_j, t_j^2) \mid j \in \mathbb{N}\} \subset \mathrm{P}(0,\alpha)$, with (x_j, t_j^2) converging to $(0,0)$. Since $D\mu(0) = L$, it follows, in particular, that

$$\lim_{r\to 0} \frac{\mu(B(0,r))}{m(B(0,r))} = L.$$

Therefore, there exists some positive number δ such that

$$\sup_{0<r<\delta} \frac{\mu(B(0,r))}{m(B(0,r))} < L+1.$$

Finiteness of the measure μ implies that for all $r \geq \delta$,

$$\frac{\mu(B(0,r))}{m(B(0,r))} \leq \frac{\mu(\mathbb{R}^n)}{m(B(0,1))\delta^n}.$$

The above two inequalities together with Lemma 3.2.6 implies that

$$\sup_{(x,t)\in \mathrm{P}(0,\alpha)} u(x,t) = \sup_{(x,t)\in \mathrm{P}(0,\alpha)} W[\mu](x,t) \leq c_\alpha M_{HL}(\mu)(0) < \infty.$$

This shows that $\{u(x_j, t_j^2)\}$ is a bounded sequence. We consider a convergent subsequence of this sequence, denote it also, for the sake of simplicity, by $\{u(x_j, t_j^2)\}$, such that

$$\lim_{j\to\infty} u(x_j, t_j^2) = L'. \tag{3.3.6}$$

It suffices to prove that L' is equal to L. Using the sequence $\{t_j\}$, we define

$$u_j(x,t) = u(t_j x, t_j^2 t), \quad (x,t) \in \mathbb{R}_+^{n+1}.$$

As we have shown in the first part, we can prove that $\{u_j\}$ is a locally bounded sequence of positive solutions of the heat equation in $\mathbb{R}_+^{n+1}$. Hence, by Lemma 3.2.7, there exists a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ which converges normally to a positive solution v of the heat equation in $\mathbb{R}_+^{n+1}$. Lemma 2.1.12, therefore, shows that there exists a unique positive measure ν on $\mathbb{R}^n$, such that $v = W[\nu]$. We now consider, for each k, the dilate μ_{j_k} of μ by t_{j_k} according to (3.2.11). An application of Lemma 3.2.9 then implies that for each k,

$$u_{j_k}(x,t) = u\left(t_{j_k}x, t_{j_k}^2 t\right) = W[\mu]\left(t_{j_k}x, t_{j_k}^2 t\right) = W[\mu_{j_k}](x,t),$$

for all $(x,t) \in \mathbb{R}_+^{n+1}$. It follows that the sequence of functions $\{W[\mu_{j_k}]\}$ converges normally to $W[\nu]$. By Lemma 3.2.4, we thus obtain weak* convergence of $\{\mu_{j_k}\}$ to ν. Since $D\mu(0) = L$, we have for any open ball $B \subset \mathbb{R}^n$,

$$\lim_{k\to\infty} \mu_{j_k}(B) = \lim_{k\to\infty} t_{j_k}{}^{-n}\mu(t_{j_k}B) = \lim_{k\to\infty} \frac{\mu(t_{j_k}B)}{m(t_{j_k}B)} m(B) = Lm(B).$$

Hence, by Lemma 3.2.5, $\nu = Lm$. As $v = W[\nu]$, it follows that

$$v(x,t) = L, \quad \text{for all } (x,t) \in \mathbb{R}_+^{n+1}.$$

This, in turn, implies that $\{u_{j_k}\}$ converges to the constant function L normally in $\mathbb{R}_+^{n+1}$. On the other hand, we note that

$$u(x_{j_k}, t_{j_k}^2) = u\left(t_{j_k}\frac{x_{j_k}}{t_{j_k}}, t_{j_k}^2\right) = u_{j_k}\left(\frac{x_{j_k}}{t_{j_k}}, 1\right).$$

Since $(x_{j_k}, t_{j_k}^2)$ belongs to the parabolic region $\mathrm{P}(0,\alpha)$, for all $k \in \mathbb{N}$, it follows that

$$\left(\frac{x_{j_k}}{t_{j_k}}, 1\right) \in \overline{B(0,\sqrt{\alpha})} \times \{1\},$$

which is a compact subset of $\mathbb{R}_+^{n+1}$. Therefore,

$$\lim_{k\to\infty} u(x_{j_k}, t_{j_k}^2) = L.$$

In view of (3.3.6), we conclude that L' is equal to L. This completes the proof. $\square$

Now, we are in a position to state and prove our main result.

Theorem 3.3.2. *Suppose that u is a positive solution of the heat equation in $\mathbb{R}^n \times (0,t_0)$, for some $t_0 \in (0,\infty]$, and that $x_0 \in \mathbb{R}^n$, $L \in [0,\infty)$. If μ is the boundary measure of u then the following statements hold.*

i) If there exists $\eta \in (0,\infty)$, such that

$$\lim_{\substack{(x,t)\to(x_0,0)\\(x,t)\in P(x_0,\eta)}} u(x,t) = L,$$

then the strong derivative of μ at x_0 is also equal to L.

ii) If the strong derivative of μ at x_0 is equal to L, then u has parabolic limit L at x_0.

Proof. As in the proof of Theorem 2.1.3, We consider the translated measure $\mu_0 = \tau_{-x_0}\mu$ (see (2.1.10)). Using translation invariance of the Lebesgue measure, it follows from the definition of strong derivative (Definition 1.0.11) that $D\mu_0(0)$ and $D\mu(x_0)$ are equal. Since $W[\mu_0]$ is given by the convolution of μ_0 with $w_{\sqrt{t}}$, as before, we have for all $(x,t) \in \mathbb{R}^n \times (0,t_0)$ (see (2.1.11)),

$$W[\mu_0](x,t) = (\tau_{-x_0}\mu * w_{\sqrt{t}})(x) = \tau_{-x_0}(\mu * w_{\sqrt{t}})(x) = W[\mu](x+x_0,t). \tag{3.3.7}$$

We fix an arbitrary positive number α. As $(x,t) \in \mathrm{P}(0,\alpha)$, if and only if $(x_0+x,t) \in \mathrm{P}(x_0,\alpha)$, one infers from (3.3.7) that

$$\lim_{\substack{(x,t)\to(0,0)\\(x,t)\in\mathrm{P}(0,\alpha)}} W[\mu_0](x,t) = \lim_{\substack{(\xi,t)\to(x_0,0)\\(\xi,t)\in\mathrm{P}(x_0,\alpha)}} W[\mu](\xi,t).$$

Hence, it suffices to prove the theorem under the assumption that x_0 is the origin. We now show that we can even take μ to be a finite measure. Let $\tilde{\mu}$ be the restriction of μ on the closed ball $\overline{B(0,1)}$. If $B(\xi,s)$ is any given open ball in $\mathbb{R}^n$, then for all positive number r smaller than $(s+\|\xi\|)^{-1}$, it follows that $rB(\xi,s)$ is a subset of $B(0,1)$. Indeed, if $x \in rB(\xi,s) = B(r\xi,rs)$, then

$$\|x\| \leq \|x-r\xi\| + \|r\xi\| < rs + r\|\xi\| < 1, \quad \text{for all } r \in (0,(s+\|\xi\|)^{-1}).$$

This in turn implies that the quantities $D\mu(0)$ and $D\tilde{\mu}(0)$ are equal. We now claim that

$$\lim_{\substack{(x,t)\to(0,0)\\(x,t)\in\mathrm{P}(0,\alpha)}} W[\mu](x,t) = \lim_{\substack{(x,t)\to(0,0)\\(x,t)\in\mathrm{P}(0,\alpha)}} W[\tilde{\mu}](x,t). \tag{3.3.8}$$

In this regard, we first observe that

$$\lim_{t\to 0}\int_{B(0,1)^c} W(x-\xi,t)\, d\mu(\xi) = 0,$$

uniformly for $x \in B(0,1/2)$.

Indeed, for $\xi \in B(0,1)^c$, and $x \in B(0,1/2)$, we have (see (3.2.3))

$$\|x-\xi\| > \frac{\|\xi\|}{2} \geq \frac{1}{2}.$$

We fix $t_1 \in (0,t_0)$. Using the inequality above and the expression of $W(x,t)$ (see 1.0.12), it follows that for all $x \in B(0,1/2)$, $t \in (0,t_1/8)$,

$$\begin{aligned}
\int_{B(0,1)^c} W(x-\xi,t)\, d\mu(\xi) &= (4\pi t)^{-\frac{n}{2}} \int_{B(0,1)^c} e^{-\frac{\|x-\xi\|^2}{8t}} e^{-\frac{\|x-\xi\|^2}{8t}}\, d\mu(\xi) \\
&\leq (4\pi t)^{-\frac{n}{2}} e^{-\frac{1}{32t}} \int_{B(0,1)^c} e^{-\frac{\|\xi\|^2}{32t}}\, d\mu(\xi) \\
&\leq (4\pi t)^{-\frac{n}{2}} e^{-\frac{1}{32t}} \int_{B(0,1)^c} e^{-\frac{\|\xi\|^2}{4t_1}}\, d\mu(\xi)
\end{aligned}$$

$$= \; t_1^{\frac{n}{2}} W[\mu](0,t_1) t^{-\frac{n}{2}} e^{-\frac{1}{32t}}.$$

Since $W[\mu](0,t_1)$ is finite, the right-hand side of the last inequality goes to zero as t tends to zero. Thus, the observation follows. Now, for $(x,t) \in \mathbb{R}^n \times (0,t_0)$,

$$W[\mu](x,t) = W[\tilde{\mu}](x,t) + \int_{B(0,1)^c} W(x-\xi,t)\, d\mu(\xi).$$

Given any positive number ϵ, we get some $\delta \in (0,t_0)$, such that for all $t \in (0,\delta)$, the integral on the right-hand side of the equation above is smaller than ϵ, for all $x \in B(0,1/2)$. On the other hand, if we choose $t \in (0,1/4\alpha)$, then it is immediate that

$$\mathtt{P}(0,\alpha) \cap \{(x,t) \in \mathbb{R}^{n+1}_+ \mid t \in (0,1/4\alpha)\} \subset B(0,1/2) \times (0,1/4\alpha).$$

Hence, for all $(x,t) \in \mathtt{P}(0,\alpha)$, with $t \in (0, \min\{\delta, 1/4\alpha\})$, we have

$$0 \leq W[\mu](x,t) - W[\tilde{\mu}](x,t) < \epsilon.$$

This proves (3.3.8). Therefore, as $\alpha \in (0,\infty)$ is arbitrary, we may and do suppose that μ is a finite measure. Using this, without loss of generality, we may also assume $t_0 = \infty$. The proof now follows from Theorem 3.3.1. □

As an immediate consequence of the theorem above we have the following.

Corollary 3.3.3. *Suppose u is a positive solution of the heat equation in $\mathbb{R}^n \times (0,t_0)$, for some $t_0 \in (0,\infty]$. If there exists $x_0 \in \mathbb{R}^n$, and $L \in [0,\infty)$, such that for some $\eta \in (0,\infty)$*

$$\lim_{\substack{(x,t)\to(x_0,0)\\(x,t)\in P(x_0,\eta)}} u(x,t) = L,$$

then for every $\alpha \in (0,\infty)$

$$\lim_{\substack{(x,t)\to(x_0,0)\\(x,t)\in P(x_0,\alpha)}} u(x,t) = L.$$

Remark 3.3.4. After preparing the final draft of the thesis, we came of the result [BC90, Theorem 5] from which Theorem 3.3.2 follows as a corollary. However, our method of proof is completely different from that of [BC90].

Chapter 4

Boundary behavior of positive solutions of the heat equation on a stratified Lie group

4.1 Introduction

In this chapter, we are concerned with the parabolic convergence of positive solutions of the heat equation on a stratified Lie group at a given boundary point. We prove that a necessary and sufficient condition for the existence of the parabolic limit of a positive solution u at a point on the boundary is the existence of the strong derivative of the boundary measure of u at that point. Moreover, the parabolic limit and the strong derivative are equal. Thus, our main result (Theorem 4.4.2) of this chapter generalizes Theorem 3.3.2. We refer the reader to Definition 4.2.19 for the relevant definitions. One of the main difficulties in this setting is that we do not have any explicit expression of the fundamental solution of the heat equation or the heat kernel. However, we do have Gaussian estimates of the heat kernels (see Theorem 4.2.11) and using these estimates we have been able to prove our results. This makes the proof of our main theorem (Theorem 4.4.2) and auxiliary results much more involved than that of their Euclidean counterparts. This chapter is organised as follows: In section 2, we will collect some basic information about stratified Lie groups and the heat equation on these groups. The proofs of the result about heat maximal functions, and other relevant lemmas are given in section 3. The statement and proof of the main theorem (Theorem 4.4.2) is given in the last section.

From now onwards, we reserve the letters c, C, C' for positive constants whose values are unimportant and can change at each occurrence, unless otherwise stated. We also use notation like C_κ to indicate the dependency on the parameter κ.

4.2 Preliminaries on stratified Lie groups

Stratified Lie groups (also known as Carnot groups) is a class of connected, simply connected, nilpotent Lie groups [CG90]. In this section, we discuss them in somewhat detail. We refer the reader to two excellent monographs [BLU07, FS82], for the extensive treatment of stratified Lie groups and analysis on these groups. Most of the material in this section is gathered from these two monographs.

A stratified Lie group $(G, \circ)$ is a connected, simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits a vector space decomposition

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_l,$$

such that

$$[V_1, V_j] = V_{j+1}, \;\; 1 \leq j < l, \qquad [V_1, V_l] = 0.$$

Here,

$$[V_1, V_j] = \text{span}_{\mathbb{R}} \, \{[X, Y] \mid X \in V_1, Y \in V_j\}.$$

Therefore, V_1 generates, $\mathfrak{g}$ as a Lie algebra. We say that G is of step l and has dim V_1 many generators. The Lie algebra $\mathfrak{g}$ is equipped with a canonical family of dilations $\{\delta_r \mid r \in (0, \infty)\}$, which are Lie algebra automorphisms defined by [FS82, P.5]

$$\delta_r \left(\sum_{j=1}^{l} X_j \right) = \sum_{j=1}^{l} r^j X_j, \quad X_j \in V_j.$$

Since $\mathfrak{g}$ is nilpotent, the exponential map, $\exp : \mathfrak{g} \to G$, is a diffeomorphism, and hence the family of dilations $\{\delta_r \mid r \in (0, \infty)\}$, defines via the exponential map a one-parameter group of automorphisms of G, which we still denote by $\{\delta_r \mid r \in (0,\infty)\}$. We fix once and for all a bi-invariant measure m on G which is the push forward of the Lebesgue measure on $\mathfrak{g}$ via the exponential map. The bi-invariant measure m on G is, in fact, the Lebesgue measure of the underlying Euclidean space. We denote by

$$Q = \sum_{j=1}^{l} j(\text{dim } V_j),$$

called the homogeneous dimension of G and by $\underline{0}$ the identity element of G. The importance of homogeneous dimension stems from the following relation

$$m\left(\delta_r(E)\right) = r^Q m(E), \tag{4.2.1}$$

which holds for all measurable sets $E \subseteq G$, and $r \in (0,\infty)$. Here,

$$\delta_r(E) = \{\delta_r(x) \mid x \in E\}.$$

To define the analogue of parabolic domain (see Definition 1.0.15) in $G \times (0,\infty)$, we need some notion of distance on G which should interact in a specified manner with the dilations. The notion of homogeneous norm on stratified Lie groups meets this requirement.

Definition 4.2.1. A homogeneous norm with respect to the family of dilations $\{\delta_r \mid r \in (0,\infty)\}$, on G is a continuous function $d : G \to [0,\infty)$, satisfying the following:

i) d is smooth on $G \setminus \{\underline{0}\}$;

ii) $d(\delta_r(x)) = rd(x)$, for all $r \in (0,\infty)$, $x \in G$;

iii) $d(x^{-1}) = d(x)$, for all $x \in G$;

iv) $d(x) = 0$, if and only if $x = \underline{0}$.

It is known [FS82, P.8-10] that homogeneous norms always exist on stratified Lie groups and for any homogeneous norm d on G, there exists a constant $C_d \in [1,\infty)$, depending only on d, such that the following quasi-triangle inequality holds.

$$d(x_1 \circ x_2) \leq C_d[d(x_1) + d(x_2)], \quad x_1 \in G,\ x_2 \in G. \tag{4.2.2}$$

Moreover, any two homogeneous norms d_1 and d_2 on G are equivalent in the following sense (see [BLU07, P.230]): there exists a positive constant C such that

$$C^{-1}d_1(x) \le d_2(x) \le Cd_1(x), \quad \text{for all } x \in G.$$

A homogeneous norm d on G defines a left invariant quasi-metric on G, denoted by $\mathbf{d}$, as follows:

$$\mathbf{d}(x_1, x_2) = d(x_1^{-1} \circ x_2), \quad x_1 \in G,\ x_2 \in G.$$

In fact, one can easily verify the following from the definition of the homogeneous norm d and from (4.2.2).

i) $\mathbf{d}(x_1, x_2) = \mathbf{d}(x_2, x_1)$, for all $x_1 \in G,\ x_2 \in G.$

ii) $\mathbf{d}(x \circ x_1, x \circ x_2) = \mathbf{d}(x_1, x_2)$, for all $x_1 \in G,\ x_2 \in G,\ x \in G.$

iii) For all $x_1 \in G$, $x_2 \in G$, $x \in G$,

$$\mathbf{d}(x_1, x_2) \le C_d\left[\mathbf{d}(x_1, x) + \mathbf{d}(x, x_2)\right]. \tag{4.2.3}$$

Remark 4.2.2 ([LD17, Proposition 3.5])**.** Every homogeneous norm on G induces the Euclidean topology on G.

Remark 4.2.3 ([BLU07, Proposition 5.15.1])**.** Let d be a homogeneous norm on G. Then it is known that for every compact set $K \subset G$, there exists a positive constant c_K (depending only on K) such that

$$(c_K)^{-1}\|x - y\| \le d(y^{-1} \circ x) \le c_K\|x - y\|^{\frac{1}{l}}, \quad \text{for all } x,\ y \in K, \tag{4.2.4}$$

where l is the step of G and $\|\cdot\|$ is the norm of the underlying Euclidean space $\mathfrak{g}$.

For a homogeneous norm d on G, the d-ball centered at $x \in G$, with radius $s \in (0, \infty)$, is defined as

$$B_d(x, s) = \{x_1 \in G \mid \mathbf{d}(x, x_1) < s\} = \{x_1 \in G \mid d(x^{-1} \circ x_1) < s\}. \tag{4.2.5}$$

It follows that $B_d(x, s)$ is the left translate by x of the ball $B_d(\underline{0}, s)$ which in turn, is the image under δ_s of the ball $B_d(\underline{0}, 1)$. This shows, using (4.2.1), that

$$m\left(B_d(x,s)\right) = m\left(B_d(\underline{0},s)\right) = m\left(B_d(\underline{0},1)\right) s^Q,$$

for all $x \in G$, $s \in (0,\infty)$. We also observe that if B is a d-ball centered at $x \in G$, with radius $s \in (0,\infty)$, then

$$\delta_r(B) = B_d(\delta_r(x), rs), \quad \text{for all} \quad r \in (0,\infty).$$

Remark 4.2.4 ([FS82, Lemma 1.4])**.** If B is the d-ball $B_d(x,s)$, then its closure

$$\overline{B} = \{x_1 \in G : \mathbf{d}(x,x_1) \leq s\} = \{x_1 \in G : d(x_1^{-1} \circ x) \leq s\}$$

is compact with respect to the Euclidean topology of G.

We recall the following formula for integration (an analogue of polar coordinate) which can be used in order to determine the integrability of functions on G ([FS82, Proposition 1.15]): for all $f \in L^1(G)$,

$$\int_G f(x)\, dm(x) = \int_0^\infty \int_\Omega f(\delta_r(\omega)) r^{Q-1}\, d\sigma(\omega)\, dr, \tag{4.2.6}$$

where

$$\Omega = \{\omega \in G \mid d(\omega) = 1\},$$

and σ is a unique positive Radon measure on Ω. For a function ψ defined on G, we define for $r \in (0,\infty)$,

$$\psi_r(x) = r^{-Q}\psi\left(\delta_{r^{-1}}(x)\right), \quad x \in G. \tag{4.2.7}$$

For a measurable function h on G and a measure μ on G, their convolution $\mu * h(x)$, at the point $x \in G$, is defined by

$$\mu * h(x) = \int_G h(\xi^{-1} \circ x)\, d\mu(\xi),$$

provided the integral converges absolutely. When $d\mu = f\, dm$, we simply denote the convolution above by $f * h(x)$. We refer to [FS82, P.15-18] for basic properties of convolution on the group G.

Remark 4.2.5.

i) It follows from (4.2.1) that if $\psi \in L^1(G)$, then for all $r \in (0, \infty)$

$$\int_G \psi_r(x)\, dm(x) = \int_G \psi(x)\, dm(x).$$

ii) Suppose that $\psi \in L^1(G)$, with

$$\int_G \psi(x)\, dm(x) = 1.$$

Then $\{\psi_r \mid r \in (0, \infty)\}$ is an approximate identity on G [FS82, Proposition 1.20]. In particular, for $f \in C_c(G)$, it follows that $f * \psi_r \to f$, as $r \to 0$, uniformly on G.

We identify $\mathfrak{g}$ as the Lie algebra of all left G-invariant vector fields on G and fix once and for all a basis $\{X_1, X_2, \cdots, X_{N_1}\}$ for V_1, with N_1 being dim V_1, which generates $\mathfrak{g}$ as a Lie algebra. The second order differential operator

$$\mathcal{L} = \sum_{j=1}^{N_1} X_j^2,$$

is called a sub-Laplacian on G.

Remark 4.2.6 ([BLU02, Theorem 2.2])**.** There exists a homogeneous norm $d_{\mathcal{L}}$ on G such that $d_{\mathcal{L}}(\cdot)^{2-Q}$ is the fundamental solution of $\mathcal{L}$.

Definition 4.2.7 ([Fol75, P.164])**.** A differential operator D acting on $C_c^\infty(G)$ is said to be homogeneous of degree λ, where $\lambda \in \mathbb{C}$, if for all $f \in C_c^\infty(G)$, and $r \in (0, \infty)$

$$D(f \circ \delta_r) = r^\lambda (Df) \circ \delta_r.$$

Remark 4.2.8. It is known that $X \in \mathfrak{g}$ is homogeneous of degree j if and only if $X \in V_j$, $1 \leq j \leq l$ (see [Fol75, P.172]). Hence, $\mathcal{L}$ is a left invariant second order differential operator on G which is homogeneous of degree two.

The heat operator $\mathcal{H}$ associated to the sub-Laplacian $\mathcal{L}$ is the differential operator

$$\mathcal{H} = \mathcal{L} - \frac{\partial}{\partial t} \tag{4.2.8}$$

on $G \times (0, \infty)$.

Remark 4.2.9. Since $X_1, X_2, \cdots, X_{N_1}$ generates $\mathfrak{g}$ as a Lie algebra, by a celebrated theorem of Hörmander [Hör67, Theorem 1.1], $\mathcal{L}$ and $\mathcal{H}$ are hypoelliptic on G and $G \times (0, \infty)$ respectively (see [BLU02]).

Hypoellipticity of $\mathcal{H}$ plays an important role in the results we are going to prove. In the following, we give some examples of stratified Lie groups. We refer the reader to [BLU07] for more examples of stratified Lie groups.

Example 4.2.10.

i) A trivial example of a stratified Lie group is the Euclidean space $\mathbb{R}^n$. The dilation δ_r is the usual isotropic dilation, that is,

$$\delta_r(x_1, \cdots, x_n) = (rx_1, \cdots, rx_n).$$

The homogeneous norm and the homogeneous dimensions are the usual Euclidean norm and usual Euclidean dimension respectively. The sub-Laplacian is the usual Laplace operator $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

ii) The simplest nontrivial example of a stratified Lie group is the Heisenberg group H^n. As a set, H^n is $\mathbb{C}^n \times \mathbb{R}$. Denoting the points of H^n by (z, s), where $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, $s \in \mathbb{R}$, we have the group law given as

$$(z, s) \circ (z', s') = \left(z + z', s + s' + \frac{1}{2} \sum_{j=1}^{n} \mathrm{Im}(z_j \overline{z_j'}) \right)$$

With the notation $z_j = x_j + y_j$, the horizontal space $V_1 = \mathbb{R}^{2n} \times \{0\}$ is spanned by the basis

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial s}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial s}$$

The one dimensional center $V_2 = \{0\} \times \mathbb{R}$ is generated by the vector field

$$S = \frac{\partial}{\partial s}.$$

The nonzero Lie brackets of the basis elements are given by

$$[X_j, Y_j] = S, \quad 1 \leq j \leq n.$$

The sub-Laplacian

$$\mathcal{L} = \sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

is known as the Kohn Laplacian in the literature. The corresponding homogeneous norm $d_{\mathcal{L}}$ (known as Korányi norm) on H^n is given by (see [BLU07, Theorem 18.3.1])

$$d_{\mathcal{L}}(z, s) = \left(|z|^4 + 16s^2\right)^{\frac{1}{4}}, \quad (z, s) \in H^n.$$

More generally, groups of Heisenberg type (also known as H-type groups) forms an important class of stratified Lie groups [BLU07, Remark 18.1.7]. We will discuss them in more detail in the last chapter.

ii) On $\mathbb{R}^4$, we consider the following group operation:

$$\begin{aligned} & (s, x_1, x_2, x_3)(t, y_1, y_2, y_3) \\ = \quad & (s + t, x_1 + y_1, x_2 + y_2 + tx_1, x_3 + y_3 + tx_2 + \frac{t^2}{2} x_1), \end{aligned}$$

which makes it a Lie group. The Lie algebra has a basis $\{Y, X_1, X_2, X_3\}$ with the following nonzero bracket relation

$$[Y, X_1] = X_2, \quad [Y, X_2] = X_3.$$

It follows that $\mathbb{R}^4$ with this bracket operation is a stratified Lie group of step three with

$$V_1 = \text{span}\{Y, X_1\}, \ V_2 = \text{span}\{X_2\}, \ V_3 = \text{span}\{X_3\}.$$

We also observe that V_3 is the center of the Lie algebra. We denote this Lie algebra by $\mathfrak{b}$. The basis elements of $\mathfrak{b}$, given above can also be viewed as the following left invariant vector fields.

$$Y = \frac{\partial}{\partial s} + x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3}, \quad X_i = \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq 3.$$

In this case a sub-Laplacian is given by

$$\mathcal{L} = Y^2 + X_1^2.$$

This is a particular case of a Kolmogorov-type group. We refer the reader to [BLU07, Section 4.1.4] for more general Kolmogorov-type groups and corresponding sub-Laplacians.

As stated before, in this chapter, we are interested in boundary behavior of positive solutions of the heat equation on stratified groups:

$$\mathcal{H}u(x,t) = 0, \qquad (x,t) \in G \times (0,\infty). \tag{4.2.9}$$

We list down some properties of the fundamental solution of the heat equation (4.2.9) on G.

Theorem 4.2.11. *The fundamental solution of $\mathcal{H}$ is given by*

$$\Gamma(x,t;\xi) := \Gamma(\xi^{-1} \circ x, t), \quad (x,t) \in G \times (0,\infty), \;\; \xi \in G,$$

where Γ is a smooth, strictly positive function on $G \times (0,\infty)$ satisfying the following properties:

(i) $\Gamma(x, t+s) = \int_G \Gamma(\xi^{-1} \circ x, t)\Gamma(\xi, s)\, dm(\xi), \quad (x,t) \in G \times (0,\infty), \;\; s \in (0,\infty).$

(ii) $\Gamma(x,t) = \Gamma(x^{-1}, t), \quad (x,t) \in G \times (0,\infty).$

(iii) $\Gamma(\delta_r(x), r^2 t) = r^{-Q}\Gamma(x,t), \;\; (x,t) \in G \times (0,\infty), \;\; r \in (0,\infty).$

(iv) $\int_G \Gamma(x,t)\, dm(x) = 1, \quad t \in (0,\infty).$

(v) There exists a positive constant c_0, depending only on $\mathcal{L}$, such that the following Gaussian estimates hold.

$$c_0^{-1} t^{-\frac{Q}{2}} \exp\left(-\frac{c_0 d_{\mathcal{L}}(x)^2}{t}\right) \leq \Gamma(x,t) \leq c_0 t^{-\frac{Q}{2}} \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{c_0 t}\right), \tag{4.2.10}$$

for every $(x,t) \in G \times (0,\infty)$.

(vi) *Given any nonnegative integers* p, q, *there exists a positive constant* $c_{p,q}$ *such that for every* $i_1, \cdots, i_p \in \{1, \cdots, N_1\}$, *we have*

$$|X_{i_1} \cdots X_{i_p} (\partial_t)^q \Gamma(x,t)| \leq c_{p,q} t^{-\frac{Q+p+2q}{2}} \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{c_0 t}\right), \tag{4.2.11}$$

for every $(x,t) \in G \times (0,\infty)$, *and the basis elements* $X_{i_1}, \cdots, X_{i_p}$ *of* V_1.

The proof of (i)-(iv) can be found in [FS82, Proposition 1.68, Corollary 8.2] and the proofs of (v), (vi) are available in [BLU02, Theorem 5.1, Theorem 5.2, Theorem 5.3]. Property (v) plays an important role in our study and we will frequently use it throughout this chapter.

For a measure μ on G, we define

$$\Gamma[\mu](x,t) = \int_G \Gamma(\xi^{-1} \circ x, t)\, d\mu(\xi), \tag{4.2.12}$$

whenever the integral above converges absolutely for $(x,t) \in G \times (0,\infty)$. If the integral above converges absolutely for all $(x,t) \in E$, where $E \subseteq G \times (0,\infty)$, we say that $\Gamma[\mu]$ is well-defined in E. We define

$$\gamma(x) = \Gamma(x,1), \quad x \in G. \tag{4.2.13}$$

Then by Therorem 4.2.11, (iii), (iv), we have

$$\Gamma(x,t) = t^{-\frac{Q}{2}} \gamma\left(\delta_{\frac{1}{\sqrt{t}}}(x)\right), \quad (x,t) \in G \times (0,\infty);$$

$$\int_G \gamma(x)\, dm(x) = 1.$$

Thus, we can rewrite (4.2.12) as follows:

$$\Gamma[\mu](x,t) = \mu * \gamma_{\sqrt{t}}(x), \tag{4.2.14}$$

provided the convolution above exists.

Remark 4.2.12. For the rest of this chapter, unless mentioned explicitly, we will always write $B(x,s)$ instead of $B_{d_{\mathcal{L}}}(x,s)$ to denote a $d_{\mathcal{L}}$-ball centered at $x \in G$, with radius $s \in (0,\infty)$.

We recall that there exists a constant $\tau \in [1, \infty)$, depending only on $\mathcal{L}$, such that

$$d_{\mathcal{L}}(x \circ \xi) \leq \tau \left[d_{\mathcal{L}}(x) + d_{\mathcal{L}}(\xi)\right], \quad x \in G, \ \xi \in G.$$

Using this we get the following useful inequality, which we will refer to as the reverse triangle inequality.

$$\mathbf{d}_{\mathcal{L}}(x, \xi) \geq \frac{1}{\tau}\mathbf{d}_{\mathcal{L}}(z, \xi) - \mathbf{d}_{\mathcal{L}}(z, x), \quad x \in G, \ \xi \in G, \ z \in G. \tag{4.2.15}$$

Suppose that d is a homogeneous norm on G. A function $\phi : G \to \mathbb{C}$, will be called d-radial if

$$\phi(x_1) = \phi(x_2), \quad \text{whenever} \ \ d(x_1) = d(x_2). \tag{4.2.16}$$

If ϕ is a d-radial function on G, for the sake of simplicity, we shall often interpret ϕ as a function on $[0, \infty)$ as follows:

$$\phi(r) = \phi(x), \quad \text{whenever} \ \ r = d(x).$$

Also, a function $\phi : G \to \mathbb{R}$, will be called d-radially decreasing if

$$\phi(x_1) \leq \phi(x_2), \quad \text{whenever} \ \ d(x_1) \geq d(x_2). \tag{4.2.17}$$

In this case, we will always assume that ϕ is bounded by $\phi(\underline{0}) \in (0, \infty)$.

Remark 4.2.13. Following [BLU07, P.247], when $d = d_{\mathcal{L}}$, a function $\phi : G \to \mathbb{R}$, satisfying (4.2.16), (4.2.17) will be called $\mathcal{L}$-radial, and $\mathcal{L}$-radially decreasing respectively.

We next prove a simple lemma regarding convolution on G.

Lemma 4.2.14. *Suppose that μ is a measure on G, and that $\phi : G \to (0, \infty)$, is a $\mathcal{L}$-radially decreasing function on G. Then finiteness of $|\mu| * \phi_{t_0}(x_0)$ for some $(x_0, t_0) \in G \times (0, \infty)$, implies the finiteness of $|\mu| * \phi_t(x)$ for all $(x, t) \in G \times (0, t_0/\tau)$.*

Proof. We take $(x, t) \in G \times (0, t_0/\tau)$, and denote the positive number $\dfrac{t_0}{t_0 - t\tau}$ by α. We note that $\alpha \in (1, \infty)$.

We write

$$\begin{aligned}
|\mu| * \phi_t(x) &= t^{-Q} \int_{B(x_0, \alpha\tau \mathbf{d}_{\mathcal{L}}(x,x_0))} \phi\left(\delta_{\frac{1}{t}}(\xi^{-1} \circ x)\right) \, d|\mu|(\xi) \\
&\quad + t^{-Q} \int_{B(x_0, \alpha\tau \mathbf{d}_{\mathcal{L}}(x,x_0))^c} \phi\left(\delta_{\frac{1}{t}}(\xi^{-1} \circ x)\right) \, d|\mu|(\xi) \\
&\le t^{-Q} \phi(\underline{0}) |\mu| \left(B(x_0, \alpha\tau \mathbf{d}_{\mathcal{L}}(x,x_0))\right) \\
&\quad + t^{-Q} \int_{B(x_0, \alpha\tau \mathbf{d}_{\mathcal{L}}(x,x_0))^c} \phi\left(\delta_{\frac{1}{t}}(\xi^{-1} \circ x)\right) \, d|\mu|(\xi). \qquad (4.2.18)
\end{aligned}$$

We note that for $\xi \in B\left(x_0, \alpha\tau \mathbf{d}_{\mathcal{L}}(x,x_0)\right)^c$,

$$\mathbf{d}_{\mathcal{L}}(x,x_0) \le \frac{1}{\alpha\tau} \mathbf{d}_{\mathcal{L}}(\xi, x_0),$$

and hence, using the reverse triangle inequality (4.2.15), we obtain for such ξ

$$\mathbf{d}_{\mathcal{L}}(\xi, x) \ge \frac{1}{\tau} \mathbf{d}_{\mathcal{L}}(\xi, x_0) - \mathbf{d}_{\mathcal{L}}(x, x_0) \ge \left(\frac{1}{\tau} - \frac{1}{\alpha\tau}\right) \mathbf{d}_{\mathcal{L}}(\xi, x_0).$$

This implies that for $\xi \in B\left(x_0, \alpha\tau \mathbf{d}_{\mathcal{L}}(x,x_0)\right)^c$,

$$\begin{aligned}
d_{\mathcal{L}}\left(\delta_{\frac{1}{t}}(\xi^{-1} \circ x)\right) &= \frac{1}{t} d_{\mathcal{L}}\left(\xi^{-1} \circ x\right) \\
&\ge \frac{1}{t}\left(\frac{1}{\tau} - \frac{1}{\alpha\tau}\right) d_{\mathcal{L}}\left(\xi^{-1} \circ x_0\right) \\
&= \frac{1}{t_0} d_{\mathcal{L}}\left(\xi^{-1} \circ x_0\right) \qquad \left(\text{as } \alpha = \frac{t_0}{t_0 - t\tau}\right) \\
&= d_{\mathcal{L}}\left(\delta_{\frac{1}{t_0}}(\xi^{-1} \circ x_0)\right).
\end{aligned}$$

Using this, and the fact that ϕ is $\mathcal{L}$-radially decreasing in (4.2.18), we get

$$|\mu| * \phi_t(x) \le t^{-Q} \phi(\underline{0}) |\mu| \left(B(x_0, \alpha\tau d_{\mathcal{L}}(x,x_0))\right) + t^{-Q} \int_G \phi\left(\delta_{\frac{1}{t_0}}(\xi^{-1} \circ x_0)\right) \, d|\mu|(\xi).$$

By our hypothesis, integral on the right-hand side is finite and hence $|\mu| * \phi_t(x)$ is finite. This completes the proof. □

Using this lemma and the Gaussian estimates (4.2.10), (4.2.11) we can prove the following.

Corollary 4.2.15. *Suppose μ is a measure on G. If $\Gamma[\mu](x_0, t_0)$ exists for some $(x_0, t_0) \in G \times (0, \infty)$, then $\Gamma[\mu]$ is well-defined in the strip $G \times (0, t_0/(c_0\tau)^2)$, where c_0 is as in (4.2.10).*

Moreover, $\Gamma[\mu]$ is a solution of the heat equation in this strip, that is,

$$\mathcal{H}u(x,t) = 0, \quad (x,t) \in G \times (0, t_0/(c_0\tau)^2).$$

Proof. As $\Gamma[\mu](x_0,t_0)$ exists, using (4.2.10) we get

$$\int_G \exp\left(-\frac{c_0 d_{\mathcal{L}}(\xi^{-1}\circ x_0)^2}{t_0}\right) d|\mu|(\xi) \leq c_0 \int_G \Gamma(\xi^{-1}\circ x_0, t_0)\, d|\mu|(\xi) < \infty. \tag{4.2.19}$$

Setting $t_1 = t_0/c_0^2$, and

$$\phi(x) = \exp\left(\frac{-d_{\mathcal{L}}(x)^2}{c_0}\right), \quad x \in G,$$

and then using (4.2.19), we note that $|\mu| * \phi_{\sqrt{t_1}}(x_0)$ is finite. Since ϕ satisfies all the requirements of Lemma 4.2.14, we thus conclude that $|\mu| * \phi_{\sqrt{t}}(x)$ is finite, for all $(x,t) \in G \times (0, t_1/\tau^2)$. It now follows from the Gaussian estimate (4.2.10) that

$$\int_G \Gamma(\xi^{-1}\circ x, t)\, d|\mu|(\xi) \leq c_0 t^{-\frac{Q}{2}} \int_G \exp\left(\frac{-d_{\mathcal{L}}(\xi^{-1}\circ x)^2}{c_0 t}\right) = c_0|\mu| * \phi_{\sqrt{t}}(x) < \infty,$$

for all $(x,t) \in G \times (0, t_0/(c_0\tau)^2)$. To prove the second part, we differentiate $\Gamma[\mu]$ in $G \times (0, t_0/(c_0\tau)^2)$ along the vector fields $X_1, \cdots, X_{N_1}, \frac{\partial}{\partial t}$ and then use the fact that Γ is a fundamental solution of $\mathcal{H}$. Differentiation under integral sign is justified because of the estimate (4.2.11). $\square$

Remark 4.2.16. For an alternative proof of the second part of Corollary 4.2.15, which uses Harnack inequality, we refer to [BU05, Lemma 2.5].

It is clear from the Gaussian estimate (4.2.10) and the integration formula in 'polar coordinate' (4.2.6) that for each $t \in (0,\infty)$, $\Gamma(\cdot,t) \in L^p(G)$, for all $p \in [1,\infty]$. Therefore, if $d\mu = f\, dm$, for some $f \in L^p(G)$, $1 \leq p \leq \infty$, then $\Gamma[\mu]$ is well-defined in $G \times (0,\infty)$, and we denote it by Γf. We recall that (see (4.2.13), (4.2.14)), for $t \in (0,\infty)$

$$\gamma = \Gamma(\cdot, 1), \quad \Gamma[\mu](\cdot,t) = \mu * \gamma_{\sqrt{t}}.$$

Thus, Remark 4.2.5 shows that $\{\gamma_{\sqrt{t}} \mid t \in (0,\infty)\}$ is an approximate identity on G. Consequently, for $f \in C_c(G)$,

$$\Gamma f(.,t) = f * \gamma_{\sqrt{t}} \to f,$$

uniformly on G, as t goes to zero. However, a stronger result is true.

Lemma 4.2.17. *If $f \in C_c(G)$, then*

$$\lim_{t\to 0} \frac{\Gamma f(.,t)}{\gamma} = \frac{f}{\gamma},$$

uniformly on G.

Proof. We assume that supp $f \subset B(\underline{0}, R)$, for some $R \in (0,\infty)$. Using (4.2.10), we obtain some positive constant C such that

$$\gamma(x) = \Gamma(x,1) \geq c_0^{-1} \exp\left(-c_0 d_{\mathcal{L}}(x)^2\right) \geq C, \quad \text{for all} \quad x \in B(\underline{0}, 2\tau R).$$

Hence, it suffices to prove that

$$\lim_{t\to 0} \frac{\Gamma f(x,t)}{\gamma(x)} = 0,$$

uniformly for $x \in B(\underline{0}, 2\tau R)^c$. We observe that

$$\begin{aligned} \frac{|\Gamma f(x,t)|}{\gamma(x)} &= \frac{1}{\gamma(x)} \left| \int_{B(\underline{0},R)} \Gamma(\xi^{-1} \circ x, t) f(\xi)\, dm(\xi) \right| \\ &\leq \frac{c_0}{t^{\frac{Q}{2}} \gamma(x)} \int_{B(\underline{0},R)} \exp\left(-\frac{d_{\mathcal{L}}(\xi^{-1} \circ x)^2}{c_0 t}\right) |f(\xi)|\, dm(\xi), \end{aligned} \tag{4.2.20}$$

where the last inequality follows from the Gaussian estimate (4.2.10). Now, for $x \in B(\underline{0}, 2\tau R)^c$, and $\xi \in B(\underline{0}, R)$, we have

$$d_{\mathcal{L}}(\xi) < R \leq \frac{d_{\mathcal{L}}(x)}{2\tau}.$$

Thus, using (4.2.15), we get that for $x \in B(\underline{0}, 2\tau R)^c$, and $\xi \in B(\underline{0}, R)$,

$$d_{\mathcal{L}}(\xi^{-1} \circ x) \geq \frac{d_{\mathcal{L}}(x)}{\tau} - d_{\mathcal{L}}(\xi) \geq \frac{d_{\mathcal{L}}(x)}{\tau} - \frac{d_{\mathcal{L}}(x)}{2\tau} = \frac{d_{\mathcal{L}}(x)}{2\tau}. \tag{4.2.21}$$

Using this observation in (4.2.20), we obtain for $x \in B(\underline{0}, 2\tau R)^c$,

$$\begin{aligned} \frac{|\Gamma f(x,t)|}{\gamma(x)} &\leq \frac{c_0}{t^{\frac{Q}{2}} \gamma(x)} \int_{B(0,R)} \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{4c_0 \tau^2 t}\right) |f(\xi)|\, dm(\xi) \\ &= c_0 \frac{\exp\left(-\frac{d_{\mathcal{L}}(x)^2}{4c_0\tau^2 t}\right)}{t^{\frac{Q}{2}} \gamma(x)} \|f\|_{L^1(G)}. \end{aligned}$$

Hence, it is enough to show that

$$\lim_{t\to 0} \frac{\exp\left(-\frac{d_{\mathcal{L}}(x)^2}{4c_0\tau^2 t}\right)}{t^{\frac{Q}{2}}\gamma(x)} = 0,$$

uniformly for $x \in B(\underline{0}, 2\tau R)^c$. But

$$\frac{\exp\left(-\frac{d_{\mathcal{L}}(x)^2}{4c_0\tau^2 t}\right)}{t^{\frac{Q}{2}}\gamma(x)} \le \frac{\exp\left(-\frac{d_{\mathcal{L}}(x)^2}{4c_0\tau^2 t}\right)}{t^{\frac{Q}{2}}c_0^{-1}\exp(-c_0 d_{\mathcal{L}}(x)^2)} = c_0 t^{-\frac{Q}{2}} \exp\left(-\left(\frac{1}{4c_0\tau^2 t} - c_0\right) d_{\mathcal{L}}(x)^2\right),$$

where the inequality follows from the Gaussian estimate (4.2.10). For $t \in \left(0, \frac{1}{4c_0^2\tau^2}\right)$, we note that $\left(\frac{1}{4c_0\tau^2 t} - c_0\right)$, is positive. Hence, for such t and for all $x \in B(\underline{0}, 2R\tau)^c$, we obtain from the last inequality that

$$\frac{\exp\left(-\frac{d_{\mathcal{L}}(x)^2}{4c_0\tau^2 t}\right)}{t^{\frac{Q}{2}}\gamma(x)} \le c_0 t^{-\frac{Q}{2}} \exp\left(-\left(\frac{1}{4c_0\tau^2 t} - c_0\right) 4\tau^2 R^2\right) \le C t^{-\frac{Q}{2}} \exp\left(-\frac{1}{c_1 t}\right),$$

for some positive constants C and c_1. The expression on the right-hand side of the inequality above goes to zero as t goes to zero. This completes the proof. □

Let $\mathcal{M}$ denote the set of all measures μ on G such that $\Gamma[\mu]$ is well-defined in $G \times (0, \infty)$. In view of Corollary 4.2.15, we have

$$\mathcal{M} = \{\mu \text{ is a measure on } G \mid \Gamma[|\mu|](\underline{0}, t) \text{ exists for all } t \in (0, \infty)\}.$$

We note that if $|\mu|(G)$ is finite, then $\mu \in \mathcal{M}$. In particular, every complex measure on G belongs to $\mathcal{M}$. We have the following observation regarding this class of measures.

Lemma 4.2.18. *If $\nu \in \mathcal{M}$, and $f \in C_c(G)$, then for each fixed $t \in (0, \infty)$*

$$\int_G \Gamma f(x,t)\, d\nu(x) = \int_G \Gamma[\nu](x,t) f(x)\, dm(x).$$

Proof. The result will follow by interchanging integrals using Fubini's theorem. In order to apply Fubini's theorem we must prove that for each fixed $t \in (0, \infty)$,

$$\int_G \int_{\text{supp } f} \Gamma(\xi^{-1} \circ x, t) |f(\xi)|\, dm(\xi)\, d|\nu|(x) < \infty.$$

We assume that supp $f \subset B(\underline{0}, R)$, for some $R \in (0, \infty)$. We fix $t \in (0, \infty)$. Then, using the Gaussian estimate (4.2.10), we obtain

$$
\begin{aligned}
I &= \int_G \int_{B(\underline{0},R)} \Gamma(\xi^{-1} \circ x, t) |f(\xi)| \, dm(\xi) \, d|\nu|(x) \\
&\leq c_0 t^{-\frac{Q}{2}} \int_G \int_{B(\underline{0},R)} \exp\left(-\frac{d_{\mathcal{L}}(\xi^{-1} \circ x)^2}{c_0 t} \right) |f(\xi)| \, dm(\xi) \, d|\nu|(x) \\
&= c_0 t^{-\frac{Q}{2}} \int_{B(\underline{0},2\tau R)} \int_{B(\underline{0},R)} \exp\left(-\frac{d_{\mathcal{L}}(\xi^{-1} \circ x)^2}{c_0 t} \right) |f(\xi)| \, dm(\xi) \, d|\nu|(x) \\
&\quad + c_0 t^{-\frac{Q}{2}} \int_{B(\underline{0},2\tau R)^c} \int_{B(\underline{0},R)} \exp\left(-\frac{d_{\mathcal{L}}(\xi^{-1} \circ x)^2}{c_0 t} \right) |f(\xi)| \, dm(\xi) \, d|\nu|(x) \\
&\leq c_0 t^{-\frac{Q}{2}} |\nu|(B(\underline{0}, 2\tau R)) \|f\|_{L^1(G)} \\
&\quad + c_0 t^{-\frac{Q}{2}} \int_{B(\underline{0},2\tau R)^c} \int_{B(\underline{0},R)} \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{4 c_0 \tau^2 t} \right) |f(\xi)| \, dm(\xi) \, d|\nu|(x),
\end{aligned}
$$

where we have used (4.2.21) in the last integral. Another application of the Gaussian estimate (4.2.10) in the last integral yields

$$
\begin{aligned}
I &\leq c_0 t^{-\frac{Q}{2}} |\nu|(B(\underline{0}, 2\tau R)) \|f\|_{L^1(G)} \\
&\quad + (4 c_0^2 \tau^2)^{\frac{Q}{2}} c_0^2 \int_{B(\underline{0},2\tau R)^c} \int_{B(\underline{0},R)} \Gamma(x, 4 c_0^2 \tau^2 t) |f(\xi)| \, dm(\xi) \, d|\nu|(x) \\
&\leq c_0 t^{-\frac{Q}{2}} |\nu|(B(\underline{0}, 2\tau R)) \|f\|_{L^1(G)} + (4 c_0^2 \tau^2)^{\frac{Q}{2}} c_0^2 \|f\|_{L^1(G)} \Gamma[|\nu|](\underline{0}, 4 c_0^2 \tau^2 t).
\end{aligned}
$$

As $\nu \in \mathcal{M}$, it follows that I is finite. This proves the lemma. □

We end this section with some important definitions that will be used in the upcoming sections as well as in the next chapter.

Definition 4.2.19.

i) For $x_0 \in G$, and $\alpha \in (0, \infty)$, we define the $\mathcal{L}$-parabolic region $\mathrm{P}(x_0, \alpha)$ with vertex at x_0 and aperture α, as follows:

$$
\begin{aligned}
\mathrm{P}(x_0, \alpha) &= \{(x, t) \in G \times (0, \infty) \mid \left(d_{\mathcal{L}}(x_0^{-1} \circ x) \right)^2 < \alpha t\} \\
&= \{(x, t) \in G \times (0, \infty) \mid (\mathbf{d}_{\mathcal{L}}(x_0, x))^2 < \alpha t\}.
\end{aligned}
$$

ii) A function u defined on $G \times (0, t_0)$, for some $t_0 \in (0, \infty]$, is said to have parabolic limit $L \in \mathbb{C}$, at $x_0 \in G$, if for each $\alpha \in (0, \infty)$

$$\lim_{\substack{t \to 0 \\ (x,t) \in \mathrm{P}(x_0, \alpha)}} u(x,t) = L.$$

iii) Given a measure μ on G, we say that μ has strong derivative $L \in \mathbb{C}$, at $x_0 \in G$, if

$$\lim_{r \to 0} \frac{\mu(x_0 \circ \delta_r(B))}{m(x_0 \circ \delta_r(B))} = L,$$

holds for every $d_{\mathcal{L}}$-ball B in G. The strong derivative of μ at x_0, if it exists, will be denoted by $D\mu(x_0)$.

iv) A sequence of functions $\{u_j \mid j \in \mathbb{N}\}$ defined on $G \times (0, \infty)$ is said to converge normally to a function u if $\{u_j\}$ converges to u uniformly on compact subsets of $G \times (0, \infty)$ (equipped with the product topology).

v) A sequence of functions $\{u_j \mid j \in \mathbb{N}\}$ defined on $G \times (0, \infty)$ is said to be locally bounded if given any compact set $K \subset G \times (0, \infty)$, there exists a positive constant C_K (depending only on K) such that for all j and for all $x \in K$

$$|u_j(x)| \leq C_K.$$

vi) A sequence of positive measures $\{\mu_j \mid j \in \mathbb{N}\}$ on G is said to converge to a positive measure μ on G in weak* if

$$\lim_{j \to \infty} \int_G \psi(y)\, d\mu_j(y) = \int_G \psi(y)\, d\mu(y), \quad \text{for all} \quad \psi \in C_c(G).$$

4.3 Some auxilary results

We start this section with the following results involving normal convergence and weak* convergence.

Lemma 4.3.1. *Suppose $\{\mu_j \mid j \in \mathbb{N}\} \subset \mathcal{M}$, and $\mu \in \mathcal{M}$, are positive measures. If $\{\Gamma[\mu_j]\}$ converges normally to $\Gamma[\mu]$, then $\{\mu_j\}$ converges to μ in weak*.*

Proof. Let $f \in C_c(G)$, be such that supp $f \subset B(\underline{0}, R)$, for some $R \in (0, \infty)$. We need to show that

$$\lim_{j\to\infty} \int_G f(x)\, d\mu_j(x) = \int_G f(x)\, d\mu(x).$$

Given $t \in (0, \infty)$, we write

$$\begin{aligned} & \int_G f(x)\, d\mu_j(x) - \int_G f(x)\, d\mu(x) \\ = & \int_G (f(x) - \Gamma f(x,t))\, d\mu_j(x) + \int_G \Gamma f(x,t)\, d\mu_j(x) - \int_G \Gamma f(x,t)\, d\mu(x) \\ & \quad + \int_G (\Gamma f(x,t) - f(x))\, d\mu(x). \end{aligned} \tag{4.3.1}$$

Fixing $\epsilon > 0$, and applying Lemma 4.2.17 we get some $t_0 \in (0, \infty)$, such that

$$\frac{|\Gamma f(x,t_0) - f(x)|}{\gamma(x)} < \epsilon, \quad \text{for all} \quad x \in G. \tag{4.3.2}$$

Using Lemma 4.2.18 in the second and third integral of (4.3.1), it follows that

$$\begin{aligned} & \left|\int_G f(x)\, d\mu_j(x) - \int_G f(x)\, d\mu(x)\right| \\ \leq & \int_G |f(x) - \Gamma f(x,t_0)|\, d\mu_j(x) + \int_{B(\underline{0},R)} |\Gamma[\mu_j](x,t_0) - \Gamma[\mu](x,t_0)||f(x)|\, dm(x) \\ & \quad + \int_G |\Gamma f(x,t_0) - f(x)|\, d\mu(x) \\ = & I_1(j) + I_2(j) + I_3. \end{aligned} \tag{4.3.3}$$

It follows from (4.3.2) that

$$I_1(j) = \int_G \frac{|\Gamma f(x,t_0) - f(x)|}{\gamma(x)} \gamma(x)\, d\mu_j(x) \leq \epsilon \int_G \gamma(x)\, d\mu_j(x) = \epsilon \Gamma[\mu_j](\underline{0}, 1),$$

for all $j \in \mathbb{N}$. Similarly, we also have that

$$I_3 \leq \epsilon \Gamma[\mu](\underline{0}, 1).$$

Since $\{\Gamma[\mu_j]\}$ converges to $\Gamma[\mu]$ normally, the sequence $\{\Gamma[\mu_j](\underline{0}, 1)\}$, in particular, is bounded. Hence, for

$$C = \sup_{j\in\mathbb{N}} \Gamma[\mu_j](\underline{0}, 1) + \Gamma[\mu](\underline{0}, 1),$$

we get that for all $j \in \mathbb{N}$,

$$I_1(j) + I_3 \leq 2C\epsilon.$$

Again using the normal convergence of $\{\Gamma[\mu_j]\}$ to $\Gamma[\mu]$, we get some $j_0 \in \mathbb{N}$, such that for all $j \geq j_0$,

$$\|\Gamma[\mu_j] - \Gamma[\mu]\|_{L^\infty(\overline{B(\underline{0},R)}\times\{t_0\})} < \epsilon.$$

This implies that for all $j \geq j_0$,

$$I_2(j) \leq \epsilon \|f\|_{L^1(G)}.$$

Hence, it follows from (4.3.3) that

$$\left| \int_G f(x) d\mu_j(x) - \int_G f(x) d\mu(x) \right| \leq \epsilon (2C + \|f\|_{L^1(G)}),$$

for all $j \geq j_0$. This completes the proof. □

In every second countable, locally compact, Hausdroff space, open sets are σ-compact. By [Rud87, Theorem 2.18], it then follows that any locally finite positive measure on $\mathbb{R}^n$ is regular. Consequently, if μ and ν are two positive measures on $\mathbb{R}^n$ (which, by our assumption, are locally finite) such that $\mu(\mathrm{B}) = \nu(\mathrm{B})$, for all Euclidean open ball B in $\mathbb{R}^n$, then $\mu = \nu$. We are now going to prove that the same conclusion can be drawn when open balls are replaced by $d_{\mathcal{L}}$-balls.

Lemma 4.3.2. *Let μ and ν be two positive measures on G. If*

$$\mu(B) = \nu(B), \tag{4.3.4}$$

for every $d_{\mathcal{L}}$-ball B in G, then $\mu = \nu$.

Proof. We set

$$\phi = m\left(B(\underline{0}, 1)\right)^{-1} \chi_{B(\underline{0},1)}.$$

Since translation and dilation of a $d_{\mathcal{L}}$-ball is again a $d_{\mathcal{L}}$-ball, it follows that for all $x \in G$ and $r \in (0, \infty)$,

$$\mu * \phi_r(x) = \nu * \phi_r(x). \tag{4.3.5}$$

On the other hand, as G is a second countable, locally compact, Hausdroff space, it follows from [Rud87, Theorem 2.18] that μ, ν are regular, and hence it suffices to show that

$$\int_G f\, d\mu = \int_G f\, d\nu, \quad \text{for all } f \in C_c(G).$$

We take $f \in C_c(G)$, with supp$f \subset B(\underline{0}, R)$, for some $R \in (0,\infty)$. We observe that for $x \in G$, $r \in (0,\infty)$

$$\begin{aligned}
f * (\mu * \phi_r)(x) &= \int_G f(y)(\mu * \phi_r)(y^{-1} \circ x)\, dm(y) \\
&= \int_G f(y) \int_G \phi_r\left(\xi^{-1} \circ (y^{-1} \circ x)\right)\, d\mu(\xi)\, dm(y) \\
&= \int_G \int_G f(y_1 \circ \xi^{-1})\phi_r(y_1^{-1} \circ x)\, d\mu(\xi)\, dm(y_1) \\
&\qquad \text{(substituting } y = y_1 \circ \xi^{-1}\text{, and using the} \\
&\qquad \text{translation invariance of } m) \\
&= \int_G f_\mu(y_1)\phi_r(y_1^{-1} \circ x)\, dm(y_1) \\
&= f_\mu * \phi_r(x), \qquad (4.3.6)
\end{aligned}$$

where

$$f_\mu(z) = \int_G f(z \circ \xi^{-1})\, d\mu(\xi), \quad z \in G. \qquad (4.3.7)$$

We now claim that f_μ is continuous at $\underline{0}$. To prove this claim, we consider a sequence $\{y_k \mid k \in \mathbb{N}\}$ in G converging to $\underline{0}$. Since the group operation and $d_{\mathcal{L}}$ are continuous, there exists some positive constant C such that

$$d_{\mathcal{L}}(y_k) \leq C, \quad \text{for all } \ k;$$

and for each $\xi \in G$, we have

$$y_k \circ \xi^{-1} \to \xi^{-1}, \quad \text{as } \ k \to \infty.$$

We note that for $\xi \in B(\underline{0}, \tau(R+C))^c$,

$$d_{\mathcal{L}}(y_k \circ \xi^{-1}) \geq \frac{1}{\tau} d_{\mathcal{L}}(\xi) - d_{\mathcal{L}}(y_k) > \frac{1}{\tau}\tau(R+C) - C = R, \quad \text{for all } k.$$

Therefore, as supp$f \subset B(\underline{0}, R)$, we can write

$$f_\mu(y_k) = \int_{B(\underline{0},\tau(R+C))} f(y_k \circ \xi^{-1})\, d\mu(\xi), \quad k \in \mathbb{N}. \tag{4.3.8}$$

By continuity of f, for each $\xi \in G$, we have

$$f(y_k \circ \xi^{-1}) \to f(\xi^{-1}), \quad \text{as} \;\; k \to \infty,$$

and hence applying dominated convergence theorem in (4.3.8), we obtain

$$f_\mu(y_k) \to \int_{B(\underline{0},\tau(R+C))} f(\xi^{-1})\, d\mu(\xi) = \int_G f(\xi^{-1})\, d\mu(\xi) = f_\mu(\underline{0}), \quad \text{as} \;\; k \to \infty.$$

This proves our claim. Let $\epsilon > 0$. Using (4.2.4) we choose some $\kappa > 0$, such that

$$|f_\mu(y) - f_\mu(\underline{0})| < \epsilon, \quad \text{for all} \;\; y \in B(\underline{0}, \kappa).$$

Hence,

$$\begin{aligned} |f_\mu * \phi_r(\underline{0}) - f_\mu(\underline{0})| &= \left| \int_G f_\mu(\xi)\phi_r(\xi^{-1})\, dm(\xi) - \int_G f_\mu(\underline{0})\phi_r(\xi^{-1})\, dm(\xi) \right| \\ &\leq \frac{1}{m(B(\underline{0}, r))} \int_{B(\underline{0},r)} |f_\mu(\xi) - f_\mu(\underline{0})|\, dm(\xi) \\ &< \epsilon, \quad \text{for all} \;\; r \in (0, \kappa). \end{aligned}$$

This, together with (4.3.6) and (4.3.7), implies that

$$f * (\mu * \phi_r)(\underline{0}) \to f_\mu(\underline{0}) = \int_G f(\xi^{-1})\, d\mu(\xi), \quad \text{as} \;\; r \to 0.$$

Similarly, we can prove that

$$f * (\nu * \phi_r)(\underline{0}) \to f_\nu(\underline{0}) = \int_G f(\xi^{-1})\, d\nu(\xi), \quad \text{as} \;\; r \to 0,$$

where f_ν is defined according to (4.3.7). Equation (4.3.5) now shows that

$$\int_G f(\xi^{-1})\, d\mu(\xi) = \int_G f(\xi^{-1})\, d\nu(\xi).$$

This completes the proof. $\square$

We now use this lemma to prove the following generalization of Lemma 3.2.5 which will be needed in the proof of our main theorem.

Lemma 4.3.3. *Suppose that $\{\mu_j \mid j \geq 1\}$, μ are positive measures on G and that $\{\mu_j\}$ converges to μ in weak*. Then for some $L \in [0,\infty)$, $\mu = Lm$ if and only if $\{\mu_j(B)\}$ converges to $Lm(B)$ for every $d_{\mathcal{L}}$-ball B in G.*

Proof. Suppose $\mu = Lm$. Fix a $d_{\mathcal{L}}$-ball B in G and $\epsilon > 0$. As $\overline{B}$ is compact with respect to the Euclidean topology, by regularity of m, there exists an open set $V \supset \overline{B}$, such that

$$m(V \setminus \overline{B}) < \epsilon.$$

Using Uryshon's lemma [Rud87, Theorem 2.12], we choose $\psi \in C_c(G)$, such that

$$0 \leq \psi(x) \leq 1, \quad \text{for all } x \in G; \quad \psi \equiv 1 \text{ on } \overline{B}; \quad \psi \equiv 0 \text{ on } V^c.$$

Then

$$\int_G \psi \, dm = \int_B \psi \, dm + \int_{V \setminus \overline{B}} \psi \, dm \leq m(B) + m(V \setminus \overline{B}) \leq m(B) + \epsilon. \tag{4.3.9}$$

Since ψ is nonnegative and identically equal to 1 on $\overline{B}$, and $\mu_j \to \mu$ in weak*,

$$\limsup_{j\to\infty} \mu_j(B) = \limsup_{j\to\infty} \int_B \psi \, d\mu_j \leq \limsup_{j\to\infty} \int_G \psi \, d\mu_j = \int_G \psi \, d\mu.$$

Using our assumption, that is, $\mu = Lm$ and (4.3.9) in the above, we get

$$\limsup_{j\to\infty} \mu_j(B) \leq L \int_G \psi \, dm \leq L(m(B) + \epsilon).$$

Since $\epsilon > 0$, is arbitrary

$$\limsup_{j\to\infty} \mu_j(B) \leq Lm(B). \tag{4.3.10}$$

Similarly, by choosing a compact set $K \subset B$, with

$$m(K) > m(B) - \epsilon \quad \text{(using Remark 4.2.2)}$$

and a function $g \in C_c(G)$ such that

$$0 \le g(x) \le 1, \quad \text{for all } x \in G; \quad g \equiv 1 \quad \text{on } K; \quad g \equiv 0 \quad \text{on } B^c,$$

we observe that

$$\int_G g\, dm \ge \int_K g\, dm = m(K) > m(B) - \epsilon.$$

Since $0 \le g \le 1$, with supp $g \subset B$, and $\mu_j \to \mu$ in weak*,

$$\liminf_{j\to\infty} \mu_j(B) \ge \liminf_{j\to\infty} \int_G g\, d\mu_j = \int_G g\, d\mu = L \int_G g\, dm > L(m(B) - \epsilon).$$

Since $\epsilon > 0$ is arbitrary

$$\liminf_{j\to\infty} \mu_j(B) \ge Lm(B).$$

Combining the inequality above with (4.3.10), we conclude that

$$\lim_{j\to\infty} \mu_j(B) = Lm(B).$$

Conversely, we suppose that

$$\lim_{j\to\infty} \mu_j(B) = Lm(B), \tag{4.3.11}$$

for every $d_{\mathcal{L}}$-ball B in G. We need to prove that $\mu = Lm$. In view of Lemma 4.3.2, it suffices to show that

$$\mu(B) = Lm(B), \quad \text{for every } d_{\mathcal{L}}\text{-ball } B \subset G.$$

The proof of this part is similar to that of the previous part. We fix $\epsilon > 0$, and a $d_{\mathcal{L}}$-ball $B = B(x_0, r)$. We denote the $d_{\mathcal{L}}$-ball centered at x_0 and radius $r + \epsilon$ by B'. Taking Remark 4.2.4 into account and applying Uryshon's lemma we get a function $f \in C_c(G)$, such that

$$0 \le f(x) \le 1, \quad \text{for all } x \in G; \quad f \equiv 1 \quad \text{on } \overline{B}; \quad f \equiv 0 \quad \text{on } G \setminus B'.$$

Using our hypothesis, namely $\mu_j \to \mu$ in weak*, and (4.3.11), above implies that

$$\mu(B) = \int_B f\, d\mu \le \int_G f\, d\mu = \lim_{j\to\infty} \int_G f\, d\mu_j \le \lim_{j\to\infty} \mu_j(B') = Lm(B') = Lc_Q(r+\epsilon)^Q,$$

where $c_Q = m(B(\underline{0}, 1))$.

Since $\epsilon > 0$ is arbitrary, we have

$$\mu(B) \leq Lm(B(0,1))r^Q = Lm(B).$$

Similarly, letting $B'' = B(x_0, r-\epsilon)$, and choosing a function $f_1 \in C_c(G)$, such that

$$0 \leq f_1(x) \leq 1, \quad \text{for all } x \in G; \quad f_1 \equiv 1 \quad \text{on } B''; \quad f_1 \equiv 0 \quad \text{on } G \setminus B;$$

we obtain

$$\mu(B) \geq \int_G f_1 \, d\mu = \lim_{j\to\infty} \int_G f_1 \, d\mu_j \geq \liminf_{j\to\infty} \int_{B''} f_1 \, d\mu_j = \liminf_{j\to\infty} \mu_j(B'').$$

Thus, (4.3.11) gives

$$\mu(B) \geq Lm(B'') = Lm(B(0,1))(r-\epsilon)^Q.$$

As $\epsilon > 0$ is arbitrary,

$$\mu(B) \geq Lm(B(0,1))r^Q = Lm(B).$$

This completes the proof. □

Remark 4.3.4. It is evident from the proofs that both Lemma 4.3.2 and Lemma 4.3.3 does not depend on any particular choice of the homogeneous norm.

Next, we shall consider various types of maximal functions on G. For a measurable function ϕ defined on G and a measure μ on G, we define the α-nontangential maximal function $M_\phi^\alpha \mu$, where $\alpha \in (0,\infty)$, and the radial maximal function $M_\phi^0 \mu$ of μ with respect to ϕ as follows ([FS82, P.62]):

$$M_\phi^\alpha \mu(x) = \sup_{\substack{(\xi,t)\in G\times(0,\infty) \\ d_{\mathcal{L}}(x^{-1}\circ\xi)<\alpha t}} |\mu * \phi_t(\xi)|, \quad x \in G;$$
$$M_\phi^0 \mu(x) = \sup_{t\in(0,\infty)} |\mu * \phi_t(x)|, \quad x \in G.$$

It is obvious that $M_\phi^0 \mu$ is pointwise dominated by $M_\phi^\alpha \mu$ for all $\alpha \in (0,\infty)$. In [FS82, Corollary 2.5], it was proved that if ϕ satisfies some polynomial decay, namely

$$|\phi(x)| \leq C(1 + d_{\mathcal{L}}(x))^{-\lambda}$$

for some positive constants C and $\lambda \in (Q, \infty)$, then M_ϕ^α is weak type $(1,1)$ and strong type (p,p), $1 < p \leq \infty$. Although Folland-Stein proved these mapping properties of M_ϕ^α for $\alpha = 1$, but their proof works for all $\alpha \in (0, \infty)$. An important special case of this type of maximal functions is the centered Hardy-Littlewood maximal function, which is obtained by taking $\phi = \chi_{B(0,1)}$ in $M_\phi^0 \mu$. We shall denote it by $M_{HL}(\mu)$. In other words,

$$M_{HL}(\mu)(x) = \sup_{r>0} \frac{|\mu(B(x,r))|}{m(B(x,r))}, \quad x \in G.$$

In the following, we shall prove a lemma regarding pointwise comparison between the centered Hardy-Littlewood maximal function and other maximal functions introduced above. We then use it to prove the corresponding result for heat maximal functions.

Lemma 4.3.5. *Let $\phi : G \to (0, \infty)$, be a $\mathcal{L}$-radial, $\mathcal{L}$-radially decreasing (see (4.2.16), (4.2.17)) and integrable function, and let μ be a positive measure on G. Then for each $\alpha \in (0, \infty)$, there exist positive constants $c_{\alpha,\phi}$ and c_ϕ such that*

$$c_\phi M_{HL}(\mu)(x_0) \leq M_\phi^0 \mu(x_0) \leq M_\phi^\alpha \mu(x_0) \leq c_{\alpha,\phi} M_{HL}(\mu)(x_0), \quad \textit{for all} \quad x_0 \in G.$$

Proof. We have already observed that the second inequality is obvious. To prove the left-most inequality we take $t \in (0, \infty)$, and note that

$$\begin{aligned}
\mu * \phi_t(x_0) &\geq \int_{B(x_0,t)} \phi_t(\xi^{-1} \circ x_0)\, d\mu(\xi) \\
&= t^{-Q} \int_{B(x_0,t)} \phi\left(\delta_{\frac{1}{t}}(\xi^{-1} \circ x_0)\right)\, d\mu(\xi) \\
&\geq t^{-Q} \int_{B(x_0,t)} \phi(1)\, d\mu(\xi) \qquad \left(\text{as } d_{\mathcal{L}}\left(\delta_{\frac{1}{t}}(\xi^{-1} \circ x_0)\right) < 1\right) \\
&= \phi(1) m(B(0,1)) \frac{\mu(B(x_0,t))}{m(B(x_0,t))}.
\end{aligned}$$

Setting $c_\phi = \phi(1) m(B(0,1))$, and then taking supremum over $t \in (0, \infty)$, on both sides of the inequality above we get

$$c_\phi M_{HL}(\mu)(x_0) \leq M_\phi^0 \mu(x_0). \tag{4.3.12}$$

To prove the right-most inequality, we take $(\xi, t) \in G \times (0, \infty)$, such that

$$d_{\mathcal{L}}(x_0^{-1} \circ \xi) < \alpha t. \tag{4.3.13}$$

Then,

$$\begin{aligned}
\mu * \phi_t(\xi) &= \int_G \phi_t(x^{-1} \circ \xi)\, d\mu(x) \\
&= t^{-Q} \int_{B(\xi,\alpha t)} \phi\left(\delta_{\frac{1}{t}}(x^{-1} \circ \xi)\right)\, d\mu(x) \\
&\quad + t^{-Q} \sum_{j=1}^{\infty} \int_{\{x \in G \mid 2^{j-1}\alpha t \le d_{\mathcal{L}}(x^{-1}\circ\xi) < 2^j \alpha t\}} \phi\left(\delta_{\frac{1}{t}}(x^{-1} \circ \xi)\right)\, d\mu(x) \\
&= I + \sum_{j=1}^{\infty} I_j. \qquad (4.3.14)
\end{aligned}$$

Since ϕ is bounded by $\phi(\underline{0})$, we get the following estimate of I:

$$I \le \phi(\underline{0}) t^{-Q} \int_{B(\xi,\alpha t)} d\mu(x) = C_{\alpha,\phi} \frac{\mu(B(\xi,\alpha t))}{m(B(x_0,\alpha t))}. \qquad (4.3.15)$$

Using (4.3.13), the quasi-triangle inequality gives

$$d_{\mathcal{L}}(x_0^{-1} \circ x) \le \tau \left(d_{\mathcal{L}}(x_0^{-1} \circ \xi) + d_{\mathcal{L}}(\xi^{-1} \circ x) \right) < \tau(\alpha t + \alpha t) = 2\tau\alpha t,$$

for all $x \in B(\xi, \alpha t)$. Consequently, $B(\xi,\alpha t) \subset B(x_0, 2\tau\alpha t)$, and hence

$$\mu(B(\xi,\alpha t)) \le \mu(B(x_0, 2\tau\alpha t)). \qquad (4.3.16)$$

Similarly,

$$\mu(B(\xi, 2^j\alpha t)) \le \mu(B(x_0, \tau(2^j+1)\alpha t)). \qquad (4.3.17)$$

Applying (4.3.16) in (4.3.15), we obtain

$$I \le C_{\alpha,\phi} \frac{\mu(B(x_0, 2\tau\alpha t))}{m(B(x_0,\alpha t))} \le C'_{\alpha,\phi} M_{HL}(\mu)(x_0). \qquad (4.3.18)$$

Now, for each $j \in \mathbb{N}$,

$$\begin{aligned}
I_j &\le t^{-Q} \int_{\{x \mid 2^{j-1}\alpha t \le d_{\mathcal{L}}(x^{-1}\circ\xi) < 2^j\alpha t\}} \phi(2^{j-1}\alpha)\, d\mu(x) \\
&\le t^{-Q} \phi(2^{j-1}\alpha) \mu(B(\xi, 2^j\alpha t)). \qquad (4.3.19)
\end{aligned}$$

In order to estimate the right-hand side of the inequality above, we use the integration formula in 'polar coordinates' given in (4.2.6) to get

$$\int_G \phi(x)\, dm(x) = \sigma(S) \int_0^\infty \phi(r) r^{Q-1}\, dr, \tag{4.3.20}$$

as ϕ is $\mathcal{L}$-radial. But ϕ is $\mathcal{L}$-radially decreasing and nonnegative. Hence,

$$\begin{aligned}
\int_\alpha^\infty \phi(r) r^{Q-1}\, dr &= \sum_{j=1}^\infty \int_{2^{j-1}\alpha}^{2^j\alpha} \phi(r) r^{Q-1}\, dr \\
&\geq \sum_{j=1}^\infty \phi(2^j\alpha) \int_{2^{j-1}\alpha}^{2^j\alpha} r^{Q-1}\, dr \\
&= \frac{\alpha^Q}{Q} \sum_{j=1}^\infty \phi(2^j\alpha)(2^{jQ} - 2^{(j-1)Q}) \\
&= \frac{(2^Q-1)\alpha^Q}{2^{3Q}Q} \sum_{j=2}^\infty \phi(2^{j-1}\alpha) 2^{(j+1)Q}.
\end{aligned}$$

Equation (4.3.20) and integrability of ϕ now imply that

$$\sum_{j=1}^\infty \phi(2^{j-1}\alpha) 2^{(j+1)Q} < \infty. \tag{4.3.21}$$

Applying (4.3.17) in (4.3.19) we obtain

$$\begin{aligned}
\sum_{j=1}^\infty I_j &\leq t^{-Q} \sum_{j=1}^\infty \phi(2^{j-1}\alpha) \mu(B(x_0, (2^j+1)\tau\alpha t)) \\
&= c'_\alpha \sum_{j=1}^\infty \phi(2^{j-1}\alpha)(2^j+1)^Q \frac{\mu(B(x_0,(2^j+1)\tau\alpha t))}{m(B(x_0,(2^j+1)\tau\alpha t))} \\
&\leq c'_\alpha \left(\sum_{j=1}^\infty \phi(2^{j-1}\alpha) 2^{(j+1)Q} \right) M_{HL}(\mu)(x_0) = c'_{\alpha,\phi} M_{HL}(\mu)(x_0).
\end{aligned}$$

In view of (4.3.21), $c'_{\alpha,\phi}$ is finite. Applying the inequality above and the inequality (4.3.18) in (4.3.14), we get that

$$\mu * \phi_t(\xi) \leq c_{\alpha,\phi} M_{HL}(\mu)(x_0).$$

Taking supremum over all $(\xi, t) \in G \times (0,\infty)$, with $d_{\mathcal{L}}(x_0,\xi) < \alpha t$, we obtain

$$M_\phi^\alpha \mu(x_0) \leq c_{\alpha,\phi} M_{HL}(\mu)(x_0). \qquad \square$$

Remark 4.3.6. It is evident from the proof of the lemma above that one can replace the homogeneous norm $d_{\mathcal{L}}$ with any other homogeneous norm. More precisely, if $\phi : G \to (0, \infty)$, satisfies (4.2.16), (4.2.17) with respect to any other homogeneous norm d, then the conclusion of Lemma 4.3.5 is true, where the Hardy-Littlewood maximal function and the α-nontangential maximal function will now be defined with respect to the d-balls instead of $d_{\mathcal{L}}$-balls.

Lemma 4.3.7. *Let $\mu \in \mathcal{M}$, be a positive measure. Then for each $\alpha \in (0, \infty)$, there exists positive constants c and c_α such that*

$$cM_{HL}(\mu)(x_0) \leq \sup_{t \in (0,\infty)} \Gamma[\mu](x_0, t^2) \leq \sup_{(x,t) \in \mathrm{P}(x_0,\alpha)} \Gamma[\mu](x, t) \leq c_\alpha M_{HL}(\mu)(x_0), \tag{4.3.22}$$

for all $x_0 \in G$.

Proof. We fix $x_0 \in G$, $\alpha \in (0, \infty)$. We recall that (see Definition 4.2.19,i))

$$\mathrm{P}(x_0, \alpha) = \{(x, t) \in G \times (0, \infty) : d_{\mathcal{L}}(x_0^{-1} \circ x)^2 < \alpha t\}.$$

The second inequality is obvious as $\{(x_0, t^2) \mid t \in (0, \infty)\}$ is contained in $\mathrm{P}(x_0, \alpha)$. To prove the left-most inequality, we take

$$\phi(x) = c_0^{-1} \exp\left(-c_0 d_{\mathcal{L}}(x)^2\right), \quad x \in G.$$

Clearly, ϕ satisfies the hypothesis of Lemma 4.3.5. By the first part of the Gaussian estimate (4.2.10), we have

$$\mu * \phi_t(x) \leq \Gamma[\mu](x, t^2), \quad \text{for all } (x, t) \in G \times (0, \infty).$$

Applying left-most inequality in Lemma 4.3.5, we obtain

$$cM_{HL}(\mu)(x_0) \leq \sup_{t>0} \mu * \phi_t(x_0) \leq \sup_{t>0} \Gamma[\mu](x_0, t^2),$$

for some positive constant c, independent of x_0. On the other hand, we consider

$$\psi(x) = c_0 \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{c_0}\right), \quad x \in G.$$

Clearly, ψ also satisfies the hypothesis of Lemma 4.3.5. Moreover, by the upper Gaussian estimate (4.2.10), we have

$$\Gamma[\mu](x,t) \leq \mu * \psi_{\sqrt{t}}(x), \quad \text{for all } (x,t) \in G \times (0,\infty). \tag{4.3.23}$$

But the right-most inequality in Lemma 4.3.5 gives us

$$\sup_{\substack{(\xi,t)\in G\times(0,\infty)\\ d_{\mathcal{L}}(x_0^{-1}\circ\xi)<\sqrt{\alpha t}}} \mu * \psi_{\sqrt{t}}(\xi) \leq c_\alpha M_{HL}(\mu)(x_0),$$

for some positive constant c_α, independent of x_0. Hence, using (4.3.23) and the definition of $\mathrm{P}(x_0,\alpha)$, we get that

$$\sup_{(\xi,t)\in \mathrm{P}(x_0,\alpha)} \Gamma[\mu](\xi,t) \leq \sup_{\substack{(\xi,t)\in G\times(0,\infty)\\ d_{\mathcal{L}}(x_0,\xi)<\sqrt{\alpha t}}} \mu * \psi_{\sqrt{t}}(\xi) \leq c_\alpha M_{HL}(\mu)(x_0).$$

This completes the proof. □

To prove our main result of this chapter we will also need an analogue of Montel's theorem for solutions of the heat equation (4.2.9). We have already observed that the heat operator $\mathcal{H} = \frac{\partial}{\partial t} - \mathcal{L}$ is hypoelliptic on $G \times (0,\infty)$ (see Remark 4.2.9). Using this hypoellipticty, we have the following Montel-type result.

Lemma 4.3.8 ([Bär13, Theorem 4]). *Let $\{u_j\}$ be a sequence of solutions of the heat equation (4.2.9) in $G\times(0,\infty)$. If $\{u_j\}$ is locally bounded then it has a subsequence which converges normally to a function v, defined on $G\times(0,\infty)$, which is also a solution of the heat equation (4.2.9).*

We have already mentioned that the positive solutions of the classical heat equation on the Euclidean upper half space are given by convolution of positive measures with the Euclidean heat kernel (see Lemma 2.1.12). In case of the heat equations on stratified Lie groups, we also have similar representation formula.

Lemma 4.3.9 ([BU05, Lemma 2.3]). *Let u be a positive solution of the heat equation $\mathcal{H}u = 0$ in the strip $G \times (0,T)$, for some $T \in (0,\infty]$. Then, there exists a unique positive measure μ*

on G such that

$$u(x,t)=\Gamma[\mu](x,t)=\int_G \Gamma(\xi^{-1}\circ x,t)\,d\mu(\xi),\quad (x,t)\in G\times(0,T).$$

The measure μ will be called the boundary measure of u. Bonfiglioli and Uguzzoni proved the above Lemma under the implicit assumption that $T\in(0,\infty)$. But the same proof will work for the case $T=\infty$. This type of representation formula has been known as Widder-type representation fromula in the literature.

Given a function F on $G\times(0,\infty)$ and $r\in(0,\infty)$, we define the parabolic dilation of F as follows

$$F_r(x,t)=F(\delta_r(x),r^2t),\quad (x,t)\in G\times(0,\infty). \tag{4.3.24}$$

Remark 4.3.10. The notion of parabolic dilation is crucial for us primarily because of the following reasons.

i) If F is a solution of the heat equation then so is F_r, for every $r\in(0,\infty)$. Indeed, $\mathcal{L}$ is homogeneous of degree two with respect to the dilations $\{\delta_r \mid r\in(0,\infty)\}$ (see Remark 4.2.8). This implies that

$$\left(\mathcal{L}-\frac{\partial}{\partial t}\right)F(\delta_r(x),r^2t)=r^2(\mathcal{L}F)(\delta_r(x),r^2t)-r^2\frac{\partial F}{\partial t}(\delta_r(x),r^2t)=0.$$

ii) $(x,t)\in \mathrm{P}(\underline{0},\alpha)$, if and only if $(\delta_r(x),r^2t)\in \mathrm{P}(\underline{0},\alpha)$, for every $r\in(0,\infty)$.

Given a measure ν on G, and $r\in(0,\infty)$, we also define the dilate ν_r of ν by

$$\nu_r(E)=r^{-Q}\nu\left(\delta_r(E)\right), \tag{4.3.25}$$

for every Borel set $E\subseteq G$. The following lemma relates the above notions of dilates.

Lemma 4.3.11. *If $\nu\in\mathcal{M}$, then for every $r\in(0,\infty)$*

$$\Gamma[\nu_r](x,t)=\Gamma[\nu]\left(\delta_r(x),r^2t\right),\quad \textit{for all}\quad (x,t)\in G\times(0,\infty).$$

Proof. For a Borel set $E \subseteq G$, using the definition of ν_r (4.3.25), it follows that

$$\int_G \chi_E \, d\nu_r = r^{-Q} \nu\left(\delta_r(E)\right) = r^{-Q} \int_G \chi_{\delta_r(E)}(x) \, d\nu(x) = r^{-Q} \int_G \chi_E\left(\delta_{r^{-1}}(x)\right) \, d\nu(x).$$

Hence, for all nonnegative measurable functions f on G, we have

$$\int_G f(x) \, d\nu_r(x) = r^{-Q} \int_G f\left(\delta_{r^{-1}}(x)\right) \, d\nu(x).$$

It now follows from the relation above that

$$\begin{aligned}
\Gamma[\nu_r](x,t) &= \int_G \Gamma(\xi^{-1} \circ x, t) \, d\nu_r(\xi) \\
&= r^{-Q} \int_G \Gamma\left(\delta_{r^{-1}}(\xi^{-1}) \circ x, t\right) \, d\nu(x) \\
&= r^{-Q} \int_G \Gamma\left(\delta_{r^{-1}}\left(\xi^{-1} \circ \delta_r(x)\right), r^{-2} r^2 t\right) \, d\nu(x) \\
&= r^{-Q} r^Q \int_G \Gamma(\xi^{-1} \circ \delta_r(x), r^2 t) \, d\nu(x) \qquad \text{(using Theorem 4.2.11, (iii))} \\
&= \Gamma[\nu]\left(\delta_r(x), r^2 t\right),
\end{aligned}$$

for all $(x,t) \in G \times (0,\infty)$. $\square$

4.4 Main theorem

We shall first prove a special case of our main result. The proof of the main result will follow by reducing matters to this special case.

Theorem 4.4.1. *Suppose that u is a positive solution of the heat equation*

$$\mathcal{H}u(x,t) = 0, \quad (x,t) \in G \times (0,\infty),$$

and that $L \in [0,\infty)$. If the boundary measure μ of u is finite, then the following statements are equivalent.

(i) u has parabolic limit L at $\underline{0}$.

(ii) μ has strong derivative L at $\underline{0}$.

Proof. We first prove that (i) implies (ii). We fix a $d_{\mathcal{L}}$-ball B_0 in G, a sequence of positive numbers $\{r_j \mid j \in \mathbb{N}\}$ converging to zero and then consider the quotient

$$L_j = \frac{\mu\left(\delta_{r_j}(B_0)\right)}{m\left(\delta_{r_j}(B_0)\right)}, \quad j \in \mathbb{N}. \tag{4.4.1}$$

Assuming (i), we will prove that $\{L_j\}$ is a bounded sequence and every convergent subsequence of $\{L_j\}$ converges to L. We first choose a positive number s such that B_0 is contained in the $d_{\mathcal{L}}$-ball $B(\underline{0}, s)$. Then, using positivity of μ and (4.2.1), we obtain for all $j \in \mathbb{N}$ that

$$L_j \leq \frac{\mu\left(\delta_{r_j}(B(\underline{0}, s))\right)}{m\left(\delta_{r_j}(B_0)\right)} = \frac{\mu\left(\delta_{r_j}(B(\underline{0}, s))\right)}{m\left(\delta_{r_j}(B(\underline{0}, s))\right)} \times \frac{m(B(\underline{0}, s))}{m(B_0)} \leq C_s M_{HL}(\mu)(\underline{0}), \tag{4.4.2}$$

where $C_s = \dfrac{m(B(\underline{0}, s))}{m(B_0)}$. Since μ is the boundary measure for u we have that

$$u(x,t) = \Gamma[\mu](x,t), \quad \text{for all} \quad (x,t) \in G \times (0, \infty).$$

By hypothesis, $u(\underline{0}, t^2)$ converges to L as t tends to zero which implies, in particular, that there exists a positive number κ such that

$$\sup_{t \in (0,\kappa)} u(\underline{0}, t^2) = \sup_{t \in (0,\kappa)} \Gamma[\mu](\underline{0}, t^2) < \infty.$$

Since μ is a finite measure, using the Gaussian estimate (4.2.10), we also have

$$\Gamma[\mu](\underline{0}, t^2) \leq c_0 t^{-Q} \int_G \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{c_0 t^2}\right) d\mu(x) \leq c_0 t^{-Q} \mu(G) \leq c_0 \kappa^{-Q} \mu(G),$$

for all $t \in [\kappa, \infty)$, and hence

$$\sup_{t \in (0,\infty)} \Gamma[\mu](\underline{0}, t^2) < \infty.$$

Inequality (4.3.22) now implies that $M_{HL}(\mu)(\underline{0})$ is finite. Boundedness of the sequence $\{L_j\}$ is now follows from the inequality (4.4.2). We take a convergent subsequence of $\{L_j\}$ and denote it also, for the sake of simplicity, by $\{L_j\}$. For $j \in \mathbb{N}$, we define

$$u_j(x,t) = u\left(\delta_{r_j}(x), r_j^2 t\right), \quad (x,t) \in G \times (0, \infty).$$

Then by Remark 4.3.10, i), $\{u_j\}$ is a sequence of solutions of the heat equation in $G\times(0,\infty)$. We claim that $\{u_j\}$ is locally bounded. To prove this claim, we choose a compact set $K\subset G\times(0,\infty)$. We consider the map

$$(x,t)\mapsto \frac{(d_{\mathcal{L}}(x))^2}{t},\quad (x,t)\in G\times(0,\infty).$$

Since $d_{\mathcal{L}}$ is continuous on G, this map is also continuous. As K is compact, image of K under this map is bounded and hence there exists a positive real number α such that

$$\frac{(d_{\mathcal{L}}(x))^2}{t}<\alpha,\quad \text{for all } (x,t)\in K.$$

In other words, $K\subset \mathrm{P}(\underline{0},\alpha)$. Using the invariance of $\mathrm{P}(\underline{0},\alpha)$ under the parabolic dilation (see Remark 4.3.10, ii)) and (4.3.22), it follows that

$$\sup_{j\in\mathbb{N}}\sup_{(x,t)\in K}u_j(x,t)\leq \sup_{(x,t)\in\mathrm{P}(\underline{0},\alpha)}u(x,t)\leq c_\alpha M_{HL}(\mu)(\underline{0})<\infty.$$

Hence, $\{u_j\}$ is locally bounded. Lemma 4.3.8 (generalization of Montel's theorem), now guarantees the existence of a subsequence $\{u_{j_k}\}$ of $\{u_j\}$, which converges normally to a positive solution v of the heat equation in $G\times(0,\infty)$. We now show that v is identically equal to L in $G\times(0,\infty)$. To show this, we take $(x_0,t_0)\in G\times(0,\infty)$, and choose a positive number η such that $(x_0,t_0)\in\mathrm{P}(\underline{0},\eta)$. Our hypothesis implies that

$$\lim_{\substack{t\to 0\\(x,t)\in\mathrm{P}(\underline{0},\eta)}}u(x,t)=L.$$

Since $\{r_{j_k}\}$ converges to zero as k goes to infinity, the equation above shows that

$$v(x_0,t_0)=\lim_{k\to\infty}u_{j_k}(x_0,t_0)=\lim_{k\to\infty}u\left(\delta_{r_{j_k}}(x_0),r_{j_k}^2t_0\right)=L,$$

as $(\delta_{r_{j_k}}(x_0),r_{j_k}^2t_0)\in\mathrm{P}(\underline{0},\eta)$, for all $k\in\mathbb{N}$. As $(x_0,t_0)\in G\times(0,\infty)$ is arbitrary, it follows that v is identically equal to L in $G\times(0,\infty)$. On the other hand, by Lemma 4.3.11, we have

$$u_{j_k}(x,t)=u\left(\delta_{r_{j_k}}(x),r_{j_k}^2t\right)=\Gamma[\mu]\left(\delta_{r_{j_k}}(x),r_{j_k}^2t\right)=\Gamma\left[\mu_{r_{j_k}}\right](x,t),\tag{4.4.3}$$

for all $(x,t)\in G\times(0,\infty)$, where $\mu_{r_{j_k}}$ is the dialte of μ according to (4.3.25).

Thus,

$$\Gamma\left[\mu_{r_{j_k}}\right] \to L = \Gamma[Lm],$$

normally as $k \to \infty$. Therefore, Lemma 4.3.1 implies that the sequence of measures $\{\mu_{r_{j_k}}\}$ converges to Lm in weak*, and hence by Lemma 4.3.3, $\{\mu_{r_{j_k}}(B)\}$ converges to $Lm(B)$ for every $d_{\mathcal{L}}$-ball $B \subset G$. Choosing $B = B_0$, it follows that

$$Lm(B_0) = \lim_{k\to\infty} \mu_{r_{j_k}}(B_0) = \lim_{k\to\infty} {r_{j_k}}^{-Q}\mu\left(\delta_{r_{j_k}}(B_0)\right) = m(B_0)\lim_{k\to\infty} \frac{\mu\left(\delta_{r_{j_k}}(B_0)\right)}{m\left(\delta_{r_{j_k}}(B_0)\right)}.$$

This implies, together with (4.4.1), that the sequence $\{L_{j_k}\}$ converges to L and hence so does $\{L_j\}$, as $\{L_j\}$ is convergent. Thus, every convergent subsequence of the bounded sequence $\{L_j\}$ converges to L. This implies that $\{L_j\}$ itself converges to L. Since B_0 and $\{r_j\}$ is arbitrary, μ has strong derivative L at $\underline{0}$.

Conversely, suppose that the strong derivative of μ at $\underline{0}$ is equal to L. We fix a positive number α and a sequence $\{(x_j, t_j^2)\} \subset \mathrm{P}(\underline{0}, \alpha)$, such that $\{t_j\}$ converges to zero. Since $D\mu(\underline{0})$ equals to L, it follows, in particular, that

$$\lim_{r\to 0}\frac{\mu(B(\underline{0}, r))}{m(B(\underline{0}, r))} = L.$$

Thus, we get for some $\kappa \in (0, \infty)$, such that

$$\sup_{r\in(0,\kappa)}\frac{\mu(B(\underline{0}, r))}{m(B(\underline{0}, r))} < L + 1.$$

Finiteness of the measure μ implies that

$$\frac{\mu(B(\underline{0}, r))}{m(B(\underline{0}, r))} \leq \frac{\mu(G)}{m(B(\underline{0}, 1))\kappa^Q},$$

for all $r \in [\kappa, \infty)$. The above inequalities, together with (4.3.22), shows that

$$\sup_{(x,t)\in\mathrm{P}(\underline{0},\alpha)} u(x,t) = \sup_{(x,t)\in\mathrm{P}(\underline{0},\alpha)} \Gamma[\mu](x,t) \leq c_\alpha M_{HL}(\mu)(\underline{0}) < \infty.$$

In particular, $\{u(x_j, t_j^2)\}$ is a bounded sequence. We now consider a convergent subsequence of this sequence, denote it also, for the sake of simplicity, by $\{u(x_j, t_j^2)\}$ such that

$$\lim_{j\to\infty} u(x_j, t_j^2) = L'. \tag{4.4.4}$$

We will prove that L' is equal to L. Using the sequence $\{t_j\}$, we consider

$$u_j(x,t) = u\left(\delta_{t_j}(x), t_j^2 t\right), \quad (x,t) \in G \times (0,\infty).$$

Arguments used in the first part of the proof shows that $\{u_j\}$ is a locally bounded sequence of positive solutions of the heat equation (4.2.9) in $G \times (0,\infty)$. Hence, by Lemma 4.3.8, there exists a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ which converges normally to a positive solution v of the heat equation in $G \times (0,\infty)$. Lemma 4.3.9 now shows that there exists a positive measure $\nu \in \mathcal{M}$, such that v is equal to $\Gamma[\nu]$. We consider the sequence of dilates $\{\mu_{j_k}\}$ of μ by $\{t_{j_k}\}$ according to (4.3.25). An application of Lemma 4.3.11 then implies that $\Gamma[\mu_{j_k}]$ equals u_{j_k}. Thus, the sequence of functions $\{\Gamma[\mu_{j_k}]\}$ converges normally to $\Gamma[\nu]$. Applying Lemma 4.3.1, we then conclude that $\{\mu_{j_k}\}$ converges to ν in weak*. Since $D\mu(\underline{0}) = L$, it follows that for any $d_{\mathcal{L}}$-ball $B \subset G$,

$$\lim_{k\to\infty} \mu_{j_k}(B) = \lim_{k\to\infty} {t_{j_k}}^{-Q} \mu(\delta_{t_{j_k}}(B)) = \lim_{k\to\infty} \frac{\mu(\delta_{t_{j_k}}(B))}{m(\delta_{t_{j_k}}(B))} m(B) = Lm(B).$$

Hence, by Lemma 4.3.3, $\nu = Lm$. As $v = \Gamma[\nu]$, it follows that

$$v(x,t) = L, \quad \text{for all} \quad (x,t) \in G \times (0,\infty).$$

This, in turn, implies that $\{u_{j_k}\}$ converges to the constant function L normally in $G \times (0,\infty)$. On the other hand, we note that

$$u(x_{j_k}, t_{j_k}^2) = u\left(\delta_{t_{j_k}}\left(\delta_{t_{j_k}^{-1}}(x_{j_k})\right), t_{j_k}^2\right) = u_{j_k}\left(\delta_{t_{j_k}^{-1}}(x_{j_k}), 1\right).$$

Since $(x_{j_k}, t_{j_k}^2)$ belongs to the parabolic region $\mathrm{P}(\underline{0}, \alpha)$, for all $k \in \mathbb{N}$, it follows that

$$\left(\delta_{t_{j_k}^{-1}}(x_{j_k}), 1\right) \in \overline{B(\underline{0}, \sqrt{\alpha})} \times \{1\},$$

which is a compact subset of $G \times (0, \infty)$. Therefore,

$$\lim_{k \to \infty} u(x_{j_k}, t_{j_k}^2) = \lim_{k \to \infty} u_{j_k}\left(\delta_{t_{j_k}^{-1}}(x_{j_k}), 1\right) = L,$$

as the convergence is uniform on $\overline{B(\underline{0}, \alpha)} \times \{1\}$. In view of (4.4.4), we can thus conclude that L' equals L. So, every convergent subsequence of the original sequence $\{u(x_j, t_j^2)\}$ converges to L. This shows that the original sequence $\{u(x_j, t_j^2)\}$ converges to L. As the positive number α and the sequence $\{(x_j, t_j^2)\} \subset \mathrm{P}(\underline{0}, \alpha)$ is arbitrary, u has parabolic limit L at $\underline{0}$. $\square$

The following is the main result of this chapter.

Theorem 4.4.2. *Suppose that u is a positive solution of the heat equation*

$$\mathcal{H}u(x, t) = 0, \quad (x, t) \in G \times (0, T),$$

for some $T \in (0, \infty]$, and that $x_0 \in G$, $L \in [0, \infty)$. If μ is the boundary measure of u then the following statements are equivalent.

(i) u has parabolic limit L at x_0.

(ii) μ has strong derivative L at x_0.

Proof. We consider the translated measure μ_0 given by

$$\mu_0(E) = \mu(x_0 \circ E),$$

for all Borel subsets $E \subset G$, where

$$x_0 \circ E = \{x_0 \circ \xi \mid \xi \in E\}.$$

Using translation invariance of m, it follows from the definition of strong derivative (see Definition 4.2.19, iii)) that $D\mu_0(\underline{0})$ and $D\mu(x_0)$ are equal. On the other hand, for a Borel set $E \subset G$,

$$\int_G \chi_E(\xi)\, d\mu_0(\xi) = \mu_0(E) = \mu(x_0 \circ E) = \int_G \chi_{(x_0 \circ E)}(\xi)\, d\mu(\xi) = \int_G \chi_E(x_0^{-1} \circ \xi)\, d\mu(\xi).$$

Hence, for $(x,t) \in G \times (0,T)$

$$\Gamma[\mu_0](x,t) = \int_G \Gamma(\xi^{-1} \circ x, t)\, d\mu_0(\xi) = \int_G \Gamma((x_0^{-1} \circ \xi)^{-1} \circ x, t)\, d\mu(\xi) = \Gamma[\mu](x_0 \circ x, t).$$

We fix an arbitrary positive number α. As $(x,t) \in \mathrm{P}(\underline{0}, \alpha)$, if and only if $(x_0 \circ x, t) \in \mathrm{P}(x_0, \alpha)$ (see the Definition 4.2.19, i)), one infers from the above that

$$\lim_{\substack{t \to 0 \\ (x,t) \in \mathrm{P}(\underline{0},\alpha)}} \Gamma[\mu_0](x,t) = \lim_{\substack{t \to 0 \\ (\xi,t) \in \mathrm{P}(x_0,\alpha)}} \Gamma[\mu](\xi,t).$$

Hence, it suffices to prove the theorem under the assumption that x_0 is the identity element $\underline{0}$. We will now show that we can take μ to be a finite measure. Let $\tilde{\mu}$ be the restriction of μ on the $d_{\mathcal{L}}$-ball $B(\underline{0}, \tau^{-1})$. Suppose $B(y,s)$ is any given $d_{\mathcal{L}}$-ball. Then for all positive number r smaller than $(\tau^2(s + d_{\mathcal{L}}(y)))^{-1}$, it follows that $\delta_r(B(y,s))$ is a subset of $B(\underline{0}, \tau^{-1})$. Indeed, for any $\xi \in \delta_r(B(y,s)) = B(\delta_r(y), rs)$,

$$d_{\mathcal{L}}(0,\xi) \le \tau\,(d_{\mathcal{L}}(0,\delta_r(y)) + d_{\mathcal{L}}(\delta_r(y),\xi)) \le \tau\,(r d_{\mathcal{L}}(y) + rs) < \tau^{-1},$$

for all $r \in (0, (\tau^2(s + d_{\mathcal{L}}(y)))^{-1})$. This in turn implies that $D\mu(\underline{0})$ and $D\tilde{\mu}(\underline{0})$ are equal. We now claim that

$$\lim_{\substack{t \to 0 \\ (x,t) \in \mathrm{P}(\underline{0},\alpha)}} \Gamma[\mu](x,t) = \lim_{\substack{t \to 0 \\ (x,t) \in \mathrm{P}(\underline{0},\alpha)}} \Gamma[\tilde{\mu}](x,t). \tag{4.4.5}$$

In this regard, we first show that

$$\lim_{t \to 0} \int_{B(\underline{0},\tau^{-1})^c} \Gamma(\xi^{-1} \circ x, t)\, d\mu(\xi) = 0, \tag{4.4.6}$$

uniformly for $x \in B(\underline{0}, 1/(2\tau^2))$. For this, we first note that for $x \in B(\underline{0}, 1/(2\tau^2))$, and $\xi \in B(\underline{0}, 1/\tau)^c$, we have

$$d_{\mathcal{L}}(x) < \frac{1}{2\tau^2} < \frac{1}{2\tau} d_{\mathcal{L}}(\xi),$$

and hence using the reverse triangle inequality (4.2.15), we get

$$d_{\mathcal{L}}(\xi^{-1} \circ x) \ge \frac{1}{\tau} d_{\mathcal{L}}(\xi) - d_{\mathcal{L}}(x) \ge \frac{d_{\mathcal{L}}(\xi)}{\tau} - \frac{d_{\mathcal{L}}(\xi)}{2\tau} = \frac{d_{\mathcal{L}}(\xi)}{2\tau} \ge \frac{1}{2\tau^2}.$$

We fix $t_0 \in (0,T)$. Using the Gaussian estimate (4.2.10) and the inequality above, we get

$$\begin{aligned}
& \int_{B(\underline{0},\tau^{-1})^c} \Gamma(\xi^{-1}\circ x,t)\,d\mu(\xi) \\
\leq\ & c_0 t^{-\frac{Q}{2}} \int_{B(\underline{0},\tau^{-1})^c} \exp\left(-\frac{\left(d_{\mathcal{L}}(\xi^{-1}\circ x)\right)^2}{c_0 t}\right) d\mu(\xi) \\
=\ & c_0 t^{-\frac{Q}{2}} \int_{B(\underline{0},\tau^{-1})^c} \exp\left(-\frac{\left(d_{\mathcal{L}}(\xi^{-1}\circ x)\right)^2}{2c_0 t}\right) \exp\left(-\frac{\left(d_{\mathcal{L}}(\xi^{-1}\circ x)\right)^2}{2c_0 t}\right) d\mu(\xi) \\
\leq\ & c_0 t^{-\frac{Q}{2}} \exp\left(-\frac{1}{8c_0\tau^4 t}\right) \int_{B(\underline{0},\tau^{-1})^c} \exp\left(-\frac{\left(d_{\mathcal{L}}(\xi^{-1}\circ x)\right)^2}{2c_0 t}\right) d\mu(\xi) \\
\leq\ & c_0 t^{-\frac{Q}{2}} \exp\left(-\frac{1}{8c_0\tau^4 t}\right) \int_{B(\underline{0},\tau^{-1})^c} \exp\left(-\frac{(d_{\mathcal{L}}(\xi))^2}{8c_0\tau^2 t}\right) d\mu(\xi) \\
\leq\ & c_0 t^{-\frac{Q}{2}} \exp\left(-\frac{1}{8c_0\tau^4 t}\right) \int_{B(\underline{0},\tau^{-1})^c} \exp\left(-\frac{2c_0(d_{\mathcal{L}}(\xi))^2}{t_0}\right) d\mu(\xi),
\end{aligned}$$

for all $t \in (0,(16c_0^2\tau^2)^{-1}t_0)$. Since μ is a positive measure and $\Gamma[\mu](\underline{0},t_0/2)$ is finite, the Gaussian estimate (4.2.10) implies that the integral on the right-hand side in the last inequality is finite. Hence, letting t goes to zero on the right-hand side in the last inequality, our desired equation (4.4.6) follows. Now,

$$\Gamma[\mu](x,t) \ = \ \Gamma[\tilde{\mu}](x,t) + \int_{B(\underline{0},\tau^{-1})^c} \Gamma(\xi^{-1}\circ x,t)\,d\mu(\xi).$$

Given any $\epsilon > 0$, using (4.4.6), we get some $t_1 \in (0,(16c_0^2\tau^2)^{-1}t_0)$, such that

$$0 \leq \Gamma[\mu](x,t) - \Gamma[\tilde{\mu}](x,t) = \int_{B(\underline{0},\tau^{-1})^c} \Gamma(\xi^{-1}\circ x,t)\,d\mu(\xi) < \epsilon,$$

for all $(x,t) \in B(\underline{0},1/(2\tau^2)) \times (0,t_1)$. On the other hand, we observe that

$$\mathrm{P}(\underline{0},\alpha) \cap \{(x,t) \mid t \in (0,1/(4\alpha\tau^4))\} \subset B(\underline{0},1/(2\tau^2)) \times (0,1/(4\alpha\tau^4)).$$

Hence, for all $(x,t) \in \mathrm{P}(\underline{0},\alpha)$, with $t \in (0,\min\{t_1,1/(4\alpha\tau^4)\})$, we have

$$\Gamma[\mu](x,t) - \Gamma[\tilde{\mu}](x,t) < \epsilon.$$

This proves (4.4.5). Therefore, as $\alpha \in (0,\infty)$ is arbitrary, we may and do suppose that μ is a finite measure. Using this, without loss of generality, we may also assume that $T = \infty$. The proof now follows from Theorem 4.4.1. $\square$

Remark 4.4.3. It is not known to us whether a result analogous to Corollary 3.2.11 holds true for solutions of the heat equation (4.2.9). However, if it turns out to be true that a nonzero solution of (4.2.9) can not vanish on any open set in $G \times (0, \infty)$, then one can actually prove a stronger version of Theorem 4.4.2 in the sense that if there exists some $\eta \in (0, \infty)$, such that

$$\lim_{\substack{t \to 0 \\ (x,t) \in \mathrm{P}(x_0, \eta)}} u(x,t) = L,$$

then μ has strong derivative L at x_0, where u is a positive solution of the heat equation and μ is the boundary measure of u.

Chapter 5

Differentiability of measures and admissible convergence on stratified Lie groups

In this chapter, we generalize a theorem of Victor L. Shapiro [Sha06] concerning nontangential convergence of the Poisson integral of an L^p-function on $\mathbb{R}^n$. Following Shapiro we introduce the notion of σ-points of a measure on a stratified Lie group and consider convolution integrals for a fairly general class of convolution kernels. We show that convolution integrals of a measure have admissible limits at σ-points of the measure. We also investigate the relationship between σ-point and strong derivative. We prove that these two notions are the same in $\mathbb{R}$.

5.1 Introduction

We recall that a point $x_0 \in \mathbb{R}^n$, is called a Lebesgue point of a measure μ on $\mathbb{R}^n$, if there exists $L \in \mathbb{C}$, such that

$$\lim_{r\to 0} \frac{|\mu - Lm|(B(x_0, r))}{m(B(0, r))} = 0. \tag{5.1.1}$$

In this case, it follows that $D_{sym}\mu(x_0)$ is equal to L. The set of all Lebesgue points of a measure μ on $\mathbb{R}^n$, is called the Lebesgue set of μ and is denoted by $\mathrm{L}_n(\mu)$ (see 1.0.17, i)). It is not very hard to see that the Lebesgue set of a measure μ includes almost all (with respect to the Lebesgue measure) points of $\mathbb{R}^n$. It is a classical result that if $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then the Poisson integral Pf of f has nontangential limit $f(x_0)$ at each Lebesgue point x_0 of f (see [SW71, Theorem 3.16]). In [Sae96, Theorem 1.5], Saeki generalized this result for more general class of kernels as well as for measures instead of L^p-functions. A natural question which arises here is the following.

Question: Does there exist $x_0 \in \mathbb{R}^n$, and $f \in L^r(\mathbb{R}^n)$, for some $r \in [1, \infty]$, such that x_0 is not a Lebesgue point of f but the Poisson integral Pf of f has nontangential limit at x_0?

As we have already mentioned in the introduction that Shapiro answered this question in the affirmative by introducing the notion of σ-point of a locally integrable function, which we recall next.

Definition 5.1.1 ([Sha06, P.3182]). A point $x_0 \in \mathbb{R}^n$, is called a σ-point of a locally integrable function f on $\mathbb{R}^n$, if for each $\epsilon > 0$, there exists $\delta > 0$, such that

$$\left| \int_{B(x,r)} \left(f(\xi) - f(x_0)\right)\, dm(\xi) \right| < \epsilon(\|x - x_0\| + r)^n,$$

whenever $\|x - x_0\| < \delta$, and $r \in (0, \delta)$.

The set of all σ-points of f is called the σ-set of f and is denoted by $\Sigma_n(f)$. As observed by Shapiro, the Lebesgue set of a locally integrable function f defined on $\mathbb{R}^n$, is contained in $\Sigma_n(f)$ (see [Sha06, P.3182]). It was also shown that the containment is strict for some particular functions. In fact, Shapiro [Sha06, Section 3] constructed a function $f \in L^p(\mathbb{R}^2)$, $p \in [1, \infty]$, such that $0 \in \Sigma_2(f)$, but it is not a Lebesgue point of f. Our main aim in this chapter is to obtain variants of the following result of Shapiro for a general convolution integral of the form $\phi[\mu]$ of some measure μ on stratified Lie groups.

Theorem 5.1.2 ([Sha06, Theorem 1]). *Let $f \in L^p(\mathbb{R}^n)$, for some $p \in [1, \infty]$. If $x_0 \in \Sigma_n(f)$, then Pf has nontangenial limit $f(x_0)$ at x_0.*

Shapiro also proved existence of nontangential limits of Gauss-Weierstrass integral of an L^p-function at σ-points of the function [Sha06, Theorem 2]. For $\mathbb{R}^n$, results of this kind has already been proved in [EH06] (a paper which generalizes earlier results in this regard proved by Brossard-Chevalier [BC90]). The author of [EH06] has dealt with differentiation of measures with respect to more general positive measures than the Lebesgue measure of $\mathbb{R}^n$. Perhaps due to this reason, the condition analogous to the comparison condition (1.0.6) used in [EH06] turned out to be stronger than what is needed for the Lebesgue measure. We will explain this difference in Example 5.4.1.

The organization of this chapter is as follows. In the next section, we will define the notion of Lebesgue point and σ-point of a measure on a stratified Lie group and prove a variant of Theorem 5.1.2. We will also discuss the relationship between the strong derivative and σ-point in section 3. In the last section, we will discuss two examples. Our first example will show that the set of all Lebesgue points of a measure on the Heisenberg group is strictly contained in that of all σ-points of the measure. The second one will show that the condition (5.4.1) analogous to the comparison condition (1.0.6) used in [EH06, Theorem 3.4] is much stronger than what is actually needed for the Lebesgue measure.

5.2 Admissible convergence of convolution integrals

Throughout this chapter, we fix a stratified Lie group G with the homogeneous norm d and identity element $\underline{0}$. We recall that G has the following vector space decomposition

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_l.$$

We denote by k, and Q the Euclidean and homogeneous dimensions of G respectively. In other words,

$$\mathrm{k} = \sum_{j=1}^{l} \text{dim } V_j; \quad Q = \sum_{j=1}^{l} j(\text{dim } V_j).$$

We denote the Euclidean norm on $\mathbb{R}^{\mathrm{k}}$ ($\cong \mathfrak{g}$ as vector spaces) by $\|\cdot\|$, and the Euclidean open ball centered at $x \in G$, and radius $r \in (0,\infty)$ by $B_e(x,r)$, that is

$$B_e(x,r) = \{y \in G \mid \|y - x\| < r\}.$$

We denote the d-ball (see 4.2.5) centered at $x \in G$, with radius $s \in (0,\infty)$ by $B_d(x,s)$. We recall that the Lebesgue measure m of $\mathbb{R}^{\mathrm{k}}$ is the Haar measure of G. Let us start by defining the notions of Lebesgue point and σ-point of a measure on a stratified Lie group.

Definition 5.2.1. Let μ be a measure on G and $x_0 \in G$.

i) The point x_0 is called a Lebesgue point of μ if there exists $L \in \mathbb{C}$, such that

$$\lim_{r\to 0} \frac{|\mu - Lm|(B_d(x_0, r))}{m(B_d(\underline{0}, r))} = 0. \tag{5.2.1}$$

As usual, we call the set of all Lebesgue points of μ as the Lebesgue set of μ, and denote it by $\mathrm{L}_G(\mu)$.

ii) The point x_0 is called a σ-point of μ if there exists $L \in \mathbb{C}$, such that for each $\epsilon > 0$, there exists $\delta > 0$, satisfying

$$|(\mu - Lm)(B_d(x, r))| < \epsilon(d(x_0^{-1} \circ x) + r)^Q,$$

whenever $d(x_0^{-1} \circ x) < \delta$, and $r \in (0, \delta)$. In this case, we will denote the complex number L by $D_\sigma\mu(x_0)$. The set of all σ-points of μ is called the σ-set of μ, and is denoted by $\Sigma_G(\mu)$.

Remark 5.2.2. We remark the following.

i) It is known that $m\left(\mathrm{L}_G(\mu)^c\right)$ is zero (see first few lines of the proof of [BU05, Thorem 2.4]). If x_0 is a Lebesgue point of μ with L as in (5.2.1), then the strong derivative (see Definition 4.2.19, iii)) of μ at x_0 exists and equals L. Indeed, we take a d-ball $B = B_d(x, t)$ in G. Then

$$\begin{aligned}
\left|\frac{\mu(x_0 \circ \delta_r(B))}{m(x_0 \circ \delta_r(B))} - L\right| &= \left|\frac{\mu\left(B_d(x_0 \circ \delta_r(x), rt\right) - Lm\left(B_d(x_0 \circ \delta_r(x), rt\right)}{m(B_d\left(\underline{0}, rt)\right)}\right| \\
&\leq \frac{|\mu - Lm|\left(B_d(x_0 \circ \delta_r(x), rt\right)}{m\left(B_d(\underline{0}, rt)\right)} \\
&\leq \frac{|\mu - Lm|\left(B_d(x_0, \tau r(t + d(x))\right)}{m\left(B_d(\underline{0}, rt)\right)} \\
&\leq \frac{|\mu - Lm|\left(B_d(x_0, \tau r(t + d(x))\right)}{m\left(B_d(\underline{0}, \tau r(t + d(x)))\right)} \times \left(\frac{\tau r(t + d(x))}{rt}\right)^Q,
\end{aligned}$$

where τ is the constant C_d in the triangle inequality (4.2.2). Using (5.2.1), we see that the right-hand side of the last inequality goes to zero as r goes to zero. As the d-ball B is arbitrary, $D\mu(x_0)$ is equal to L.

ii) For a measure μ on G, we have the containment

$$\mathrm{L}_G(\mu) \subset \Sigma_G(\mu).$$

Moreover,

$$D_\sigma\mu(x_0) = D\mu(x_0), \quad \text{for all} \;\; x_0 \in \mathrm{L}_G(\mu).$$

To see this, we take $x_0 \in \mathrm{L}_G(\mu)$, and fix $\epsilon > 0$. By the definition of Lebesgue point (5.2.1), there exists $\delta > 0$, such that

$$|\mu - Lm|(B_d(x_0, r)) < \frac{\epsilon}{\tau^Q m(B_d(0,1))} m(B_d(x_0, r)) = \frac{\epsilon}{\tau^Q} r^Q,$$

whenever $r \in (0, \delta)$, where $L = D\mu(x_0)$. This implies that

$$\begin{aligned} |(\mu - Lm)(B_d(x,r))| &\leq |\mu - Lm|(B_d(x,r)) \\ &\leq |\mu - Lm| \left(B_d(x_0, \tau(d(x_0^{-1} \circ x) + r))\right) \\ &< \epsilon(d(x_0^{-1} \circ x) + r)^Q, \end{aligned}$$

whenever $r + d(x_0^{-1} \circ x) < \delta$. This shows that $x_0 \in \Sigma_G(\mu)$, and that

$$D_\sigma\mu(x_0) = L = D\mu(x_0).$$

We define a notion analogous to nontangential convergence in the context of stratified Lie groups as follows:

Definition 5.2.3. A function u defined on $G \times (0, t_0)$, for some $t_0 \in (0, \infty]$, is said to have admissible limit $L \in \mathbb{C}$, at $x_0 \in G$, if for each $\alpha \in (0, \infty)$,

$$\lim_{\substack{t \to 0 \\ (x,t) \in \mathrm{S}(x_0, \alpha)}} u(x,t) = L,$$

where

$$\begin{aligned} \mathrm{S}(x_0, \alpha) &= \{(x,t) \in S \mid d(x_0^{-1} \circ x) < \alpha t\} \\ &= \{(x,t) \in S \mid \mathbf{d}(x_0, x) < \alpha t\}. \end{aligned} \tag{5.2.2}$$

is called the admissible domain with vertex at x_0 and aperture α.

Given a measure μ and a complex-valued function ϕ on G, we define the convolution integral $\phi[\mu](x,t)$ by

$$\phi[\mu](x,t) = \mu * \phi_t(x) = t^{-Q} \int_G \phi\left(\delta_{\frac{1}{t}}(\xi^{-1} \circ x)\right) \, d\mu(\xi), \tag{5.2.3}$$

whenever the integral converges absolutely for $(x,t) \in \mathbb{R}^{n+1}_+$. If the integral above converges absolutely for all $(x,t) \in E$, where $E \subseteq G \times (0,\infty)$, we say that $\phi[\mu]$ is well-defined in E.

Remark 5.2.4. From the proof of Lemma 4.2.14 it is clear that if μ is a measure on G, and $\phi : G \to (0,\infty)$, is a d-radially decreasing function on G (see (4.2.17)), then finiteness of $|\mu| * \phi_{t_0}(x_0)$ for some $(x_0,t_0) \in G \times (0,\infty)$, implies that $\phi[\mu]$ is well-defined in $G \times (0, t_0/\tau)$.

Although σ-point seems to be a natural generalization of the Lebesgue point, it does not reflect the inherent notion of admissible convergence. To bring that out, we introduce the notion of χ-point. This will help us to understand the characteristic of the σ-point in light of the theme of this chapter.

Definition 5.2.5. Let μ be a measure on G. A point $x_0 \in G$, is called a χ-point of μ if there exists $L \in \mathbb{C}$, such that for every $\alpha \in (0,\infty)$,

$$\lim_{\substack{r \to 0 \\ (x,r) \in \mathrm{S}(x_0,\alpha)}} \frac{\mu(B_d(x,r))}{m(B_d(x,r))} = L.$$

In this case, we will denote the complex number L by $D_\chi \mu(x_0)$. The set of all χ-points of μ is called the χ-set of μ, and is denoted by $X_G(\mu)$.

Remark 5.2.6. As

$$\mu * (\chi_{B_d(\underline{0},1)})_r(x) = \frac{\mu(B_d(x,r))}{m(B_d(x,r))}, \quad (x,r) \in (0,\infty),$$

the χ-points of μ are precisely those points where the convolution integral $\chi_{B(0,1)}[\mu]$ has admissible limit.

Lemma 5.2.7. *For a measure μ on G, we have*

$$\Sigma_G(\mu) \subseteq X_G(\mu).$$

Moreover,

$$D_\chi\mu(x_0) = D_\sigma\mu(x_0), \quad \textit{whenever } x_0 \in \Sigma_G(\mu).$$

Proof. Let $x_0 \in \Sigma_G(\mu)$. We fix $\epsilon > 0$, and $\alpha_0 \geq 1$. Then there exists some $\delta > 0$, such that

$$|(\mu - D_\sigma\mu(x_0)m)(B_d(x,r))| < \frac{m(B(0,1))\epsilon}{(2\alpha_0)^Q}(d(x_0^{-1}\circ x) + r)^Q, \tag{5.2.4}$$

whenever $d(x_0^{-1}\circ x) < \delta$, and $r \in (0,\delta)$.

Now, for any $(x,r) \in \mathrm{S}(x_0,\alpha_0)$, we have

$$\begin{aligned}
&\left|\frac{\mu(B_d(x,r))}{m(B_d(x,r))} - D_\sigma\mu(x_0)\right| \\
= \;& \frac{|(\mu - D_\sigma\mu(x_0)m)(B_d(x,r))|}{m(B_d(x,r))} \\
= \;& \frac{|(\mu - D_\sigma\mu(x_0)m)(B_d(x,r))|}{m(B_d(\underline{0},1))(\frac{r}{2}+\frac{r}{2})^Q} \\
\leq \;& \frac{|(\mu - D_\sigma\mu(x_0)m)(B(x,r))|}{m(B(\underline{0},1))(\frac{d(x_0^{-1}\circ x)}{2\alpha_0}+\frac{r}{2})^Q} && \text{(as } (x,r) \in S(x_0,\alpha_0)\text{)} \\
= \;& \frac{(2\alpha_0)^Q}{m(B_d(\underline{0},1))}\frac{|(\mu - D_\sigma\mu(x_0)m)(B_d(x,r))|}{(d(x_0^{-1}\circ x)+\alpha_0 r)^Q} \\
\leq \;& \frac{(2\alpha_0)^Q}{m(B_d(\underline{0},1))}\frac{|(\mu - D_\sigma\mu(x_0)m)(B_d(x,r))|}{(d(x_0^{-1}\circ x)+r)^Q},
\end{aligned}$$

where the last inequality follows from the fact that $\alpha_0 \geq 1$. Since

$$d(x_0^{-1}\circ x) < \alpha_0 r < \delta,$$

whenever $r \in (0,\delta/\alpha_0)$, it follows from (5.2.4) that

$$\left|\frac{\mu(B_d(x,r))}{m(B_d(x,r))} - D_\sigma\mu(x_0)\right| < \epsilon,$$

whenever $(x,r) \in \mathrm{S}(x_0,\alpha_0)$, with $r \in (0,\delta/\alpha_0)$. As $\alpha_0 \geq 1$, is arbitrary, we have x_0 is a χ-point of μ with

$$D_\chi\mu(x_0) = D_\sigma\mu(x_0). \qquad \square$$

We say that a function $\phi : G \to (0,\infty)$, satisfies the comparison condition if

$$\sup \left\{ \frac{\phi_t(x)}{\phi(x)} \mid t \in (0,1),\ d(x) \geq 1 \right\} < \infty. \tag{5.2.5}$$

Remark 5.2.8. As in the case of Euclidean spaces (see Example 2.1.2), one can show that the following functions satisfy the comparison condition (5.2.5).

$$\frac{1}{(1+d(x)^2)^{\alpha}\log(2+d(x)^{\kappa})}, \quad e^{-\epsilon d(x)^{\beta}}, \quad x \in G,$$

where $\alpha \geq [Q/2,\infty)$, $\kappa \in [0,\infty)$, and ϵ, β are positive numbers.

The following lemma shows that with the aid of the comparison condition (5.2.5) it is sufficient to discuss admissible convergence of convolution integrals of measures with finite total variation.

Lemma 5.2.9. *Suppose that $\phi : G \to (0,\infty)$, is a d-radial, d-radially decreasing, integrable function. Furthermore, suppose that ϕ satisfies the comparison condition (5.2.5) and that μ is a measure on G, such that $|\mu| * \phi_{t_0}(\underline{0})$ is finite for some $t_0 \in (0,\infty)$. Let $\tilde{\mu}$ be the restriction of μ on the d-ball $B_d(\underline{0}, t_0/\tau)$. Then we have the following.*

i) For all $\alpha \in (0,\infty)$,

$$\lim_{\substack{t\to 0\\ (x,t)\in \mathrm{S}(\underline{0},\alpha)}} \mu * \phi_t(x) = \lim_{\substack{t\to 0\\ (x,t)\in \mathrm{S}(\underline{0},\alpha)}} \tilde{\mu} * \phi_t(x), \tag{5.2.6}$$

provided one of the limits exist.

ii) If $\underline{0}$ is a σ-point of μ, then $\underline{0}$ is also a σ-point of $\tilde{\mu}$ and vice versa. In either case,

$$D_{\sigma}\mu(\underline{0}) = D_{\sigma}\tilde{\mu}(\underline{0}).$$

iii) If $\underline{0}$ is a χ-point of μ, then $\underline{0}$ is also a χ-point of $\tilde{\mu}$ and vice versa. In either case,

$$D_{\chi}\mu(\underline{0}) = D_{\chi}\tilde{\mu}(\underline{0}).$$

Proof. In view of Remark 5.2.4, without loss of generality we assume that $t_0/\tau < 1$. We write for $t \in (0, t_0/\tau)$, $x \in G$,

$$\mu * \phi_t(x) = \tilde{\mu} * \phi_t(x) + \int_{\{\xi \in G \mid d(\xi) \geq \frac{t_0}{\tau}\}} \phi_t(\xi^{-1} \circ x)\, d\mu(\xi). \tag{5.2.7}$$

Since ϕ is a d-radial, d-radially decreasing function, using integration formula in polar coordinate (4.2.6) we have for any $r \in (0, \infty)$,

$$\int_{\{x \in G \mid r/2 \leq d(x) \leq r\}} \phi(x)\, dx = C_Q \int_{r/2}^{r} \phi(s) s^{Q-1}\, ds \geq C_Q \phi(r) \int_{r/2}^{r} s^{Q-1}\, ds = C_Q' r^Q \phi(r).$$

Since ϕ is an integrable function, the integral on the left-hand side converges to zero as r goes to zero and infinity. Hence, it follows that

$$\lim_{d(x) \to 0} d(x)^Q \phi(x) = \lim_{d(x) \to \infty} d(x)^Q \phi(x) = 0. \tag{5.2.8}$$

We denote the integral appearing on the right-hand side of (5.2.7) by $I(x, t)$. We fix $\alpha \in (0, \infty)$. We observe that for $(x, t) \in \mathrm{S}(\underline{0}, \alpha)$, and $d(\xi) \geq t_0/\tau$,

$$d(x) < \alpha t < \alpha \frac{t_0}{2\tau^2 \alpha} \leq \frac{d(\xi)}{2\tau}, \quad \text{for all} \quad 0 < t < \min\left\{\frac{1}{2}, \frac{t_0}{2\tau^2\alpha}\right\}.$$

Using this and the reverse triangle inequality (4.2.15), we get that for $d(\xi) > t_0/\tau$, and $(x, t) \in \mathrm{S}(0, \alpha) \cap \left(G \times (0, \min\{\frac{1}{2}, \frac{t_0}{2\tau^2\alpha}\})\right)$,

$$d(\xi^{-1} \circ x) \geq \frac{d(\xi)}{\tau} - d(x) \geq \frac{d(\xi)}{\tau} - \frac{d(\xi)}{2\tau} = \frac{d(\xi)}{2\tau}.$$

Therefore, using the fact that ϕ is d-radially decreasing, we obtain for $(x, t) \in \mathrm{S}(0, \alpha) \cap \left(G \times (0, \min\{\frac{1}{2}, \frac{t_0}{2\tau^2\alpha}\})\right)$,

$$\begin{aligned}
\left| I\left(x, \frac{tt_0}{\tau^2}\right) \right| &= \left(\frac{tt_0}{\tau^2}\right)^{-Q} \left| \int_{\{\xi \in G \mid d(\xi) \geq \frac{t_0}{\tau}\}} \phi\left(\delta_{\frac{\tau^2}{tt_0}}\left(\xi^{-1} \circ x\right)\right) d\mu(\xi) \right| \\
&\leq \left(\frac{tt_0}{\tau^2}\right)^{-Q} \int_{\{\xi \in G \mid d(\xi) \geq \frac{t_0}{\tau}\}} \phi\left(\delta_{\frac{\tau^2}{tt_0}}\left(\xi^{-1} \circ x\right)\right) d|\mu|(\xi) \\
&\leq \left(\frac{tt_0}{\tau^2}\right)^{-Q} \int_{\{\xi \in G \mid d(\xi) \geq \frac{t_0}{\tau}\}} \phi\left(\delta_{\frac{\tau}{2tt_0}}(\xi)\right) d|\mu|(\xi)
\end{aligned}$$

$$= \tau^{2Q} \int_{\{\xi \in G \mid d(\xi) \geq \frac{t_0}{\tau}\}} \frac{\left(\frac{d(\xi)}{tt_0}\right)^Q \phi\left(\delta_{\frac{\tau}{2tt_0}}(\xi)\right)}{d(\xi)^Q \phi_{t_0}(\xi)} \times \phi_{t_0}(\xi)\, d|\mu|(\xi). \tag{5.2.9}$$

From (5.2.8) we get that

$$\lim_{t\to 0} \left(\frac{d(\xi)}{tt_0}\right)^Q \phi\left(\delta_{\frac{\tau}{2tt_0}}(\xi)\right) = 0,$$

for each fixed $\xi \in G$. On the other hand, by the comparison condition (5.2.5), there exists some positive constant C such that

$$\frac{\left(\frac{d(\xi)}{tt_0}\right)^Q \phi\left(\delta_{\frac{\tau}{2tt_0}}(\xi)\right)}{d(\xi)^Q \phi_{t_0}(\xi)} = 2^Q \frac{\phi_{2t}\left(\delta_{\frac{\tau}{t_0}}(\xi)\right)}{\phi\left(\delta_{\frac{1}{t_0}}(\xi)\right)} \leq 2^Q \frac{\phi_{2t}\left(\delta_{\frac{\tau}{t_0}}(\xi)\right)}{\phi\left(\delta_{\frac{\tau}{t_0}}(\xi)\right)} \leq C,$$

for all $d(\xi) \geq t_0/\tau$, $t \in (0, 1/2)$, as $\tau \geq 1$. Since $|\mu| * \phi_{t_0}(\underline{0})$ is finite, that is, $\phi_{t_0} \in L^1(\mathbb{R}^n, d|\mu|)$, by the dominated convergence theorem, it follows from (5.2.9) that

$$\lim_{\substack{t\to 0 \\ (x,t)\in \mathrm{S}(\underline{0},\alpha)}} \left| I\left(x, \frac{tt_0}{\tau^2}\right)\right| = 0. \tag{5.2.10}$$

We note that

$$\begin{aligned}
\lim_{\substack{t\to 0 \\ (x,t)\in \mathrm{S}(\underline{0},\alpha)}} I\left(x, \frac{\tau^2 t}{t_0} \times \frac{t_0}{\tau^2}\right) &= \lim_{\substack{t\to 0 \\ (x,t)\in \mathrm{S}(\underline{0},\alpha)}} I(x,t) \\
&= \lim_{\substack{t\to 0 \\ (x,t)\in \mathrm{S}(\underline{0},\alpha)}} \int_{\{\xi \in G \mid d(\xi) \geq \frac{t_0}{\tau}\}} \phi_t(\xi^{-1} \circ x)\, d\mu(\xi).
\end{aligned}$$

We have assumed that $t_0/\tau \in (0,1)$. Since $\tau \in [1, \infty)$, we thus have $\frac{\tau^2}{t_0} \in (1, \infty)$. Therefore, $\left(x, \frac{\tau^2 t}{t_0}\right) \in \mathrm{S}(\underline{0}, \alpha)$, whenever $(x,t) \in \mathrm{S}(\underline{0},\alpha)$. Hence, using (5.2.10) in the last equation, we obtain

$$\lim_{\substack{t\to 0 \\ (x,t)\in \mathrm{S}(\underline{0},\alpha)}} \int_{\{\xi \in G \mid d(\xi) \geq \frac{t_0}{\tau}\}} \phi_t(\xi^{-1} \circ x)\, d\mu(\xi) = 0. \tag{5.2.11}$$

In view of (5.2.7), this proves $i)$.

We now prove $ii)$. Suppose that $\underline{0}$ is a σ-point of μ with

$$D_\sigma \mu(\underline{0}) = L.$$

We fix $\epsilon > 0$. Then it follows from the definition of σ-point that there exists $\delta \in (0, t_0/(2\tau^2))$, such that for $d(x) < \delta$, and $r \in (0, \delta)$

$$|(\mu - Lm)(B_d(x,r))| < \epsilon(d(x) + r)^Q. \tag{5.2.12}$$

On the other hand, for $d(x) < \delta$, and $r \in (0, \delta)$, we have

$$B_d(x,r) \subset B_d(\underline{0}, 2\tau\delta) \subset B_d(\underline{0}, t_0/\tau),$$

as $\delta \in (0, t_0/(2\tau^2))$. Using the definition of $\tilde{\mu}$ in (5.2.12), we get that

$$|(\tilde{\mu} - Lm)(B_d(x,r))| = |(\mu - Lm)(B_d(x,r))| < \epsilon(d(x) + r)^Q,$$

whenever $d(x) < \delta$, and $r \in (0, \delta)$. This shows that the $\underline{0}$ is a σ-point of $\tilde{\mu}$. Moreover,

$$D_\sigma \tilde{\mu}(\underline{0}) = L = D_\sigma \mu(\underline{0}).$$

Proof of the converse implication is similar.

To prove $iii)$, we fix $\alpha \in (0, \infty)$. We observe that for $(x, r) \in \mathrm{S}(\underline{0}, \alpha)$, and $\xi \in B_d(x, r)$

$$d(\xi) \leq \tau \left(d(x) + d(x^{-1} \circ \xi) \right) < \tau(r + \alpha r) < \frac{t_0}{\tau},$$

whenever $r \in \left(0, \dfrac{t_0}{\tau^2(\alpha+1)}\right)$. This shows that

$$\mu(B_d(x,r)) = \tilde{\mu}(B_d(x,r)),$$

for all $(x, r) \in \mathrm{S}(\underline{0}, \alpha)$, with $r \in \left(0, \dfrac{t_0}{\tau^2(\alpha+1)}\right)$. This proves $iii)$. □

Before proceeding to our next lemma, we recall that [Rud87, P.37], a real-valued function f on a topological space X is said to be lower semicontinuous if $\{x \in X : f(x) > s\}$ is open for every real number s.

Lemma 5.2.10. *Assume that $\phi : G \to [0, \infty)$, is a d-radial, d-radially decreasing, nonzero integrable function. If ϕ is lower semicontinuous then, for each $t \in (0, \phi(\underline{0}))$,*

$$\mathsf{B}_t = \{x \in G \mid \phi(x) > t\},$$

is a d-ball centred at $\underline{0}$ with some finite radius $\theta(t)$, which is a measurable function of t.

Proof. We fix $t \in (0, \phi(\underline{0}))$. Since ϕ is integrable, there exists $x_0 \in G$, such that $\phi(x_0) \le t$. For any $x \in \mathsf{B}_t$,

$$\phi(x) > t \ge \phi(x_0).$$

As ϕ is d-radial and d-radially decreasing, the inequality above implies that $d(x)$ is smaller than $d(x_0)$, and hence B_t is a bounded set. By the lower semicontinuity of ϕ, B_t is open. As $\underline{0} \in \mathsf{B}_t$, there exists some $s \in (0, \infty)$, such that

$$\overline{B_e(\underline{0}, s)} \subset \mathsf{B}_t.$$

Using Remark 4.2.3, we can find a positive number r_s such that

$$\overline{B_d(\underline{0}, r_s)} \subset \overline{B_e(\underline{0}, s)} \subset \mathsf{B}_t.$$

It now follows that

$$\theta(t) = \sup\{r > 0 \mid \overline{B_d(\underline{0}, r)} \subseteq \mathsf{B}_t\} \in (0, \infty).$$

We claim that $B_d(\underline{0}, \theta(t))$ is contained in B_t. To prove this claim, we choose $x \in B_d(\underline{0}, \theta(t))$. By the definition of $\theta(t)$, we get some $r_0 \in (d(x), \theta(t))$, such that

$$x \in \overline{B_d(\underline{0}, r_0)} \subseteq \mathsf{B}_t.$$

This proves our claim. We now observe that if $y \in \mathsf{B}_t \setminus \{\underline{0}\}$, then for all $\xi \in \overline{B_d(\underline{0}, d(y))}$, we have

$$\phi(\xi) \ge \phi(y) > t.$$

Thus,

$$\overline{B_d(\underline{0}, d(y))} \subseteq \mathsf{B}_t, \quad \text{for all} \quad y \in \mathsf{B}_t \setminus \{\underline{0}\}. \tag{5.2.13}$$

By the definition of $\theta(t)$, it follows from (5.2.13) that

$$d(y) \leq \theta(t), \quad \text{for all} \ \ y \in \mathsf{B}_t \setminus \{\underline{0}\}.$$

Consequently,

$$B_d(\underline{0}, \theta(t)) \subseteq B_t \subseteq \overline{B_d(\underline{0}, \theta(t))}. \tag{5.2.14}$$

Suppose that there exists some $\xi \in \mathsf{B}_t \setminus B_d(\underline{0}, \theta(t))$. Then by (5.2.14), $d(\xi)$ is equal to $\theta(t)$. Since ϕ is d-radial, this implies that

$$\mathsf{B}_t = \overline{B_d(\underline{0}, \theta(t))}.$$

Hence, as B_t is open, it follows from (5.2.14) that

$$\mathsf{B}_t = B_d(\underline{0}, \theta(t)). \qquad \square$$

As $s > t$ implies that $\mathsf{B}_s \subseteq \mathsf{B}_t$, it follows that θ is a decreasing function on $(0, \phi(\underline{0}))$, and hence measurable.

Remark 5.2.11. It follows from the proof above that if ϕ is not assumed to be lower semi-continuous, then B_t may turn out to be a closed ball centered at origin. This can be seen from the following example. We define $\phi : G \to (0, \infty)$, by

$$\phi(x) = \begin{cases} e^{-d(x)}, & d(x) \leq 1 \\ e^{-2d(x)}, & d(x) > 1. \end{cases}$$

Then for any $t \in (e^{-2}, e^{-1})$, we have

$$\mathsf{B}_t = \overline{B_d(\underline{0}, 1)}.$$

We are now ready to present the main result of this chapter.

Theorem 5.2.12. *Suppose that $\phi : G \to (0, \infty)$, satisfies the following conditions:*

1. *ϕ is a d-radial, d-radially decreasing, lower semicontinuous function with*
$$\int_G \phi(x)\, dm(x) = 1.$$

2. *ϕ satisfies the comparison condition (5.2.5).*

Suppose μ is a measure on G such that $|\mu| * \phi_{t_0}(x_1)$ is finite for some $x_1 \in G$, and $t_0 \in (0, \infty)$. If $x_0 \in X_G(\mu)$, with
$$D_\chi \mu(x_0) = L,$$
then $\phi[\mu]$ has admissible limit L at x_0.

Proof. As in the proof of Theorem 4.4.2, we consider the translated measure μ_0 given by
$$\mu_0(E) = \mu(x_0 \circ E),$$
for all Borel subsets $E \subset G$. Then for all $(x, t) \in G \times (0, t_0/\tau)$, we have
$$\mu_0 * \psi_t(x) = \int_G \psi_t(\xi^{-1} \circ x) d\mu_0(\xi) = \int_G \psi_t\left((x_0^{-1} \circ \xi)^{-1} \circ x\right) d\mu(\xi) = \mu * \psi_t(x_0 \circ x),$$
where ψ is either ϕ or $\chi_{B_d(0,1)}$. We fix an arbitrary positive number α. As $(x, t) \in \mathrm{S}(\underline{0}, \alpha)$, if and only if $(x_0 \circ x, t) \in \mathrm{S}(x_0, \alpha)$, We conclude from the equation above that
$$\lim_{\substack{t \to 0 \\ (x,t) \in \mathrm{S}(\underline{0}, \alpha)}} \psi[\mu_0](x, t) = \lim_{\substack{t \to 0 \\ (\xi,t) \in \mathrm{S}(x_0, \alpha)}} \psi[\mu](\xi, t),$$
provided one of the limits exist. Hence, it suffices to prove the theorem under the assumption that x_0 is the identity element $\underline{0}$. Applying Lemma 5.2.9 i), and iii) we can restrict μ on $B_d(\underline{0}, t_0/\tau)$, if necessary, to assume that $|\mu|(G)$ is finite. Since $D_\chi \mu(\underline{0}) = L$, we have, in particular, that
$$\lim_{r \to 0} \frac{\mu(B_d(\underline{0}, r))}{m(B_d(\underline{0}, r))} = L.$$
Therefore, there exists a positive number r_0 such that
$$\frac{|\mu(B_d(\underline{0}, r))|}{m(B_d(\underline{0}, r))} < L + 1, \quad \text{for all} \quad r \in (0, r_0).$$

Using finiteness of $|\mu|$, we get that

$$\frac{|\mu(B_d(\underline{0},r))|}{m(B_d(\underline{0},r))} \leq \frac{|\mu|(B_d(\underline{0},r))}{m(B_d(\underline{0},r))} \leq \frac{|\mu|(G)}{m(B_d(\underline{0},r_0))}, \quad \text{for all} \quad r \in [r_0,\infty).$$

Combining the above inequalities, we obtain

$$M_{HL}\mu(\underline{0}) = \sup_{r>0} \frac{|\mu(B_d(\underline{0},r))|}{m(B_d(\underline{0},r))} < \infty. \tag{5.2.15}$$

For each $t \in (0,\phi(\underline{0}))$, by Lemma 5.2.10, B_t is a d-ball centered at $\underline{0}$ with radius $\theta(t)$. We also note that for any $(x,r) \in G\times(0,\infty)$, and $t \in (0,\phi(\underline{0}))$,

$$\left\{\xi \in G \mid \phi\left(\delta_{\frac{1}{r}}(\xi^{-1}\circ x)\right) > t\right\} = B_d(x, r\theta(t)).$$

Let $\{(x_k,t_k) \mid k \in \mathbb{N}\}$ be a sequence in $\mathrm{S}(\underline{0},\alpha)$ such that $\{t_k\}$ converges to zero as k goes to infinity. Without loss of generality, we assume that $t_k \in (0,t_0/\tau)$, for all k. We have

$$\mu * \phi_{t_k}(x_k) = t_k^{-Q}\int_G \phi\left(\delta_{\frac{1}{t_k}}(\xi^{-1}\circ x_k)\right)\,d\mu(\xi) = t_k^{-Q}\int_G \left(\int_0^{\phi\left(\delta_{\frac{1}{t_k}}(\xi^{-1}\circ x_k)\right)} ds\right)\,d\mu(\xi).$$

As $|\mu| * \phi_t(x)$ is finite for all $(x,t) \in G\times(0,t_0/\tau)$, applying Fubini's theorem on the right hand side of the last equality, we obtain

$$\begin{aligned}
\mu * \phi_{t_k}(x_k) &= t_k^{-Q}\int_0^{\phi(\underline{0})} \mu\left(\left\{\xi \in G : \phi\left(\delta_{\frac{1}{t_k}}(\xi^{-1}\circ x_k)\right) > s\right\}\right)\,ds \\
&= t_k^{-Q}\int_0^{\phi(\underline{0})} \mu\left(B_d(x_k,t_k\theta(s))\right)\,ds \\
&= m(B(\underline{0},1))\int_0^{\phi(\underline{0})} \frac{\mu\left(B_d(x_k,t_k\theta(s))\right)}{m\left(B_d(x_k,t_k\theta(s))\right)}\theta(s)^Q\,ds.
\end{aligned} \tag{5.2.16}$$

Since $D_\chi\mu(\underline{0}) = L$, it follows from the definition of χ-point that for each $s \in (0,\phi(\underline{0}))$, the integrand in (5.2.16) has limit $Lm(B(\underline{0},1))\theta(s)^Q$, as k goes to infinity, because $(x_k,t_k) \in \mathrm{S}(\underline{0},\alpha)$, for all k. Moreover, using (5.2.15), the integrand in (5.2.16) is bounded by the function

$$s \mapsto m(B(\underline{0},1))M_{HL}\mu(\underline{0})\theta(s)^Q, \quad s \in (0,\phi(\underline{0})).$$

In order to apply the dominated convergence theorem in (5.2.16), we need to show that this function is integrable in $(0,\phi(\underline{0}))$. For this, it is enough to show that the function $s \mapsto \theta(s)^Q$, is integrable in $(0, \phi(\underline{0}))$. Using a well-known formula involving distribution functions [Rud87, Theorem 8.16], we observe that

$$\begin{aligned} 1 = \int_G \phi(x)\, dm(x) &= \int_0^{\phi(\underline{0})} m\left(\{x \in G \mid \phi(x) > s\}\right)\, ds \\ &= \int_0^{\phi(\underline{0})} m(\mathsf{B}_s)\, ds \\ &= m(B_d(\underline{0}, 1)) \int_0^{\phi(\underline{0})} \theta(s)^Q\, ds. \end{aligned} \tag{5.2.17}$$

Hence, applying the dominated convergence theorem it follows from (5.2.16) that

$$\lim_{k\to\infty} \phi[\mu](x_k, t_k) = \lim_{k\to\infty} \mu * \phi_{t_k}(x_k) = Lm(B(\underline{0}, 1)) \int_0^{\phi(\underline{0})} \theta(s)^Q\, ds = L.$$

This completes the proof. □

In view of Lemma 5.2.7, we have the following corollary of Theorem 5.2.12, which can be thought of as an analogue of the result of Shapiro (Theorem 5.1.2) for stratified Lie groups.

Theorem 5.2.13. *Suppose that ϕ and μ is as in Theorem 5.2.12. If $x_0 \in \Sigma_G(\mu)$, with*

$$D_\sigma \mu(x_0) = L,$$

then $\phi[\mu]$ has admissible limit L at x_0.

When we specialize to the case of $G = \mathbb{R}^n$, taking ϕ to be the Poisson kernel or Gauss-Weierstrass kernel we recover results of Shapiro alluded to in the introduction (see Theorem 5.1.2).

Corollary 5.2.14. *Suppose that μ is a measure on $\mathbb{R}^n$, with well-defined Poisson integral $P[\mu]$. If $x_0 \in \Sigma_n(\mu)$, with $D_\sigma\mu(x_0) = L \in \mathbb{C}$, then both the Poisson integral $P[\mu]$, and the Gauss-Weierstrass integral $W[\mu]$ has nontangential limit L at x_0.*

Remark 5.2.15. We are not in a position to make any claim regarding parabolic convergence or admissible convergence of $\Gamma[\mu]$ at σ-points of μ because the heat kernel Γ (see Theorem 4.2.11, and (4.2.12)) may not be a d-radial function. However, such convergences do hold at Lebesgue points of μ (see Corollary 5.2.20).

Remark 5.2.16. As in the case of Saeki's result (Theorem 1.0.5), Theorem 5.2.12 also fails in the absence of the comparison condition (5.2.5). This can be seen by extending the example given by Saeki [Sae96, Remark 1.6] in the setting of stratified Lie groups. Indeed, suppose that $\phi : G \to (0, \infty)$, satisfies the condition (1) of Theorem 5.2.12 but ϕ does not satisfy the comparison condition (5.2.5), that is,

$$\sup \left\{ \frac{\phi_t(x)}{\phi(x)} \mid t \in (0,1),\ d(x) \geq 1 \right\} = \infty.$$

Then for each $k \in \mathbb{N}$, there exists $t_k \in (0,1)$, $x_k \in B_d(\underline{0}, 1)^c$, such that

$$\frac{\phi_{t_k}(x_k)}{\phi(x_k)} = \frac{\phi\left(\delta_{\frac{1}{t_k}}(x_k)\right)}{t_k^Q \phi(x_k)} > k^3. \tag{5.2.18}$$

As $t_k \in (0,1)$, $d(x_k) \geq 1$, we have that

$$d\left(\delta_{\frac{1}{t_k}}(x_k)\right) > d(x_k).$$

Using this, and the fact that ϕ is d-radially decreasing we observe from (5.2.18) that for all k

$$k^3 < \frac{\phi\left(\delta_{\frac{1}{t_k}}(x_k)\right)}{t_k^Q \phi(x_k)} \leq \frac{\phi(x_k)}{t_k^Q \phi(x_k)} = \frac{1}{t_k^Q}.$$

This shows that $t_k \to 0$, as k goes to infinity. Now, we consider the measure μ on G given by

$$\mu = \sum_{k=1}^{\infty} \frac{1}{k^2 \phi(x_k)} \nu_{x_k},$$

where ν_{x_k} is the Dirac measure concentrated at x_k. We note that

$$\mu * \phi(\underline{0}) = \int_G \phi(\xi)\, d\mu(\xi) = \sum_{k=1}^{\infty} \frac{1}{k^2 \phi(x_k)} \phi(x_k) = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

As $x_k \in B_d(\underline{0}, 1)^c$, for each k, $\underline{0}$ is a Lebesgue point of μ with $D\mu(\underline{0})$ being equal to zero, and hence $\underline{0} \in X_G(\mu)$. On the other hand, for each k, we have

$$\phi[\mu](\underline{0}, t_k) = t_k^{-Q} \int_G \phi\left(\delta_{\frac{1}{t_k}}(\xi)\right) d\mu(\xi) = \sum_{j=1}^{\infty} \frac{\phi\left(\delta_{\frac{1}{t_k}}(x_j)\right)}{t_k^Q j^2 \phi(x_j)} \geq \frac{\phi\left(\delta_{\frac{1}{t_k}}(x_k)\right)}{t_k^Q \phi(x_k)} \frac{1}{k^2} \geq k.$$

This shows that $\phi[\mu]$ has no admissible limit at $\underline{0}$.

However, we next show that the comparison condition (5.2.5) in Theorem 5.2.12 can be dropped by imposing some growth condition on μ. More precisely, we have the following.

Theorem 5.2.17. *Let $\phi : G \to [0, \infty)$, be a nonzero, radial, radially decreasing, lower semicontinuous function with*

$$\int_G \phi(x)\, dm(x) = 1.$$

Suppose that μ is a measure on G, such that

$$|\mu(B_d(0, r))| = O(r^Q), \quad as\ r \to \infty, \tag{5.2.19}$$

and that $\mu * \phi_{t_0}(x_1)$ is finite for some $(x_1, t_0) \in G \times (0, \infty)$. If $x_0 \in X_G(\mu)$, with $D_\chi \mu(x_0)$ being equal to L, then $\phi[\mu]$ has admissible limit L at x_0.

Proof. As before, without loss of generality, we assume that $x_0 = \underline{0}$. We will use the same notation as in the proof of Theorem 5.2.12. From the proof of Theorem 5.2.12, we observe that it suffices to prove that $M_{HL}\mu(\underline{0})$ is finite and then the rest of the arguments remain same. As $D_\chi \mu(\underline{0})$ is L, it follows that $D\mu(\underline{0})$ is also equal to L, and hence there exists a positive number r_0 such that

$$\frac{|\mu(B_d(\underline{0}, r))|}{m(B_d(\underline{0}, r))} < L + 1, \quad \text{for all } r \in (0, r_0).$$

Using (5.2.19), we get two positive constants M_0 and R_0 such that

$$\frac{|\mu(B_d(\underline{0}, r))|}{m(B_d(\underline{0}, r))} < M_0, \quad \text{for all } r \geq R_0.$$

Finally, for all $r \in (r_0, R_0)$

$$\frac{|\mu(B_d(\underline{0}, r))|}{m(B_d(\underline{0}, r))} \leq \frac{|\mu|(B_d(\underline{0}, R_0))}{m(B_d(\underline{0}, r_0))}.$$

From the last three inequalities and using the fact that $|\mu|$ is locally finite, we conclude that

$$M_{HL}\mu(\underline{0}) = \sup_{r>0} \frac{|\mu(B_d(\underline{0}, r))|}{m(B_d(\underline{0}, r))} < \infty. \qquad \square$$

Remark 5.2.18. i) If μ is an absolutely continuous measure with L^p density, then μ satisfies the growth condition (5.2.19). Indeed, if $d\mu = f\,dm$, with $f \in L^p(G)$, $p \in (1,\infty]$, then by the Hölder's inequality we have

$$\begin{aligned}|\mu|(B_d(\underline{0},r)) = \int_{B_d(\underline{0},r)} |f|\,dm &\leq \|f\|_{L^p(G)}\left(m(B_d(\underline{0},1))\right)^{\frac{1}{p'}} r^{\frac{Q}{p'}} \\ &\leq \|f\|_{L^p(G)}\left(m(B_d(\underline{0},1))\right)^{\frac{1}{p'}} r^{Q},\end{aligned}$$

for all $r > 1$, where p' is the conjugate exponent of p. If $d\mu = f\,dm$, with $f \in L^1(G)$, then $|\mu|(G)$ is finite. Hence, in this case the comparison condition (5.2.5) in Theorem 5.2.12 is not necessary.

ii) We can drop the assumption that ϕ is lower semicontinuous from Theorem 5.2.12 and Theorem 5.2.17 if the measure μ is absolutely continuous with respect to the Lebesgue measure m. This follows from the following observation.

$$\mu\left(B_d(x,r)\right) = \mu\left(\overline{B_d(x,r)}\right), \quad x \in G,\ r > 0.$$

We can also do this in Theorem 5.2.12 if we are concerned about the admissible convergence at Lebesgue points. The following theorem can be seen as an extension of Saeki's result [Sae96, Theorem 1.5] mentioned in the introduction for stratified Lie groups.

Theorem 5.2.19. *Suppose that $\phi : G \to (0,\infty)$, is a d-radial, d-radially decreasing, integrable function that satisfies the comparison condition (5.2.5), and that μ is a measure on G such that $|\mu| * \phi_{t_0}(x_1)$ is finite for some $x_1 \in G$, and $t_0 \in (0,\infty)$. Then, for each measurable function ψ on G, with $|\psi| \leq \phi$,*

$$\int_G \psi(x)\,dm(x) = 1, \tag{5.2.20}$$

and each Lebesgue point x_0 of μ, $\psi[\mu]$ has admissible limit $D\mu(x_0)$ at x_0.

Proof. We set $L = D\mu(x_0)$. As usual, without loss of generality, we assume that $x_0 = \underline{0}$. We fix $\alpha \in (0,\infty)$. Then, by the proof of Lemma 5.2.9 (see (5.2.11)),

$$\lim_{\substack{t\to 0\\ (x,t)\in \mathrm{S}(\underline{0},\alpha)}} \int_{\{\xi \in G \mid d(\xi) \geq \frac{t_0}{\tau}\}} \left|\psi_t(\xi^{-1}\circ x)\right|\,d|\mu|(\xi)$$

$$\leq \lim_{\substack{t\to 0 \\ (x,t)\in \mathrm{S}(\underline{0},\alpha)}} \int_{\{\xi \in G \mid d(\xi) \geq \frac{t_0}{\tau}\}} \phi_t(\xi^{-1}\circ x)\, d|\mu|(\xi) = 0.$$

Thus, it is enough to prove the assertion of the theorem under the assumption that μ is supported on the d-ball $B_d(\underline{0}, t_0/\tau)$. Thus, using the fact that $\underline{0}$ is a Lebesgue point of μ, we get

$$M_{HL}(|\mu - Lm|)(\underline{0}) = \sup_{r>0} \frac{|\mu - Lm|(B_d(\underline{0}, r))}{m(B_d(\underline{0}, r))} < \infty. \tag{5.2.21}$$

Let $\{(x_k, t_k) \mid k \in \mathbb{N}\}$ be a sequence in $\mathrm{S}(\underline{0}, \alpha)$ such that $\{t_k\}$ converges to zero as k goes to infinity. Without loss of generality, we assume that $t_k \in (0, t_0/\tau)$, for all k. Using (5.2.20), we can write

$$\begin{aligned}
|\psi[\mu](x_k, t_k) - L| &= \left| \int_G \psi_{t_k}(\xi^{-1}\circ x_k)\, d\mu(\xi) - L\int_G \psi_{t_k}(\xi^{-1}\circ x_k)\, dm(\xi)\right| \\
&\leq \int_G \left|\psi_{t_k}(\xi^{-1}\circ x_k)\right|\, d|\mu - Lm|(\xi) \\
&\leq \int_G \phi_{t_k}(\xi^{-1}\circ x_k)\, d|\mu - Lm|(\xi).
\end{aligned} \tag{5.2.22}$$

We observe from the proof of Lemma 5.2.10 that for each $t \in (0, \phi(\underline{0}))$, the set $\{x \in G \mid \phi(x) > t\}$ is either $B_d(\underline{0}, \theta(t))$ or $\overline{B_d(\underline{0}, \theta(t))}$, for some positive number $\theta(t)$. Consequently, the set $\{\xi \in G \mid \phi_r(\xi^{-1}\circ x) > t\}$ is either $B_d(x, r\theta(t))$ or $\overline{B_d(x, r\theta(t))}$, for any $r > 0$. Proceeding as in the proof of Theorem 5.2.12, we have

$$\begin{aligned}
&\int_G \phi_{t_k}(\xi^{-1}\circ x_k)\, d|\mu - Lm|(\xi) \\
&= t_k^{-Q} \int_0^{\phi(\underline{0})} |\mu - Lm| \left(\left\{ \xi \in G : \phi\left(\delta_{\frac{1}{t_k}}(\xi^{-1}\circ x)\right) > s \right\}\right) ds \\
&\leq t_k^{-Q} \int_0^{\phi(\underline{0})} |\mu - Lm| \left(\overline{B_d(x_k, t_k\theta(s))}\right) ds \\
&\leq t_k^{-Q} \int_0^{\phi(\underline{0})} |\mu - Lm| \left(B_d(\underline{0}, \tau(t_k\theta(s) + \alpha t_k))\right) ds \\
&= \tau^Q m(B_d(\underline{0}, 1)) \int_0^{\phi(\underline{0})} \frac{|\mu - Lm| \left(B_d(\underline{0}, \tau(t_k\theta(s) + \alpha t_k))\right)}{m\left(B_d(\underline{0}, \tau(t_k\theta(s) + \alpha t_k))\right)} \left(\theta(s) + \alpha\right)^Q ds
\end{aligned} \tag{5.2.23}$$

It follows from the definition of Lebesgue point (Definition 5.2.1, i)) that for each $s \in (0, \phi(\underline{0}))$, the integrand in (5.2.23) goes to zero as k tends to infinity. Moreover, using (5.2.21), the integrand is bounded by the function

$$s \mapsto \tau^Q m(B_d(\underline{0}, 1)) M_{HL}(|\mu - Lm|)(\underline{0}) \left(\theta(s) + \alpha\right)^Q, \quad s \in (0, \phi(\underline{0})),$$

which is integrable as ϕ is integrable (see 5.2.17). In view of (5.2.22), applying dominated convergence theorem in (5.2.23) we get

$$\lim_{k\to\infty} \psi[\mu](x_k, t_k) = L. \qquad \square$$

Using the Gaussian estimate (4.2.10) and taking

$$\phi(x) = \exp\left(-\frac{d_{\mathcal{L}}(x)^2}{c_0}\right), \quad x \in G;$$

we obtain the following corollary, which was proved in [BU05, Theorem 2.4].

Corollary 5.2.20. *Let μ be a measure on G such that $\Gamma[\mu](x_0, t_0)$ exists (see (4.2.12)) for some $(x_0, t_0) \in G \times (0, \infty)$. Then, the parabolic limit as well as the admissible limit of $\Gamma[\mu]$ is $D\mu(x)$ at every $x \in \mathrm{L}_G(\mu)$.*

Proof. We fix $x \in \mathrm{L}_G(\mu)$, and $\alpha \in (0, \infty)$. By recalling (see 4.2.14) the fact that $\Gamma[\mu](\cdot, t) = \mu * \gamma_{\sqrt{t}}$, it directly follows from Theorem 5.2.19 that $\Gamma[\mu]$ has parabolic limit $D\mu(x)$ at x. Therefore,

$$\lim_{\substack{t\to 0 \\ d(x^{-1}\circ\xi)<\sqrt{\alpha t}}} \Gamma[\mu](\xi, t) = D\mu(x). \tag{5.2.24}$$

On the other hand, we have the following containment.

$$\mathrm{S}(x_0, \alpha) \cap G \times (0, 1/\alpha) \subset \{(\xi, t) \in G \times (0, \infty) \mid d(x^{-1} \circ \xi) < \sqrt{\alpha t},\ t \in (0, 1/\alpha)\}.$$

Using this containment together with (5.2.24), we conclude that $\Gamma[\mu]$ has admissible limit $D\mu(x)$ at x. $\square$

5.3 σ-point and strong derivative

In this section, we will discuss the relationship between σ-point of a measure and the notion of strong derivative. For a measure μ on G we denote the set of all points where the strong derivative of μ exists by $S_G(\mu)$.

Lemma 5.3.1. *Let μ be a measure on G. If $x_0 \in G$, is a χ-point of μ with*

$$D_\chi \mu(x_0) = L,$$

then the strong derivative of μ at x_0 exists and is also equal to L. In particular, $X_G(\mu) \subseteq S_G(\mu)$.

Proof. We take a d-ball $B = B_d(x, s)$ in G. As x_0 is a χ-point of μ,

$$\lim_{\substack{r \to 0 \\ (\xi, r) \in \mathrm{S}(x_0, \alpha)}} \frac{\mu(B_d(\xi, r))}{m(B_d(\xi, r))} = L,$$

for all $\alpha \in (0, \infty)$. We choose $\alpha_0 \in (0, \infty)$, such that $(x, s) \in \mathrm{S}(\underline{0}, \alpha_0)$. Then for each $r \in (0, \infty)$, $(x_0 \circ \delta_r(x), rs)$ belongs to $\mathrm{S}(x_0, \alpha_0)$. Therefore,

$$\lim_{r \to 0} \frac{\mu(x_0 \circ \delta_r(B))}{m(\delta_r(B))} = \lim_{r \to 0} \frac{\mu(B_d(x_0 \circ \delta_r(x), rs))}{m(B_d(x_0 \circ \delta_r(x), rs))} = L.$$

Since B is an arbitrary d-ball in G, the strong derivative of μ at x_0 exists and is equal to L. $\square$

Combining Lemma 5.2.7, Remark 5.2.2 and Lemma 5.3.1, we have the following.

Corollary 5.3.2. *Let μ be a measure on G. Then*

$$\mathrm{L}_G(\mu) \subseteq \Sigma_G(\mu) \subseteq X_G(\mu) \subseteq S_G(\mu).$$

Remark 5.3.3. We will see in the next section that the first containment is strict. However, at the moment, it is not known to us which of the above set containments are strict. Nevertheless, it shows that there there exists a positive measure μ and $x_0 \in G$, which is not a Lebesgue measure but the strong derivative of μ at x_0 exists.

In the case of $\mathbb{R}$, Shapiro mentioned that being a σ-point of an absolutely continuous measure is equivalent to being a point of differentiability of its distribution function. We next show that this is true for any measure.

Theorem 5.3.4. *Let μ be a measure on $\mathbb{R}$, with distribution function F. Then F is differentiable at $x_0 \in \mathbb{R}$, if and only if x_0 is a σ-point of μ. In either case,*

$$F'(x_0) = D_\sigma \mu(x_0).$$

Proof. Suppose that F is differentiable at $x_0 \in \mathbb{R}$, with $F'(x_0) = L$. We fix $\epsilon > 0$, and then choose $\delta > 0$, such that

$$\left| \frac{F(x_0+h) - F(x_0)}{h} - L \right| < \epsilon, \quad \text{whenever} \quad |h| < \delta. \tag{5.3.1}$$

For $x \in \mathbb{R}$, and $r \in (0, \infty)$, with $|(x - x_0) + r| < \delta$, and $|(x - x_0) - r| < \delta$, we have

$$\begin{aligned}
& |(\mu - Lm)\left((x-r, x+r)\right)| \\
= \ & |F(x+r) - F(x-r) - 2rL| \\
= \ & \left| \frac{F(x_0 + x - x_0 + r) - F(x_0)}{x - x_0 + r} \times (x - x_0 + r) - (x - x_0 + r)L \right. \\
& \quad \left. +(x - x_0 - r)L - \frac{F(x_0 + x - x_0 - r) - F(x_0)}{x - x_0 - r} \times (x - x_0 - r) \right| \\
\leq \ & |x - x_0 + r| \left| \frac{F(x_0 + x - x_0 + r) - F(x_0)}{x - x_0 + r} - L \right| \\
& \quad +|x - x_0 - r| \left| \frac{F(x_0 + x - x_0 - r) - F(x_0)}{x - x_0 - r} - L \right| \\
< \ & |x - x_0 + r|\epsilon + |x - x_0 - r|\epsilon,
\end{aligned}$$

where the last inequality follows from (5.3.1). This implies that

$$|(\mu - Lm)(B(x,r))| < 2\epsilon(|x - x_0| + r),$$

whenever $|x - x_0| < \delta/2$, and $r < \delta/2$. Thus, x_0 is a σ-point of μ with $D_\sigma\mu(x_0) = L$.

Conversely, we assume that x_0 is a σ-point of μ with $D_\sigma\mu(x_0) = L$, and fix $\epsilon > 0$. Then there exists $\delta > 0$, such that

$$|(\mu - Lm)((x-r, x+r))| < \epsilon(|x - x_0| + r), \tag{5.3.2}$$

whenever $|x - x_0| < \delta$, and $r < \delta$.

Taking $x = x_0 + r$, where r is a positive number in (5.3.2), we obtain

$$\begin{aligned} |\mu((x_0, x_0 + 2r)) - 2rL| &= |F(x_0 + 2r) - F(x_0) - 2rL| \\ &= 2r \left| \frac{F(x_0 + 2r) - F(x_0)}{2r} - L \right| \\ &< 2r\epsilon, \end{aligned}$$

whenever $r \in (0, \delta)$. This shows that

$$F'(x_0+) = L.$$

Similarly, by taking $x = x_0 - r$, with r being positive, in (5.3.2), we get that

$$F'(x_0-) = L.$$

Hence, F is differentiable at x_0 with $F'(x_0)$ being equal to L. □

Combining the theorem above with Corollary 5.3.2 and the result relating strong derivative of a measure and derivative of its distribution function (Theorem 3.1.4), we have the following.

Corollary 5.3.5. *Suppose that μ is a measure on $\mathbb{R}$. Then*

$$\Sigma_1(\mu) = X_1(\mu) = S_1(\mu).$$

Moreover,

$$D_\sigma\mu(x) = D_\chi\mu(x) = D\mu(x), \quad \text{for all } x \in \Sigma_1(\mu).$$

Remark 5.3.6. i) As we have mentioned before, it is not known to us at the moment whether for a measure μ on $\mathbb{R}^n$, $n > 1$, there is an equality among the sets $\Sigma_n(\mu)$, $X_n(\mu)$ and $S_n(\mu)$. It would be surprising if such a result is true in higher dimensions.

A heuristic reasoning behind this is the following observation. Suppose μ is a measure on $\mathbb{R}^n$.

a) If the origin is a σ-point of μ, then

$$(\mu - D_\sigma(0)m)(B(x, r)) \to 0, \quad \text{as } (x, r) \to (0, 0).$$

b) If the origin is a χ-point of μ, then for each $\alpha \in (0,\infty)$

$$(\mu - D_\chi(0)m)(B(x,r)) \to 0, \quad \text{as } (x,r) \to (0,0), \text{ within } S(0,\alpha).$$

c) The existence of the strong derivative at the origin only ensures

$$(\mu - D\mu(0)m)(B(x,r)) \to 0, \quad \text{as } (x,r) \to (0,0),$$

along the ray $\{(rx_0, rt_0) \mid r \in (0,\infty)\}$, for every $(x_0,t_0) \in \mathbb{R}^{n+1}_+$.

ii) It is not known whether the condition $x_0 \in X_G(\mu)$, in Theorem 5.2.12 can be replaced by seemingly weaker condition $x_0 \in S_G(\mu)$ (even for a positive measure μ). Corollary 5.3.5, in this regard, shows that this is the case when $G = \mathbb{R}$. However, if $x_0 \in S_G(\mu)$, then the following convergence result for $\phi[\mu]$ holds true.

Theorem 5.3.7. *Let ϕ and μ be as in Theorem 5.2.12. If $D\mu(x_0)$ equals $L \in \mathbb{C}$, then for each $(\xi,\eta) \in G \times (0,\infty)$*

$$\lim_{r\to 0} \phi[\mu](x_0 \circ \delta_r(\xi), r\eta) = L.$$

Proof. As in the proof of Theorem 4.4.2, we can reduce matter to the case $x_0 = \underline{0}$. Let $\tilde{\mu}$ be the restriction of μ on the ball $B_d(\underline{0}, t_0/\tau)$. If $B_d(y,s)$ is any given d-ball, then for all $r \in (0, t_0\tau^{-2}(s + d(y))^{-1})$, $\delta_r(B_d(y,s))$ is contained in $B_d(\underline{0}, t_0/\tau)$. This in turn implies that $D\mu(\underline{0})$ and $D\tilde{\mu}(\underline{0})$ are equal. We note that for each fixed $(\xi,\eta) \in G \times (0,\infty)$, there exists a positive number α such that

$$\{(\delta_r(\xi), r\eta) \mid r \in (0,\infty)\} \subset \mathrm{S}(\underline{0},\alpha).$$

Thus, in view of Lemma 5.2.9, i), without loss of generality, we can assume that $|\mu|(G)$ is finite. We will use the same notation as in the proof of Theorem 5.2.12. Since $D\mu(\underline{0})$ is equal to L, it follows that $M\mu(\underline{0})$ is finite (see the argument preceding equation (5.2.15)). We fix $(\xi,\eta) \in G \times (0,\infty)$, and choose a sequence $\{r_k \mid k \in \mathbb{N}\}$ of positive numbers converging to zero. Substituting $x_k = \delta_{r_k}(\xi)$, $t_k = r_k\eta$, in the equation (5.2.16), we obtain

$$\phi[\mu](r_k\xi, r_k\eta) = m(B(\underline{0},1)) \int_0^{\phi(\underline{0})} \frac{\mu\left(B_d(\delta_{r_k}(\xi), r_k\eta\theta(s))\right)}{m\left(B_d(\delta_{r_k}(\xi), r_k\eta\theta(s))\right)} \theta(s)^Q \, ds. \tag{5.3.3}$$

As $D\mu(\underline{0})$ equals L, using the definition of strong derivative, we observe that for each fixed $s \in (0, \phi(\underline{0}))$,

$$\lim_{k\to\infty} m(B(\underline{0},1))\frac{\mu\left(B_d(\delta_{r_k}(\xi), r_k\eta\theta(s))\right)}{m\left(B_d(\delta_{r_k}(\xi), r_k\eta\theta(s))\right)}\theta(s)^Q = Lm(B_d(\underline{0},1))\theta(s)^Q. \tag{5.3.4}$$

Also, the integrand in (5.3.3) is bounded by the function

$$s \mapsto m(B_d(\underline{0},1))M_{HL}\mu(\underline{0})\theta(s)^Q, \quad s \in (0, \phi(\underline{0})).$$

We have seen in the proof of Theorem 5.2.12 that this function is integrable in $(0, \phi(\underline{0}))$, with

$$m(B(\underline{0},1))\int_0^{\phi(\underline{0})} \theta(s)^Q\, ds = 1.$$

In view of (5.3.4), we can now apply dominated convergence theorem on the right-hand side of (5.3.3) to complete the proof. □

Remark 5.3.8. i) As $\underline{0}$ is a Lebesgue point of the measure μ constructed in Remark 5.2.16, comparison condition (5.2.5) is also necessary in Theorem 5.2.19 and Theorem 5.3.7. However, with a similar argument as in the proof of the Theorem 5.2.17, we can also drop the comparison condition (5.2.5) in Theorem 5.2.19 and Theorem 5.3.7, by imposing the growth condition (5.2.19) on μ.

ii) When $G = \mathbb{R}^n$, the theorem above says that if $D\mu(x_0) = L$, then $\phi[\mu](x,t)$ has limit L as $(x,t) \to (x_0, 0)$, along each ray through $(x_0, 0)$ in $\mathbb{R}^{n+1}_+$.

5.4 Two examples

In this section, we shall discuss two examples regarding Theorem 5.2.12. Our first example deals with the comparison condition (5.2.5), and second one with the construction of σ-point which is not a Lebesgue point.

Example 5.4.1. In [EH06, Theorem 3.4], El-Hosseiny have proved a result analogous to Theorem 5.2.12 for $G = \mathbb{R}^n$, under the following condition, among others, on ϕ.

$$\lim_{t\to 0} \sup_{\|x\|\geq 1} \frac{\phi_t(\alpha x)}{\phi(x)} = 0, \quad \text{for all } \alpha > 0. \tag{5.4.1}$$

We now show by an example that there exists $\phi : \mathbb{R}^n \to (0,\infty)$, such that ϕ satisfies all the conditions of Theorem 5.2.12 but not (5.4.1). We first note that

$$\int_{\|x\|\geq 2} \frac{1}{\|x\|^n \left(\log(\|x\|)\right)^2}\, dx = \Omega_{n-1} \int_2^\infty \frac{1}{r\left(\log r\right)^2}\, dr < \infty,$$

where Ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$. We define

$$\phi(x) = \begin{cases} \frac{c_n}{2^n(\log 2)^2}, & \|x\| \leq 2 \\ \frac{c_n}{\|x\|^n(\log\|x\|)^2}, & \|x\| > 2, \end{cases}$$

where c_n is a normalizing constant so that $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$. Clearly, ϕ is radial, radially decreasing and continuous. Now, for each fixed $t \in (0,1)$, we have for $\alpha \in (0,1]$

$$\begin{aligned} \sup_{\|x\|\geq 2} \frac{\phi_t(\alpha x)}{\phi(x)} &\geq \sup_{\|x\|\geq 2} \frac{\phi_t(x)}{\phi(x)} \\ &= \sup_{\|x\|\geq 2} t^{-n} \frac{\|x\|^n\left(\log\|x\|\right)^2}{\frac{\|x\|^n}{t^n}\left(\log\frac{\|x\|}{t}\right)^2} \\ &= \sup_{\|x\|\geq 2} \left(\frac{\log\|x\|}{\log\|x\| - \log t}\right)^2 \\ &= 1. \end{aligned} \tag{5.4.2}$$

Similarly, for each fixed $t \in (0,1)$, we have for $\alpha > 1$

$$\sup_{\|x\|\geq 2} \frac{\phi_t(\alpha x)}{\phi(x)} = \frac{1}{\alpha^n} \sup_{\|x\|\geq 2} \left(\frac{\log\|x\|}{\log\|x\| + \log\alpha - \log t}\right)^2 = \frac{1}{\alpha^n}.$$

This, together with (5.4.2), shows that ϕ does not satisfies (5.4.1).

On the other hand,

$$\begin{aligned}
\sup_{t\in(0,1)}\sup_{1\le\|x\|\le 2}\frac{\phi_t(x)}{\phi(x)} &\le \sup_{t\in(0,1/2)}\sup_{1\le\|x\|\le 2}\frac{\phi_t(x)}{\phi(x)}+\sup_{t\in[1/2,1)}\sup_{1\le\|x\|\le 2}\frac{\phi_t(x)}{\phi(x)}\\
&= \sup_{t\in(0,1/2)}\sup_{1\le\|x\|\le 2} t^{-n}\frac{\|x\|^n\left(\log\|x\|\right)^2}{\frac{\|x\|^n}{t^n}\left(\log\frac{\|x\|}{t}\right)^2}+\sup_{t\in[1/2,1)}\sup_{1\le\|x\|\le 2} t^{-n}\frac{\phi\left(\frac{x}{t}\right)}{\phi(x)}\\
&= \sup_{t\in(0,1/2)}\sup_{1\le\|x\|\le 2}\left(\frac{\log\|x\|}{\log\|x\|-\log t}\right)^2+\sup_{t\in[1/2,1)}\sup_{1\le\|x\|\le 2} t^{-n}\frac{\phi\left(\frac{x}{t}\right)}{\phi(x)}\\
&\le 1+2^n\frac{\phi(0)}{\phi(2)},
\end{aligned}$$

where the last inequality follows from the fact that for each $t\in(0,1/2)$, $-\log t$ is positive, and ϕ is radially decreasing. Combining this with (5.4.2), we conclude that ϕ satisfies (5.2.5), and hence ϕ satisfies all the conditions of Theorem 5.2.12.

Example 5.4.2. In this example, we show that there exist absolutely continuous measures μ for which the containment $\mathrm{L}_G(\mu)\subseteq\Sigma_G(\mu)$ is strict. We construct such measures on the Heisenberg group H^1. We recall that $H^1=\mathbb{R}^2\times\mathbb{R}$ has homogeneous dimension 4 (see Example 4.2.10, ii)), and the group law

$$(x,y,t)\circ(x',y',t')=\left(x+x',y+y',t+t'+\frac{1}{2}(x'y-xy')\right).$$

We consider the following homogeneous norm on H^1:

$$d(x,y,t)=\left((x^2+y^2)^2+t^2\right)^{\frac{1}{4}},\quad (x,y,t)\in H^1.$$

We will use Shapiro's construction to produce a function f on H^1 such that $(0,0,0)$ is a σ-point of f, but $(0,0,0)$ is not a Lebesgue point of f. Shapiro [Sha06, P.3185] began with constructing an odd function $g:\mathbb{R}\to[-1,1]$, satisfying the following.

i) g is continuous everywhere except at 0, with $g(0)=0$.

ii) For all $s\in(0,1]$,

$$s^{-1}\int_0^s|g(t)|\,dt\ge\frac{1}{6}. \tag{5.4.3}$$

Shapiro then considered the definite integral G of g

$$G(s) = \int_0^s g(t)\,dt, \quad s \in \mathbb{R},$$

and proved that G has the following properties.

i) G is differentiable everywhere and even. Moreover,

$$G'(s) = g(s), \quad \text{for all} \;\; s \in \mathbb{R}. \tag{5.4.4}$$

ii) For all s with $|s| \in (0,1]$,

$$\frac{|G(s)|}{|s|} \leq |s|. \tag{5.4.5}$$

We now define the function $f : H^1 \to [-1,1]$, for our example as follows:

$$f(x,y,t) = \begin{cases} g(x), & \text{for} \;\; d(x,y,t) \leq 10 \\ 0, & \text{for} \;\; d(x,y,t) > 10. \end{cases}$$

It is clear that $f \in L^p(H^1)$, for any $p \in [1,\infty]$. For $r \in (0,1)$, we define

$$\mathrm{Q}(r) = \{(x,y,t) \in H^1 \mid |x| < r, \; |y| < r, \; |t| < r^2\}.$$

It is evident that $\mathrm{Q}(r) \subset B_d((0,0,0), 5^{\frac{1}{4}}r)$. Therefore, we get for $r \in (0,1)$

$$\begin{aligned} r^{-4} \int_{B_d((0,0,0),5^{\frac{1}{4}}r)} |f(x,y,t)|\,dxdydt &\geq r^{-4} \int_{\mathrm{Q}(r)} |f(x,y,t)|\,dxdydt \\ &= r^{-4} \int_{-r^2}^{r^2} \int_{-r}^{r} \int_{-r}^{r} |g(x)|\,dxdydt \\ &= 4r^{-1} \int_{-r}^{r} |g(x)|\,dx \\ &= 8r^{-1} \int_0^r |g(x)|\,dx \\ &\geq \frac{4}{3}, \end{aligned}$$

where the last inequality follows from (5.4.3). This shows that $(0,0,0)$ is not a Lebesgue point of f, as $f(0,0,0) = 0$. For the second part we need the following version of divergence theorem.

Lemma 5.4.3 ([Pfe87, Corollary 7.4]). *Let* M *be an* k*-dimensional compact oriented manifold, and let* ω *be a continuous* $(k-1)$*-form on* M *which is differentiable in* $\mathrm{M}-\partial\mathrm{M}$*. Then* $d\omega$ *is integrable in* M *and*

$$\int_{\mathrm{M}} d\omega = \int_{\partial\mathrm{M}} \omega.$$

We shall apply this version of divergence theorem on the following manifolds.

$$\overline{B_d\left((x,y,t),r\right)} = \{(u,v,s) \in H^1 \mid d\left((x,y,t)^{-1} \circ (u,v,s)\right) \leq r\}.$$

We define a function $F : H^1 \to \mathbb{R}$, as follows:

$$F(x,y,t) = G(x). \tag{5.4.6}$$

It follows from (5.4.4) that F has a total derivative at each point of H^1. Moreover,

$$\begin{aligned} \frac{\partial F}{\partial x}(x,y,t) &= G'(x) = g(x) = f(x,y,t), \\ \frac{\partial F}{\partial y}(x,y,t) &= 0, \\ \frac{\partial F}{\partial t}(x,y,t) &= 0,. \end{aligned} \tag{5.4.7}$$

whenever $d(x,y,t) \leq 2$. We note that

$$\partial\overline{B_d\left((x,y,t),r\right)} = \{(u,v,s) \in H^1 \mid h_{(x,y,t)}(u,v,s) = 0\},$$

where for $(u,v,s) \in H^1$,

$$h_{(x,y,t)}(u,v,s) = \left((u-x)^2 + (v-y)^2\right)^2 + \left(s - t + \frac{1}{2}(xv - yu)\right)^2 - r^4.$$

We have

$$\begin{aligned} \frac{\partial h_{(x,y,t)}}{\partial u}(u,v,s) &= 4\left((u-x)^2 + (v-y)^2\right)(u-x) - y\left(s - t + \frac{1}{2}(xv-yu)\right); \\ \frac{\partial h_{(x,y,t)}}{\partial v}(u,v,s) &= 4\left((u-x)^2 + (v-y)^2\right)(v-y) + x\left(s - t + \frac{1}{2}(xv-yu)\right); \\ \frac{\partial h_{(x,y,t)}}{\partial s}(u,v,s) &= 2\left(s - t + \frac{1}{2}(xv-yu)\right). \end{aligned}$$

It is clear that the partial derivatives of $h_{(x,y,t)}$ can not be simultaneously vanishing on $\partial\overline{B_d\left((x,y,t),r\right)}$. Thus, applying Lemma 5.4.3, we obtain for $d(x,y,t)<1$, $r<1$,

$$\int_{\overline{B_d((x,y,t),r)}} \mathrm{div}\,(F,0,0)\,dm = \int_{\partial\overline{B_d((x,y,t),r)}} (F,0,0)\cdot n\,dS,$$

where n is the outward unit normal to the surface $\partial\overline{B_d\left((x,y,t),r\right)}$ and dS is the surface measure on $\partial\overline{B_d\left((x,y,t),r\right)}$. Using (5.4.7), we obtain from the equation above that

$$\int_{B_d((x,y,t),r)} f(x,y,t)\,dxdydt \quad = \quad \int_{\partial\overline{B_d((x,y,t),r)}} (F,0,0)\cdot n\,dS. \tag{5.4.8}$$

We note that

$$\partial\overline{B_d\left((x,y,t),r\right)} = (x,y,t)\circ\partial\overline{B_d\left((0,0,0),r\right)}.$$

We have the following parametrization of $\partial\overline{B_d\left((0,0,0),r\right)}$ (see [GS94, P.133]).

$$\partial\overline{B_d\left((0,0,0),r\right)} = \{\Psi(\phi,\theta)\mid \phi\in(0,\pi),\ \theta\in(0,2\pi)\},$$

where

$$\Psi(\phi,\theta) = (r\sqrt{\sin\phi}\sin\theta, r\sqrt{\sin\phi}\cos\theta, r^2\cos\phi).$$

Using this we get the following parametrization of $\partial\overline{B_d\left((x,y,t),r\right)}$.

$$\partial\overline{B_d\left((x,y,t),r\right)} = \{\Psi_{(x,y,t)}(\phi,\theta)\mid \phi\in(0,\pi),\ \theta\in(0,2\pi)\},$$

where for $\phi\in(0,\pi)$, $\theta\in(0,2\pi)$

$$\begin{aligned}&\Psi_{(x,y,t)}(\phi,\theta)\\ =\ &\left(x+r\sqrt{\sin\phi}\sin\theta, y+r\sqrt{\sin\phi}\cos\theta, t+r^2\cos\phi-\frac{r}{2}\sqrt{\sin\phi}(x\cos\theta-y\sin\theta)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\partial\Psi_{(x,y,t)}}{\partial\phi}(\phi,\theta)\\ =\ &\left(r\frac{\cos\phi}{2\sqrt{\sin\phi}}\sin\theta, r\frac{\cos\phi}{2\sqrt{\sin\phi}}\cos\theta, -r^2\sin\phi-\frac{r\cos\phi}{4\sqrt{\sin\phi}}(x\cos\theta-y\sin\theta)\right);\end{aligned}$$

and

$$\frac{\partial \Psi_{(x,y,t)}}{\partial \theta}(\phi,\theta) = \left(r\sqrt{\sin\phi}\cos\theta, -r\sqrt{\sin\phi}\sin\theta, \frac{r}{2}\sqrt{\sin\phi}(x\sin\theta + y\cos\theta) \right).$$

To evaluate the right-hand side of (5.4.8), we need only the first coordinate of

$$\begin{aligned}
&\frac{\partial \Psi_{(x,y,t)}}{\partial \phi} \times \frac{\partial \Psi_{(x,y,t)}}{\partial \theta}(\phi,\theta) \\
= &\begin{vmatrix} i & j & k \\ r\frac{\cos\phi}{2\sqrt{\sin\phi}}\sin\theta & r\frac{\cos\phi}{2\sqrt{\sin\phi}}\cos\theta & -r^2\sin\phi - \frac{r\cos\phi}{4\sqrt{\sin\phi}}(x\cos\theta - y\sin\theta) \\ r\sqrt{\sin\phi}\cos\theta & -r\sqrt{\sin\phi}\sin\theta & \frac{r}{2}\sqrt{\sin\phi}(x\sin\theta + y\cos\theta) \end{vmatrix},
\end{aligned}$$

which is equal to

$$\frac{r^2}{4}y\cos\phi - r^3\sin^{\frac{3}{2}}\phi\sin\theta.$$

Using this, together with the definition of F (see (5.4.6)) in (5.4.8), we obtain from (5.4.8) that

$$\begin{aligned}
&\int_{B_d((x,y,t),r)} f(x,y,t)\, dxdydt \\
= &\int_0^{2\pi}\int_0^{\pi} G(x + r\sqrt{\sin\phi}\sin\theta)\left(\frac{r^2}{4}y\cos\phi - r^3\sin^{\frac{3}{2}}\phi\sin\theta\right)\, d\phi\, d\theta.
\end{aligned}$$

As $d(x,y,t)$ is bigger than $|x|$, we have

$$|x + r\sqrt{\sin\phi}\sin\theta| \le d(x,y,t) + r.$$

Hence, by the estimate (5.4.5), we get for all $(x,y,t) \in H^1$, $r > 0$, with $d(x,y,t) + r \in (0,1)$,

$$\begin{aligned}
&\left|\int_{B_d((x,y,t),r)} f(x,y,t)\, dxdydt\right| \\
\le\ &\int_0^{2\pi}\int_0^{\pi} |x + r\sqrt{\sin\phi}\sin\theta|^2 \left|\frac{r^2}{4}y\cos\phi - r^3\sin^{\frac{3}{2}}\phi\sin\theta\right|\, d\phi\, d\theta \\
\le\ &2\pi^2(d(x,y,t)+r)^2\left(\frac{r^2}{4}|y| + r^3\right) \\
\le\ &2\pi^2(d(x,y,t)+r)^2\left(r^2 d(x,y,t) + r^3\right) \\
=\ &2\pi^2 r^2(d(x,y,t)+r)^3 \\
\le\ &2\pi^2 r(d(x,y,t)+r)^4.
\end{aligned}$$

Thus, for a given $\epsilon > 0$, choosing $\delta = \min\{\frac{\epsilon}{2\pi^2}, 1\}$, yields

$$\left| \int_{B_d((x,y,t),r)} f(x,y,t)\, dxdydt \right| \leq \epsilon (d(x,y,t)+r)^4,$$

whenever $0 < (d(x,y,t)+r) < \delta$. This shows that $(0,0,0)$ is a σ-point of f, as $f(0,0,0) = 0$. We get our desired measure by taking $d\mu = f\, dm$.

Since $|f|$ is bounded by one, we have

$$f_1 = (f+1)\chi_{B_d((0,0,0),10)} \geq 0,$$

and $f_1 \in L^p(H^1)$, for any $p \in [1, \infty]$. One can easily check that $(0,0,0)$ is a σ-point of f_1, but $(0,0,0)$ is not a Lebesgue point of f_1. Setting $d\nu = f_1\, dm$, we obtain a positive measure ν such that $L_{H^1}(\nu) \subsetneq \Sigma_{H^1}(\nu)$.

Chapter 6

Admissible convergence of positive eigenfunctions on Harmonic NA groups

In this chapter, we extend the result of Ramey and Ullrich (Theorem 1.0.13) alluded to in the introduction. We show that a positive eigenfunction u of $\mathcal{L}$ with eigenvalue $\beta^2 - \rho^2$, where $\beta \in (0, \infty)$, has admissible limit in the sense of Korányi, precisely at those boundary points where the strong derivative of the boundary measure of u exists. Moreover, the admissible limit and the strong derivative are the same.

6.1 Introduction

We start by recalling the following result regarding boundary behavior of positive eigenfunctions of the Laplace-Beltrami operator on real hyperbolic spaces. Combining Saeki's result (Theorem 1.0.5) and Theorem 2.2.4, we obtain the following equivalence of boundary behavior of a positive eigenfunction in $\mathbb{H}^l$ along the normal at a given boundary point with the existence of the symmetric derivative of its boundary measure at that point.

Theorem 6.1.1. *Let u be a positive eigenfunction of the Laplace-Beltrami operator $\Delta_{\mathbb{H}^l}$ on real hyperbolic space $\mathbb{H}^l$, $l \geq 2$, with eigenvalue $\beta^2 - \rho^2$, where $\beta \in (0, \infty)$, and $\rho = (l-1)/2$. Suppose that $x_0 \in \mathbb{R}^{l-1}$, and $L \in [0, \infty)$. If the boundary measure of u is μ, then*

$$D_{sym}\mu(x_0) = L,$$

if and only if

$$\lim_{y\to 0} y^{\beta-\rho}u(x_0,y) = L.$$

We refer the reader to Lemma 2.2.2 and (1.0.1) for the meaning of boundary measure u, and the definition of the symmetric derivative.

Surprisingly, as shown by Rudin [Rud08, Example 5.4.13], the natural analogue of Theorem 6.1.1 is not true for complex hyperbolic spaces. Rudin considered the ball model

$$B_2 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

of the 4-dimensional complex hyperbolic space equipped with the standard Riemannian metric

$$dz = (1 - |z_1|^2 - |z_2|^2)^{-3} dx_1 dy_1 dx_2 dy_2, \quad z_k = x_k + iy_k,\ \ k = 1, 2.$$

Rudin showed that there is a positive harmonic function u in B_2 with boundary measure μ defined on the boundary

$$\partial B_2 = \{(\zeta, \eta) \in \mathbb{C}^2 : |\zeta|^2 + |\eta|^2 = 1\},$$

such that the symmetric derivative $\mathrm{D}_{sym}\mu(e_1)$ of μ at $e_1 = (1, 0) \in \partial B_2$, is equal to 2, but the limit of u along the normal at e_1 is equal to 4, that is,

$$\lim_{r\to 1} u(re_1) = 4.$$

Here, the symmetric derivative $\mathrm{D}_{sym}\nu(\zeta)$ of a measure ν defined on ∂B_2 at $\zeta \in \partial B_2$, is given by

$$\mathrm{D}_{sym}u(\zeta) = \lim_{r\to 0} \frac{\nu\left(\mathrm{Q}(\zeta, r)\right)}{\upsilon\left(\mathrm{Q}(\zeta, r)\right)},$$

where

$$\mathrm{Q}(\zeta, r) = \{\eta \in \partial B_2 \mid |1 - \langle \zeta, \eta \rangle| < r\}, \quad r > 0,$$

and υ is the rotation-invariant normalized positive Borel measure on ∂B_2.

Therefore, it is reasonable to enquire for different types of boundary behavior of positive harmonic functions on complex hyperbolic spaces other than the radial behavior mentioned above. It was Korányi [Kor69b], who first investigated a different type of boundary behavior, known as the admissible convergence, of harmonic functions on complex hyperbolic spaces. Complex hyperbolic spaces are prototypical examples of Riemannian symmetric spaces of noncompact type with real rank one. In [Kor69a], Korányi extended the notion of admissible convergence from complex hyperbolic spaces to Riemannian symmetric spaces of noncompact type and proved a Fatou-type theorem regarding admissible convergence of the Poisson integral of an integrable function almost everywhere on the Furstenberg boundary. For the definition of admissible convergence we refer the reader to Definition 6.2.8, i). There is an extensive literature concerning extension and generalization of this result (see for example [KT81], [Mic73], [Sch85], [Sjö84] and references therein). This version of Fatou theorem regarding almost everywhere admissible convergence was further extended by Michelson [Mic73] for eigenfunctions of the Laplace-Beltrami operator on Riemannian symmetric spaces of noncompact type. However, this body of literature does not seem to contain any result relating the notion of admissible convergence of positive eigenfunctions of the Laplace-Beltrami operator and differentiation property of boundary measures on Riemannian symmetric spaces of noncompact type at a given point on the relevant boundary. This motivated us to search for an analogue of the result of Ramey and Ullrich (Theorem 1.0.13) for positive eigenfunctions of the Laplace-Beltrami operator on a class of noncompact Riemannian manifolds which includes all Riemannian symmetric spaces of noncompact type with real rank one (excluding real hyperbolic spaces), namely, the harmonic NA groups (also known as Damek-Ricci spaces), by replacing the nontangential convergence with the admissible convergence.

Let us now briefly describe the main result of this chapter. Let u be a positive eigenfunction of the Laplace-Beltrami operator $\mathcal{L}$ on S, a Harmonic NA group, with eigenvalue $\beta^2 - \rho^2$, where $\beta \in (0, \infty)$. Here, N is a group of Heisenberg type, $A = (0, \infty)$ acts on N as nonisotropic dilation and $S = NA$ is the semidirect product under the action of dilation. For unexplained notions and terminologies we refer the reader to Section 2. It is known that S is a Riemannian manifold with respect to a metric which is left-invariant under the action of S (see [ADY96]). By a result in [DR92] (see Lemma 6.3.5), u is essentially given by a Poisson type integral of a positive measure μ defined on N. As N is a stratified Lie group, we have the notion of strong derivative of a measure on N (see Definition 4.2.19, iii)). With the aid of these basic notions, the main result (Theorem 6.4.2) of this chapter says that given a point $n_0 \in N$, the positive eigenfunction u of the Laplace-Beltrami operator has admissible limit

L at n_0 if and only if the strong derivative of μ at n_0 is equal to L. Though our proof is modeled on the proof given by Ramey and Ullrich, the main contrast with the Euclidean case is that the Poisson kernel is not a function of homogeneous norm. Here also, the result of Bär [Bär13, Theorem 4] on generalization of Montel's theorem plays an important role in the proof of the main theorem.

This chapter is organised as follows. In section 2, we will discuss some basic information about Harmonic NA groups, generalized Poisson kernel and Poisson integral on these groups. In section 3 we prove some results which are crucial for the proof of the main theorem. The statement and proof of the main theorem (Theorem 6.4.2) is given in the last section.

6.2 Preliminaries on Harmonic NA groups

A harmonic NA group is a solvable Lie group as well as a Harmonic manifold. Their distinguished prototypes are the Riemannian symmetric spaces of noncompact type with real rank one. However, the Riemannian symmetric spaces of noncompact type with real rank one form a very small subclass in the class of Harmonic NA groups [ADY96, 1.10]. In the following, we discuss them in detail. Most of these material can be found in [FS82, ADY96, ACDB97].

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra equipped with an inner product $\langle , \rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement. We say that $\mathfrak{n}$ is a H-type algebra if for every $Z \in \mathfrak{z}$, the map $J_Z : \mathfrak{v} \to \mathfrak{v}$, defined by

$$\langle J_Z X, Y \rangle = \langle [X, Y], Z \rangle, \quad X, Y \in \mathfrak{v},$$

satisfies the condition

$$J_Z^2 = -|Z|^2 I_{\mathfrak{v}},$$

where $I_{\mathfrak{v}}$ is the identity operator on $\mathfrak{v}$. A connected and simply connected nilpotent Lie group N is called a group of Heisenberg type or an H-type group if its Lie algebra is a H-type algebra. Since $\mathfrak{n}$ is nilpotent, the exponential map is a diffeomorphism and hence we can parametrize elements of $N = \exp \mathfrak{n}$ by (X, Z) for $X \in \mathfrak{v}$, $Z \in \mathfrak{z}$. It follows from the

Baker-Campbell-Hausdorff formula that the group law of N is given by

$$nn_1 = (X, Z)(X_1, Z_1) = (X + X_1, Z + Z_1 + \frac{1}{2}[X, X_1]),$$

for $n = (X, Z) \in N, n_1 = (X_1, Z_1) \in N$. When $\mathfrak{z} = \mathbb{R}$, $\mathfrak{v} = \mathbb{R}^{2l}$, and for $s \in \mathbb{R}$, $J_s : \mathbb{R}^{2l} \to \mathbb{R}^{2l}$ is given by

$$J_s(x, y) = (-sy, sx), \quad x \in \mathbb{R}^l,\ y \in \mathbb{R}^l,$$

then we get the Heisenberg group H^l which is the prototype of a H-type group. If G is a connected noncompact rank one semisimple Lie group with finite center, then it follows that the nilpotent Lie group N which appears in the Iwasawa decomposition of G, is a H-type group. As N is a strtified Lie group, we recall that the Lebesgue measure $dXdZ$ is a Haar measure on N and we denote it by m. The multiplicative group $A = (0, \infty)$ acts on the H-type group N by nonisotropic dilation:

$$\delta_a(n) = \delta_a(X, Z) = (\sqrt{a}X, aZ), \quad a \in A,\ n = (X, Z) \in N. \tag{6.2.1}$$

A Harmonic NA group S is the semidirect product of a H-type group N and A under the above action. Thus, the multiplication on S is given by

$$(X, Z, a)(X', Z', a') = (X + \sqrt{a}X', Z + aZ' + \frac{1}{2}\sqrt{a}[X, X'], aa').$$

Then S is a solvable, connected and simply connected Lie group having Lie algebra $\mathfrak{s} = \mathfrak{n}\oplus\mathfrak{z}\oplus\mathbb{R}$ with Lie bracket

$$[(X, Z, u), (X', Z', u')] = \left(\frac{1}{2}uX' - \frac{1}{2}u'X, uZ' - u'Z + [X, X'], 0\right). \tag{6.2.2}$$

We shall write $(n, a) = (X, Z, a)$ for the element $(\exp(X + Z), a)$, $a \in A$, $X \in \mathfrak{v}$, $Z \in \mathfrak{z}$. We note that for any $Z \in \mathfrak{z}$, with $\|Z\| = 1$,

$$J_Z^2 = -I_{\mathfrak{v}},$$

and hence $\mathfrak{v}$ is even dimensional. We suppose that $dim\,\mathfrak{v} = 2p$, $dim\,\mathfrak{z} = k$. Then $Q = p+k$ is the homogeneous dimension of N with respect to $\{\delta_a \mid a \in A\}$. We recall that the importance of homogeneous dimension stems from the following relation

$$m\left(\delta_a(E)\right) = a^Q m(E), \tag{6.2.3}$$

which holds for all measurable sets $E \subseteq N$, and $a \in A$. For convenience, we shall also use the notation $\rho = Q/2$. We denote by e the identity element $(\underline{0}, 1)$ of S, where $\underline{0}$, 1 are the identity elements of N and A respectively. We note that ρ corresponds to the half-sum of positive roots when $S = G/K$, is a rank one symmetric space of noncompact type. The group S is equipped with the left-invariant Riemannian metric induced by

$$\langle (X, Z, l), (X', Z', l') \rangle_S = \langle X, X' \rangle + \langle Z, Z' \rangle + ll'$$

on $\mathfrak{s}$. The associated left-invariant Haar measure dx on S is given by

$$dx = a^{-Q-1} dX\, dZ\, da,$$

where dX, dZ, da are the Lebesgue measures on $\mathfrak{v}$, $\mathfrak{z}$, A respectively. We have discussed about stratified Lie groups in Chapter 4 and H-type groups are a special case of them. However, in the following, we quickly recollect some of the basic tools for doing analysis on H-type groups for the sake of completeness. Being a stratified Lie group, N always admits homogeneous norms with respect to the family of dilations $\{\delta_a \mid a \in A\}$. We refer the reader to Definition 4.2.1 for definition of homogeneous norm. In this chapter, we will work with the following homogeneous norm [DK16, P.1918]:

$$d(n) = d(X, Z) = \left(\|X\|^4 + 16\|Z\|^2\right)^{\frac{1}{2}}, \quad n = (X, Z) \in N, \tag{6.2.4}$$

where $\|X\|$, $\|Z\|$ are usual Euclidean norms of $X \in \mathfrak{v} \cong \mathbb{R}^{2p}$ and $Z \in \mathfrak{z} \cong \mathbb{R}^k$ respectively. We recall that there exists a positive constant $\tau \in [1, \infty)$, such that

$$d(nn_1) \leq \tau\left[d(n) + d(n_1)\right], \quad n \in N,\ n_1 \in N. \tag{6.2.5}$$

As in Chapter 4, we denote by $\mathbf{d}$, the left invariant quasi-metric on N induces by d, that is,

$$\mathbf{d}(n_1, n_2) = d(n_1^{-1} n_2), \quad n_1 \in N, \ n_2 \in N.$$

Then $\mathbf{d}$ satisfies the following quasi-triangle inequality.

$$\mathbf{d}(n_1, n_2) \leq \tau \left[\mathbf{d}(n_1, n) + \mathbf{d}(n, n_2)\right], \quad \text{for all } n_1, \ n_2, \ n \in N. \tag{6.2.6}$$

For $n \in N$ and $r \in (0, \infty)$, the d-ball centered at n with radius r will be denoted by $\mathrm{B}(n, r)$. In other words,

$$\mathrm{B}(n, r) = \{n_1 \in N \mid \mathbf{d}(n, n_1) < r\} = \{n_1 \in N \mid d(n^{-1} n_1) < r\}.$$

If $B = \mathrm{B}(n, t)$, for some $n \in N$, $t \in (0, \infty)$, then it follows that

$$\delta_a(B) = \mathrm{B}(\delta_a(n), at), \quad \text{for all } a \in A.$$

We recall the following formula for integration (an analogue of polar coordinate) which can be used in order to determine the integrability of functions on N: for all $g \in L^1(N)$,

$$\int_N g(n)\, dm(n) n = \int_0^\infty \int_\Omega g(\delta_r(\omega)) r^{Q-1}\, d\sigma(\omega)\, dr, \tag{6.2.7}$$

where $\Omega = \{\omega \in N \mid d(\omega) = 1\}$ and σ is a unique positive Radon measure on Ω. For a function ψ defined on N, we define for $a \in A$,

$$\psi_a(n) = a^{-Q} \psi\left(\delta_{\frac{1}{a}}(n)\right), \quad n \in N. \tag{6.2.8}$$

If g is a measurable function on N and μ is a measure on N, their convolution $\mu * g(n)$ is defined by

$$\mu * g(n) = \int_N g(n_1^{-1} n)\, d\mu(n_1),$$

provided the integrals converges. When $d\mu = f\, dm$, we simply denote the above convolution by $f * g$. If $\psi \in L^1(N)$, with

$$\int_G \psi(n)\, dm(n) = 1,$$

then $\{\psi_a \mid a \in A\}$ forms an approximate identity on N.

We now describe the Laplace-Beltrami operator on S. Let $\{e_i \mid 1 \leq i \leq 2p\}$, $\{e_r \mid 2p+1 \leq r \leq 2p+k\}$, $\{e_0\}$ be an orthonormal basis of $\mathfrak{s}$ corresponding to the decomposition $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{z} \oplus \mathbb{R}$. We denote by E_l the left-invariant vector field on S determined by e_l, $0 \leq l \leq 2p+k$. Damek [Dam87, Theorem 2.1] (see also [DR92, P.234]) showed that the Laplace-Beltrami operator $\mathcal{L}$ associated to the left-invariant metric $\langle , \rangle_S$ has the form

$$\mathcal{L} = \sum_{l=0}^{2p+k} E_l^2 - QE_0.$$

Let ∂_i, ∂_r, ∂_a be the partial derivatives for the system of coordinates (X_i, Z_r, a) corresponding to (e_i, e_r, e_0). Applying the definition of vector fields:

$$E_l f(X,Z,a) = \frac{d}{dt} f\left((X,Z,a)\exp(tE_l)\right)|_{t=0},$$

one can show that

$$\begin{aligned}
&E_0 = a\partial_a;\\
&E_i = a\partial_i + \frac{a}{2}\sum_{r=2p+1}^{2p+k} \langle [X,e_i], e_r\rangle \partial_r, \qquad 1 \leq i \leq 2p;\\
&E_r = a^2\partial_r, \quad 2p+1 \leq r \leq 2p+k.
\end{aligned}$$

Using these expressions, $\mathcal{L}$ can be written as [DR92, P.234]

$$\mathcal{L} = a^2\partial_a^2 + \mathcal{L}_a + (1-Q)a\partial_a, \tag{6.2.9}$$

where

$$\mathcal{L}_a = a(a + \frac{1}{4}\|X\|^2)\sum_{r=2p+1}^{2p+k} \partial_r^2 + a\sum_{i=1}^{2p} \partial_i^2 + a^2 \sum_{2p+1}^{2p+k}\sum_{i=1}^{2p} \langle [X,e_i], e_r\rangle \partial_r\partial_i. \tag{6.2.10}$$

The formula for the Poisson kernel $\mathcal{P} : S \times N \to (0,\infty)$, corresponding to $\mathcal{L}$ is given by [Dam87, Theorem 2.2] (see also [ACDB97, P.409])

$$\mathcal{P}(x,n) = P_a(n_1^{-1}n), \quad x = (n_1, a) \in S, \;\; n \in N,$$

where P is the function on N defined by

$$P(n) = P(X,Z) = \frac{c_{p,k}}{\left(\left(1+\frac{\|X\|^2}{4}\right)^2 + \|Z\|^2\right)^Q}, \quad n=(X,Z)\in N, \tag{6.2.11}$$

and $c_{p,k}$ is a positive constant so that

$$\int_N P(n)\, dm(n) = 1.$$

Using the notion of dilation of a function (6.2.8) we get that

$$P_a(n) = P_a(X,Z) = \frac{c_{p,k}\, a^Q}{\left(\left(a+\frac{\|X\|^2}{4}\right)^2 + \|Z\|^2\right)^Q}, \quad n=(X,Z)\in N,\ \ a\in A. \tag{6.2.12}$$

Remark 4.2.5 implies that $\{P_a \mid a\in A\}$, is an approximate identity on N. Expanding the square involving $\|X\|$ and a in the denominator of the right-hand side of (6.2.12), and then making use of the expression (6.2.4) of d, we obtain the following alternative formula of the Poisson kernel.

$$P_a(n) = P_a(X,Z) = \frac{16^Q c_{p,k}\, a^Q}{(16a^2 + 8a\|X\|^2 + d(X,Z)^2)^Q}, \tag{6.2.13}$$

for $n=(X,Z)\in N,\ \ a\in A$.

Remark 6.2.1. We list down the following properties of the function P_a which can be derived from (6.2.13) and (6.2.7).

i) $P_a(n) = P_a(n^{-1})$, for all $n\in N,\ a\in A$.

ii) For $1\le r\le\infty$, $P_a\in L^r(N)$, for all $a\in A$.

It turns out that general eigenfunctions of $\mathcal{L}$ can be obtained by considering the complex power of the Poisson kernel. For $\lambda\in\mathbb{C}$, the λ-Poisson kernel is defined as

$$\mathcal{P}_\lambda(x,n) = \left[\frac{\mathcal{P}(x,n)}{P(\underline{0})}\right]^{\frac{1}{2}-\frac{i\lambda}{Q}} = \left[\frac{P_a(n_1^{-1}n)}{P(\underline{0})}\right]^{\frac{1}{2}-\frac{i\lambda}{Q}}, \quad x=(n_1,a)\in S,\ \ n\in N. \tag{6.2.14}$$

We note from the expression of the function P given in (6.2.11) that $P(\underline{0}) = c_{p,k}$.

It is well-known that for $\lambda \in \mathbb{C}$, the function $\mathcal{P}_\lambda(.,n)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $-(\lambda^2+\rho^2)$, for each fixed $n \in N$, [ADY96, P.654], that is, for each $n \in N$

$$\mathcal{L}\mathcal{P}_\lambda(\cdot,n) = -(\lambda^2+\rho^2)\mathcal{P}_\lambda(\cdot,n).$$

We note from above that $\mathcal{P}_{i\rho} = \mathcal{P}$, which is annihilated by $\mathcal{L}$ and hence

$$\{\mathcal{P}_{i\rho}((\underline{0},a),\cdot) \mid a \in A\} = \{P_a \mid a \in A\},$$

is an approximate identity on N. We observe from the expression of P_a given in (6.2.13) and the integration formula in polar form (6.2.7) that if $\mathrm{Im}(\lambda) \in (0,\infty)$, then $\mathcal{P}_\lambda(x,.) \in L^r(N)$, $r \in [1,\infty]$, for each $x=(n_1,a) \in S$. Indeed,

$$\begin{aligned}
\int_N |\mathcal{P}_\lambda(x,n)|^r\, dm(n) &\leq C_{\lambda,r}\int_N \frac{1}{\left(16a^2+d(n_1^{-1}n)^2\right)^{r\left(\frac{Q}{2}+\mathrm{Im}(\lambda)\right)}}\, dm(n)\\
&= C_{\lambda,r}\int_N \frac{1}{(16a^2+d(n)^2)^{r\left(\frac{Q}{2}+\mathrm{Im}(\lambda)\right)}}\, dm(n)\\
&= C_{\lambda,r}\sigma(\Omega)\int_0^\infty \frac{t^{Q-1}}{(16a^2+t^2)^{r\left(\frac{Q}{2}+\mathrm{Im}(\lambda)\right)}}\, dt\\
&< \infty. \qquad (6.2.15)
\end{aligned}$$

We also have the following important formula [Kum16, Lemma 2.3],

$$\begin{aligned}
\int_N \mathcal{P}_\lambda(e,n)\, dm(n) &= \int_N \left[\frac{P(n)}{P(\underline{0})}\right]^{\frac{1}{2}-\frac{i\lambda}{Q}} dm(n)\\
&= \frac{\mathbf{c}(-\lambda)}{c_{p,k}}, \qquad \mathrm{Im}(\lambda) \in (0,\infty), \qquad (6.2.16)
\end{aligned}$$

where $\mathbf{c}(\lambda)$ generalizes Harish-Chandra $\mathbf{c}$-function and is given by

$$\mathbf{c}(\lambda) = \frac{2^{Q-2i\lambda}\Gamma(2i\lambda)\Gamma\left(\frac{2p+k+1}{2}\right)}{\Gamma\left(\frac{Q}{2}+i\lambda\right)\Gamma\left(\frac{p+1}{2}+i\lambda\right)}, \qquad \mathrm{Im}(\lambda) < 0. \qquad (6.2.17)$$

From the above formula it follows that the $\mathbf{c}$-function has no pole or zero in $\{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) < 0\}$. Therefore, using (6.2.16), we can normalize $\mathcal{P}_\lambda$, for $\mathrm{Im}(\lambda) \in (0,\infty)$, to define

$$\tilde{\mathcal{P}}_\lambda(x,n) = C_\lambda \mathcal{P}_\lambda(x,n), \quad x \in S,\ n \in N, \qquad (6.2.18)$$

where $C_\lambda = c_{p,k}\mathbf{c}(-\lambda)^{-1}$. For $\mathrm{Im}(\lambda) \in (0,\infty)$, the λ-Poisson transform of a measure μ on N is defined by

$$\mathcal{P}_\lambda[\mu](n,a) = \int_N \tilde{\mathcal{P}}_\lambda((n,a),n')\, d\mu(n'), \tag{6.2.19}$$

whenever the integral converges absolutely for every $(n,a) \in S$. In this case, we say that the λ-Poisson transform $\mathcal{P}_\lambda[\mu]$ of μ is well-defined. If $d\mu = f dm$ for some $f \in L^r(N)$, where $r \in [1,\infty]$, then $\mathcal{P}_\lambda[\mu]$ is well-defined and we denote it by $\mathcal{P}_\lambda f$. Since for each $n \in N$, $\tilde{\mathcal{P}}_\lambda(.,n)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $-(\lambda^2+\rho^2)$, it follows that $\mathcal{P}_\lambda[\mu]$ is also an eigenfunction of $\mathcal{L}$ with the same eigenvalue, provided $\mathcal{P}_\lambda[\mu]$ is well-defined. Using the definition of $\mathcal{P}_\lambda$ given in (6.2.14) and the relation (6.2.18), we make the following important observation for $\mathrm{Im}(\lambda) \in (0,\infty)$, $x = (n_1,a) \in S$ and $n \in N$.

$$\begin{aligned}
\tilde{\mathcal{P}}_\lambda(x,n) &= C_\lambda \mathcal{P}_\lambda(x,n) \\
&= C_\lambda \left[P(\underline{0})^{-1} P_a(n_1^{-1}n)\right]^{\frac{1}{2}-\frac{i\lambda}{Q}} \\
&= C_\lambda \left[P(\underline{0})^{-1} a^{-Q} P\left(\delta_{a^{-1}}(n_1^{-1}n)\right)\right]^{\frac{1}{2}-\frac{i\lambda}{Q}} \\
&= C_\lambda a^{\frac{Q}{2}+i\lambda} a^{-Q} \left[P(\underline{0})^{-1} P\left(\delta_{a^{-1}}(n_1^{-1}n)\right)\right]^{\frac{1}{2}-\frac{i\lambda}{Q}} \\
&= a^{\frac{Q}{2}+i\lambda} a^{-Q} q^\lambda\left(\delta_{a^{-1}}(n_1^{-1}n)\right) \\
&= a^{\frac{Q}{2}+i\lambda} q_a^\lambda(n_1^{-1}n),
\end{aligned} \tag{6.2.20}$$

where for $\mathrm{Im}(\lambda) \in (0,\infty)$,

$$\begin{aligned}
q^\lambda(n) &= C_\lambda \left[P(\underline{0})^{-1} P(n)\right]^{\frac{1}{2}-\frac{i\lambda}{Q}}; \\
q_a^\lambda(n) &= a^{-Q} q^\lambda\left(\delta_{a^{-1}}(n)\right), \quad n \in N,\ a \in A.
\end{aligned} \tag{6.2.21}$$

We will need more explicit expression of the function q_a^λ, $\mathrm{Im}(\lambda) \in (0,\infty)$, which can be obtained from the expression (6.2.13) of the function P_a.

$$q_a^\lambda(n) = c_\lambda \frac{a^{-2i\lambda}}{(16a^2 + 8a\|X\|^2 + d(n)^2)^{\frac{Q}{2}-i\lambda}}, \quad n = (X,Z) \in N,\ a \in A, \tag{6.2.22}$$

where $c_\lambda = 16^{\rho - i\lambda} c_{p,k}\mathbf{c}(-\lambda)^{-1}$. It is clear from (6.2.20) and the definition of q^λ that if $\mathrm{Im}(\lambda) \in (0,\infty)$, then $q^\lambda \in L^r(N)$, for all $r \in [1,\infty]$ (as, by (6.2.15), $\mathcal{P}_\lambda(e,\cdot)$ is so), and that

$$\int_N q^\lambda(n)\, dm(n) = 1.$$

It thus follow that $\{q_a^\lambda \mid a \in A\}$, is an approximate identity on N. For a measure μ on N and $\mathrm{Im}(\lambda) \in (0,\infty)$, we define the convolution integral

$$\mathcal{Q}_\lambda[\mu](n,a) = \mu * q_a^\lambda(n) = a^{-Q} \int_N q^\lambda \left(\delta_{a^{-1}}(n_1^{-1}n)\right) \, d\mu(n_1), \tag{6.2.23}$$

whenever the integral converges absolutely for every $(n,a) \in S$, and say that $\mathcal{Q}_\lambda[\mu]$ is well-defined. If $d\mu = f dm$ for some $f \in L^r(N)$, where $r \in [1,\infty]$, then $\mathcal{Q}_\lambda[\mu]$ is well-defined and we denote it by $\mathcal{Q}_\lambda f$. From the definition of the λ-Poisson integral (6.2.19) and (6.2.20), it follows that for a measure μ with well-defined $\mathcal{P}_\lambda[\mu]$, where $\mathrm{Im}(\lambda) \in (0,\infty)$, we have

$$\mathcal{P}_\lambda[\mu](n,a) = a^{\frac{Q}{2}+i\lambda} \mathcal{Q}_\lambda[\mu](n,a), \quad \text{for all } (n,a) \in S. \tag{6.2.24}$$

Remark 6.2.2. $\mathcal{P}_\lambda[\mu]$ is a convolution but not with an approximate identity on N. However, that is the case with $\mathcal{Q}_\lambda[\mu]$. Since $\{q_a^\lambda \mid a \in A\}$ is an approximate identity on N, it follows from (6.2.24) that for $f \in C_c(N)$, and $\mathrm{Im}(\lambda) \in (0,\infty)$,

$$\lim_{a\to 0} \frac{\mathcal{P}_\lambda f(n,a)}{a^{\frac{Q}{2}+i\lambda}} = \lim_{a\to 0} \mathcal{Q}_\lambda f(n,a) = f(n),$$

uniformly for $n \in N$.

In this chapter we will be interested only in the case $\lambda = i\beta$, for $\beta \in (0,\infty)$. Using the formula of P_a given in (6.2.13) we can explicitly write down the expression of $\mathcal{P}_{i\beta}[\mu]$ for a suitable measure μ. In this regard, we define the following class of measures on N.

Definition 6.2.3. We fix some $\beta \in (0,\infty)$, and denote by M_β the set of all measures μ on N such that $\mathcal{P}_{i\beta}[\mu]$ (equivalently $\mathcal{Q}_{i\beta}[\mu]$) is well-defined.

If $\mu \in M_\beta$, then using (6.2.22), we have for all $n = (X,Z) \in N$, $a \in A$

$$\begin{aligned} \mathcal{Q}_{i\beta}[\mu](n,a) &= a^{-Q} \int_N q^{i\beta} \left(\delta_{a^{-1}}(n_1^{-1}n)\right) \, d\mu(n_1) \\ &= \int_N \frac{c_\beta \, a^{2\beta}}{\left(16a^2 + 8a\|X - X_1\|^2 + d\left((X_1,Z_1)^{-1}(X,Z)\right)^2\right)^{\rho+\beta}} \, d\mu(X_1,Z_1), \end{aligned} \tag{6.2.25}$$

where

$$c_\beta = 16^{\rho+\beta} C_{i\beta} = 16^{\rho+\beta} c_{p,k} \mathbf{c}(-i\beta)^{-1} > 0,$$

follows from the expression of the $\mathbf{c}$-function given in (6.2.17). The following elementary lemma gives a lower bound for the Poisson kernel $P_a(n)$, for large values of $d(n)$.

Lemma 6.2.4. *If $R \in [1, \infty)$, then for each $a \in A$, there exists a positive constant C_a such that*

$$\frac{1}{16a^2 + 8a\|X\|^2 + d(n)^2} \geq \frac{C_a}{16a^2 + \frac{d(n)^2}{4\tau^2}},$$

for all $n = (X, Z) \in \mathrm{B}(\underline{0}, 2R)^c$.

Proof. We take $a \in A$, and consider the quotient

$$T(n) = \frac{16a^2 + d(n)^2 + 8a\|X\|^2}{16a^2 + \frac{d(n)^2}{4\tau^2}}, \quad n \in N.$$

For $n = (X, Z) \in N$, with $d(n) > 2R$, we have from (6.2.4)

$$d(n)^2 \geq \|X\|^4 \geq \|X\|^2, \qquad \text{if} \quad \|X\| > 1,$$

and hence for $\|X\| > 1$,

$$T(n) \leq 4\tau^2 \left(\frac{16a^2}{d(n)^2} + 1 + 8a\frac{\|X\|^2}{d(n)^2} \right) \leq 4\tau^2 \left(\frac{4a^2}{R^2} + 1 + 8a \right).$$

On the other hand, if $\|X\| \leq 1$, then

$$T(n) \leq 4\tau^2 \left(\frac{16a^2}{d(n)^2} + 1 + 8a\frac{\|X\|^2}{d(n)^2} \right) \leq 4\tau^2 \left(\frac{4a^2}{R^2} + 1 + \frac{8a}{4R^2} \right).$$

As $R \in [1, \infty)$, combining both the inequalities we get that for all $n = (X, Z) \in \mathrm{B}(\underline{0}, 2R)^c$

$$T(n) \leq 4\tau^2(4a^2 + 1 + 8a). \tag{6.2.26}$$

The result follows by setting $C_a = (4\tau^2(4a^2 + 1 + 8a))^{-1}$. □

Lemma 6.2.5. *Let $\beta \in (0, \infty)$, and let μ be a positive measure on N such that $\mu * q_a^{i\beta}(\underline{0})$ is finite for some $a \in A$. Then*

$$\int_N \left(16a^2 + \frac{d(n)^2}{4\tau^2} \right)^{-\beta-\rho} d\mu(n) < \infty.$$

Proof. Since the integrand is a continuous function on N, it suffices to show that

$$\int_{\mathrm{B}(\underline{0},2)^c}\left(16a^2+\frac{d(n)^2}{4\tau^2}\right)^{-\beta-\rho} d\mu(n)<\infty.$$

Using Lemma 6.2.4 and the explicit expression of q_a^λ, for $\lambda=i\beta$ (see 6.2.22)), we obtain

$$\begin{aligned} & a^{2\beta}\int_{\mathrm{B}(\underline{0},2)^c}\left(16a^2+\frac{d(n)^2}{4\tau^2}\right)^{-\beta-\rho} d\mu(n) \\ \leq\ & C_{a,\beta}\, a^{2\beta}\int_{\mathrm{B}(\underline{0},2)^c}\left(16a^2+d(n)^2+8a\|X\|^2\right)^{-\beta-\rho} d\mu(n) \\ =\ & C'_{a,\beta}\, \mu * q_a^{i\beta}(\underline{0})<\infty. \end{aligned}$$

This completes the proof. □

We have observed in Remark 6.2.2 that $\mathcal{Q}_{i\beta}f(\cdot,a)\to f$, as $a\to 0$, uniformly on N, whenever $f\in C_c(N)$, and $\beta\in(0,\infty)$. However, a stronger result is true.

Lemma 6.2.6. *If $\beta\in(0,\infty)$ and $f\in C_c(N)$, then*

$$\frac{\mathcal{Q}_{i\beta}f(\cdot,a)}{q^{i\beta}}\to\frac{f}{q^{i\beta}},$$

uniformly on N as $a\to 0$.

Proof. We assume that $\operatorname{supp} f\subset \mathrm{B}(\underline{0},R)$ for some $R\in(1,\infty)$. Since $q^{i\beta}$ is a strictly positive continuous function on N, it follows that $\frac{1}{q^{i\beta}}$ is bounded in $\overline{\mathrm{B}(\underline{0},2\tau R)}$. In view of Remark 6.2.2, it suffices to prove that

$$\frac{\mathcal{Q}_{i\beta}f(n,a)}{q^{i\beta}(n)}\to 0,$$

uniformly for $n\in\mathrm{B}(\underline{0},2\tau R)^c$, as a goes to zero. From the quasi-triangle inequality (6.2.5) for d, we have

$$d(n_1^{-1}n)\geq\frac{d(n)}{\tau}-d(n_1),\quad \text{for all } n\in N,\ n_1\in N. \tag{6.2.27}$$

For $n\in\mathrm{B}(\underline{0},2\tau R)^c$, and $n_1\in\mathrm{B}(\underline{0},R)$, we have

$$d(n_1)<R\leq\frac{d(n)}{2\tau}.$$

Hence, for all $n \in \mathrm{B}(\underline{0}, 2\tau R)^c$, and $n_1 \in \mathrm{B}(\underline{0}, R)$,

$$d(n_1^{-1}n) \geq \frac{d(n)}{\tau} - \frac{d(n)}{2\tau} = \frac{d(n)}{2\tau}. \tag{6.2.28}$$

From the expression of $\mathcal{Q}_{i\beta}f$ (see 6.2.25) and the inequality (6.2.28) above, it follows that for $n = (X, Z) \in \mathrm{B}(\underline{0}, 2\tau R)^c$, $a \in A$,

$$\begin{aligned}
& |\mathcal{Q}_{i\beta}f(n,a)| \\
\leq \;& c_\beta\, a^{2\beta} \int_{\mathrm{B}(\underline{0},R)} \frac{|f(X_1, Z_1)|}{\left(16a^2 + 8a\|X - X_1\|^2 + d\left((X_1,Z_1)^{-1}(X,Z)\right)^2\right)^{\rho+\beta}}\, dX_1 dZ_1 \\
\leq \;& c_\beta\, a^{2\beta} \int_{\mathrm{B}(\underline{0},R)} \frac{|f(X_1, Z_1)|}{\left(16a^2 + 8a\|X - X_1\|^2 + \frac{d(X,Z)^2}{4\tau^2}\right)^{\beta+\rho}}\, dX_1 dZ_1 \\
\leq \;& c_\beta\, a^{2\beta} \int_{\mathrm{B}(\underline{0},R)} \frac{|f(X_1, Z_1)|}{\left(16a^2 + \frac{d(X,Z)^2}{4\tau^2}\right)^{\beta+\rho}}\, dX_1 dZ_1.
\end{aligned}$$

Using the expression of $q^{i\beta}$ given in (6.2.22) and Lemma 6.2.4 (as $\tau \geq 1$) for $a = 1$, it follows from the inequality above that for all $n = (X, Z) \in \mathrm{B}(\underline{0}, 2\tau R)^c$,

$$\begin{aligned}
\frac{|\mathcal{Q}_{i\beta}f(n,a)|}{q^{i\beta}(n)} &\leq c_\beta\, a^{2\beta} \left(\frac{16 + 8\|X\|^2 + d(n)^2}{16a^2 + \frac{d(n)^2}{4\tau^2}}\right)^{\beta+\rho} \int_{\mathrm{B}(\underline{0},R)} |f(n_1)|\, dm(n_1) \\
&\leq c_{\beta,\tau}\, a^{2\beta} \left(\frac{16 + \frac{d(n)^2}{4\tau^2}}{16a^2 + \frac{d(n)^2}{4\tau^2}}\right)^{\beta+\rho} \int_{\mathrm{B}(\underline{0},R)} |f(n_1)|\, dm(n_1) \\
&\leq c_{\beta,\tau}\, a^{2\beta} \left(\frac{64\tau^2}{d(n)^2} + 1\right)^{\beta+\rho} \|f\|_{L^1(N)} \\
&\leq C_R\, a^{2\beta} \|f\|_{L^1(N)},
\end{aligned}$$

as $d(n)^2 > 2\tau R$. Letting $a \to 0$, in the last inequality, we complete the proof. □

Recall that for $\beta \in (0, \infty)$, M_β denotes the set of all measures μ on N such that $\mathcal{Q}_{i\beta}[\mu]$ is well-defined (see the paragraph after Remark 6.2.2). It is clear that $L^r(N) \subset M_\beta$, for all $r \in [1, \infty]$. We also note that if $|\mu|(N)$ is finite, then $\mu \in M_\beta$. In particular, every complex measure μ on N belongs to M_β. We have the following observation regarding this class of measures.

Lemma 6.2.7. *Suppose that $\beta \in (0,\infty)$. If $\nu \in M_\beta$, and $f \in C_c(N)$, then for each fixed $a \in A$,*

$$\int_N \mathcal{Q}_{i\beta} f(n,a)\, d\nu(n) = \int_N \mathcal{Q}_{i\beta}[\nu](n,a) f(n)\, dm(n).$$

Proof. The result will follow by interchanging integrals using Fubini's theorem. In order to apply Fubini's theorem we have to show that for each fixed $a \in A$,

$$\int_N \int_{\text{supp } f} q_a^{i\beta}(n_1^{-1}n)|f(n_1)|\, dm(n_1)\, d|\nu|(n) < \infty.$$

We asuume that supp $f \subset \mathrm{B}(\underline{0}, R)$, for some $R \in (1,\infty)$. We fix $a \in A$. Then

$$\begin{aligned}
I &= \int_N \int_{\mathrm{B}(\underline{0},R)} q_a^{i\beta}(n_1^{-1}n)|f(n_1)|\, dm(n_1)\, d|\nu|(n) \\
&= c_\beta\, a^{2\beta} \int_{\mathrm{B}(\underline{0},2\tau R)} \int_{\mathrm{B}(\underline{0},R)} \frac{|f(X_1,Z_1)|\, dm(X_1,Z_1)\, d|\nu|(X,Z)}{\left(16a^2 + 8a\|X - X_1\|^2 + d\left((X_1,Z_1)^{-1}(X,Z)\right)^2\right)^{\rho+\beta}} \\
&\quad + c_\beta\, a^{2\beta} \int_{\mathrm{B}(\underline{0},2\tau R)^c} \int_{\mathrm{B}(\underline{0},R)} \frac{|f(X_1,Z_1)|\, dm(X_1,Z_1)\, d|\nu|(X,Z)}{\left(16a^2 + 8a\|X - X_1\|^2 + d\left((X_1,Z_1)^{-1}(X,Z)\right)^2\right)^{\rho+\beta}} \\
&\leq c_\beta\, a^{2\beta}(16a^2)^{-\beta-\rho}|\nu|(\mathrm{B}(\underline{0},2\tau R))\|f\|_{L^1(N)} \\
&\quad + c_\beta\, a^{2\beta} \int_{\mathrm{B}(\underline{0},2\tau R)^c} \int_{\mathrm{B}(\underline{0},R)} \frac{|f(X_1,Z_1)|\, dm(X_1,Z_1)\, d|\nu|(X,Z)}{\left(16a^2 + 8a\|X - X_1\|^2 + \frac{d(X,Z)^2}{4\tau^2}\right)^{\beta+\rho}} \\
&\qquad\qquad \text{(using (6.2.28) in the second integral)} \\
&\leq c_\beta\, a^{2\beta}(16a^2)^{-\beta-\rho}|\nu|(\mathrm{B}(\underline{0},2\tau R))\|f\|_{L^1(N)} \\
&\quad + c_\beta\, a^{2\beta}\|f\|_{L^1(N)} \int_{\mathrm{B}(\underline{0},2\tau R)^c} \left(16a^2 + \frac{d(X,Z)^2}{4\tau^2}\right)^{-\beta-\rho} d|\nu|(X,Z).
\end{aligned}$$

As $\nu \in M_\beta$, we have finiteness of the quantity $|\nu| * q_a^{i\beta}(\underline{0})$. Lemma 6.2.5 now implies that the integral on the right-hand side of the inequality above is finite. This completes the proof. □

We end this section with the following important definitions which constitute the heart of the matter.

Definition 6.2.8.

i) A function u defined on S is said to have admissible limit $L \in \mathbb{C}$, at $n_0 \in N$, if for each $\alpha \in (0,\infty)$,

$$\lim_{\substack{a \to 0 \\ (n,a) \in \Gamma_\alpha(n_0)}} u(n,a) = L,$$

where

$$\begin{aligned}\Gamma_\alpha(n_0) &= \{(n,a)\in S \mid d(n_0^{-1}n) < \alpha a\} \\ &= \{(n,a)\in S \mid \mathbf{d}(n_0,n) < \alpha a\}. \end{aligned} \tag{6.2.29}$$

is called the admissible domain with vertex at n_0 and aperture α.

ii) For a differential operator D on S, a smooth function u on S satisfying $Du = 0$ is said to be a D-harmonic function.

For the notion of admissible convergence in the context of Riemannian symmetric spaces of noncompact type with real rank one we refer the reader to [KP76, P.158].

6.3 Some auxilary results

We start with the following result which relates the weak* convergence of the sequence of positive measures $\{\mu_j \mid j \in \mathbb{N}\}$ with the normal convergence of the sequence of functions $\{\mathcal{Q}_{i\beta}[\mu]\ j \in \mathbb{N}\}$. This result is analogous to Lemma 4.3.1.

Lemma 6.3.1. *Let $\beta \in (0,\infty)$. Suppose that $\{\mu_j \mid j \in \mathbb{N}\} \subset M_\beta$, and $\mu \in M_\beta$, are positive measures. If $\{\mathcal{Q}_{i\beta}[\mu_j]\}$ converges normally to $\mathcal{Q}_{i\beta}[\mu]$, then $\{\mu_j\}$ converges to μ in weak*.*

Proof. We have to show that if $f \in C_c(N)$, then

$$\lim_{j\to\infty} \int_N f(n)\, d\mu_j(n) = \int_N f(n)\, d\mu(n).$$

We assume that $f \in C_c(N)$, with supp $f \subset \mathrm{B}(\underline{0}, R)$, for some $R \in (1,\infty)$. For any $a \in A$, we write

$$\begin{aligned} &\int_N f(n)\, d\mu_j(n) - \int_N f(n)\, d\mu(n) \\ = \ &\int_N (f(n) - \mathcal{Q}_{i\beta} f(n,a))\, d\mu_j(n) + \int_N \mathcal{Q}_{i\beta} f(n,a)\, d\mu_j(n) - \int_N \mathcal{Q}_{i\beta} f(n,a)\, d\mu(n) \\ &\quad + \int_N \left(\mathcal{Q}_{i\beta} f(n,a) - f(n)\right)\, d\mu(n). \end{aligned} \tag{6.3.1}$$

Given a positive number ϵ, by Lemma 6.2.6 we get some $a_0 \in (0, \infty)$, such that

$$\frac{|\mathcal{Q}_{i\beta} f(n, a_0) - f(n)|}{q^{i\beta}(n)} < \epsilon, \quad \text{for all} \quad n \in N. \tag{6.3.2}$$

Using Lemma 6.2.7, it follows that

$$\begin{aligned} & \int_N \mathcal{Q}_{i\beta} f(n, a)\, d\mu_j(n) - \int_N \mathcal{Q}_{i\beta} f(n, a)\, d\mu(n) \\ = & \int_N \left(\mathcal{Q}_{i\beta}[\mu_j](n, a) - \mathcal{Q}_{i\beta}[\mu](n, a)\right) f(n)\, dm(n). \end{aligned}$$

Applying the relation above for $a = a_0$ in (6.3.1) we get that

$$\begin{aligned} & \left| \int_N f(n)\, d\mu_j(n) - \int_N f(n)\, d\mu(n) \right| \\ \leq & \int_N |f(n) - \mathcal{Q}_{i\beta} f(n, a_0)|\, d\mu_j(n) + \int_{\mathrm{B}(\underline{0}, R)} |\mathcal{Q}_{i\beta}[\mu_j](n, a_0) - \mathcal{Q}_{i\beta}[\mu](n, a_0)|\, |f(n)|\, dm(n) \\ & \quad + \int_N |f(n) - \mathcal{Q}_{i\beta} f(n, a_0)|\, d\mu(n) \\ = & \; I_1(j) + I_2(j) + I_3. \end{aligned} \tag{6.3.3}$$

In order to estimate $I_1(j)$, we use (6.3.2) to get

$$I_1(j) = \int_N \frac{|f(n) - \mathcal{Q}_{i\beta} f(n, a_0)|}{q^{i\beta}(n)} q^{i\beta}(n)\, d\mu_j(n) < \epsilon \int_N q^{i\beta}(n)\, d\mu_j(n) = \epsilon \mathcal{Q}_{i\beta}[\mu_j](e).$$

Similarly, we can prove that

$$I_3 \leq \epsilon \mathcal{Q}_{i\beta}[\mu](e).$$

Since $\{\mathcal{Q}_{i\beta}[\mu_j]\}$ converges to $\mathcal{Q}_{i\beta}[\mu]$ normally, the sequence $\{\mathcal{Q}_{i\beta}[\mu_j](e)\}$, in particular, is bounded. Hence, setting

$$C = \sup_{j \in \mathbb{N}} \mathcal{Q}_{i\beta}[\mu_j](e) + \mathcal{Q}_{i\beta}[\mu](e),$$

we get that for all $j \in \mathbb{N}$

$$I_1(j) + I_3 \leq 2C\epsilon.$$

Again using the hypothesis that $\{\mathcal{Q}_{i\beta}[\mu_j]\}$ converges normally to $\mathcal{Q}_{i\beta}[\mu]$, we get some $j_0 \in \mathbb{N}$, such that for all $j \geq j_0$,

$$\|\mathcal{Q}_{i\beta}[\mu_j] - \mathcal{Q}_{i\beta}[\mu]\|_{L^\infty\left(\overline{\mathrm{B}(\underline{0}, R)} \times \{a_0\}\right)} < \epsilon.$$

This implies that for all $j \geq j_0$,

$$I_2(j) \leq \epsilon \|f\|_{L^1(N)}.$$

Hence, it follows from (6.3.3) that

$$\left| \int_N f(n)\, d\mu_j(n) - \int_N f(n)\, d\mu(n) \right| \leq \epsilon (2C + \|f\|_{L^1(N)}),$$

for all $j \geq j_0$. This completes the prove. $\square$

We shall next prove a result regarding pointwise comparison between the Hardy-Littlewood maximal function of a positive measure on N and Poisson maximal functions of the same measure. We recall that for a positive measure μ on N, the Hardy-Littlewood maximal function $M_{HL}(\mu)$ of μ is given by

$$M_{HL}(\mu)(n) = \sup_{r>0} \frac{\mu(\mathrm{B}(n,r))}{m(\mathrm{B}(n,r))}, \quad n \in N.$$

The following result is analogous to Lemma 4.3.7 and the proof is also similar.

Lemma 6.3.2. *If $\mu \in M_\beta$ is a positive measure, for some $\beta \in (0,\infty)$, and $\alpha \in (0,\infty)$, then there exist positive constants C_β and $C_{\alpha,\beta}$ such that for all $n_0 \in N$,*

$$C_\beta M_{HL}(\mu)(n_0) \leq \sup_{a \in A} \mathcal{Q}_{i\beta}[\mu](n_0, a) \leq \sup_{(n,a)\in\Gamma_\alpha(n_0)} \mathcal{Q}_{i\beta}[\mu](n,a) \leq C_{\alpha,\beta} M_{HL}(\mu)(n_0).$$

Proof. We fix an $n_0 = (X_0, Z_0) \in N$, and note that the second inequality follows from the definition of supremum. To prove the left-most inequality we take $a \in A$. Using the expression of $\mathcal{Q}_{i\beta}[\mu]$ given in (6.2.25), note that

$$\begin{aligned}
&\mathcal{Q}_{i\beta}[\mu](n_0, a) \\
&= c_\beta\, a^{-Q} \int_N \frac{1}{\left(16 + 8\frac{\|X_0 - X\|^2}{a} + \frac{d((X,Z)^{-1}(X_0,Z_0))^2}{a^2}\right)^{\rho+\beta}}\, d\mu(X_1, Z_1) \\
&\geq c_\beta\, a^{-Q} \int_{\mathrm{B}(n_0,a)} \frac{1}{\left(16 + 8\frac{\|X_0 - X\|^2}{a} + \frac{d((X,Z)^{-1}(X_0,Z_0))^2}{a^2}\right)^{\rho+\beta}}\, d\mu(X_1, Z_1). \qquad (6.3.4)
\end{aligned}$$

For $(X,Z) \in \mathrm{B}(n_0, a)$,

$$d\left((X,Z)^{-1}(X_0,Z_0)\right)^2 = \|X_0 - X\|^4 + 16\|Z - Z_0\|^2 < a^2,$$

and hence

$$\|X_0 - X\|^2 < a.$$

Consequently, for all $(X, Z) \in \mathrm{B}(n_0, a)$

$$\frac{c_\beta}{\left(16 + 8\frac{\|X_0-X\|^2}{a} + \frac{d((X,Z)^{-1}(X_0,Z_0))^2}{a^2}\right)^{\rho+\beta}} \geq \frac{c_\beta}{(16+8+1)^{\rho+\beta}} = C'_\beta.$$

Using this observation in (6.3.4), we get

$$\mathcal{Q}_{i\beta}[\mu](n_0, a) \geq C'_\beta a^{-Q} \mu(\mathrm{B}(n_0, a)) = C_\beta \frac{\mu(\mathrm{B}(n_0, a))}{m(\mathrm{B}(n_0, a))}.$$

Taking supremum over $a > 0$, on both sides of the inequality above, we get

$$C_\beta M_{HL}(\mu)(n_0) \leq \sup_{a>0} \mathcal{Q}_{i\beta}[\mu](n_0, a). \tag{6.3.5}$$

To prove the right-most inequality, we define

$$\phi(n) = \frac{c_\beta}{(16 + d(n)^2)^{\rho+\beta}}, \quad n \in N.$$

Then, as in the proof of the second part of Lemma 4.3.5 (see Remark 4.3.6), we can show that

$$\sup_{\substack{(n,a)\in N\times(0,\infty)\\ d(n_0,n)<\alpha a}} \mu * \phi_a(n) \leq C_{\alpha,\beta} M_{HL}(\mu)(n_0). \tag{6.3.6}$$

But it follows from the expression of $q_a^{i\beta}$ (for $a = 1$, see (6.2.22)) that

$$q^{i\beta}(n) \leq \phi(n), \quad \text{for all } n \in N.$$

Hence,

$$\sup_{(n,a)\in\Gamma_\alpha(n_0)} \mathcal{Q}_{i\beta}[\mu](n, a) = \sup_{(n,a)\in\Gamma_\alpha(n_0)} \mu * q_a^{i\beta}(n) \leq \sup_{\substack{(n,a)\in N\times(0,\infty)\\ d(n_0,n)<\alpha a}} \mu * \phi_a(n) \tag{6.3.7}$$

The right-most inequality now follows by combining (6.3.7) and (6.3.6). $\square$

Given $\beta \in (0, \infty)$, we define a second order differential operator $\mathcal{L}^\beta$ on S (see [DK16, Theorem 3.2]), having the same second order term as the Laplace-Beltrami operator $\mathcal{L}$, by the formula

$$\mathcal{L}^{\beta} = a^2\partial_a^2 + \mathcal{L}_a + (1-2\beta)a\partial_a.$$

We recall that the Laplace-Beltrami operator $\mathcal{L}$ is given by

$$\mathcal{L} = a^2\partial_a^2 + \mathcal{L}_a + (1-Q)a\partial_a.$$

Thus, when $\beta = \rho = Q/2$, we recover $\mathcal{L}$. We note that

$$\mathcal{L}^{\beta} - \mathcal{L} = 2(\rho-\beta)a\partial_a = 2(\rho-\beta)E_0. \tag{6.3.8}$$

We recall that $E_0 = a\partial_a$ is the left-invariant vector field on S corresponding to the basis element $e_0 = (0,0,1)$ of $\mathfrak{s}$ and hence $\mathcal{L}^{\beta}$ is left S-invariant. The following lemma shows that there is a one to one correspondence between the eigenfunctions of $\mathcal{L}$ with eigenvalue $\beta^2-\rho^2$ and $\mathcal{L}^{\beta}$-harmonic functions (see Definition 6.2.8, ii)).

Lemma 6.3.3. *Let $\beta \in (0,\infty)$, and let u be a smooth function on S. Then u is an eigenfunction of $\mathcal{L}$ with eigenvalue $\beta^2-\rho^2$, if and only if the function $(n,a) \mapsto a^{\beta-\rho}u(n,a)$, is $\mathcal{L}^{\beta}$-harmonic.*

Proof. If $\beta = \rho$, then there is nothing to prove. So, we assume that $\beta \neq \rho$. We set

$$F(n,a) = a^{\beta-\rho}u(n,a), \quad (n,a) \in S.$$

Suppose u is an eigenfunction of $\mathcal{L}$ with eigenvalue $\beta^2-\rho^2$. We note that

$$\mathcal{L}(a^{\rho-\beta}F) = \mathcal{L}u = (\beta^2-\rho^2)u = (\beta^2-\rho^2)a^{\rho-\beta}F.$$

Since $\mathcal{L}_a$ does not have any term involving ∂_a (see (6.2.10)), expanding the left-hand side of the equation above we obtain

$$\begin{aligned}(\beta^2-\rho^2)a^{\rho-\beta}F &= \Big((1-2\rho)a\partial_a + a^2\partial_a^2 + \mathcal{L}_a\Big)(a^{\rho-\beta}F) \\ &= (1-2\rho)a\Big((\rho-\beta)a^{\rho-\beta-1}F + a^{\rho-\beta}\partial_a F\Big) \\ &\quad + a^2\Big((\rho-\beta)(\rho-\beta-1)a^{\rho-\beta-2}F + 2(\rho-\beta)a^{\rho-\beta-1}\partial_a F + a^{\rho-\beta}\partial_a^2 F\Big)\end{aligned}$$

$$
\begin{aligned}
&+ a^{\rho-\beta}\mathcal{L}_a F \\
&= a^{\rho-\beta}\left((1-2\rho)(\rho-\beta) + (\rho-\beta)(\rho-\beta-1)\right) F \\
&\quad + a^{\rho-\beta}\left((1-2\rho) + 2(\rho-\beta)\right) a\partial_a F + a^{\rho-\beta}\left(a^2\partial_a^2 F + \mathcal{L}_a F\right) \\
&= a^{\rho-\beta}\left(\beta^2 - \rho^2 + (1-2\beta)a\partial_a + a^2\partial_a^2 + \mathcal{L}_a\right) F.
\end{aligned}
$$

Canceling the term $(\beta^2-\rho^2)a^{\rho-\beta}F$ from both sides of the equation above shows that F is $\mathcal{L}^\beta$-harmonic.

Conversely, suppose that F is $\mathcal{L}^\beta$-harmonic. Using the definition of $\mathcal{L}^\beta$ and F, we can write

$$\left(a^2\partial_a^2 + \mathcal{L}_a + (1-2\beta)a\partial_a\right)(a^{\beta-\rho}u) = 0.$$

Expanding the left-hand side of the equation above as before, we get

$$(-\beta^2+\rho^2)u + \left((1-Q)a\partial_a + a^2\partial_a^2 + \mathcal{L}_a\right)u = 0.$$

Hence, u is an eigenfunction of $\mathcal{L}$ with eigenvalue $\beta^2-\rho^2$. □

From (6.3.8) we see that $\mathcal{L}^\beta$ and $\mathcal{L}$ differs by a first order term and hence $\mathcal{L}^\beta$ is an elliptic operator. Thus, applying Bär's result on generalization of Montel's theorem [Bär13, Theorem 4], we get the following result.

Lemma 6.3.4. *Let $\beta \in (0,\infty)$, and let $\{F_j\}$ be a sequence of $\mathcal{L}^\beta$-harmonic functions on S. If $\{F_j\}$ is locally bounded then it has a subsequence which converges normally to a $\mathcal{L}^\beta$-harmonic function F.*

We recall that if $\beta \in (0,\infty)$, and $\mu \in M_\beta$, is a positive measure then $\mathcal{P}_{i\beta}[\mu]$ is a positive eigenfunction of the Laplace-Beltrami operator $\mathcal{L}$ with eigenvalue $\beta^2-\rho^2$. Characterization of such positive eigenfunctions was proved by Damek and Ricci in [DR92, Theorem 7.11].

Lemma 6.3.5. *Suppose u is a positive eigenfunction of the Laplace-Beltrami operator $\mathcal{L}$ on the Harmonic NA group S with eigenvalue $\beta^2-\rho^2$, for some $\beta \in (0,\infty)$. Then there exists a unique positive measure μ (known as the boundary measure of u) on N and a unique nonnegative constant C such that*

$$u(n,a) = Ca^{\beta+\rho} + \mathcal{P}_{i\beta}[\mu](n,a), \quad \textit{for all } (n,a) \in S. \tag{6.3.9}$$

Remark 6.3.6. The result of Damek and Ricci is valid for all positive eigenfunctions of $\mathcal{L}$, namely, positive eigenfunctions with eigenvalue $\beta^2 - \rho^2$, $\beta \in \mathbb{R}$. However, the results we are going to prove, will not apply when $\beta \in (-\infty, 0]$.

Next, we consider the natural action of the subgroup A on S (see (6.2.1)):

$$r \cdot (n, a) = (\delta_r(n), ra), \quad r \in A, \ (n, a) \in S. \tag{6.3.10}$$

Remark 6.3.7. We note that for each $\alpha \in (0, \infty)$, the admissible domain $\Gamma_\alpha(\underline{0})$ is invariant under the above action of A. Indeed, if $(n, a) \in \Gamma_\alpha(\underline{0})$, then for every $r \in (0, \infty)$,

$$d(\delta_r(n)) = rd(n) < \alpha ra.$$

Given a function F on S and $r \in (0, \infty)$, we define the dilation F_r of F by

$$F_r(n, a) = F\left(\delta_r(n), ra\right), \quad (n, a) \in S.$$

Given a measure ν on N and $r \in (0, \infty)$, we recall that the dilate ν_r of ν is defined by

$$\nu_r(E) = r^{-Q} \nu\left(\delta_r(E)\right), \tag{6.3.11}$$

for every Borel set $E \subseteq N$.

Lemma 6.3.8. *Let $\beta \in (0, \infty)$. If F is an $\mathcal{L}^\beta$-harmonic functions on S then so is F_r, for every $r \in (0, \infty)$.*

Proof. Observe that F_r is the left translation of F by $(\underline{0}, r) \in S$. The proof now follows trivially as $\mathcal{L}^\beta$ is left S-invariant. □

Lemma 6.3.9. *Let $\beta \in (0, \infty)$. If $\nu \in M_\beta$, then for each $r \in (0, \infty)$,*

$$\mathcal{Q}_{i\beta}[\nu_r](n, a) = \mathcal{Q}_{i\beta}[\nu](\delta_r(n), ra), \quad \textit{for all} \ \ (n, a) \in S.$$

Proof. As in the proof of Lemma 4.3.11, it follows from the definition of ν_r (6.3.11), we have for all nonnegative measurable functions f on N that

$$\int_N f(n)\, d\nu_r(n) = r^{-Q} \int_N f\left(\delta_{r^{-1}}(n)\right)\, d\nu(n).$$

Thus, from the definition of $\mathcal{Q}_{i\beta}$ (see 6.2.23) we get that for all $(n,a) \in S$,

$$\begin{aligned}
\mathcal{Q}_{i\beta}[\nu_r](n,a) &= a^{-Q} \int_N q^{i\beta}\left(\delta_{a^{-1}}(n_1^{-1}n)\right)\, d\nu_r(n_1) \\
&= a^{-Q} r^{-Q} \int_N q^{i\beta}\left(\delta_{a^{-1}}\left(\delta_{r^{-1}}(n_1^{-1})n\right)\right)\, d\nu(n_1) \\
&= (ra)^{-Q} \int_N q^{i\beta}\left(\delta_{a^{-1}}\left(\delta_{r^{-1}}\left(n_1^{-1}\delta_r(n)\right)\right)\right)\, d\nu(n_1) \\
&= (ra)^{-Q} \int_N q^{i\beta}\left(\delta_{(ra)^{-1}}\left(n_1^{-1}\delta_r(n)\right)\right)\, d\nu(n_1) \\
&= \mathcal{Q}_{i\beta}[\nu](\delta_r(n), ra). \quad \square
\end{aligned}$$

6.4 Main theorem

We shall first prove a special case of our main result. The proof of the main result will then follow by reducing matters to this special case.

Theorem 6.4.1. *Suppose that u is a positive eigenfunction of the Laplace-Beltrami operator $\mathcal{L}$ on S with eigenvalue $\beta^2 - \rho^2$, where $\beta \in (0,\infty)$, and that $L \in [0,\infty)$. If the boundary measure μ of u is finite then the following statements hold.*

(i) If there exists $\theta \in (0,\infty)$, such that

$$\lim_{\substack{a\to 0\\ (n,a)\in\Gamma_\theta(\underline{0})}} a^{\beta-\rho}u(n,a) = L, \tag{6.4.1}$$

then $D\mu(\underline{0}) = L$.

(ii) If $D\mu(\underline{0}) = L$, then the function

$$(n,a) \mapsto a^{\beta-\rho}u(n,a)$$

has admissible limit L at $\underline{0}$.

Proof. We first prove (i). We choose a d-ball $B_0 \subset N$, a sequence of positive numbers $\{r_j \mid j \in \mathbb{N}\}$ converging to zero and consider the quotient

$$L_j = \frac{\mu\left(\delta_{r_j}(B_0)\right)}{m\left(\delta_{r_j}(B_0)\right)}, \quad j \in \mathbb{N}. \tag{6.4.2}$$

Assuming (6.4.1), we will prove that $\{L_j\}$ is a bounded sequence and every convergent subsequence of $\{L_j\}$ converges to L. We first choose a positive number s such that B_0 is contained in the d-ball $\mathrm{B}(\underline{0}, s)$. Then, exactly as in chapter 4 (see (4.4.2)), we get

$$L_j \leq \frac{m(\mathrm{B}(\underline{0}, s))}{m(B_0)} M_{HL}(\mu). \tag{6.4.3}$$

Thus, to show $\{L_j\}$ is a bounded sequence, it suffices to show that $M_{HL}(\mu)(\underline{0})$ is finite. Since μ is the boundary measure for u we have from (6.3.9) and the relation between $\mathcal{P}_{i\beta}$ and $\mathcal{Q}_{i\beta}$ given in (6.2.24) (putting $\lambda = i\beta$) that

$$a^{\beta-\rho} u(n, a) = C a^{2\beta} + \mathcal{Q}_{i\beta}[\mu](n, a), \quad \text{for all} \quad (n, a) \in S, \tag{6.4.4}$$

for some nonnegative constant C. Since we are interested in the limit as a tends to zero, we may and do assume that C is zero. Therefore, we can rewrite (6.4.1) as

$$\lim_{\substack{a \to 0 \\ (n,a) \in \Gamma_\theta(\underline{0})}} \mathcal{Q}_{i\beta}[\mu](n, a) = L.$$

This implies, in particular, that

$$\lim_{a \to 0} \mathcal{Q}_{i\beta}[\mu](\underline{0}, a) = L,$$

and hence there exists a positive number κ such that

$$\sup_{0 < a < \kappa} \mathcal{Q}_{i\beta}[\mu](\underline{0}, a) < \infty.$$

Since μ is a finite positive measure, using boundedness of the function $q^{i\beta}$, we also have that for all $a \in [\kappa, \infty)$,

$$\mathcal{Q}_{i\beta}[\mu](\underline{0}, a) = a^{-Q} \int_N q^{i\beta}\left(\delta_{a^{-1}}(n_1^{-1})\right) d\mu(n_1) \leq C' a^{-Q} \int_N d\mu(n_1) \leq C' \kappa^{-Q} \mu(N).$$

Combining the above two inequalities, we obtain

$$\sup_{a\in A} \mathcal{Q}_{i\beta}[\mu](\underline{0},a) < \infty.$$

Lemma 6.3.2 now implies that $M_{HL}(\mu)(\underline{0})$ is finite. Boundedness of the sequence $\{L_j\}$ is now a consequence of the inequality (6.4.3). We now choose a convergent subsequence of $\{L_j\}$ and denote it again, for the sake of simplicity, by $\{L_j\}$. For each $j\in\mathbb{N}$, we define a function F_j on S by

$$F_j(n,a) = F\left(\delta_{r_j}(n), r_j a\right), \qquad (n,a)\in S,$$

where, as in Lemma 6.3.3,

$$F(n,a) = a^{\beta-\rho}u(n,a) = \mathcal{Q}_{i\beta}[\mu](n,a), \tag{6.4.5}$$

where the second equality follows from (6.4.4) as C has been assumed to be zero. Lemma 6.3.3 now implies that F is $\mathcal{L}^\beta$-harmonic and hence by Lemma 6.3.8, F_j is $\mathcal{L}^\beta$-harmonic for each $j\in\mathbb{N}$. We now claim that $\{F_j\}$ is locally bounded. To prove this claim, we choose a compact set $K\subset S$. Then there exists a positive number α such that K is contained in $\Gamma_\alpha(\underline{0})$. Indeed, we consider the map

$$(n,a)\mapsto \frac{d(n)}{a}, \quad (n,a)\in S.$$

Using continuity of this map and compactness of K, we get some positive number α such that

$$\frac{d(n)}{a} < \alpha, \quad \text{for all} \quad (n,a)\in K,$$

that is, $K\subset\Gamma_\alpha(\underline{0})$. Using the invariance of $\Gamma_\alpha(\underline{0})$ under the action (6.3.10) (see Remark 6.3.7) and Lemma 6.3.2, we obtain that

$$\sup_j \sup_{(n,a)\in\Gamma_\alpha(\underline{0})} F_j(n,a) = \sup_{(n,a)\in\Gamma_\alpha(\underline{0})} F(n,a) = \sup_{(n,a)\in\Gamma_\alpha(\underline{0})} \mathcal{Q}_{i\beta}[\mu](n,a) \le c_\alpha M_{HL}(\mu)(\underline{0}).$$

As $M_{HL}(\mu)(\underline{0})$ is a finite quantity and $K\subset\Gamma_\alpha(\underline{0})$, this implies that

$$\sup_j \sup_{(n,a)\in K} F_j(n,a) < \infty.$$

Hence, $\{F_j\}$ is a locally bounded sequence of $\mathcal{L}^\beta$-harmonic functions on S. Applying Lemma 6.3.4 (generalization of Montel's theorem), we extract a subsequence $\{F_{j_k}\}$ of F_j which converges normally to a $\mathcal{L}^\beta$-harmonic function g on S. We now show that g is identically equal to L in $\Gamma_\theta(\underline{0})$. To show this, we take $(n_0, a_0) \in \Gamma_\theta(\underline{0})$. Our hypothesis (6.4.1) and the defining equation (6.4.5) of F implies that

$$\lim_{\substack{a \to 0 \\ (n,a) \in \Gamma_\theta(\underline{0})}} F(n,a) = L.$$

Since $\{r_{j_k}\}$ converges to zero as k goes to infinity, and $(\delta_{r_{j_k}}(n_0), r_{j_k} a_0) \in \Gamma_\theta(\underline{0})$, for each $k \in \mathbb{N}$, the equation above shows that

$$g(n_0, a_0) = \lim_{k \to \infty} F_{j_k}(n_0, a_0) = \lim_{k \to \infty} F\left(\delta_{r_{j_k}}(n_0), r_{j_k} a_0\right) = L.$$

So, g is the constant function L on $\Gamma_\theta(\underline{0})$. Since $\mathcal{L}^\beta$ is elliptic with real analytic coefficients, it follows that g is real analytic. Therefore, as $\Gamma_\theta(\underline{0})$ is open in S,

$$g(n,a) = L, \quad \text{for all} \quad (n,a) \in S. \tag{6.4.6}$$

We now consider the dilate $\mu_{r_{j_k}}$ of μ according to (6.3.11). By Lemma 6.3.9, we have that

$$F_{j_k}(n,a) = F\left(\delta_{r_{j_k}}(n), r_{j_k} a\right) = \mathcal{Q}_{i\beta}[\mu]\left(\delta_{r_{j_k}}(n), r_{j_k} a\right) = \mathcal{Q}_{i\beta}\left[\mu_{r_{j_k}}\right](n,a), \tag{6.4.7}$$

for all $(n,a) \in S$. It now follows from (6.4.6) and (6.4.7) that $\mathcal{Q}_{i\beta}[\mu_{r_{j_k}}]$ converges normally to the constant function L, which is same as $\mathcal{Q}_{i\beta}[Lm]$. Lemma 6.3.1 then implies that the sequence of positive measures $\{\mu_{r_{j_k}}\}$ converges to the positive measure Lm in weak*. We then apply Lemma 4.3.3 to conclude that $\{\mu_{r_{j_k}}(B)\}$ converges to $Lm(B)$ for every d-ball $B \subset N$. Choosing $B = B_0$, it follows that

$$Lm(B_0) = \lim_{k \to \infty} \mu_{r_{j_k}}(B_0) = \lim_{k \to \infty} {r_{j_k}}^{-Q} \mu\left(\delta_{r_{j_k}}(B_0)\right) = m(B_0) \lim_{k \to \infty} \frac{\mu\left(\delta_{r_{j_k}}(B_0)\right)}{m\left(\delta_{r_{j_k}}(B_0)\right)}.$$

This implies, together with (6.4.2), that the sequence $\{L_{j_k}\}$ converges to L and hence so does $\{L_j\}$, as $\{L_j\}$ is convergent. Thus, every convergent subsequence of the bounded sequence $\{L_j\}$ converges to L. This implies that $\{L_j\}$ itself converges to L. Since the d-ball B_0 and the sequence $\{r_j\}$ are arbitrary, it follows that μ has strong derivative L at $\underline{0}$.

We now prove (ii). We suppose that $D\mu(\underline{0})$ is equal to L. Since the admissible limit of the function $(n,a) \mapsto a^{2\beta}$ at $\underline{0}$ is zero, we assume, as before, that C is zero in (6.4.4). We need to prove that the admissible limit of the function

$$F(n,a) = a^{\beta-\rho} u(n,a) = \mathcal{Q}_{i\beta}[\mu](n,a), \ \ (n,a) \in S,$$

at $\underline{0}$ is equal to L. We fix a positive number α and a sequence $\{(n_j,a_j) \mid j \in \mathbb{N}\} \subset \Gamma_\alpha(\underline{0})$ such that $\{a_j\}$ converges to zero. Since $D\mu(\underline{0})$ is equal to L, it follows, in particular, that

$$\lim_{r\to 0} \frac{\mu(\mathrm{B}(\underline{0},r))}{m(\mathrm{B}(\underline{0},r))} = L.$$

Therefore, there exists some positive number κ such that

$$\sup_{0<r<\kappa} \frac{\mu(\mathrm{B}(\underline{0},r))}{m(\mathrm{B}(\underline{0},r))} < L+1.$$

Finiteness of the measure μ implies that

$$\sup_{r\geq\kappa} \frac{\mu(\mathrm{B}(\underline{0},r))}{m(\mathrm{B}(\underline{0},r))} \leq \frac{\mu(N)}{m(\mathrm{B}(\underline{0},1))\kappa^Q}.$$

The above two inequalities together with Lemma 6.3.2 implies that

$$\sup_{(n,a)\in\Gamma_\alpha(\underline{0})} F(n,a) = \sup_{(n,a)\in\Gamma_\alpha(\underline{0})} \mathcal{Q}_{i\beta}[\mu](n,a) \leq C_{\alpha,\beta} M_{HL}(\mu)(\underline{0}) < \infty.$$

In particular, $\{F(n_j,a_j)\}$ is a bounded sequence. We consider a convergent subsequence of this sequence, denote it also, for the sake of simplicity, by $\{F(n_j,a_j)\}$ such that

$$\lim_{j\to\infty} F(n_j,a_j) = L'. \tag{6.4.8}$$

It suffices to prove that L' is equal to L. Using the sequence $\{a_j\}$, we define for each $j \in \mathbb{N}$,

$$F_j(n,a) = F\left(\delta_{a_j}(n), a_j a\right), \ \ (n,a) \in S.$$

As we have shown in the first part, we can prove that $\{F_j\}$ is a locally bounded sequence of

$\mathcal{L}^\beta$-harmonic functions on S. Hence, by Lemma 6.3.4, there exists a subsequence $\{F_{j_k}\}$ of $\{F_j\}$ which converges normally to a positive $\mathcal{L}^\beta$-harmonic function g on S. By defining

$$v(n,a) = a^{\rho-\beta} g(n,a), \quad (n,a) \in S,$$

we get from Lemma 6.3.3 that v is a positive eigenfunction of $\mathcal{L}$ with eigenvalue $\beta^2 - \rho^2$. Hence, by Lemma 6.3.5 and the relation (6.2.24), there exists a unique positive measure ν on N and a nonnegative constant C' such that

$$v(n,a) = C' a^{\beta+\rho} + a^{\rho-\beta} \mathcal{Q}_{i\beta}[\nu](n,a), \quad \text{for all } (n,a) \in S.$$

This implies that

$$g(n,a) = C' a^{2\beta} + \mathcal{Q}_{i\beta}[\nu](n,a), \quad \text{for all } (n,a) \in S. \tag{6.4.9}$$

Applying Lemma 6.3.2 once again, we observe that

$$\sup_{a\in A}\sup_j F_j(\underline{0},a) = \sup_{a\in A} F(\underline{0},a) = \mathcal{Q}_{i\beta}[\mu](\underline{0},a) \le c_\alpha M_{HL}(\mu)(\underline{0}) < \infty.$$

This shows that

$$\sup_{a\in A} g(\underline{0},a) < \infty,$$

an hence we must have $C' = 0$ in (6.4.9). Considering the dilate μ_{j_k} of μ according to (6.3.11) by a_{j_k}, we see by using Lemma 6.3.9 that for all $(n,a) \in S$

$$F_{j_k}(n,a) = F\left(\delta_{a_{j_k}}(n), a_{j_k}a\right) = \mathcal{Q}_{i\beta}[\mu]\left(\delta_{a_{j_k}}(n), a_{j_k}a\right) = \mathcal{Q}_{i\beta}[\mu_{j_k}](n,a).$$

Therefore, in view of (6.4.9), we conclude that $\mathcal{Q}_{i\beta}[\mu_{j_k}]$ converges to $\mathcal{Q}_{i\beta}[\nu]$ normally on S. By Lemma 6.3.1, we thus obtain weak* convergence of $\{\mu_{j_k}\}$ to ν. Since $D\mu(\underline{0}) = L$, it follows that for any d-ball $B \subset N$,

$$\lim_{k\to\infty} \mu_{j_k}(B) = \lim_{k\to\infty} {a_{j_k}}^{-Q} \mu(\delta_{a_{j_k}}(B)) = \lim_{k\to\infty} \frac{\mu(\delta_{a_{j_k}}(B))}{m(\delta_{a_{j_k}}(B))} m(B) = Lm(B).$$

Hence by Lemma 4.3.3, $\nu = Lm$. As $g = \mathcal{Q}_{i\beta}[\nu]$, it follows that

$$g(n,a) = L, \quad \text{for all} \quad (n,a) \in S.$$

This, in turn, implies that $\{F_{j_k}\}$ converges to the constant function L normally on S. On the other hand, we note that

$$F(n_{j_k}, a_{j_k}) = F\left(\delta_{a_{j_k}}\left(\delta_{a_{j_k}^{-1}}(n_{j_k})\right), a_{j_k}\right) = F_{j_k}\left(\delta_{a_{j_k}^{-1}}(n_{j_k}), 1\right).$$

As (n_{j_k}, a_{j_k}) belongs to the admissible region $\Gamma_\alpha(\underline{0})$, for all $k \in \mathbb{N}$, it follows that

$$\left(\delta_{a_{j_k}^{-1}}(n_{j_k}), 1\right) \in \overline{\mathrm{B}(\underline{0}, \alpha)} \times \{1\},$$

which is a compact subset of S. Therefore,

$$\lim_{k\to\infty} F(n_{j_k}, a_{j_k}) = \lim_{k\to\infty} F_{j_k}\left(\delta_{a_{j_k}^{-1}}(n_{j_k}), 1\right) = L,$$

as the convergence is uniform on $\overline{\mathrm{B}(\underline{0}, \alpha)} \times \{1\}$. In view of (6.4.8), we can thus conclude that $L' = L$. This completes the proof. $\square$

We now state and prove our main result.

Theorem 6.4.2. *Suppose that u is a positive eigenfunction of $\mathcal{L}$ in S with eigenvalue $\beta^2 - \rho^2$, where $\beta \in (0,\infty)$, and that $n_0 \in N$, $L \in [0,\infty)$. If μ is the boundary measure of u then the following statements hold.*

(i) If there exists $\theta \in (0,\infty)$, such that

$$\lim_{\substack{a\to 0 \\ (n,a)\in\Gamma_\theta(n_0)}} a^{\beta-\rho} u(n,a) = L,$$

then $D\mu(n_0) = L$.

(ii) If $D\mu(n_0) = L$, then the function $(n,a) \mapsto a^{\beta-\rho} u(n,a)$ has admissible limit L at n_0.

Proof. As in the proof of Theorem 4.4.2, we consider the translated measure $\mu_0 = \tau_{n_0}\mu$, where

$$\mu_0(E) = \mu(n_0 E),$$

for all Borel subsets $E \subseteq N$. In the proof of Theorem 4.4.2, we have seen that $D\mu_0(\underline{0})$ and $D\mu(n_0)$ are equal. As in the previous theorem, we may and do suppose that $C = 0$ in (6.4.4). Thus, we can rewrite the representation formula (6.4.4) for u as

$$a^{\beta-\rho}u(n,a) = \mathcal{Q}_{i\beta}[\mu](n,a) = \mu_0 * q_a^{i\beta}, \quad (n,a) \in S.$$

It also follows, as in the proof of Theorem 4.4.2, that

$$\mathcal{Q}_{i\beta}[\mu_0](n,a) = (\tau_{n_0}\mu) * q_a^{i\beta}(n) = \mu * q_a^{i\beta}(n_0 n) = \mathcal{Q}_{i\beta}[\mu](n_0 n, a),$$

for all $(n,a) \in S$. We fix an arbitrary positive number α. Using the left-invariance of the quasi-metric $\mathbf{d}$ in the definition of admissible domain (see Definition 6.2.8, i)), we have $(n,a) \in \Gamma_\alpha(\underline{0})$ if and only if $(n_0 n, a) \in \Gamma_\alpha(n_0)$. Thus, we conclude from the last equation that

$$\lim_{\substack{a\to 0\\(n,a)\in\Gamma_\alpha(\underline{0})}} \mathcal{Q}_{i\beta}[\mu_0](n,a) = \lim_{\substack{a\to 0\\(n,a)\in\Gamma_\alpha(n_0)}} \mathcal{Q}_{i\beta}[\mu](n,a).$$

Hence, it suffices to prove the theorem under the assumption that $n_0 = \underline{0}$. We now show that we can even take μ to be finite. Let $\tilde{\mu}$ be the restriction of μ on $\overline{\mathrm{B}(\underline{0},\tau^{-1})}$. Suppose $\mathrm{B}(n,s)$ is any given d-ball in N. Then, for all $r \in (0, [\tau^2(s + d(n))]^{-1})$, it follows that $\delta_r(\mathrm{B}(n,s))$ is a subset of $\mathrm{B}(\underline{0},\tau^{-1})$. Indeed, if $n_1 \in \delta_r(\mathrm{B}(n,s)) = \mathrm{B}(\delta_r(n), rs)$, then we have

$$d(\underline{0}, n_1) \le \tau\left[d(\underline{0}, \delta_r(n)) + d(\delta_r(n), n_1)\right] \le \tau\left[r d(n) + rs\right] < \tau^{-1}.$$

This implies that $D\mu(\underline{0})$ and $D\tilde{\mu}(\underline{0})$ are equal. We now claim that

$$\lim_{\substack{a\to 0\\(n,a)\in\Gamma_\alpha(\underline{0})}} \mathcal{Q}_{i\beta}[\mu](n,a) = \lim_{\substack{a\to 0\\(n,a)\in\Gamma_\alpha(\underline{0})}} \mathcal{Q}_{i\beta}[\tilde{\mu}](n,a), \tag{6.4.10}$$

provided one of the limits exist. In order to prove this claim, we first observe that

$$\lim_{a\to 0} \int_{\mathrm{B}(\underline{0},\tau^{-1})^c} q_a^{i\beta}(n_1^{-1}n)\, d\mu(n_1) = 0, \tag{6.4.11}$$

uniformly for $n \in \mathrm{B}(\underline{0}, 1/(2\tau^2))$. To prove this observation, we first note that for $n \in \mathrm{B}(\underline{0}, 1/(2\tau^2))$ and $n_1 \in \mathrm{B}(\underline{0}, \tau^{-1})^c$,

$$d(n) < (2\tau^2)^{-1} = (2\tau)^{-1}\tau^{-1} \leq (2\tau)^{-1} d(n_1).$$

Thus, using (6.2.27) and the inequality above, we obtain for $n \in \mathrm{B}(\underline{0}, 1/(2\tau^2))$ and $n_1 \in \mathrm{B}(\underline{0}, \tau^{-1})^c$,

$$d(n_1^{-1}n) = d(n^{-1}n_1) \geq \frac{1}{\tau} d(n_1) - d(n) \geq \frac{1}{\tau} d(n_1) - \frac{d(n_1)}{2\tau} = \frac{d(n_1)}{2\tau}. \tag{6.4.12}$$

Recalling the expression (6.2.22) of $q_a^{i\beta}$, we get for $n = (X, Z) \in \mathrm{B}(\underline{0}, 1/(2\tau^2))$

$$\begin{aligned}
& \int_{\mathrm{B}(\underline{0},\tau^{-1})^c} q_a^{i\beta}(n_1^{-1}n)\, d\mu(n_1) \\
= & \int_{\mathrm{B}(\underline{0},\tau^{-1})^c} \frac{c_\beta\, a^{2\beta}}{\left(16a^2 + 8a\|X - X_1\|^2 + d\left((X_1, Z_1)^{-1}(X, Z)\right)\right)^{\rho+\beta}}\, d\mu(X_1, Z_1) \\
\leq & \; c_\beta\, a^{2\beta} \int_{\mathrm{B}(\underline{0},\tau^{-1})^c} \frac{1}{\left(16a^2 + d(n_1^{-1}n)^2\right)^{\beta+\rho}}\, d\mu(n_1) \\
\leq & \; c_\beta\, a^{2\beta} \int_{\mathrm{B}(\underline{0},\tau^{-1})^c} \frac{1}{\left(16a^2 + \frac{d(n_1)^2}{4\tau^2}\right)^{\beta+\rho}}\, d\mu(n_1) \quad \text{(using the inequality (6.4.12))} \\
= & \; c_\beta\, a^{2\beta} \int_{\mathrm{B}(\underline{0},\tau^{-1})^c} \left(\frac{16 + \frac{d(n_1)^2}{4\tau^2}}{16a^2 + \frac{d(n_1)^2}{4\tau^2}}\right)^{\beta+\rho} \frac{1}{\left(16 + \frac{d(n_1)^2}{4\tau^2}\right)^{\beta+\rho}}\, d\mu(n_1) \\
\leq & \; c_\beta\, a^{2\beta} \int_{\mathrm{B}(\underline{0},\tau^{-1})^c} \left(\frac{64\tau^2}{d(n_1)^2} + 1\right)^{\beta+\rho} \frac{1}{\left(16 + \frac{d(n_1)^2}{4\tau^2}\right)^{\beta+\rho}}\, d\mu(n_1) \\
\leq & \; c_{\beta,\tau}\, a^{2\beta} \int_{\mathrm{B}(\underline{0},\tau^{-1})^c} \frac{1}{\left(16 + \frac{d(n_1)^2}{4\tau^2}\right)^{\beta+\rho}}\, d\mu(n_1).
\end{aligned}$$

Applying Lemma 6.2.5 for $a = 1$, the integral in the last inequality is finite. Hence, as $\beta \in (0, \infty)$, letting a go to zero on the right-hand side of the last inequality, (6.4.11) follows. Now,

$$\mathcal{Q}_{i\beta}[\mu](n, a) = \mathcal{Q}_{i\beta}[\tilde{\mu}](n, a) + \int_{\mathrm{B}(\underline{0},\tau^{-1})^c} q_a^{i\beta}(n_1^{-1}n)\, d\mu(n_1).$$

We take a positive number ϵ. By (6.4.11), we get some positive number a_1 such that for all $(n, a) \in \mathrm{B}(\underline{0}, 1/(2\tau^2)) \times (0, a_1)$,

$$0 \leq \mathcal{Q}_{i\beta}[\mu](n, a) - \mathcal{Q}_{i\beta}[\tilde{\mu}](n, a) = \int_{\mathrm{B}(\underline{0},\tau^{-1})^c} q_a^{i\beta}(n_1^{-1}n)\, d\mu(n_1) < \epsilon.$$

On the other hand, we note that

$$\Gamma_\alpha(\underline{0}) \cap \{(n,a) \in S \mid a < 1/(2\alpha\tau^2)\} \subset \mathrm{B}\left(\underline{0}, 1/(2\tau^2)\right) \times \{(n,a) \in S \mid a < 1/(2\alpha\tau^2)\}.$$

Hence, for all $(n,a) \in \Gamma_\alpha(\underline{0})$ with $a < \min\{a_1, 1/(2\alpha\tau^2)\}$, we have

$$\mathcal{Q}_{i\beta}[\mu](n,a) - \mathcal{Q}_{i\beta}[\tilde{\mu}](n,a) < \epsilon.$$

This proves (6.4.10). Therefore, as $\alpha \in (0,\infty)$ is arbitrary, we may and do suppose that μ is a finite measure. The proof now follows from Theorem 6.4.1. $\square$

As an immediate consequence of Theorem 6.4.2 we have the following.

Corollary 6.4.3. *Suppose that u is a positive eigenfunction of $\mathcal{L}$ on S with eigenvalue $\beta^2-\rho^2$, where $\beta \in (0,\infty)$, and that $n_0 \in N$, $L \in [0,\infty)$. If for some $\theta \in (0,\infty)$,*

$$\lim_{\substack{a\to 0\\ (n,a)\in\Gamma_\theta(n_0)}} a^{\beta-\rho} u(n,a) = L,$$

then for every $\alpha \in (0,\infty)$,

$$\lim_{\substack{a\to 0\\ (n,a)\in\Gamma_\alpha(n_0)}} a^{\beta-\rho} u(n,a) = L.$$

Remark 6.4.4. Since N has been assumed to be noncommutative, the class of harmonic NA groups don't contain the real hyperbolic spaces. However, the obvious analogue of Theorem 6.4.2 for real hyperbiloc spaces $\mathbb{H}^l = \{(x,y) \mid x \in \mathbb{R}^{l-1}, y \in (0,\infty)\}$, $l \geq 2$, also holds true. We note for $\mathbb{H}^l$, $l \geq 2$, we have $Q = 2\rho = l-1$, and

$$\mathcal{L} = y^2\left(\Delta_{\mathbb{R}^{l-1}} + \frac{\partial^2}{\partial y^2}\right) - (l-2)y\frac{\partial}{\partial y},$$

$$\mathcal{L}^\beta = y^2\left(\Delta_{\mathbb{R}^{l-1}} + \frac{\partial^2}{\partial y^2}\right) - (2\beta-1)y\frac{\partial}{\partial y}, \quad \beta \in (0,\infty),$$

with (see (2.2.1))

$$P(x) = c_l(1+\|x\|^2)^{-(l-1)}, \quad x \in \mathbb{R}^{l-1}.$$

The proof then follows simply by rewritting the proof of Theorem 6.4.2.

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List of Publications

1. Swagato K. Ray, Jayanta Sarkar

   Fatou theorem and its converse for positive eigenfunctions of the Laplace-Beltrami operator on Harmonic NA groups

   Preprint : https://arxiv.org/abs/2105.04964

2. Jayanta Sarkar

   On parabolic convergence of positive solutions of the heat equation

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3. Jayanta Sarkar

   On Pointwise converse of Fatou's theorem for Euclidean and Real hyperbolic spaces

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   To appear in *Israel Journal of Mathematics*

4. Jayanta Sarkar

   Boundary behavior of positive solutions of the heat equation on a stratified Lie group

   Preprint : https://arxiv.org/abs/2101.03977

5. Jayanta Sarkar

   A note on σ-point and nontangential convergence

   Preprint : https://arxiv.org/abs/2101.05660