PROJECT REPORT

on

RAINBOW EDGE COLORING

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This is to certify that the dissertation entitled **'Rainbow Edge Coloring'** submitted by Sudipta Ghosh to Indian Statistical Institute, Kolkata, in partial fulfillment for the award of the degree of Master of Technology in Computer Science is a bonafied record of work carried out by him under my supervision and guidance. The dissertation has fulfilled all the requirements as per the regulations of this institute and in my opinion, has recorded the standard needed for submission.

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Date

Dr. Sourav Chakraborty

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Abstract

Graph coloring is a well known problem with wide-ranging applications. The vertex and edge coloring problems have been studied in various models of computation. Rainbow coloring is a type of edge coloring that also acts as a connectivity measure for graphs. It was first introduced by Chartrand et al. in 2008.In 2011 Chakrobarty et al. proved that, it NP-Hard to compute rainbow connection number of a graph.

In this thesis first we have define some notation for graph and rainbow coloring. Then we do a literature overview of the results about rainbow coloring. In the final part we have proved that, if G is a square of tree, then $rc(G) \in \{diam(G), diam(G)+1\}$,and the corresponding optimal rainbow coloring can be found in the time that is linear in the size of G.

Contents

1 Introduction

Graph coloring is an ubiquitous problem in computer science and has widespread practical applications. The problem of graph coloring can be defined as the assignment of colors to different elements of the graph, provided certain constraints are satisfied. The various graph coloring problems that has been widely studied is vertex coloring, edge coloring and rainbow coloring.

Vertex coloring is the assignment of colors to vertices of the graph with the constraint that adjacent vertices do not get the same colors. Edge coloring is a graph coloring problem where you assign colors to the edges of the graph such that edges incident on the same vertex get assigned different colors.

Rainbow connectivity is a graph coloring problem that is also a connectivity measure for graphs. It was introduced by Chartrand et al. in 2008 [7]. Rainbow coloring is a special type of edge coloring where, for every pair of vertices in the graph, there should exist a path connecting the pair where every edge gets assigned a distinct color. The minimum number of colors required to make a graph rainbow connected, is known as rainbow connection number.

In addition to being a natural combinatorial measure, rainbow connectivity can be motivated by its interesting interpretation in the area of networking. Suppose that G represents a network (e.g., a cellular network). We wish to route messages between any two vertices in a pipeline, and require that each link on the route between the vertices (namely, each edge on the path) is assigned a distinct channel (e.g. a distinct frequency). Clearly, we want to minimize the number of distinct channels that we use in our network. This number is precisely rainbow connection number of G or $rc(G)$.

In the first paper on rainbow coloring [7], Chartrand et al. studied rainbow connection number of various class of graphs. In 2008, Caro et al. [3] conjectured that computing rainbow connection number of a graph is a NP-Hard problem. This conjecture is proved by Chakrobarty et al. in 2011 [4].

Let $T = (V_T, E_G)$ be a tree. A square of tree is a graph $G = (V_G, E_G)$, where $V_G = V_T$ and the two vertex is connected in G if the distance between them in T is ≤ 2.

In this thesis we have proved the following theorem.

Theorem 1.1. *If G is a square of tree, then $rc(G) \in \{diam(G), diam(G)+1\}$,and the corresponding optimal rainbow coloring can be found in the time that is linear in the size of G.*

This work has been generalised to higher power of trees by Diptiman Ghosh in his M.Tech. thesis.

1.1 Thesis Outline

We started with defining (various types of) rainbow coloring and graph classes in Chapter 2. In Chapter 3, we have a literature review of the work have been done on this topic. Here we have given already proven results on bound for general graph to find rainbow connection number. In this chapter we have also pointed out some results on rainbow connection number of various graph classes and the time complexity to decide rainbow connection number.

In Chapter 4, we have proved our results Theorem 1.1 on rainbow connection number of square of trees.

2 Preliminary and Definition

The concept of rainbow connection was introduced by Chartrand et al [7] in 2008. It is interesting and recently quite a lot papers have been published about it.

Definition 2.1. *[7] Let $G = (V,E)$ be a graph and $c : E \to \{1,2,3,...,r\}, r \in \mathbb{N}$, where adjacent edge can be colored same.For any two arbitrary vertices u and v, if $\exists$ a path between u and v such that every edge in that path is of different color, then that path is called $rainbow\ path$ and u and v is called $rainbow\ connected$. If for every pair of vertices in a graph is rainbow connected, then that graph is called $rainbow\ connected\ graph$. The minimum number of colors needed to make a graph rainbow connected is rainbow connection number of that graph denoted as $rc(G)$.*

To understand rainbow coloring, we first need to understand some basic definitions about graph.

2.1 Basic definitions and notations

Definition 2.2. *The* eccentricity *of a vertex v is $ecc(v) := \max_{x \in V(G)} d(v,x)$. The* radius *of G is $rad(G) := \min_{x \in V(G)} ecc(x)$. The* diameret *of G is $diam(G) := \max_{x \in V(G)} ecc(x)$.*

Definition 2.3. *A center of a graph G is a vertex c for which $eccentricity(c)$ in minimum and equal to radius of G.*

Definition 2.4. *For a graph G, a set $D \subseteq V(G)$ is called a k-*step dominating set *of G, if every $u \in D$ and $v \in V(G)$, $d(u,v) \le k$. Further, if D induces a connected sub-graph of G, it is called a* connected k-step dominating set *of G.*

Definition 2.5. *A dominating set D in a graph G is called a* two-way dominating set *if every pendant vertex of G is included in D. In addition, if D induces a connected sub-graph of G, we call D a* connected two-way dominating set.

Next we will define various types of graph product for which rainbow connection number has been studied in various paper.

Definition 2.6. *Given two graphs G and H, the Cartesian product of G and H, denoted by $G\Box H$, is defined as follows: $V(G\Box H) = V(G)\times V(H)$. Two distinct vertices $[g_1,h_1]$ and $[g_2,h_2]$ of $G\Box H$ are adjacent if and only if either $g_1 = g_2$ and $(h_1,h_2)\in E(H)$ or $h_1 = h_2$ and $(g_1,g_2)\in E(G)$.*

Definition 2.7. *Given two graphs G and H, the lexicographic product of G and H, denoted by $G\circ H$, is defined as follows: $V(G\circ H) = V(G)\times V(H)$. Two distinct vertices $[g_1,h_1]$ and $[g_2,h_2]$ of $G\circ H$ are adjacent if and only if either $(g_1,g_2)\in E(G)$ or $g_1 = g_2$ and $(h_1,h_2)\in E(H)$.*

Definition 2.8. *Given two graphs G and H, the strong product of G and H, denoted by $G\boxtimes H$, is defined as follows: $V(G\boxtimes H) = V(G)\times V(H)$. Two distinct vertices $[g_1,h_1]$ and $[g_2,h_2]$ of $G\boxtimes H$ are adjacent if and only if one of the three conditions hold:*

1. *$g_1 = g_2$ and $(h_1,h_2)\in E(H)$ or*
2. *$h_1 = h_2$ and $(g_1,g_2)\in E(G)$ or*
3. *$(g_1,g_2)\in E(G)$ and $(h_1,h_2)\in E(H)$.*

Definition 2.9. *The* k-th Power *of a graph, denoted by G^k where $k\geq 1$, is defined as follows: $V(G^k) = V(G)$. Two vertices u and v are adjacent in $V(G^k)$ if and only if the distance between vertices u and v in G, i.e., $dist_G(u,v)\leq k$.*

Definition 2.10. *Given graphs G and H with vertex sets $V(G) = \{g_i : 0\leq i\leq |G|-1\}$ and $V(H) = \{h_i : 0\leq i\leq |H|-1\}$ respectively. We define a decomposition of $G\Box H$ as follows: For $0\leq j\leq |H|-1$, define induced subgraphs, G_j, with vertex sets, $V(G_j) = \{[g_i,h_j] : 0\leq i\leq |G|-1\}$. Similarly, for $0\leq i\leq |G|-1$, define induced subgraphs, H_i, with vertex sets, $V(H_i) = \{[g_i,h_j] : 0\leq j\leq |H|-1\}$. Then we have the following:*

1. *For* $0 \leq j \leq |H|-1$*,* G_j *is isomorphic to* G *and for* $0 \leq i \leq |G|-1$*,* H_i *is isomorphic to* H*.*

2. *For* $0 \leq i < j \leq |H|-1$*,* $V(G_i) \cap V(G_j) = \emptyset$ *and hence* $E(G_i) \cap E(G_j) = \emptyset$*.*

3. *For* $0 \leq k < l \leq |G|-1$*,* $V(H_k) \cap V(H_l) = \emptyset$ *and hence* $E(H_k) \cap E(H_l) = \emptyset$*.*

4. *For* $0 \leq j \leq |H|-1$ *and* $0 \leq i \leq |G|-1$*,* $V(G_j) \cap V(H_i) = [g_i, h_j]$ *and* $E(G_j) \cap E(H_i) = \emptyset$*.*

We call $G_1, G_2, \ldots, G_{|H|-1}, H_1, H_2, \ldots, H_{|G|-1}$ *as the* ***(G,H)-Decomposition*** *of* $G \square H$*.*

2.2 Various Types of Rainbow Coloring

There are various types of rainbow coloring studied in various papers.

It is natural to ask whether the shortest path between every pair of vertices is rainbow connected or not.

Definition 2.11. *[7] For all* $u, v \in V(G)$*, if* $\exists$ *a rainbow path between* u *and* v *of length* $d(u,v)$*, then the graph is called strongly rainbow connected. The minimum number of colors required to get a strongly rainbow connected is called the strong rainbow connection number* $src(G)$ *of* G *.*

In rainbow coloring we need to find one rainbow path between every vertices. In a natural generalization, we can find k disjoint path between every vertices for $k \geq 1$. This is also first studied by Chartrand et al. [8] in 2009.

Definition 2.12. *[8] For an* $l-connected$ *graph* G *and an integer* k *with* $1 \leq k \leq l$*, the rainbow* $k-connectivity$ $rc_k(G)$ *of* G *is the minimum integer* j *for which there exist a edge-coloring of* G *with* j *colors such that every two distinct vertices of* G *are connected by* k *internally disjoint rainbow paths.*

From definition it is clear that, $rc_1(G) = rc(G)$.

There may be many shortest path between two vertices in a graph. It is natural to ask if all of them are rainbow path. This generalization of strong rainbow coloring was first studied by Chandran et al. [11] in 2018.

Definition 2.13. *[11] Very strong rainbow connection number $vsrc(G)$ of a graph G, which is the smallest number of colors for which there exists a coloring of $E(G)$ such that, for every pair of vertices and every shortest path P between them, all edges of P receive different colors.*

Now we will define general case of rainbow connection number, d-local rainbow connection number.

Definition 2.14. *A d-local rainbow coloring is an edge coloring such that any two vertices with distance at most d can be connected by a rainbow path, and we define d-local rainbow connection number $lrc_d(G)$ as the smallest number of colors in such a coloring.This generalizes rainbow connection numbers, which are the special case $d = diam(G)$. Similarly, we define d-local strong rainbow coloring and d-local strong rainbow connection number $lsrc_d(G)$ by replacing the word "path" with "geodesic".*

Similar to edges, we can color the vertices to get a path by vertices of different color. This varient of rainbow color first studied by Krivelevich and Yuster.

Definition 2.15. *[10] A vertex-colored graph G is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex connection of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected.*

Rainbow color have been studied also for directed graph as well. As an analogous setting for digraphs, Dorbec et al. proposed the concept of rainbow connection for digraphs [9], and Alva-Samos and Montellano-Ballesteros introduced the concept of the strong rainbow connection for digraphs in [1] . For directed graph, rainbow connection number is denoted by $\overrightarrow{rc}(D)$ and strong rainbow connection number is denoted by $\overrightarrow{src}(D)$.

2.3 Various Graph Classes

In the first part of the thesis, we have done a literature review for the work has been done in this area. For that we need to understand various graph classes for which the rainbow connection number has been studied.

Definition 2.16. *An independent triple of vertices x, y, z in a graph G is an* asteroidal triple (AT)*, if between every pair of vertices in the triple, there is a path that does not contain any neighbour of the third. A graph without asteroidal triples is called an* AT-free graph.

Definition 2.17. *A graph G is a* threshold graph*, if there exists a weight function $w : V(G) \to \mathbb{R}$ and a real constant t such that two vertices $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$.*

Definition 2.18. *A bipartite graph $G(A, B)$ is called a* chain graph *if the vertices of A can be ordered as $A = (a_1, a_2, \cdots, a_k)$ such that $N(a_1) \subseteq N(a_2) \subseteq \cdots \subseteq N(a_k)$.*

Definition 2.19. *An interval graph is an undirected graph where each vertex represents an interval in real line and two vertex is connected by an edge if the corresponding intervals has non-empty intersection.*

3 Literature Review

There is a obvious bound for rainbow connection number of a graph.

Theorem 3.1. *For any graph G*

$$diam(G) \leq rc(G) \leq src(G) \leq m = |E(G)|.$$

3.1 Results on bounds for general graphs

In this part we will study some published results on bound of rainbow connection number for any graph.

Theorem 3.2. *[7] Let a and b be positive integers with $a \geq 4$ and $b \geq \frac{5a-6}{3}$. Then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$.*

Theorem 3.3. *[5] If D is a connected two-way dominating set in a graph G, then*

$$rc(G) \leq rc(G[D]) + 3,$$

where $G[D]$ is is induced sub-graph by D in G.

Theorem 3.4. *[5] For every connected graph G of order n and minimum degree δ,*

$$rc(G) \leq \frac{3n}{\delta + 1} + 3.$$

Moreover, for every $\delta \geq 2$, there exist infinitely many graphs G such that $rc(G) \geq \frac{3(n-2)}{\delta+1} - 1$.

3.2 Results for various graph classes

Chartrand et al. proved the following results about rainbow connection number and strong rainbow connection number of bipartite and k-partite graph.

Theorem 3.5. *[7] For integers s and t with $2 \le s \le t$,*

$$rc(K_{s,t}) = min\{\lceil \sqrt[s]{t} \rceil, 4\}$$

Theorem 3.6. *[7] Let $G = K_{n_1,n_2,...,n_k}$ be a complete k-partite graph, wherek ≥ 3 and $n_1 \le n_2 \le ... \le n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then*

$$rc(G) = \begin{cases} 1 & \text{if } n_k = 1 \\ 2 & \text{if } n_k \ge 2 \text{ and } s > t \\ min\ \{\ \lceil \sqrt[s]{t} \rceil, 3\} & s \le t \end{cases}$$

Theorem 3.7. *[7] For integers s and t with $1 \le s \le t$,*

$$src(K_{s,t}) = \lceil \sqrt[s]{t} \rceil$$

Theorem 3.8. *[7] Let $G = K_{n_1,n_2,...,n_k}$ be a complete k-partite graph, where $k \ge 3$ and $n_1 \le n_2 \le ... \le n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then*

$$src(G) = \begin{cases} 1 & \text{if } n_k = 1 \\ 2 & \text{if } n_k \ge 2 \text{ and } s > t \\ \lceil \sqrt[s]{t} \rceil & s \le t \end{cases}$$

L. Sunil Chandran et al. proved results on bounds of various types of graph classes.

Theorem 3.9. *[5] Let G be a connected graph with $\delta(G) \ge 2$. Then,*

(i) *if G is an interval graph, $diam(G) \le rc(G) \le diam(G) + 1$,*

(ii) *if G is AT-free, $diam(G) \le rc(G) \le diam(G) + 3$,*

(iii) *if G is a threshold graph, $diam(G) \le rc(G) \le 3$,*

(iv) *if* G *is a chain graph,* $diam(G) \leq rc(G) \leq 4$,

(v) *if* G *is a circular arc graph,* $diam(G) \leq rc(G) \leq diam(G)+4$.

Moreover, there exist interval graphs, threshold graphs and chain graphs with minimum degree at least 2 *and rainbow connection number equal to the corresponding upper bound above. There exists an AT-free graph* G *with minimum degree at least* 2 *and* $rc(G) = diam(G)+2$, *which is* 1 *less than the upper bound above.*

Theorem 3.10. *[5] If* G *is a bridge-less chordal graph, then* $rc(G) \leq 3.rad(G)$. *Moreover, there exists a bridge-less chordal graph with* $rc(G) = 3.rad(G)$.

Manu Basavaraju et al. proved the following results about various types of graph product in [2].

Theorem 3.11. *[2] If* G *and* H *are two connected, non-trivial graphs then* $rad(G\Box H) \leq rc(G\Box H) \leq 2*rad(G\Box H)$. *The bounds are tight. Note that* $rad(G\Box H) = rad(G)+rad(H)$.

Theorem 3.12. *[2] Given two non-trivial graphs* G *and* H *such that* G *is connected we have the following:*

1. *If* $rad(G\circ H) \geq 2$ *then* $rad(G\circ H) \leq rc(G\circ H) \leq 2*rad(G\circ H)$. *This bound is tight.*
2. *If* $rad(G\circ H) = 1$ *then* $1 \leq rc(G\circ H) \leq 3$. *This bound is tight.*

Theorem 3.13. *[2] If* G *and* H *are two connected, non-trivial graphs then* $rad(G\boxtimes H) \leq rc(G\boxtimes H) \leq 2*rad(G\boxtimes H)+2$. *The upper bound is tight up to an additive constant* 2. *Note that* $rad(G\boxtimes H) = max\{rad(G), rad(H)\}$.

Theorem 3.14. *[2] If* G *is a connected graph then* $rad(G^k) \leq rc(G^k) \leq 2*rad(G^k)+1$ *for all* $k \geq 2$. *The upper bound is tight up to an additive constant of* 1. *Note that* $rad(G^k) = \left\lceil \frac{rad(G)}{k} \right\rceil$.

3.3 Results on Hardness and Complexity

It is natural to ask the time complexity to computing rainbow connection number of any general graph. Chakraborty et al. solved the conjectures posed by Caro et al. in [3] and proved the following complexity results:

Theorem 3.15. *[4] Given a graph G, deciding if $rc(G) = 2$ is NP-Complete. In particular, computing $rc(G)$ is NP-Hard.*

In the same paper Chakraborty et al. also proved the following about the checking if a given coloring is rainbow coloring or not.

Theorem 3.16. *[4] The following problem is NP-Complete: Given an edge-colored graph G, check whether the given coloring makes G rainbow connected.*

Theorem 3.17. *[4] Given an edge-colored graph G and a pair of vertices s and t, deciding if s and t are connected by a rainbow path is NP-Complete.*

3.4 Rainbow coloring for power of trees

In the paper "Algorithms for the rainbow vertex coloring problem on graph classes" [12] Lima et al. proved the following results about rainbow vertex connection number of power of trees.

Theorem 3.18. *[12] If G is a power of a tree, then $rvc(G) \in \{diam(G) - 1, diam(G)\}$, and the corresponding optimal rainbow vertex coloring can be found in time that is linear in the size of G.*

But we have not found any results on the rainbow connection number of power of trees. So we try to find the answer of the following question.

Question: *What is the rainbow connection number for power of trees, i.e., if G is a power of tree, then $rc(G) =?$*

4 Edge Rainbow Coloring for Squares of Trees (Our main contribution)

In this section we present the proof of Theorem 1.1. But let us first recall the definition of power of a graph.

Definition 4.1. *The* k-th Power *of a graph, denoted by G^k where $k \geq 1$, is defined as follows: $V(G^k) = V(G)$. Two vertices u and v are adjacent in $V(G^k)$ if and only if the distance between vertices u and v in G, i.e., $dist_G(u,v) \leq k$.*

So, the two vertices in a tree T is connected by a path of length $\leq k$, then they are connected by an edge in T^k.

Also, recall the definition of center of a graph.

Definition 4.2. *The* eccentricity *of a vertex v is $ecc(v) := \max_{x \in V(G)} d(v,x)$.*

A center *of a graph G is a vertex c for which eccentricity ($ecc(c)$) in minimum and equal to radius of G.*

We can define branches of a tree as follows:

Definition 4.3. *Let T be a tree, and z is the center of T. Let $e = zv$ be an edge that is incident to z, with v not in the center. When e is removed from the tree, the tree will fall apart in two parts, a branch is the part that does not contain z. If the center of T contains only one vertex, the number of branches equals the degree of z.*

Now we will define *Layer of each vertices* and *Subbranch of a branch* in a tree.

Definition 4.4 (Layer $\ell(v)$)**.** *We define layer i as the set of all vertices with distance $\lfloor \frac{diam(T)}{2} \rfloor - i$ to the center of T. For a vertex v, we write $\ell(v)$ for the layer that it is contained in, so $\ell(v) = \lfloor \frac{diam(T)}{2} \rfloor - d$, where d is the distance of v to the center of T.*

Also, we use the term *single edge* for an edge $\{u,v\}$ if $\ell(u) = \ell(v) + 1$ and *double egde* if $\ell(u) = \ell(v) + 2$, assuming $\ell(u) \geq \ell(v)$.

Definition 4.5 (Subbranch)**.** *Let v be a vertex in a branch B and v has degree more than two. Suppose the edge between B and the center has removed. Let $e = uv$ be an edge, where $\ell(v) < \ell(u)$. If we remove the edge e, the branch will fall apart in two parts. The part which does not contain a vertex of minimum layer is called subbranch. If both of them contain vertex of minimum layer, then one of them is subbranch.*

We start the proof of Theorem 1.1 by proof a bunch of lemmas. First we will prove that, for tree with single vertex as center and number of branches of maximum length ≥ 3. Then we will prove for tree with single vertex as center and exactly two branches. In the last lemma, we will prove for tree with double vertex as center.

Lemma 1. *Suppose T is a tree with single vertex as center and $diam(T) \geq 6$ and exactly three branches from center with maximum length. Then $src(T^2) \geq rc(T^2) \geq diam(T^2)+1$.*

Proof. Suppose B_1, B_2, B_3 are three branches with maximum length from center. Suppose v_1, v_2, v_3 are three farthest leaves from center in B_1, B_2, B_3. There exists a unique shortest path P from v_1 to v_2 that will use $diam(T^2)$ edges . Let the edges are $e_1, e_2, ..., e_k$. Similarly,there exists a unique shortest path Q from v_1 to v_3 that will use $diam(T^2)$ edges. Let the edges are $f_1, f_2, ..., f_k$. Note that, $e_1 = f_1, e_2 = f_2, ...e_j = f_j$ where $j = \lfloor \frac{diam(T^2)}{2} \rfloor$. That is, both path will use same edges in B_1. The unique shortest path R in T^2 from v_2 to v_3 will use the path $e_n, e_{(k-1)}, ..., e_{(j+1)}, f_{(j+1)}, ..., f_k$.

We give a proof by contradiction. Let c be a rainbow vertex coloring that uses at most $diam(T^2)$ colors. Notice that the paths P, Q, and R have length $diam(T^2)$. Therefore, for each of these paths, all edges are assigned different colors and all colors appear in the path.

Since the first j edges of the paths P and Q are equal, we see that the colors used for $e_{j+1}, \ldots, e_k$ are the same as the colors used for $f_{j+1}, \ldots, f_k$. Since $diam(T) \geq 6$, $\{e_{j+1}, \ldots, e_k\}$ and $\{f_{j+1}, \ldots, f_k\}$ are non-empty. Hence, there is a color that appears more than once in R, which yields a contradiction.

We conclude that $rc(T^2) \geq diam(T^2) + 1$.

Also, since we have $src(T^2) \geq rc(T^2)$, we have $src(T^2) \geq rc(T^2) \geq diam(T^2) + 1$.

□

Lemma 2. *If T is a tree with single center and $diam(T) \geq 6$ and at least three branches from center with maximum length. Then $rc(T^2) = diam(T^2) + 1$.*

Proof. Suppose $D = \lfloor diam(T)/2 \rfloor$

Consider the coloring:

$$c(v_i v_j) = \begin{cases} c & \text{if } \ell(\mathrm{v}_i) = \mathrm{D} \text{ and } \ell(\mathrm{v}_j) = \mathrm{D} - 1 \\ c & \text{if } \ell(\mathrm{v}_i) = \ell(\mathrm{v}_j) \\ \ell(\mathrm{v}_j) & \text{if } \ell(\mathrm{v}_i) \neq \mathrm{D} \text{ and } \ell(\mathrm{v}_i) = 1 + \ell(\mathrm{v}_j) \\ \ell(\mathrm{v}_j) + 1 & \text{if } \ell(\mathrm{v}_i) = 2 + \ell(\mathrm{v}_j) \\ \mathrm{c} & \text{otherwise} \end{cases}$$

where c is a unique color different from all the color.

Couple of things to notice in the coloring procedure are

- if a vertex is in odd layer, then the double edges connected to it is of even color. And if a vertex is in even layer, then the double edges connected to it is of odd color.
- if a vertex is in odd layer, then the single edge towards center is of odd color. And if a vertex is in even layer, then the single edge towards center is of even color.

Note that, total layers are $\frac{diam(T)}{2} + 1 = diam(T^2) + 1$ (since $\frac{diam(T)}{2} = diam(T^2)$, namely $0, 1, ..., diam(T^2)$. But we didn't use $diam(T^2)$ as a color and use c as a unique color. We total number of used color is $diam(T^2) + 1$.

<u>Claim:</u> This is a rainbow coloring i.e. ∃ rainbow path between every pair of vertices.

<u>Proof:</u> Suppose z is the center and u,v be any two vertices such that one of them in in odd layer and other one is in even layer ,say, $\ell(u)$ is even and $\ell(v)$ is odd. Also suppose if

$path_1$ is the path from u to z using vertices in even layers and $path_2$ is the path from v to z using vertices odd layers. From the coloring, it is clear that we will use odd colored edges in $path_1$ and even colored edges in $path_2$. Then union of $path_1$ and reversed of $path_2$ is the rainbow path between u and v.

If both of u and v is in odd layer, then in $path_1$ first use a single edge towards center then use double edges to follow even layer vertices.

If both of u and v is in even layer, then in $path_2$ first use a single edge towards center then use double edges to follow odd layer vertices.

■

Combining this with Lemma 5.1, we conclude that $rc(T^2) = diam(T^2) + 1$.

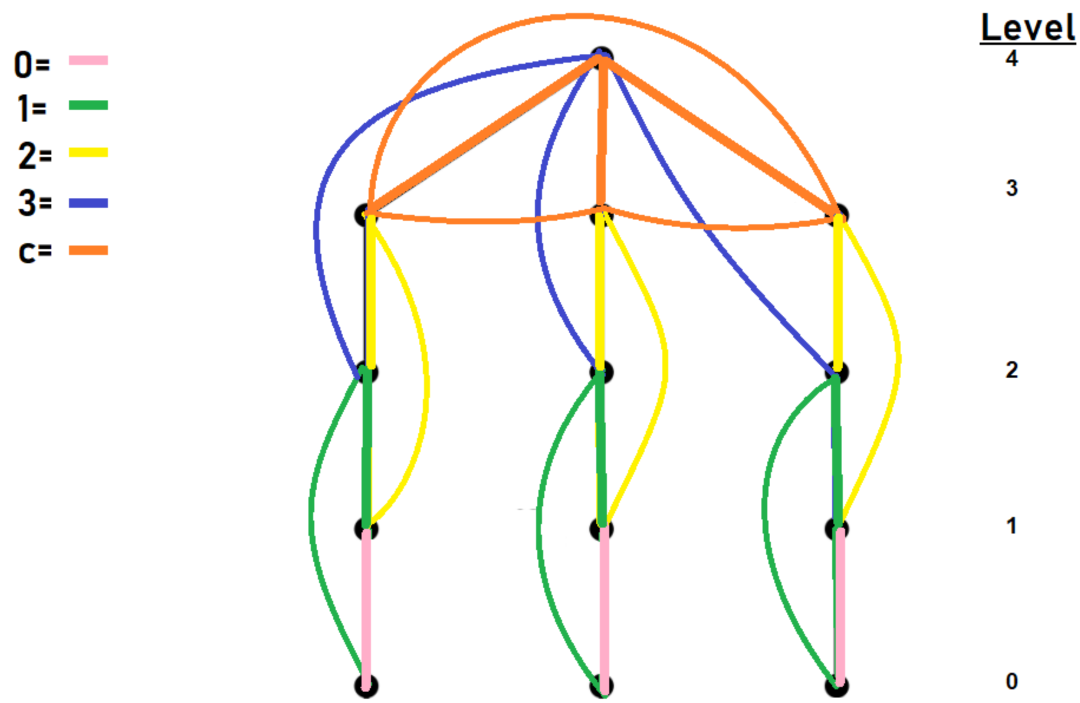

Figure 1: Square of tree with three branches of maximum length

□

Lemma 3. *If T is a tree with one centers and $diam(T) \geq 6$ and there are exactly two branches from center with maximum length. Then $rc(T^2) = diam(T^2)$.*

Proof. Suppose two maximum branches are B_1 and B_2. One of other branches is B_3. All Other branches will be colored as B_3. B_i' is subbranch in B_i.

Let $\ell(v_i) \leq \ell(v_j)$. First let $diam(T^2)$ is even.

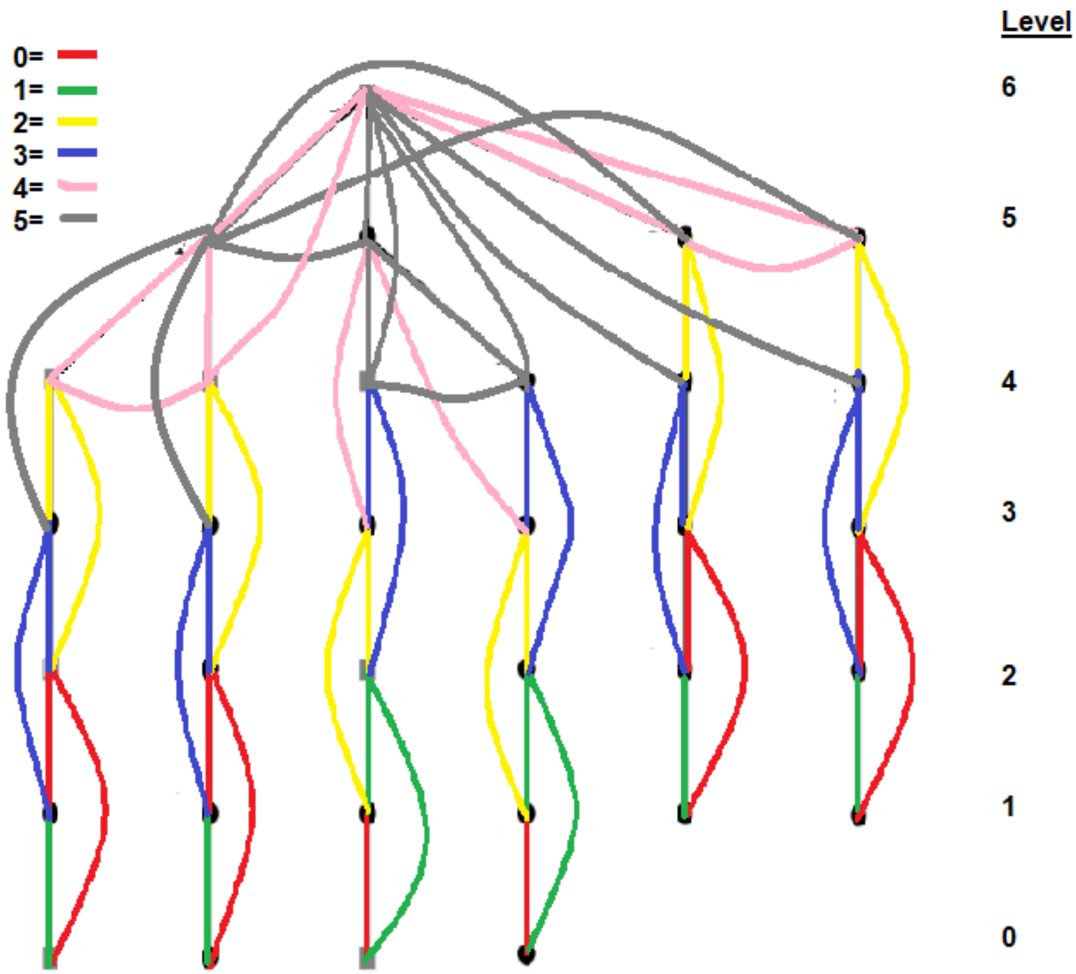

Figure 2: Square of tree with two branches of maximum length and even diameter

$$
c(v_iv_j)=\begin{cases}
\mathrm{diam(T^2)}-2 & v_i,v_j\in B_1,\ \ell(v_i)=diam(T^2)-2 \text{ and } \ell(v_j)=1+\ell(v_i)\\
\mathrm{diam(T^2)}-2 & v_i\in B_1,v_j\in B_1',\ \ell(v_i)=diam(T^2)-2 \text{ and } \ell(v_j)=\ell(v_i)\\
\mathrm{diam(T^2)}-2 & v_i\in B_1,v_j \text{ is center, and } \ell(v_j)=1+\ell(v_i)\\
\ell(v_i) & v_i,v_j\in B_1,\ \ell(v_i) \text{ is even and } \ell(v_j)=2+\ell(v_i)\\
\ell(v_j) & v_i,v_j\in B_1,\ \ell(v_i) \text{ is odd and } \ell(v_j)=2+\ell(v_i)\\
\ell(v_i)-1 & v_i,v_j\in B_1,\ \ell(v_i) \text{ is odd and } \ell(v_j)=1+\ell(v_i)\\
\ell(v_j) & v_i,v_j\in B_1,\ \ell(v_i) \text{ is even and } \ell(v_j)=1+\ell(v_i)\\
\mathrm{diam(T^2)}-1 & v_i,v_j\in B_2,\ \ell(v_i)=diam(T^2)-2 \text{ and } \ell(v_j)=1+\ell(v_i)\\
\mathrm{diam(T^2)}-1 & v_i\in B_2,v_j\in B_2',\ \ell(v_i)=diam(T^2)-2 \text{ and } \ell(v_j)=\ell(v_i)\\
\mathrm{diam(T^2)}-1 & v_i\in B_2,v_j \text{ is center, and } \ell(v_j)=1+\ell(v_i)\\
\ell(v_i)+1 & v_i,v_j\in B_2 \text{ and } \ell(v_j)=2+\ell(v_i)\\
\ell(v_i) & v_i,v_j\in B_2 \text{ and } \ell(v_j)=1+\ell(v_i)\\
\ell(v_i)+1 & v_i,v_j\in B_3,\ \ell(v_i) \text{ is even and } \ell(v_j)=2+\ell(v_i)\\
\ell(v_i)-1 & v_i,v_j\in B_3,\ \ell(v_i) \text{ is odd and } \ell(v_j)=2+\ell(v_i)\\
\ell(v_i) & v_i,v_j\in B_3,\ \ell(v_i) \text{ is odd and } \ell(v_j)=1+\ell(v_i)\\
\ell(v_i)-2 & v_i,v_j\in B_3,\ \ell(v_i) \text{ is even and } \ell(v_j)=1+\ell(v_i)\\
\mathrm{diam(T^2)}-2 & v_i\in B_3,\ v_j \text{ is center, and } \ell(v_j)=1+\ell(v_i)\\
\mathrm{diam(T^2)}-1 & v_i\in B_i,\ v_j\in B_j, i<j, i<3 \text{ and } \ell(v_j)=\ell(v_i)\\
\mathrm{diam(T^2)}-2 & v_i\in B_i,\ v_j\in B_j, i<j, i\geq 3 \text{ and } \ell(v_j)=\ell(v_i)\\
\mathrm{diam(T^2)}-1 & \text{otherwise}
\end{cases}
$$

Also color of edges in B_i' will be same as B_i.

For B_1, we have use one color to go one level to alother level. So, total color used in B_1 is $\frac{diam(T)}{2} = diam(T^2)$. Same set of colors have been used in B_2 and B_3. So, total number of colors remain same i.e. $diam(T^2)$.

Couple of things to notice in the coloring procedure of B_1 are

- if a vertex is in odd layer, then the double edges connected to it is of odd color. And if a vertex is in even layer, then the double edges connected to it is of even color.
- if a vertex is in odd layer, then the single edge towards center is of even color. And if a vertex is in even layer, then the single edge towards center is of odd color.

Same thing we can point out for B_3 and opposite thing can be pointed out for B_2.

Now we show that there is a rainbow path between every pair of vertices. Let u and v be any two vertices.

Case 1. $u \in B_1$ and $v \in B_1'$.

Let B_1' is started from $(diam(T^2) - 1)$ layer.

Subcase (i): $\ell(u)$ is even, $\ell(v)$ is odd.

Use vertices in even layers in B_1 by taking edges of even colors and use vertices in odd layers in B_1' by taking edges of odd colors. Use the edge of color $diam(T^2) - 2$ in B_1 to reach the common ancestor.

Subcase (ii): $\ell(u)$ is odd, $\ell(v)$ is odd.

First take a single edge of even color in B_1, then follow the path as $\ell(u)$ is even, $\ell(v)$ is odd.

Subcase (iii): $\ell(u)$ is odd, $\ell(v)$ is even.

Use vertices in odd layers in B_1 by taking edges of odd colors and use vertices in even layers in B_1' by taking edges of even colors. Use the edge of color $(diam(T^2) - 2)$ in B_1' to reach the common ancestor.

Subcase (iv): $\ell(u)$ is even, $\ell(v)$ is even.

First take a single edge of odd color in B_1, then follow the path as $\ell(u)$ is odd, $\ell(v)$ is even. Use the equal layer edge if v is next to common ancestor.

Case 2. $u \in B_2$ and $v \in B'_2$.

Similar to case 1.

Case 3. $u \in B_1$ and $v \in B_2$.

Use vertices in even layers in B_1 by taking even colored edges to reach the center, then use vertices in even layers in B_2 by taking odd colored edges.

Case 4. $u \in B_1$ and $v \in B_3$.

Use vertices in even layers in B_1 by taking even colored edges to reach the center, then use vertices in even layers in B_3 by taking odd colored edges.

If the vertex in B_3 is next to center, then in B_1 first go to the vertex at layer $(diam(T^2)-1)$ by taking single edge of color $(diam(T^2)-2)$, then use the same layer edge to reach the destination.

Case 5. $u \in B_2$ and $v \in B_3$.

Use vertices in even layers in B_2 by taking odd colored edges to reach the center, then use vertices in odd layers in B_3 by taking even colored edges.

Case 6. $u \in B_3$ and $v \in B_4$.

Use odd color edges in B_3 and even color edges in B_4.

Use vertices in even layers in B_3 by taking odd colored edges to reach the center, then use vertices in odd layers in B_4 by taking even colored edges.

Case 7. $u \in B_3$ and $v \in B'_3$.

Use path same as if $u \in B_3$ and $v \in B_4$ by first go to center. Then travel along B_3 until B'_3 started. Then travel along B'_3 to reach destination.

Similarly, when $diam(T^2)$ is odd, we can color the tree. In that case, we will replace the color $(diam(T^2)-2)$ by $(diam(T^2)-1)$.

□

Lemma 4. *If T is a tree with two centers and $diam(T) \geq 5$. Then $rc(T^2) = diam(T^2)$.*

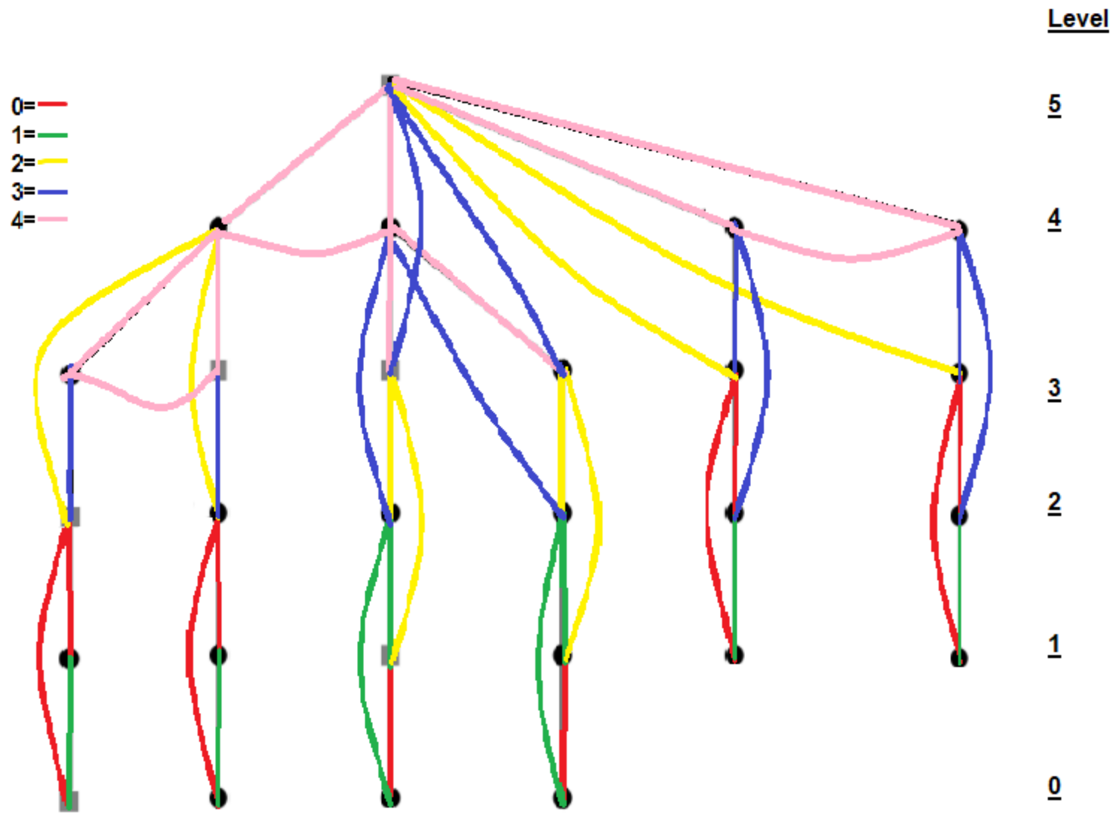

Figure 3: Square of tree with two branches of maximum length and odd diameter

Proof. Consider the following coloring c:

$$c(v_i v_j) = \begin{cases} diam(\mathrm{T}^2) - 1 & \text{if one endpoint is a center and } \ell(\mathrm{v}_i) = 1 + \ell(\mathrm{v}_j) \\ diam(\mathrm{T}^2) - 1 & \text{if } v_i \text{ and } v_j \text{ are centers} \\ \ell(\mathrm{v}_j) & \text{if no endpoint is center and } \ell(\mathrm{v}_i) = 1 + \ell(\mathrm{v}_j) \\ \ell(\mathrm{v}_j) + 1 & \text{if } \ell(\mathrm{v}_i) = 2 + \ell(\mathrm{v}_j) \\ diam(\mathrm{T}^2) - 1 & \text{else} \end{cases}$$

Here we have use $diam(T^2)$ colors namely $0, 1, ..., diam(T^2) - 1$.

Couple of things to notice in the coloring procedure are

- if a vertex is in odd layer, then the double edges connected to it is of even color. And if a vertex is in even layer, then the double edges connected to it is of odd color.
- if a vertex is in odd layer, then the single edge towards center is of odd color. And if a vertex is in even layer, then the single edge towards center is of even color.

Suppose z_1 and z_2 are centers. If two vertices are in two different branches, then in one branch travel via even layered vertices using odd colored edges and in other branch travel via odd layered vertices using even colored edges. And use a $diam(T^2) - 1$ colored edge to connect them. Since centers will be either in even layer or in odd layer, this double edge must be connected with a center. So its color will be different from all other edges of the path. If two vertices are in two different subbranch of in same branch, consider as two different branch.

Now suppose two vertices are next to two centers. Suppose u is in branch of z_1 and v is in branch of z_2. Then the rainbow path between u and v will be : Use $diam(T^2) - 1$ colored edge to go z_2 from u and then use next double edge to go a vertex in the branch from z_2 and then use single edge to reach v.

Note that, in one branch we will use even colored edges and in one branch we will use odd colored edges. So the path will be rainbow path.

□

Proof of 1.1. Suppose $G = T^2$. If T is unknown, it can be computed in linear time [6] . First, we compute the center of T, and then we distinguish cases as in Lemmas 2, 3, 4. This costs linear time. In each of those lemmas, an optimal coloring is given that can be computed in linear time.

□

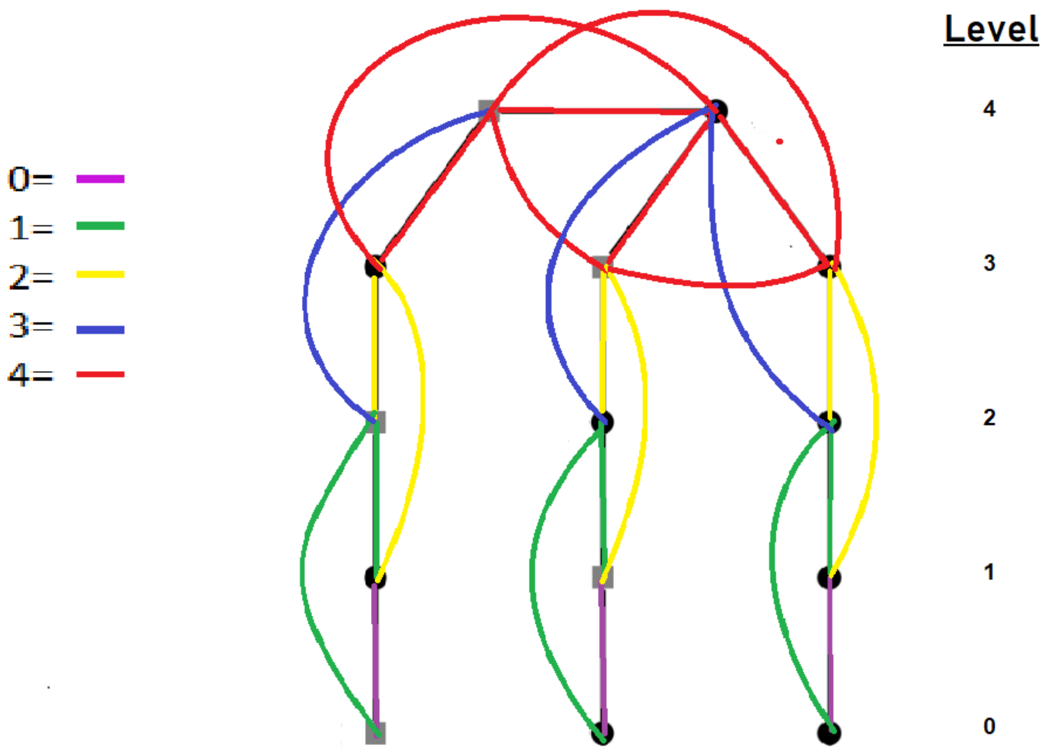

Figure 4: Square of tree with two center

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5 Conclusion and Future Work

In this work we have found rainbow connection number of power of tree. It will be interesting to find rainbow connection number and rainbow vertex connection number of power of various other classes. Also some researchers are working on rainbow coloring in random graph and online streaming graph.

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