A Last Progeny Modified Branching Random Walk

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A Last Progeny Modified Branching Random Walk

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Dedicated to Ma and Baba

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Chapter 1

Introduction

1.1 Branching Random Walk

Branching random walk (BRW) was introduced by Hammersley [23] in the early '70s. Over the last five decades, it has received a lot of attention from various researchers in probability theory and statistical physics. The model, as such, is very simple to describe.

It starts with one particle at the origin. After a unit amount of time, the particle dies and gives birth to a number of similar particles, which are placed at possibly different locations on the real line $\mathbb{R}$. These particles at possibly different places on $\mathbb{R}$ form the so-called first generation of the process and can be described through a point process, say Z on $\mathbb{R}$. After another unit time, each of the particles in the first generation behaves independently and identically as that of the parent, that is, it dies, but before that, it produces a bunch of offspring particles which are displaced by independent copies of Z. Further, the particles in generation one behave independently but identically of one another. The process then continues in the next unit of time and so on.

If we denote the number of particles in generation n by N_n, then from the definition, it follows that $\{N_n\}_{n\geq 0}$ is a Galton-Watson branching process with progeny distribution given by $N := Z(\mathbb{R})$. So the backbone of the process is a branching process tree with weighted edges, where the weights represent the displacements of the particles relative to their respective parents. We write $|v| = n$ if an individual v is in the n-th generation, and $S(v)$ denotes its position, which is the sum of all the displacements the particle v and its ancestors have received. The stochastic process $\{S(v) : |v| = n\}_{n\geq 0}$ is typically referred to as the classical *branching random walk (BRW)*.

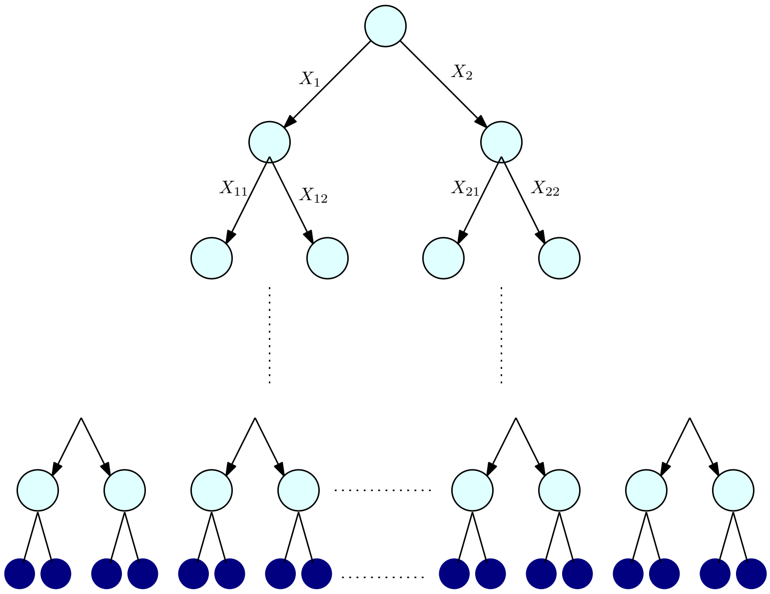

Figure 1.1: I.I.D. Gaussian displacement binary BRW

To illustrate this, we consider a specific example. Let $N = 2$ with probability one and $Z = \delta_{\xi_1} + \delta_{\xi_2}$, where ξ_j's are i.i.d. $\mathrm{N}(0, 1)$. As displayed in Figure 1.1, the backbone of this process is just a binary branching tree with weighted edges, and the weights on the edges are i.i.d. $\mathrm{N}(0, 1)$ random variables. At time n, there are a total of 2^n particles. Each particle is positioned at a random distance from the origin with distribution the same as the position of a random walker starting at the origin and taking n i.i.d. standard Gaussian steps. Note that the positions of the particles at the same generation are identical, but they need not be independent. In fact, for two particles u and v at the n-th generation, if we denote $u \wedge v$ as their least common ancestor in the associated tree, then

$$\mathrm{Cov}\left(S(u), S(v)\right) = |u \wedge v| \,.$$

Further, this leads to the following observation.

$$\#\left\{w : |w| = n, \mathrm{Cov}\left(S(u), S(w)\right) \geq k\right\} = 2^{n-k},$$

for any non-negative integer $k \leq n$. This observation is often referred to as "branching random walk is *log-correlated*".

1.2 The Modification

In this work, we consider a modified version of the classical BRW. The modification is done at the last generation where we add *i.i.d.* displacements of a specific form. Since the modifications have been done only at the last generation, we call this model a *last progeny modified branching random walk* or abbreviate it as *LPM-BRW*.

In our model, we introduce two parameters. One is a non-negative real number, which we denote by $\theta > 0$. The other one is a positively supported distribution, which we will denote by $\mu \in \mathcal{P}(\bar{\mathbb{R}}_+)$. The parameter θ should be thought of as a *scaling parameter* for the extra displacement we give to each individual at the n-th generation. This extra displacement is as follows. At a generation $n \geq 1$, we give additional displacements to each of the particles at the generation n, which are of the form $\frac{1}{\theta} A_v := \frac{1}{\theta} (\log Y_v - \log E_v)$, where $\{Y_v\}_{|v|=n}$ are i.i.d. μ, while $\{E_v\}_{|v|=n}$ are i.i.d. Exponential (1) and they are independent of each other and also of the process $\big(S(u)\big)_{|u| \leq n}$. We denote by $R_n^*(\theta, \mu)$ the maximum position of this *last progeny modified branching random walk (LPM-BRW)*, i.e.,

$$R_n^*(\theta, \mu) := \max_{|v|=n} \left\{ S(v) + \frac{1}{\theta} \log(Y_v / E_v) \right\}. \tag{1.1}$$

If the parameters θ and μ are clear from the context, then we will simply write this as R_n^*. A schematic of the process is given in Figure 1.2.

1.3 Motivation

Our main motivation to study this new LPM-BRW model is what we will see in the sequel that, there is a nice *coupling* of R_n^* with a *linear statistic* associated with BRW (see Theorem 2.3.2 for details). For such statistic, asymptotics can be computed using various martingale techniques, some of which are known. This novel connection is indeed the reason why the model intrigued us. Our model is one example where this

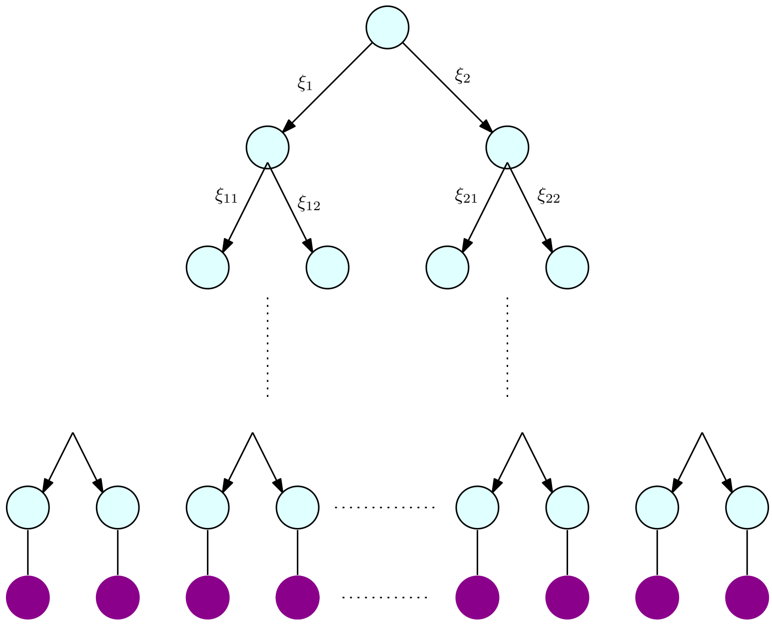

Figure 1.2: Last progeny modified branching random walk (LPM-BRW)

coupling technique works. We believe (also see Chapter 2) that this connection is a novel mathematical tool that has the potential for many more applications.

The other motivation and perhaps more straightforward one, is to be able to compare our results with the existing ones in the context of the classical BRW (such as, asymptotics derived in [2]). We see a difference appears in the constant factor in front of the Bramson correction (see Theorem 3.5.3), but the final weak limit remains the same. This in turn shows that the centered asymptotic results are heavily dependent on the displacements given at the end nodes, but not the limit. While doing this comparison, we also have been able to get the exact constant for the centered limit which was earlier not known (see Remark 3.5.5 for the details).

1.4 Survey of Known Results

1.4.1 Almost Sure Asymptotic Limit

Hammersley [23] introduced the BRW model as the first-death problem in an age-dependent branching process. In his model, he considered the tree to be a family tree, and the edge weights were the lifespan of a person. And he asked at what point in time a death first occurs to some member of the n-th generation. Although he needed to study the minimum of a branching random walk, the minimum can be converted into the negative of the new maximum by changing the sign of the increments. So the study of the minimum and the maximum are essentially the same. We will denote the position of the *right-most particle* in the generation n by $R_n \equiv R_n(Z)$. The following result was proved by Hammersley.

Theorem 1.4.1 (Hammersley [23])**.** *If $\mathbb{P}(N = 0) = 0$, then there exists a constant γ such that*

$$\frac{R_n}{n} \xrightarrow{p} \gamma.$$

Hammersley proved this using the properties of *super-convolutive semigroups*. Later, it was also proved using Liggett's version of *Kingman's subadditive ergodic theorem* (see Zeitouni [34] for the details).

Kingman [25] showed that the convergence in Theorem 1.4.1 is almost sure under certain assumptions and also calculated γ explicitly. In his model, the 'lifespans' are always non-negative and so after changing the sign, the underlying point process Z is supported on $(-\infty, 0]$. We define

$$\nu(t) := \log \mathbb{E}\left[\int_{\mathbb{R}} e^{tx}\, Z(dx)\right], \tag{1.2}$$

and we denote by ν^* the Fenchel-Legendre transform of ν, i.e.,

$$\nu^*(a) := \sup_{\theta\in\mathbb{R}} \left\{a\theta - \nu(\theta)\right\}. \tag{1.3}$$

Kingman assumed that there exists $\theta > 0$ such that $0 < \nu(\theta) < \infty$. This implies $\nu(0) > 0$ and $\nu(t) < \infty$ for all $t \geq \theta$. He showed that

Theorem 1.4.2 (Kingman [25])**.** *Under the above assumptions, on the event of survival of the branching process,*

$$\frac{R_n}{n} \to \gamma \;\; a.s.,$$

where $\gamma := \sup\left\{a : \nu^*(a) < 0\right\}$.

Relaxing the assumption in Kingman's work [25] that Z is supported on $(-\infty, 0]$, Biggins [9] showed that

Theorem 1.4.3 (Biggins [9])**.** *If $\nu(\theta) < \infty$ for some $\theta > 0$, then on the event of survival of the branching process,*

$$\frac{R_n}{n} \to \gamma \text{ a.s.},$$

where γ is as in Theorem 1.4.2.

1.4.2 Centered Asymptotic Limits

From historical point-of-view, it is interesting to note here that Biggins [9] wrote:

> "Of course pride of place in the open problems goes to establishing more detailed results than Theorem 1.4.3 of the kinds that are already available for branching Brownian motion."

Indeed, McKean [30] showed that for similar continuous time version with *Branching Brownian Motion (BBM)*, the maximum position, when centered by its median, converges weakly to a travelling wave solution. Later Bramson [16, 14] gave detailed order of the centering and showed that an "extra" logarithmic term appears, which later was termed as the *Bramson correction*. Later Lalley and Sellke [27] gave a different probabilistic interpretation of the travelling wave limit through certain conditional limit theorem and using a new concept called the *derivative martingales*.

To obtain the centered asymptotic limits for BRW, Bachmann [6] considered the case in which the underlying point process Z consists of N i.i.d. random variables from some log-concave distribution and N satisfies $\mathbb{P}(N \geq 1) = 1$, $\mathbb{P}(N = 1) < 1$ and $\mathbb{E}[N] < \infty$. He showed that

Theorem 1.4.4 (Bachmann [6])**.** *Under the above assumptions, there exists a strictly monotone, continuous distribution function G (with $G(0) = 1/2$, in order that G is unique) such that for any $\alpha \in (0, 1)$ and $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(R_n \leq q_n^\alpha + x\right) = G\left(q_\alpha + x\right),$$

where q_n^α and q_α are the α-quantiles of the random variables R_n and the distribution function G, respectively.

Using recursive distributional equations, Aldous and Bandyopadhyay [4] proved the tightness of the centered maximum. They assumed that $\mathbb{P}(N \geq 1) = 1$, $\mathbb{P}(N = 1) < 1$, and $\nu(t) < \infty$ for some $t > 0$, and proved

Theorem 1.4.5 (Aldous and Bandyopadhyay [4])**.** *If*

$$\big(median(R_{n+1}) - median(R_n), n \geq 0\big) \text{ is bounded above,}$$

then

$$\big(R_n - median(R_n), n \geq 1\big) \text{ is tight.}$$

A similar result was also obtained by Bramson and Zeitouni [15]. They studied the branching random walk where the point process Z consists of N i.i.d. random variables from a distribution F for which there exist $a > 0$ and $M_0 > 0$ satisfying

$$F(x + M) \leq e^{-aM} F(x)$$

for all $x \geq 0$ and $M \geq M_0$. They assumed that $\mathbb{P}(N \geq 1) = 1$, $\mathbb{P}(N = 1) < 1$, and $\mathbb{E}[N^t] < \infty$ for some $t > 0$, and showed

Theorem 1.4.6 (Bramson and Zeitouni [15])**.** *Under the above assumptions,*

$$\big(R_n - median(R_n), n \geq 1\big) \text{ is tight.}$$

Note that the above results tell us about the asymptotics of R_n shifted by its median or the α-th quantile, but none of them provides an exact formula for the median or the α-th quantile.

The second-order term for R_n was then obtained in 2009 by two independent groups of researchers, Hu and Shi, and Addario-Berry and Reed. Hu and Shi [24] showed that if one assumes that $\mathbb{E}[N^{1+\delta}]$, $\nu(-\delta)$, and $\nu(1 + \delta)$ are finite for some $\delta > 0$, $\mathbb{E}[N] > 1$, and $\nu(1) = \nu'(1) = 0$, then

Theorem 1.4.7 (Hu and Shi [24]). *Under the above assumptions, conditionally on the system's survival*

$$\liminf_{n\to\infty} \frac{R_n}{\log n} = -\frac{3}{2} \ \textit{a.s.};$$

$$\limsup_{n\to\infty} \frac{R_n}{\log n} = -\frac{1}{2} \ \textit{a.s.; and}$$

$$\frac{R_n}{\log n} \xrightarrow{p} -\frac{3}{2}.$$

Addario-Berry and Reed [1], on the other hand, considered a supercritical branching random walk where the underlying point process Z consists of N i.i.d. random variables. They proved

Theorem 1.4.8 (Addario-Berry and Reed [1]). *Suppose that the following conditions hold:*

- *there exists an integer $d \geq 2$ such that $\mathbb{P}(N \leq d) = 1$;*
- *there exists $\vartheta > 0$ such that $\nu(-\vartheta) < \infty$; and*
- *there exists $\theta_0 > 0$ in the interior of the set $\{t : \nu(t) < \infty\}$ satisfying $\nu(\theta_0) = \theta_0 \nu'(\theta_0)$.*

Let $\mathfrak{S}$ be the event that the branching random walk survives. Then

$$\mathbb{E}\left[R_n | \mathfrak{S}\right] = \frac{\nu(\theta_0)}{\theta_0} n - \frac{3}{2\theta_0} \log n + O(1),$$

and there exist constants $C > 0$, $\delta > 0$ depending only on the increment distribution, such that for all $x \in \mathbb{R}$

$$\mathbb{P}\left(\left| R_n - \mathbb{E}\left[R_n | \mathfrak{S}\right] \right| \geq x \,\middle|\, \mathfrak{S} \right) \leq C e^{-\delta x}.$$

This immediately implies that under the above assumptions, conditionally on the system's survival,

$$R_n - \frac{\nu(\theta_0)}{\theta_0} n + \frac{3}{2\theta_0} \log n \ \text{ is tight.}$$

Finally, Aïdékon [2] proved that the centered maximum converges in law to a randomly shifted Gumbel distribution when $\mathbb{E}[N] > 1$ and there is a $\theta_0 \in (0, \infty)$ satisfying $\nu(\theta_0) = \theta_0 \nu'(\theta_0)$. Under some mild conditions on the underlying progeny point process, he showed that

Theorem 1.4.9 (Aïdékon [2]). *There exists a finite positive constant c such that for all $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(R_n - \frac{\nu(\theta_0)}{\theta_0} n + \frac{3}{2\theta_0} \log n \le x \right) = \mathbb{E}\left[e^{-cD_{\theta_0}^{\infty} e^{-\theta_0 x}} \right],$$

where $D_{\theta_0}^{\infty} > 0$ on the event of systems survival and it is also the a.s. limit of the derivative martingale $\{D_n\}_{n\ge 1}$ defined as

$$D_n = -\sum_{|v|=n} \left(\theta_0 S(v) - n\nu(\theta_0)\right) e^{\left(\theta_0 S(v) - n\nu(\theta_0)\right)}.$$

This essentially settles the long-standing open problem of Biggins [9].

1.4.3 Brunet-Derrida Type Result

The point process convergence for the classical BRW was proved by Madaule [29]. He showed that

Theorem 1.4.10 (Madaule [29]). *Under the assumptions of Theorem 1.4.9, conditioning on the set of non-extinction, the centered point process at the n-th generation, denoted by*

$$Z_n = \sum_{|v|=n} \delta_{\left\{\theta_0 S(v) - \log E_v - n\nu(\theta_0) + \frac{3}{2}\log n - \log D_{\theta_0}^{\infty}\right\}},$$

converges in distribution to a decorated Poisson point process, and the limiting point process is independent of $D_{\theta_0}^{\infty}$.

Following is the definition of a *decorated Poisson point process*:

Definition (Decorated Poisson Point Process). *Let $\mathfrak{Z} = \sum_{i\ge 1} \delta_{\zeta_i}$ be a Poisson point process on $\mathbb{R}$ with intensity measure $\lambda e^{-x}\, dx$ for some $\lambda > 0$. Then, independently for each point ζ_i, we replace it with a point process χ_i shifted by ζ_i, where $\{\chi_i\}_{i\ge 1}$ are* *independent copies of a point process χ and are also independent of the Poisson point process $\mathfrak{Z}$. Then the resulting point process*

$$\mathfrak{X} = \sum_{i \geq 1} \mathcal{T}_{\zeta_i}(\chi_i)$$

is called the Poisson point process $\mathfrak{Z}$ decorated by χ. Here, $\mathcal{T}_x$ represents the translation by x.

1.4.4 Large Deviations

Gantert and Höfelsauer [21] calculated the large deviations for the laws of $\{R_n/n\}_{n\geq 1}$. They considered the case where the point process Z consists of N i.i.d. copies of some random variable X, whose moment-generating function is finite in a neighbourhood of 0. They showed that if $\mathbb{E}[N] > 1$, and the process satisfies both the Kesten-Stigum and the Schröder conditions, i.e., $\mathbb{E}[N \log N] < \infty$ and $\mathbb{P}(N \leq 1) > 0$, then

Theorem 1.4.11 (Gantert and Höfelsauer [21])**.** *Under the above assumptions, conditionally on the event of survival of the branching process, the laws of $\{R_n/n\}_{n\geq 1}$ satisfy the large deviation principle with the rate function*

$$\Phi(x) = \begin{cases} I(x) - \log \mathbb{E}[N] & \text{if } x > \gamma; \\ 0 & \text{if } x = \gamma; \\ \inf\limits_{0<t\leq 1} \left\{ \rho t + tI\left(\frac{x-(1-t)\gamma}{t}\right)\right\} & \text{if } x < \gamma, \end{cases}$$

where $\rho = -\log \mathbb{P}(N = 1)$, and I is the rate function for the random walk whose increments are i.i.d. copies of X.

As mentioned in Gantert and Höfelsauer [21], the Schröder condition is only required for the lower large deviations, i.e., $x < \gamma$. That means the upper large deviations remain the same even if we omit the Schröder condition. However, the result is incomplete since it does not provide the lower large deviation rate function in this case.

1.4.5 Time Inhomogeneous Setup

The time inhomogeneous setup was studied by Fang and Zeitouni [20]. In their setup, the underlying point process consists of two i.i.d. mean zero Gaussian random variables, whose variances vary with generations. For each $n \geq 1$, the variance is σ_1^2 in the first $\lfloor n/2 \rfloor$ many generations and in the remaining $\lceil n/2 \rceil$ many generations, it is σ_2^2. Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and $\lceil x \rceil$ represents the smallest integer greater than or equal to x. They showed that

Theorem 1.4.12 (Fang and Zeitouni [20]). *When $\sigma_1^2 < \sigma_2^2$,*

$$R_n - \left(\sqrt{\left(\sigma_1^2 + \sigma_2^2\right) \log 2}\right) n + \frac{\sqrt{\sigma_1^2 + \sigma_2^2}}{4\sqrt{\log 2}} \log n \ \text{ is tight.}$$

Theorem 1.4.13 (Fang and Zeitouni [20]). *When $\sigma_1^2 > \sigma_2^2$,*

$$R_n - \frac{\sqrt{2 \log 2}\,(\sigma_1 + \sigma_2)}{2} n + \frac{3\,(\sigma_1 + \sigma_2)}{2\sqrt{2 \log 2}} \log n \ \text{ is tight.}$$

Note that in this section, we have only stated the results that are relevant to our work. There are many other very important and interesting results that are not mentioned here.

1.5 Outline

The thesis is organized as follows. Chapter 2 provides our most important tool: the coupling between the maximum statistic and a linear statistic. Chapter 3 contains the asymptotics of the maximum. In Chapter 4, we present Brunet-Derrida type results for our LPM-BRW model. The large deviation rate functions are discussed in Chapter 5. Finally, in Chapter 6, we provide the results in the time inhomogeneous setup.

Chapter 2

Coupling between a Maximum and a Linear Statistic [1]

2.1 Introduction

In this chapter, we first define a few operators on the space of probabilities which we use to develop a novel mathematical tool. This novel technique allows us to obtain a very simple coupling between the right-most position of our LPM-BRW model with a more well-studied statistical quantity known as the smoothing transform. This coupling helps us to get the results related to our LPM-BRW model in the subsequent chapters.

2.2 A Few Operators

Let $Z = \sum_{j \geq 1} \delta_{\xi_j}$ be a point process on $\mathbb{R}$. In the sequel, we denote $\mathcal{P}(A)$ as the set of all probabilities on a measurable space $(A, \mathcal{A})$, $\bar{\mathbb{R}} = [-\infty, \infty]$ and $\bar{\mathbb{R}}_+ = [0, \infty]$. We define the following operators.

[1] This chapter is based on Section 3 of the paper entitled "*Right-most position of a last progeny modified branching random walk*" [7].

2.2.1 Maximum Operator

Definition 2.2.1. *The maximum operator* $M_Z : \mathcal{P}\left(\bar{\mathbb{R}}\right) \to \mathcal{P}\left(\bar{\mathbb{R}}\right)$ *is defined as*

$$M_Z(\eta) = dist\left(\max_j \left\{\xi_j + X_j\right\}\right),$$

where $\{X_j\}_{j\geq 1}$ *are i.i.d.* $\eta \in \mathcal{P}\left(\bar{\mathbb{R}}\right)$ *and are independent of the point process* Z*.*

We denote $M_Z^n(\eta)$ as the n-th iterate of $M_Z(\eta)$. Suppose we add i.i.d. displacements from η to each of the particles at the n-th generation, where the added displacements are also independent of the BRW. Then as illustrated in Figure 2.1, the distribution of the new maximum will be $M_Z^n(\eta)$.

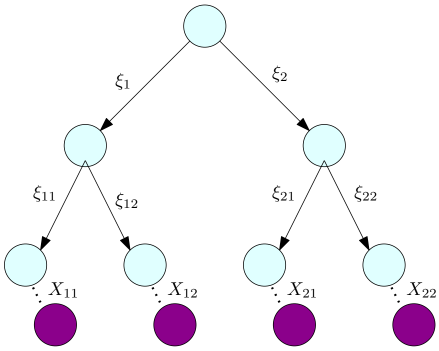

Figure 2.1: Illustration of the maximum operator

Proposition 2.2.1. *For any* $n \in \mathbb{N}$*,*

$$M_Z^n(\eta) = dist\left(\max_{|v|=n} \left\{S(v) + X_v\right\}\right),$$

where $\{X_v\}_{|v|=n}$ *are i.i.d.* $\eta \in \mathcal{P}\left(\bar{\mathbb{R}}\right)$ *and are independent of the BRW.*

Proof. It follows from the definition that

$$\left\{S(v) : |v| = 1\right\} \stackrel{d}{=} \left\{\xi_j : j \geq 1\right\}. \tag{2.1}$$

So, the proposition holds trivially for n=1. Suppose the proposition holds for $n = k - 1$ for some $k \in \mathbb{N}$. Observe that

$$\max_{|v|=k} \left\{S(v) + X_v\right\} = \max_{|u|=1} \left\{ \max_{|v|=k, u<v} \left\{S(v) + X_v\right\} \right\}$$
$$= \max_{|u|=1} \left\{ S(u) + \max_{|v|=k, u<v} \left\{S(v) - S(u) + X_v\right\} \right\}. \tag{2.2}$$

Here $u < v$ means v is a descendant of u. Now, for $|u| = 1$, we have

$$\left\{S(v) - S(u) : |v| = k, u < v\right\} \stackrel{d}{=} \left\{S(v) : |v| = k - 1\right\}. \tag{2.3}$$

Thus

$$\mathfrak{T}_u := \max_{|v|=k, u<v} \left\{S(v) - S(u) + X_v\right\} \sim M_Z^{k-1}(\eta).$$

Also, note that $\mathfrak{T}_u$'s are i.i.d. and are independent of $\left\{S(u) : |u| = 1\right\}$. Therefore by (2.2),

$$\max_{|v|=k} \left\{S(v) + X_v\right\} = \max_{|u|=1} \left\{S(u) + \mathfrak{T}_u\right\} \sim M_Z \circ M_Z^{k-1}(\eta) = M_Z^k(\eta).$$

Hence, by induction, the proposition holds for any $n \in \mathbb{N}$. □

Remark 2.2.1: Note that if we take $\eta = \delta_0$, we get that $R_n \sim M_Z^n(\delta_0)$, and $R_n^* \sim M_Z^n(\eta)$ if η is the distribution of $\frac{1}{\theta}\log(Y/E)$, where $Y \sim \mu$, $E \sim$ Exponential (1), and they are independent of each other. Also, note that in the expression of M_Z, if we replace the maximum by sum and the addition by multiplication (one way of doing that is exponentiation), we will get the operator discussed next.

2.2.2 Smoothing Operator

Definition 2.2.2. *The smoothing operator* $L_Z : \mathcal{P}(\bar{\mathbb{R}}_+) \to \mathcal{P}(\bar{\mathbb{R}}_+)$ *is defined by*

$$L_Z(\mu) = dist\left(\sum_{j\geq 1} e^{\xi_j} Y_j\right),$$

where $\{Y_j\}_{j\geq 1}$ *are i.i.d.* $\mu \in \mathcal{P}(\bar{\mathbb{R}}_+)$ *and are independent of the point process* Z*.*

If we denote $L_Z^n(\mu)$ as the n-th iterate of $L_Z(\mu)$, then we have the following.

Proposition 2.2.2. *For any $n \in \mathbb{N}$,*

$$L_Z^n(\mu) = dist\left(\sum_{|v|=n} e^{S(v)} Y_v\right),$$

where $\{Y_v\}_{|v|=n}$ are i.i.d. $\mu \in \mathcal{P}(\bar{\mathbb{R}}_+)$ and are independent of the BRW.

Proof. Note that by (2.1), the proposition holds trivially for n=1. Suppose it holds for $n = k-1$ for some $k \in \mathbb{N}$. Observe that

$$\sum_{|v|=k} e^{S(v)} Y_v = \sum_{|u|=1} \sum_{|v|=k, u<v} e^{S(v)} Y_v = \sum_{|u|=1} e^{S(u)} \left(\sum_{|v|=k, u<v} e^{S(v)-S(u)} Y_v\right). \tag{2.4}$$

Also, by (2.3),

$$\mathfrak{T}'_u := \sum_{|v|=k, u<v} e^{S(v)-S(u)} Y_v \sim L_Z^{k-1}(\mu).$$

Note that $\mathfrak{T}'_u$'s are i.i.d. and are independent of $\{S(u) : |u| = 1\}$. Therefore by (2.4),

$$\sum_{|v|=n} e^{S(v)} Y_v = \sum_{|u|=1} e^{S(u)} \mathfrak{T}'_u \sim L_Z \circ L_Z^{k-1}(\mu) = L_Z^k(\mu).$$

Hence, by induction, the proposition holds for any $n \in \mathbb{N}$. □

2.2.3 Link Operator

Definition 2.2.3. *The link operator $\mathcal{E} : \mathcal{P}(\bar{\mathbb{R}}_+) \to \mathcal{P}(\bar{\mathbb{R}})$ is defined by*

$$\mathcal{E}(\mu) = dist\left(\log \frac{Y}{E}\right),$$

where $E \sim Exponential(1)$ and $Y \sim \mu \in \mathcal{P}(\bar{\mathbb{R}}_+)$ and they are independent.

2.2.4 Scaling and Centering Operator

Definition 2.2.4. *For $a \geq 0$ and $b \in \mathbb{R}$, the operator $\Xi_{a,b}$ on the set of all point processes is defined by*

$$\Xi_{a,b}(\mathcal{Z}) = \sum_{j \geq 1} \delta_{a\zeta_j - b},$$

where $\mathcal{Z} = \sum_{j \geq 1} \delta_{\zeta_j}$. Sometimes we may denote $\Xi_{a,0}$ by Ξ_a for notational simplicity.

2.3 Transforming Relationship

The following result is one of the most important observations, and it links the operators defined above.

Theorem 2.3.1 (Transforming Relationship)**.** *For all $k \in \mathbb{N}$,*

$$M_Z^k \circ \mathcal{E} = \mathcal{E} \circ L_Z^k.$$

Remark 2.3.1: The above theorem is a formalization of the intuitive idea that under the link operator $\mathcal{E}$, a $(\text{Max}, +)$-type algebra on general random variables gets rightly converted to the usual $(+, \cdot)$-type algebra on non-negative random variables.

Proof. Let $Z = \sum_{j \geq 1} \delta_{\xi_j}$, $\{E_j\}_{j \geq 1}$ are i.i.d. Exponential (1), $\{Y_j\}_{j \geq 1}$ are i.i.d. μ, and they are independent of each other. Now,

$$\begin{aligned} M_Z \circ \mathcal{E}(\mu) &= \text{dist}\left(\max_j \left(\xi_j + \log \frac{Y_j}{E_j}\right)\right) \\ &= \text{dist}\left(\max_j \left(\log \frac{e^{\xi_j} Y_j}{E_j}\right)\right) = \text{dist}\left(-\log\left(\min_j \frac{E_j}{e^{\xi_j} Y_j}\right)\right). \end{aligned} \tag{2.5}$$

Let $\mathcal{A}$ be the σ-algebra generated by Z and $\{Y_j\}_{j \geq 1}$. So, given $\mathcal{A}$, $\left\{\frac{E_j}{e^{\xi_j} Y_j}\right\}_{j \geq 1}$ are conditionally independent and

$$\frac{E_j}{e^{\xi_j} Y_j} \,\bigg|\, \mathcal{A} \sim \text{Exponential}\left(e^{\xi_j} Y_j\right).$$

This implies

$$\min_j \frac{E_j}{e^{\xi_j} Y_j} \,\Big|\, \mathcal{A} \sim \text{Exponential}\left(\sum_{j\geq 1} e^{\xi_j} Y_j\right).$$

We also have

$$\frac{E_1}{\sum_{j\geq 1} e^{\xi_j} Y_j} \,\Big|\, \mathcal{A} \sim \text{Exponential}\left(\sum_{j\geq 1} e^{\xi_j} Y_j\right).$$

Therefore

$$\min_j \frac{E_j}{e^{\xi_j} Y_j} \stackrel{d}{=} \frac{E_1}{\sum_{j\geq 1} e^{\xi_j} Y_j}. \tag{2.6}$$

Applying this to (2.5), we get

$$\begin{aligned} M_Z \circ \mathcal{E}(\mu) &= \text{dist}\left(-\log\left(\min_j \frac{E_j}{e^{\xi_j} Y_j}\right)\right) \\ &= \text{dist}\left(-\log\left(\frac{E_1}{\sum_{j\geq 1} e^{\xi_j} Y_j}\right)\right) = \mathcal{E} \circ L_Z(\mu). \end{aligned} \tag{2.7}$$

Therefore, by induction, for all $k \in \mathbb{N}$,

$$M_Z^k \circ \mathcal{E}(\mu) = \mathcal{E} \circ L_Z^k(\mu).$$

□

Remark 2.3.2: An alternative proof of Theorem 2.3.1 is as follows. Let $\{\pi_i\}_{i\geq 1}$ be an i.i.d. sequence of Poisson point processes with intensity $e^{-x}\,dx$. Now, we consider the superposition

$$\mathfrak{X} = \sum_{j\geq 1} \mathcal{T}_{\xi_j + \log Y_j}\left(\pi_j\right), \tag{2.8}$$

where $\mathcal{T}_x$ is translation by x. In equation (2.8), $\{\xi_j\}_{j\geq 1}$ are as earlier, $\{Y_j\}_{j\geq 1}$ are i.i.d. μ, and these two sequences are independent of each other and also independent of $\{\pi_j\}_{j\geq 1}$. Note that $\max \pi_1$ is Gumbel-distributed, i.e., distributed as $-\log E_1$. Hence,

$$\max \mathfrak{X} \stackrel{d}{=} \max_j \left(\xi_j + \log Y_j - \log E_j\right). \tag{2.9}$$

On the other hand, each $\mathcal{T}_{\xi_j+\log Y_j}(\pi_j)$ is a Poisson point process with intensity $e^{\xi_j - x} Y_j\, dx$, and thus, $\mathfrak{X}$ is a Poisson point process with intensity $\left(\sum_{j\geq 1} e^{\xi_j} Y_j\right) e^{-x}\, dx$. Therefore,

$$\max \mathfrak{X} \stackrel{d}{=} \log\left(\sum_{j\geq 1} e^{\xi_j} Y_j\right) + \max \pi_1 \stackrel{d}{=} \log\left(\sum_{j\geq 1} e^{\xi_j} Y_j\right) - \log E_1. \tag{2.10}$$

Now by combining (2.9) and (2.10) and then using induction we get the required result. This alternative proof was indicated by an anonymous referee.

As an immediate corollary of the above theorem, we get a very useful coupling between the LPM-BRW and the linear statistic associated with the *linear operator*.

Theorem 2.3.2. *Let $\theta > 0$ and $\mu \in \mathcal{P}(\bar{\mathbb{R}}_+)$. Then for any $n \geq 1$,*

$$\theta R_n^*(\theta, \mu) \stackrel{d}{=} \log Y_n^{\mu}(\theta) - \log E,$$

where $Y_n^{\mu}(\theta) := \sum_{|v|=n} e^{\theta S(v)} Y_v$, $\{Y_v\}_{|v|=n}$ are i.i.d. μ, $E \sim$ Exponential(1), and $\{Y_v\}_{|v|=n}$ and E are independent of each other and also of the BRW.

Proof. From (1.1) together with Proposition 2.2.1, it follows that

$$\theta R_n^*(\theta, \mu) = \max_{|v|=n} \left\{\theta S(v) + \log(Y_v/E_v)\right\} \sim M^n_{\Xi_\theta(Z)} \circ \mathcal{E}(\mu). \tag{2.11}$$

Therefore, by applying Theorem 2.3.1 and Proposition 2.2.2, we get

$$\begin{aligned} \operatorname{dist}\left(\theta R_n^*(\theta, \mu)\right) &= M^n_{\Xi_\theta(Z)} \circ \mathcal{E}(\mu) \\ &= \mathcal{E} \circ L^n_{\Xi_\theta(Z)}(\mu) = \operatorname{dist}\left(\log Y_n^{\mu} - \log E\right). \end{aligned}$$

□

Chapter 3

Asymptotics of the Right-most Position [1]

3.1 Introduction

In this chapter, we first present and prove some asymptotic results about the associated linear statistic, which we later use together with the coupling technique mentioned in Chapter 2 to obtain the asymptotics of R_n^*.

3.2 Assumptions

Before we state our assumptions, we introduce the following important quantity. For a point process $Z = \sum_{j=1}^{N} \delta_{\xi_j}$, we will write

$$m(t) := \mathbb{E}\left[\int_{\mathbb{R}} e^{tx}\, Z(dx)\right] = \mathbb{E}\left[\sum_{j=1}^{N} e^{t\xi_j}\right],$$

where $t \in \mathbb{R}$, whenever the expectation exists. Naturally, m is the *moment-generating function* of the point process Z.

[1]This chapter is based on Sections 2.1, 2.2, 4, 5.1, 5.2, 5.3 and 5.4 of the paper entitled "*Right-most position of a last progeny modified branching random walk*" [7].

We now state our main assumptions. Throughout this work, we will assume the following three conditions hold:

(**A1**) $m(t)$ is finite for all $t \in (-\vartheta, \infty)$ for some $\vartheta > 0$.

(**A2**) The point process Z is *non-trivial*, and the *extinction probability* of the underlying *branching process* is 0. In other words, $\mathbb{P}(N = 1) < 1$, $\mathbb{P}(Z(\{a\}) = N) < 1$ for any $a \in \mathbb{R}$, and $\mathbb{P}(N \geq 1) = 1$.

(**A3**) N has finite $(1 + p)$-th moment for some $p > 0$.

Remark 3.2.1: (**A1**) implies that m is infinitely differentiable on $(-\vartheta, \infty)$. Together with (**A3**), it also implies that there exists $q > 0$ such that for all $t \in [0, \infty)$

$$\mathbb{E}\left[\left(\int_{\mathbb{R}} e^{tx}\, Z(dx)\right)^{1+q}\right] < \infty. \tag{3.1}$$

To see this, observe that

$$\int_{\mathbb{R}} e^{tx}\, Z(dx) \leq N e^{\max_{j=1}^{N} t\xi_j}. \tag{3.2}$$

This, together with Hölder's inequality, implies

$$\begin{aligned}
\mathbb{E}\left[\left(\int_{\mathbb{R}} e^{tx}\, Z(dx)\right)^{1+q}\right] &\leq \mathbb{E}\left[N^{1+q} \cdot e^{(1+q)\left(\max_{j=1}^{N} t\xi_j\right)}\right] \\
&\leq \left(\mathbb{E}\left[N^{(1+q)^2}\right]\right)^{\frac{1}{1+q}} \cdot \left(\mathbb{E}\left[e^{\frac{(1+q)^2}{q}\left(\max_{j=1}^{N} t\xi_j\right)}\right]\right)^{\frac{q}{1+q}} \\
&\leq \left(\mathbb{E}\left[N^{(1+q)^2}\right]\right)^{\frac{1}{1+q}} \cdot \left(\mathbb{E}\left[\int_{\mathbb{R}} e^{\frac{(1+q)^2}{q}tx}\, Z(dx)\right]\right)^{\frac{q}{1+q}} \\
&= \left(\mathbb{E}\left[N^{(1+q)^2}\right]\right)^{\frac{1}{1+q}} \cdot \left(m\left(t(1+q)^2/q\right)\right)^{\frac{q}{1+q}}.
\end{aligned}$$

Then, by choosing q such that $(1 + q)^2 \leq 1 + p$, one gets (3.1).

3.3 A Specific Scaling Constant θ_0

We define

$$\nu(t) \equiv \nu_Z(t) := \log\big(m(t)\big) = \log \mathbb{E}\left[\int_{\mathbb{R}} e^{tx}\, Z(dx)\right]$$

for $t \in \mathbb{R}$, whenever $m\,(t)$ is defined. Note that under assumptions (**A1**) and (**A2**), ν is strictly convex in $(-\vartheta, \infty)$. Although this is a well-known fact, we are unable to find an exact reference to it. So we give a proof of this.

Proposition 3.3.1. *$\nu(t)$ is strictly convex in $(-\vartheta, \infty)$.*

Proof. From assumption (**A1**), we know that

$$m(t) = \mathbb{E}\left[\int_{\mathbb{R}} e^{tx}\, Z(dx)\right] < \infty,$$

for all $t \in (-\vartheta, \infty)$. Therefore using the dominated convergence theorem, we have for all $t \in (-\vartheta, \infty)$,

$$m'(t) = \mathbb{E}\left[\int_{\mathbb{R}} x e^{tx}\, Z(dx)\right] < \infty,$$

and

$$m''(t) = \mathbb{E}\left[\int_{\mathbb{R}} x^2 e^{tx}\, Z(dx)\right] < \infty.$$

From assumption (**A2**), we have that $\mathbb{P}(Z(\{a\}) = N) < 1$ for all $a \in \mathbb{R}$. Therefore for all $a \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}} (x-a)^2 e^{tx}\, Z(dx)\right] > 0 \\
\Rightarrow\ &\mathbb{E}\left[\int_{\mathbb{R}} x^2 e^{tx}\, Z(dx)\right] - 2a\mathbb{E}\left[\int_{\mathbb{R}} x e^{tx}\, Z(dx)\right] + a^2\mathbb{E}\left[\int_{\mathbb{R}} e^{tx}\, Z(dx)\right] > 0 \\
\Rightarrow\ &\mathbb{E}\left[\int_{\mathbb{R}} x^2 e^{tx}\, Z(dx)\right] \cdot \mathbb{E}\left[\int_{\mathbb{R}} e^{tx}\, Z(dx)\right] > \left(\mathbb{E}\left[\int_{\mathbb{R}} x e^{tx}\, Z(dx)\right]\right)^2 \\
\Rightarrow\ &m''(t) \cdot m(t) > \big(m'(t)\big)^2.
\end{aligned}$$

Hence for all $t \in (-\vartheta, \infty)$,

$$\nu''(t) = \frac{m''(t) \cdot m(t) - \big(m'(t)\big)^2}{\big(m(t)\big)^2} > 0.$$

This proves the proposition. □

We now define a constant related to the underlying driving point process Z, which we denote by θ_0. Let

$$\theta_0 := \inf \left\{ \theta > 0 : \frac{\nu(\theta)}{\theta} = \nu'(\theta) \right\}. \tag{3.3}$$

The fact that $\nu(\theta)$ is strictly convex ensures that the above set is at most singleton. If it is a singleton, then as illustrated in Figure 3.1, θ_0 is the unique point in $(0, \infty)$ such that a tangent from the origin to the graph of $\nu(\theta)$ touches the graph at $\theta = \theta_0$. And if it is empty, then by definition θ_0 takes value ∞, and there does not exist any tangent from the origin to the graph of $\nu(\theta)$ on the right half-plane.

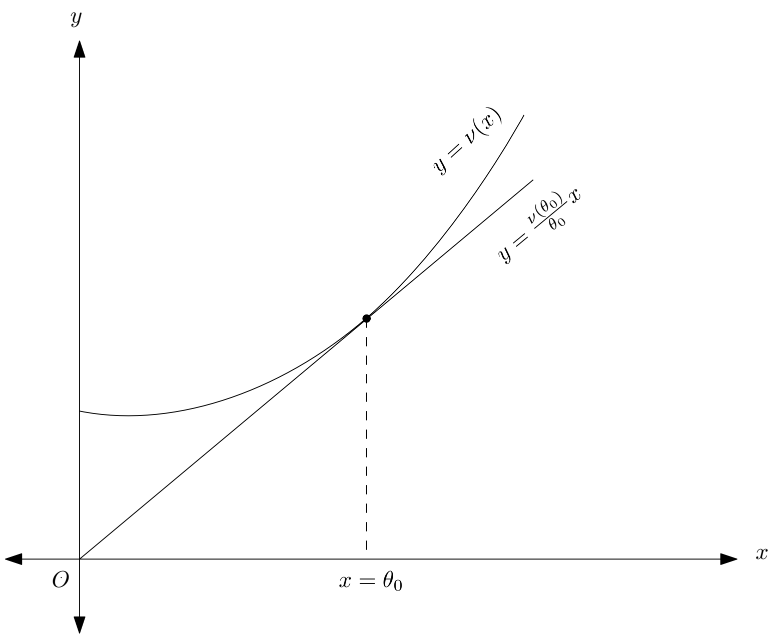

Figure 3.1: Construction of the quantity θ_0

Remark 3.3.1: It is worth noting that $\nu(\theta)/\theta$ is strictly decreasing for $\theta \in (0, \theta_0)$ and strictly increasing for $\theta \in (\theta_0, \infty)$. Therefore, as shown in Figure 3.2, when θ_0 is finite, it is the unique point of minimum for $\nu(\theta)/\theta$.

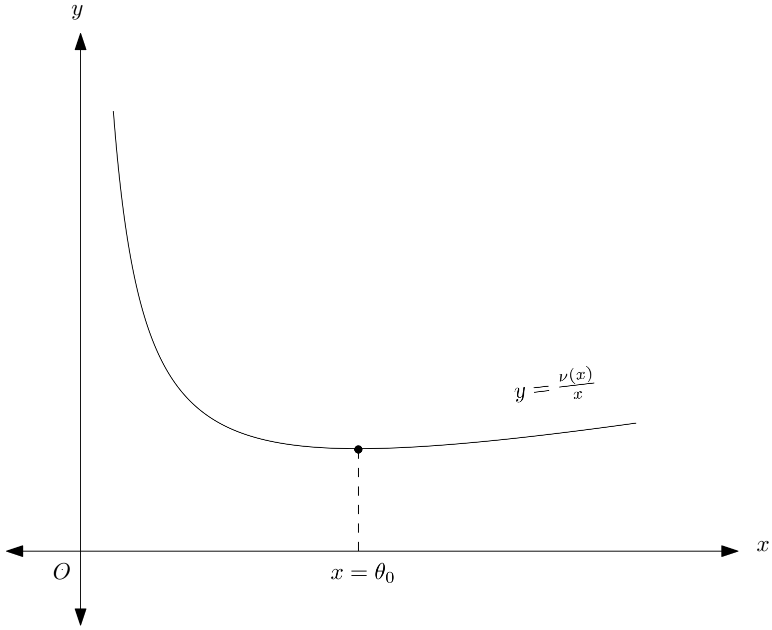

Figure 3.2: Graph of $y = \nu(x)/x$

Remark 3.3.2: Note that

$$\frac{\nu(\theta)}{\theta} = \lim_{n\to\infty} \frac{1}{n\theta} \log \mathbb{E}\left[W_n(\theta)\right],$$

where $W_n(\theta) = W_n(\theta, 0)$ is as defined in (3.5). $\nu(\theta)/\theta$ is often referred as the *"annealed free energy"*. Further, using Jensen's inequality, it is easy to see that, the so-called *"quenched free energy"*, say $F(\theta)$, defined below satisfies the following inequality

$$F(\theta) := \lim_{n\to\infty} \frac{1}{n\theta} \mathbb{E}\left[\log W_n(\theta)\right] \leq \frac{\nu(\theta)}{\theta}.$$

Whether θ_0 is finite or infinite can be characterized by the following proposition.

Proposition 3.3.2. *$\theta_0 < \infty$ iff*

$$\lim_{\theta\to\infty} \nu(\theta) - \theta \left(\lim_{x\to\infty} \nu'(x) \right) < 0.$$

Proof. Let $f : (0, \infty) \to \mathbb{R}$ be a function defined as

$$f(\theta) = \nu(\theta) - \theta \left(\lim_{x\to\infty} \nu'(x) \right).$$

Since ν is strictly convex,

$$f'(\theta) = \nu'(\theta) - \left(\lim_{x\to\infty} \nu'(x)\right) < 0,$$

for all $\theta \in (0,\infty)$. Therefore f is strictly decreasing in $(0,\infty)$, and hence $\lim_{\theta\to\infty} f(\theta)$ exists.

(Only if part). If $\lim_{\theta\to\infty} f(\theta) \geq 0$, then for all $\theta \in (0,\infty)$,

$$\nu(\theta) - \theta\nu'(\theta) > \nu(\theta) - \theta\left(\lim_{x\to\infty} \nu'(x)\right) > 0,$$

which implies $\theta_0 = \infty$.

(If part). Now, suppose $\theta_0 = \infty$, i.e., there does not exist $\theta \in (0,\infty)$ satisfying $\nu(\theta) = \theta\nu'(\theta)$. Since ν and ν' are continuous and by assumption (**A2**), $\nu(0) > 0 \cdot \nu'(0)$, we have $\nu(\theta) > \theta\nu'(\theta)$ for all $\theta \in (0,\infty)$. Let T_x be the tangent to the graph of ν at the point x for some $x \in (0,\infty)$, i.e.,

$$T_x(\theta) = (\theta - x)\nu'(x) + \nu(x).$$

Since ν is convex, we have

$$\nu(\theta) \geq T_x(\theta) = \theta\nu'(x) + \nu(x) - x\nu'(x) > \theta\nu'(x),$$

for all $\theta \in (0,\infty)$. Therefore letting $x \to \infty$, we obtain

$$\nu(\theta) \geq \theta\left(\lim_{x\to\infty} \nu'(x)\right),$$

for all $\theta \in (0,\infty)$, which implies $\lim_{\theta\to\infty} f(\theta) \geq 0$. This completes the proof. □

Remark 3.3.3: An alternative proof for the "if part" is as follows. If $\lim_{\theta\to\infty} f(\theta) < 0$, then there exists $\theta \in (0,\infty)$ such that $f(\theta) < 0$, which implies

$$\frac{\nu(\theta)}{\theta} < \lim_{x\to\infty} \nu'(x) = \lim_{x\to\infty} \frac{\nu(x)}{x}.$$

Therefore, by Remark 3.3.1, $\theta_0 < \infty$.

Remark 3.3.4: It is to be noted that θ_0 is always finite if $\lim_{x\to\infty} \nu'(x) = \infty$.

3.4 Few Auxiliary Results on the Linear Statistic

In this section, we provide a few convergence results related to the *linear operator*, L_Z^n, as defined in the Section 2.2.2 and associated *linear statistic*, which is defined in the sequel (see equation (3.5)).

We start by observing that if we consider the point process $\Xi_{\theta,\nu_Z(\theta)}(Z)$, then

$$\nu_{\Xi_{\theta,\nu_Z(\theta)}(Z)}(\alpha) = \log \mathbb{E}\left[\int_{\mathbb{R}} e^{\alpha\theta x - \alpha\nu_Z(\theta)}\, Z(dx)\right] = \nu_Z(\alpha\theta) - \alpha\nu_Z(\theta).$$

Differentiating this with respect to α, we get

$$\nu'_{\Xi_{\theta,\nu_Z(\theta)}(Z)}(\alpha) = \theta\nu'_Z(\alpha\theta) - \nu_Z(\theta).$$

So, by taking $\alpha = 1$, we obtain $\nu_{\Xi_{\theta,\nu_Z(\theta)}(Z)}(1) = 0$, and

$$\nu'_{\Xi_{\theta,\nu_Z(\theta)}(Z)}(1) = \theta\nu'_Z(\theta) - \nu_Z(\theta) \begin{cases} > 0 & \text{if } \theta_0 < \theta < \infty; \\ = 0 & \text{if } \theta = \theta_0 < \infty; \\ < 0 & \text{if } \theta < \theta_0 \leq \infty. \end{cases}$$

Therefore, using Theorem 1.6 of Liu [28], we get

$$L^n_{\Xi_{\theta,\nu_Z(\theta)}(Z)}(\mu) \xrightarrow{w} \begin{cases} \delta_0 & \text{if } \theta = \theta_0 < \infty; \\ \mu_\theta^\infty & \text{if } \theta < \theta_0 \leq \infty, \end{cases} \tag{3.4}$$

where for all $\theta < \theta_0$, $\mu_\theta^\infty \neq \delta_0$ is a fixed point of $L_{\Xi_{\theta,\nu_Z(\theta)}(Z)}$ and have same mean as μ. Note that as μ_θ^∞ is a fixed point of $L_{\Xi_{\theta,\nu_Z(\theta)}(Z)}$, we have

$$\mu_\theta^\infty(\{0\}) = \mathbb{E}\left[\mu_\theta^\infty(\{0\})^N\right].$$

Together with assumption (**A2**), this implies for all $\theta < \theta_0$, $\mu_\theta^\infty(\{0\}) \in \{0,1\}$, and hence $\mu_\theta^\infty(\{0\}) = 0$.

We now define the *linear statistic* associated with the linear operator L_Z^n.

$$W_n(a,b) := \sum_{|v|=n} e^{aS(v)-nb}. \tag{3.5}$$

To simplify the notations, sometimes we may write $W_n(a,0)$ as $W_n(a)$. From Proposition 2.2.2, we get that

$$L^n_{\Xi_{a,b}(Z)}(\delta_1) = \mathrm{dist}\left(W_n(a,b)\right).$$

Since $\{W_n(\theta,\nu_Z(\theta))\}_{n\geq 1}$ is a non-negative martingale, it converges a.s. Therefore (3.4) implies that almost surely,

$$W_n(\theta,\nu_Z(\theta)) \to \begin{cases} 0 & \text{if } \theta = \theta_0 < \infty; \\ D_\theta^\infty & \text{if } \theta < \theta_0 \leq \infty, \end{cases} \tag{3.6}$$

for some positive random variable D_θ^∞ with $\mathbb{E}[D_\theta^\infty] = 1$, and the distribution of D_θ^∞ is a solution to the following *linear recursive distributional equation (RDE)*

$$\Delta \stackrel{d}{=} \sum_{|v|=1} e^{\theta S(v)-\nu(\theta)}\Delta_v, \tag{3.7}$$

where Δ_v are i.i.d. and have the same distribution as that of Δ. Biggins and Kyprianou [12] have shown that under the assumptions in Section 3.2, the solutions to the linear RDE are unique up to a scale factor whenever they exist. Therefore D_θ^∞ is the unique solution to the linear RDE (3.7) with mean 1.

The following proposition provides convergence results of $W_n(a,b)$ for various values of a and b.

Proposition 3.4.1. *For any $a > 0$ and $b \in \mathbb{R}$, almost surely*

$$W_n(a,b) \to \begin{cases} 0 & \text{if } a < \theta_0,\, b > \nu(a); & (i) \\ D_a^\infty & \text{if } a < \theta_0,\, b = \nu(a); & (ii) \\ \infty & \text{if } a < \theta_0,\, b < \nu(a); & (iii) \\ 0 & \text{if } \theta_0 < \infty,\, a \geq \theta_0,\, b \geq a\nu(\theta_0)/\theta_0; & (iv) \\ \infty & \text{if } \theta_0 < \infty,\, a \geq \theta_0,\, b < a\nu(\theta_0)/\theta_0. & (v) \end{cases}$$

To prove this proposition, we use the following elementary result. We provide the proof for the sake of completeness.

Lemma 3.4.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a continuously differentiable convex function and $\mathbb{S}$ be a convex subset of $[0,\infty) \times \mathbb{R}$ satisfying*

- *$(x,y) \in \mathbb{S}$ for all $0 < x < x_0$ and $y > f(x)$ and*
- *$(x,y) \notin \mathbb{S}$ for all $0 < x < x_0$ and $y < f(x)$,*

for some $x_0 > 0$. Then

$$\mathbb{S} \subseteq \left\{(x,y) : y \geq T_{x_0}(x)\right\},$$

where $T_{x_0}(\cdot)$ denotes the tangent to f at x_0.

Proof. We define a function $g : [0,\infty) \to \bar{\mathbb{R}}$ as

$$g(x) = \inf\left\{y : (x,y) \in \mathbb{S}\right\}.$$

We first show that g is convex. Take any x_1, x_2 such that $g(x_1)$, $g(x_2) < \infty$. By definition of g, for every $\epsilon > 0$, there exist $y_1 < g(x_1) + \epsilon$ and $y_2 < g(x_2) + \epsilon$ such that $(x_1,y_1), (x_2,y_2) \in \mathbb{S}$. So for any $\alpha \in (0,1)$, $(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) \in \mathbb{S}$. Therefore

$$g(\alpha x_1 + (1-\alpha)x_2) \leq \alpha y_1 + (1-\alpha)y_2 < \alpha g(x_1) + (1-\alpha)g(x_2) + \epsilon.$$

As $\epsilon > 0$ is arbitrary, we have

$$g(\alpha x_1 + (1-\alpha)x_2) \leq \alpha g(x_1) + (1-\alpha)g(x_2),$$

and this is true for all $\alpha \in (0,1)$. Therefore g is convex.

Let $T_x(.)$ be the tangent to f at x. As f is continuously differentiable, T_x converges pointwise to T_{x_0} as $x \to x_0$. Note that $g = f$ in $(0, x_0)$. Therefore for all $x \in (0, x_0)$, T_x is also tangent to g at x. As g is convex, $g \geq T_x$ for all $x \in (0, x_0)$. Hence $g \geq T_{x_0}$. This completes the proof. □

Proof of Proposition 3.4.1. Proof of (i),(ii) and (iii). Noting that

$$W_n(a,b) = W_n(a,\nu(a)) \cdot e^{n(\nu(a)-b)}$$

(i), (ii) and (iii) follow from (3.6).

Proof of (iv). For $a \geq \theta_0$, we have

$$\begin{aligned}
W_n(a,b) &= \sum_{|v|=n} e^{aS(v)-nb} \\
&\leq \left(\sum_{|v|=n} e^{\left(aS(v)-nb\right)\theta_0/a} \right)^{a/\theta_0} \\
&= W_n\left(\theta_0, b\theta_0/a\right)^{a/\theta_0} \\
&= \left(W_n\left(\theta_0, \nu(\theta_0)\right) \cdot e^{n\left(\nu(\theta_0)-b\theta_0/a\right)} \right)^{a/\theta_0}
\end{aligned}$$

Since $W_n(a,b)$ is non-negative, using (3.6), we get that for $a \geq \theta_0$ and $b\theta_0/a \geq \nu(\theta_0)$,

$$W_n(a,b) \to 0 \text{ a.s.}$$

Proof of (v). Using (i) and (iii), we know that there exists $\mathcal{N} \subset \Omega$ with $\mathbb{P}(\mathcal{N}) = 0$ such that for all $\omega \notin \mathcal{N}$ and $(a, b) \in [(0, \theta_0) \times \mathbb{R}] \cap \mathbb{Q}^2$,

$$W_n(a,b)(\omega) \to \begin{cases} 0 & \text{if } b > \nu(a); \\ \infty & \text{if } b < \nu(a). \end{cases}$$

For any $\omega \notin \mathcal{N}$ and any subsequence $\{n_k\}$, we define

$$\mathbb{S}\left(\{n_k\}, \omega\right) = \left\{ (c,d) : \limsup_{k\to\infty} W_{n_k}(c,d)(\omega) < \infty \right\}.$$

Now, suppose $(c_1, d_1), (c_2, d_2) \in \mathbb{S}\left(\{n_k\}, \omega\right)$. Then for any $\alpha \in (0, 1)$,

$$\begin{aligned}
&\limsup_{k\to\infty} W_{n_k}\left(\alpha c_1 + (1-\alpha)c_2, \alpha d_1 + (1-\alpha) d_2\right)(\omega) \\
&= \limsup_{k\to\infty} \sum_{|v|=n_k} \exp\Big(\alpha\left[c_1 S(v)(\omega) - n_k d_1\right] + (1-\alpha)\left[c_2 S(v)(\omega) - n_k d_2\right]\Big) \\
&\le \alpha \left[\limsup_{k\to\infty} \sum_{|v|=n_k} \exp\left(c_1 S(v)(\omega) - n_k d_1\right)\right] \\
&\qquad + (1-\alpha)\left[\limsup_{k\to\infty} \sum_{|v|=n_k} \exp\left(c_2 S(v)(\omega) - n_k d_2\right)\right] \\
&= \alpha\left[\limsup_{k\to\infty} W_{n_k}(c_1,d_1)(\omega)\right] + (1-\alpha)\left[\limsup_{k\to\infty} W_{n_k}(c_2,d_2)(\omega)\right] < \infty.
\end{aligned}$$

Therefore $\mathbb{S}\left(\{n_k\}, \omega\right)$ is convex. As $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, the conditions in Lemma 3.4.1 hold for the convex function ν, the convex set $\mathbb{S}\left(\{n_k\}, \omega\right)$, and the point θ_0. Thus for any $a \ge \theta_0$ and any $b < a\nu(\theta_0)/\theta_0$, we have $(a, b) \notin \mathbb{S}\left(\{n_k\}, \omega\right)$, which implies

$$\limsup_{k\to\infty} W_{n_k}(a,b)(\omega) = \infty.$$

This holds for all subsequences $\{n_k\}$ and all $\omega \notin \mathcal{N}$. Hence for all $a \ge \theta_0$ and all $b < a\nu(\theta_0)/\theta_0$, we have

$$W_n(a,b) \to \infty \text{ a.s.}$$

□

The following corollary is a simple consequence of Proposition 3.4.1.

Corollary 3.4.1. *Almost surely*

$$\frac{\log W_n(\theta)}{n\theta} \to \begin{cases} \frac{\nu(\theta)}{\theta} & \text{if } \theta < \theta_0 \leq \infty; \\ \frac{\nu(\theta_0)}{\theta_0} & \text{if } \theta_0 \leq \theta < \infty. \end{cases}$$

Proof. From the definition of W_n in (3.5), it follows that

$$\log W_n(\theta, b) = \log W_n(\theta) - nb. \tag{3.8}$$

Therefore, by applying Proposition 3.4.1, we get that for $\theta < \theta_0 \leq \infty$, almost surely

$$\log W_n(\theta) - nb \to \begin{cases} -\infty & \text{if } b > \nu(\theta); \\ \infty & \text{if } b < \nu(\theta); \end{cases} \tag{3.9}$$

and similarly for $\theta_0 \leq \theta < \infty$, almost surely

$$\log W_n(\theta) - nb \to \begin{cases} -\infty & \text{if } b > \theta\nu(\theta_0)/\theta_0; \\ \infty & \text{if } b > \theta\nu(\theta_0)/\theta_0. \end{cases} \tag{3.10}$$

Combining (3.9) and (3.10) proves the corollary. □

Remark 3.4.1: To understand why the limit in Corollary 3.4.1 becomes constant for $\theta \geq \theta_0$, let us consider

$$\mathfrak{F}(\theta) = \lim_{n\to\infty} \frac{\log W_n(\theta)}{n\theta}.$$

Since $\left[W_n(\theta)\right]^{1/\theta}$ is non-increasing in θ, so is $\mathfrak{F}(\theta)$. Now by the Cauchy–Schwarz inequality, we get that for any $\theta_1, \theta_2 > 0$,

$$\left(W_n(\theta_1 + \theta_2)\right)^2 \leq W_n(2\theta_1) \cdot W_n(2\theta_2).$$

Since dyadic rational numbers are dense in the real numbers, this gives us that

$$W_n(\alpha\theta_1 + (1-\alpha)\theta_2) \leq W_n(\theta_1)^{\alpha} \cdot W_n(\theta_2)^{1-\alpha},$$

which means that $\log W_n(\theta)$ is convex in θ, and therefore so is $\theta\mathfrak{F}(\theta)$. Now, for $\theta < \theta_0$, $\mathfrak{F}(\theta) = \nu(\theta)/\theta$. So by Remark 3.3.1, the left-derivative of $\mathfrak{F}$ is 0 at θ_0. Hence the right-derivative is greater than or equal to 0 at θ_0, by convexity of the function $\theta \mapsto \theta\mathfrak{F}(\theta)$. Using again this convexity, it is now easy to show that $\mathfrak{F}'(\theta) \geq 0$ for all $\theta \geq \theta_0$, hence $\mathfrak{F}(\theta) \geq \mathfrak{F}(\theta_0)$ for all $\theta \geq \theta_0$. But since $\mathfrak{F}$ is non-increasing, it has to be constant for $\theta \geq \theta_0$.

Our next result gives a relation between $W_n(\theta)$ and $Y_n^\mu(\theta)$, where $Y_n^\mu(\theta)$ is as in Theorem 2.3.2.

Proposition 3.4.2. *For any $\theta \in (0, \theta_0)$ and also for $\theta = \theta_0 < \infty$,*

$$\frac{Y_n^\mu(\theta)}{W_n(\theta)} \xrightarrow{p} \langle \mu \rangle,$$

where $\langle \mu \rangle$ is the mean of μ.

Proof. Before proving this proposition, we first quote a result of Biggins and Kyprianou [10], which is a particular case of Lemma 2.2 in Kurtz [26].

Lemma 3.4.2. *Suppose $\{c_i\}$ is a sequence of nonnegative constants satisfying $\sum_i c_i = 1$, with $a = \max_i c_i$. Suppose $\{Y_i\}$ are independent identically distributed copies of a random variable Y with $\mathbb{E}[|Y|] < \infty$ and $\mathbb{E}[Y] = 0$. Then, for $\varepsilon < 1/2$,*

$$\mathbb{P}\left(\left|\sum_{i=1} c_i Y_i\right| > \varepsilon\right) \leq \frac{2}{\varepsilon^2}\left(\int_0^{1/a} at \cdot \mathbb{P}(|Y| > t)\, dt + \int_{1/a}^{\infty} \mathbb{P}(|Y| > t)\, dt\right).$$

Now, observe that

$$\frac{Y_n^\mu(\theta)}{W_n(\theta)} - \langle \mu \rangle = \sum_{|v|=n} \left(\frac{e^{\theta S(v)}}{\sum_{|u|=n} e^{\theta S(u)}}\right) \big(Y_v - \langle \mu \rangle\big). \tag{3.11}$$

We define

$$M_n(\theta) := \max_{|v|=n} \frac{e^{\theta S(v)}}{\sum_{|u|=n} e^{\theta S(u)}} = \frac{e^{\theta R_n}}{W_n(\theta)}.$$

For $\theta \in (0, \theta_0)$, we choose any $\theta_1 \in (\theta, \theta_0)$. Then we get

$$M_n(\theta) \le \frac{\left[W_n(\theta_1)\right]^{\theta/\theta_1}}{W_n(\theta)} \le \frac{\left[W_n(\theta_1, \nu(\theta_1))\right]^{\theta/\theta_1} \cdot e^{-n\theta\left(\frac{\nu(\theta)}{\theta} - \frac{\nu(\theta_1)}{\theta_1}\right)}}{W_n(\theta, \nu(\theta))} \tag{3.12}$$

Since ν is strictly convex, $\nu(\theta)/\theta$ is strictly decreasing for $\theta \in (0, \theta_0)$. Therefore using Proposition 3.4.1, we get

$$M_n(\theta) \to 0 \text{ a.s.} \tag{3.13}$$

For $\theta = \theta_0 < \infty$, by choosing $\theta_2 \in (\theta_0, \infty)$, we obtain

$$M_n(\theta_0) \le \frac{\left[W_n\left(\theta_2\right)\right]^{\theta_0/\theta_2}}{W_n(\theta_0)} = \frac{\left[n^{\theta_2/\theta_0} W_n\left(\theta_2, \theta_2\nu(\theta_0)/\theta_0\right)\right]^{\theta_0/\theta_2}}{nW_n(\theta_0, \nu(\theta_0))}. \tag{3.14}$$

Aïdékon and Shi [3] showed that when $\theta_0 < \infty$, under the assumptions in Section 3.2, there exists a positive random variable $D^\infty_{\theta_0}$ such that

$$\sqrt{n}\, W_n(\theta_0, \nu(\theta_0)) \xrightarrow{p} \left(\frac{2}{\pi\sigma^2}\right)^{1/2} D^\infty_{\theta_0}. \tag{3.15}$$

The details of the random variable $D^\infty_{\theta_0}$ and the constant σ^2 have been discussed in Remark 3.4.2. Hu and Shi [24] proved that if $\theta_0 < \infty$, then for any $\theta \in (\theta_0, \infty)$,

$$\frac{1}{\log n} \log W_n\left(\theta, \frac{\theta\nu(\theta_0)}{\theta_0}\right) \xrightarrow{p} -\frac{3\theta}{2\theta_0}, \tag{3.16}$$

under the assumptions in Section 3.2 .

By applying (3.15) and (3.16) to (3.14), we get that for $\theta_0 < \infty$,

$$M_n(\theta_0) \xrightarrow{p} 0. \tag{3.17}$$

Thus, by combining (3.13) and (3.17) we obtain that for any $\theta \in (0, \theta_0)$ and also for $\theta = \theta_0 < \infty$,

$$M_n(\theta) \xrightarrow{p} 0. \tag{3.18}$$

Now, let $\mathcal{F}$ be the σ-algebra generated by the BRW. By Lemma 3.4.2, for every $\varepsilon \in (0, 1/2)$, we have

$$\mathbb{P}\left(\left|\frac{Y_n^{\mu}(\theta)}{W_n(\theta)} - \langle\mu\rangle\right| > \varepsilon \,\middle|\, \mathcal{F}\right)$$
$$\leq \frac{2}{\varepsilon^2}\left(\int_0^{\frac{1}{M_n(\theta)}} M_n(\theta)t \cdot \mathbb{P}\left(\left|Y - \langle\mu\rangle\right| > t\right)\, dt + \int_{\frac{1}{M_n(\theta)}}^{\infty} \mathbb{P}\left(\left|Y - \langle\mu\rangle\right| > t\right)\, dt\right),$$

which by (3.18) and the dominated convergence theorem, converges to 0 in probability as $n \to \infty$. Then by taking expectation and using the dominated convergence theorem again, we get

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{Y_n^{\mu}(\theta)}{W_n(\theta)} - \langle\mu\rangle\right| > \varepsilon\right) = 0.$$

This completes the proof. □

Remark 3.4.2: The constant σ^2 in Equation 3.15 is defined as

$$\sigma^2 := \mathbb{E}\left[\sum_{|v|=1} \left(\theta_0 S(v) - \nu(\theta_0)\right)^2 e^{\theta_0 S(v) - \nu(\theta_0)}\right].$$

The random variable $D_{\theta_0}^{\infty}$ is the almost sure limit of a *derivative martingale* defined by

$$D_n := -\sum_{|v|=n} (\theta_0 S(v) - \nu(\theta_0)n) e^{\theta_0 S(v) - \nu(\theta_0)n}.$$

The same derivative martingale also appears in Biggins and Kyprianou [11]. $D_{\theta_0}^{\infty} > 0$ a.s. under the assumptions in Section 3.2 and is a solution to a *linear RDE* given by

$$\Delta \stackrel{d}{=} \sum_{|v|=1} e^{\theta_0 S(v) - \nu(\theta_0)} \Delta_v, \tag{3.19}$$

where Δ_v are i.i.d. and have the same distribution as that of Δ.

3.5 Convergence Results

We classify our model in three different classes depending on whether the (scale) parameter θ is *below*, *equal* or *above* the quantity θ_0. We term these as *below the boundary case*, *the boundary case*, and *above the boundary case*, respectively.

3.5.1 Almost Sure Asymptotic Limit

The following result is a *strong law of large number*-type result, which is similar to Theorem 1.4.3.

Theorem 3.5.1. *For every non-negatively supported probability $\mu \neq \delta_0$ that admits a finite mean, almost surely*

$$\frac{R_n^*(\theta,\mu)}{n} \to \begin{cases} \frac{\nu(\theta)}{\theta} & \text{if } \theta < \theta_0 \leq \infty; \\ \frac{\nu(\theta_0)}{\theta_0} & \text{if } \theta_0 \leq \theta < \infty. \end{cases} \tag{3.20}$$

Remark 3.5.1: Note that the almost sure limit remains same as $\frac{\nu(\theta_0)}{\theta_0}$ for the *boundary case* and also in *above the boundary case.*

Proof. ***(Upper bound).*** We denote $\beta = \min(\theta, \theta_0)$. Using Markov's inequality, we have that for every $\epsilon > 0$,

$$\mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} - \frac{\nu(\beta)}{\beta} > \epsilon\right) \leq e^{-n\left(\beta\epsilon+\nu(\beta)\right)/2} \cdot \mathbb{E}\left[e^{\beta R_n^*(\theta,\mu)/2}\right].$$

Now, using Theorem 2.3.2, we have

$$\mathbb{E}\left[e^{\beta R_n^*(\theta,\mu)/2}\right] = \mathbb{E}\left[\left(\sum_{|v|=n} e^{\theta S(v)} Y_v\right)^{\beta/(2\theta)}\right] \cdot \mathbb{E}\left[E^{-\beta/(2\theta)}\right],$$

where $E \sim$ Exponential (1). Using a subadditive inequality and then using Jensen's inequality, we get that

$$\mathbb{E}\left[e^{\beta R_n^*(\theta,\mu)/2}\right] \leq \mathbb{E}\left[\sqrt{\sum_{|v|=n} e^{\beta S(v)} Y_v^{\beta/\theta}}\right] \cdot \Gamma\left(1 - \frac{\beta}{2\theta}\right)$$

$$\leq \sqrt{\mathbb{E}\left[\sum_{|v|=n} e^{\beta S(v)} Y_v^{\beta/\theta}\right]} \cdot \Gamma\left(1 - \frac{\beta}{2\theta}\right) = \sqrt{e^{n\nu(\beta)} \cdot \langle\mu\rangle_{\beta/\theta}} \cdot \Gamma\left(1 - \frac{\beta}{2\theta}\right),$$

where $\langle\mu\rangle_{\beta/\theta}$ is the (β/θ)-th moment of μ. So for every $\epsilon > 0$, we have

$$\mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} - \frac{\nu(\beta)}{\beta} > \epsilon\right) \leq \sqrt{\langle\mu\rangle_{\beta/\theta}} \cdot \Gamma\left(\frac{1}{2}\right) \cdot e^{-n\beta\epsilon/2},$$

which implies

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} - \frac{\nu(\beta)}{\beta} > \epsilon\right) < \infty. \tag{3.21}$$

Since $\epsilon > 0$ is arbitrary, using the Borel–Cantelli lemma, we obtain for all $\theta > 0$, almost surely

$$\limsup_{n\to\infty} \frac{R_n^*(\theta,\mu)}{n} \leq \begin{cases} \frac{\nu(\theta)}{\theta} & \text{if } \theta < \theta_0 \leq \infty; \\ \frac{\nu(\theta_0)}{\theta_0} & \text{if } \theta_0 \leq \theta < \infty. \end{cases} \tag{3.22}$$

(Lower bound). For u such that $|u| = m \leq n$, we define

$$R_{n-m}^{*(u)} \equiv R_{n-m}^{*(u)}(\theta,\mu) := \max_{|v|=n, u<v} \left\{S(v) + \frac{1}{\theta}\log(Y_v/E_v)\right\} - S(u). \tag{3.23}$$

Here $u < v$ means v is a descendant of u. Note that $\{R_{n-m}^{*(u)}\}_{|u|=m}$ are i.i.d. copies of R_{n-m}^* and are independent of the BRW up to generation m. Now, the definition of R_n^* in (1.1) implies that

$$\begin{aligned} R_n^*(\theta,\mu) &= \max_{|u|=m}\left\{\max_{|v|=n,u<v}\left\{S(v) + \frac{1}{\theta}\log(Y_v/E_v)\right\}\right\} \\ &= \max_{|u|=m}\left\{S(u) + R_{n-m}^{*(u)}(\theta,\mu)\right\} \\ &\geq S(\tilde{u}_m) + \max_{|u|=m} R_{n-m}^{*(u)}(\theta,\mu), \end{aligned} \tag{3.24}$$

where $\tilde{u}_m \equiv \tilde{u}_m(\theta, \mu) := \arg\max_{|u|=m} R^{*(u)}_{n-m}(\theta, \mu)$.

The boundary and below the boundary case ($\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$). For any $\epsilon \in (0,1)$ and for $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$, using (3.24) together with Markov's inequality, we get

$$
\begin{aligned}
&\mathbb{P}\left(\frac{R^*_n(\theta,\mu)}{n} - \frac{\nu(\theta)}{\theta} < -\epsilon\right) \\
&\leq \mathbb{P}\left(S(\tilde{u}_{\lfloor\sqrt{n}\rfloor}) + \max_{|u|=\lfloor\sqrt{n}\rfloor} R^{*(u)}_{n-\lfloor\sqrt{n}\rfloor}(\theta,\mu) < n\left(\frac{\nu(\theta)}{\theta} - \epsilon\right)\right) \\
&\leq \mathbb{P}\left(\max_{|u|=\lfloor\sqrt{n}\rfloor} R^{*(u)}_{n-\lfloor\sqrt{n}\rfloor}(\theta,\mu) < n\left(\frac{\nu(\theta)}{\theta} - \frac{\epsilon}{2}\right)\right) + \mathbb{P}\left(S(\tilde{u}_{\lfloor\sqrt{n}\rfloor}) < -\frac{n\epsilon}{2}\right) \\
&\leq \mathbb{E}\left[\mathbb{P}\left(R^*_{n-\lfloor\sqrt{n}\rfloor}(\theta,\mu) < n\left(\frac{\nu(\theta)}{\theta} - \frac{\epsilon}{2}\right)\right)^{N_{\lfloor\sqrt{n}\rfloor}}\right] + e^{-n\epsilon\vartheta/4}\cdot\mathbb{E}\left[e^{-\vartheta S(\tilde{u}_{\lfloor\sqrt{n}\rfloor})/2}\right]. \quad (3.25)
\end{aligned}
$$

Here $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x, N_k represents the total number of particles at generation k, and ϑ is as in assumption (**A1**). The combination of Theorem 2.3.2, Proposition 3.4.2, and Corollary 3.4.1 gives us that for any $\theta < \theta_0 \leq \infty$ and also for $\theta = \theta_0 < \infty$,

$$
\frac{R^*_n(\theta,\mu)}{n} \xrightarrow{p} \frac{\nu(\theta)}{\theta}. \quad (3.26)
$$

Therefore for all large enough n,

$$
\mathbb{P}\left(R^*_{n-[\sqrt{n}]}(\theta,\mu) < n\left(\frac{\nu(\theta)}{\theta} - \frac{\epsilon}{2}\right)\right) < \epsilon. \quad (3.27)
$$

Since $\mathbb{P}(N=0)=0$, $N_{\lfloor\sqrt{n}\rfloor} < n$ implies that at least $\lfloor\sqrt{n}\rfloor - \lceil\log_2 n\rceil$ many particles have given birth to a single offspring, and therefore

$$
\mathbb{P}\left(N_{[\sqrt{n}]} < n\right) \leq \left(\mathbb{P}(N=1)\right)^{[\sqrt{n}]-\lceil\log_2 n\rceil}. \quad (3.28)
$$

For the second term on the right-hand side of (3.25), we have

$$
\mathbb{E}\left[e^{-\vartheta S(\tilde{u}_{[\sqrt{n}]})/2}\right] \leq \mathbb{E}\left[W_{[\sqrt{n}]}(-\vartheta/2)\right] = e^{[\sqrt{n}]\nu(-\vartheta/2)}. \quad (3.29)
$$

By combining (3.25), (3.27), (3.28), and (3.29), we have for all large enough n,

$$\mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} - \frac{\nu(\theta)}{\theta} < -\epsilon\right)$$
$$\leq \epsilon^n + \big(\mathbb{P}(N=1)\big)^{[\sqrt{n}]-\lceil \log_2 n\rceil} + e^{-n\epsilon\vartheta/4+[\sqrt{n}]\nu(-\vartheta/2)},$$

which implies for all $\epsilon \in (0,1)$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} - \frac{\nu(\theta)}{\theta} < -\epsilon\right) < \infty. \tag{3.30}$$

So by using the Borel–Cantelli lemma, we obtain that for $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$,

$$\liminf_{n\to\infty} \frac{R_n^*(\theta,\mu)}{n} \geq \frac{\nu(\theta)}{\theta} \text{ a.s.} \tag{3.31}$$

Above the boundary case ($\theta_0 < \theta < \infty$). To get the lower bound for $\theta_0 < \theta < \infty$, we need the following result, the proof of this is given at the end of this proof.

Proposition 3.5.1. *For every non-negatively supported probability $\mu \neq \delta_0$ that admits a finite mean, almost surely*

$$\frac{\log Y_n^\mu(\theta)}{n\theta} \to \begin{cases} \frac{\nu(\theta)}{\theta} & \text{if } \theta < \theta_0 \leq \infty; \\ \frac{\nu(\theta_0)}{\theta_0} & \text{if } \theta_0 \leq \theta < \infty. \end{cases}$$

Now observe that,

$$\theta R_n^*(\theta,\mu) = \max_{|v|=n} \big\{\theta S(v) + \log Y_v - \log E_v\big\}$$
$$\geq \max_{|v|=n} \big\{\theta S(v) + \log Y_v\big\} - \log E_{v_n}, \tag{3.32}$$

where $v_n \equiv v_n(\theta,\mu) := \arg\max_{|v|=n} \big\{\theta S(v) + \log Y_v\big\}$. Also, observe

$$Y_n^\mu(\theta+\theta_0) = \sum_{|v|=n} e^{(\theta+\theta_0)S(v)} Y_v \leq W_n(\theta_0) \cdot e^{\max_{|v|=n}\{\theta S(v)+\log Y_v\}}. \tag{3.33}$$

Therefore we have

$$\frac{\theta R_n^*(\theta,\mu)}{n} \geq \frac{\log Y_n^\mu(\theta+\theta_0)}{n} - \frac{\log W_n(\theta_0)}{n} - \frac{\log E_{v_n}}{n}. \tag{3.34}$$

Since $\mathbb{E}\left[|\log E_{v_n}|\right]$ is finite, the Borel–Cantelli lemma implies that the last terms on the right-hand side of (3.34) converges to 0 almost surely. By Corollary 3.4.1 and Proposition 3.5.1, the first and the second terms almost surely converges to $(\theta+\theta_0)\nu(\theta_0)/\theta_0$ and $\nu(\theta_0)$, respectively. Thus, we obtain that for $\theta_0 < \theta < \infty$,

$$\liminf_{n\to\infty} \frac{R_n^*(\theta,\mu)}{n} \geq \frac{\nu(\theta_0)}{\theta_0} \text{ a.s.} \tag{3.35}$$

Combining (3.22), (3.31), and (3.35) completes the proof. □

Proof of Proposition 3.5.1. Notice that Proposition 3.5.1 is only required to prove the lower bound of Theorem 3.5.1 above the boundary case. We can therefore use the results proved in the proof of the remaining cases to prove this proposition.

Now, Theorem 2.3.2 says that

$$\theta R_n^*(\theta,\mu) \stackrel{d}{=} \log Y_n^\mu(\theta) - \log E.$$

Since $\mathbb{E}\left[|\log E|\right] < \infty$, (3.21) and (3.30), together with the Borel-Cantelli lemma, imply that for $\theta < \theta_0 \leq \infty$ and also for $\theta = \theta_0 < \infty$,

$$\frac{\log Y_n^\mu(\theta)}{n\theta} \to \frac{\nu(\theta)}{\theta} \text{ a.s.,} \tag{3.36}$$

and for $\theta_0 < \theta < \infty$,

$$\limsup_{n\to\infty} \frac{\log Y_n^\mu(\theta)}{n\theta} \leq \frac{\nu(\theta_0)}{\theta_0} \text{ a.s.} \tag{3.37}$$

So for any $a > 0$ and $b \in \mathbb{R}$, we have almost surely

$$Y_n^\mu(a) \cdot e^{-nb} \to \begin{cases} 0 & \text{if } a < \theta_0,\ b > \nu(a); \\ \infty & \text{if } a < \theta_0,\ b < \nu(a); \\ 0 & \text{if } \theta_0 < \infty,\ a \geq \theta_0,\ b > a\nu(\theta_0)/\theta_0. \end{cases} \tag{3.38}$$

Now, the exact similar argument as in the proof of Proposition 3.4.1 (v) suggests that for $\theta_0 < \infty$, $a \geq \theta_0$ and $b < a\nu(\theta_0)/\theta_0$,

$$Y_n^{\mu}(a) \cdot e^{-nb} \to \infty \text{ a.s.} \tag{3.39}$$

This, together with (3.38), implies that for $\theta_0 < \theta < \infty$,

$$\frac{\log Y_n^{\mu}(\theta)}{n\theta} \to \frac{\nu(\theta_0)}{\theta_0} \text{ a.s.} \tag{3.40}$$

Therefore, combining (3.36) and (3.40) proves the proposition. □

It is worth mentioning that Corollary 3.4.1 gives an alternative proof of the following well-known result of Biggins [9].

Theorem 3.5.2. *Almost surely*

$$\frac{R_n}{n} \to \begin{cases} \nu'(\theta_0) & \text{if } \theta_0 < \infty; \\ \lim_{\theta\to\infty} \nu'(\theta) & \text{if } \theta_0 = \infty. \end{cases}$$

An alternative proof. From the definition, it follows that

$$W_n(\theta) = \sum_{|v|=n} e^{\theta S(v)} \geq e^{\theta R_n}.$$

Also,

$$\begin{aligned} W_n(2\theta) = \sum_{|v|=n} e^{2\theta S(v)} &\leq \sum_{|v|=n} e^{\theta R_n + \theta S(v)} \\ &= e^{\theta R_n} \left(\sum_{|v|=n} e^{\theta S(v)} \right) = e^{\theta R_n} \cdot W_n(\theta). \end{aligned}$$

Therefore we have for any $\theta > 0$,

$$\frac{W_n(2\theta)}{W_n(\theta)} \leq e^{\theta R_n} \leq W_n(\theta). \tag{3.41}$$

This implies

$$\frac{\log W_n(2\theta)}{n\theta} - \frac{\log W_n(\theta)}{n\theta} \leq \frac{R_n}{n} \leq \frac{\log W_n(\theta)}{n\theta}. \tag{3.42}$$

If $\theta_0 < \infty$, then for any $\theta \in (\theta_0, \infty)$, letting $n \to \infty$ and using Corollary 3.4.1, we get

$$2\left(\frac{\nu(\theta_0)}{\theta_0}\right) - \frac{\nu(\theta_0)}{\theta_0} \le \liminf_{n\to\infty} \frac{R_n}{n} \le \limsup_{n\to\infty} \frac{R_n}{n} \le \frac{\nu(\theta_0)}{\theta_0} \qquad \text{a.s.},$$

which implies almost surely

$$\frac{R_n}{n} \to \frac{\nu(\theta_0)}{\theta_0} = \nu'(\theta_0). \tag{3.43}$$

Now, suppose $\theta_0 = \infty$. By Corollary 3.4.1, letting $n \to \infty$ and then letting $\theta \to \infty$, we obtain

$$\lim_{\theta\to\infty} \frac{\nu(2\theta)}{\theta} - \frac{\nu(\theta)}{\theta} \le \liminf_{n\to\infty} \frac{R_n}{n} \le \limsup_{n\to\infty} \frac{R_n}{n} \le \lim_{\theta\to\infty} \frac{\nu(\theta)}{\theta} \qquad \text{a.s.}$$

Since ν is convex, we know that

$$\lim_{\theta\to\infty} \frac{\nu(\theta)}{\theta} = \lim_{\theta\to\infty} \nu'(\theta), \qquad \text{and also} \qquad \lim_{\theta\to\infty} \frac{\nu(2\theta) - \nu(\theta)}{\theta} = \lim_{\theta\to\infty} \nu'(\theta).$$

Therefore, almost surely

$$\frac{R_n}{n} \to \lim_{\theta\to\infty} \nu'(\theta). \tag{3.44}$$

Combining (3.43) and (3.44) completes the proof. □

Remark 3.5.2: As pointed out by an anonymous referee, this alternative proof has a direct relation with the proof by Kingman [25], where Kingman used the fact that $\mathfrak{F}'(\theta_0) = 0$, where $\mathfrak{F}$ is as in Remark 3.4.1. As explained in Remark 3.4.1, Corollary 3.4.1 relies on the fact that $\mathfrak{F}'(\theta_0) = 0$, which we have also used in the alternative proof.

3.5.2 Centered Asymptotic Limits

The centered asymptotic limits vary in the three different cases depending on the value of the parameter θ as described above. We thus state the results separately for the three cases.

The Boundary case $(\theta = \theta_0 < \infty)$

Theorem 3.5.3. *Assume that μ admits a finite mean, then there exists a random variable $H^{\infty}_{\theta_0}$, which may depend on θ_0, such that,*

$$R^*_n - \frac{\nu(\theta_0)}{\theta_0} n + \frac{1}{2\theta_0} \log n \xrightarrow{d} H^{\infty}_{\theta_0} + \frac{1}{\theta_0} \log\langle\mu\rangle. \tag{3.45}$$

Remark 3.5.3: Notice that the coefficient for the linear term is exactly the same as that of the centering of R_n as proved by Aïdékon [2]. However, the coefficient for the logarithmic term is 1/3-rd of that of the centering of R_n as proved by Aïdékon [2]. The limiting distribution is also similar to that obtained by Aïdékon [2], which is a randomly shifted *Gumbel distribution*.

Proof. As mentioned in (3.15), for $\theta_0 < \infty$, Aïdékon and Shi [3] have shown that under the assumptions in Section 3.2,

$$\sqrt{n}\, W_n(\theta_0) \cdot e^{-n\nu(\theta_0)} = \sqrt{n}\, W_n(\theta_0, \nu(\theta_0)) \xrightarrow{p} \left(\frac{2}{\pi\sigma^2}\right)^{1/2} D^{\infty}_{\theta_0}.$$

This, together with Proposition 3.4.2, implies that

$$\sqrt{n}\, Y^{\mu}_n(\theta_0) \cdot e^{-n\nu(\theta_0)} \xrightarrow{p} \left(\frac{2}{\pi\sigma^2}\right)^{1/2} \cdot D^{\infty}_{\theta_0} \cdot \langle\mu\rangle. \tag{3.46}$$

Therefore, taking the logarithm of both sides and then applying Theorem 2.3.2, we obtain the result in (3.45) with

$$H^{\infty}_{\theta_0} = \frac{1}{\theta_0}\left[\log D^{\infty}_{\theta_0} + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right) - \log E\right], \tag{3.47}$$

where $E \sim \text{Exponential}\,(1)$ and is independent of the BRW. □

As we have seen in the proof of the above theorem, we have a slightly stronger result, which is as follows:

Theorem 3.5.4. *Assume that μ admits a finite mean. Let*

$$\hat{H}_{\theta_0}^{\infty} = \frac{1}{\theta_0}\left[\log D_{\theta_0}^{\infty} + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)\right], \tag{3.48}$$

where

$$D_{\theta_0}^{\infty} \stackrel{a.s.}{=\!=} \lim_{n\to\infty} -\frac{1}{m(\theta_0)^n}\sum_{|v|=n}\left(\theta_0 S(v) - n\nu(\theta_0)\right)e^{\theta_0 S(v)}, \tag{3.49}$$

$$\sigma^2 := \mathbb{E}\left[\frac{1}{m(\theta_0)}\sum_{|v|=1}\left(\theta_0 S(v) - \nu(\theta_0)\right)^2 e^{\theta_0 S(v)}\right]. \tag{3.50}$$

Then

$$R_n^* - \frac{\nu(\theta_0)}{\theta_0}n + \frac{1}{2\theta_0}\log n - \hat{H}_{\theta_0}^{\infty} \xrightarrow{d} \frac{1}{\theta_0}\left[\log\langle\mu\rangle - \log E\right], \tag{3.51}$$

where $E \sim$ Exponential(1).

Remark 3.5.4: We note here that the $H_{\theta_0}^{\infty}$ in Theorem 3.5.3 has the same distribution as $\hat{H}_{\theta_0}^{\infty} - \frac{1}{\theta_0}\log E$, where $E \sim$ Exponential(1) and is independent of $\hat{H}_{\theta_0}^{\infty}$.

Remark 3.5.5: One advantage of the above result is that we have been able to identify the exact constant for the limit, which turns out to be $\frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)$, where σ^2 is given in the equation (3.50). As far as we know, this was not discovered in any of the earlier works.

Proof. Observe that for any $\theta > 0$,

$$\begin{aligned}\theta R_n^*(\theta,\mu) - \log Y_n^{\mu}(\theta) &= \max_{|v|=n}\left(\theta S(v) + \log Y_v - \log E_v\right) - \log\left(\sum_{|u|=n} e^{\theta S(u)} Y_u\right) \\ &= -\log\left(\min_{|v|=n} E_v\left(\frac{e^{\theta S(v)}Y_v}{\sum_{|u|=n} e^{\theta S(u)}Y_u}\right)^{-1}\right)\end{aligned} \tag{3.52}$$

Let $\mathcal{A}$ be the σ-algebra generated by the BRW and $\{Y_v\}_{|v|=n}$. So, given $\mathcal{A}$, the random variables

$$\left\{E_v\left(\frac{e^{\theta S(v)}Y_v}{\sum_{|u|=n} e^{\theta S(u)}Y_u}\right)^{-1}\right\}_{|v|=n}$$

are conditionally independent and

$$E_v\left(\frac{e^{\theta S(v)}Y_v}{\sum_{|u|=n} e^{\theta S(u)}Y_u}\right)^{-1}\Bigg|\, \mathcal{A} \sim \text{Exponential}\left(\frac{e^{\theta S(v)}Y_v}{\sum_{|u|=n} e^{\theta S(u)}Y_u}\right).$$

Therefore

$$\min_{|v|=n} E_v\left(\frac{e^{\theta S(v)}Y_v}{\sum_{|u|=n} e^{\theta S(u)}Y_u}\right)^{-1}\Bigg|\, \mathcal{A} \sim \text{Exponential}\,(1),$$

which is independent of $\mathcal{A}$. Hence, we obtain

$$\min_{|v|=n} E_v\left(\frac{e^{\theta S(v)}Y_v}{\sum_{|u|=n} e^{\theta S(u)}Y_u}\right)^{-1} = E, \tag{3.53}$$

where $E \sim \text{Exponential}\,(1)$ and is independent of $\mathcal{A}$. By applying the above equation to (3.52), we get

$$\theta R_n^*(\theta,\mu) - \log Y_n^\mu(\theta) = -\log E. \tag{3.54}$$

This, together with (3.46), gives us the required result. □

Below the Boundary case ($\theta < \theta_0 \leq \infty$)

Theorem 3.5.5. *Assume that μ admits finite mean, then for $\theta < \theta_0 \leq \infty$, there exists a random variable H_θ^∞, which may depend on θ, such that,*

$$R_n^* - \frac{\nu(\theta)}{\theta}n \xrightarrow{d} H_\theta^\infty + \frac{1}{\theta}\log\langle\mu\rangle. \tag{3.55}$$

Remark 3.5.6: We note that in this case the *logarithmic correction* disappears.

Proof. As mentioned in (3.6), we have for any $\theta < \theta_0 \leq \infty$,

$$W_n(\theta)\cdot e^{-n\nu(\theta)} = W_n(\theta,\nu(\theta)) \to D_\theta^\infty \text{ a.s.} \tag{3.56}$$

Together with Proposition 3.4.2, this suggests that

$$Y_n^\mu(\theta)\cdot e^{-n\nu(\theta)} \xrightarrow{p} D_\theta^\infty \cdot \langle\mu\rangle. \tag{3.57}$$

Thus, taking the logarithm of both sides and then applying Theorem 2.3.2, we get the result in (3.55) with

$$H_\theta^\infty = \frac{1}{\theta}\left[\log D_\theta^\infty - \log E\right], \tag{3.58}$$

where $E \sim \text{Exponential}\,(1)$ and is independent of the BRW. □

Once again, just like in the boundary case, here too, we have a slightly stronger result, which is as follows:

Theorem 3.5.6. *Assume that μ admits a finite mean. Let*

$$\hat{H}_\theta^\infty = \frac{1}{\theta}\log D_\theta^\infty, \tag{3.59}$$

where D_θ^∞ is the mean 1 solution of the following linear RDE

$$\Delta \stackrel{d}{=} \sum_{|v|=1} e^{\theta S(v) - \nu(\theta)}\Delta_v, \tag{3.60}$$

where Δ_v are i.i.d. and has the same distribution as that of Δ; and $E \sim$ Exponential$\,(1)$. Then

$$R_n^* - \frac{\nu(\theta)}{\theta}n - \hat{H}_\theta^\infty \xrightarrow{d} \frac{1}{\theta}\left[\log\langle\mu\rangle - \log E\right]. \tag{3.61}$$

Remark 3.5.7: It is to be noted that the random variable H_θ^∞ in Theorem 3.5.5 has the same distribution as $\hat{H}_\theta^\infty - \frac{1}{\theta}\log E$, where $E \sim \text{Exponential}\,(1)$ and is independent of $\hat{H}_\theta^\infty$.

Proof. Combining (3.54) and (3.57) completes the proof. □

Above the Boundary case ($\theta_0 < \theta < \infty$)

Theorem 3.5.7. *Suppose $\mu = \delta_1$,*

$$\frac{R_n^* - \frac{\nu(\theta_0)}{\theta_0}n}{\log n} \xrightarrow{p} -\frac{3}{2\theta_0}. \tag{3.62}$$

Remark 3.5.8: We would like to point out here that this is not the best result for this case. For technical reasons, which will be clear from the proof, we have only been able to prove it for $\mu = \delta_1$. Also, the result is unsatisfactory as it is not a centered limit as in the other two cases. However, we note that we now capture the right constant for the *logarithmic correction.*

Proof. Hu and Shi [24] have proved that under the assumptions in Section 3.2, for $\theta_0 < \theta < \infty$,

$$\frac{1}{\log n}\left(\log W_n(\theta) - \frac{\nu(\theta_0)}{\theta_0}\theta n\right) \xrightarrow{p} -\frac{3\theta}{2\theta_0}. \tag{3.63}$$

Since $W_n(\theta) = Y_n^{\delta_1}(\theta)$, using Theorem 2.3.2, we get the required result. □

3.6 A Specific Example

We consider the modified version of the *i.i.d. Gaussian displacement binary BRW*, where $N = 2$ with probability one, $Z = \delta_{\xi_1} + \delta_{\xi_2}$ with ξ_j's i.i.d. $\mathrm{N}(0,1)$, and $\mu = \delta_1$. In that case,

$$\nu(\theta) = \log \mathbb{E}\left[e^{\theta\xi_1} + e^{\theta\xi_2}\right] = \log\left(2e^{\frac{\theta^2}{2}}\right) = \log 2 + \frac{\theta^2}{2}.$$

Differentiating with respect to θ, we obtain

$$\nu'(\theta) = \theta.$$

From definition in (3.3), it follows that

$$\log 2 + \frac{\theta_0^2}{2} = \theta_0^2 \;\Rightarrow\; \theta_0 = \sqrt{2\log 2}$$

Therefore, in view of Theorem 3.5.3, there exists a random variable, say, G_∞ with a randomly shifted Gumbel distribution such that

$$R_n^*(\sqrt{2\log 2}, \delta_1) - \sqrt{2\log 2}\, n + \frac{1}{2\sqrt{2\log 2}} \log n \xrightarrow{d} G_\infty.$$

The corresponding result for R_n derived from Aïdékon's [2] work is as follows.

$$R_n - \sqrt{2\log 2}\, n + \frac{3}{2\sqrt{2\log 2}} \log n \xrightarrow{d} G'_\infty,$$

where G'_∞ is a randomly shifted Gumbel-distributed random variable.

Chapter 4

Brunet-Derrida Type Results [1]

4.1 Introduction

In this chapter, we present results of the type Brunet and Derrida [17] for convergence of the extremal point processes. Their conjecture for the classical BRW was proved by Madaule [29]. Here we present similar results for our LPM-BRW. It is to be noted that the convergence of the point processes mentioned here is under the vague convergence topology on the set of all counting measures on $\mathbb{R}$.

Throughout this chapter, we will work under the assumptions (**A1**), (**A2**) and (**A3**) as stated in Section 3.2.

4.2 Few Technical Results on Point Processes

In this section, we prove two technical facts, which we use to prove our main results.

Lemma 4.2.1. *Let* $\left\{E_{i,n} : 1 \le i \le m_n, n \ge 1\right\}$ *be an array of independent random variables with* $E_{i,n} \sim$ *Exponential*$(\lambda_{i,n})$. *Suppose for all* $n \ge 1$, $\sum_{i=1}^{m_n} \lambda_{i,n} = 1$, *and*

[1] This chapter is based on Sections 2.3, 5.5 and 5.6 of the paper entitled "*Right-most position of a last progeny modified branching random walk*" [7].

$\lim_{n\to\infty} \max_{i=1}^{m_n} \lambda_{i,n} = 0$. *Then as* $n \to \infty$, *the point process*

$$\sum_{i=1}^{m_n} \delta_{E_{i,n}} \xrightarrow{d} \mathcal{N},$$

where $\mathcal{N}$ *is a homogeneous Poisson point process on* $\mathbb{R}_+$ *with intensity* 1.

Proof. Since $\sum_{i=1}^{m_n} \lambda_{i,n} = 1$, $\lim_{n\to\infty} \max_{i=1}^{m_n} \lambda_{i,n} = 0$ means $\lim_{n\to\infty} m_n = \infty$. We fix any $k \in \mathbb{N}$. For $m_n > k$, we denote $E_{(1,n)}, E_{(2,n)}, \ldots, E_{(k,n)}$ as the first k order statistics of $\{E_{i,n} : 1 \le i \le m_n\}$. Then for $0 \le x_1 \le x_2 \le \cdots \le x_k$, the joint density function of $E_{(1,n)}, E_{(2,n)}, \ldots, E_{(k,n)}$ is given by

$$\begin{aligned}
&f_{E_{(1,n)},E_{(2,n)},\ldots,E_{(k,n)}}(x_1, x_2, \ldots, x_k) \\
&= \sum_{\substack{(j_1,j_2,\ldots,j_k) \\ \in \mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n} e^{-\lambda_{j_i,n} x_i} \right) \cdot \left(\prod_{\substack{l=1 \\ l \ne j_1,j_2,\ldots,j_k}}^{n} e^{-\lambda_{l,n} x_k} \right) \\
&= \sum_{\substack{(j_1,j_2,\ldots,j_k) \\ \in \mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n} \right) \cdot \exp\left(-\sum_{i=1}^{k} \lambda_{j_i,n} x_i - \sum_{\substack{l=1 \\ l \ne j_1,j_2,\ldots,j_k}}^{n} \lambda_{l,n} x_k \right), \qquad (4.1)
\end{aligned}$$

where

$$\mathcal{S}(k,n) = \left\{ (j_1, j_2, \ldots, j_k) \in \{1, 2, \ldots, m_n\}^k : j_r \ne j_t \text{ for all } 1 \le r < t \le k \right\}.$$

Now, we use the following transformation.

$$Y_{1,n} = E_{(1,n)}, \quad Y_{2,n} = E_{(2,n)} - E_{(1,n)}, \quad \ldots, \quad Y_{k,n} = E_{(k,n)} - E_{(k-1,n)}.$$

The Jacobian of the above transformation is 1. Therefore for any non-negative real numbers $y_1, y_2, \ldots, y_k$, the joint density function of $Y_{1,n}, Y_{2,n}, \ldots Y_{k,n}$ is given by

$$f_{Y_{1,n},Y_{2,n},\ldots Y_{k,n}}(y_1, y_2, \ldots, y_k)$$

$$= \sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) \cdot \exp\left(-\sum_{i=1}^{k} \lambda_{j_i,n}\left(\sum_{r=1}^{i} y_r\right) - \sum_{\substack{l=1\\ l\neq j_1,j_2,\ldots,j_k}}^{n} \lambda_{l,n}\left(\sum_{r=1}^{k} y_r\right)\right)$$

$$= \exp\left(-\sum_{r=1}^{k} y_r\right) \cdot \left[\sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) \cdot \exp\left(\sum_{r=2}^{k}\left(\sum_{i=1}^{r-1} \lambda_{j_i,n}\right) y_r\right)\right]. \tag{4.2}$$

Now, observe that

$$\sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) \leq \left(\sum_{i=1}^{m_n} \lambda_{i,n}\right)^k$$

$$\leq \sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) + \binom{k}{2}\left(\sum_{i=1}^{m_n} \lambda_{i,n}^2\right)\left(\sum_{i=1}^{m_n} \lambda_{i,n}\right)^{k-2}.$$

Note that $\sum_{i=1}^{m_n} \lambda_{i,n} = 1$. So, if we write $\lambda_n^* = \max_{i=1}^{m_n} \lambda_{i,n}$, we obtain

$$\sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) \leq 1 \leq \sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) + \binom{k}{2}\lambda_n^*.$$

Therefore we have

$$\lim_{n\to\infty} \sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) = 1. \tag{4.3}$$

also, observe that

$$\sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) \leq \sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) \cdot \exp\left(\sum_{r=2}^{k}\left(\sum_{i=1}^{r-1} \lambda_{j_i,n}\right) y_r\right)$$

$$\leq \sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) \cdot \exp\left(\sum_{r=2}^{k} (r-1)\, y_r \lambda_n^*\right).$$

Thus letting $n \to \infty$ and then applying (4.3), we get

$$\lim_{n\to\infty} \sum_{\substack{(j_1,j_2,\ldots,j_k)\\ \in\mathcal{S}(k,n)}} \left(\prod_{i=1}^{k} \lambda_{j_i,n}\right) \cdot \exp\left(\sum_{r=2}^{k}\left(\sum_{i=1}^{r-1}\lambda_{j_i,n}\right) y_r\right) = 1 \tag{4.4}$$

Applying this to (4.2), gives us

$$\lim_{n\to\infty} f_{Y_{1,n},Y_{2,n},\ldots Y_{k,n}}(y_1, y_2, \ldots, y_k) = \exp\left(-\sum_{r=1}^{k} y_r\right). \tag{4.5}$$

If we write the k-th order statistic of the homogeneous Poisson point process $\mathcal{N}$ as $Q_{(k)}$, then the joint density function of $Q_{(1)}, Q_{(2)} - Q_{(1)}, \ldots, Q_{(k)} - Q_{(k-1)}$ is given by

$$f_{Q_{(1)},Q_{(2)}-Q_{(1)},\ldots,Q_{(k)}-Q_{(k-1)}}(y_1, y_2, \ldots, y_k) = \exp\left(-\sum_{r=1}^{k} y_r\right).$$

Therefore by Scheffe's Theorem, we get

$$\left(Y_{1,n}, Y_{2,n}, \ldots Y_{k,n}\right) \xrightarrow{d} \left(Q_{(1)}, Q_{(2)} - Q_{(1)}, \ldots, Q_{(k)} - Q_{(k-1)}\right).$$

Equivalently, for any $k \in \mathbb{N}$,

$$\left(E_{(1,n)}, E_{(2,n)}, \ldots, E_{(k,n)}\right) \xrightarrow{d} \left(Q_{(1)}, Q_{(2)}, \ldots, Q_{(k)}\right). \tag{4.6}$$

Now, take any continuous function f that vanishes outside a bounded set, say, $[0, M]$. We fix any $\epsilon > 0$, and choose k_0 large enough so that $\mathbb{P}\left(\mathcal{N}\left([0, M]\right) \geq k_0\right) < \epsilon$. Observe that for any $x \in \mathbb{R}$,

$$\left|\mathbb{P}\left(\int_0^\infty f\, d\mathcal{N} \leq x\right) - \mathbb{P}\left(\sum_{i=1}^{k_0} f\left(Q_{(i)}\right) \leq x\right)\right| \leq \mathbb{P}\left(\mathcal{N}\left([0, M]\right) \geq k_0\right) < \epsilon. \tag{4.7}$$

By (4.6), we obtain

$$\limsup_{n\to\infty}\left|\mathbb{P}\left(\int_0^\infty f\,d\left(\sum_{i=1}^{m_n}\delta_{E_{i,n}}\right)\le x\right)-\mathbb{P}\left(\sum_{i=1}^{k_0} f\left(E_{(i,n)}\right)\le x\right)\right|$$

$$\le \lim_{n\to\infty}\mathbb{P}\left(E_{(k_0,n)}\le M\right)=\mathbb{P}\left(Q_{(k_0)}\le M\right)<\epsilon. \tag{4.8}$$

In view of (4.6), we also get that

$$\lim_{n\to\infty}\mathbb{P}\left(\sum_{i=1}^{k_0} f\left(E_{(i,n)}\right)\le x\right)=\mathbb{P}\left(\sum_{i=1}^{k_0} f\left(Q_{(i)}\right)\le x\right). \tag{4.9}$$

Thus, by combining (4.7), (4.8), and (4.9), and letting $\epsilon\to 0$, we finally obtain

$$\int_0^\infty f\,d\left(\sum_{i=1}^{m_n}\delta_{E_{i,n}}\right)\xrightarrow{d}\int_0^\infty f\,d\mathcal{N}. \tag{4.10}$$

This, together with Proposition 11.1.VII of Daley and Vere-Jones [18], gives

$$\sum_{i=1}^{m_n}\delta_{E_{i,n}}\xrightarrow{d}\mathcal{N}.$$

This completes the proof. □

Remark 4.2.1: An alternative derivation of Lemma 4.2.1 can be obtained using Rényi's representation [31] and the generalized version of it by Tikhov (see equation (3) of Tikhov [33]). The details of such a derivation are similar to the proof presented here. For the sake of completeness, we have provided our own proof.

Lemma 4.2.2. *If $\mathcal{N}=\sum_{j\ge 1}\delta_{\mathfrak{z}_j}$ is a homogeneous Poisson point process on $\mathbb{R}_+$ with intensity 1, then $\mathcal{Y}=\sum_{j\ge 1}\delta_{-\log\mathfrak{z}_j}$ is an inhomogeneous Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}\,dx$.*

Proof. For any Borel set $B\subset\mathbb{R}$, we have that

$$\mathcal{Y}(B)=\mathcal{N}(\{e^{-x}: x\in B\})\sim \text{Poisson}(\Lambda_B), \tag{4.11}$$

where

$$\Lambda_B = \int_{\{e^{-x}:x\in B\}} dx = \int_B e^{-x}\, dx.$$

Also, note that for any disjoint Borel sets $B_1, B_2, \ldots, B_k$, the random variables $\mathcal{N}(\{e^{-x} : x \in B_1\}), \mathcal{N}(\{e^{-x} : x \in B_2\}), \ldots, \mathcal{N}(\{e^{-x} : x \in B_k\})$ are independent, which means $\mathcal{Y}(B_1), \mathcal{Y}(B_2), \ldots, \mathcal{Y}(B_k)$ are also independent. This, together with (4.11), completes the proof. □

4.3 Convergence Results

For any $\theta < \theta_0 \le \infty$, we define

$$Z_n(\theta) := \sum_{|v|=n} \delta_{\left\{\theta S(v) - \log E_v - n\nu(\theta) - \log D_\theta^\infty\right\}}, \tag{4.12}$$

where D_θ^∞ is defined in the Theorem 3.5.6. Also for $\theta = \theta_0 < \infty$, we define

$$Z_n(\theta_0) := \sum_{|v|=n} \delta_{\left\{\theta_0 S(v) - \log E_v - n\nu(\theta_0) + \frac{1}{2}\log n - \log D_{\theta_0}^\infty - \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)\right\}}, \tag{4.13}$$

where $D_{\theta_0}^\infty$ and σ^2 are as in Theorem 3.5.4.

Our first result is the weak convergence of the point processes $\left(Z_n(\theta)\right)_{n\ge 0}$.

Theorem 4.3.1. *For $\theta < \theta_0 \le \infty$ or $\theta = \theta_0 < \infty$,*

$$Z_n(\theta) \xrightarrow{d} \mathcal{Y},$$

where $\mathcal{Y}$ is a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}\, dx$.

Proof. Let $\mathcal{F}$ be the σ-algebra generated by the BRW. We know that conditioned on $\mathcal{F}$, $\left\{E_v W_n(\theta) e^{-\theta S(v)} : |v| = n, n \ge 1\right\}$ are independent, and

$$E_v W_n(\theta) e^{-\theta S(v)} \Big| \mathcal{F} \sim \text{ Exponential } \left(\frac{e^{\theta S(v)}}{W_n(\theta)}\right).$$

Note that

$$\sum_{|v|=n} \frac{e^{\theta S(v)}}{W_n(\theta)} = 1,$$

and by (3.18), we also have that for $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$,

$$\max_{|v|=n} \frac{e^{\theta S(v)}}{W_n(\theta)} \xrightarrow{p} 0.$$

Therefore by Lemma 4.2.1, for any positive integer k, Borel sets $B_1, B_2, \ldots, B_k$ and non-negative integers $t_1, t_2, \ldots, t_k$, we have

$$\mathbb{P}\left(\sum_{|v|=n} \delta_{E_v W_n(\theta) e^{-\theta S(v)}}(B_1) = t_1, \ldots, \sum_{|v|=n} \delta_{E_v W_n(\theta) e^{-\theta S(v)}}(B_k) = t_k \middle| \mathcal{F} \right)$$

$$\xrightarrow{p} \mathbb{P}\left(\mathcal{N}(B_1) = t_1, \ldots, \mathcal{N}(B_k) = t_k\right), \quad (4.14)$$

where $\mathcal{N}$ is a homogeneous Poisson point process on $\mathbb{R}_+$ with intensity 1. Therefore, using the dominated convergence theorem, we get

$$\mathbb{P}\left(\sum_{|v|=n} \delta_{E_v W_n(\theta) e^{-\theta S(v)}}(B_1) = t_1, \ldots, \sum_{|v|=n} \delta_{E_v W_n(\theta) e^{-\theta S(v)}}(B_k) = t_k \right)$$

$$\to \mathbb{P}\left(\mathcal{N}(B_1) = t_1, \ldots, \mathcal{N}(B_k) = t_k\right). \quad (4.15)$$

or equivalently (see Theorem 11.1.VII of Daley and Vere-Jones [18]),

$$\sum_{|v|=n} \delta_{E_v W_n(\theta) e^{-\theta S(v)}} \xrightarrow{d} \mathcal{N}. \quad (4.16)$$

Since $-\log(.)$ is continuous and therefore Borel measurable, (4.16) suggests that

$$\mathcal{U}_n(\theta) := \sum_{|v|=n} \delta_{\theta S(v) - \log E_v - \log W_n(\theta)} \xrightarrow{d} \mathcal{Y}, \quad (4.17)$$

where $\mathcal{Y}$ is a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}\,dx$. To simplify the notations, for all $\theta < \theta_0 \leq \infty$, we denote

$$A_n(\theta) := n\nu(\theta) + \log D_\theta^\infty,$$

and for $\theta = \theta_0 < \infty$, we denote

$$A_n(\theta_0) := n\nu(\theta_0) - \frac{1}{2}\log n + \log D_{\theta_0}^{\infty} + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right).$$

Recall that by (3.6) and (3.15), for $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$,

$$A_n(\theta) - \log W_n(\theta) \xrightarrow{p} 0. \tag{4.18}$$

Now, take any positive integer k, non-negative integers $\{t_i\}_{i=1}^k$, and extended real numbers $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ with $a_i < b_i$ for all i. We choose $\delta \in \Big(0, \min_{i=1}^k (b_i - a_i)/2\Big)$. Then, we have

$$\begin{aligned}
&\mathbb{P}\Big(\mathcal{U}_n(\theta)\left((a_1-\delta, b_1+\delta)\right) \leq t_1, \ldots, \mathcal{U}_n(\theta)\left((a_k-\delta, b_k+\delta)\right) \leq t_k\Big) \\
&\qquad - \mathbb{P}\left(\left|A_n(\theta) - \log W_n(\theta)\right| > \delta\right) \\
&\leq \mathbb{P}\Big(Z_n(\theta)\left((a_1, b_1)\right) \leq t_1, \ldots, Z_n(\theta)\left((a_k, b_k)\right) \leq t_k\Big) \\
&\leq \mathbb{P}\Big(\mathcal{U}_n(\theta)\left((a_1+\delta, b_1-\delta)\right) \leq t_1, \ldots, \mathcal{U}_n(\theta)\left((a_k+\delta, b_k-\delta)\right) \leq t_k\Big) \\
&\qquad + \mathbb{P}\left(\left|A_n(\theta) - \log W_n(\theta)\right| > \delta\right).
\end{aligned} \tag{4.19}$$

Now, by (4.17), we have $\mathcal{U}_n(\theta) \xrightarrow{d} \mathcal{Y}$, which by Lemma 4.2.2 is a Poisson point process with intensity $e^{-x}\,dx$ and is therefore continuous. Thus, by allowing $n \to \infty$ and then letting $\delta \to 0$, from inequality (4.19) we get that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\Big(Z_n(\theta)\left((a_1, b_1)\right) \leq t_1, \ldots, Z_n(\theta)\left((a_k, b_k)\right) \leq t_k\Big) \\
&\qquad\qquad = \mathbb{P}\Big(\mathcal{Y}\left((a_1, b_1)\right) \leq t_1, \ldots, \mathcal{Y}\left((a_k, b_k)\right) \leq t_k\Big),
\end{aligned}$$

or equivalently, $Z_n(\theta) \xrightarrow{d} \mathcal{Y}$. This, together with Lemma 4.2.2, completes the proof. □

Following is a slightly weaker version of the above theorem, which is essentially a point process convergence of the appropriately centered LPM-BRW model.

Theorem 4.3.2. *For $\theta < \theta_0 \leq \infty$,*

$$\sum_{|v|=n} \delta_{\left\{\theta S(v) - \log E_v - n\nu(\theta)\right\}} \xrightarrow{d} \sum_{j\geq 1} \delta_{\zeta_j + \log D_\theta^\infty},$$

and for $\theta = \theta_0 < \infty$,

$$\sum_{|v|=n} \delta_{\left\{\theta_0 S(v) - \log E_v - n\nu(\theta_0) + \frac{1}{2}\log n\right\}} \xrightarrow{d} \sum_{j\geq 1} \delta_{\zeta_j + \log D_{\theta_0}^\infty + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)},$$

where $\mathcal{Y} = \sum_{j\geq 1} \delta_{\zeta_j}$ is a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}\,dx$, which is independent of the BRW.

Proof. For $\theta < \theta_0 \leq \infty$, we define

$$\tilde{\mathcal{U}}_n(\theta) := \sum_{|v|=n} \delta_{\theta S(v) - \log E_v - \log W_n(\theta) + \log D_\theta^\infty}.$$

and for $\theta = \theta_0 < \infty$, we define

$$\tilde{\mathcal{U}}_n(\theta_0) := \sum_{|v|=n} \delta_{\theta_0 S(v) - \log E_v - \log W_n(\theta_0) + \log D_{\theta_0}^\infty + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)}.$$

By an argument similar to that of the proof of Theorem 4.3.1, we get that for $\theta < \theta_0 \leq \infty$,

$$\tilde{\mathcal{U}}_n(\theta) \xrightarrow{d} \sum_{j\geq 1} \delta_{\zeta_j + \log D_\theta^\infty}.$$

and for $\theta = \theta_0 < \infty$,

$$\tilde{\mathcal{U}}_n(\theta_0) \xrightarrow{d} \sum_{j\geq 1} \delta_{\zeta_j + \log D_{\theta_0}^\infty + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)},$$

where $\mathcal{Y} = \sum_{j\geq 1} \delta_{\zeta_j}$ is a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}\,dx$, which is independent of the BRW. Now, for $\theta < \theta_0 \leq \infty$, we write

$$\tilde{Z}_n(\theta) := \sum_{|v|=n} \delta_{\left\{\theta S(v) - \log E_v - n\nu(\theta)\right\}},$$

and for $\theta = \theta_0 < \infty$, we write

$$\tilde{Z}_n(\theta_0) := \sum_{|v|=n} \delta_{\left\{\theta_0 S(v) - \log E_v - n\nu(\theta_0) + \frac{1}{2}\log n\right\}}.$$

Then by replacing $\mathcal{U}_n$ by $\tilde{\mathcal{U}}_n$ and Z_n by $\tilde{Z}_n$ in inequality (4.19) and then letting $n \to \infty$ and $\delta \to 0$ gives us the required result. □

Now, we denote $\mathcal{Y}_{\max}$ as the right-most position of the point process $\mathcal{Y}$, and we write $\overline{\mathcal{Y}}$ as the point process $\mathcal{Y}$ seen from its right-most position, that is,

$$\overline{\mathcal{Y}} = \sum_{j\geq 1} \delta_{\zeta_j - \mathcal{Y}_{\max}}.$$

Following result is an immediate corollary of the above theorem, which confirms that the *Brunet-Derrida Conjecture* holds for our model when $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$.

Theorem 4.3.3. *For $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$,*

$$\sum_{|v|=n} \delta_{\left\{\theta S(v) - \log E_v - \theta R_n^*(\theta,\delta_1)\right\}} \xrightarrow{d} \overline{\mathcal{Y}}.$$

Remark 4.3.1: As in Theorem 1.4.10, Madaule [29] showed the convergence of the centered point process obtained in the classical setup to a decorated Poisson point process. However, unlike in our case, he could not explicitly derive the limiting point process. Although it is to be noted that in our case the decoration disappears. This is because for the boundary and below the boundary cases,

$$\max_{|v|=n} \frac{e^{\theta S(v)}}{W_n(\theta)} \xrightarrow{p} 0. \tag{4.20}$$

However, (4.20) does not hold for above the boundary case. This added complication is the main reason that the results for above the boundary case remains open.

Remark 4.3.2: The point process $\overline{\mathcal{Y}}$ can be described explicitly in the following way: Let $\mathcal{N} = \sum_{j\geq 1} \delta_{\mathfrak{z}_j}$ be a homogeneous Poisson point process on $\mathbb{R}_+$ with intensity 1 and

$E \sim$ Exponential (1) be independent of $\mathcal{N}$. Then

$$\overline{\mathcal{Y}} \stackrel{d}{=} \delta_0 + \sum_{j \geq 1} \delta_{-\log\left(1+(\mathfrak{z}_j/E)\right)}.$$

Chapter 5

Large Deviations [1]

5.1 Introduction

In Chapter 3, we have learned that for any $\theta > 0$, $R_n^*(\theta, \mu)/n$ converges almost surely to a finite constant, as mentioned in (3.20). If we call the limit as $c(\theta)$, then we have

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} > x\right) = 0 \text{ for } x > c(\theta); \text{ and also}$$
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} < x\right) = 0 \text{ for } x < c(\theta).$$

This chapter investigates the exponential decay rates of these probabilities, which is in essence a problem of *large deviation principle (LDP)*. To study them, we need slightly more solid assumptions about the model, which are as follows.

(**A1′**) The progeny point process Z consists of N i.i.d. copies of a random variable, say X, whose moment-generating function is finite everywhere, i.e., for all $\lambda \in \mathbb{R}$,

$$m_X(\lambda) = \mathbb{E}\left[e^{\lambda X}\right] < \infty.$$

[1] This chapter is based on the paper entitled "*Large deviations for the right-most position of a last progeny modified branching random walk*" [22].

(**A2′**) The point process Z is non-trivial, i.e., $\mathbb{P}(N=1)<1$ and $\mathbb{P}(X=t)<1$ for any $t\in\mathbb{R}$. The extinction probability of the underlying branching process is zero, i.e., $\mathbb{P}(N=0)=0$.

(**A3′**) N has finite $(1+p)$-th moment for some $p>0$.

(**A4′**) For all $k\in\mathbb{Z}$,
$$\int_0^\infty x^k\,d\mu(x)<\infty.$$

Note that under assumption (**A1′**), the moment-generating function of the progeny point process Z
$$m(\lambda)=\mathbb{E}\left[\sum_{i=1}^N e^{\lambda X_i}\right]=m_X(\lambda)\cdot\mathbb{E}[N].$$
So if we denote $\phi(\lambda):=\log m_X(\lambda)$, then we have
$$\nu(\lambda)=\phi(\lambda)+\log\mathbb{E}[N]. \tag{5.1}$$

5.2 The Rate Function

Let $\{X_n\}_{n\geq 1}$ be i.i.d. copies of X. We define $\mathsf{S}_n:=\sum_{i=1}^n X_i$. It follows from Cramér's theorem (see Dembo and Zeitouni [19]) that the laws of $\{\mathsf{S}_n/n\}_{n\geq 1}$ satisfy the large deviation principle with the rate function
$$I(x):=\sup_{\lambda\in\mathbb{R}}\lambda x-\phi(\lambda). \tag{5.2}$$

Since ν is strictly convex and differentiable, so is ϕ. Thus, using Theorem 1 of Rockafellar [32], we obtain that $I(x)$ is strictly convex and differentiable on the interior of its effective domain $\mathcal{D}_I:=\{x\in\mathbb{R}:I(x)<\infty\}$ with $I'(x)=(\phi')^{-1}(x)$. This implies
$$I'\big(\mathbb{E}[X]\big)=0,\qquad\text{and}\qquad\lim_{x\downarrow\inf\mathcal{D}_I}I'(x)=-\infty.$$

Therefore, whenever $\rho:=-\log\mathbb{P}(N=1)$ is finite, there exists a unique point $a_\theta^\rho\in\big(\inf\mathcal{D}_I,\mathbb{E}[X]\big)$ such that a tangent from the point $\big(c(\theta),0\big)$ to the graph of $I(x)+\rho$

touches the graph at $x = a_\theta^\rho$, i.e., a_θ^ρ satisfies

$$\frac{I\left(a_\theta^\rho\right)+\rho}{a_\theta^\rho - c(\theta)} = I'\left(a_\theta^\rho\right). \tag{5.3}$$

We denote $d(\theta) := \max\left\{c(\theta), \phi'(\theta)\right\}$. Then we have

Theorem 5.2.1. *The laws of* $\{R_n^*(\theta,\mu)/n\}_{n\geq 1}$ *satisfy the large deviation principle with the rate function*

$$\Psi_\theta(x) := \begin{cases} \theta x - \phi(\theta) - \log \mathbb{E}[N] & \text{if } x > d(\theta); & (i) \\ I(x) - \log \mathbb{E}[N] & \text{if } c(\theta) < x \leq d(\theta); & (ii) \\ 0 & \text{if } x = c(\theta); & (iii) \\ I'\left(a_\theta^\rho\right)\left(x - c(\theta)\right) & \text{if } a_\theta^\rho \leq x < c(\theta) \text{ and } \rho < \infty; & (iv) \\ I(x) + \rho & \text{if } x < a_\theta^\rho \text{ and } \rho < \infty; & (v) \\ \infty & \text{if } x < c(\theta) \text{ and } \rho = \infty. & (vi) \end{cases}$$

We refer to Figure 5.1 for an illustration of the theorem.

To prove Theorem 5.2.1, we need the following lemma, which provides LDP for each of the branches of the LPM-BRW.

Lemma 5.2.1. *Let* $Y \sim \mu$ *and* $E \sim$ *Exponential*(1) *be independent of each other and also independent of the random variables* $\{X_n\}_{n\geq 1}$*. Then, for any* $\theta > 0$*, the laws of* $\left\{\frac{\mathsf{S}_n}{n} + \frac{1}{n\theta}\log(Y/E)\right\}_{n\geq 1}$ *satisfy the large deviation principle with the rate function*

$$I_\theta(x) := \begin{cases} I(x) & \text{if } x \leq \phi'(\theta); \\ \theta x - \phi(\theta) & \text{if } x \geq \phi'(\theta). \end{cases}$$

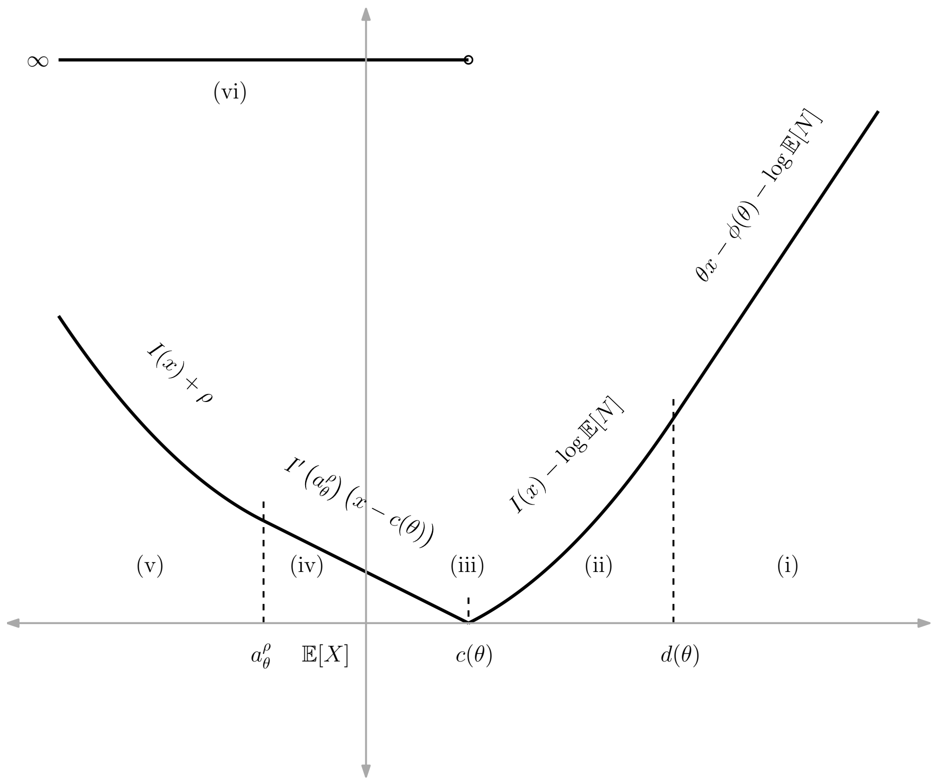

Figure 5.1: Illustration of Theorem 5.2.1

Proof. For each $\theta > 0$ and $\lambda \in \mathbb{R}$, we define

$$\begin{aligned}\Upsilon_\theta(\lambda) &:= \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{\lambda \mathsf{S}_n + \frac{\lambda}{\theta}\log(Y/E)}\right] \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(e^{n\phi(\lambda)} \cdot \mathbb{E}\left[Y^{\lambda/\theta}\right] \cdot \mathbb{E}\left[E^{-\lambda/\theta}\right]\right) = \begin{cases} \phi(\lambda) & \text{if } \lambda < \theta; \\ \infty & \text{if } \lambda \geq \theta. \end{cases}\end{aligned}$$

Its Fenchel-Legendre transform is

$$\Upsilon_\theta^*(x) := \sup_{\lambda\in\mathbb{R}} \lambda x - \Upsilon_\theta(\lambda) = \sup_{\lambda<\theta} \lambda x - \phi(\lambda) = I_\theta(x).$$

Since 0 belongs to the interior of the set $\{\lambda \in \mathbb{R} : \Upsilon_\theta(\lambda) < \infty\}$, it follows from the Gärtner-Ellis theorem (see Dembo and Zeitouni [19]) that for any closed set F,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\mathsf{S}_n}{n} + \frac{1}{n\theta}\log(Y/E) \in F\right) \leq -\inf_{x\in F} I_\theta(x), \tag{5.4}$$

and for any open set G,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\mathsf{S}_n}{n} + \frac{1}{n\theta}\log(Y/E) \in G\right) \geq - \inf_{x\in G,\, x<\phi'(\theta)} I_\theta(x). \tag{5.5}$$

Note that since Y is a positive random variable, there exists a constant $\alpha > 0$ such that $\mathbb{P}(Y > \alpha) > 0$. Now, for any $x \geq \phi'(\theta)$, we have

$$\mathbb{P}\left(\frac{\mathsf{S}_n}{n} + \frac{1}{n\theta}\log(Y/E) > x\right) \geq \mathbb{P}\left(\mathsf{S}_n > n\phi'(\theta)\right) \cdot \mathbb{P}(Y > \alpha) \cdot \mathbb{P}\left(E < \alpha e^{-n\theta\left(x-\phi'(\theta)\right)}\right).$$

Therefore using Cramér's theorem, we get

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\mathsf{S}_n}{n} + \frac{1}{n\theta}\log(Y/E) > x\right) \geq -I\left(\phi'(\theta)\right) - \theta\left(x - \phi'(\theta)\right) = -I_\theta(x). \tag{5.6}$$

Combining (5.5) and (5.6), we obtain that for any open set G,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\mathsf{S}_n}{n} + \frac{1}{n\theta}\log(Y/E) \in G\right) \geq - \inf_{x\in G} I_\theta(x).$$

This, together with (5.4), completes the proof. □

Now we have all the machinery to prove the theorem.

Proof of Theorem 5.2.1. Proof of Part (vi). Recall that as in (3.24), we have

$$R_n^*(\theta,\mu) \geq S(\tilde{u}_m) + \max_{|u|=m} R_{n-m}^{*(u)}(\theta,\mu).$$

Since $\{S(u)\}_{|u|=m}$ are identically distributed and are independent of $\{R_{n-m}^{*(u)}\}_{|u|=m}$, we also have

$$S(\tilde{u}_m) \overset{d}{=\!=} \mathsf{S}_m. \tag{5.7}$$

Now, for any $x < c(\theta)$ and $\epsilon \in \left(0, c(\theta) - x\right)$, an argument exactly similar to that in (3.25) gives us

$$\mathbb{P}(R_n^* < nx) \leq \mathbb{E}\left[\mathbb{P}\left(R_{n-\lfloor\sqrt{n}\rfloor}^* < n(x+\epsilon)\right)^{N_{\lfloor\sqrt{n}\rfloor}}\right] + \mathbb{P}\left(\mathsf{S}_{\lfloor\sqrt{n}\rfloor} < -n\epsilon\right). \tag{5.8}$$

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and N_k represents the total number of particles at generation k. Note that N_k is at least 2^k since $\mathbb{P}(N=1)=0$. Now since $x+\epsilon < c(\theta)$, which is the almost sure limit of $R^*_{n-\lfloor\sqrt{n}\rfloor}/n$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[\mathbb{P}\left(R^*_{n-\lfloor\sqrt{n}\rfloor} < n(x+\epsilon)\right)^{N_{\lfloor\sqrt{n}\rfloor}}\right]$$
$$\leq \lim_{n\to\infty} \frac{2^{\lfloor\sqrt{n}\rfloor}}{n} \log \mathbb{P}\left(R^*_{n-\lfloor\sqrt{n}\rfloor} < n(x+\epsilon)\right) = -\infty. \tag{5.9}$$

Let $\{t_n\}_{n\geq 1}$ be a non-negative real sequence increasing to ∞ such that $\phi(-t_n) \leq \log n$. Such a sequence exists since $\phi(\lambda) < \infty$ for all $\lambda \leq 0$. We can construct such a sequence by the following recursive relation

$$t_1 = 0, \qquad \text{and for all } n \geq 2, \quad t_n = \begin{cases} t_{n-1}+1 & \text{if } \phi(-t_{n-1}-1) \leq \log n; \\ t_{n-1} & \text{otherwise.} \end{cases}$$

Then using Markov's inequality we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\mathsf{S}_{\lfloor\sqrt{n}\rfloor} < -n\epsilon\right) \leq \lim_{n\to\infty} \frac{1}{n} \log\left(e^{-nt_n\epsilon} \cdot \mathbb{E}\left[e^{-t_n \mathsf{S}_{\lfloor\sqrt{n}\rfloor}}\right]\right)$$
$$= \lim_{n\to\infty} -t_n\epsilon + \frac{\lfloor\sqrt{n}\rfloor \phi(-t_n)}{n} = -\infty. \tag{5.10}$$

Therefore, by combining (5.8), (5.9), and (5.10), we get that for $\rho = \infty$ and all $x < c(\theta)$,

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}(R^*_n < nx) = \infty. \tag{5.11}$$

□

Proof of Parts (iv) & (v). ***(Lower bound).*** Take any $x < c(\theta)$ and $t \in (0,1]$. Observe that for $|v| = \lceil tn \rceil$ and $\epsilon > 0$,

$$\mathbb{P}(R^*_n < nx) \geq \mathbb{P}\left(S(v) + R^{*(v)}_{\lfloor(1-t)n\rfloor} < nx, N_{\lceil tn\rceil} = 1\right)$$
$$\geq \mathbb{P}\left(N_{\lceil tn\rceil} = 1\right) \cdot \mathbb{P}\left(R^*_{\lfloor(1-t)n\rfloor} < n(1-t)(c(\theta)+\epsilon)\right)$$
$$\cdot\, \mathbb{P}\left(\mathsf{S}_{\lceil tn\rceil} < nx - n(1-t)(c(\theta)+\epsilon)\right). \tag{5.12}$$

Here $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Also, note that $N_{\lceil tn \rceil}$, $S(v)$ and $R^{*(v)}_{\lfloor (1-t)n \rfloor}$ are independent of each other, which implied the last inequality. For the first term on the right-hand side, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(N_{\lceil tn \rceil} = 1\right) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(N=1)^{\lceil tn \rceil} = -\rho t. \tag{5.13}$$

For $t = 1$, the second term equals $\mathbb{P}(Y < E) > 0$, and for $t \in (0,1)$, $c(\theta)$ is the almost sure limit of $R^*_{\lfloor (1-t)n \rfloor} / \big(n(1-t)\big)$. Therefore for all $t \in (0,1]$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(R^*_{\lfloor (1-t)n \rfloor} < n(1-t)(c(\theta)+\epsilon)\right) = 0. \tag{5.14}$$

Finally, for the last term, using Cramér's theorem, we get

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\mathsf{S}_{\lceil tn \rceil} < nx - n(1-t)(c(\theta)+\epsilon)\right) = -tI\left(\frac{x-(1-t)(c(\theta)+\epsilon)}{t}\right), \tag{5.15}$$

whenever

$$0 < t \le f(x) := \min\left\{1, \frac{c(\theta)-x}{c(\theta)-\mathbb{E}[X]}\right\}.$$

So, by combining (5.12), (5.13), (5.14), and (5.15), and allowing $\epsilon \downarrow 0$, we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(R^*_n < nx) \ge -\inf_{0<t\le f(x)} \left\{\rho t + tI\left(\frac{x-(1-t)c(\theta)}{t}\right)\right\}.$$

Since $I\left(\big(x-(1-t)c(\theta)\big)/t\right)$ is non-decreasing for $t \ge \big(c(\theta)-x\big)/\big(c(\theta)-\mathbb{E}[X]\big)$, the above inequality implies

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(R^*_n < nx) \ge -\inf_{0<t\le 1} \left\{\rho t + tI\left(\frac{x-(1-t)c(\theta)}{t}\right)\right\}. \tag{5.16}$$

(Upper bound). Now, we fix any $k \in \mathbb{N}$ and define $n_i = \lfloor nif(x)/k \rfloor$ for all $i = 0, 1, 2, \ldots, k$. Since $N_{n_0} = N_0 = 1$, for any $n \ge 2$, we have

$$\begin{aligned}\mathbb{P}(R^*_n < nx) = &\sum_{i=0}^{k-2} \mathbb{P}\left(N_{n_i} < n^2, N_{n_{i+1}} \ge n^2\right) \cdot \mathbb{P}\left(R^*_n < nx \big| N_{n_i} < n^2, N_{n_{i+1}} \ge n^2\right) \\ &+ \mathbb{P}\left(N_{n_{k-1}} < n^2\right) \cdot \mathbb{P}\left(R^*_n < nx \big| N_{n_{k-1}} < n^2\right).\end{aligned} \tag{5.17}$$

Using Theorem 2.5 of Gantert and Höfelsauer [21], we get that for $1 \leq i \leq k-1$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(N_{n_i} < n^2) = -\frac{if(x)\rho}{k}. \tag{5.18}$$

On the other hand, using inequality (3.24), we have for all $\epsilon > 0$ and $0 \leq i \leq k-2$,

$$\begin{aligned}
&\mathbb{P}\left(R_n^* < nx \middle| N_{n_i} < n^2, N_{n_{i+1}} \geq n^2\right) \\
&\leq \mathbb{P}\left(S(\tilde{u}_{n_{i+1}}) < nx - (n-n_{i+1})(c(\theta)-\epsilon) \middle| N_{n_i} < n^2, N_{n_{i+1}} \geq n^2\right) \\
&\qquad + \mathbb{P}\left(\max_{|u|=n_{i+1}} R_{n-n_{i+1}}^{*(u)} < (n-n_{i+1})(c(\theta)-\epsilon) \middle| N_{n_i} < n^2, N_{n_{i+1}} \geq n^2\right) \\
&\leq \mathbb{P}\left(\mathsf{S}_{n_{i+1}} < nx - (n-n_{i+1})(c(\theta)-\epsilon)\right) + \mathbb{P}\left(R_{n-n_{i+1}}^* < (n-n_{i+1})(c(\theta)-\epsilon)\right)^{n^2}.
\end{aligned} \tag{5.19}$$

Notice that $S(\tilde{u}_{n_{i+1}})$ is independent of N_{n_i} and $N_{n_{i+1}}$, and by (5.7), it has the same distribution as $\mathsf{S}_{n_{i+1}}$, which implied the last inequality. Now, we know that $c(\theta)$ is the almost sure limit of $R_{n-n_{i+1}}^*/(n-n_{i+1})$. Therefore for any $i = 0, 1, 2, \ldots, k-2$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(R_{n-n_{i+1}}^* < (n-n_{i+1})(c(\theta)-\epsilon)\right)^{n^2} = -\infty.$$

Thus, from (5.19), we get that for any $i = 0, 1, 2, \ldots, k-2$ and $\epsilon > 0$ small enough,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(R_n^* < nx \middle| N_{n_i} < n^2, N_{n_{i+1}} \geq n^2\right) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\mathsf{S}_{n_{i+1}} < nx - (n-n_{i+1})(c(\theta)-\epsilon)\right) \\
&= -\frac{(i+1)f(x)}{k} \cdot I\left(\frac{x - \left(1 - \frac{(i+1)f(x)}{k}\right)(c(\theta)-\epsilon)}{\frac{(i+1)f(x)}{k}}\right).
\end{aligned} \tag{5.20}$$

For the last term of (5.17), if $f(x) = \big(c(\theta)-x\big) / \big(c(\theta)-\mathbb{E}[X]\big)$, we trivially have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(R_n^* < nx \middle| N_{n_{k-1}} < n^2\right) \leq 0 = -f(x) \cdot I\left(\mathbb{E}[X]\right).$$

and if $f(x) = 1$, we have $x \le \mathbb{E}[X]$. In that case, from Lemma 5.2.1, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(R_n^* < nx \middle| N_{n_{k-1}} < n^2 \right)$$
$$\le \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\mathsf{S}_n + \frac{1}{\theta} \log(Y/E) < nx \right) = -I(x).$$

Combining the above two inequalities, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(R_n^* < nx \middle| N_{n_{k-1}} < n^2 \right) \le -f(x) \cdot I\left(\frac{x - (1 - f(x))c(\theta)}{f(x)} \right). \tag{5.21}$$

Therefore, by combining (5.17), (5.18), (5.20), and (5.21), and then allowing $\epsilon \downarrow 0$ and $k \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(R_n^* < nx) \le - \inf_{0<t\le 1} \left\{ \rho t + tI\left(\frac{x - (1-t)c(\theta)}{t} \right) \right\}. \tag{5.22}$$

This, together with (5.16), implies that for any $x < c(\theta)$ and $\rho < \infty$,

$$\begin{aligned} \lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}(R_n^* < nx) &= \inf_{0<t\le 1} \left\{ \rho t + tI\left(\frac{x - (1-t)c(\theta)}{t} \right) \right\} \\ &= \inf_{y \le x} \left\{ (\rho + I(y)) \frac{c(\theta) - x}{c(\theta) - y} \right\} \\ &= \big(c(\theta) - x\big) \left(\inf_{y\le x} \left\{ \frac{\rho + I(y)}{c(\theta) - y} \right\} \right) \\ &= \begin{cases} I'\left(a_\theta^\rho\right)\big(x - c(\theta)\big) & \text{if } a_\theta^\rho \le x < c(\theta); \\ I(x) + \rho & \text{if } x < a_\theta^\rho. \end{cases} \end{aligned} \tag{5.23}$$

□

Proof of Part (iii). This part follows from (3.20). □

Proof of Part (ii). Note that $c(\theta) < d(\theta)$ means $d(\theta) = \phi'(\theta) = \nu'(\theta)$. Therefore, $c(\theta) < d(\theta)$ occurs iff $\theta_0 < \infty$ and $\theta > \theta_0$. So this part is only relevant for this range of θ.

(Upper bound). Take any $x \in (c(\theta), d(\theta)]$ and observe that

$$\mathbb{P}(R_n^* > nx) = \mathbb{E}\left[\mathbb{P}(R_n^* > nx|N_n)\right] \le \mathbb{E}\left[N_n \cdot \mathbb{P}\left(\mathsf{S}_n + \frac{1}{\theta}\log(Y/E) > nx\right)\right]$$
$$= \left(\mathbb{E}[N]\right)^n \cdot \mathbb{P}\left(\mathsf{S}_n + \frac{1}{\theta}\log(Y/E) > nx\right).$$

Since $\phi'(\theta) \ge x > c(\theta) > \mathbb{E}[X]$, using Lemma 5.2.1, we get

$$\limsup_{n\to\infty} \frac{1}{n}\log\mathbb{P}(R_n^* > nx) \le \log\mathbb{E}[N] - I(x). \tag{5.24}$$

(Lower bound). For any $\alpha \in (0,1)$, using inequality (3.24), we obtain

$$\begin{aligned}\mathbb{P}(R_n^* > nx) &\ge \mathbb{P}\left(S(\tilde{u}_{\lfloor \alpha n\rfloor}) + \max_{|u|=\lfloor \alpha n\rfloor} R^{*(u)}_{\lceil (1-\alpha)n\rceil} > nx\right)\\
&\ge \mathbb{P}\left(S(\tilde{u}_{\lfloor \alpha n\rfloor}) > \lfloor \alpha n\rfloor x\right)\cdot\mathbb{P}\left(\max_{|u|=\lfloor \alpha n\rfloor} R^{*(u)}_{\lceil (1-\alpha)n\rceil} > \lceil (1-\alpha)n\rceil x\right)\\
&\ge \mathbb{P}\left(\mathsf{S}_{\lfloor \alpha n\rfloor} > \lfloor \alpha n\rfloor x\right)\cdot\mathbb{P}\left(N_{\lfloor \alpha n\rfloor} > \frac{1}{2}\cdot\mathbb{E}[N]^{\lfloor \alpha n\rfloor}\right)\\
&\qquad\cdot\mathbb{P}\left(\max_{|u|=\lfloor \alpha n\rfloor} R^{*(u)}_{\lceil (1-\alpha)n\rceil} > \lceil (1-\alpha)n\rceil x\,\middle|\, N_{\lfloor \alpha n\rfloor} > \frac{1}{2}\cdot\mathbb{E}[N]^{\lfloor \alpha n\rfloor}\right)\\
&\ge \mathbb{P}\left(\mathsf{S}_{\lfloor \alpha n\rfloor} > \lfloor \alpha n\rfloor x\right)\cdot\mathbb{P}\left(N_{\lfloor \alpha n\rfloor} > \frac{1}{2}\cdot\mathbb{E}[N]^{\lfloor \alpha n\rfloor}\right)\\
&\qquad\cdot\left(1-\left(1-\mathbb{P}\left(R^*_{\lceil (1-\alpha)n\rceil} > \lceil (1-\alpha)n\rceil x\right)\right)^{\frac{1}{2}\cdot\mathbb{E}[N]^{\lfloor \alpha n\rfloor}}\right).\end{aligned}$$

Note that for any $a \in [0,1]$ and $t \ge 2$,

$$1-(1-a)^t - at + a^2t^2 = \int_0^a\int_0^r \left(2t^2 - t(t-1)(1-s)^{t-2}\right) ds\, dr \ge 0$$

$$\Rightarrow 1-(1-a)^t \ge at(1-at).$$

Therefore, for all large enough n, we get

$$\mathbb{P}(R_n^* > nx) \geq \mathbb{P}\left(\mathsf{S}_{\lfloor \alpha n \rfloor} > \lfloor \alpha n \rfloor x\right) \cdot \mathbb{P}\left(N_{\lfloor \alpha n \rfloor} > \frac{1}{2} \cdot \mathbb{E}[N]^{\lfloor \alpha n \rfloor}\right)$$
$$\cdot \frac{1}{2} \cdot \mathbb{E}[N]^{\lfloor \alpha n \rfloor} \cdot \mathbb{P}\left(R^*_{\lceil (1-\alpha) n \rceil} > \lceil (1-\alpha) n \rceil x\right)$$
$$\cdot \left(1 - \frac{1}{2} \cdot \mathbb{E}[N]^{\lfloor \alpha n \rfloor} \cdot \mathbb{P}\left(R^*_{\lceil (1-\alpha) n \rceil} > \lceil (1-\alpha) n \rceil x\right)\right). \tag{5.25}$$

Note that since $c(\theta) = \nu(\theta_0)/\theta_0$, we have

$$I(x) = \sup_{\lambda \in \mathbb{R}} \lambda x - \phi(\lambda) \geq \theta_0 x - \phi(\theta_0) = \theta_0 \left(x - c(\theta)\right) + \log \mathbb{E}[N].$$

Now, for all $x \in (c(\theta), d(\theta)]$, we choose α_x such that

$$o < \alpha_x < \frac{\theta_0 \left(x - c(\theta)\right)}{\theta_0 \left(x - c(\theta)\right) + \log \mathbb{E}[N]},$$

which ensures $(1 - \alpha_x) I(x) > \log \mathbb{E}[N]$. Together with (5.24), this implies

$$\lim_{n \to \infty} \mathbb{E}[N]^{\lfloor \alpha_x n \rfloor} \cdot \mathbb{P}\left(R^*_{\lceil (1-\alpha_x) n \rceil} > \lceil (1-\alpha_x) n \rceil x\right) = 0.$$

Therefore, for $\alpha = \alpha_x$, the last term on the right-hand side of (5.25) tends to 1, as n tends to ∞. Also, assumption (**A3′**) implies that (see Athreya and Ney [5])

$$\lim_{n \to \infty} \mathbb{P}\left(N_{\lfloor \alpha_x n \rfloor} > \frac{1}{2} \cdot \mathbb{E}[N]^{\lfloor \alpha_x n \rfloor}\right) > 0.$$

Thus inequality (5.25) indicates

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(R_n^* > nx) \geq \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\mathsf{S}_{\lfloor \alpha_x n \rfloor} > \lfloor \alpha_x n \rfloor x\right) + \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}[N]^{\lfloor \alpha_x n \rfloor}$$
$$+ \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(R^*_{\lceil (1-\alpha_x) n \rceil} > \lceil (1-\alpha_x) n \rceil x\right).$$

Together with Cramér's theorem, this implies

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(R_n^* > nx) \geq \alpha_x \big(\log \mathbb{E}[N] - I(x)\big) + (1-\alpha_x) \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(R_n^* > nx). \tag{5.26}$$

Since $I(x)$ is finite for $x \in \big(c(\theta), d(\theta)\big] = \big(\phi'(\theta_0), \phi'(\theta)\big]$, using Lemma 5.2.1, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(R_n^* > nx) \geq \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\mathsf{S}_n + \frac{1}{\theta} \log(Y/E) > nx\right) = -I(x) > -\infty.$$

So, from (5.26), we get

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(R_n^* > nx) \geq \log \mathbb{E}[N] - I(x). \tag{5.27}$$

Combining (5.24) and (5.27), we obtain that for all $x \in \big(c(\theta), d(\theta)\big]$,

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}\left(R_n^* > nx\right) = I(x) - \log \mathbb{E}[N]. \tag{5.28}$$

□

Proof of Part (i). ***(Upper bound).*** Using Markov's inequality, we obtain that for any $x \in \mathbb{R}$ and any $\lambda < \theta$,

$$\begin{aligned}\mathbb{P}(R_n^* > nx) \leq e^{-n\lambda x} \cdot \mathbb{E}\left[e^{\lambda R_n^*}\right] &\leq e^{-n\lambda x} \cdot \mathbb{E}\left[\sum_{|v|=n} e^{\lambda S(v)} Y_v^{\lambda/\theta} E_v^{-\lambda/\theta}\right] \\ &= e^{-n\lambda x} \cdot \mathbb{E}[N]^n \cdot e^{n\phi(\lambda)} \cdot \mathbb{E}\left[Y^{\lambda/\theta}\right] \cdot \Gamma\left(1 - \frac{\lambda}{\theta}\right).\end{aligned}$$

Since this inequality holds for all $\lambda < \theta$, we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(R_n^* > nx) &\leq \lim_{\lambda \uparrow \theta} -\lambda x + \phi(\lambda) + \log \mathbb{E}[N] \\ &= -\theta x + \phi(\theta) + \log \mathbb{E}[N]. \end{aligned} \tag{5.29}$$

(Lower bound). For any $x > d(\theta)$ and any $\epsilon > 0$, from Theorem 2.3.2, we get

$$\begin{aligned}\mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} > x\right) &= \mathbb{P}\left(\frac{\log Y_n^\mu(\theta)}{n\theta} - \frac{\log E}{n\theta} > x\right) \\ &\geq \mathbb{P}\left(\frac{\log Y_n^\mu(\theta)}{n\theta} > d(\theta) - 2\epsilon\right) \cdot \mathbb{P}\left(-\frac{\log E}{n\theta} > x - d(\theta) + 2\epsilon\right).\end{aligned} \tag{5.30}$$

Now, take any $\theta_1 \geq \theta$ and denote μ_1 as the distribution of $Y^{\theta_1/\theta}$. Then we have

$$\left(Y_n^\mu(\theta)\right)^{1/\theta} = \left(\sum_{|v|=n} e^{\theta S(v)} Y_v\right)^{1/\theta} \geq \left(\sum_{|v|=n} e^{\theta_1 S(v)} Y_v^{\theta_1/\theta}\right)^{1/\theta_1} = \left(Y_n^{\mu_1}(\theta_1)\right)^{1/\theta_1}. \tag{5.31}$$

From Theorem 2.3.2, we also have

$$\begin{aligned}\mathbb{P}\left(\frac{R_n^*(\theta_1,\mu_1)}{n} > d(\theta) - \epsilon\right) &= \mathbb{P}\left(\frac{\log Y_n^{\mu_1}(\theta_1)}{n\theta_1} - \frac{\log E}{n\theta_1} > d(\theta) - \epsilon\right) \\ &\leq \mathbb{P}\left(\frac{\log Y_n^{\mu_1}(\theta_1)}{n\theta_1} > d(\theta) - 2\epsilon\right) + \mathbb{P}\left(-\frac{\log E}{n\theta_1} > \epsilon\right).\end{aligned} \tag{5.32}$$

Therefore, by combining (5.30), (5.31), and (5.32), we obtain

$$\begin{aligned}\mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} > x\right) \geq &\left(\mathbb{P}\left(\frac{R_n^*(\theta_1,\mu_1)}{n} > d(\theta) - \epsilon\right) - \mathbb{P}\left(-\frac{\log E}{n\theta_1} > \epsilon\right)\right) \\ &\cdot \mathbb{P}\left(-\frac{\log E}{n\theta} > x - d(\theta) + 2\epsilon\right).\end{aligned} \tag{5.33}$$

Observe that for any $t > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(-\log E > nt) = \lim_{n\to\infty} \frac{1}{n} \log\left(1 - e^{-e^{-nt}}\right) = -t. \tag{5.34}$$

Now, for $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$, we take $\theta_1 = \theta$. In that case, $d(\theta) = c(\theta)$, which implies

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{R_n^*(\theta_1,\mu_1)}{n} > d(\theta) - \epsilon\right) = 0.$$

As a result, in view of (5.33) and (5.34), we get that for $\theta < \theta_0$ or $\theta = \theta_0 < \infty$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} > x\right) \geq -\theta\left(x - d(\theta) + 2\epsilon\right). \tag{5.35}$$

For $\theta_0 < \theta < \infty$, we know that $c(\theta) < d(\theta)$. So choosing $\epsilon < d(\theta) - c(\theta)$, by part (ii) of the theorem, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{R_n^*(\theta_1,\mu_1)}{n} > d(\theta) - \epsilon\right) = -\Psi_{\theta_1}\left(d(\theta) - \epsilon\right) = -\Psi_\theta\left(d(\theta) - \epsilon\right).$$

Now, we choose θ_1 large enough such that $\theta_1\epsilon > \Psi_\theta\left(d(\theta) - \epsilon\right)$, which ensures

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\mathbb{P}\left(\frac{R_n^*(\theta_1,\mu_1)}{n} > d(\theta) - \epsilon\right) - \mathbb{P}\left(-\frac{\log E}{n\theta_1} > \epsilon\right)\right) = -\Psi_\theta\left(d(\theta) - \epsilon\right).$$

Together with (5.33) and (5.34), this implies that for $\theta_0 < \theta < \infty$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} > x\right) \geq -\Psi_\theta\left(d(\theta) - \epsilon\right) - \theta\left(x - d(\theta) + 2\epsilon\right). \tag{5.36}$$

Since $\epsilon > 0$ can be chosen arbitrarily small and Ψ_θ is continuous in $[c(\theta), \infty)$, by combining (5.35) and (5.36), we get that for any $\theta > 0$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} > x\right) \geq -\Psi_\theta\left(d(\theta)\right) - \theta\left(x - d(\theta)\right) = -\Psi_\theta\left(x\right). \tag{5.37}$$

Thus, by combining (5.29) and (5.37), we finally obtain that for any $x > d(\theta)$,

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}\left(R_n^* > nx\right) = \theta x - \phi(\theta) - \log \mathbb{E}[N]. \tag{5.38}$$

□

This completes the proof of Theorem 5.2.1. □

5.3 Comparison with Branching Random Walk

While proving the above result, we also observe that we can complete the LDP for $\{R_n/n\}_{n\geq 1}$, which was proved by Gantert and Höfelsauer [21] but only partially (see Theorem 1.4.11).

Theorem 5.3.1. *The laws of* $\{R_n/n\}_{n\geq 1}$ *satisfy the large deviation principle with the rate function*

$$\Phi(x) := \begin{cases} I(x) - \log \mathbb{E}[N] & \text{if } x > c(\theta_0); & \text{(i)} \\ 0 & \text{if } x = c(\theta_0); & \text{(ii)} \\ I'\big(a^{\rho}_{\theta_0}\big)\big(x - c(\theta_0)\big) & \text{if } a^{\rho}_{\theta_0} \leq x < c(\theta_0) \text{ and } \rho < \infty; & \text{(iii)} \\ I(x) + \rho & \text{if } x < a^{\rho}_{\theta_0} \text{ and } \rho < \infty; & \text{(iv)} \\ \infty & \text{if } x < c(\theta_0) \text{ and } \rho = \infty. & \text{(v)} \end{cases}$$

Remark 5.3.1: The parts (i), (ii), (iii), and (iv) of Theorem 5.3.1 were proved by Gantert and Höfelsauer [21] as mentioned in Theorem 1.4.11. Part (v) was unsolved, which we prove here. Also, parts (iii) and (iv) of Theorem 5.3.1 calculated by Gantert and Höfelsauer [21] have been simplified here.

Proof. For (iii) and (iv), the expression in Gantert and Höfelsauer [21] can be simplified as we did in equation (5.23). The proof of (v) is essentially the proof of the part (vi) of Theorem 5.2.1 verbatim. □

Remark 5.3.2: Note that assumption (**A3′**) was only required for the almost sure convergence of $R^*_n(\theta, \mu)/n$ and therefore is not required to prove part (v) of Theorem 5.3.1. But we do need $\mathbb{E}[N \log N] < \infty$ for the remaining parts, as shown in Gantert and Höfelsauer [21].

We observe that for $\theta_0 \leq \theta < \infty$, the lower large deviations for the laws of $\{R_n/n\}_{n\geq 1}$ and $\{R^*_n(\theta, \mu)/n\}_{n\geq 1}$ coincide.

For $\theta_0 < \theta < \infty$, the upper large deviations for the laws of $\{R_n^*(\theta,\mu)/n\}_{n\geq 1}$ agrees with that of $\{R_n/n\}_{n\geq 1}$ up to $\phi'(\theta)$.

5.4 Specific Examples

To illustrate Theorem 5.2.1, we consider two specific examples. Our first example is when N takes value 2 with probability 1, $X \sim \mathrm{N}(0,1)$, $\theta = 3$, and $\mu = \delta_1$. Then, as displayed in Figure 5.2, the large deviation rate function for the laws of $\{R_n^*(3,\delta_1)/n\}_{n\geq 1}$ is

$$f_1(x) = \begin{cases} 3x - \frac{9}{2} - \log 2 & \text{if } x \geq 3; \\ \frac{x^2}{2} - \log 2 & \text{if } \sqrt{2\log 2} \leq x \leq 3; \\ \infty & \text{if } x < \sqrt{2\log 2}. \end{cases}$$

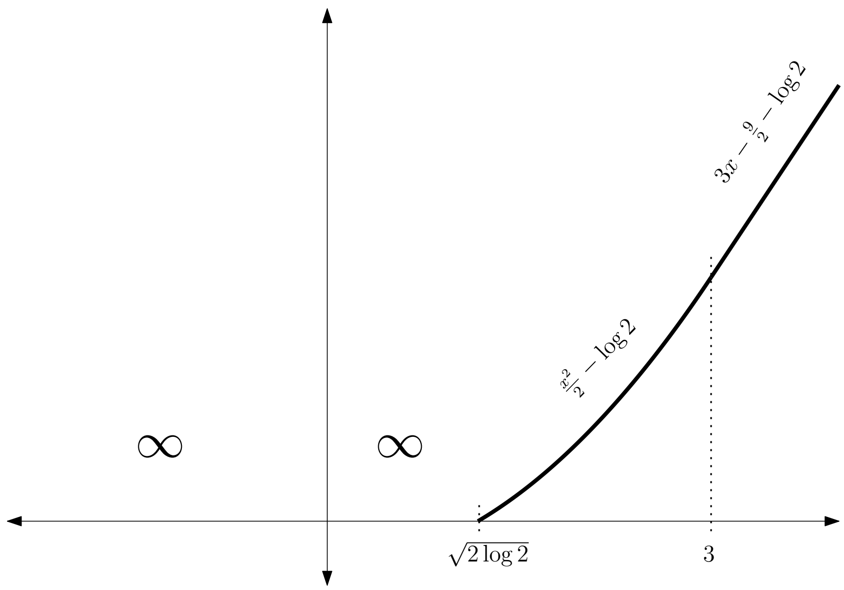

Figure 5.2: Graph of f_1

On the other hand, if N takes the value 1 with probability 1/2 and 3 with probability 1/2, and X, θ, and μ are as in the previous example, then, as demonstrated in Figure 5.3,

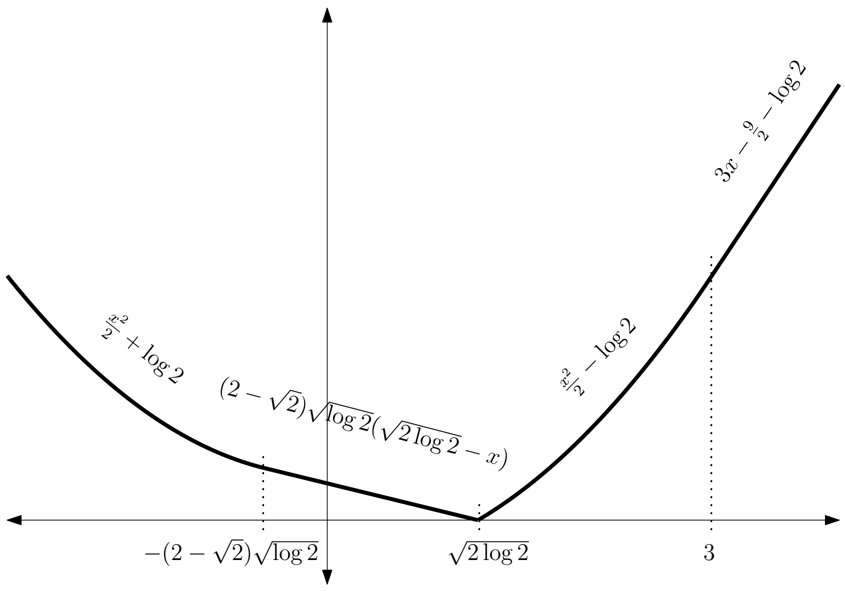

Figure 5.3: Graph of f_2

the large deviation rate function for the laws of $\{R_n^*(3, \delta_1)/n\}_{n \geq 1}$ is

$$f_2(x) = \begin{cases} 3x - \frac{9}{2} - \log 2 & \text{if } x \geq 3; \\ \frac{x^2}{2} - \log 2 & \text{if } \sqrt{2\log 2} \leq x \leq 3; \\ \big(2-\sqrt{2}\big)\sqrt{\log 2}\big(\sqrt{2\log 2} - x\big) & \text{if } -\big(2-\sqrt{2}\big)\sqrt{\log 2} \leq x \leq \sqrt{2\log 2}; \\ \frac{x^2}{2} + \log 2 & \text{if } x \leq -\big(2-\sqrt{2}\big)\sqrt{\log 2}. \end{cases}$$

Notice that the upper large deviations coincide because $\mathbb{E}[N]$ remains the same in both examples. However, since $\mathbb{P}(N = 1)$ is different, the lower large deviations do not match.

Chapter 6

Time Inhomogeneous Setup [1]

6.1 Introduction

In this chapter, we consider a modification of time inhomogeneous branching random walk, where the driving increment distribution changes over time macroscopically and we give certain i.i.d. displacements to all the particles at the n-th generation. We call this process *last progeny modified time inhomogeneous branching random walk (LPMTI-BRW).*

6.1.1 Model

We fix $k \in \mathbb{N}$. For each $i = 1, 2, \ldots, k$, we let Z_i be a point process with $N_i := Z_i(\mathbb{R}) < \infty$ almost surely and q_i be a sequence of integers satisfying $\sum_{i=1}^{k} q_i(n) = n$, and we write $t_m = \sum_{i=1}^{m} q_i(n)$. A *time inhomogeneous branching random walk (TI-BRW)* is a discrete-time stochastic process that can be described for each $n \geq 1$ as follows:

At the 0-th generation, we start with an initial particle at the origin. At time $t \in (t_{m-1}, t_m]$, each of the particles at generation $(t-1)$ gives birth to a random number of offspring distributed according to N_m. The offspring are then given random displacements independently and according to a copy of the point process Z_m.

[1] This chapter is based on the paper entitled "*Right-most position of a last progeny modified time inhomogeneous branching random walk*" [8].

As earlier, $S(v)$ denotes the position of a particle v, which is the sum of all the displacements the particle v and its ancestors have received. We shall call the process $\{S(v) : |v| = t, 0 \leq t \leq n, n \geq 1\}$ a *time inhomogeneous branching random walk (TI-BRW)*.

We then modify this process by giving further displacements to all particles of the n-th generation. The additional displacements are of the form $\frac{1}{\theta}(-\log E_v)$, where $\{E_v\}_{|v|=n}$ are i.i.d. Exponential (1) and are independent of the process $\{S(v) : |v| \leq n\}$. We call this new process *last progeny modified time inhomogeneous branching random walk (LPMTI-BRW)*. We denote by $R_n^*(\theta)$ the maximum position of this LPMTI-BRW.

6.1.2 Assumptions

We first introduce the following important quantities. For each point process Z_i with $1 \leq i \leq k$, we define

$$\nu_i(a) := \log \mathbb{E}\left[\int_{\mathbb{R}} e^{ax}\, Z_i(dx)\right], \tag{6.1}$$

for $a \in \mathbb{R}$, whenever the expectations exist. Needless to say that for each $i = 1, 2, \ldots, k$, ν_i is the logarithm of the moment-generating function of the point process Z_i.

Throughout this chapter, we assume that all the Z_i's satisfy the assumptions in Section 3.2, i.e., for each $i = 1, 2, \ldots, k$,

(A1″) $\nu_i(a)$ is finite for all $a \in (-\vartheta, \infty)$ for some $\vartheta > 0$.

(A2″) The point process Z_i is *non-trivial*, and the *extinction probability* of the underlying *branching process* is 0, i.e., $\mathbb{P}(N_i = 1) < 1$, $\mathbb{P}(Z_i(\{a\}) = N_i) < 1$ for any $a \in \mathbb{R}$, and $\mathbb{P}(N_i \geq 1) = 1$.

(A3″) N_i has finite $(1 + p)$-th moment for some $p > 0$.

6.2 An Auxiliary Result on the Linear Statistic

As done in the homogeneous setup, here too, we introduce some specific quantities. For each $i = 1, 2, \ldots, k$, we define

$$\theta_{(i)} := \inf \left\{ a > 0 : \frac{\nu_i(a)}{a} = \nu_i'(a) \right\}. \tag{6.2}$$

We also define the linear statistic

$$W_n^{\mathrm{TI}}(\theta)(q_1(n), q_2(n) \ldots, q_k(n), Z_1, Z_2 \ldots, Z_k) := \sum_{|v|=n} e^{\theta S(v)}, \tag{6.3}$$

in the TI-BRW, where the underlying progeny point process is Z_1 for the first $q_1(n)$ generations, Z_2 for the next $q_2(n)$ generations, etc., and Z_k for the last $q_k(n)$ generations.

Then we have

Lemma 6.2.1. *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$ and also for $\theta = \theta_{(1)} < \min_{i \ne 1} \theta_{(i)} \le \infty$,*

$$\frac{W_n^{TI}(\theta)(q_1(n), \ldots, q_k(n), Z_1, \ldots, Z_k) \cdot e^{-\sum_{i=1}^k q_i(n)\nu_i(\theta)}}{W_{q_1(n)}^{TI}(\theta)(q_1(n), Z_1) \cdot e^{-q_1(n)\nu_1(\theta)}} \xrightarrow{p} 1$$

Proof. Without loss of generality we can assume that $\nu_i(\theta) = 0$ for all $i = 1, 2, \ldots, k$. This can be made to satisfy by centering each point process Z_i by $\nu_i(\theta)$.

We prove the lemma by induction on k. We note that for $k = 1$, the lemma holds trivially. We assume that the lemma holds for $k = m - 1$ for some $m \in \mathbb{N}$.

Now, we take $k = m$. For each v such that $|v| = q_1(n)$, we define

$$\overline{W}_{n,v}^{\mathrm{TI}}(\theta)(q_1(n), q_2(n) \ldots, q_m(n), Z_1, Z_2 \ldots, Z_m) = \sum_{|u|=n, v<u} e^{\theta\left(S(u)-S(v)\right)}, \tag{6.4}$$

in the TI-BRW, where the underlying progeny point process is Z_1 for the first $q_1(n)$ generations, Z_2 for the next $q_2(n)$ generations, etc., and Z_m for the last $q_m(n)$ generations. Notice that $\left\{\overline{W}_{n,v}^{\mathrm{TI}}(\theta)(q_1(n), q_2(n) \ldots, q_m(n), Z_1, Z_2 \ldots, Z_m)\right\}_{|v|=q_1(n)}$ are i.i.d.

and have the same distribution as $W^{\text{TI}}_{n-q_1(n)}(\theta)(q_2(n)\ldots,q_m(n),Z_2\ldots,Z_m)$. Now, since $\theta<\min_{i\neq 1}\theta_{(i)}$, by our induction hypothesis,

$$\frac{W^{\text{TI}}_{n-q_1(n)}(\theta)(q_2(n)\ldots,q_m(n),Z_2\ldots,Z_m)}{W^{\text{TI}}_{q_2(n)}(\theta)(q_2(n),Z_2)}\xrightarrow{p}1. \tag{6.5}$$

Note that $W^{\text{TI}}_{q_2(n)}(\theta)(q_2(n),Z_2)$ is essentially the linear statistic in the homogeneous setup where the underlying progeny point process is Z_2, and so by Proposition 3.4.1 (ii), it converges almost surely to some $D^{\infty}_{\theta,(2)}$ which has mean 1. This, together with (6.5), implies

$$W^{\text{TI}}_{n-q_1(n)}(\theta)(q_2(n)\ldots,q_m(n),Z_2\ldots,Z_m)\xrightarrow{p}D^{\infty}_{\theta,(2)}.$$

Since $\nu_i(\theta)=0$ for all $i=1,2,\ldots,m$, $W^{\text{TI}}_{n-q_1(n)}(\theta)(q_2(n)\ldots,q_m(n),Z_2\ldots,Z_m)$ also has mean 1, and therefore we get

$$W^{\text{TI}}_{n-q_1(n)}(\theta)(q_2(n)\ldots,q_m(n),Z_2\ldots,Z_m)\xrightarrow{L_1}D^{\infty}_{\theta,(2)}. \tag{6.6}$$

Now, in the TI-BRW, where the underlying progeny point process is Z_1 for the first $q_1(n)$ generations, Z_2 for the next $q_2(n)$ generations, etc., and Z_m for the last $q_m(n)$ generations, we observe that

$$\begin{aligned}&\frac{W^{\text{TI}}_{n}(\theta)(q_1(n),\ldots,q_m(n),Z_1,\ldots,Z_m)}{W^{\text{TI}}_{q_1(n)}(\theta)(q_1(n),Z_1)}-1\\ =&\sum_{|v|=q_1(n)}\frac{e^{\theta S(v)}}{\sum_{|u|=q_1(n)}e^{\theta S(u)}}\left(\overline{W}^{\text{TI}}_{n,v}(\theta)(q_1(n),q_2(n)\ldots,q_m(n),Z_1,Z_2\ldots,Z_m)-1\right).\end{aligned} \tag{6.7}$$

Since the TI-BRW is homogeneous up to $q_1(n)$-th generation and $\theta\leq\theta_{(1)}$, by (3.18), we get

$$M_{q_1(n)}(\theta)=\max_{|v|=q_1(n)}\frac{e^{\theta S(v)}}{\sum_{|u|=q_1(n)}e^{\theta S(u)}}\xrightarrow{p}0.$$

Let $\mathcal{F}_{q_1(n)}$ be the σ-field generated by $\{S(v) : |v| \leq q_1(n)\}$. Then by Lemma 3.4.2, for every $\varepsilon \in (0, 1/2)$, we have

$$\mathbb{P}\left(\left|\frac{W_n^{\mathrm{TI}}(\theta)(q_1(n),\ldots,q_m(n),Z_1,\ldots,Z_m)}{W_{q_1(n)}^{\mathrm{TI}}(\theta)(q_1(n),Z_1)} - 1\right| > \varepsilon \,\middle|\, \mathcal{F}_n\right)$$

$$\leq \frac{2}{\varepsilon^2}\left(\int_0^{\frac{1}{M_{q_1(n)}(\theta)}} M_{q_1(n)}(\theta)t \cdot \mathbb{P}\left(\left|W_{n-q_1(n)}^{\mathrm{TI}}(\theta)(q_2(n)\ldots,q_m(n),Z_2\ldots,Z_m) - 1\right| > t\right)\,dt\right.$$

$$\left.+ \int_{\frac{1}{M_{q_1(n)}(\theta)}}^{\infty} \mathbb{P}\left(\left|W_{n-q_1(n)}^{\mathrm{TI}}(\theta)(q_2(n)\ldots,q_m(n),Z_2\ldots,Z_m) - 1\right| > t\right)\,dt\right)$$

$$\leq \frac{2}{\varepsilon^2}\left(\int_0^{\infty} \mathbb{P}\left(\left|W_{n-q_1(n)}^{\mathrm{TI}}(\theta)(q_2(n)\ldots,q_m(n),Z_2\ldots,Z_m) - D_{\theta,(2)}^{\infty}\right| > t/2\right)\,dt\right.$$

$$+ \int_0^{\frac{1}{M_{q_1(n)}(\theta)}} M_{q_1(n)}(\theta)t \cdot \mathbb{P}\left(\left|D_{\theta,(2)}^{\infty} - 1\right| > t/2\right)\,dt$$

$$\left.+ \int_{\frac{1}{M_{q_1(n)}(\theta)}}^{\infty} \mathbb{P}\left(\left|D_{\theta,(2)}^{\infty} - 1\right| > t/2\right)\,dt\right) \tag{6.8}$$

By using the dominated convergence theorem, the second and the third term on the right-hand side of (6.8) converges to 0 as $n \to \infty$, and by (6.6), the first term also tends to 0 as $n \to \infty$. Then by taking expectation and using the dominated convergence theorem again, we get

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{W_n^{\mathrm{TI}}(\theta)(q_1(n),\ldots,q_m(n),Z_1,\ldots,Z_m)}{W_{q_1(n)}^{\mathrm{TI}}(\theta)(q_1(n),Z_1)} - 1\right| > \varepsilon\right) = 0,$$

which implies

$$\frac{W_n^{\mathrm{TI}}(\theta)(q_1(n),\ldots,q_m(n),Z_1,\ldots,Z_m)}{W_{q_1(n)}^{\mathrm{TI}}(\theta)(q_1(n),Z_1)} \xrightarrow{p} 1. \tag{6.9}$$

So, if the lemma holds for $k = m - 1$, it also holds for $k = m$. Therefore, by using induction we complete the proof. □

6.3 Convergence Results

6.3.1 Asymptotic limits

Our first two results are centered asymptotic limits of the right-most position, which are similar to Theorems 3.5.3 and 3.5.5 in the homogeneous setup.

Theorem 6.3.1. *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$, there exists a random variable $H^{\infty}_{\theta,(1)}$ depending only on θ and Z_1, such that,*

$$R_n^*(\theta) - \sum_{i=1}^{k} \frac{q_i(n)\nu_i(\theta)}{\theta} \xrightarrow{d} H^{\infty}_{\theta,(1)}. \tag{6.10}$$

Theorem 6.3.2. *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$ and $\theta_{(1)} < \min_{i \ne 1} \theta_{(i)} \le \infty$, then there exists a random variable $H^{\infty}_{\theta_{(1)},(1)}$ depending only on Z_1, such that,*

$$R_n^*\left(\theta_{(1)}\right) - \sum_{i=1}^{k} \frac{q_i(n)\nu_i\left(\theta_{(1)}\right)}{\theta_{(1)}} + \frac{1}{2\theta_{(1)}} \log\left(q_1(n)\right) \xrightarrow{d} H^{\infty}_{\theta_{(1)},(1)}. \tag{6.11}$$

Remark 6.3.1: It is very interesting to note that the centered asymptotic limit only depends on the point process of the first set of displacements. More interestingly, the result is valid as long as $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$. In particular, the rate of divergence of $q_1(n)$ can be very slow but we will still have the centered asymptotic limit depending only on the distribution of Z_1. Thus our model LPMTI-BRW may be used as a very efficient "*statistical sheave*" to filter out the distribution of the first set of displacements (may be thought as the "*signal*") from a number of others which may be considered as "*noise*" and of much larger in numbers compared to that of the "*signal*". We thus feel this result may have greater statistical significance.

Proof of Theorems 6.3.1 amd 6.3.2. Notice that an argument similar to that in the proof of Theorem 2.3.2 gives us

$$\theta R_n^* \stackrel{d}{=} W_n^{\mathrm{TI}}(\theta)(q_1(n), q_2(n) \ldots, q_k(n), Z_1, Z_2 \ldots, Z_k) - \log E, \tag{6.12}$$

where $E \sim$ Exponential (1) and is independent of the TI-BRW.

Since the TI-BRW is homogeneous up to $q_1(n)$-th generation, Proposition 3.4.1 (ii) implies that for any $\theta < \min_i \theta_{(i)} \leq \infty$,

$$W^{\text{TI}}_{q_1(n)}(\theta)(q_1(n), Z_1) \cdot e^{-q_1(n)\nu_1(\theta)} \xrightarrow{p} D^{\infty}_{\theta,(1)}. \tag{6.13}$$

where $D^{\infty}_{\theta,(1)}$ is a positive random variable with mean 1, whose distribution depends only on θ and Z_1. Similarly, (3.15) suggests that for $\theta = \theta_{(1)} < \min_{i\neq 1} \theta_{(i)} \leq \infty$,

$$\sqrt{q_1(n)}\, W^{\text{TI}}_{q_1(n)}(\theta)(q_1(n), Z_1) \cdot e^{-q_1(n)\nu_1(\theta)} \xrightarrow{p} \left(\frac{2}{\pi\sigma_1^2}\right)^{1/2} D^{\infty}_{\theta_{(1)},(1)}, \tag{6.14}$$

where $D^{\infty}_{\theta_{(1)},(1)}$ is also a positive random variable, whose distribution depends only on Z_1. Here, $D^{\infty}_{\theta,(1)}$, $D^{\infty}_{\theta_{(1)},(1)}$, and σ_1^2 are the quantities exactly similar to D^{∞}_{θ}, $D^{\infty}_{\theta_0}$, and σ^2, respectively, discussed in Section 3.4.

Now, by combining (6.12), (6.13), (6.14), and Lemma 6.2.1, we get the required results with

$$H^{\infty}_{\theta,(1)} = \frac{1}{\theta}\left[\log D^{\infty}_{\theta,(1)} - \log E\right], \tag{6.15}$$

and

$$H^{\infty}_{\theta_{(1)},(1)} = \frac{1}{\theta_{(1)}}\left[\log D^{\infty}_{\theta_{(1)},(1)} + \frac{1}{2}\log\left(\frac{2}{\pi\sigma_1^2}\right) - \log E\right], \tag{6.16}$$

where $E \sim$ Exponential (1) and is independent of the TI-BRW. □

Once again, just like in the homogeneous setup, here too, we have a slightly stronger result. As in Theorem 3.5.6, we let

$$\hat{H}^{\infty}_{\theta,(1)} = \frac{1}{\theta}\log D^{\infty}_{\theta,(1)},$$

where $D^{\infty}_{\theta,(1)}$ is the unique solution of the following *linear recursive distributional equation* with mean 1.

$$\Delta \stackrel{\text{d}}{=\!=} \sum_{|v|=1} e^{\theta S(v)-\nu_1(\theta)}\Delta_v, \tag{6.17}$$

where Δ_v are i.i.d. and have the same distribution as that of Δ. As in Theorem 3.5.4, we also let

$$\hat{H}^{\infty}_{\theta_{(1)},(1)} = \frac{1}{\theta_{(1)}}\left[\log D^{\infty}_{\theta_{(1)},(1)} + \frac{1}{2}\log\left(\frac{2}{\pi\sigma_1^2}\right)\right],$$

where

$$D^{\infty}_{\theta_{(1)},(1)} \stackrel{\text{a.s.}}{=\!=} \lim_{n\to\infty} -\sum_{|v|=q_1(n)}\left(\theta_{(1)}S_v - q_1(n)\nu\left(\theta_{(1)}\right)\right)e^{\theta_{(1)}S_v - q_1(n)\nu(\theta_{(1)})}, \tag{6.18}$$

$$\sigma_1^2 := \mathbb{E}\left[\sum_{|v|=1}\left(\theta_{(1)}S_v - \nu\left(\theta_{(1)}\right)\right)^2 e^{\theta_{(1)}S_v-\nu(\theta_{(1)})}\right]. \tag{6.19}$$

Then we have

Theorem 6.3.3. *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$,*

$$R_n^*(\theta) - \sum_{i=1}^{k}\frac{q_i(n)\nu_i(\theta)}{\theta} - \hat{H}^{\infty}_{\theta,(1)} \stackrel{d}{\longrightarrow} -\log E, \tag{6.20}$$

where $E \sim$ Exponential(1).

Theorem 6.3.4. *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$ and $\theta_{(1)} < \min_{i\neq 1}\theta_{(i)} \le \infty$, then*

$$R_n^*\left(\theta_{(1)}\right) - \sum_{i=1}^{k}\frac{q_i(n)\nu_i\left(\theta_{(1)}\right)}{\theta_{(1)}} + \frac{1}{2\theta_{(1)}}\log\left(q_1(n)\right) - \hat{H}^{\infty}_{\theta_{(1)},(1)} \stackrel{d}{\longrightarrow} -\log E, \tag{6.21}$$

where $E \sim$ Exponential(1).

Remark 6.3.2: Note that $H^{\infty}_{\theta,(1)}$ in Theorem 6.3.1 has the same distribution as $\hat{H}^{\infty}_{\theta,(1)} - \log E$, where $E \sim$ Exponential(1) and is independent of $\hat{H}^{\infty}_{\theta,(1)}$. Similarly, $H^{\infty}_{\theta_{(1)},(1)}$ in Theorem 6.3.2 has the same distribution as $\hat{H}^{\infty}_{\theta_{(1)},(1)} - \log E$, where $E \sim$ Exponential(1) and is independent of $\hat{H}^{\infty}_{\theta_{(1)},(1)}$.

Proof of Theorems 6.3.3 and 6.3.4. An argument exactly similar to that in the proof of Theorem 3.5.4 yields that for any $\theta > 0$ in the time inhomogeneous setup,

$$\theta R_n^* - \log W_n^{\mathrm{TI}}(\theta)(q_1(n), q_2(n)\ldots, q_k(n), Z_1, Z_2\ldots, Z_k) \stackrel{d}{=\!=} \log E, \tag{6.22}$$

where $E \sim \text{Exponential}\,(1)$.

Now, by combining Lemma 6.2.1, (6.13), (6.14), and (6.22) we obtain the required results. □

As a corollary of the above results, we obtain that if the centering term converges after dividing by n, then R_n^*/n has a limit in probability. In particular, we have the following result:

Theorem 6.3.5. *If for all $1 \le i \le k$, $q_i(n) \longrightarrow \infty$ satisfying $\lim_{n\to\infty} q_i(n)/n = \alpha_i \ge 0$, then for any $\theta < \min_i \theta_{(i)} \le \infty$ and also for $\theta = \theta_{(1)} < \min_{i\neq 1} \theta_{(i)} \le \infty$,*

$$\frac{R_n^*(\theta)}{n} \xrightarrow{p} \sum_{i=1}^{k} \frac{\alpha_i \nu_i\,(\theta)}{\theta}. \tag{6.23}$$

6.3.2 Brunet-Derrida type results

Here we present results of the type Brunet and Derrida [17] for our LPMTI-BRW.

For any $\theta < \min_i \theta_{(i)} \le \infty$, we define

$$Z_n(\theta) = \sum_{|v|=n} \delta_{\left\{\theta S(v) - \log E_v - \sum_{i=1}^{k} q_i(n)\nu_i(\theta) - \theta \hat{H}^{\infty}_{\theta,(1)}\right\}}, \tag{6.24}$$

and for $\theta_{(1)} < \min_{i\neq 1} \theta_{(i)} \le \infty$, we define

$$Z_n\big(\theta_{(1)}\big) = \sum_{|v|=n} \delta_{\left\{\theta_{(1)} S(v) - \log E_v - \sum_{i=1}^{k} q_i(n)\nu_i(\theta_{(1)}) + \frac{1}{2}\log\left(q_1(n)\right) - \theta_{(1)} \hat{H}^{\infty}_{\theta_{(1)},(1)}\right\}}, \tag{6.25}$$

where $\hat{H}^{\infty}_{\theta,(1)}$ and $\hat{H}^{\infty}_{\theta_{(1)},(1)}$ are as in Theorems 6.3.3 and 6.3.4. Our first result is the weak convergence of the point processes $\left(Z_n\,(\theta)\right)_{n\ge 0}$, which is similar to Theorem 4.3.1 in the homogeneous setup.

Theorem 6.3.6. *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$ and also for $\theta = \theta_{(1)} < \min_{i\neq 1} \theta_{(i)} \le \infty$,*

$$Z_n(\theta) \xrightarrow{d} \mathcal{Y},$$

where $\mathcal{Y}$ is a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}\,dx$.

Remark 6.3.3: Notice that the limiting point process $\mathcal{Y}$ in Theorem 6.3.6 and the limiting point process $\mathcal{Y}$ in Theorem 4.3.1 are exactly the same. This tells us that just like the maximum, the limiting point process also remains unaffected by inhomogeneity.

The following is a slightly weaker version of the above theorem.

Theorem 6.3.7. *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$,*

$$\sum_{|v|=n} \delta_{\left\{\theta S(v) - \log E_v - \sum_{i=1}^k q_i(n)\nu_i(\theta)\right\}} \xrightarrow{d} \sum_{j\ge1} \delta_{\zeta_j + \theta \hat{H}^\infty_{\theta,(1)}}, \tag{6.26}$$

and for $\theta_{(1)} < \min_{i\neq1} \theta_{(i)} \le \infty$,

$$\sum_{|v|=n} \delta_{\left\{\theta_{(1)} S_v - \log E_v - \sum_{i=1}^k q_i(n)\nu_i(\theta_{(1)}) + \frac{1}{2}\log\left(q_1(n)\right)\right\}} \xrightarrow{d} \sum_{j\ge1} \delta_{\zeta_j + \theta_{(1)} \hat{H}^\infty_{\theta_{(1)},(1)}}, \tag{6.27}$$

where $\mathcal{Y} = \sum_{j\ge1} \delta_{\zeta_j}$ is a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}\,dx$, which is independent of the process $\{S(v) : |v| \le n\}$.

Proof of Theorems 6.3.6 and 6.3.7. A similar argument as in the proof of Theorem 4.3.1 and Theorem 4.3.2, together with (6.13), (6.14) and Lemma 6.2.1, yields Theorems 6.3.6 and 6.3.7. □

Let $\mathcal{Y}_{\max}$ be the right-most position of the point process $\mathcal{Y}$, and $\overline{\mathcal{Y}}$ be the point process $\mathcal{Y}$ viewed from its right-most position, i.e.,

$$\overline{\mathcal{Y}} = \sum_{j\ge1} \delta_{\zeta_j - \mathcal{Y}_{\max}}.$$

Then as a corollary of the above theorem, we get the following result, which confirms the validity of the *Brunet-Derrida Conjecture* for LPMTI-BRW for $\theta < \min_i \theta_{(i)} \le \infty$ as well as for for $\theta = \theta_{(1)} < \min_{i\neq1} \theta_{(i)} \le \infty$.

Theorem 6.3.8. *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$ and also for $\theta = \theta_{(1)} < \min_{i \ne 1} \theta_{(i)} \le \infty$,*

$$\sum_{|v|=n} \delta_{\left\{\theta S(v) - \log E_v - \theta R_n^*(\theta)\right\}} \xrightarrow{d} \overline{\mathcal{Y}}.$$

Remark 6.3.4: Notice again that the limiting point process $\overline{\mathcal{Y}}$ in Theorem 6.3.8 and the limiting point process $\overline{\mathcal{Y}}$ in Theorem 4.3.3 are exactly the same.

6.4 A Specific Example

In this section, we consider a *time inhomogeneous Gaussian displacement binary BRW*, which is a specific example of inhomogeneous BRW introduced by Fang and Zeitouni [20]. Here we shall consider the last progeny modified version of the same example. To be precise, let $Z_1 = \delta_{\xi_{11}} + \delta_{\xi_{12}}$, $Z_2 = \delta_{\xi_{21}} + \delta_{\xi_{22}}$, ξ_{11}, ξ_{12} are i.i.d. $\mathrm{N}(0, \sigma_1^2)$, ξ_{21}, ξ_{22} are i.i.d. $\mathrm{N}(0, \sigma_2^2)$ and $q_1(n) = q_2(n) = n/2$. In this case, we have

$$\nu_1(t) = \log 2 + \frac{\sigma_1^2 t^2}{2} \quad \text{and} \quad \nu_2(t) = \log 2 + \frac{\sigma_2^2 t^2}{2},$$

and

$$\theta_1 = \frac{\sqrt{2\log 2}}{\sigma_1} \quad \text{and} \quad \theta_2 = \frac{\sqrt{2\log 2}}{\sigma_2}.$$

Therefore by the Theorem 6.3.2, we obtain that

Theorem 6.4.1. *Assume $\sigma_1 > \sigma_2$, then the following sequence of random variables*

$$R_n^*\left(\frac{\sqrt{2\log 2}}{\sigma_1}\right) - n\left(\sigma_1\sqrt{\frac{\log 2}{2}} + \frac{\sqrt{2\log 2}}{4\sigma_1}\left(\sigma_1^2 + \sigma_2^2\right)\right) + \log n\left(\frac{\sigma_1}{2\sqrt{2\log 2}}\right)$$

converges in distribution to a non-trivial distribution which depends only on σ_1.

As a comparison, we note that in Fang and Zeitouni [20], it is shown that for this example, when $\sigma_1 > \sigma_2$, the following sequence of random variables

$$R_n - n\left((\sigma_1 + \sigma_2)\sqrt{\frac{\log 2}{2}}\right) + \log n\left(\frac{3(\sigma_1 + \sigma_2)}{2\sqrt{2\log 2}}\right)$$

is tight.

Thus for our model, we have been able to establish more than Fang and Zeitouni [20] as we obtain a weak limit for the right-most position of the LMPTI-BRW after an appropriate centering. However, we only have this for the case when $\sigma_1 > \sigma_2$. As mentioned in Theorem 1.4.12, the other case when $\sigma_1 < \sigma_2$ has also been worked out by Fang and Zeitouni [20], and the tightness of the right-most position has been proved with an appropriate centering. Similar results were studied by Bovier and Hartung [13] for the two-speed BBM, where the initial speed σ_1 changes to σ_2 after some time.

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