Tridiagonal shifts and Analytic perturbations

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Contents

Notations & Abbreviations

$\mathbb{N}$	Set of all Natural numbers.
$\mathbb{Z}_+$	the set of all non-negative integers.
$\mathbb{C}$	Set of all complex numbers.
$H, E, \mathcal{E}, \mathcal{H}, \mathcal{W}$	Hilbert spaces.
$\mathbb{C}f_0$	the linear subspace generated by the single vector $f_0 \in \mathcal{H}$.
$[E]_T$	smallest closed linear subspace containing E.
$\langle , \rangle$	The inner product of a Hilbert space.
$\mathcal{O}(\Omega, \mathcal{E})$	The set of all holomorphic functions on $\Omega \subseteq \mathbb{C}$ to $\mathcal{E}$.
$\mathcal{O}(\Omega)$	The set of all holomorphic functions on $\Omega \subseteq \mathbb{C}$ to $\mathbb{C}$.
$\mathbb{D}$	The open unit disk in the complex plane.
$\mathcal{H}_k$	The reproduncing kernel Hilbert space with kernel k.
$\mathbb{C}[z]$	The ring of all polynomials over $\mathbb{C}$.
$H^2(\mathbb{D})$	the Hardy space over $\mathbb{D}$
$H^\infty(\mathbb{D})$	The algebra of multipliers of the Hardy space.
T^*	the adjoint of the operator T on a Hilbert space.
I	the identity operator on a Hilbert space.
$\mathcal{B}(\mathcal{H})$	the algebra of all bounded linear operators on $\mathcal{H}$.

Chapter 1

Introduction

This thesis deals with the merge of a number of operator and function theoretic concepts, namely, reproducing kernels, Aluthge transforms, left invertible operators, compact perturbations of isometries, and invariant subspaces of finite rank perturbations of isometries. Our study intends to contribute equally to these subjects.

The main contributions of this thesis are:

1. *Tridiagonal kernels and left-invertible operators with applications to Aluthge transform:* We study left-invertible operators on tridiagonal spaces and present computational approach to the theory of Aluthge transform. Given scalars $a_n(\neq 0)$ and b_n, $n \geq 0$, the tridiagonal kernel or band kernel with bandwidth 1 is the positive definite kernel k on the open unit disc $\mathbb{D}$ defined by

$$k(z,w) = \sum_{n=0}^{\infty} \Big((a_n + b_n z)z^n\Big)\Big((\bar{a}_n + \bar{b}_n \bar{w})\bar{w}^n\Big) \qquad (z, w \in \mathbb{D}).$$

   This defines a reproducing kernel Hilbert space $\mathcal{H}_k$ (known as tridiagonal space) of analytic functions on $\mathbb{D}$ with $\{(a_n + b_n z)z^n\}_{n=0}^{\infty}$ as an orthonormal basis. We consider shift operators M_z on $\mathcal{H}_k$ and prove that M_z is left-invertible if and only if $\{|a_n/a_{n+1}|\}_{n\geq 0}$ is bounded away from zero. We find that, unlike the case of weighted shifts, Shimorin's models for left-invertible operators fail to bring to the foreground the tridiagonal structure of shifts. In fact, the tridiagonal structure of a kernel k, as above, is preserved under Shimorin model if and only if $b_0 = 0$ or that M_z is a weighted shift. We prove concrete classification results concerning invariance of tridiagonality of kernels, Shimorin models, and positive operators.

   We also develop a computational approach to Aluthge transforms of shifts. Curiously, in contrast to direct kernel space techniques, often Shimorin models fails to yield tridiagonal Aluthge transforms of shifts defined on tridiagonal spaces.

2. *Invariant subspaces of analytic perturbations:* By analytic perturbations, we refer to shifts that are finite rank perturbations of the form $M_z + F$, where M_z is the unilateral shift and F is a finite rank operator on the Hardy space over the open unit disc. Here shift refers to the multiplication operator M_z on some analytic reproducing kernel Hilbert space. Here, we first isolate a natural class of finite rank operators for which the corresponding perturbations are analytic, and then we present a complete classification of invariant subspaces of those analytic perturbations. We also exhibit some instructive examples and point out several distinctive properties (like cyclicity, essential normality, hyponormality, etc.) of analytic perturbations.

3. *Tridiagonal shifts as compact + isometry:* We consider the tridiagonal kernel k on $\mathbb{D}$ as above. Denote by M_z the multiplication operator on the reproducing kernel Hilbert space corresponding to the kernel k. Assume that M_z is left-invertible. We prove that $M_z =$ compact + isometry if and only if
$$\Big|\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}\Big| \to 0,$$
and
$$\Big|\frac{a_n}{a_{n+1}}\Big| \to 1.$$

4. *Left-invertibility of rank-one perturbations:* For each isometry V acting on some Hilbert space and a pair of vectors f and g in the same Hilbert space, we associate a nonnegative number $c(V; f, g)$ defined by
$$c(V; f, g) = (\|f\|^2 - \|V^* f\|^2)\|g\|^2 + |1 + \langle V^* f, g\rangle|^2.$$
We prove that the rank-one perturbation $V + f \otimes g$ is left-invertible if and only if
$$c(V; f, g) \neq 0.$$
We also consider examples of rank-one perturbations of isometries that are shift on some Hilbert space of analytic functions. Here, shift refers to the operator of multiplication by the coordinate function z. Finally, we examine $D + f \otimes g$, where D is a diagonal operator with nonzero diagonal entries and f and g are vectors with nonzero Fourier coefficients. We prove that $D + f \otimes g$ is left-invertible if and only if $D + f \otimes g$ is invertible.

In this thesis, all Hilbert spaces will be separable and over $\mathbb{C}$. Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ will denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We will simply write $\mathcal{B}(\mathcal{H})$ whenever $\mathcal{H} = \mathcal{K}$.

Let us now elaborate on the above content chapter-wise. This thesis contains four independent but closely related chapters (excluding the preliminary chapter):

Chapter 3: Tridiagonal kernels and left-invertible operators with applications to Aluthge transform.

The theory of left-invertible weighted shifts or multiplication operators M_z on "diagonal" reproducing kernel Hilbert spaces is one of the most useful in operator theory, function theory, and operator algebras (see the classic by Shields [54]). Given a bounded sequence of positive real numbers $w = \{w_n\}_{n\geq 0}$, and an orthonormal basis $\{e_n\}_{n\geq 0}$ of an infinite-dimensional Hilbert space $\mathcal{H}$ (complex separable), the operator S_w defined by

$$S_w e_n = w_n e_{n+1} \qquad (n \geq 0),$$

is called a *weighted shift* with weights $\{w_n\}_{n\geq 0}$. In this case, S_w is bounded ($S_w \in \mathcal{B}(\mathcal{H})$ in short) and $\|S_w\| = \sup_n w_n$. If the sequence $\{w_n\}_{n\geq 0}$ is bounded away from zero, then S_w is a left-invertible but non-invertible operator. Note that the multiplication operator M_z on (most of the) diagonal reproducing kernel Hilbert spaces is the function theoretic counterpart of left-invertible weighted shifts which includes the Dirichlet shift, the Hardy shift, and the weighted and unweighted Bergman shifts, etc.

The main focus of this chapter (as well as some other parts of this thesis) is to study shifts on the "next best" concrete analytic kernels, namely, tridiagonal kernels. This notion was introduced by Adams and McGuire [3] in 2001 (also see the motivating paper by Adams, McGuire and Paulsen [4]). However, in spite of its natural appearance and potential applications, far less attention has been paid to the use of tridiagonal kernels in the aforementioned subjects. On the other hand, Shimorin [56] developed the idea of analytic models of left-invertible operators at about the same time as Adams and McGuire, which has been put forth as a key model for left-invertible operators by a number of researchers [20, 35, 58, 48].

In this chapter we consider the next level of shifts on tridiagonal spaces, namely left-invertible shifts on tridiagonal spaces. We also discuss the pending and inevitable comparisons between Shimorin's analytic models of left-invertible operators and Adams and McGuire's theory of left-invertible shifts on tridiagonal spaces. In particular (and curiously enough), we find that, unlike the case of weighted shifts, Shimorin models fail to bring to the foreground the tridiagonal structure of shifts. We resolve this dilemma by presenting a complete classification of tridiagonal kernels that are preserved under Shimorin models.

We also prove a number of results concerning left-invertible properties of shifts on tridiagonal spaces, new tridiagonal spaces from the old, classifications of quasinormal operators, rank-one perturbations of left inverses, a computational approach to Aluthge transforms of shifts, etc. Again, curiously enough, some of our definite computations in the setting of tridiagonal kernels verify that the direct reproducing kernel Hilbert space technique is somewhat more powerful than Shimorin models. We also provide a family of instructive examples and supporting counterexamples. We believe that some of our results and approaches may be of independent interest and may find additional applications.

To demonstrate the main contribution of this chapter, it is now necessary to disambiguate central concepts. Needless to say, the theory of reproducing kernel Hilbert spaces will play a central role in this thesis. Briefly stated, the essential idea of reproducing kernel Hilbert space [10] is to single out the role of positive definiteness of inner products, multipliers and bounded point evaluations of function Hilbert spaces. We denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disc in $\mathbb{C}$. Let $\mathcal{E}$ be a Hilbert space. A function $k : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E})$ is called an *analytic kernel* if k is positive definite, that is,

$$\sum_{i,j=1}^{n} \langle k(z_i, z_j)\eta_j, \eta_i\rangle_{\mathcal{E}} \geq 0,$$

for all $\{z_i\}_{i=1}^n \subseteq \mathbb{D}$, $\{\eta_i\}_{i=1}^n \subseteq \mathcal{E}$ and $n \in \mathbb{N}$, and k analytic in the first variable. In this case there exists a Hilbert space $\mathcal{H}_k$, which we call *analytic reproducing kernel Hilbert space* (*analytic Hilbert space*, in short), of $\mathcal{E}$-valued analytic functions on $\mathbb{D}$ such that $\{k(\cdot, w)\eta : w \in \mathbb{D}, \eta \in \mathcal{E}\}$ is a total set in $\mathcal{H}_k$ with the *reproducing* property $\langle f, k(\cdot, w)\eta\rangle_{\mathcal{H}_k} = \langle f(w), \eta\rangle_{\mathcal{E}}$ for all $f \in \mathcal{H}_k$, $w \in \mathbb{D}$, and $\eta \in \mathcal{E}$. The *shift operator* on $\mathcal{H}_k$ is the multiplication operator M_z (which will be assumed to be bounded) defined by

$$(M_z f)(w) = wf(w) \qquad (f \in \mathcal{H}_k, w \in \mathbb{D}).$$

Note that there exist $C_{mn} \in \mathcal{B}(\mathcal{E})$ such that $k(z, w) = \sum_{m,n=0}^{\infty} C_{mn} z^m \bar{w}^n$, $z, w \in \mathbb{D}$. We say that $\mathcal{H}_k$ is a diagonal reproducing kernel Hilbert space (and k is a *diagonal kernel*) if $C_{mn} = 0$ for all $|m - n| \geq 1$. We say that k is a *tridiagonal kernel* (or *band kernel with bandwidth* 1) if

$$C_{mn} = 0 \qquad (|m - n| \geq 2).$$

In this case, we say that $\mathcal{H}_k$ is a *tridiagonal space*. Now let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be a sequences of scalars. In this thesis, we will always assume that $a_n \neq 0$, for all $n \geq 0$. Set

$$f_n(z) = (a_n + b_n z)z^n \qquad (n \geq 0).$$

Assume that $\{f_n\}_{n\geq 0}$ is an orthonormal basis of an analytic Hilbert space $\mathcal{H}_k$. Then $\mathcal{H}_k$ is a tridiagonal space, as the well known fact from the reproducing kernel theory implies that

$$k(z, w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} \qquad (z, w \in \mathbb{D}). \tag{1.0.1}$$

A linear operator V on $\mathcal{H}$ is an *isometry* if $\|Vh\| = \|h\|$ for all $h \in \mathcal{H}$, or equivalently

$$V^*V = I_{\mathcal{H}}.$$

Along this line, left-invertible operators (also known as, by a slight abuse of terminology, "operators close to an isometry" [56]) are also natural examples of noncompact operators: $T \in \mathcal{B}(\mathcal{H})$ is *left-invertible* if T is bounded below, that is, there exists $\epsilon > 0$ such that $\|Th\| \geq \epsilon\|h\|$ for all $h \in \mathcal{H}$, or equivalently, there exists $S \in \mathcal{B}(\mathcal{H})$ such that

$$ST = I_{\mathcal{H}}.$$

We now turn to Shimorin's analytic model of left-invertible operators [56], which says that if $T \in \mathcal{B}(\mathcal{H})$ is left-invertible and analytic (that is, $\cap_{n=0}^{\infty} T^n\mathcal{H} = \{0\}$), then there exists an analytic Hilbert space $\mathcal{H}_k (\subseteq \mathcal{O}(\mathbb{D}, \mathcal{W}))$ such that T and M_z on $\mathcal{H}_k$ are unitarily equivalent, where $\mathcal{W} = \ker T^* = \mathcal{H} \ominus T\mathcal{H}$ is the *wandering subspace* of T, and $\mathcal{O}(\mathbb{D}, \mathcal{W})$ is the set of $\mathcal{W}$-valued analytic functions on $\mathbb{D}$. The *Shimorin kernel* k is explicit which involves the *Shimorin left inverse*

$$L_T = (T^*T)^{-1}T^*, \tag{1.0.2}$$

of T. The representation of the Shimorin kernel is useful in studying wandering subspaces of invariant subspaces of weighted shifts [55, 56]. See [38, Chapter 6] and [52] in the context of the wandering subspace problem, and [48] and the extensive list of references therein for recent developments and implementations of Shimorin models.

An *analytic tridiagonal kernel* is a scalar kernel k as in (1.0.1) such that $\mathbb{C}[z] \subseteq \mathcal{H}_k$, and

$$\sup_{n\geq 0}\left|\frac{a_n}{a_{n+1}}\right| < \infty \quad \text{and} \quad \limsup_{n\geq 0}\left|\frac{b_n}{a_{n+1}}\right| < 1,$$

(which ensures that M_z on $\mathcal{H}_k$ is bounded) and $\{|\frac{a_n}{a_{n+1}}|\}_{n\geq 0}$ is bounded away from zero. An analytic Hilbert space is called *analytic tridiagonal space* if the kernel function is an analytic tridiagonal kernel.

Now we turn to Aluthge transforms. The notion of Aluthge transforms was introduced by Aluthge [7] in his study of p-hyponormal operators. Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$, and let $T = U|T|$ be the polar decomposition of T. Here, and throughout this note, $|T| = (T^*T)^{\frac{1}{2}}$ and U is the unique partial isometry such that $\ker U = \ker T$. The *Aluthge transform* of T is the bounded linear operator

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

The Aluthge transform of $\tilde{T}$ turns T into a more "normal" operator while keeping intact the basic spectral properties of T [40]. Evidently, the main difficulty associated with $\tilde{T}$ is to compute or represent the positive part $|T|$. This is certainly not true for weighted shifts: Since $|S_w| = \text{diag}(w_0, w_1, w_2, \ldots)$, it follows that $\tilde{S_w} = S_{\sqrt{w}}$, where

$$\sqrt{w} := \{\sqrt{w_0w_1}, \sqrt{w_1w_2}, \ldots\}.$$

Therefore, $\tilde{S}_w$ is also a weighted shift, namely $S_{\sqrt{w}}$.

We prove the following set of results:

(a) Weighted shifts behave well under Shimorin's analytic models.

(b) $\{|\frac{a_n}{a_{n+1}}|\}_{n\geq 0}$ is bounded away from zero is equivalent to the fact that M_z on $\mathcal{H}_k$ is left-invertible.

(c) Representations of Shimorin left inverses of shifts on analytic tridiagonal spaces.

(d) Shimorin kernels do not necessarily preserve the tridiagonal structure of kernels. However, it does for a kernel k of the form (1.0.1) if and only if M_z on $\mathcal{H}_k$ is a weighted shift or
$$b_0 = 0.$$

(e) Classification of positive operators P on a tridiagonal space $\mathcal{H}_k$ such that $K(z,w) := \langle Pk(\cdot,w), k(\cdot,z)\rangle_{\mathcal{H}_k}$ defines a tridiagonal kernel on $\mathbb{D}$. More specifically, if
$$P = \begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{03} & \cdots \\ \bar{c}_{01} & c_{11} & c_{12} & c_{13} & \ddots \\ \bar{c}_{02} & \bar{c}_{12} & c_{22} & c_{23} & \ddots \\ \bar{c}_{03} & \bar{c}_{13} & \bar{c}_{23} & c_{33} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$
denote the matrix representation of P with respect to the basis $\{(a_n + b_n z)z^n\}_{n\geq 0}$ of $\mathcal{H}_k$, then the kernel K is tridiagonal if and only if $c_{0n} = (-1)^{n-1}\frac{\bar{b}_1\cdots\bar{b}_{n-1}}{\bar{a}_2\cdots\bar{a}_n}c_{01}$, $n \geq 2$, and $c_{mn} = (-1)^{n-m-1}\frac{\bar{b}_{m+1}\cdots\bar{b}_{n-1}}{\bar{a}_{m+2}\cdots\bar{a}_n}c_{m,m+1}$ for all $1 \leq m \leq n-2$.

(f) Suppose M_z is non-normal on an analytic tridiagonal space $\mathcal{H}_k$. Denote by $P_{\mathbb{C}f_0}$ the orthogonal projection of $\mathcal{H}_k$ onto $\mathbb{C}f_0$. Then M_z is quasinormal if and only if there exists $r > 0$ such that
$$M_z^* M_z - M_z M_z^* = rP_{\mathbb{C}f_0}.$$

(g) Computation of $\tilde{M}_z$, where M_z is a left-invertible shift on some analytic Hilbert space $\mathcal{H}_k$. We prove that $\tilde{M}_z$ is also a left-invertible shift on some analytic Hilbert space $\mathcal{H}_{\tilde{k}}$. The kernel $\tilde{k}$ can be obtained either via Shimorin's model, which we call the *Shimorin-Aluthge kernel of* M_z, or by a direct approach, which we call the *standard Aluthge kernel of* M_z. We prove that if $\mathbb{C}[z] \subseteq \mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D})$, then L_{M_z} and $L_{\tilde{M}_z}$ are similar up to the perturbation of an operator of rank at most one. Moreover, in this setting Shimorin-Aluthge kernels are somewhat more explicit.

(h) We consider truncated spaces (subclass of analytic tridiagonal spaces) in order to pinpoint more definite results, instructive examples, and counterexamples. The computational advantage of a truncated space is that it annihilate a rank one operator associated with L_{M_z} of the shift M_z. As a result, in this case we are able to prove a complete classification of tridiagonal Shimorin-Aluthge kernels of shifts.

(i) We also comment on the assumptions in the definition of truncated kernels. We point out, at the other extreme, if one consider a (non-truncated) tridiagonal kernel k with

$$b_0 = b_1 = 1 \text{ or } b_0 = 1,$$

and all other b_i's are equal to 0, then the standard Aluthge kernel of M_z is a tridiagonal but the Shimorin-Aluthge kernel of M_z is not.

We remark that some of the observations outlined in this chapter are based on several more general results that have an independent interest in broader operator theory and function theoretic contexts.

Chapter 4: Invariant subspaces of analytic perturbations.

Note that the main aim of perturbation theory is to study (and also compare the properties of)

$$S = T + F,$$

where T is a tractable operator (like unitary, normal, isometry, self-adjoint, etc.) and F is a finite rank (or compact, Hilbert–Schmidt, Schatten-von Neumann class, etc.) operator on some Hilbert space.

In this chapter, we propose an analytic approach to perturbation theory, namely, we study analytic perturbations of the unilateral shift on the Hardy space $H^2(\mathbb{D})$. Recall that the unilateral shift is an isometry on $H^2(\mathbb{D})$, which is also the most well-known example of a non-normal operator on infinite-dimensional Hilbert spaces. From this point of view (and also as a part of the main motivations), we examine the above-mentioned problem by replacing the normal operator with the unilateral shift. More specifically, along with other natural properties, we deal with closed invariant subspaces of "shift" operators of the form

$$S_n = M_z + F,$$

where M_z denotes the unilateral shift and F is a finite rank operator (of rank $\leq n$) on $H^2(\mathbb{D})$. We call a bounded linear operator S acting on a Hilbert space a *shift* if S is unitarily equivalent to M_z on some analytic Hilbert space, where M_z denote the multiplication operator by the coordinate function z. In this chapter, analytic Hilbert spaces will refer to reproducing kernel Hilbert spaces of analytic functions on $\mathbb{D}$. The unilateral shift M_z on $H^2(\mathbb{D})$ is a natural example (which is also a model example of isometry) of shift.

Now the classification of invariant subspaces of the unilateral shift is completely known, thanks to the classical work of Beurling [15]: A nonzero closed subspace $\mathcal{M} \subseteq$

$H^2(\mathbb{D})$ is invariant under M_z if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that

$$\mathcal{M} = \theta H^2(\mathbb{D}).$$

We use the standard notation $H^\infty(\mathbb{D})$ to denote the Banach algebra of all bounded analytic functions on $\mathbb{D}$.

In this chapter, we first introduce a class of finite rank operators F (we call them *n-perturbations*) on $H^2(\mathbb{D})$ for which the corresponding perturbations $S_n = M_z + F$ are shifts (we call them *n-shifts*). Then we present a complete classification of S_n-invariant closed subspaces of $H^2(\mathbb{D})$. Note again that S_n is unitarily equivalent to the multiplication operator M_z on some analytic Hilbert space.

Our central result of this chapter is the following invariant subspace theorem:

Theorem 1.0.1. *Let $S_n = M_z + F$ on $H^2(\mathbb{D})$ be an n-shift, and let $\mathcal{M}$ be a nonzero closed subspace of $H^2(\mathbb{D})$. Then $\mathcal{M}$ is invariant under S_n if and only if there exist an inner function $\theta \in H^\infty(\mathbb{D})$ and polynomials $\{p_i, q_i\}_{i=0}^{n-1} \subseteq \mathbb{C}[z]$ such that*

$$\mathcal{M} = (\mathbb{C}\varphi_0 \oplus \mathbb{C}\varphi_1 \oplus \cdots \oplus \mathbb{C}\varphi_{n-1}) \oplus z^n \theta H^2(\mathbb{D}),$$

where $\varphi_i = z^i p_i \theta - q_i$ for all $i = 0, \ldots, n-1$, and

$$S_n \varphi_j \in (\mathbb{C}\varphi_{j+1} \oplus \cdots \oplus \mathbb{C}\varphi_{n-1}) \oplus z^n \theta H^2(\mathbb{D}),$$

for all $j = 0, \ldots, n-2$, and $S_n \varphi_{n-1} = z^n p_{n-1} \theta$.

The above classification is based on a result of independent interest:

Theorem 1.0.2. *If $\mathcal{M}$ is a nonzero closed S_n-invariant subspace of $H^2(\mathbb{D})$, then*

$$dim(\mathcal{M} \ominus S_n \mathcal{M}) = 1.$$

Clearly, this is a Burling-type property of S_n-invariant subspaces.

We remark that a priori examples of n-shifts may seem counter-intuitive because of the intricate structure of perturbed linear operators. Subsequently, we put special emphasis on natural examples of n-shifts, and as interesting as it may seem, analytic spaces corresponding to truncated tridiagonal kernels or band kernels with bandwidth 1 give several natural examples of n-shifts. In the special case when S_n is unitarily equivalent to a shift on an analytic space corresponding to a band truncated kernel with bandwidth 1, we prove that the invariant subspaces of S_n are also hyperinvariant. Our proof of this fact follows a classical route: computation of commutants of shifts. In general, it is a difficult problem to compute the commutant of a shift (even for weighted shifts). However, in our band truncated kernel case, we are able to explicitly compute the commutant of n-shifts:

$$\{S_n\}' = \{T_\varphi + N : \varphi \in H^\infty(\mathbb{D}), \text{rank} N \leq n\},$$

where T_φ denotes the analytic Toeplitz operator with symbol $\varphi \in H^\infty(\mathbb{D})$, and N admits an explicit (and restricted) representation. We also present concrete examples of 1-shifts on tridiagonal kernel spaces with special emphasis on cyclicity of invariant subspaces. For instance, a simple example of S_1-shift brings out the following distinctive properties:

1. $[S_1^*, S_1] := S_1^* S_1 - S_1 S_1^*$ is of finite rank (in particular, S_1 is essentially normal).
2. S_1 is not subnormal (and, more curiously, not even hyponormal).
3. Invariant subspaces of S_1 are cyclic.

We remark that perturbations of concrete operators (with some analytic flavor) have been also studied in different contexts by other authors. For instance, see [34, 44, 53], and notably Clark [21].

Chapter 5: Tridiagonal shifts as compact + isometry.

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is called *semi-Fredholm* if the range space ran$T$ is closed and at least one of the spaces $\ker T$ and $\ker T^*$ is of finite dimension. If T is semi-Fredholm then

$$ind(T) = \dim \ker T - \dim \ker T^*,$$

is called the *index* of T. We shall always assume that our Hilbert spaces are separable and over $\mathbb{C}$. The starting point of our present note is the following classification of compact perturbations of isometries [33, page 191]:

Theorem 1.0.3 (Fillmore, Stampfli, and Williams)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then T = compact + isometry if and only if $I - T^*T$ is compact and T is semi-Fredholm with $ind(T) \leq 0$.*

Here we are interested in a quantitative version of the above theorem. For instance, consider a bounded sequence of non-zero scalars $\{w_n\}_{n\geq 0}$ and an infinite-dimensional Hilbert space $\mathcal{H}$ with an orthonormal basis $\{e_n\}_{n\geq 0}$. Then the weighted shift S_w defined by

$$S_w(e_n) = w_n e_{n+1} \qquad (n \geq 0),$$

is in $\mathcal{B}(\mathcal{H})$ with $\|S_w\| = \sup_n |w_n|$. Let the weight sequence $\{w_n\}$ be bounded away from zero. Since $\ker S_w = \{0\}$ and $\ker S_w^* = \{e_0\}$, it follows that S_w is semi-Fredholm and $\text{ind}(S_w) = -1$. Moreover, using the fact that $S_w^* e_0 = 0$ and $S_w^* e_n = \bar{w}_{n-1} e_{n-1}$, $n \geq 1$, it follows that

$$I - S_w^* S_w = \text{diag}(1 - |w_0|^2, 1 - |w_1|^2, \dots).$$

Theorem 5.0.1 then readily implies that

$$\lim_{n\to\infty} |w_n| = 1 \text{ if and only if } S_w = \text{compact} + \text{isometry}. \tag{1.0.3}$$

We note that since in this case the weight sequence is bounded away from zero, S_w is necessarily left-invertible.

Also note that S_w is a concrete example of a left-invertible shift on an analytic Hilbert space. A standard computation now reveals that S_w, under some appropriate assumption on the weight sequence $\{w_n\}_{n\geq 0}$ [54, proposition 7], is unitarily equivalent to M_z on a diagonal space. Therefore (5.0.1) yields a quantitative classification of shifts on diagonal spaces that are compact perturbations of isometries. This motivates the following natural question:

QUESTION 1. *Is it possible to find a quantitative classification of left-invertible shifts on analytic Hilbert spaces that are compact perturbations of isometries?*

The main purpose of this chapter is to provide an answer to the above question for the case of M_z on tridiagonal spaces. As earlier, we fix sequences of scalars $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ with the assumption that $a_n \neq 0$, $n \geq 0$. We set

$$f_n(z) = (a_n + b_n z)z^n \qquad (n \geq 0),$$

and consider the Hilbert space $\mathcal{H}_k$ with $\{f_n\}_{n\geq 0}$ as an orthonormal basis. Then $\mathcal{H}_k$ is a tridiagonal space corresponding to the tridiagonal kernel

$$k(z, w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} \qquad (z, w \in \mathbb{D}).$$

We assume that $\{|\frac{a_n}{a_{n+1}}|\}_{n\geq 0}$ is bounded away from zero and

$$\sup_{n\geq 0} |\frac{a_n}{a_{n+1}}| < \infty \text{ and } \limsup_{n\geq 0} |\frac{b_n}{a_{n+1}}| < 1.$$

Recall that the latter two assumptions ensure that M_z on $\mathcal{H}_k$ is bounded, whereas the first assumption implies that M_z is left-invertible. In this case we also call M_z a *tridiagonal shift*.

The following is the answer to Question 3 for tridiagonal shifts (as well as the main theorem of this chapter):

Theorem 1.0.4. *Let M_z be the tridiagonal shift on $\mathcal{H}_k$. Then M_z = compact+ isometry if and only if $|\frac{a_n}{a_{n+1}}| \to 1$ and $|\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}| \to 0$.*

We also offer a general (but abstract) classification of shifts that are compact perturbations of isometries.

Proposition 1.0.5. *Let $\mathcal{H}_k$ be an analytic Hilbert space. Suppose the shift M_z on $\mathcal{H}_k$ is left-invertible and of finite index. Define C on $\mathcal{H}_k$ by*

$$(Cf)(w) = \langle f, (1 - z\bar{w})k(\cdot, w)\rangle_{\mathcal{H}_k} \qquad (f \in \mathcal{H}_k, w \in \mathbb{D}).$$

Then M_z = compact + isometry if and only if C defines a compact operator on $\mathcal{H}_k$.

Chapter 6: Left-invertibility of rank-one perturbations.

Rank-one operators are the simplest as well as easy to spot among all bounded linear operators on Hilbert spaces. Indeed, for each pair of nonzero vectors f and g in a Hilbert space $\mathcal{H}$, one can associate a *rank-one* operator $f \otimes g \in \mathcal{B}(\mathcal{H})$ defined by

$$(f \otimes g)h = \langle h, g\rangle f \qquad (h \in \mathcal{H}).$$

These are the only operators whose range spaces are one-dimensional. Here $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. Note that finite-rank operators, that is, linear sums of rank-one operators are norm dense in the ideal of compact operators, where one of the most important and natural examples of a noncompact operator is an isometry. The intent of this chapter is to make a modest contribution to the delicate structure of rank-one perturbations of bounded linear operators [41]. More specifically, this chapter aims to introduce some methods for the left-invertibility of rank-one perturbations of isometries and, to some extent, diagonal operators. The following is the central question that interests us:

QUESTION 2. *Find necessary and sufficient conditions for left-invertibility of the rank-one perturbation $V + f \otimes g$, where $V \in \mathcal{B}(\mathcal{H})$ is an isometry or a diagonal operator and f and g are vectors in $\mathcal{H}$.*

The answer to this question is completely known for isometries. Given an isometry $V \in \mathcal{B}(\mathcal{H})$ and vectors $f, g \in \mathcal{H}$, the perturbation $X = V + f \otimes g$ is an isometry if and only if there exist a unit vector $h \in \mathcal{H}$ and a scalar α of modulus one such that $f = (\alpha - 1)h$ and $g = V^*h$. In other words, a rank-one perturbation X of the isometry V is an isometry if and only if there exists a unit vector $f \in \mathcal{H}$ and a scalar α of modulus one such that

$$X = V + (\alpha - 1)f \otimes V^*f. \tag{1.0.4}$$

This result is due to Nakamura [44, 43] (and also see [53]). For more on rank-one perturbations of isometries and related studies, we refer the reader to [13, 22, 21, 31, 34] and also [39].

In this chapter, we extend the above idea to a more general setting of left-invertibility of rank-one perturbations of isometries. In this case, however, left-invertibility of rank-one perturbations of isometries completely relies on certain real numbers. More specifically, given an isometry $V \in \mathcal{B}(\mathcal{H})$ and a pair of vectors f and g in $\mathcal{H}$, we associate a real number $c(V; f, g)$ defined by

$$c(V; f, g) = (\|f\|^2 - \|V^*f\|^2)\|g\|^2 + |1 + \langle V^*f, g\rangle|^2. \tag{1.0.5}$$

This is the number which precisely determine the left-invertibility of $V + f \otimes g$:

Theorem 1.0.6. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry, and let f and g be vectors in $\mathcal{H}$. Then $V + f \otimes g$ is left-invertible if and only if*

$$c(V; f, g) \neq 0.$$

Note that since V is an isometry, we have $\|V^*f\| \leq \|f\|$, and hence, the quantity $c(V; f, g)$ is always nonnegative. Therefore, the condition $c(V; f, g) \neq 0$ in the above theorem can be rephrased as saying that $c(V; f, g) > 0$, or equivalently, $\|V^*f\| < \|f\|$ or $1 + \langle V^*f, g\rangle \neq 0$. However, in what follows, we will keep the constant $c(V; f, g)$ in our consideration. Not only $c(V; f, g)$ plays a direct role in the proof of the above theorem but this quantity also appears in the explicit representation of a left inverse of a left-invertible perturbation.

The following conclusion is a simple variation of the above theorem:

Corollary 1.0.7. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry, and let f and g be vectors in $\mathcal{H}$. Then $V + f \otimes g$ is not left-invertible if and only if*

$$\|V^*f\| = \|f\| \text{ and } \langle V^*f, g\rangle = -1.$$

The above theorem also provides us with a rich source of natural examples of left-invertible operators. For instance, let us denote by $\mathbb{D}$ the open unit disc in $\mathbb{C}$. Consider the shift M_z on the $\mathcal{E}$-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ over $\mathbb{D}$, where $\mathcal{E}$ is a Hilbert space. Then for any

$$\eta \in \ker M_z^* = \mathcal{E} \subseteq H^2_{\mathcal{E}}(\mathbb{D}),$$

and nonzero vector $g \in H^2_{\mathcal{E}}(\mathbb{D})$, the rank-one perturbation $M_z + \eta \otimes g$ is left-invertible. A similar conclusion holds if $f, g \in H^2(\mathbb{D})$ and

$$\langle M_z^* f, g\rangle \neq -1.$$

We discuss a follow-up question: Characterizations of shifts that are rank-one perturbations of isometries. Here a shift refers to the multiplication operator M_z on some Hilbert space of analytic functions (that is, a reproducing kernel Hilbert space) on a domain in $\mathbb{C}$. Note, however, that our analysis will be mostly limited to the level of elementary examples.

We also study rank-one perturbations of diagonal operators. It is well known that the structure of rank-one perturbations of diagonal operators is also complicated (cf.

[6, 31, 32, 39]). Moreover, comparison between perturbations of diagonal operators and that of isometries is perhaps inevitable if one views diagonals as normal operators and isometries as one of the best tractable non-normal operators. Here we consider $D + f \otimes g$ on some Hilbert space $\mathcal{H}$, where D is a diagonal operator with nonzero diagonal entries with respect to an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of $\mathcal{H}$. We also assume that the Fourier coefficients of f and g with respect to $\{e_n\}_{n=0}^{\infty}$ are nonzero. We prove:

Theorem 1.0.8. *$D + f \otimes g$ is left-invertible if and only if $D + f \otimes g$ is invertible.*

We observe that the parameterized spaces considered in the work of Davidson, Paulsen, Raghupathi and Singh [25] is connected to rank-one perturbations of isometries. We compute $c(V; f, g)$ when $V + f \otimes g$ is an isometry and make some further comments on rank-one perturbations of diagonal operators.

Finally, we remark that the last two decades have witnessed more intense interest in the theory of left-invertible operators starting from the work of Shimorin [56]. For instance, see [49] and references therein. For a more recent account of Shimorin's approach in the context of analytic model theory, invariant subspaces, and wandering subspaces in several variables, we refer the reader to Eschmeier [27] (also see [16] as part of the motivation), Eschmeier and Langendörfer [28], and Eschmeier and Toth [30]. Also see the monograph by Eschmeier and Putinar [29] for the general framework and motivation.

Chapter 2

Preliminaries

In this chapter, we will present the basic notions that will be used in the following chapters. In this thesis, all Hilbert spaces will be separable and over $\mathbb{C}$. Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ will denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We will simply write $\mathcal{B}(\mathcal{H})$ whenever $\mathcal{H} = \mathcal{K}$.

2.1 Reproducing Kernel Hilbert spaces

We begin with reproducing kernel Hilbert spaces and some basic operator theory. Briefly stated, the essential idea of reproducing kernel Hilbert space is to single out the role of positive definiteness of inner products, multipliers and bounded point evaluations of function Hilbert spaces. We refer the reader to Aronszajn [10] and the monographs [6, 26] for reproducing kernels, and the classics [36, 37, 42] for operator theory.

Definition 2.1.1. *Let $\mathcal{E}$ be a Hilbert space, and let X be a non-empty set. A function $k : X \times X \to \mathcal{B}(\mathcal{E})$ is called a reproducing kernel (or simply a kernel) if*

$$\sum_{i,j=1}^{m} \langle K(z_i, z_j)\eta_j, \eta_i\rangle_{\mathcal{E}} \geq 0,$$

for all $\{z_1, \ldots, z_m\} \subseteq X$, $\{\eta_1, \ldots, \eta_m\} \subseteq \mathcal{E}$ and $m \geq 1$.

Reproducing kernels are naturally attached with function Hilbert spaces known as reproducing kernel Hilbert spaces. Let k be a $\mathcal{B}(\mathcal{E})$-valued kernel function, and let $\mathcal{H}_k$ be the closure of the linear space

$$\left\{\sum_{i=1}^{m} k(\cdot, z_i)\eta_i : z \in \Omega, \;\; \eta \in \mathcal{E} \;\text{ and }\; m \in \mathbb{N}\right\},$$

with respect to the inner product

$$\langle k(\cdot, w)\eta, k(\cdot, z)\zeta\rangle := \langle k(z, w)\eta, \zeta\rangle_{\mathcal{E}},$$

for all $z, w \in \Omega$ and $\eta, \zeta \in \mathcal{E}$. Then $\mathcal{H}_k$ is a Hilbert space of $\mathcal{E}$-valued functions on Ω and

$$\mathcal{H}_k = \overline{\text{span}}\{k(\cdot, w)\eta : \eta \in \mathcal{E}, w \in \Omega\}.$$

Then we have the *reproducing property*

$$\langle f, k(\cdot, w)\eta\rangle = \langle f(w), \eta\rangle_{\mathcal{E}},$$

for all $w \in \Omega$, $f \in \mathcal{H}_k$ and $\eta \in \mathcal{E}$. Let $\mathcal{H}_k$ be $\mathcal{E}$-valued reproducing kernel Hilbert space corresponding to a $\mathcal{B}(\mathcal{E})$-valued kernel function k. Given $w \in \Omega$, consider the evaluation operator $ev_w : \mathcal{H}_k \to \mathcal{E}$ defined by

$$ev_w(f) = f(w) \qquad (f \in \mathcal{H}_K).$$

It then follows that

$$k(z, w) = ev_z \circ ev_w^* \qquad (z, w \in \Omega)$$

From now on, we assume that Ω a domain in $\mathbb{C}$. However, we will mostly deal with $\Omega = \mathbb{D}$ case, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 2.1.2. *The kernel k is said to be analytic if k is analytic in the first variables. If k is analytic, then we call $\mathcal{H}_k$ as an analytic reproducing kernel Hilbert space (or analytic Hilbert space, in short).*

By the definition of kernel functions, if k is analytic, then $\mathcal{H}_k$ is a reproducing kernel Hilbert space of analytic functions on Ω. In what follows, we will deal with $\mathcal{H}_K$ such that $z\mathcal{H}_k \subseteq \mathcal{H}_k$. In this case

$$(M_z f)(w) = wf(w) \qquad (w \in \Omega, f \in \mathcal{H}_k).$$

defines a bounded linear operator M_z on $\mathcal{H}_k$.

Definition 2.1.3. *The bounded linear operator M_z on $\mathcal{H}_k$ is called a shift operator (or simply a shift).*

If M_z on $\mathcal{H}_k$ is a shift, then it is easy to verify that

$$M_z^*(k(\cdot, w)\eta) = \overline{w}k(\cdot, w)\eta$$

for all $w \in \Omega$ and $\eta \in \mathcal{E}$. The following is a list of familiar reproducing kernel Hilbert spaces:

Example 2.1.4. *1. The Hardy space $H^2(\mathbb{D})$ over the open unit disc $\mathbb{D}$ is a reproducing kernel Hilbert space with the Szegö kernel*

$$\mathbb{S}(z,w) = (1 - z\overline{w})^{-1} \qquad (z, w \in \mathbb{D}).$$

2. Let $\alpha > 1$. The weighted Bergman space $L^2_{a,\alpha}(\mathbb{D})$ is a reproducing kernel Hilbert space corresponding to the weighted Bergman kernel

$$k_{L^2_{a,\alpha}(\mathbb{D})}(z,w) = (1 - z\bar{w})^{-\alpha}, \qquad (z, w \in \mathbb{D}).$$

3. The Dirichlet space $\mathcal{D}(\mathbb{D})$ is the reproducing kernel Hilbert space corresponding to the Dirichlet kernel

$$k_{\mathcal{D}(\mathbb{D})}(z,w) = 1 + log\frac{1}{1 - z\bar{w}} \qquad (z, w \in \mathbb{D}).$$

2.2 Multipliers

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces, and let $k_i : \Omega \times \Omega \to \mathcal{B}(\mathcal{E}_i)$, $i = 1, 2$, be kernel functions. A function $\varphi : \Omega \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ is said to be a *multiplier* if

$$\varphi \mathcal{H}_{k_1} \subseteq \mathcal{H}_{k_2}.$$

Denote by $\mathcal{M}(\mathcal{H}_{k_1}, \mathcal{H}_{k_2})$ the set of all multipliers, that is

$$\mathcal{M}(\mathcal{H}_{k_1}, \mathcal{H}_{k_2}) = \{\varphi : \Omega \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2) : \varphi \mathcal{H}_{k_1} \subseteq \mathcal{H}_{k_2}\}.$$

By the closed graph theorem, a multiplier $\varphi \in \mathcal{M}(\mathcal{H}_{k_1}, \mathcal{H}_{k_2})$ defines a bounded linear operator $M_\varphi : \mathcal{H}_{k_1} \to \mathcal{H}_{k_2}$, where

$$(M_\varphi f)(w) = (\varphi f)(w) = \varphi(w) f(w),$$

for all $f \in \mathcal{H}_{k_1}$ and $w \in \Omega$. Moreover, if $\varphi \in \mathcal{M}(\mathcal{H}_{k_1}, \mathcal{H}_{k_2})$, it then follows that

$$M_\varphi^*(k_2(\cdot, w)\eta) = \varphi(w)^* k_1(\cdot, \omega)\eta,$$

for all $w \in \Omega$ and $\eta \in \mathcal{E}$. If $k_1 = k_2$, the we simply write $\mathcal{M}(\mathcal{H}_{k_1}, \mathcal{H}_{k_1})$ as $\mathcal{M}(\mathcal{H}_{k_1})$. It is well known that

$$\mathcal{M}(H^2(\mathbb{D})) = H^\infty(\mathbb{D}),$$

where $H^\infty(\mathbb{D})$ denotes the Banach algebra of all bounded analytic functions on $\mathbb{D}$. On the other hand, the multiplier algebra $\mathcal{M}(\mathcal{D}(\mathbb{D}))$ of the Dirichlet space is a proper subalgebra of $H^\infty(\mathbb{D})$.

2.3 Tridiagonal kernels

In this subsection, we study the "next best" concrete analytic kernels (after the diagonal kernels), namely, tridiagonal kernels. This notion was introduced by Adams and McGuire [3] (also see the motivating paper by Adams, McGuire and Paulsen [4]).

Let $k : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E})$ be an analytic kernel. Then there exist $C_{mn} \in \mathcal{B}(\mathcal{E})$ such that

$$k(z, w) = \sum_{m,n=0}^{\infty} C_{mn} z^m \bar{w}^n \qquad (z, w \in \mathbb{D}).$$

We say that $\mathcal{H}_k$ is a *diagonal reproducing kernel Hilbert space* (and k is a *diagonal kernel*) if

$$C_{mn} = 0 \qquad (|m - n| \geq 1).$$

We say that k is a *tridiagonal kernel* (or *band kernel with bandwidth* 1) if

$$C_{mn} = 0 \qquad (|m - n| \geq 2).$$

In this case, we say that $\mathcal{H}_k$ is a *tridiagonal space.*

Following Adams and McGuire [3], in the following, we construct a large class of tridiagonal kernels. Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be a sequences of scalars. In this thesis, we will always assume that $a_n \neq 0$, for all $n \geq 0$. Set

$$f_n(z) = (a_n + b_n z) z^n \qquad (n \geq 0).$$

Assume that $\{f_n\}_{n\geq 0}$ is an orthonormal basis of an analytic Hilbert space $\mathcal{H}_k$. Then $\mathcal{H}_k$ is a tridiagonal space, as the well known fact from the reproducing kernel theory implies that

$$k(z, w) = \sum_{n=0}^{\infty} f_n(z) \overline{f_n(w)} \qquad (z, w \in \mathbb{D}).$$

Moreover, if

$$\sup_{n\geq 0} \left| \frac{a_n}{a_{n+1}} \right| < \infty \quad \text{and} \quad \limsup_{n\geq 0} \left| \frac{b_n}{a_{n+1}} \right| < 1,$$

then the shift M_z is a bounded linear operator on $\mathcal{H}_k$ [3, Theorem 5]. If $b_n = 0$ for all $n \geq 0$, then k is a diagonal kernel, and M_z on $\mathcal{H}_k$ is a weighted shift operator.

2.4 Left-invertible operators

Recall that a linear operator V on $\mathcal{H}$ is an *isometry* if $\|Vh\| = \|h\|$ for all $h \in \mathcal{H}$, or equivalently

$$V^*V = I_{\mathcal{H}}.$$

Along this line, left-invertible operators (also known as, by a slight abuse of terminology, "operators close to an isometry" [56]) are also natural examples of noncompact operators (so long as the ambient Hilbert space is infinite dimensional): $T \in \mathcal{B}(\mathcal{H})$ is *left-invertible* if T is bounded below, that is, there exists $\epsilon > 0$ such that

$$\|Th\| \geq \epsilon \|h\| \qquad (h \in \mathcal{H}),$$

or equivalently, there exists $S \in \mathcal{B}(\mathcal{H})$ such that

$$ST = I_{\mathcal{H}}.$$

This is also equivalent to the invertibility condition of T^*T. Let $T \in \mathcal{B}(\mathcal{H})$ be a left-invertible operator. We use the fact that T^*T is invertible to see that $(T^*T)^{-1}T^*$ is a left inverse of T. We call

$$L_T := (T^*T)^{-1}T^*,$$

the *Shimorin left inverse*, to distinguish it from other left inverses of T. Note that $(TL_T)^2 = TL_T = (TL_T)^*)$, that is, TL_T is an orthogonal projection. Moreover, if $T^*f = 0$ for some $f \in \mathcal{H}$, then $(I - TL_T)f = f$. On the other hand, if $(I - TL_T)f = f$ for some $f \in \mathcal{H}$, then $TL_T f = 0$ and hence $T^*TL_T f = 0$, which implies that $T^*f = 0$. Therefore, $I - TL_T$ is the orthogonal projection onto $\ker T^*$, that is

$$I - TL_T = P_{\ker T^*}.$$

Now we briefly describe the construction of Shimorin's analytic models of left-invertible operators. Following Shimorin, a bounded linear operator $X \in \mathcal{B}(\mathcal{H})$ is *analytic* if

$$\bigcap_{n=0}^{\infty} X^n \mathcal{H} = \{0\}$$

Note that from the viewpoint of analytic Hilbert spaces:

Lemma 2.4.1. *Shifts are always analytic.*

Proof. Indeed, let $\mathcal{H}_k \subseteq \mathcal{O}(\Omega, \mathcal{E})$, where $\Omega \subseteq \mathbb{C}$ is a domain, and suppose the shift M_z is bounded on $\mathcal{H}_k$. If $f \in \bigcap_{n=0}^{\infty} M_z^n \mathcal{H}_k$, then for each $n \geq 0$, there exists $g_n \in \mathcal{H}_k$ such that $f = z^n g_n$. Since Ω is a domain and f is analytic on Ω, we see that $f \equiv 0$, that is, $\bigcap_{n=0}^{\infty} M_z^n \mathcal{H}_k = \{0\}$. □

Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded below operator. Set

$$\mathcal{W} = \ker T^* = \mathcal{H} \ominus T\mathcal{H},$$

and $\Omega = \{z \in \mathbb{C} : |z| < \frac{1}{r(L_T)}\}$, where $r(L_T)$ is the spectral radius of L_T. Then

$$k_T(z,w) = P_{\mathcal{W}}(I - zL_T)^{-1}(I - \bar{w}L_T^*)^{-1}|_{\mathcal{W}} \qquad (z, w \in \Omega), \tag{2.4.1}$$

defines a $\mathcal{B}(\mathcal{W})$-valued analytic kernel $k_T : \Omega \times \Omega \to \mathcal{B}(\mathcal{W})$, which we call the *Shimorin kernel* of T (see [56, Corollary 2.14]). We lose no generality by assuming, as we shall do, that $\Omega = \mathbb{D}$. If, in addition, T is analytic, then the unitary $U : \mathcal{H} \to \mathcal{H}_k$ defined by

$$(Uf)(z) = \sum_{n=0}^{\infty} (P_{\mathcal{W}} L_T^n f) z^n \qquad (f \in \mathcal{H}, z \in \mathbb{D}), \tag{2.4.2}$$

satisfies $UT = M_z U$ [56]. More precisely, we have the following result:

Theorem 2.4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an analytic left-invertible operator. Then T on $\mathcal{H}$ and M_z on $\mathcal{H}_{k_T}$ are unitarily equivalent.*

Denote by $P_{\mathcal{W}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{W} = \ker T^*$. From the above discussion, it follows that

$$P_{\mathcal{W}} = I_{\mathcal{H}} - TL_T.$$

This equality plays an important role (in the sense of Wold decomposition of left-invertible operators) in the proof of the above theorem.

2.5 Aluthge transforms

The notion of Aluthge transforms was introduced by Aluthge [7] in his study of p-hyponormal operators. Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$, and let $T = U|T|$ be the polar decomposition of T. Here, and throughout, $|T| = (T^*T)^{\frac{1}{2}}$ and U is the unique partial isometry such that $\ker U = \ker T$. The *Aluthge transform* of T is the bounded linear operator

$$\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}.$$

The Aluthge transform of $\tilde{T}$ turns T into a more "normal" operator while keeping intact the basic spectral properties of T [40]. Evidently, the main difficulty associated with $\tilde{T}$ is to compute or represent the positive part $|T|$. This is certainly not true for weighted shifts: Since $|S_\alpha| = \mathrm{diag}(\alpha_0, \alpha_1, \alpha_2, \ldots)$, it follows that $\tilde{S_\alpha} = S_{\sqrt{\alpha}}$, where

$$\sqrt{\alpha} := \{\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \ldots\}.$$

Therefore, $\tilde{S}_\alpha$ is also a weighted shift, namely $S_{\sqrt{\alpha}}$.

2.6 Beurling theorem

Recall that a bounded analytic function $\theta \in H^\infty(\mathbb{D})$ is said to be inner if

$$|\theta(z)| = 1 \qquad (z \in \mathbb{T} \; a.e.)$$

The celebrated Beurling theorem [15] states: A non-zero closed subspace $\mathcal{S}$ of $H^2(\mathbb{D})$ is invariant under M_z if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that

$$\mathcal{S} = \theta H^2(\mathbb{D}).$$

Note also that it follows (or the other way around) in particular from the above representation of $\mathcal{S}$ that

$$\mathcal{S} \ominus z\mathcal{S} = \theta\mathbb{C},$$

and so

$$\mathcal{S} = \bigoplus_{m=0}^{\infty} z^m(\mathcal{S} \ominus z\mathcal{S}).$$

In particular, we have:

(1) $\mathcal{S}$ is singly generated, and

(2) $\mathcal{S} \cap H^\infty(\mathbb{D}) \neq \{0\}$.

Chapter 3

Tridiagonal kernels and left-invertible operators with applications to Aluthge transforms

In this chapter, we study left-invertible shifts on tridiagonal spaces. We also discuss the pending and inevitable comparisons between Shimorin's analytic models of left-invertible operators and Adams and McGuire's theory of left-invertible shifts on tridiagonal spaces. We prove that unlike the case of weighted shifts, Shimorin models fail to bring to the foreground the tridiagonal structure of shifts. We resolve this dilemma by presenting a complete classification of tridiagonal kernels that are preserved under Shimorin models.

We also prove a number of results concerning left-invertible properties of shifts on tridiagonal spaces, new tridiagonal spaces from the old, classifications of quasinormal operators, rank-one perturbations of left inverses, a computational approach to Aluthge transforms of shifts, etc. Again, curiously enough, some of our definite computations in the setting of tridiagonal kernels verify that the direct reproducing kernel Hilbert space technique is somewhat more powerful than Shimorin models. We also provide a family of instructive examples and supporting counterexamples.

We remark that some of the observations outlined in Subsections 3.6 and 3.7 are based on several more general results that have an independent interest in broader operator theory and function theoretic contexts.

3.1 Preparatory results and examples

In this subsection, we set up some definitions, collect some known facts about tridiagonal reproducing kernel Hilbert spaces and Shimorin analytic models, and observe some auxiliary results which are needed throughout the chapter. We also explain the idea of Shimorin with the example of diagonal kernels (or equivalently, weighted shifts).

We start with tridiagonal spaces. Here we avoid finer technicalities [3] and introduce only the necessary features of tridiagonal spaces. Let $\mathcal{E}$ be a Hilbert space, k be a $\mathcal{B}(\mathcal{E})$-valued analytic kernel on $\mathbb{D}$, and let $\mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D}, \mathcal{E})$ be the corresponding reproducing kernel Hilbert space. Then there exists a sequence $\{C_{mn}\}_{m,n\geq 0} \subseteq \mathcal{B}(\mathcal{E})$ such that

$$k(z,w) = \sum_{m,n=0}^{\infty} C_{mn} z^m \bar{w}^n \qquad (z, w \in \mathbb{D}).$$

Recall that k is a *tridiagonal kernel* if $C_{mn} = 0$, $|m-n| \geq 2$. We say that $\mathcal{H}_k$ is a *tridiagonal space* if k is tridiagonal. We now single out two natural tridiagonal spaces.

Definition 3.1.1. *A tridiagonal space $\mathcal{H}_k$ is called semi-analytic tridiagonal space if $\mathbb{C}[z] \subseteq \mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D})$, and there exist scalars $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, $a_n \neq 0$ for all $n \geq 0$, such that*

$$\sup_{n\geq 0} \left| \frac{a_n}{a_{n+1}} \right| < \infty \quad \text{and} \quad \limsup_{n\geq 0} \left| \frac{b_n}{a_{n+1}} \right| < 1, \tag{3.1.1}$$

and $\{f_n\}_{n\geq 0}$ is an orthonormal basis of $\mathcal{H}_k$, where

$$f_n(z) = (a_n + b_n z) z^n \qquad (n \geq 0). \tag{3.1.2}$$

Note that the conditions in (3.1.1) ensure that the shift M_z is a bounded linear operator on $\mathcal{H}_k$ [3, Theorem 5]. We refer the reader to [3, Theorem 2] on the containment of polynomials.

Definition 3.1.2. *A semi-analytic tridiagonal space $\mathcal{H}_k$ is said to be analytic tridiagonal space if the sequence $\{|\frac{a_n}{a_{n+1}}|\}_{n\geq 0}$ is bounded away from zero, that is, there exists $\epsilon > 0$ such that*

$$\left| \frac{a_n}{a_{n+1}} \right| > \epsilon \qquad (n \geq 0). \tag{3.1.3}$$

A scalar kernel k is called *semi-analytic (analytic) tridiagonal kernel* if the corresponding reproducing kernel Hilbert space $\mathcal{H}_k$ is a semi-analytic (an analytic) tridiagonal space.

It is important to note that (3.1.3) is essential for left invertibility of M_z. As we will see in Theorem 3.2.5, if $\mathcal{H}_k (\supseteq \mathbb{C}[z])$ is a tridiagonal space corresponding to the orthonormal basis $\{f_n\}_{n\geq 0}$ and if $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ satisfies the conditions in (3.1.1), then condition (3.1.3) is equivalent to the left invertibility of M_z on $\mathcal{H}_k$. Also recall that the weighted shift S_α with weights $\{\alpha_n\}_{n\geq 0}$ is bounded if and only if $\sup_{n\geq 0}\alpha_n < \infty$. In this case, S_α is left-invertible if and only if $\{\alpha_n\}_{n\geq 0}$ is bounded away from zero (cf. Proposition 3.1.7). By translating this into the setting of analytic Hilbert spaces [54, Proposition 7], it is clear that if $b_n = 0$, $n \geq 0$, then (3.1.3) is a necessary and sufficient condition for left invertibility of shifts on diagonal kernels.

Suppose k is a semi-analytic tridiagonal kernel. Note that $k(z,w) = \sum_{n=0}^\infty f_n(z)\overline{f_n(w)}$. Now fix $n \geq 0$, and write $z^n = \sum_{m=0}^\infty \alpha_m f_m$ for some $\alpha_m \in \mathbb{C}$, $m \geq 0$. Then

$$z^n = \alpha_0 a_0 + \sum_{m=1}^\infty (\alpha_{m-1}b_{m-1} + \alpha_m a_m) z^m.$$

Thus comparing coefficients, we have $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0$, and $\alpha_n = \frac{1}{a_n}$, as a_i's are non-zero scalars. Since $\alpha_{n+j-1}b_{n+j-1} + \alpha_{n+j}a_{n+j} = 0$, it follows that $\alpha_{n+j} = -\frac{\alpha_{n+j-1}b_{n+j-1}}{a_{n+j}}$, and thus $\alpha_{n+j} = \frac{(-1)^j}{a_n}\frac{b_n b_{n+1}\cdots b_{n+j-1}}{a_{n+1}\cdots a_{n+j}}$ for all $j \geq 1$. This implies

$$z^n = \frac{1}{a_n}\sum_{m=0}^\infty (-1)^m \Big(\frac{\prod_{j=0}^{m-1} b_{n+j}}{\prod_{j=0}^{m-1} a_{n+j+1}}\Big) f_{n+m} \qquad (n \geq 0), \tag{3.1.4}$$

where $\prod_{j=0}^{-1} x_{n+j} := 1$. With this, we now proceed to compute M_z [3, Section 3]. Let $n \geq 0$. Then $M_z f_n = a_n z^{n+1} + b_n z^{n+2}$ implies that

$$M_z f_n = \frac{a_n}{a_{n+1}} f_{n+1} + (b_n - \frac{a_n b_{n+1}}{a_{n+1}}) z^{n+2} = \frac{a_n}{a_{n+1}} f_{n+1} + a_{n+2}(\frac{b_n}{a_{n+2}} - \frac{a_n}{a_{n+1}}\frac{b_{n+1}}{a_{n+2}}) z^{n+2},$$

that is

$$M_z f_n = \frac{a_n}{a_{n+1}} f_{n+1} + a_{n+2} c_n z^{n+2}, \tag{3.1.5}$$

where

$$c_n = \frac{a_n}{a_{n+2}}\Big(\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}\Big) \qquad (n \geq 0). \tag{3.1.6}$$

Then (3.1.4) implies that

$$M_z f_n = \Big(\frac{a_n}{a_{n+1}}\Big) f_{n+1} + c_n \sum_{m=0}^\infty (-1)^m \Big(\frac{\prod_{j=0}^{m-1} b_{n+2+j}}{\prod_{j=0}^{m-1} a_{n+3+j}}\Big) f_{n+2+m} \qquad (n \geq 0), \tag{3.1.7}$$

and hence, with respect to the orthonormal basis $\{f_n\}_{n\geq 0}$, we have (also see [3, Page 729])

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \frac{a_0}{a_1} & 0 & 0 & 0 & \ddots \\ c_0 & \frac{a_1}{a_2} & 0 & 0 & \ddots \\ \frac{-c_0 b_2}{a_3} & c_1 & \frac{a_2}{a_3} & 0 & \ddots \\ \frac{c_0 b_2 b_3}{a_3 a_4} & \frac{-c_1 b_3}{a_4} & c_2 & \frac{a_3}{a_4} & \ddots \\ \frac{-c_0 b_2 b_3 b_4}{a_3 a_4 a_5} & \frac{c_1 b_3 b_4}{a_4 a_5} & \frac{-c_2 b_4}{a_5} & c_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{3.1.8}$$

The matrix representation of the conjugate of M_z is going to be useful in what follows:

$$[M_z^*] = \begin{bmatrix} 0 & \frac{\bar{a}_0}{\bar{a}_1} & \bar{c}_0 & \frac{-\bar{c}_0 \bar{b}_2}{\bar{a}_3} & \frac{-\bar{c}_0 \bar{b}_2 \bar{b}_3}{\bar{a}_3 \bar{a}_4} & \dots \\ 0 & 0 & \frac{\bar{a}_1}{\bar{a}_2} & \bar{c}_1 & \frac{-\bar{c}_1 \bar{b}_3}{\bar{a}_4} & \ddots \\ 0 & 0 & 0 & \frac{\bar{a}_2}{\bar{a}_3} & \bar{c}_2 & \ddots \\ 0 & 0 & 0 & 0 & \frac{\bar{a}_3}{\bar{a}_4} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{3.1.9}$$

In particular, M_z is a weighted shift if $c_n = 0$ for all $n \geq 0$. Also, by (3.1.6), we have $c_n = 0$ if and only if $\frac{b_{n+1}}{a_{n+1}} = \frac{b_n}{a_n}$, $n \geq 0$. Therefore, we have the following observation:

Lemma 3.1.3. *The shift M_z on a semi-analytic tridiagonal space $\mathcal{H}_k$ is a weighted shift if $c_n = 0$ for all $n \geq 0$, or, equivalently, $\{\frac{b_n}{a_n}\}_{n \geq 0}$ is a constant sequence.*

The proof of the following lemma uses the assumption that $\mathbb{C}[z] \subseteq \mathcal{H}_k$.

Lemma 3.1.4. *If $\mathcal{H}_k$ is a semi-analytic tridiagonal space, then* $\ker M_z^* = \mathbb{C} f_0$.

Proof. Clearly, (3.1.9) implies that $f_0 \in \ker M_z^*$. On the other hand, from $\mathbb{C}[z] \subseteq \mathcal{H}_k$ we deduce that $f_n = M_z(a_n z^{n-1} + b_n z^n) \in \operatorname{ran} M_z$ for all $n \geq 1$, and hence $\operatorname{span}\{f_n : n \geq 1\} \subseteq \operatorname{ran} M_z$. The result now follows from the fact that $\mathbb{C} f_0 = (\operatorname{span}\{f_n : n \geq 1\})^{\perp} \supseteq \ker M_z^*$. $\square$

Now we briefly describe the construction of Shimorin's analytic models of left-invertible operators. Let $\mathcal{H}$ be a Hilbert space, and let $T \in \mathcal{B}(\mathcal{H})$. We say that T is *left-invertible* if there exists $X \in \mathcal{B}(\mathcal{H})$ such that $XT = I_{\mathcal{H}}$. It is easy to check that this equivalently means that T is bounded below, which is also equivalent to the invertibility of T^*T. Following Shimorin, a bounded linear operator $X \in \mathcal{B}(\mathcal{H})$ is *analytic* if

$$\bigcap_{n=0}^{\infty} X^n \mathcal{H} = \{0\} \tag{3.1.10}$$

Note that from the viewpoint of analytic Hilbert spaces, shifts are always analytic. Indeed, let $\mathcal{H}_k \subseteq \mathcal{O}(\Omega, \mathcal{E})$, where $\Omega \subseteq \mathbb{C}$ is a domain, and suppose the shift M_z is bounded on $\mathcal{H}_k$. If $f \in \bigcap_{n=0}^{\infty} M_z^n \mathcal{H}_k$, then for each $n \geq 0$, there exists $g_n \in \mathcal{H}_k$ such that $f = z^n g_n$. Since Ω is a domain and f is analytic on Ω, we see that $f \equiv 0$, that is, $\bigcap_{n=0}^{\infty} M_z^n \mathcal{H}_k = \{0\}$.

Now let $T \in \mathcal{B}(\mathcal{H})$ be a bounded below operator. We call $L_T := (T^*T)^{-1}T^*$ the *Shimorin left inverse*, to distinguish it from other left inverses of T. Set

$$\mathcal{W} = \ker T^* = \mathcal{H} \ominus T\mathcal{H},$$

and $\Omega = \{z \in \mathbb{C} : |z| < \frac{1}{r(L_T)}\}$, where $r(L_T)$ is the spectral radius of L_T. Then

$$k_T(z,w) = P_{\mathcal{W}}(I - zL_T)^{-1}(I - \bar{w}L_T^*)^{-1}|_{\mathcal{W}} \qquad (z, w \in \Omega), \tag{3.1.11}$$

defines a $\mathcal{B}(\mathcal{W})$-valued analytic kernel $k_T : \Omega \times \Omega \to \mathcal{B}(\mathcal{W})$, which we call the *Shimorin kernel* of T (see [56, Corollary 2.14]). We lose no generality by assuming, as we shall do, that $\Omega = \mathbb{D}$. If, in addition, T is analytic, then the unitary $U : \mathcal{H} \to \mathcal{H}_k$ defined by

$$(Uf)(z) = \sum_{n=0}^{\infty} (P_{\mathcal{W}} L_T^n f) z^n \qquad (f \in \mathcal{H}, z \in \mathbb{D}), \tag{3.1.12}$$

satisfies $UT = M_z U$ [56]. More precisely, we have the following result:

Theorem 3.1.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be an analytic left-invertible operator. Then T on $\mathcal{H}$ and M_z on $\mathcal{H}_{k_T}$ are unitarily equivalent.*

Denote by $P_{\mathcal{W}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{W} = \ker T^*$. It follows that

$$P_{\mathcal{W}} = I_{\mathcal{H}} - TL_T, \tag{3.1.13}$$

This plays an important role (in the sense of Wold decomposition of left-invertible operators) in the proof of the above theorem. The following equality will be very useful in what follows.

Lemma 3.1.6. *If T is a left-invertible operator on $\mathcal{H}$, then $L_T L_T^* = |T|^{-2}$.*

Proof. This follows from the fact that $L_T L_T^* = (T^*T)^{-1}T^*T(T^*T)^{-1} = (T^*T)^{-1}$. □

In the case of left-invertible weighted shifts S_α, it is known that the shift M_z on $\mathcal{H}_{k_{S_\alpha}}$ corresponding to the Shimorin kernel k_{S_α} is also a weighted shift (for instance, see [48, Example 5.2] in the context of bilateral weighted shifts). Nonetheless, we sketch the proof here for the sake of completeness.

Proposition 3.1.7. *Let S_α be the weighted shift with weights $\{\alpha_n\}_{n\geq 0}$. If $\{\alpha_n\}_{n\geq 0}$ is bounded away from zero, then S_α is left-invertible, and the Shimorin kernel k_{S_α} is diagonal.*

Proof. Let $\{e_n\}_{n\geq 0}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$, and let $S_\alpha e_n = \alpha_n e_{n+1}$ for all $n \geq 0$. Observe that $S_\alpha^* e_n = \alpha_{n-1}e_{n-1}$, $n \geq 1$, and $S_\alpha^* e_0 = 0$. Then $\mathcal{W} = \ker S_\alpha^* = \mathbb{C}e_0$, and $S_\alpha^* S_\alpha e_n = \alpha_n^2 e_n$ for all $n \geq 0$. Since $S_\alpha^* S_\alpha$ is a diagonal operator and $\{\alpha_n\}_{n\geq 0}$ is bounded away from zero, it follows that $S_\alpha^* S_\alpha$ is invertible, and hence S_α is left-invertible. Then the Shimorin left inverse $L_{S_\alpha} := (S_\alpha^* S_\alpha)^{-1} S_\alpha^*$ is given by

$$L_{S_\alpha} e_n = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{\alpha_{n-1}} e_{n-1} & \text{if } n \geq 1. \end{cases} \tag{3.1.14}$$

Therefore, L_{S_α} is the backward shift, and

$$L_{S_\alpha}^m e_n = \begin{cases} 0 & \text{if } m > n \\ \frac{1}{\alpha_0 \cdots \alpha_{n-1}} e_0 & \text{if } m = n \\ \frac{1}{\alpha_{n-1} \cdots \alpha_{n-m}} e_{n-m} & \text{if } m < n, \end{cases} \tag{3.1.15}$$

for all $m \geq 1$. Moreover, $L_{S_\alpha}^{*m} e_n = \frac{1}{\alpha_n \alpha_{n+1} \cdots \alpha_{n+m-1}} e_{n+m}$ for all $n \geq 0$ and $m \geq 1$. In particular, $L_{S_\alpha}^{*m} e_0 = \frac{1}{\alpha_0 \alpha_1 \cdots \alpha_{m-1}} e_m$, $m \geq 1$, and thus, for each $(m, n) \neq (0, 0)$, we have clearly

$$P_{\mathcal{W}} L_{S_\alpha}^m L_{S_\alpha}^{*n} e_0 = \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{(\alpha_0 \cdots \alpha_{n-1})^2} e_0 & \text{if } m = n. \end{cases}$$

This immediately gives $k_{S_\alpha}(z, w) = \sum_{n=0}^\infty (P_{\mathcal{W}} L_{S_\alpha}^n L_{S_\alpha}^{*n}|_{\mathcal{W}})(z\bar{w})^n$ for all $z, w \in \mathbb{D}$, where $\mathcal{W} = \mathbb{C}e_0$. In particular, the Shimorin kernel k_{S_α} is a diagonal kernel. Finally, identifying $\mathcal{W}$ with $\mathbb{C}$ and setting $\beta_n = \frac{1}{\alpha_0 \cdots \alpha_{n-1}}$, $n \geq 1$, we get

$$k_{S_\alpha}(z, w) = 1 + \sum_{n=1}^\infty \frac{1}{\beta_n^2} (z\bar{w})^n \qquad (z, w \in \mathbb{D}).$$

$\square$

Notice in the above, the Shimorin left inverse L_{S_α} is the backward shift corresponding to the weight sequence $\{\frac{1}{\alpha_n}\}_{n\geq 0}$, that is,

$$L_{S_\alpha} = \begin{bmatrix} 0 & \frac{1}{\alpha_0} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\alpha_1} & 0 & \ddots \\ 0 & 0 & 0 & \frac{1}{\alpha_2} & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

In the setting of Proposition 3.1.7, we now turn to the unitary map $U : \mathcal{H} \to \mathcal{H}_{k_{S_\alpha}}$, where $\mathcal{H}_{k_{S_\alpha}} \subseteq \mathcal{O}(\mathbb{D}, \mathcal{W})$, and $(Uf)(z) = \sum_{n=0}^\infty (P_{\mathcal{W}} L_{S_\alpha}^n f) z^n$ for all $f \in \mathcal{H}$ and $z \in \mathbb{D}$ (see (3.1.12)). Set $f_n = Ue_n$, $n \geq 0$. Since $\mathcal{W} = \mathbb{C}e_0$, (3.1.14) yields $f_0 = Ue_0 = P_{\mathcal{W}} e_0 = e_0$.

On the other hand, if $n \geq 1$, then (3.1.15) implies that

$$P_{\mathcal{W}} L_{S_\alpha}^m e_n = \begin{cases} \frac{1}{\beta_n} e_0 & \text{if } m = n \\ 0 & \text{otherwise,} \end{cases}$$

and hence $f_n = \frac{1}{\beta_n} z^n e_0$. Therefore $\{e_0\} \cup \{\frac{1}{\beta_n} z^n e_0\}_{n\geq 1}$ is the orthonormal basis of $\mathcal{H}_{k_{S_\alpha}}$ corresponding to U. Moreover, for each $n \geq 1$, we have

$$M_z(\frac{1}{\beta_n} z^n e_0) = \frac{1}{\beta_n} z^{n+1} e_0 = \alpha_n \frac{1}{\beta_{n+1}} z^{n+1} e_0 = \alpha_n(\frac{1}{\beta_{n+1}} z^{n+1} e_0),$$

and hence M_z on $\mathcal{H}_{k_{S_\alpha}}$ is also a weighted shift with the same weights $\{\alpha_n\}_{n\geq 0}$.

3.2 Tridiagonal spaces and left-invertibility

The main contribution of this section is the left invertibility and representations of Shimorin left inverses of shifts on tridiagonal reproducing kernel Hilbert spaces. Recall that the conditions in (3.1.1) ensures that the shift M_z is bounded on the semi-analytic tridiagonal space $\mathcal{H}_k$. Here we use the remaining condition (3.1.3) to prove that M_z is left-invertible.

Before we state and prove the result, we need to construct a specific bounded linear operator. The choice of this operator is not accidental, as we will see in Theorem 3.2.4 that it is nothing but the Shimorin left inverse of M_z. For each $n \geq 1$, set

$$d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}. \tag{3.2.1}$$

Proposition 3.2.1. *Let k be an analytic tridiagonal kernel corresponding to the orthonormal basis $\{f_n\}_{n\geq 0}$, where $f_n(z) = (a_n + b_z z)z^n$, $n \geq 0$. Then the linear operator L represented by*

$$[L] = \begin{bmatrix} 0 & \frac{a_1}{a_0} & 0 & 0 & 0 & \cdots \\ 0 & d_1 & \frac{a_2}{a_1} & 0 & 0 & \ddots \\ 0 & \frac{-d_1 b_1}{a_2} & d_2 & \frac{a_3}{a_2} & 0 & \ddots \\ 0 & \frac{d_1 b_1 b_2}{a_2 a_3} & \frac{-d_2 b_2}{a_3} & d_3 & \frac{a_4}{a_3} & \ddots \\ 0 & \frac{-d_1 b_1 b_2 b_3}{a_2 a_3 a_4} & \frac{d_2 b_2 b_3}{a_3 a_4} & \frac{-d_3 b_3}{a_4} & d_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

with respect to the orthonormal basis $\{f_n\}_{n\geq 0}$ defines a bounded linear operator on $\mathcal{H}_k$.

Proof. For each $n \geq 1$, we have clearly $d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}} = \frac{a_{n+1}}{a_n}\frac{b_n}{a_{n+1}} - \frac{a_n}{a_{n-1}}\frac{b_{n-1}}{a_n}$, and hence

$$|d_n| \leq \Big|\frac{a_{n+1}}{a_n}\Big|\Big|\frac{b_n}{a_{n+1}}\Big| + \Big|\frac{a_n}{a_{n-1}}\Big|\Big|\frac{b_{n-1}}{a_n}\Big|.$$

Since $\{|\frac{a_n}{a_{n+1}}|\}_{n\geq 0}$ is bounded away from zero (see (3.1.3)), we have that $\sup_{n\geq 0}|\frac{a_{n+1}}{a_n}| < \infty$. This and the second assumption then imply that $\{d_n\}$ is a bounded sequence.

Let S denote the matrix obtained from $[L]$ by deleting all but the superdiagonal elements of $[L]$. Similarly, L_0 denote the matrix obtained from $[L]$ by deleting all but the diagonal elements of $[L]$, and in general, assume that L_i denote the matrix obtained from $[L]$ by deleting all but the i-th subdiagonal of $[L]$, $i = 0, 1, 2 \ldots$. Since

$$L = S + \sum_{i\geq 0} L_i,$$

it clearly suffices to prove that S and $\{L_i\}_{i\geq 0}$ are bounded, and $S + \sum_{i\geq 0} L_i$ is absolutely convergent. Note that $\|S\| = \sup_{n\geq 0}|\frac{a_{n+1}}{a_n}| < \infty$. Moreover, our assumption $\limsup_{n\geq 0}|\frac{b_n}{a_{n+1}}| < 1$ implies that there exist $r < 1$ and $n_0 \in \mathbb{N}$ such that

$$\Big|\frac{b_n}{a_{n+1}}\Big| < r \qquad (n \geq n_0).$$

Set

$$M = \sup_{n\geq 1}\Big\{\Big|\frac{b_n}{a_{n+1}}\Big|, |d_n|\Big\}.$$

Then $\|L_i\| \leq M^{i+1}$ for all $i = 0, \ldots, n_0$, and

$$\|L_i\| \leq M^{n_0+1} r^{i-n_0} \qquad (i > n_0),$$

from which it follows that

$$\begin{aligned}
\|S\| + \sum_{i\geq 0}\|L_i\| &= \sup_{n\geq 0}\Big|\frac{a_{n+1}}{a_n}\Big| + \sum_{0\leq i\leq n_0}\|L_i\| + \sum_{i\geq n_0+1}\|L_i\| \\
&\leq \sup_{n\geq 0}\Big|\frac{a_{n+1}}{a_n}\Big| + \sum_{0\leq i\leq n_0}\|L_i\| + M^{n_0+1}\Big(\sum_{i\geq n_0+1} r^{i-n_0}\Big) \\
&\leq \sup_{n\geq 0}\Big|\frac{a_{n+1}}{a_n}\Big| + \sum_{0\leq i\leq n_0}\|L_i\| + M^{n_0+1}\frac{r}{1-r},
\end{aligned}$$

and completes the proof of the theorem. □

We are now ready to prove that M_z is left-invertible.

Theorem 3.2.2. *In the setting of Proposition 3.2.1, we have $LM_z = I_{\mathcal{H}_k}$.*

Proof. We consider the matrix representations of M_z and L as in (3.1.8) and Proposition 3.2.1, respectively. Let $[L][M_z] = (\alpha_{mn})_{m,n\geq 0}$. Clearly it suffices to prove that $\alpha_{mn} =$

δ_{mn}. It is easy to see that $\alpha_{m,m+k} = 0$ for all $k \geq 1$. Now by (3.1.6), we have

$$c_n = -\frac{a_n}{a_{n+2}} d_{n+1} \qquad (n \geq 0). \tag{3.2.2}$$

Note that the n-th column, $n \geq 0$, of $[M_z]$ is the transpose of

$$\Big(\underbrace{0, \ldots, 0}_{n+1}, \frac{a_n}{a_{n+1}}, c_n, -\frac{c_n b_{n+2}}{a_{n+3}}, \ldots, (-1)^{m-n-2} \frac{c_n b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m}, (-1)^{m-n-1} \frac{c_n b_{n+2} \cdots b_m}{a_{n+3} \cdots a_{m+1}}, \ldots\Big),$$

and the m-th row, $m \geq 0$, of $[L]$ is given by

$$\Big(0, (-1)^{m-1} \frac{d_1 b_1 \cdots b_{m-1}}{a_2 \cdots a_m}, (-1)^{m-2} \frac{d_2 b_2 \cdots b_{m-1}}{a_3 \cdots a_m}, (-1)^{m-3} \frac{d_3 b_3 \cdots b_{m-1}}{a_4 \cdots a_m}, \ldots$$
$$\ldots, \frac{-d_{m-1} b_{m-1}}{a_m}, d_m, \frac{a_{m+1}}{a_m}, 0, 0, \ldots\Big).$$

Now, if $n \leq (m-2)$, then the α_{mn} (the (m,n)-th entry of $[L][M_z]$) is given by

$$\begin{aligned}
\alpha_{mn} &= (-1)^{m-n-1} \frac{d_{n+1} b_{n+1} \cdots b_{m-1}}{a_{n+2} \cdots a_m} \frac{a_n}{a_{n+1}} + (-1)^{m-n-2} \frac{d_{n+2} b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m} c_n \\
&\quad + (-1)^{m-n-3} \frac{d_{n+3} b_{n+3} \cdots b_{m-1}}{a_{n+4} \cdots a_m} (-c_n \frac{b_{n+2}}{a_{n+3}}) + \cdots + (-\frac{d_{m-1} b_{m-1}}{a_m})(-1)^{m-n-3} \times \\
&\quad c_n \frac{b_{n+2} \cdots b_{m-2}}{a_{n+3} \cdots a_{m-1}} + d_m (-1)^{m-n-2} c_n \frac{b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m} + \frac{a_{m+1}}{a_m} (-1)^{m-n-1} c_n \frac{b_{n+2} \cdots b_m}{a_{n+3} \cdots a_m a_{m+1}},
\end{aligned}$$

and hence, using (3.2.2), we obtain

$$\begin{aligned}
\alpha_{mn} &= (-1)^{m-n-1} d_{n+1} \frac{a_n b_{n+1} \cdots b_{m-1}}{a_{n+1} a_{n+2} \cdots a_m} + (-1)^{m-n-2} (-\frac{a_n}{a_{n+2}} d_{n+1}) \frac{d_{n+2} b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m} + \\
&\quad (-1)^{m-n-2} (-\frac{a_n}{a_{n+2}} d_{n+1}) (\frac{b_{n+2}}{a_{n+3}}) (\frac{d_{n+3} b_{n+3} \cdots b_{m-1}}{a_{n+4} \cdots a_m}) + \cdots + \\
&\quad \cdots + (-1)^{m-n-2} (-\frac{a_n}{a_{n+2}} d_{n+1}) \frac{d_{m-1} b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m} + \\
&\quad (-1)^{m-n-2} (-\frac{a_n}{a_{n+2}} d_{n+1}) \frac{d_m b_{n+2} \cdots b_{m-1}}{a_{n+3} \cdots a_m} + (-1)^{m-n-1} (-\frac{a_n}{a_{n+2}} d_{n+1}) (\frac{b_{n+2} \cdots b_m}{a_{n+3} \cdots a_m^2}) \\
&= (-1)^{m-n-1} d_{n+1} \Big(\frac{a_n b_{n+1} \cdots b_{m-1}}{a_{n+1} a_{n+2} \cdots a_m} + \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} d_{n+2} + \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} d_{n+3} + \\
&\quad \cdots + \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} d_{m-1} + \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} d_m - \frac{a_n b_{n+2} \cdots b_m}{a_{n+2} a_{n+3} \cdots a_m^2} \Big) \\
&= (-1)^{m-n-1} d_{n+1} \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} \left(\frac{b_{n+1}}{a_{n+1}} + (d_{n+2} + d_{n+3} + \cdots + d_{m-1} + d_m) - \frac{b_m}{a_m} \right).
\end{aligned}$$

Recall from (3.2.1) that $d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}$, $n \geq 1$. Then

$$\alpha_{mn} = (-1)^{m-n-1} d_{n+1} \frac{a_n b_{n+2} \cdots b_{m-1}}{a_{n+2} a_{n+3} \cdots a_m} \left((\frac{b_{n+1}}{a_{n+1}} - \frac{b_m}{a_m}) + (\frac{b_m}{a_m} - \frac{b_{n+1}}{a_{n+1}}) \right) = 0.$$

For the case $n = m - 1$, we have

$$\alpha_{m,m-1} = d_m\Big(\frac{a_{m-1}}{a_m}\Big) + \frac{a_{m+1}}{a_m}(c_{m-1}) = \Big(\frac{a_{m-1}}{a_m}\Big)d_m + \frac{a_{m+1}}{a_m}\Big(-\frac{a_{m-1}}{a_{m+1}}d_m\Big) = 0,$$

and finally, $\alpha_{mm} = (\frac{a_{m+1}}{a_m})(\frac{a_m}{a_{m+1}}) = 1$ completes the proof. □

In view of Theorem 3.2.2, let us point out, in particular (see the discussion following (3.1.10)), that shifts on analytic tridiagonal spaces are always analytic:

Proposition 3.2.3. *If k is an analytic tridiagonal kernel, then M_z is an analytic left-invertible operator on $\mathcal{H}_k$.*

Now let $\mathcal{H}_k$ be an analytic tridiagonal space. Our aim is to compute the Shimorin left inverse $L_{M_z} = (M_z^* M_z)^{-1} M_z^*$ of M_z on $\mathcal{H}_k$. What we prove in fact is that L in Proposition 3.2.1 is the Shimorin left inverse of M_z. First note that

$$L_{M_z} z^n = z^{n-1} \qquad (n \geq 1). \tag{3.2.3}$$

Indeed, $L_{M_z} z^n = (M_z^* M_z)^{-1} M_z^* M_z z^{n-1} = (M_z^* M_z)^{-1}(M_z^* M_z) z^{n-1}$. Therefore, L_{M_z} is the backward shift on $\mathcal{H}_k$ (a well known fact about Shimorin left inverses). On the other hand, by Lemma 3.1.4 we have $L_{M_z} f_0 = (M_z^* M_z)^{-1} M_z^* f_0 = 0$, and hence $L_{M_z} f_0 = 0$, which in particular yields

$$L_{M_z} 1 = -\frac{b_0}{a_0}. \tag{3.2.4}$$

Let $n \geq 1$. Using (3.2.1), we have $L_{M_z} f_n = L_{M_z}(a_n z^n + b_n z^{n+1}) = a_n z^{n-1} + b_n z^n$, which implies

$$L_{M_z} f_n = \frac{a_n}{a_{n-1}}(a_{n-1} z^{n-1} + b_{n-1} z^n) + \Big(b_n - \frac{a_n b_{n-1}}{a_{n-1}}\Big) z^n = \frac{a_n}{a_{n-1}} f_{n-1} + d_n a_n z^n,$$

and hence $L_{M_z} f_n = \frac{a_n}{a_{n-1}} f_{n-1} + d_n(a_n z^n + b_n z^{n+1}) - d_n b_n z^{n+1}$. By (3.1.4), we have

$$L_{M_z} f_n = \frac{a_n}{a_{n-1}} f_{n-1} + d_n f_n - d_n \Big(\sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=0}^m b_{n+j}}{\prod_{j=0}^m a_{n+1+j}} f_{n+1+m} \Big).$$

This is precisely the left inverse L of M_z in Proposition 3.2.1. Whence the next statement:

Theorem 3.2.4. *Let $\mathcal{H}_k$ be an analytic tridiagonal space. If L is as in Proposition 3.2.1, then the Shimorin left inverse L_{M_z} of M_z is given by $L_{M_z} = L$. In particular, $L_{M_z} f_0 = 0$, and*

$$L_{M_z} f_n = \frac{a_n}{a_{n-1}} f_{n-1} + d_n f_n - d_n \Big(\sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=0}^m b_{n+j}}{\prod_{j=0}^m a_{n+1+j}} f_{n+1+m} \Big) \qquad (n \geq 1),$$

where $d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}$ *for all* $n \geq 1$*. Moreover, the matrix representation of* L_{M_z} *with respect to the orthonormal basis* $\{f_n\}_{n\geq 0}$ *is given by*

$$[L_{M_z}] = \begin{bmatrix} 0 & \frac{a_1}{a_0} & 0 & 0 & 0 & \dots \\ 0 & d_1 & \frac{a_2}{a_1} & 0 & 0 & \ddots \\ 0 & \frac{-d_1 b_1}{a_2} & d_2 & \frac{a_3}{a_2} & 0 & \ddots \\ 0 & \frac{d_1 b_1 b_2}{a_2 a_3} & \frac{-d_2 b_2}{a_3} & d_3 & \frac{a_4}{a_3} & \ddots \\ 0 & \frac{-d_1 b_1 b_2 b_3}{a_2 a_3 a_4} & \frac{d_2 b_2 b_3}{a_3 a_4} & \frac{-d_3 b_3}{a_4} & d_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Next we verify that the bounded away assumption of $\{|\frac{a_n}{a_{n+1}}|\}_{n\geq 0}$ in (3.1.3) is also a necessary condition for left-invertible shifts.

Theorem 3.2.5. *Let* $\mathcal{H}_k$ *be a semi-analytic tridiagonal space corresponding to the orthonormal basis* $\{f_n\}_{n\geq 0}$*, where* $f_n(z) = (a_n + b_n z)z^n$*,* $n \geq 0$*. Then* M_z *is left-invertible if and only if* $\{|\frac{a_n}{a_{n+1}}|\}_{n\geq 0}$ *is bounded away from zero, or equivalently,* $\mathcal{H}_k$ *is an analytic tridiagonal space.*

Proof. In view of Theorem 3.2.2 we only need to prove the necessary part. Consider the Shimorin left inverse $L_{M_z} = (M_z^* M_z)^{-1} M_z^*$. Using the fact that $\mathbb{C}[z] \subseteq \mathcal{H}_k$, one can show, along the similar line of computation preceding Theorem 3.2.4 (note that, by assumption, L_{M_z} is bounded), that the matrix representation of L_{M_z} with respect to the orthonormal basis $\{f_n\}_{n\geq 0}$ is precisely given by the one in Theorem 3.2.4. Then for each $n \geq 0$, we have

$$\|(M_z^* M_z)^{-1} M_z^*\|_{\mathcal{B}(\mathcal{H}_k)} \geq \|(M_z^* M_z)^{-1} M_z^* f_n\|_{\mathcal{H}_k} \geq \left|\frac{a_{n+1}}{a_n}\right|,$$

which implies that

$$\left|\frac{a_n}{a_{n+1}}\right| \geq \frac{1}{\|(M_z^* M_z)^{-1} M_z^*\|_{\mathcal{B}(\mathcal{H}_k)}},$$

and hence the sequence is bounded away from zero. □

3.3 Tridiagonal Shimorin models

As emphasized already in Proposition 3.1.7 that if k is a diagonal kernel, then k_{M_z} is also a diagonal kernel. However, as we will see in the example below, Shimorin kernels are not compatible with tridiagonal kernels. This consequently motivates one to ask: How to determine whether or not the Shimorin kernel k_{M_z} of a tridiagonal kernel k is also tridiagonal? We have a complete answer to this question: k_{M_z} is tridiagonal if and only if $b_0 = 0$ or that M_z is a weighted shift on $\mathcal{H}_k$. This is the main content of this section.

Example 3.3.1. *Let $a_n = 1$ for all $n \geq 0$, $b_0 = \frac{1}{2}$, and let $b_n = 0$ for all $n \geq 1$. Let $\mathcal{H}_k$ denote the analytic tridiagonal space corresponding to the orthonormal basis $\{f_n\}_{n\geq 0}$, where $f_n = (a_n + b_n z)z^n$ for all $n \geq 0$. Since $f_0 = 1 + \frac{1}{2}z$ and $f_n = z^n$ for all $n \geq 1$, by (3.1.8), we have*

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \ddots \\ \frac{1}{2} & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Let $a_n = 1$ for all $n \geq 0$, $b_0 = \frac{1}{2}$, and let $b_n = 0$ for all $n \geq 1$. Let $\mathcal{H}_k$ denote the analytic tridiagonal space corresponding to the orthonormal basis $\{f_n\}_{n\geq 0}$, where $f_n = (a_n + b_n z)z^n$ for all $n \geq 0$. Since $f_0 = 1 + \frac{1}{2}z$ and $f_n = z^n$ for all $n \geq 1$, by (3.1.8), we have

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \ddots \\ \frac{1}{2} & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

By Theorem 3.2.4, the Shimorin left inverse $L_{M_z} = (M_z^* M_z)^{-1} M_z^*$ is given by

$$L_{M_z} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & \frac{-1}{2} & 1 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 1 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Recall, in this case, that $\mathcal{W} = \mathbb{C}f_0$. It is easy to check that $L_{M_z} f_1 = f_0 - \frac{1}{2}f_1$, $L^*_{M_z} f_0 = f_1$, $L^*_{M_z} f_1 = -\frac{1}{2}f_1 + f_2$, and $L^*_{M_z} f_2 = f_3$. Then

$$L^{*3}_{M_z} f_0 = -\frac{1}{2} L^*_{M_z} f_1 + L^*_{M_z} f_2 = \frac{1}{4} f_1 - \frac{1}{2} f_2 + f_3,$$

and hence $P_{\mathcal{W}} L_{M_z} L^{*3}_{M_z} f_0 = \frac{1}{4} P_{\mathcal{W}}(L_{M_z} f_1)$, as $P_{\mathcal{W}} L_{M_z} f_j = 0$ for all $j \neq 1$. Consequently

$$P_{\mathcal{W}} L_{M_z} L^{*3}_{M_z} f_0 = \frac{1}{4} f_0 \neq 0,$$

which implies that the Shimorin kernel k_{M_z}, as defined in (3.1.11), is not a tridiagonal kernel.

Throughout this section, $\mathcal{H}_k$ will be an analytic tridiagonal space corresponding to the orthonormal basis $\{f_n\}_{n\geq 0}$, where $f_n(z) = (a_n + b_n z)z^n$, $n \geq 0$. Recall that the

Shimorin kernel $k_{M_z} : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{W})$ is given by (see (3.1.11) and also Theorem 3.1.5)

$$k_{M_z}(z, w) = P_{\mathcal{W}}(I - zL_{M_z})^{-1}(I - \bar{w}L^*_{M_z})^{-1}|_{\mathcal{W}} \qquad (z, w \in \mathbb{D}).$$

Here, of course, $\mathcal{W} = \mathbb{C}f_0$, the one-dimensional space generated by the vector f_0. So one may regard k_{M_z} as a scalar kernel. We are now ready for the main result of this section.

Theorem 3.3.2. *The Shimorin kernel k_{M_z} of M_z is tridiagonal if and only if M_z on $\mathcal{H}_k$ is a weighted shift or*

$$b_0 = 0.$$

Proof. We split the proof into several steps.

Step 1: We first denote $L_{M_z} = L$ and

$$X_{mn} = P_{\mathcal{W}} L^m L^{*n}|_{\mathcal{W}} \qquad (m, n \geq 0),$$

for simplicity. First observe that Theorem 3.2.4 implies that $L^m f_0 = 0$, $m \geq 1$, and hence, $X_{m0} = 0 = X^*_{m0} = X_{0m}$ for all $m \geq 1$. Then the formal matrix representation of the Shimorin kernel k_{M_z} is given by

$$[k_{M_z}] = \begin{bmatrix} I_{\mathcal{W}} & 0 & 0 & 0 & \dots \\ 0 & X_{11} & X_{12} & X_{13} & \dots \\ 0 & X^*_{12} & X_{22} & X_{23} & \dots \\ 0 & X^*_{13} & X^*_{23} & X_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{3.3.1}$$

Clearly, in view of the above, k_{M_z} is tridiagonal if and only if $X_{mn} f_0 = 0$ for all $m, n \neq 0$ and $|m - n| \geq 2$.

Step 2: In this step we aim to compute matrix representations of L^p and L^{*p}, $p \geq 1$, with respect to the orthonormal basis $\{f_n\}_{n \geq 0}$. The matrix representation of $[L]$ in Theorem 3.2.4 is instructive. It also follows that

$$[L^*] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{\bar{a}_1}{\bar{a}_0} & \bar{d}_1 & \frac{-\bar{d}_1 \bar{b}_1}{\bar{a}_2} & \frac{\bar{d}_1 \bar{b}_1 \bar{b}_2}{\bar{a}_2 \bar{a}_3} & \frac{-\bar{d}_1 \bar{b}_1 \bar{b}_2 \bar{b}_3}{\bar{a}_2 \bar{a}_3 \bar{a}_4} & \dots \\ 0 & \frac{\bar{a}_2}{\bar{a}_1} & \bar{d}_2 & \frac{-\bar{d}_2 \bar{b}_2}{\bar{a}_3} & \frac{\bar{d}_2 \bar{b}_2 \bar{b}_3}{\bar{a}_3 \bar{a}_4} & \ddots \\ 0 & 0 & \frac{\bar{a}_3}{\bar{a}_2} & \bar{d}_3 & \frac{-\bar{d}_3 \bar{b}_3}{\bar{a}_4} & \ddots \\ 0 & 0 & 0 & \frac{\bar{a}_4}{\bar{a}_3} & \bar{d}_4 & \ddots \\ 0 & 0 & 0 & 0 & \frac{\bar{a}_5}{\bar{a}_4} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{3.3.2}$$

Here we redo the construction taking into account the general $p \geq 1$, and proceed as in the proof of Theorem 3.2.4. However, the proof is by no means the same and the general case is quite involved. Assume that $n \geq 1$. We need to consider two cases: $n \geq p$ and $n \leq p-1$. Suppose $n \geq p$. By (3.2.3) and (3.2.4), we have

$$L^p f_n = a_n L^p z^n + b_n L^p z^{n+1} = a_n z^{n-p} + b_n z^{n-p+1},$$

which implies

$$L^p f_n = \frac{a_n}{a_{n-p}}(a_{n-p}z^{n-p}+b_{n-p}z^{n-p+1})+(b_n-\frac{a_n}{a_{n-p}}b_{n-p})z^{n-p+1} = \frac{a_n}{a_{n-p}}f_{n-p}+d_n^{(p)}z^{n-p+1},$$

where

$$d_n^{(p)} = b_n - \frac{a_n}{a_{n-p}}b_{n-p} \qquad (n \geq p). \tag{3.3.3}$$

Hence by (3.1.4)

$$L^p f_n = \frac{a_n}{a_{n-p}}f_{n-p} + \frac{d_n^{(p)}}{a_{n-p+1}}\Big(f_{n-p+1} - \frac{b_{n-p+1}}{a_{n-p+2}}f_{n-p+2} + \frac{b_{n-p+1}b_{n-p+2}}{a_{n-p+2}a_{n-p+3}}f_{n-p+3} - \cdots\Big),$$

that is

$$L^p f_n = \frac{a_n}{a_{n-p}}f_{n-p} + \frac{d_n^{(p)}}{a_{n-p+1}}\sum_{m=0}^{\infty}(-1)^m\Big(\frac{\prod_{j=0}^{m-1} b_{n-p+j+1}}{\prod_{j=0}^{m-1} a_{n-p+j+2}}\Big)f_{n-p+m+1},$$

for all $n \geq p$. Here and in what follows, we define $\prod_{j=0}^{-1} x_j := 1$.

We now let $p = 1$ and $n = 1$. Then by Theorem 3.2.4, we have

$$Lf_1 = \frac{a_1}{a_0}f_0 + d_1 f_1 + (-\frac{d_1 b_1}{a_2})f_2 + (\frac{d_1 b_1 b_2}{a_2 a_3})f_3 + \cdots. \tag{3.3.4}$$

Finally, let $1 \leq n \leq p-1$. Then $p > 1$, and again by (3.2.3) and (3.2.4), we have

$$L^p f_n = L^p(a_n z^n + b_n z^{n+1}) = a_n L^{p-n}1 + b_n L^{p-n-1}1 = a_n\Big(\frac{-b_0}{a_0}\Big)^{p-n} + b_n\Big(\frac{-b_0}{a_0}\Big)^{p-n-1},$$

and hence $L^p f_n = a_n\Big(\frac{-b_0}{a_0}\Big)^{p-n-1}\Big[\frac{b_n}{a_n} - \frac{b_0}{a_0}\Big]$. We set

$$\beta_n = \frac{b_n}{a_n} - \frac{b_0}{a_0} \qquad (n \geq 1), \tag{3.3.5}$$

and

$$\beta_n^{(p)} = a_n\Big(\frac{-b_0}{a_0}\Big)^{p-n-1}\beta_n \qquad (1 \leq n \leq p-1). \tag{3.3.6}$$

Then $L^p f_n = \beta_n^{(p)}$ and (3.1.4) implies that

$$L^p(f_n) = \frac{\beta_n^{(p)}}{a_0}\sum_{m=0}^{\infty}(-1)^m\Big(\frac{\Pi_{j=0}^{m-1}b_j}{\Pi_{j=0}^{m-1}a_{j+1}}\Big)f_m,$$

for all $1 \leq n \leq p-1$. Then

$$[L^2] = \begin{bmatrix} 0 & \frac{\beta_1^{(2)}}{a_0} & \frac{a_2}{a_0} & 0 & 0 & \cdots \\ 0 & -\frac{\beta_1^{(2)} b_0}{a_0 a_1} & \frac{d_2^{(2)}}{a_1} & \frac{a_3}{a_1} & 0 & \ddots \\ 0 & \frac{\beta_1^{(2)} b_0 b_1}{a_0 a_1 a_2} & -\frac{d_2^{(2)} b_1}{a_1 a_2} & \frac{d_3^{(2)}}{a_2} & \frac{a_4}{a_2} & \ddots \\ 0 & -\frac{\beta_1^{(2)} b_0 b_1 b_2}{a_0 a_1 a_2 a_3} & \frac{d_2^{(2)} b_1 b_2}{a_1 a_2 a_3} & -\frac{d_3^{(2)} b_2}{a_2 a_3} & \frac{d_4^{(2)}}{a_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \tag{3.3.7}$$

and in general, for each $p \geq 2$, we have

$$[L^p] = \begin{bmatrix} 0 & \frac{\beta_1^{(p)}}{a_0} & \frac{\beta_2^{(p)}}{a_0} & \cdots & \frac{\beta_{p-1}^{(p)}}{a_0} & \frac{a_p}{a_0} & 0 & 0 & \cdots \\ 0 & -\frac{\beta_1^{(p)} b_0}{a_0 a_1} & -\frac{\beta_2^{(p)} b_0}{a_0 a_1} & \cdots & -\frac{\beta_{p-1}^{(p)} b_0}{a_0 a_1} & \frac{d_p^{(p)}}{a_1} & \frac{a_{p+1}}{a_1} & 0 & \ddots \\ 0 & \frac{\beta_1^{(p)} b_0 b_1}{a_0 a_1 a_2} & \frac{\beta_2^{(p)} b_0 b_1}{a_0 a_1 a_2} & \cdots & \frac{\beta_{p-1}^{(p)} b_0 b_1}{a_0 a_1 a_2} & -\frac{d_p^{(p)} b_1}{a_1 a_2} & \frac{d_{p+1}^{(p)}}{a_2} & \frac{a_{p+2}}{a_2} & \ddots \\ 0 & -\frac{\beta_1^{(p)} b_0 b_1 b_2}{a_0 a_1 a_2 a_3} & -\frac{\beta_2^{(p)} b_0 b_1 b_2}{a_0 a_1 a_2 a_3} & \cdots & -\frac{\beta_{p-1}^{(p)} b_0 b_1 b_2}{a_0 a_1 a_2 a_3} & \frac{d_p^{(p)} b_1 b_2}{a_1 a_2 a_3} & -\frac{d_{p+1}^{(p)} b_2}{a_2 a_3} & \frac{d_{p+2}^{(p)}}{a_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{3.3.8}$$

Hence, for each $p \geq 2$, we have

$$[L^{*p}] = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{\bar{\beta}_1^{(p)}}{\bar{a}_0} & -\frac{\bar{\beta}_1^{(p)} \bar{b}_0}{\bar{a}_0 \bar{a}_1} & \frac{\bar{\beta}_1^{(p)} \bar{b}_0 \bar{b}_1}{\bar{a}_0 \bar{a}_1 \bar{a}_2} & -\frac{\bar{\beta}_1^{(p)} \bar{b}_0 \bar{b}_1 \bar{b}_2}{\bar{a}_0 \bar{a}_1 \bar{a}_2 \bar{a}_3} & \ddots \\ \frac{\bar{\beta}_2^{(p)}}{\bar{a}_0} & -\frac{\bar{\beta}_2^{(p)} \bar{b}_0}{\bar{a}_0 \bar{a}_1} & \frac{\bar{\beta}_2^{(p)} \bar{b}_0 \bar{b}_1}{\bar{a}_0 \bar{a}_1 \bar{a}_2} & -\frac{\bar{\beta}_2^{(p)} \bar{b}_0 \bar{b}_1 \bar{b}_2}{\bar{a}_0 \bar{a}_1 \bar{a}_2 \bar{a}_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{\bar{\beta}_{p-1}^{(p)}}{\bar{a}_0} & -\frac{\bar{\beta}_{p-1}^{(p)} \bar{b}_0}{\bar{a}_0 \bar{a}_1} & \frac{\bar{\beta}_{p-1}^{(p)} \bar{b}_0 \bar{b}_1}{\bar{a}_0 \bar{a}_1 \bar{a}_2} & -\frac{\bar{\beta}_{p-1}^{(p)} \bar{b}_0 \bar{b}_1 \bar{b}_2}{\bar{a}_0 \bar{a}_1 \bar{a}_2 \bar{a}_3} & \ddots \\ \frac{\bar{a}_p}{\bar{a}_0} & \frac{\bar{d}_p^{(p)}}{\bar{a}_1} & -\frac{\bar{d}_p^{(p)} \bar{b}_1}{\bar{a}_1 \bar{a}_2} & \frac{\bar{d}_p^{(p)} \bar{b}_1 \bar{b}_2}{\bar{a}_1 \bar{a}_2 \bar{a}_3} & \ddots \\ 0 & \frac{\bar{a}_{p+1}}{\bar{a}_1} & \frac{\bar{d}_{p+1}^{(p)}}{\bar{a}_2} & -\frac{\bar{d}_{p+1}^{(p)} \bar{b}_2}{\bar{a}_2 \bar{a}_3} & \ddots \\ 0 & 0 & \frac{\bar{a}_{p+2}}{\bar{a}_2} & \frac{\bar{d}_{p+2}^{(p)}}{\bar{a}_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.3.9}$$

Step 3: We now identify condition on the sequence $\{\beta_n^{(n+2)}\}_{n\geq 1}$ implied by the requirement that $X_{m,m+2} = 0$, $m \geq 1$. Before proceeding further, we record here the following crucial observation: Suppose $\beta_n^{(p)} = 0$ for some p and n such that $1 \leq n \leq p-1$. Then by (3.3.6), we have

$$\beta_n^{(q)} = 0 \qquad (q \geq p). \tag{3.3.10}$$

Now assume $m \geq 1$. The matrix representation in (3.3.9) implies

$$L^{*m+2} f_0 = \frac{1}{\bar{a}_0}\Big(\bar{\beta}_1^{(m+2)} f_1 + \bar{\beta}_2^{(m+2)} f_2 + \cdots + \bar{\beta}_{m+1}^{(m+2)} f_{m+1} + \bar{a}_{m+2} f_{m+2}\Big). \tag{3.3.11}$$

Observe that, by Theorem 3.2.4, we have

$$P_{\mathcal{W}}L(f_i)=\begin{cases}\frac{a_1}{a_0}f_0 & \text{if } i=1\\ 0 & \text{if } i\neq 1.\end{cases}$$

Let us now assume that $m\geq 2$. Then (3.3.8) implies

$$P_{\mathcal{W}}L^m(f_i)=\begin{cases}\frac{\beta_i^{(m)}}{a_0}f_0 & \text{if } 1\leq i\leq m-1\\ \frac{a_m}{a_0}f_0 & \text{if } i=m\\ 0 & \text{if } i\geq m+1.\end{cases} \tag{3.3.12}$$

Since $X_{m,m+2}=P_{\mathcal{W}}L^mL^{*m+2}|_{\mathcal{W}}$, this yields

$$X_{m,m+2}f_0=\frac{1}{|a_0|^2}\left(\bar{\beta}_1^{(m+2)}\beta_1^{(m)}+\bar{\beta}_2^{(m+2)}\beta_2^{(m)}+\cdots+\bar{\beta}_{m-1}^{(m+2)}\beta_{m-1}^{(m)}+\bar{\beta}_m^{(m+2)}a_m\right)f_0. \tag{3.3.13}$$

In particular, if $m=1$, then we have

$$X_{13}f_0=\frac{1}{\bar{a}_0}\left(\bar{\beta}_1^{(3)}\frac{a_1}{a_0}\right)f_0,$$

and hence $X_{13}=0$ if and only if $\beta_1^{(3)}=0$. By (3.3.13), applied with $m=2$ we have

$$X_{24}f_0=\frac{1}{|a_0|^2}\left(\bar{\beta}_1^{(4)}\beta_1^{(2)}+\bar{\beta}_2^{(4)}a_2\right)f_0.$$

Assume that $\beta_1^{(3)}=0$. By (3.3.10), we have $\beta_1^{(4)}=0$, and, consequently

$$X_{24}f_0=\bar{\beta}_2^{(4)}\frac{a_2}{|a_0|^2}f_0.$$

Hence we obtain $X_{24}=0$ if and only if $\beta_2^{(4)}=0$. Therefore, if $X_{m,m+2}=0$ for all $m\geq 1$, then by induction, it follows that $\beta_m^{(m+2)}=0$ for all $m\geq 1$. The converse also follows from the above computation.

Thus we have proved: $X_{m,m+2}=0$ *for all* $m\geq 1$ *if and only if* $\beta_m^{(m+2)}=0$ *for all* $m\geq 1$.

Step 4: Our aim is to prove the following claim: *Suppose* $X_{i,i+2}=0$ *for all* $i=1,\ldots,m$, *and* $m\geq 1$. *Then* $X_{mn}=0$ *for all* $n=m+3,m+4,\ldots$, *and* $m\geq 1$.

To this end, let $n=m+j$ and $j\geq 3$. Then the matrix representation in (3.3.9) (or the equality (3.3.11)) implies

$$L^{*n}f_0=\frac{1}{\bar{a}_0}\left(\bar{\beta}_1^{(n)}f_1+\bar{\beta}_2^{(n)}f_2+\cdots+\bar{\beta}_{n-1}^{(n)}f_{n-1}+\bar{a}_nf_n\right),$$

and then

$$P_{\mathcal{W}} L^m L^{*n} f_0 = \Big(\frac{1}{\bar{a}_0}\sum_{i=1}^{n-1}\bar{\beta}_i^{(n)} P_{\mathcal{W}} L^m(f_i)\Big) + \frac{\bar{a}_n}{\bar{a}_0} P_{\mathcal{W}} L^m f_n = \frac{1}{\bar{a}_0}\sum_{i=1}^{m}\bar{\beta}_i^{(n)} P_{\mathcal{W}} L^m(f_i),$$

since $P_{\mathcal{W}} L^m f_i = 0$, $i > m$, which follows from the matrix representation of L^m in (3.3.8). Hence by (3.3.12) (or directly from (3.3.8)), we have

$$P_{\mathcal{W}} L^m L^{*n} f_0 = \frac{1}{|a_0|^2}\Big(\bar{\beta}_1^{(n)}\beta_1^{(m)} + \bar{\beta}_2^{(n)}\beta_2^{(m)} + \cdots + \bar{\beta}_{m-1}^{(n)}\beta_{m-1}^{(m)} + a_m\bar{\beta}_m^{(n)}\Big) f_0.$$

Now note that $X_{i,i+2} = 0$, that is, $\beta_i^{(i+2)} = 0$, $i = 1, \ldots, m$, by assumption. Since $i + 2 \leq m + j$ for all $i = 1, \ldots, m$, by (3.3.10), we have

$$\beta_i^{(n)} = \beta_i^{(m+j)} = 0 \qquad (i = 1, \ldots, m).$$

Hence $P_{\mathcal{W}} L^m L^{*n} f_0 = 0$, that is, $X_{m,m+i} = 0$, $i = 3, 4, \ldots$, which proves the claim.

Step 5: So far all we have proved is that $X_{mn} = 0$ for all $|m - n| \geq 2$ if and only if $\beta_m^{(m+2)} = 0$ for all $m \geq 1$. Now, by (3.3.6) and (3.3.5), we have

$$\beta_n^{(n+2)} = a_n\Big(-\frac{b_0}{a_0}\Big)\beta_n,$$

where $\beta_n = \frac{b_n}{a_n} - \frac{b_0}{a_0}$ for all $n \geq 1$. Thus $\beta_n^{(n+2)} = 0$ for all $n \geq 1$ if and only if $b_0 = 0$ or $\beta_n = 0$ for all $n \geq 1$. On the other hand, Lemma 3.1.3 implies that $\beta_n = 0$ for all $n \geq 1$ if and only if M_z is a weighted shift.

Finally, by Proposition 3.1.7, we know that if M_z is a left-invertible weighted shift, then the Shimorin kernel is also a diagonal kernel. This completes the proof of Theorem 3.3.2. $\square$

3.4 Positive operators and tridiagonal kernels

Our aim is to classify positive operators P on a tridiagonal space $\mathcal{H}_k$ such that

$$\mathbb{D} \times \mathbb{D} \ni (z, w) \mapsto \langle Pk(\cdot, w), k(\cdot, z)\rangle_{\mathcal{H}_k},$$

is also a tridiagonal kernel. While this problem is of independent interest, the motivation for our interest in this question also comes from Theorem 3.6.7 (also see the paragraph preceding Corollary 3.8.2). We start with a simple example.

Example 3.4.1. *We consider the same example as in Example 3.3.1. Note that M_z is left-invertible and not a weighted shift with respect to the orthonormal basis $\{f_n\}_{n\geq 0}$ of*

$\mathcal{H}_k$. Then by Lemma 3.1.6, we have

$$|M_z|^{-2} = L_{M_z} L_{M_z}^* = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \dots \\ -\frac{1}{2} & \frac{5}{4} & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let

$$|M_z|^{-1} = \begin{bmatrix} \alpha & \beta & 0 & 0 & \dots \\ \beta & \gamma & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ is the positive square root of $\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{bmatrix}$. A straightforward calculation shows that $\frac{\alpha}{2} + \beta \neq 0$. Define $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ by

$$K(z,w) = \langle |M_z|^{-1} k(\cdot, w), k(\cdot, z)\rangle_{\mathcal{H}_k} \qquad (z, w \in \mathbb{D}).$$

A simple computation then shows that

$$K(z,w) = \alpha + (\frac{\alpha}{2} + \beta)\bar{w} + (\frac{\alpha}{2} + \beta)z + (\frac{\alpha}{4} + \beta + \gamma)z\bar{w} + \sum_{n\geq 2} z^n \bar{w}^n,$$

that is, K is also a tridiagonal kernel.

The following is a complete classification of positive operators P for which $(z,w) \mapsto \langle Pk(\cdot, w), k(\cdot, z)\rangle_{\mathcal{H}_k}$ defines a tridiagonal kernel.

Theorem 3.4.2. *Let $\mathcal{H}_k$ be a tridiagonal space corresponding to the orthonormal basis $f_n(z) = (a_n + b_n z)z^n$, $n \geq 0$. Let P be a positive operator on $\mathcal{H}_k$ with matrix representation*

$$P = \begin{bmatrix} c_{00} & c_{01} & c_{02} & c_{03} & \dots \\ \bar{c}_{01} & c_{11} & c_{12} & c_{13} & \ddots \\ \bar{c}_{02} & \bar{c}_{12} & c_{22} & c_{23} & \ddots \\ \bar{c}_{03} & \bar{c}_{13} & \bar{c}_{23} & c_{33} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

with respect to the basis $\{f_n\}_{n\geq 0}$. Then the positive definite scalar kernel K, defined by

$$K(z,w) = \langle Pk(\cdot, w), k(\cdot, z)\rangle_{\mathcal{H}_k} \qquad (z, w \in \mathbb{D}),$$

is tridiagonal if and only if

$$c_{0n} = (-1)^{n-1}\frac{\bar{b}_1\cdots\bar{b}_{n-1}}{\bar{a}_2\cdots\bar{a}_n}c_{01} \qquad (n \geq 2),$$

and

$$c_{mn} = (-1)^{n-m-1}\frac{\bar{b}_{m+1}\cdots\bar{b}_{n-1}}{\bar{a}_{m+2}\cdots\bar{a}_n}c_{m,m+1} \qquad (1 \leq m \leq n-2).$$

Equivalently, K is tridiagonal if and only if

$$P = \begin{bmatrix} c_{00} & c_{01} & -\frac{\bar{b}_1}{\bar{a}_2}c_{01} & \frac{\bar{b}_1\bar{b}_2}{\bar{a}_2\bar{a}_3}c_{01} & \cdots \\ \bar{c}_{01} & c_{11} & c_{12} & -\frac{\bar{b}_2}{\bar{a}_3}c_{12} & \ddots \\ -\frac{b_1}{a_2}\bar{c}_{01} & \bar{c}_{12} & c_{22} & c_{23} & \ddots \\ \frac{b_1b_2}{a_2a_3}\bar{c}_{01} & -\frac{b_2}{a_3}\bar{c}_{12} & \bar{c}_{23} & c_{33} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Proof. Note, for each $w \in \mathbb{D}$, by (1.0.1), we have $k(\cdot, w) = \sum_{m=0}^{\infty} \overline{f_m(w)} f_m$, and thus

$$Pk(\cdot,w) = \sum_{m=0}^{\infty}\Big(\sum_{n=0}^{m-1}\bar{c}_{nm}\overline{f_n(w)} + \sum_{n=m}^{\infty}c_{mn}\overline{f_n(w)}\Big)f_m,$$

where $\sum_{n=0}^{-1} x_n := 0$. Then

$$\begin{aligned}\langle Pk(\cdot,w), k(\cdot,z)\rangle_{\mathcal{H}_k} &= \sum_{m=0}^{\infty} f_m(z)\Big(\sum_{n=0}^{m-1}\bar{c}_{nm}\overline{f_n(w)} + \sum_{n=m}^{\infty}c_{mn}\overline{f_n(w)}\Big) \\ &= \sum_{m=0}^{\infty}(a_m z^m + b_m z^{m+1})\Big(\sum_{n=0}^{m-1}\bar{c}_{nm}(\bar{a}_n\bar{w}^n + \bar{b}_n\bar{w}^{n+1}) \\ &\qquad + \sum_{n=m}^{\infty}c_{mn}(\bar{a}_n\bar{w}^n + \bar{b}_n\bar{w}^{n+1})\Big) \\ &= \sum_{m,n\geq 0}\alpha_{mn}z^m\bar{w}^n,\end{aligned}$$

where α_{mn} denotes the coefficient of $z^m\bar{w}^n$, $m,n \geq 0$. Our interest here is to compute α_{mn}, $|m-n| \geq 2$. Clearly, $\alpha_{mn} = \bar{\alpha}_{nm}$ for all $m, n \geq 0$, and

$$\alpha_{0n} = a_0(\bar{a}_n c_{0n} + \bar{b}_{n-1}c_{0,n-1}) \qquad (n \geq 2), \tag{3.4.1}$$

and

$$\alpha_{mn} = a_m\Big(\bar{a}_n c_{mn} + \bar{b}_{n-1}c_{m,n-1}\Big) + b_{m-1}\Big(\bar{a}_n c_{m-1,n} + \bar{b}_{n-1}c_{m-1,n-1}\Big) \qquad (1 \leq m < n). \tag{3.4.2}$$

Suppose $n \geq 2$. By (3.4.1), $\alpha_{0n} = 0$ if and only if $c_{0n} = -\frac{\bar{b}_{n-1}}{\bar{a}_n} c_{0,n-1}$. In particular, if $n = 2$, then $c_{02} = -\frac{\bar{b}_1}{\bar{a}_2} c_{01}$, and hence, by (3.4.1) again, we have

$$c_{0n} = (-1)^{n-1} \frac{\prod_{i=1}^{n-1} \bar{b}_i}{\prod_{i=2}^{n} \bar{a}_i} c_{01} \qquad (n \geq 2).$$

Therefore, $\alpha_{0n} = 0$ for all $n \geq 2$ if and only if the above identity hold for all $n \geq 2$.

Next we want to consider the case $m, n \neq 0$ and $|m - n| \geq 2$. Assume that $n \geq 3$. Then (3.4.2) along with (3.4.1) implies

$$\alpha_{1n} = a_1(\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1,n-1}) + b_0(\bar{a}_n c_{0n} + \bar{b}_{n-1} c_{0,n-1}) = a_1(\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1,n-1}) + \frac{b_0}{a_0} \alpha_{0n}.$$

Therefore, if $\alpha_{0n} = 0$ for all $n \geq 3$, then $\alpha_{1n} = a_1(\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1,n-1})$. Hence $\alpha_{1n} = 0$ if and only if $\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1,n-1} = 0$, which is equivalent to

$$c_{1n} = -\frac{\bar{b}_{n-1}}{\bar{a}_n} c_{1,n-1}.$$

Therefore, under the assumption that $\alpha_{1n} = 0$ and $n \geq 4$, (3.4.2) along with (3.4.1) implies

$$\alpha_{2n} = a_2(\bar{a}_n c_{2n} + \bar{b}_{n-1} c_{2,n-1}) + b_1(\bar{a}_n c_{1n} + \bar{b}_{n-1} c_{1,n-1}) = a_2(\bar{a}_n c_{2n} + \bar{b}_{n-1} c_{2,n-1}).$$

Then $\alpha_{2n} = 0$, $n \geq 4$, if and only if $c_{2n} = -\frac{\bar{b}_{n-1}}{\bar{a}_n} c_{2,n-1}$. Consequently, by induction, for all $m, n \neq 0$ and $|m-n| \geq 2$, we have that $\alpha_{mn} = 0$ if and only if $\bar{a}_n c_{mn} + \bar{b}_{n-1} c_{m,n-1} = 0$, or equivalently

$$c_{mn} = -\frac{\bar{b}_{n-1}}{\bar{a}_n} c_{m,n-1}.$$

Finally, observe that $c_{mn} = (-1)^{n-m-1} \frac{\bar{b}_{n-1} \cdots \bar{b}_{m+1}}{\bar{a}_n \cdots \bar{a}_{m+2}} c_{m,m+1}$ for all $1 \leq m \leq n-2$. This completes the proof of the theorem. □

We will return to this in Theorem 3.7.3 and Corollary 3.7.4.

3.5 Quasinormal operators

A bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *quasinormal* if T^*T and T commutes, that is

$$[T^*, T]T = 0,$$

where $[T^*, T] = T^*T - TT^*$ is the commutator of T. In this section, we present a complete classification of quasinormality of M_z on analytic tridiagonal spaces. Here, however, we do not need to assume that M_z is left-invertible.

To motivate our result on quasinormality, we first consider the known case of weighted shifts. Recall that the weighted shift S_α corresponding to the weight sequence (of positive real numbers) $\{\alpha_n\}_{n\geq 0}$ is given by $S_\alpha e_n = \alpha_n e_{n+1}$ for all $n \geq 0$. Then (see the proof of Proposition 3.1.7)

$$S_\alpha S_\alpha^* e_{n+1} = \alpha_n^2 e_{n+1},$$

and hence $(S_\alpha^* S_\alpha - S_\alpha S_\alpha^*)S_\alpha = 0$ if and only if $(S_\alpha^* S_\alpha - S_\alpha S_\alpha^*)S_\alpha e_n = 0$ for all $n \geq 0$, which is equivalent to

$$\alpha_n(\alpha_{n+1}^2 - \alpha_n^2) = 0,$$

for all n. Thus, we have proved [37, Problem 139]:

Lemma 3.5.1. *The weighted shift S_α is quasinormal if and only if the weight sequence $\{\alpha_n\}_{n\geq 0}$ is a constant sequence.*

Now we turn to M_z on a semi-analytic tridiagonal space $\mathcal{H}_k$. Suppose $[M_z^*, M_z] = rP_{f_0}$, where r is a non-negative real number and P_{f_0} denote the orthogonal projection of $\mathcal{H}_k$ onto the one dimensional space $\mathbb{C}f_0$. Then $[M_z^*, M_z]M_z = rP_{f_0}M_z$ implies that

$$([M_z^*, M_z]M_z)f_n = rP_{f_0}(zf_n).$$

Now by (3.1.7) we have

$$zf_n = \sum_{i=n+1}^{\infty} \beta_i f_i,$$

for some scalar $\beta_i \in \mathbb{C}$, $i \geq n+1$. Note that $\beta_{n+1} = \frac{a_n}{a_{n+1}} \neq 0$. This shows that $P_{f_0}(zf_n) = 0$, and hence

$$([M_z^*, M_z]M_z)f_n = 0 \qquad (n \geq 0),$$

that is, M_z is quasinormal. Conversely, assume that M_z is a non-normal and quasinormal operator. Then $[M_z^*, M_z]M_z = 0$ implies that $\text{ran}M_z \subseteq \ker[M_z^*, M_z]$, and therefore, by Lemma 3.1.4, we have

$$\mathbb{C}f_0 = \ker M_z^* \supseteq \overline{ran}[M_z^*, M_z].$$

Clearly this implies $[M_z^*, M_z] = rP_{f_0}$ for some non-zero scalar r. Then

$$r\|f_0\|^2 = \langle rP_{f_0}f_0, f_0\rangle_{\mathcal{H}_k} = \langle [M_z^*, M_z]f_0, f_0\rangle_{\mathcal{H}_k} = \|M_z f_0\|^2 - \|M_z^* f_0\|^2 = \|M_z f_0\|^2,$$

as $M_z^* f_0 = 0$, which implies

$$r = \frac{\|M_z f_0\|^2}{\|f_0\|^2} > 0.$$

Thus, we have proved:

Theorem 3.5.2. *Let $\mathcal{H}_k$ be a semi-analytic tridiagonal space. Assume that M_z is a non-normal operator on $\mathcal{H}_k$. Then M_z is quasinormal if and only if there exists a positive* *real number r such that*

$$M_z^* M_z - M_z M_z^* = r P_{f_0},$$

where P_{f_0} denote the orthogonal projection of $\mathcal{H}_k$ onto the one dimensional space $\mathbb{C} f_0$.

In more algebraic terms this result can be formulated as follows: First we recall the matrix representation of M_z (see (3.1.8))

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \frac{a_0}{a_1} & 0 & 0 & 0 & \ddots \\ c_0 & \frac{a_1}{a_2} & 0 & 0 & \ddots \\ \frac{-c_0 b_2}{a_3} & c_1 & \frac{a_2}{a_3} & 0 & \ddots \\ \frac{c_0 b_2 b_3}{a_3 a_4} & \frac{-c_1 b_3}{a_4} & c_2 & \frac{a_3}{a_4} & \ddots \\ \frac{-c_0 b_2 b_3 b_4}{a_3 a_4 a_5} & \frac{c_1 b_3 b_4}{a_4 a_5} & \frac{-c_2 b_4}{a_5} & c_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

For each $n \geq 0$, we denote by R_n and C_n the n-th row and n-th column, respectively, of $[M_z]$. We then identify each of these column and row vectors with elements in $\mathcal{H}_k$. Then $R_n, C_n \in \mathcal{H}_k$, $n \geq 0$. Using the matrix representation $[M_z^*]$ (see (3.1.9)) and $[M_z]$, we get

$$\langle R_0, R_n \rangle_{\mathcal{H}_k} = 0,$$

for all $n \geq 0$, and, consequently

$$\Big[[M_z^*, M_z]\Big] = \begin{bmatrix} \langle C_0, C_0 \rangle_{\mathcal{H}_k} & \langle C_1, C_0 \rangle_{\mathcal{H}_k} & \langle C_2, C_0 \rangle_{\mathcal{H}_k} & \cdots \\ \langle C_0, C_1 \rangle_{\mathcal{H}_k} & \langle C_1, C_1 \rangle_{\mathcal{H}_k} - \langle R_1, R_1 \rangle_{\mathcal{H}_k} & \langle C_2, C_1 \rangle_{\mathcal{H}_k} - \langle R_1, R_2 \rangle_{\mathcal{H}_k} & \cdots \\ \langle C_0, C_2 \rangle_{\mathcal{H}_k} & \langle C_1, C_2 \rangle_{\mathcal{H}_k} - \langle R_2, R_1 \rangle_{\mathcal{H}_k} & \langle C_2, C_2 \rangle_{\mathcal{H}_k} - \langle R_2, R_2 \rangle_{\mathcal{H}_k} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore:

Corollary 3.5.3. *Let $\mathcal{H}_k$ be a semi-analytic tridiagonal space. Then M_z on $\mathcal{H}_k$ is quasinormal if and only if $\langle C_0, C_0 \rangle_{\mathcal{H}_k} = r$ and*

$$\langle C_0, C_i \rangle_{\mathcal{H}_k} = 0 \qquad (i \geq 1),$$

and

$$\langle C_n, C_m \rangle_{\mathcal{H}_k} - \langle R_m, R_n \rangle_{\mathcal{H}_k} = 0,$$

for all $1 \leq m \leq n$.

It is easy to see that a quasinormal operator is always subnormal [37]. However, a complete classification of subnormality of M_z on tridiagonal spaces is rather more subtle and not quite as clear-cut as in the quasinormal situation. In fact the general classification of subnormality of M_z on tridiagonal spaces is not known (however, see [2]).

3.6 Aluthge transforms of shifts

Recall that the *Aluthge transform* of an operator $T \in \mathcal{B}(\mathcal{H})$ is the bounded linear operator

$$\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}.$$

In this section, we prove that the Aluthge transform of a left-invertible shift on an analytic Hilbert space is again an explicit shift on some analytic Hilbert space. We present two approaches to this problem, one based on Shimorin's analytic models of left-invertible operators and one is based on rather direct reproducing kernel Hilbert space techniques.

We begin with the following simple fact concerning Aluthge transforms of left-invertible operators:

Lemma 3.6.1. *If T is a left-invertible operator on $\mathcal{H}$, then*

$$\tilde{T} = |T|^{\frac{1}{2}} T |T|^{-\frac{1}{2}},$$

and $\ker \tilde{T}^* = |T|^{-\frac{1}{2}} \ker T^*$. *In particular, $\tilde{T}$ is similar to T.*

Proof. Indeed, $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} = |T|^{\frac{1}{2}} (U|T|) |T|^{-\frac{1}{2}} = |T|^{\frac{1}{2}} T |T|^{-\frac{1}{2}}$, as T^*T is invertible. The second equality follows from the first. □

Suppose in addition that T is a shift on an analytic Hilbert space. In Theorem 3.6.3 (under an additional assumption that T is analytic), and then in Theorem 3.6.7 again, we prove that $\tilde{T}$, up to unitary equivalence, is also a shift on an explicit analytic Hilbert space. In connection with Lemma 3.1.6, we now prove the following:

Proposition 3.6.2. *If T is a left-invertible operator on $\mathcal{H}$, then the Shimorin left inverse $L_{\tilde{T}}$ of the Aluthge transform $\tilde{T}$ is given by*

$$L_{\tilde{T}} = |T|^{\frac{1}{2}} \Big((L_T |T| T)^{-1} L_T \Big) |T|^{\frac{1}{2}} = |T|^{\frac{1}{2}} \Big((T^* |T| T)^{-1} T^* \Big) |T|^{\frac{1}{2}}.$$

Proof. Note that by Lemma 3.6.1, we have $\tilde{T}^* \tilde{T} = |T|^{-\frac{1}{2}} (T^* |T| T) |T|^{-\frac{1}{2}}$. Since $T^*|T|T$ is invertible, it follows that $(\tilde{T}^* \tilde{T})^{-1} = |T|^{\frac{1}{2}} (T^* |T| T)^{-1} |T|^{\frac{1}{2}}$. Then

$$L_{\tilde{T}} = (\tilde{T}^* \tilde{T})^{-1} \tilde{T}^* = (|T|^{\frac{1}{2}} (T^* |T| T)^{-1} |T|^{\frac{1}{2}}) |T|^{-\frac{1}{2}} T^* |T|^{\frac{1}{2}} = |T|^{\frac{1}{2}} \Big((T^* |T| T)^{-1} T^* \Big) |T|^{\frac{1}{2}}.$$

On the other hand, since $T^* = |T|^2 L_T$, we have $T^*|T|T = |T|^2 L_T|T|T$, and hence

$$(T^*|T|T)^{-1} = (L_T|T|T)^{-1}|T|^{-2}.$$

Therefore, $(\tilde{T}^*\tilde{T})^{-1} = |T|^{\frac{1}{2}}(L_T|T|T)^{-1}|T|^{-\frac{3}{2}}$, which gives

$$L_{\tilde{T}} = (\tilde{T}^*\tilde{T})^{-1}\tilde{T}^* = |T|^{\frac{1}{2}}(L_T|T|T)^{-1}|T|^{-2}(T^*|T|^{\frac{1}{2}}) = |T|^{\frac{1}{2}}(L_T|T|T)^{-1}L_T|T|^{\frac{1}{2}},$$

and completes the proof. $\square$

Then the above, along with Theorem 3.1.5 and Lemma 3.6.1 implies the following:

Theorem 3.6.3. *Let $\mathcal{E}$ be a Hilbert space, and let $k : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E})$ be an analytic kernel. Suppose M_z is left-invertible on $\mathcal{H}_k$. Then the Aluthge transform $\tilde{M}_z$ is unitarily equivalent to the shift M_z on $\mathcal{H}_{\tilde{k}} \subseteq \mathcal{O}(\mathbb{D}, \tilde{\mathcal{W}})$, where*

$$\tilde{k}(z,w) = P_{\tilde{\mathcal{W}}}(I - zL)^{-1}(I - \bar{w}L^*)^{-1}|_{\tilde{\mathcal{W}}} \qquad (z, w \in \mathbb{D}),$$

and $\tilde{\mathcal{W}} = \ker \tilde{M}_z^* = |M_z|^{-\frac{1}{2}} \ker M_z^*$, and

$$L = |M_z|^{\frac{1}{2}}((L_{M_z}|M_z|M_z)^{-1}L_{M_z})|M_z|^{\frac{1}{2}}.$$

Definition 3.6.4. *The kernel $\tilde{k}$ is called the Shimorin-Aluthge kernel of M_z.*

Under some additional assumptions on scalar-valued analytic kernels, we now prove that, up to similarity and a perturbation of an operator of rank at most one, $L_{\tilde{M}_z}$ and L_{M_z} are the same. As far as concrete examples are concerned, these assumptions are indispensable and natural (cf. Lemma 3.1.4).

Theorem 3.6.5. *Let $k : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be an analytic kernel, $\mathbb{C}[z] \subseteq \mathcal{H}_k$, and let $\{f_n\} \subseteq \mathbb{C}[z]$ be an orthonormal basis of $\mathcal{H}_k$. Assume that M_z on $\mathcal{H}_k$ is left-invertible, $\ker M_z^* = \mathbb{C}f_0$, and*

$$f_n \in \mathit{span}\{z^m : m \geq 1\} \qquad (n \geq 1).$$

Then $L_{\tilde{M}_z}$ and L_{M_z} are similar up to the perturbation of an operator of rank at most one.

Proof. Since $\ker M_z^* = \mathbb{C}f_0$, $L_{M_z}f_0 = 0$ and $L_{M_z}z^n = L_{M_z}M_z(z^{n-1}) = z^{n-1}$, by the definition of L_{M_z}. This implies $L_{M_z}z^n = z^{n-1}$, $n \geq 1$ (also see (3.2.3)). In particular, $L_{M_z}f_n \in \mathbb{C}[z]$ for all $n \geq 0$. Moreover, for each $n \geq 1$, we have

$$\begin{aligned} L_{\tilde{M}_z}(|M_z|^{\frac{1}{2}}z^n) &= |M_z|^{\frac{1}{2}}((L_{M_z}|M_z|M_z)^{-1}L_{M_z})|M_z|z^n \\ &= |M_z|^{\frac{1}{2}}(L_{M_z}|M_z|M_z)^{-1}(L_{M_z}|M_z|M_z)z^{n-1}, \end{aligned}$$

that is, $L_{\tilde{M}_z}(|M_z|^{\frac{1}{2}}z^n) = |M_z|^{\frac{1}{2}}z^{n-1}$. Therefore, we have

$$(|M_z|^{-\frac{1}{2}}L_{\tilde{M}_z}|M_z|^{\frac{1}{2}})z^n = L_{M_z}z^n = z^{n-1} \qquad (n \geq 1).$$

Then $(|M_z|^{-\frac{1}{2}}L_{\tilde{M}_z}|M_z|^{\frac{1}{2}} - L_{M_z})f_n = 0$ for all $n \geq 1$, which gives

$$(|M_z|^{-\frac{1}{2}}L_{\tilde{M}_z}|M_z|^{\frac{1}{2}} - L_{M_z})|_{\overline{span}\{f_n : n\geq 1\}} = 0.$$

Finally, we have clearly $(|M_z|^{-\frac{1}{2}}L_{\tilde{M}_z}|M_z|^{\frac{1}{2}} - L_{M_z})f_0 = (|M_z|^{-\frac{1}{2}}L_{\tilde{M}_z}|M_z|^{\frac{1}{2}})f_0$, and hence

$$F := |M_z|^{-\frac{1}{2}}L_{\tilde{M}_z}|M_z|^{\frac{1}{2}} - L_{M_z}, \tag{3.6.1}$$

is of rank at most one, and consequently $L_{\tilde{M}_z}|M_z|^{\frac{1}{2}} = |M_z|^{\frac{1}{2}}(L_{M_z} + F)$. This completes the proof of the theorem. □

The following analysis of F, defined as in (3.6.1), will be useful in what follows. Note that

$$L_{\tilde{M}_z}|M_z|^{\frac{1}{2}} = |M_z|^{\frac{1}{2}}(L_{M_z} + F). \tag{3.6.2}$$

Let $g \in \mathcal{H}_k$. Clearly, since $L_{M_z}f_0 = 0$, we have $Fg = \langle g, f_0\rangle_{\mathcal{H}_k}(|M_z|^{-\frac{1}{2}}L_{\tilde{M}_z}|M_z|^{\frac{1}{2}}f_0)$. Then Lemma 3.1.6 implies that

$$Fg = \langle g, f_0\rangle_{\mathcal{H}_k}((M_z^*|M_z|M_z)^{-1}M_z^*|M_z|f_0) \qquad (g \in \mathcal{H}_k). \tag{3.6.3}$$

As we will see in Section 3.7, the appearance of the finite rank operator F causes severe computational difficulties for Shimorin-Aluthge kernels of shifts. On the other hand, combining Theorem 3.1.5, Proposition 3.6.2 and (3.6.2), we have:

Theorem 3.6.6. *In the setting of Theorem 3.6.5, the Aluthge transform $\tilde{M}_z$ of M_z on $\mathcal{H}_k$ is unitarily equivalent to the shift M_z on $\mathcal{H}_{\tilde{k}}$, where*

$$\tilde{k}(z,w) = P_{\mathcal{W}}(I - zL)^{-1}(I - \bar{w}L^*)^{-1}|_{\mathcal{W}},$$

$\mathcal{W} = |M_z|^{-\frac{1}{2}}\ker M_z^* = \mathbb{C}(|M_z|^{-\frac{1}{2}}f_0)$, *and*

$$L = |M_z|^{\frac{1}{2}}(L_{M_z} + F)|M_z|^{-\frac{1}{2}},$$

and $Fg = \langle g, f_0\rangle_{\mathcal{H}_k}((M_z^*|M_z|M_z)^{-1}M_z^*|M_z|f_0)$ *for all* $g \in \mathcal{H}_k$.

We now revisit Theorem 3.6.3 from a direct reproducing kernel Hilbert space standpoint. Indeed, there is a rather more concrete proof of Theorem 3.6.3 which avoids using the analytic model of left-invertible operators. In this case, also, the reproducing kernel of the corresponding Aluthge transform is explicit. Part of the proof follows the same line of argumentation as the proof of reproducing kernel property of range spaces (cf. [4]). To the reader's benefit, we include all necessary details.

Theorem 3.6.7. *Let $\mathcal{E}$ be a Hilbert space, and let $k : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E})$ be an analytic kernel. Assume that the shift M_z is left-invertible on $\mathcal{H}_k$. Then*

$$\langle \tilde{k}(z,w)\eta, \zeta\rangle_{\mathcal{E}} = \langle |M_z|^{-1}(k(\cdot,w)\eta), k(\cdot,z)\zeta)\rangle_{\mathcal{H}_k} \qquad (z,w \in \mathbb{D}, \eta,\zeta \in \mathcal{E}),$$

defines a kernel $\tilde{k} : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E})$. Moreover, the shift M_z on $\mathcal{H}_{\tilde{k}}$ defines a bounded linear operator, and there exists a unitary $U : \mathcal{H}_k \to \mathcal{H}_{\tilde{k}}$ such that $U\tilde{M}_z = M_z U$.

Proof. Define $\tilde{\mathcal{H}} = |M_z|^{-\frac{1}{2}}\mathcal{H}_k$. Then $\tilde{\mathcal{H}}(= \mathcal{H}_k)$ is an $\mathcal{E}$-valued function Hilbert space endowed with the inner product $\langle |M_z|^{-\frac{1}{2}} f, |M_z|^{-\frac{1}{2}} g\rangle_{\tilde{\mathcal{H}}} = \langle f, g\rangle_{\mathcal{H}_k}$ for all $f, g \in \mathcal{H}_k$. For each $f \in \mathcal{H}_k$, $w \in \mathbb{D}$ and $\eta \in \mathcal{E}$, we have

$$\langle |M_z|^{-\frac{1}{2}} f, |M_z|^{-1}(k(\cdot,w)\eta)\rangle_{\tilde{\mathcal{H}}} = \langle f, |M_z|^{-\frac{1}{2}}(k(\cdot,w)\eta)\rangle_{\mathcal{H}_k} = \langle |M_z|^{-\frac{1}{2}} f, k(\cdot,w)\eta\rangle_{\mathcal{H}_k},$$

and hence, by the reproducing property of $\mathcal{H}_k$, it follows that

$$\langle |M_z|^{-\frac{1}{2}} f, |M_z|^{-1}(k(\cdot,w)\eta)\rangle_{\tilde{\mathcal{H}}} = \langle (|M_z|^{-\frac{1}{2}} f)(w), \eta\rangle_{\mathcal{E}}. \tag{3.6.4}$$

This says that $\{|M_z|^{-1}(k(\cdot,w)\eta) : w \in \mathbb{D}, \eta \in \mathcal{E}\}$ reproduces the values of functions in $\tilde{\mathcal{H}}$, and furthermore, the evaluation operator $ev_w : \tilde{\mathcal{H}} \to \mathcal{E}$ is continuous. Indeed

$$\begin{aligned}
|\langle ev_w(|M_z|^{-\frac{1}{2}} f), \eta\rangle_{\mathcal{E}}| &= |\langle (|M_z|^{-\frac{1}{2}} f)(w), \eta\rangle_{\mathcal{E}}| \\
&= |\langle |M_z|^{-\frac{1}{2}} f, |M_z|^{-1}(k(\cdot,w)\eta)\rangle_{\tilde{\mathcal{H}}}| \\
&\leq \||M_z|^{-\frac{1}{2}} f\|_{\tilde{\mathcal{H}}} \||M_z|^{-1}(k(\cdot,w)\eta)\|_{\tilde{\mathcal{H}}} \\
&= \||M_z|^{-\frac{1}{2}} f\|_{\tilde{\mathcal{H}}}\ \||M_z|^{-\frac{1}{2}}(k(\cdot,w)\eta)\|_{\mathcal{H}_k}.
\end{aligned}$$

Since $\|k(\cdot,w)\eta\|^2_{\mathcal{H}_k} = \langle k(\cdot,w)\eta, k(\cdot,w)\eta\rangle_{\mathcal{H}_k} = \langle k(w,w)\eta,\eta\rangle_{\mathcal{E}} = \|k(w,w)^{\frac{1}{2}}\eta\|^2_{\mathcal{E}}$, it follows that

$$\begin{aligned}
\||M_z|^{-\frac{1}{2}}(k(\cdot,w)\eta)\|_{\mathcal{H}_k} &\leq \||M_z|^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{H}_k)} \|k(\cdot,w)\eta\|_{\mathcal{H}_k} \\
&= \||M_z|^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{H}_k)}\ \|k(w,w)^{\frac{1}{2}}\eta\|_{\mathcal{E}} \\
&\leq \||M_z|^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{H}_k)}\ \|k(w,w)^{\frac{1}{2}}\|_{\mathcal{B}(\mathcal{E})}\ \|\eta\|_{\mathcal{E}},
\end{aligned}$$

which implies that

$$|\langle ev_w(|M_z|^{-\frac{1}{2}} f), \eta\rangle_{\mathcal{E}}| \leq (\||M_z|^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{H}_k)} \|k(w,w)^{\frac{1}{2}}\|_{\mathcal{B}(\mathcal{E})}) \||M_z|^{-\frac{1}{2}} f\|_{\tilde{\mathcal{H}}}\ \|\eta\|_{\mathcal{E}}.$$

Therefore $\tilde{\mathcal{H}}$ is an $\mathcal{E}$-valued reproducing kernel Hilbert space corresponding to the kernel function

$$\tilde{k}(z,w) = ev_z \circ ev_w^* \qquad (z,w \in \mathbb{D}).$$

Clearly, (3.6.4) implies that $ev_w^*\eta = |M_z|^{-1}(k(\cdot, w)\eta)$ for all $w \in \mathbb{D}$ and $\eta \in \mathcal{E}$. Since $\langle \tilde{k}(z,w)\eta, \zeta\rangle_{\mathcal{E}} = \langle ev_w^*\eta, ev_z^*\zeta\rangle_{\mathcal{E}}$, it follows that

$$\begin{aligned}\langle \tilde{k}(z,w)\eta, \zeta\rangle_{\mathcal{E}} &= \langle |M_z|^{-1}(k(\cdot, w)\eta), |M_z|^{-1}(k(\cdot, z)\zeta)\rangle_{\tilde{\mathcal{H}}}\\ &= \langle |M_z|^{-\frac{1}{2}}(k(\cdot, w)\eta), |M_z|^{-\frac{1}{2}}(k(\cdot, z)\zeta)\rangle_{\mathcal{H}_k},\end{aligned}$$

that is, $\langle \tilde{k}(z,w)\eta, \zeta\rangle_{\mathcal{E}} = \langle |M_z|^{-1}(k(\cdot, w)\eta), k(\cdot, z)\zeta)\rangle_{\mathcal{H}_k}$, $z, w \in \mathbb{D}, \eta, \zeta \in \mathcal{E}$. Therefore, as a reproducing kernel Hilbert space corresponding to the kernel $\tilde{k}$, we have $\mathcal{H}_{\tilde{k}} = \tilde{\mathcal{H}}$. Define the unitary map $U : \mathcal{H}_k \to \mathcal{H}_{\tilde{k}}$ by

$$Uh = |M_z|^{-\frac{1}{2}}h \qquad (h \in \mathcal{H}_k),$$

and recall from Lemma 3.6.1 that $\tilde{M}_z^* = |M_z|^{-\frac{1}{2}}M_z^*|M_z|^{\frac{1}{2}}$. Let $f \in \mathcal{H}_k$, $w \in \mathbb{D}$, and let $\eta \in \mathcal{E}$. Then

$$\begin{aligned}\langle (U\tilde{M}_zU^*(|M_z|^{-\frac{1}{2}}f))(w), \eta\rangle_{\mathcal{E}} &= \langle U\tilde{M}_zU^*(|M_z|^{-\frac{1}{2}}f), |M_z|^{-1}(k(\cdot, w)\eta)\rangle_{\mathcal{H}_{\tilde{k}}}\\ &= \langle \tilde{M}_zU^*(|M_z|^{-\frac{1}{2}}f), |M_z|^{-\frac{1}{2}}(k(\cdot, w)\eta)\rangle_{\mathcal{H}_k}\\ &= \langle f, \tilde{M}_z^*|M_z|^{-\frac{1}{2}}(k(\cdot, w)\eta)\rangle_{\mathcal{H}_k}\\ &= \langle f, |M_z|^{-\frac{1}{2}}M_z^*(k(\cdot, w)\eta)\rangle_{\mathcal{H}_k}.\end{aligned}$$

But since $M_z^*(k(\cdot, w)\eta) = \bar{w}k(\cdot, w)\eta$, we have

$$\langle (U\tilde{M}_zU^*(|M_z|^{-\frac{1}{2}}f))(w), \eta\rangle_{\mathcal{E}} = w\langle f, |M_z|^{-\frac{1}{2}}(k(\cdot, w)\eta)\rangle_{\mathcal{H}_k} = \langle w(|M_z|^{-\frac{1}{2}}f))(w), \eta\rangle_{\mathcal{E}},$$

which implies that $U\tilde{M}_zU^*(|M_z|^{-\frac{1}{2}}f) = z(|M_z|^{-\frac{1}{2}}f)$ for all $f \in \mathcal{H}_k$. Thus the shift M_z on $\mathcal{H}_{\tilde{k}}$ is a bounded linear operator and $U\tilde{M}_z = M_zU$. □

Definition 3.6.8. *The kernel $\tilde{k}$ is called the standard Aluthge kernel of M_z.*

In particular, if k is a scalar-valued kernel, then $\tilde{k}(\cdot, w) = U(|M_z|^{-\frac{1}{2}}k(\cdot, w))$ and

$$\tilde{k}(z,w) = \langle |M_z|^{-1}k(\cdot, w), k(\cdot, z)\rangle_{\mathcal{H}_k} \qquad (z, w \in \mathbb{D}).$$

Therefore, if the shift on a tridiagonal space $\mathcal{H}_k$ is left-invertible, then there are two ways to compute the Aluthge kernel $\tilde{k}$: use Theorem 3.6.3, or use the one above. However, it is curious to note that, from a general computational point of view, neither approach is completely satisfactory and definite. On the other hand, often the standard Aluthge kernel approach (and sometimes both standard Aluthge kernel and Shimorin-Aluthge kernel methods) lead to satisfactory results. We will discuss this in the following section.

3.7 Truncated tridiagonal kernels

In this section, we introduce a (perhaps both deliberate and accidental) class of analytic tridiagonal kernels from a computational point of view. Let $\mathcal{H}_k$ be an analytic tridiagonal space corresponding to the kernel

$$k(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} \qquad (z, w \in \mathbb{D}),$$

where $f_n = (a_n + b_n z)z^n$, $n \geq 0$. Suppose $r \geq 2$ is a natural number. We say that k is a *truncated tridiagonal kernel of order* r (in short, *truncated kernel of order* r) if

$$b_n = 0 \qquad (n \neq 2, 3, \ldots, r).$$

We say that an analytic tridiagonal space $\mathcal{H}_k$ is *truncated space of order* r if k is a truncated kernel of order r. Note that there are no restrictions imposed on the scalars $b_2, \ldots, b_r$.

Let $\mathcal{H}_k$ be a truncated space of order r. Then $\tilde{M_z}$ is unitarily equivalent to M_z on $\mathcal{H}_{\tilde{k}}$, where $\tilde{k}$ is either the Shimorin-Aluthge kernel or the standard Aluthge kernel of M_z as in Theorem 3.6.3 and Theorem 3.6.7, respectively. Here our aim is to compute the Shimorin-Aluthge kernel of M_z. More specifically, we classify all truncated kernels k such that the Shimorin-Aluthge kernel $\tilde{k}$ of M_z is tridiagonal. We begin by computing $|M_z|^{-1}$.

Lemma 3.7.1. *If $\mathcal{H}_k$ is a truncated space of order r, then*

$$\left[|M_z|^{-1}\right] = \begin{bmatrix} |\frac{a_1}{a_0}| & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & c_{11} & c_{12} & \cdots & c_{1,r+1} & 0 & 0 & \ddots \\ 0 & \bar{c}_{12} & c_{22} & \cdots & c_{2,r+1} & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots \\ 0 & \bar{c}_{1,r+1} & \bar{c}_{2,r+1} & \cdots & c_{r+1,r+1} & 0 & 0 & \ddots \\ 0 & 0 & 0 & \cdots & 0 & |\frac{a_{r+3}}{a_{r+2}}| & 0 & \ddots \\ 0 & 0 & 0 & \cdots & 0 & 0 & |\frac{a_{r+4}}{a_{r+3}}| & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

with respect to the orthonormal basis $\{f_n\}_{n\geq 0}$.

Proof. For each $n \geq 1$, by the definition of d_n from (3.2.1), we have $d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}$, and hence $d_1 = d_{r+i} = 0$, $i = 2, 3, \ldots$. Then Theorem 3.2.4 tells us that

$$[L_{M_z}] = \begin{bmatrix} 0 & \frac{a_1}{a_0} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{a_2}{a_1} & \cdots & 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & d_2 & \cdots & 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & (-1)^{r-2}\frac{d_2 b_2 \cdots b_{r-1}}{a_3 \cdots a_r} & \cdots & d_r & \frac{a_{r+1}}{a_r} & 0 & 0 & \ddots \\ 0 & 0 & (-1)^{r-1}\frac{d_2 b_2 \cdots b_r}{a_3 \cdots a_r a_{r+1}} & \cdots & -\frac{d_r b_r}{a_{r+1}} & d_{r+1} & \frac{a_{r+2}}{a_{r+1}} & 0 & \ddots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{a_{r+3}}{a_{r+2}} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Now, by Lemma 3.1.6, $|M_z|^{-2} = L_{M_z} L_{M_z}^*$, which implies

$$\left[|M_z|^{-2}\right] = \begin{bmatrix} |\frac{a_1}{a_0}|^2 & 0 & 0 \\ 0 & A_{r+1}^2 & 0 \\ 0 & 0 & D^2 \end{bmatrix},$$

where $D^2 = \operatorname{diag}\left(\left|\frac{a_{r+3}}{a_{r+2}}\right|^2, \left|\frac{a_{r+4}}{a_{r+3}}\right|^2, \ldots\right)$ and A_{r+1}^2 is a positive definite matrix of order $r+1$. Using this, one easily completes the proof. $\square$

From the computational point of view, it is useful to observe that $A_{r+1}^2 = L_{r+1} L_{r+1}^*$, where

$$L_{r+1} = \begin{bmatrix} \frac{a_2}{a_1} & 0 & 0 & 0 & 0 \\ d_2 & \frac{a_3}{a_2} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{r-2}\frac{d_2 b_2 \cdots b_{r-1}}{a_3 \cdots a_r} & (-1)^{r-3}\frac{d_3 b_3 \cdots b_{r-1}}{a_4 \cdots a_r} & \cdots & \frac{a_{r+1}}{a_r} & 0 \\ (-1)^{r-1}\frac{d_2 b_2 \cdots b_r}{a_3 \cdots a_r a_{r+1}} & (-1)^{r-2}\frac{d_3 b_3 \cdots b_r}{a_4 \cdots a_r a_{r+1}} & \cdots & d_{r+1} & \frac{a_{r+2}}{a_{r+1}} \end{bmatrix}.$$

In other words, A_{r+1}^2 admits a lower-upper triangular factorization. This is closely related to the Cholesky factorizations/decompositions of positive-definite matrices in the setting of infinite dimensional Hilbert spaces (see [4] and [47]).

We recall from Theorem 3.6.6 that the Shimorin-Aluthge kernel of M_z is given by

$$\tilde{k}(z, w) = P_{\tilde{\mathcal{W}}}(I - zL_{\tilde{M}_z})^{-1}(I - \bar{w}L_{\tilde{M}_z}^*)^{-1}|_{\tilde{\mathcal{W}}} \qquad (z, w \in \mathbb{D}),$$

where $\tilde{\mathcal{W}} = |M_z|^{-\frac{1}{2}} \ker M_z^*$, and

$$L_{\tilde{M}_z} = |M_z|^{\frac{1}{2}}(L_{M_z} + F)|M_z|^{-\frac{1}{2}}, \tag{3.7.1}$$

and $Fg = \langle g, f_0\rangle_{\mathcal{H}_k}\Big((M_z^*|M_z|M_z)^{-1}M_z^*|M_z|f_0\Big)$, $g \in \mathcal{H}_k$. We now come to the key point.

Lemma 3.7.2. *If k is a truncated kernel, then $F = 0$ and $L_{\tilde{M}_z}|M_z|^{\frac{1}{2}} = |M_z|^{\frac{1}{2}}L_{M_z}$.*

Proof. The matrix representation of $|M_z|^{-1}$ in Lemma 3.7.1 implies that $|M_z|f_0 = |\frac{a_0}{a_1}|f_0$, and hence

$$M_z^*|M_z|f_0 = \left|\frac{a_0}{a_1}\right|M_z^*f_0 = 0,$$

by Lemma 3.1.4. Therefore, the proof follows from the definition of F and (3.7.1). □

We are finally ready to state and prove the result we are aiming for.

Theorem 3.7.3. *Let $\mathcal{H}_k$ be a truncated space of order r. Then the Shimorin-Aluthge kernel is tridiagonal if and only if*

$$c_{mn} = (-1)^{n-m-1}\frac{\bar{b}_{m+1}\cdots\bar{b}_{n-1}}{\bar{a}_{m+2}\cdots\bar{a}_n}c_{m,m+1},$$

for all $1 \le m \le n-2$ and $3 \le n \le r+1$, where c_{mn} are the entries of the middle block submatrix of order $r+1$ of $\left[|M_z|^{-1}\right]$ in Lemma 3.7.1.

Proof. We split the proof into several steps.

Step 1: First observe that $\tilde{k}(z,w) = \sum_{m,n=0}^{\infty}\tilde{X}_{mn}z^m\bar{w}^n$, where $\tilde{X}_{mn} = P_{\tilde{\mathcal{W}}}L_{\tilde{M}_z}^m L_{\tilde{M}_z}^{*n}|_{\tilde{\mathcal{W}}}$ for all $m, n \ge 0$. Now Lemma 3.7.2 implies that $L_{\tilde{M}_z}^m L_{\tilde{M}_z}^{*n} = |M_z|^{\frac{1}{2}}L_{M_z}^m|M_z|^{-1}L_{M_z}^{*n}|M_z|^{\frac{1}{2}}$, and $P_{\tilde{\mathcal{W}}} = I - \tilde{M}_zL_{\tilde{M}_z}$ by (3.1.13). Since $\tilde{M}_z = |M_z|^{\frac{1}{2}}M_z|M_z|^{-\frac{1}{2}}$ and $L_{\tilde{M}_z} = |M_z|^{\frac{1}{2}}L_{M_z}|M_z|^{-\frac{1}{2}}$, we have

$$P_{\tilde{\mathcal{W}}} = |M_z|^{\frac{1}{2}}(I - M_zL_{M_z})|M_z|^{-\frac{1}{2}} = |M_z|^{\frac{1}{2}}P_{\mathcal{W}}|M_z|^{-\frac{1}{2}}$$

that is, $P_{\tilde{\mathcal{W}}}|M_z|^{\frac{1}{2}} = |M_z|^{\frac{1}{2}}P_{\mathcal{W}}$, which implies

$$\tilde{X}_{mn} = |M_z|^{\frac{1}{2}}P_{\mathcal{W}}L_{M_z}^m|M_z|^{-1}L_{M_z}^{*n}|_{\mathcal{W}} \qquad (m, n \ge 0). \tag{3.7.2}$$

As a passing remark, we note that the above equality holds so long as the finite rank operator $F = 0$ (this observation also will be used in Example 3.8.1).

Step 2: Now we compute the matrix representation of $L_{M_z}^p$, $p \ge 1$. By Theorem 3.2.4, we have

$$[L_{M_z}] = \begin{bmatrix} 0 & \frac{a_1}{a_0} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{a_2}{a_1} & 0 & 0 & \ddots \\ 0 & 0 & d_2 & \frac{a_3}{a_2} & 0 & \ddots \\ 0 & 0 & \frac{-d_2b_2}{a_3} & d_3 & \frac{a_4}{a_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

In particular, this yields

$$P_{\mathcal{W}} L_{M_z} f_j = \begin{cases} \frac{a_1}{a_0} f_0 & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now we let $p \geq 2$. Recall from (3.3.6) the definition $\beta_n^{(p)} = a_n \left(\frac{-b_0}{a_0}\right)^{p-n-1} \beta_n$ for all $n = 1, \ldots, p-1$, where $\beta_n = \frac{b_n}{a_n} - \frac{b_0}{a_0}$. Since $b_0 = 0$, we have $\beta_n^{(p)} = 0$, $1 \leq n < p-1$, and

$$\beta_{p-1}^{(p)} = a_{p-1}\beta_{p-1} = a_{p-1}\left(\frac{b_{p-1}}{a_{p-1}} - \frac{b_0}{a_0}\right),$$

that is, $\beta_{p-1}^{(p)} = b_{p-1}$ for all $p \geq 2$. In particular, since $b_1 = 0$, we have $\beta_1^{(2)} = b_1 = 0$. Also recall from (3.3.3) the definition $d_n^{(p)} = b_n - \frac{a_n}{a_{n-p}} b_{n-p}$, $n \geq p$. Therefore, by (3.3.7), the matrix representation of $L_{M_z}^2$ is given by

$$[L_{M_z}^2] = \begin{bmatrix} 0 & 0 & \frac{a_2}{a_0} & 0 & 0 & \cdots \\ 0 & 0 & \frac{d_2^{(2)}}{a_1} & \frac{a_3}{a_1} & 0 & \ddots \\ 0 & 0 & 0 & \frac{d_3^{(2)}}{a_2} & \frac{a_4}{a_2} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \end{bmatrix},$$

and in general, by (3.3.8), we have

$$[L_{M_z}^p] = \begin{bmatrix} 0 & \cdots & 0 & \frac{b_{p-1}}{a_0} & \frac{a_p}{a_0} & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \frac{d_p^{(p)}}{a_1} & \frac{a_{p+1}}{a_1} & 0 & \ddots \\ 0 & \cdots & 0 & 0 & 0 & \frac{d_{p+1}^{(p)}}{a_2} & \frac{a_{p+2}}{a_2} & \ddots \\ 0 & \cdots & 0 & 0 & 0 & -\frac{d_{p+1}^{(p)} b_2}{a_2 a_3} & \frac{d_{p+2}^{(p)}}{a_3} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad (p \geq 2). \tag{3.7.3}$$

Then

$$[L_{M_z}^{*p}] = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \frac{\bar{b}_{p-1}}{\bar{a}_0} & 0 & 0 & 0 & \ddots \\ \frac{\bar{a}_p}{\bar{a}_0} & \frac{\bar{d}_p^{(p)}}{\bar{a}_1} & 0 & 0 & \ddots \\ 0 & \frac{\bar{a}_{p+1}}{\bar{a}_1} & \frac{\bar{d}_{p+1}^{(p)}}{\bar{a}_2} & -\frac{\bar{d}_{p+1}^{(p)} \bar{b}_2}{\bar{a}_2 \bar{a}_3} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad (p \geq 2). \tag{3.7.4}$$

Step 3: We prove that $\tilde{X}_{0n} = |M_z|^{\frac{1}{2}} P_{\mathcal{W}} |M_z|^{-1} L_{M_z}^{*n}|_{\mathcal{W}} = 0$ for all $n \geq 1$. In what follows, the above matrix representations and the one of $|M_z|^{-1}$ in Lemma 3.7.1 will be used repeatedly. By (3.3.2), we have $L_{M_z}^* f_0 = \frac{\bar{a}_1}{\bar{a}_0} f_1$, and hence

$$\tilde{X}_{01} f_0 = |M_z|^{\frac{1}{2}} P_{\mathcal{W}} |M_z|^{-1} L_{M_z}^* f_0 = |M_z|^{\frac{1}{2}} P_{\mathcal{W}} (\frac{\bar{a_1}}{\bar{a_0}} [c_{11} f_1 + \bar{c}_{12} f_2 + \cdots]) = 0.$$

On the other hand, if $n \geq 2$, then $L_{M_z}^{*n} f_0 = \frac{\bar{b}_{n-1}}{\bar{a}_0} f_{n-1} + \frac{\bar{a}_n}{\bar{a}_0} f_n$, and hence $|M_z|^{-1} f_0 \perp L_{M_z}^{*n} f_0$. This implies that $\tilde{X}_{0n} = 0$ for all $n \geq 2$. Therefore, all entries in the first row (and hence, also in the first column) of the formal matrix representation of $\tilde{k}(z, w)$ are zero except the $(0, 0)$-th entry (which is $I_{\mathcal{W}}$). Hence (see also (3.3.1))

$$\left[\tilde{k}(z,w)\right] = \begin{bmatrix} I_{\tilde{\mathcal{W}}} & 0 & 0 & 0 & \cdots \\ 0 & \tilde{X}_{11} & \tilde{X}_{12} & \tilde{X}_{13} & \ddots \\ 0 & \tilde{X}_{12}^* & \tilde{X}_{22} & \tilde{X}_{23} & \ddots \\ 0 & \tilde{X}_{13}^* & \tilde{X}_{23}^* & \tilde{X}_{33} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Step 4: Our only interest here is to analyze the finite rank (of rank at most one) operator $\tilde{X}_{m,m+k}$, $m \geq 1$, $k \geq 2$. The matrix representation in (3.7.4) implies

$$L_{M_z}^{*m+k} f_0 = \frac{1}{\bar{a}_0} (\bar{b}_{m+k-1} f_{m+k-1} + \bar{a}_{m+k} f_{m+k}), \tag{3.7.5}$$

and hence

$$|M_z|^{-1} L_{M_z}^{*m+k} f_0 = \frac{1}{\bar{a}_0} (\bar{b}_{m+k-1} |M_z|^{-1} f_{m+k-1} + \bar{a}_{m+k} |M_z|^{-1} f_{m+k}). \tag{3.7.6}$$

There are three cases to be considered:

Case I $(m+k = r+2)$*:* Note that $b_{r+1} = 0$. Then $|M_z|^{-1} L_{M_z}^{*r+2} f_0 = \frac{1}{\bar{a}_0} (\bar{a}_{r+2} |M_z|^{-1} f_{r+2})$, by (3.7.6), and thus

$$L_{M_z}^m |M_z|^{-1} L_{M_z}^{*r+2} f_0 = \frac{\bar{a}_{r+2}}{\bar{a}_0} L_{M_z}^m |M_z|^{-1} f_{r+2} = \frac{\bar{a}_{r+2}}{\bar{a}_0} \left| \frac{a_{r+3}}{a_{r+2}} \right| L_{M_z}^m f_{r+2}.$$

By (3.7.3), we have $P_{\mathcal{W}} L_{M_z}^m f_{r+2} = P_{\mathcal{W}} L_{M_z}^m f_{m+k} = 0$ (note that $k \geq 2$), and hence

$$P_{\mathcal{W}} L_{M_z}^m |M_z|^{-1} L_{M_z}^{*r+2} f_0 = 0,$$

that is, $\tilde{X}_{m,m+k} = 0$. It is easy to check that the equality also holds for $m = 1$.

Case II $(m + k - 1 \geq r + 2)$*:* In this case, $b_{m+k-1} = 0$ and

$$|M_z|^{-1} f_{m+k} = \left| \frac{a_{m+k+1}}{a_{m+k}} \right| f_{m+k}.$$

Again, by (3.7.3), we have $P_{\mathcal{W}} L_{M_z}^m f_{m+k} = 0$, $k \geq 2$, and hence in this case also $\tilde{X}_{m,m+k} = 0$. Again, it is easy to check that the equality holds for $m = 1$.

Case III $(m+k<r+2)$*:* We again stress that $m\geq 1$ and $k\geq 2$. It is useful to observe, by virtue of (3.7.3) (also see (3.3.12)), that

$$P_{\mathcal{W}}L_{M_z}^m f_j=\begin{cases}\frac{b_{m-1}}{a_0}f_0 & \text{if } j=m-1\\ \frac{a_m}{a_0}f_0 & \text{if } j=m\\ 0 & \text{otherwise.}\end{cases}$$

Now set $s=m+k-1$. The matrix representation of $|M_z|^{-1}$ in Lemma 3.7.1 implies that

$$|M_z|^{-1}f_s=c_{1s}f_1+c_{2s}f_2+\cdots+c_{ss}f_s+\bar{c}_{s,s+1}f_{s+1}+\cdots+\bar{c}_{s,r+1}f_{r+1}.$$

By (3.7.3) and the above equality, we have

$$P_{\mathcal{W}}L_{M_z}^m|M_z|^{-1}f_s=(c_{m-1,s}\frac{b_{m-1}}{a_0}+c_{m,s}\frac{a_m}{a_0})f_0. \tag{3.7.7}$$

Next, set $t=m+k$. Again, the matrix representation of $|M_z|^{-1}$ in Lemma 3.7.1 implies that

$$|M_z|^{-1}f_t=c_{1t}f_1+c_{2t}f_2+\cdots+c_{tt}f_t+\bar{c}_{t,t+1}f_{t+1}+\cdots+\bar{c}_{t,r+1}f_{r+1},$$

and, again, by (3.7.3) and the above equality, we have

$$P_{\mathcal{W}}L_{M_z}^m|M_z|^{-1}f_t=(c_{m-1,t}\frac{b_{m-1}}{a_0}+c_{m,t}\frac{a_m}{a_0})f_0. \tag{3.7.8}$$

It is easy to see that the equalities (3.7.7) and (3.7.8) also holds for $m=1$. The equality in (3.7.5) becomes

$$|M_z|^{-1}L_{M_z}^{*m+k}f_0=\frac{1}{\bar{a}_0}(\bar{b}_s|M_z|^{-1}f_s+\bar{a}_t|M_z|^{-1}f_t),$$

and hence, the one in (3.7.6) implies

$$P_{\mathcal{W}}L_{M_z}^m|M_z|^{-1}L_{M_z}^{*m+k}f_0=\frac{1}{|a_0|^2}[\bar{b}_s(c_{m-1,s}b_{m-1}+c_{m,s}a_m)+\bar{a}_t(c_{m-1,t}b_{m-1}+c_{m,t}a_m)]f_0.$$

This shows that $P_{\mathcal{W}}L_{M_z}^m|M_z|^{-1}L_{M_z}^{*m+k}f_0=0$ if and only if

$$\bar{b}_s(c_{m-1,s}b_{m-1}+c_{m,s}a_m)+\bar{a}_t(c_{m-1,t}b_{m-1}+c_{m,t}a_m)=0.$$

Step 5: So far all we have proved is that $\tilde{k}$ is tridiagonal if and only if

$$b_{m-1}(\bar{b}_{m+k-1}c_{m-1,m+k-1}+\bar{a}_{m+k}c_{m-1,m+k})+a_m(\bar{b}_{m+k-1}c_{m,m+k-1}+\bar{a}_{m+k}c_{m,m+k})=0, \tag{3.7.9}$$

for all $m\geq 1$, $k\geq 2$ and $m+k<r+2$.

If $m = 1$, then using the fact that $b_0 = 0$, we have $c_{1,k+1} = -\frac{\bar{b}_k}{\bar{a}_{1+k}} c_{1,k}$, $2 \leq k < r+1$, and hence

$$c_{1n} = (-1)^{n-2} \frac{\prod_{i=2}^{n-1} \bar{b}_i}{\prod_{i=3}^{n} \bar{a}_i} c_{12} \qquad (3 \leq n \leq r+1).$$

Similarly, if $m = 2$, then (3.7.9) together with the assumption that $b_1 = 0$ implies that

$$c_{2n} = (-1)^{n-3} \frac{\prod_{i=3}^{n-1} \bar{b}_i}{\prod_{i=4}^{n} \bar{a}_i} c_{23} \qquad (4 \leq n \leq r+1). \tag{3.7.10}$$

Next, if $m = 3$, then (3.7.9) again implies

$$b_2(\bar{b}_{k+2} c_{2,k+2} + \bar{a}_{k+3} c_{2,k+3}) + a_3(\bar{b}_{k+2} c_{3,k+2} + \bar{a}_{k+3} c_{3,k+3}) = 0 \qquad (k < r-1).$$

On the other hand, by (3.7.10), we have $c_{2,k+3} = -\frac{\bar{b}_{k+2}}{\bar{a}_{k+3}} c_{2,k+2}$, and hence $\bar{b}_{k+2} c_{3,k+2} + \bar{a}_{k+3} c_{3,k+3} = 0$, which implies $c_{3,k+3} = -\frac{\bar{b}_{k+2}}{\bar{a}_{k+3}} c_{3,k+2}$, $k < r-1$. Now, evidently the recursive situation is exactly the same as that of the proof of Theorem 3.4.2 (more specifically, see (3.4.2)). This completes the proof of the theorem. $\square$

As is clear by now, by virtue of Theorem 3.4.2, the classification criterion of the above theorem is also a classification criterion of tridiagonality of standard Aluthge kernels. Therefore, we have the following:

Corollary 3.7.4. *If $\mathcal{H}_k$ is a truncated space, then the Shimorin-Aluthge kernel of M_z is tridiagonal if and only if the standard Aluthge kernel of M_z is tridiagonal.*

3.8 Final comments and results

First we comment on the assumptions in the definition of truncated kernels (see Section 3.7). The main advantage of the truncated space corresponding to a truncated kernel is that $F = 0$, where F is the finite rank operator as in (3.6.3). In this case, as already pointed out, we have $L_{\tilde{M}_z} = |M_z|^{\frac{1}{2}} L_{M_z} |M_z|^{-\frac{1}{2}}$. This brings a big cut down in computation. On the other hand, quite curiously, if

$$b_0 = b_1 = 1 \text{ or } b_0 = 1,$$

and all other b_i's are equal to 0, then the corresponding standard Aluthge kernel of M_z is tridiagonal kernel but the corresponding Shimorin-Aluthge kernel of M_z is not a tridiagonal kernel. Since computations are rather complicated in the presence of F, we only present the result for the following (convincing) case:

Example 3.8.1. *Let $a_n = b_0 = b_1 = 1$ and $b_m = 0$ for all $n \geq 0$ and $m \geq 2$. Let $\mathcal{H}_k$ denote the tridiagonal space corresponding to the basis $\{(a_n + b_n z)z^n\}_{n \geq 0}$. By* (3.1.8) *and Theorem 3.2.4, we have*

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \ddots \\ 0 & 1 & 0 & 0 & 0 & \ddots \\ 0 & 1 & 1 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad [L_{M_z}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \ddots \\ 0 & 0 & -1 & 1 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

respectively. Hence, applying $L_{M_z} L_{M_z}^* = |M_z|^{-2}$ (see Lemma 3.1.6) to this, we obtain

$$|M_z|^{-2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \cdots & \\ 0 & 1 & -1 & 0 & 0 & \ddots \\ 0 & -1 & 2 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \end{bmatrix}.$$

Suppose $\alpha = \frac{3+\sqrt{5}}{2}$ and $\beta = \frac{3-\sqrt{5}}{2}$. It is useful to observe that $(1-\alpha)(1-\beta)+1 = 0$. Set $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{\frac{1}{2}}$, where $a = \frac{1}{\sqrt{5}}[\sqrt{\alpha}(1-\beta) - \sqrt{\beta}(1-\alpha)]$ and $b = \frac{1}{\sqrt{5}}[-\sqrt{\alpha} + \sqrt{\beta}]$, and $c = \frac{1}{\sqrt{5}}[-\sqrt{\alpha}(1-\alpha) + \sqrt{\beta}(1-\beta)]$. Clearly

$$|M_z|^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

From this it follows that $|M_z| f_0 = f_0$, and hence the finite rank operator F, as in (3.6.3), is given by

$$Fg = \langle g, f_0 \rangle_{\mathcal{H}_k} \Big((M_z^* |M_z| M_z)^{-1} M_z^* |M_z| f_0 \Big) = 0 \qquad (g \in \mathcal{H}_k).$$

Then $F = 0$, and hence (3.6.2) implies that $L_{\tilde{M}_z} = |M_z|^{\frac{1}{2}} L_{M_z} |M_z|^{-\frac{1}{2}}$. By (3.7.2) (also see Step 1 in the proof of Theorem 3.7.3), the coefficient of $z^m \bar{w}^n$ of the Shimorin-Aluthge kernel $\tilde{k}$ is given by $\tilde{X}_{mn} = |M_z|^{\frac{1}{2}} P_{\mathcal{W}} L_{M_z}^m |M_z|^{-1} L_{M_z}^{*n}|_{\mathcal{W}}$, $m, n \geq 0$. We compute *the coefficient of $z\bar{w}^3$ as*

$$\begin{aligned}
P_{\mathcal{W}} L_{M_z} |M_z|^{-1} L_{M_z}^{*3} f_0 &= P_{\mathcal{W}} L_{M_z} |M_z|^{-1} L_{M_z}^{*2} f_1 \\
&= P_{\mathcal{W}} L_{M_z} |M_z|^{-1} L_{M_z}^{*} f_2 \\
&= P_{\mathcal{W}} L_{M_z} |M_z|^{-1} (-f_2 + f_3) \\
&= P_{\mathcal{W}} L_{M_z} (-b f_1 - c f_2 + f_3) \\
&= P_{\mathcal{W}} L_{M_z} (-b f_1) \\
&= -b f_0.
\end{aligned}$$

But $b = \frac{1}{\sqrt{5}}[-\sqrt{\alpha}+\sqrt{\beta}] \neq 0$, and hence $\tilde{X}_{13} \neq 0$. This implies that the Shimorin-Aluthge kernel is not tridiagonal. On the other hand, the matrix representation of $|M_z|^{-1}$ implies right away that the standard Aluthge kernel is tridiagonal (see Theorem 3.4.2).

Now we return to standard Aluthge kernels of shifts (see the definition following Theorem 3.6.7). Let $\mathcal{H}_k \subseteq \mathcal{O}(\mathbb{D})$ be a reproducing kernel Hilbert space. Suppose M_z on $\mathcal{H}_k$ is left-invertible. Then Theorem 3.6.7 says that $\tilde{M}_z$ and M_z on $\mathcal{H}_{\tilde{k}} (\subseteq \mathcal{O}(\mathbb{D}))$ are unitarily equivalent, where

$$\tilde{k}(z,w) := \langle |M_z|^{-1} k(\cdot,w), k(\cdot,z)\rangle_{\mathcal{H}_k} = \Big(|M_z|^{-1} k(\cdot,w)\Big)(z),$$

for all $z, w \in \mathbb{D}$. In the following, as a direct application of Theorem 3.4.2, we address the issue of tridiagonal representation of the shift M_z on $\mathcal{H}_k$.

Corollary 3.8.2. *In the setting of Theorem 3.6.7, assume in addition that $\mathcal{E} = \mathbb{C}$ and $\mathcal{H}_{\tilde{k}}$ is a tridiagonal space with respect to the orthonormal basis $\{f_n\}_{n\geq 0}$, where $f_n(z) = (a_n + b_n z)z^n$, $n \geq 0$. Then $\mathcal{H}_k$ is a tridiagonal space if and only if*

$$U|M_z|U^* = \begin{bmatrix}
c_{00} & c_{01} & -\frac{\bar{b}_1}{\bar{a}_2} c_{01} & \frac{\bar{b}_1 \bar{b}_2}{\bar{a}_2 \bar{a}_3} c_{01} & \cdots \\
\bar{c}_{01} & c_{11} & c_{12} & -\frac{\bar{b}_2}{\bar{a}_3} c_{12} & \ddots \\
-\frac{b_1}{a_2} \bar{c}_{01} & \bar{c}_{12} & c_{22} & c_{23} & \ddots \\
\frac{b_1 b_2}{a_2 a_3} \bar{c}_{01} & -\frac{b_2}{a_3} \bar{c}_{12} & \bar{c}_{23} & c_{33} & \ddots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix},$$

with respect to the basis $\{f_n\}_{n\geq 0}$.

Proof. Recall from Theorem 3.6.7 that $\mathcal{H}_{\tilde{k}} = |M_z|^{-\frac{1}{2}} \mathcal{H}_k$ and $Uh = |M_z|^{-\frac{1}{2}} h$, $h \in \mathcal{H}_k$, defines the intertwining unitary. Set $P := U|M_z|U^*$. Then $P \in \mathcal{B}(\mathcal{H}_{\tilde{k}})$ is a positive operator, and for any $z, w \in \mathbb{D}$, we have

$$\begin{aligned}
\langle P\tilde{k}(\cdot,w), \tilde{k}(\cdot,z)\rangle_{\mathcal{H}_{\tilde{k}}} &= \langle |M_z| U^* \tilde{k}(\cdot,w), U^* \tilde{k}(\cdot,z)\rangle_{\mathcal{H}_k} \\
&= \langle |M_z| |M_z|^{-\frac{1}{2}} k(\cdot,w), |M_z|^{-\frac{1}{2}} k(\cdot,z)\rangle_{\mathcal{H}_k} \\
&= \langle k(\cdot,w), k(\cdot,z)\rangle_{\mathcal{H}_k},
\end{aligned}$$

as $U\Big(|M_z|^{-\frac{1}{2}}k(\cdot,w)\Big) = \tilde{k}(\cdot,w)$. Hence $k(z,w) = \langle P\tilde{k}(\cdot,w), \tilde{k}(\cdot,z)\rangle_{\mathcal{H}_{\tilde{k}}}$, $z,w \in \mathbb{D}$. The result now follows from Theorem 3.4.2. □

In particular, if $\tilde{k}$ is a tridiagonal kernel, then for k to be a tridiagonal kernel, it is necessary (as well as sufficient) that $U|M_z|U^*$ is of the form as in the above statement.

We conclude this chapter with the following curious observation which stems from the matrix representations of Shimorin left inverses of shifts on analytic tridiagonal spaces (see Theorem 3.2.4). Let $\mathcal{H}_k$ be an analytic tridiagonal space. Recall that L_{M_z} denotes the Shimorin left inverse of M_z. By Lemma 3.1.6, we have $|M_z|^{-2} = L_{M_z}L_{M_z}^*$. From the matrix representation of L_{M_z} in Theorem 3.2.4, one can check that the matrix representation of $|M_z|^{-2}$ satisfies the conclusion of Theorem 3.4.2. Consequently, the positive definite scalar kernel

$$K(z,w) = \langle |M_z|^{-2}k(\cdot,w), k(\cdot,z)\rangle_{\mathcal{H}_k} \qquad (z,w \in \mathbb{D}),$$

is a tridiagonal kernel. On the other hand, consider

$$a_n = \begin{cases} 2 & \text{if } n = 2 \\ 1 & \text{otherwise,} \end{cases} \quad \text{and } b_n = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then the shift M_z on the analytic tridiagonal space $\mathcal{H}_k$ corresponding to the orthonormal basis $\{f_n\}_{n\geq 0}$, where $f_n(z) = (a_n + b_n z)z^n$, $n \geq 0$, is left-invertible. However, a moderate computation reveals that the matrix representation of $|M_z|^{-1}$ does not satisfy the conclusion of Theorem 3.4.2. In other words, the positive definite scalar kernel

$$K(z,w) = \langle |M_z|^{-1}k(\cdot,w), k(\cdot,z)\rangle_{\mathcal{H}_k} \qquad (z,w \in \mathbb{D}),$$

is not a tridiagonal kernel.

Chapter 4

Invariant subspaces of analytic perturbations

In this chapter, we first introduce a class of finite rank operators F (we call them *n-perturbations*) on $H^2(\mathbb{D})$ for which the corresponding perturbations $S_n = M_z + F$ are shifts (we call them *n-shifts*). Then we present a complete classification of S_n-invariant closed subspaces of $H^2(\mathbb{D})$. Note that S_n is unitarily equivalent to the multiplication operator M_z on some analytic Hilbert space.

We remark that a priori examples of n-shifts may seem counter-intuitive because of the intricate structure of perturbed linear operators. Subsequently, we put special emphasis on natural examples of n-shifts, and as interesting as it may seem, analytic spaces corresponding to (truncated) tridiagonal kernels or band kernels with bandwidth 1 give several natural examples of n-shifts. In the special case when S_n is unitarily equivalent to a shift on an analytic space corresponding to a band truncated kernel with bandwidth 1, we prove that the invariant subspaces of S_n are also hyperinvariant. Our proof of this fact follows a classical route: computation of commutants of shifts. In general, it is a difficult problem to compute the commutant of a shift (even for weighted shifts). However, in our band truncated kernel case, we are able to explicitly compute the commutant of n-shifts:

$$\{S_n\}' = \{T_\varphi + N : \varphi \in H^\infty(\mathbb{D}), \text{rank} N \leq n\},$$

where T_φ denotes the analytic Toeplitz operator with symbol $\varphi \in H^\infty(\mathbb{D})$, and N admits an explicit (and restricted) representation. We also present concrete examples of 1-shifts on tridiagonal kernel spaces with special emphasis on cyclicity of invariant subspaces. For instance, a simple example of S_1-shift brings out the following distinctive properties:

1. $[S_1^*, S_1] := S_1^* S_1 - S_1 S_1^*$ is of finite rank (in particular, S_1 is essentially normal).
2. S_1 is not subnormal (and, more curiously, not even hyponormal).

3. Invariant subspaces of S_1 are cyclic.

We believe that these observations along with the classification of invariant subspaces of shifts on tridiagonal spaces are of independent interest beside their application to the theory of perturbed operators. Finally, we remark that perturbations of concrete operators (with some analytic flavor) have been also studied in different contexts by other authors. For instance, see [34, 44, 53], and notably Clark [21].

4.1 n-shifts

This section introduces the central concept of this chapter, namely, analytic perturbations or n-shifts. We also explore some basic properties of n-shifts.

for all $f \in \mathcal{H}_k$, $w \in \mathbb{D}$, and $\eta \in \mathcal{E}$. We now present the formal definition of shift operators:

Definition 4.1.1. *The shift on $\mathcal{H}_k$ is the multiplication operator M_z defined by $(M_z f)(w) = wf(w)$ for all $f \in \mathcal{H}_k$ and $w \in \mathbb{D}$.*

In what follows, we will be mostly concerned with bounded shifts. Therefore, we always assume that M_z is bounded. Note that, in the scalar-valued case, that is, when $\mathcal{E} = \mathbb{C}$, the positivity condition of the kernel becomes

$$\sum_{i,j=1}^{m} \bar{c}_i c_j k(z_i, z_j) \geq 0,$$

for all $\{z_1, \ldots, z_m\} \subseteq \mathbb{D}$, $\{\eta_1, \ldots, \eta_m\} \subseteq \mathcal{E}$ and $m \geq 1$. The simplest example of an analytic kernel is the *Szegö kernel* $\mathbb{S}$ on $\mathbb{D}$, where

$$\mathbb{S}(z, w) = (1 - z\bar{w})^{-1} \qquad (z, w \in \mathbb{D}).$$

The analytic space corresponding to the Szegö kernel is the well-known (scalar-valued) Hardy space $H^2(\mathbb{D})$, where the shift M_z on $H^2(\mathbb{D})$ is known as the *unilateral shift* (of multiplicity one). Also, recall that the unilateral shift M_z on $H^2(\mathbb{D})$ is the model operator for contractions on Hilbert spaces (in the sense of basic building blocks [21]).

We also record the key terms of the agreement: $X_1 \in \mathcal{B}(\mathcal{H}_1)$ and $X_2 \in \mathcal{B}(\mathcal{H}_2)$ are the same means there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $UX_1 = X_2U$, that is, X_1 and X_2 are unitarily equivalent. Therefore, $X \in \mathcal{B}(\mathcal{H})$ is a shift if there exists an analytic Hilbert space $\mathcal{H}_k$ such that the shift M_z on $\mathcal{H}_k$ and X are unitarily equivalent. Finally, we are ready to introduce the central objects of this chapter:

Definition 4.1.2 (n-shifts)**.** *A linear operator F on $H^2(\mathbb{D})$ is called an n-perturbation if* *(i)* $Fz^m = 0$ *for all* $m \geq n$*,*

(ii) $F(z^m H^2(\mathbb{D})) \subseteq z^{m+1}\mathbb{C}[z]$ *for all* $m \geq 0$*, and*

(iii) $M_z + F$ *is left-invertible.*

We call $S_n = M_z + F$ *the* n*-shift corresponding to the* n*-perturbation* F *(or simply* n*-shift if* F *is clear from the context).*

It follows that an n-perturbation is of rank m for some $m \leq n$. In fact, it is easy to see that the rank of the 2-perturbation

$$Fz^m = \begin{cases} z^2 & \text{if } m = 0, 1 \\ 0 & \text{otherwise,} \end{cases}$$

is precisely 1. Moreover, $S_2 = M_z + F$ is a 2-shift. Indeed, since $S_2^* S_2 = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \oplus I_{z^2 H^2(\mathbb{D})}$ on $H^2(\mathbb{D}) = \mathbb{C} \oplus \mathbb{C}z \oplus z^2 H^2(\mathbb{D})$, it follows that $S_2^* S_2$ is invertible, and hence S_2 is left-invertible. Now we justify Definition 4.1.2 by showing that an n-shift is indeed a shift.

Lemma 4.1.3. *Let* F *be an* n*-perturbation. If* $S_n = M_z + F$*, then:*

(i) $F(z^m f) = 0$ *for each* $m \geq n$ *and* $f \in H^2(\mathbb{D})$*.*

(ii) For each $f \in H^2(\mathbb{D})$ *and* $m \geq 1$*, there exists* $p \in \mathbb{C}[z]$*, depending on both* f *and* m*, such that*

$$S_n^m f = z^m (f + p).$$

(iii) S_n *is a shift on some analytic Hilbert space.*

Proof. Part (i) immediately follows from the fact that $F(z^m p) = 0$ for all $p \in \mathbb{C}[z]$. Since by assumption $F(z^m H^2(\mathbb{D})) \subseteq z^{m+1}\mathbb{C}[z]$, $m \geq 0$, for each $f \in H^2(\mathbb{D})$, there exists a polynomial $p_f \in \mathbb{C}[z]$ such that $Ff = zp_f$. Then

$$S_n f = (M_z + F)f = zf + zp_f = z(f + p_f),$$

and hence, there exists $q_f \in \mathbb{C}[z]$ such that

$$S_n^2 f = (M_z + F)(z(f + p_f)) = z^2(f + p_f) + z^2 q_f = z^2(f + p_f + q_f).$$

The second assertion now follows by the principle of mathematical induction. To prove the last assertion, we use (ii) to conclude that

$$S_n^m H^2(\mathbb{D}) \subseteq z^m H^2(\mathbb{D}) \qquad (m \geq 0). \tag{4.1.1}$$

Since we know that M_z on $H^2(\mathbb{D})$ is pure, that is, $\cap_{m\geq 0} z^m H^2(\mathbb{D}) = \{0\}$, the above inclusion implies that

$$\cap_{m\geq 0} S_n^m H^2(\mathbb{D}) \subseteq \cap_{m\geq 0} z^m H^2(\mathbb{D}) = \{0\}.$$

Using this and the left invertibility of S_n, it follows that S_n on $H^2(\mathbb{D})$ is a shift. $\square$

Note that the following standard fact [56] has been used in the above proof: If $T \in \mathcal{B}(\mathcal{H})$ is a left-invertible operator and if $\cap_{m=0}^{\infty} T^m \mathcal{H} = \{0\}$, then T is unitarily equivalent to the shift M_z on some $\mathcal{W}$-valued analytic Hilbert space, where $\mathcal{W} = \mathcal{H} \ominus T\mathcal{H}$. In the present case, if

$$\mathcal{W} = \ker S_n^* = \ker(M_z + F)^*,$$

then S_n on $H^2(\mathbb{D})$ is unitarily equivalent to M_z on some $\mathcal{W}$-valued analytic Hilbert space $\mathcal{H}_k$ over $\mathbb{D}$. Here the kernel function k is explicit [56, Corollary 2.14] and involves a specific left inverse of S_n (namely, $(S_n^* S_n)^{-1} S_n^*$), but we will not need this.

Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Given a vector $f \in \mathcal{H}$, let $[f]_T$ denote the T-cyclic closed subspace generated by f, that is

$$[f]_T = \text{clos}\, \{p(T)f : p \in \mathbb{C}[z]\}.$$

Lemma 4.1.4. *If $f \in H^2(\mathbb{D})$ is a nonzero function, then $[f]_{S_n}$ contains a nontrivial closed M_z-invariant subspace of $H^2(\mathbb{D})$.*

Proof. Suppose $g \in H^2(\mathbb{D})$. By part (ii) of Lemma 4.1.3, we already know that $S_n^n g = z^n(g + p)$ for some $p \in \mathbb{C}[z]$. Then part (i) of the same lemma implies that

$$S_n^{n+1} g = (M_z + F)(z^n g + z^n p) = M_z(z^n g + z^n p) = M_z(S_n^n g).$$

Then, by induction, we have $S_n^m g = M_z^{m-n}(S_n^n g)$, and hence

$$S_n^m = M_z^{m-n} S_n^n \qquad (m \geq n+1). \tag{4.1.2}$$

In particular, if f is nonzero in $H^2(\mathbb{D})$, then $[S_n^n f]_{M_z}$ is an M_z-invariant closed subspace of $[f]_{S_n}$. $\square$

In the context of the equality (4.1.2), note in general that

$$[M_z^{m-n}, S_n^n] = M_z^{m-n} S_n^n - S_n^n M_z^{m-n} \neq 0 \qquad (m \geq n+1).$$

4.2 Invariant subspaces

In this section, we will prove the central result of this chapter: a complete classification of n-shift invariant closed subspaces of $H^2(\mathbb{D})$. However, as a first step, we need to prove a Beurling type property of invariant subspaces of n-shifts. We recall that if $\mathcal{S}$ is a nonzero closed M_z-invariant subspace of $H^2(\mathbb{D})$, then

$$\dim\,(\mathcal{S} \ominus z\mathcal{S}) = 1.$$

This is an easy consequence of the Beurling theorem (or, one way to prove the Beurling theorem). In the following, we prove a similar result for S_n-invariant closed subspaces of $H^2(\mathbb{D})$.

Theorem 4.2.1. *If $\mathcal{M} \subseteq H^2(\mathbb{D})$ is a nonzero closed S_n-invariant subspace, then*

$$dim(\mathcal{M} \ominus S_n\mathcal{M}) = 1.$$

Proof. Suppose if possible that $\mathcal{M} \ominus S_n\mathcal{M} = \{0\}$. Since S_n is left-invertible, it follows that

$$S_n^m\mathcal{M} = \mathcal{M} \qquad (m \geq 1),$$

which implies that

$$\mathcal{M} = \cap_{m\geq 1} S_n^m\mathcal{M} \subseteq \cap_{m\geq 1} S_n^m H^2(\mathbb{D}) \subseteq \cap_{m\geq 1} z^m H^2(\mathbb{D}) = \{0\},$$

where the second inclusion follows from (4.1.1). This contradiction shows that $\mathcal{M} \ominus S_n\mathcal{M} \neq \{0\}$. Now suppose that $f, g \in \mathcal{M} \ominus S_n\mathcal{M}$ be unit vectors. If possible, assume that f and g are orthogonal, that is, $\langle f, g\rangle = 0$. We claim that

$$[f]_{S_n} \cap [g]_{S_n} = \{0\}.$$

To prove this, first we pick a nonzero vector $h \in [f]_{S_n} \cap [g]_{S_n}$. Then there exist sequences of polynomials $\{p_m\}_{m\geq 1}$ and $\{q_m\}_{m\geq 1}$ such that

$$h = \lim_{m\to\infty}(p_m(S_n)f) = \lim_{m\to\infty}(q_m(S_n)g). \tag{4.2.1}$$

For each $m \geq 1$, we let

$$p_m(z) = \alpha_{m,0} + \alpha_{m,1}z + \cdots + \alpha_{m,t_m}z^{t_m},$$

and

$$q_m(z) = \beta_{m,0} + \beta_{m,1}z + \cdots + \beta_{m,l_m}z^{l_m},$$

where t_m and l_m are in $\mathbb{N}$ and $m \geq 1$. Now $S_n^l g \in S_n\mathcal{M}$ for all $l \geq 1$, together with $\langle g, f\rangle = 0$ implies that $\langle q_m(S_n)g, f\rangle = 0$ for all $m \geq 1$. Therefore

$$\langle h, f\rangle = \langle \lim_{m\to\infty} p_m(S_n)f, f\rangle = \langle \lim_{m\to\infty} q_m(S_n)g, f\rangle = \lim_{m\to\infty}\langle q_m(S_n)g, f\rangle = 0,$$

that is, $\langle h, f\rangle = 0$, where, on the other hand

$$\langle h, f\rangle = \langle \lim_{m\to\infty} p_m(S_n)f, f\rangle = \lim_{m\to\infty}\langle p_m(S_n)f, f\rangle = \lim_{m\to\infty}\langle \alpha_{m,0} f, f\rangle,$$

as $S_n^l f \in S_n\mathcal{M}$ for all $l \geq 1$, and $f \perp S_n\mathcal{M}$. We immediately deduce that

$$\lim_{m\to\infty} \alpha_{m,0} = 0.$$

Thus we obtain

$$h = \lim_{m\to\infty}((\alpha_{m,1}S_n + \cdots + \alpha_{m,t_m}S_n^{t_m})f).$$

Since $\langle S_n^k g, g\rangle = 0$ and $\langle S_n^l f, g\rangle = 0$ for all $k, l \geq 1$, repeating the same argument as above, we have $\langle h, g\rangle = 0$ and

$$\lim_{m\to\infty} \beta_{m,0} = 0,$$

and consequently

$$h = \lim_{m\to\infty}((\beta_{m,1}S_n + \cdots + \beta_{m,l_m}S_n^{l_m})g).$$

Thus we obtain

$$\lim_{m\to\infty}((\alpha_{m,1}S_n + \cdots + \alpha_{m,t_m}S_n^{t_m})f) = \lim_{m\to\infty}((\beta_{m,1}S_n + \cdots + \beta_{m,l_m}S_n^{l_m})g).$$

Multiplying both sides by a left inverse of S_n (for instance, $(S_n^*S_n)^{-1}S_n^*$ is a left inverse of S_n [56]) then gives

$$\begin{aligned} h_1 :&= \lim_{m\to\infty}((\alpha_{m,1} + \alpha_{m,2}S_n + \cdots + \alpha_{m,t_m}S_n^{t_m-1})f) \\ &= \lim_{m\to\infty}((\beta_{m,1} + \beta_{m,2}S_n + \cdots + \beta_{m,l_m}S_n^{l_m-1})g). \end{aligned}$$

We are now in exactly the same situation as in (4.2.1). Proceeding as above, we then have

$$\lim_{m\to\infty} \alpha_{m,1} = \lim_{m\to\infty} \beta_{m,1} = 0.$$

Arguing similarly, it will follow by induction that

$$\lim_{m\to\infty} \alpha_{m,t} = \lim_{m\to\infty} \beta_{m,l} = 0.$$

for all $t = 0, 1, \ldots, t_m$, and $l = 0, 1, \ldots, l_m$, and $m \geq 1$, and so $h = 0$. This contradiction shows that

$$[f]_{S_n} \cap [g]_{S_n} = \{0\}.$$

Now by Lemma 4.1.4 and the classical Beurling theorem, we know that $\theta_1 H^2(\mathbb{D}) \subseteq [f]_{S_n}$ and $\theta_2 H^2(\mathbb{D}) \subseteq [g]_{S_n}$ for some inner functions θ_1 and θ_2 in $H^\infty(\mathbb{D})$. Since

$$\theta_1\theta_2 \in \theta_1 H^2(\mathbb{D}) \cap \theta_2 H^2(\mathbb{D}) \subseteq [f]_{S_n} \cap [g]_{S_n},$$

it follows that $\theta_1 H^2(\mathbb{D}) \cap \theta_2 H^2(\mathbb{D}) \neq \{0\}$, which contradicts the fact that $[f]_{S_n} \cap [g]_{S_n} = \{0\}$. Therefore, $\dim(\mathcal{M} \ominus S_n\mathcal{M}) = 1$, and completes the proof of the theorem. □

Note that the final part of the above proof uses the classical Beurling theorem : If $\mathcal{M}$ is a nonzero M_z-invariant closed subspace of $H^2(\mathbb{D})$, then there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that $\mathcal{M} = [\theta]_{M_z}$. We will return to the issue of cyclic invariant subspaces of 1-shifts in Section 4.5, and here we proceed to state and prove our general invariant subspace theorem.

Theorem 4.2.2. *Let F be an n-perturbation on $H^2(\mathbb{D})$, and let $\mathcal{M}$ be a nonzero closed subspace of $H^2(\mathbb{D})$. Then $\mathcal{M}$ is invariant under $S_n = M_z + F$ if and only if there exist an inner function $\theta \in H^\infty(\mathbb{D})$ and polynomials $\{p_i, q_i\}_{i=0}^{n-1} \subseteq \mathbb{C}[z]$ such that*

$$\mathcal{M} = (\mathbb{C}\varphi_0 \oplus \mathbb{C}\varphi_1 \oplus \cdots \oplus \mathbb{C}\varphi_{n-1}) \oplus z^n\theta H^2(\mathbb{D}),$$

where $\varphi_i = z^i p_i\theta - q_i$ for all $i = 0, \ldots, n-1$, and

$$S_n\varphi_j \in (\mathbb{C}\varphi_{j+1} \oplus \cdots \oplus \mathbb{C}\varphi_{n-1}) \oplus z^n\theta H^2(\mathbb{D}),$$

for all $j = 0, \ldots, n-2$, and $S_n\varphi_{n-1} = z^n p_{n-1}\theta$.

Proof. Let $\mathcal{M}$ be a nonzero closed subspace of $H^2(\mathbb{D})$. Observe that

$$S_n(z^n f) = (M_z + F)(z^n f) = z^{n+1} f + F(z^n f) = z^{n+1} f,$$

for all $f \in H^2(\mathbb{D})$, where the last equality follows from Lemma 4.1.3. Therefore

$$M_z^{m+n} = S_n^m M_z^n \qquad (m \geq 1). \tag{4.2.2}$$

To prove the sufficient part, we see, by (4.2.2), that

$$S_n(z^n\theta f) = z^{n+1}\theta f \in z^n\theta H^2(\mathbb{D}),$$

for all $f \in H^2(\mathbb{D})$, and hence $S_n(z^n\theta H^2(\mathbb{D})) \subseteq z^n\theta H^2(\mathbb{D})$. This and the remaining assumptions then implies that $S_n\mathcal{M} \subseteq \mathcal{M}$.

For the converse direction, assume that $S_n\mathcal{M} \subseteq \mathcal{M}$. Theorem 4.2.1 then implies

$$\mathcal{M} = \mathbb{C}\varphi_0 \oplus S_n\mathcal{M},$$

for some nonzero vector $\varphi_0 \in \mathcal{M} \ominus S_n\mathcal{M}$. Since $\mathcal{M}$ is closed and S_n is left invertible, it follows that $S_n\mathcal{M}$ is also a nonzero closed S_n-invariant subspace of $H^2(\mathbb{D})$. By Theorem 4.2.1 again, we have

$$\mathcal{M} = \mathbb{C}\varphi_0 \oplus (\mathbb{C}\varphi_1 \oplus S_n^2\mathcal{M}),$$

for some nonzero vector $\varphi_1 \in S_n\mathcal{M} \ominus S_n(S_n\mathcal{M})$. Continuing exactly in the same way, by induction, we find $\varphi_i \in S_n^i\mathcal{M} \ominus S_n^{i+1}\mathcal{M}$, $i = 0, 1, \ldots, n-1$, such that

$$\mathcal{M} = (\mathbb{C}\varphi_0 \oplus \mathbb{C}\varphi_1 \oplus \cdots \oplus \mathbb{C}\varphi_{j-1}) \oplus S_n^j\mathcal{M},$$

for all $j = 1, \ldots, n$. In particular, $\mathcal{M} = (\mathbb{C}\varphi_0 \oplus \mathbb{C}\varphi_1 \oplus \cdots \oplus \mathbb{C}\varphi_{n-1}) \oplus S_n^n\mathcal{M}$. Now, by (4.1.2), we have $M_z(S_n^n f) = S_n^{n+1} f$, $f \in \mathcal{M}$, which implies that $M_z(S_n^n\mathcal{M}) \subseteq S_n^n\mathcal{M}$, that is, $S_n^n\mathcal{M}$ is a closed nonzero M_z-invariant subspace of $H^2(\mathbb{D})$. By the Beurling theorem this implies that $S_n^n\mathcal{M} = \tilde{\theta}H^2(\mathbb{D})$ for some inner function $\tilde{\theta} \in H^\infty(\mathbb{D})$. Since each element in $S_n^n\mathcal{M}$ has a zero of order at least n at $z = 0$ (see part (ii) of Lemma 4.1.3), it follows that $\tilde{\theta} = z^n\theta$ for some inner function $\theta \in H^\infty(\mathbb{D})$. Thus

$$S_n^n\mathcal{M} = z^n\theta H^2(\mathbb{D}), \tag{4.2.3}$$

and hence

$$\mathcal{M} = (\mathbb{C}\varphi_0 \oplus \mathbb{C}\varphi_1 \oplus \cdots \oplus \mathbb{C}\varphi_{n-1}) \oplus z^n\theta H^2(\mathbb{D}),$$

for some inner function $\theta \in H^\infty(\mathbb{D})$. Fix an $i \in \{0, 1, \ldots, n-1\}$. Since $\varphi_i \in S_n^i\mathcal{M} \ominus S_n^{i+1}\mathcal{M}$, by construction, we have $\varphi_i \in S_n^i\mathcal{M}$, and hence (4.2.3) implies

$$S^{n-i}\varphi_i \in S_n^n\mathcal{M} = z^n\theta H^2(\mathbb{D}).$$

Therefore, there exists $h_i \in H^2(\mathbb{D})$ such that

$$S_n^{n-i}\varphi_i = z^n\theta h_i. \tag{4.2.4}$$

By part (ii) of Lemma 4.1.3, there exists a polynomial $q_i \in \mathbb{C}[z]$ such that $S_n^{n-i}\varphi_i = z^{n-i}(\varphi_i + q_i)$. Then

$$\varphi_i + q_i = z^i\theta h_i. \tag{4.2.5}$$

Since $\varphi_i \perp S_n^n\mathcal{M} = z^n\theta H^2(\mathbb{D})$, by construction, for each $l \geq 0$, we have

$$\langle z^i\theta h_i, z^{n+l}\theta\rangle = \langle \varphi_i + q_i, z^{n+l}\theta\rangle = \langle q_i, z^{n+l}\theta\rangle,$$

which, along with $\langle z^i\theta h_i, z^{n+l}\theta\rangle = \langle h_i, z^{n+l-i}\rangle$, implies that

$$\langle h_i, z^{n+l-i}\rangle = \langle q_i, z^{n+l}\theta\rangle.$$

Finally, using the fact that q_i is a polynomial, we conclude that for each $i = 0, \ldots, n-1$, there exists a natural number n_i such that $\langle h_i, z^t\rangle = 0$ for all $t \geq n_i$, and hence $p_i := h_i$ is a polynomial. This completes the proof. $\square$

From the final part of the above proof, we note that $h_i := p_i$ is a polynomial. Therefore, by (4.2.4) and (4.2.5), there exist polynomials $p_i, q_i \in \mathbb{C}[z]$ such that $\varphi_i = z^i p_i \theta - q_i$, and

$$S_n^{n-i}\varphi_i = z^n p_i \theta \qquad (i = 0, 1, \ldots, n-1). \tag{4.2.6}$$

The description of invariant subspaces of S_n as in the above theorem appears to be satisfactory and complete. However, a more detailed delicacy is hidden in the structure of polynomials $\{p_i, q_i\}_{i=0}^{n-1}$ and the finite rank operator F. In fact, without much control over these polynomials (and/or the finite rank operator F), hardly much can be said about the other basic properties of n-shift invariant subspaces. For instance:

When an n-shift invariant subspace is cyclic?

Needless to say, the cyclicity property of shift operators is a complex problem. We will return to this issue in Section 4.5, and refer [1, 17] for some modern development of cyclic vectors of shift invariant subspaces of function Hilbert spaces.

4.3 Commutants

In this section, we compute commutants of n-shifts on analytic Hilbert spaces corresponding to truncated tridiagonal kernels. The concept of tridiagonal kernels or band kernels with bandwidth one in the context of analytic Hilbert spaces was introduced in [3, 4]. Note that shifts on analytic Hilbert spaces corresponding to band kernels with bandwidth one are the next best examples of shifts after the weighted shifts.

The following definition is a variant of truncated tridiagonal kernels which is also motivated by a similar (but not exactly the same) concept of kernels in the context of Shimorin's analytic models [24].

Definition 4.3.1. *Let $\mathcal{H}_k$ be an analytic Hilbert space corresponding to an analytic kernel $k : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$. We say that $\mathcal{H}_k$ is a truncated space (and k is a truncated kernel) if:*

(i) $\mathbb{C}[z] \subseteq \mathcal{H}_k$,

(ii) the shift M_z is bounded on $\mathcal{H}_k$, and

(iii) $\{f_m\}_{m\geq 0}$ forms an orthonormal basis of $\mathcal{H}_k$, where $f_m = (a_m + b_m z)z^m$, $m \geq 0$, for some scalars $\{a_m\}_{m\geq 0}$ and $\{b_m\}_{m\geq 0}$ such that $a_s \neq 0$ for all $s \geq 0$, and $b_t = 0$ for all $t \geq n$.

Note that in the above definition, n is a fixed natural number. Also, in this case, the kernel function k is given by

$$k(z,w) = \sum_{m=0}^{\infty} f_m(z)\overline{f_m(w)} \qquad (z,w \in \mathbb{D}).$$

If, in addition, $\{|\frac{a_m}{a_{m+1}}|\}_{m\geq 0}$ is bounded away from zero, then M_z on $\mathcal{H}_k$ is left-invertible [24, Theorem 3.5]. Clearly, the above representation of k justifies the use of the term tridiagonal kernel.

Throughout this section, we will assume that $a_m = 1$ for all $m \geq 0$. Using the orthonormal basis $\{f_m = (1+b_m z)z^m\}_{m\geq 0}$ of $\mathcal{H}_k$, a simple calculation reveals that (cf. [3, Section 3] or [24, Section 2])

$$z^m = \sum_{t=0}^{\infty}(-1)^t\Big(\prod_{j=0}^{t-1} b_{m+j}\Big) f_{m+t} \qquad (m \geq 0), \tag{4.3.1}$$

where $\Pi_{j=0}^{-1} x_{m+j} := 1$. Since $b_m = 0$, $m \geq n$, we have $\prod_{j=0}^{t-1} b_{m+j} = 0$ for all $t \geq n+1$. In particular, the above is a finite sum. We set

$$c_{m,p} = b_m - b_{m+p}, \tag{4.3.2}$$

for all $m \geq 0$ and $p \geq 1$. Clearly, $c_{m,p} = 0$ for all $m \geq n$. Now $M_z f_m = z^{m+1} + b_m z^{m+2}$ implies that

$$z f_m = f_{m+1} + (b_m - b_{m+1})z^{m+2} = f_{m+1} + c_{m,1} z^{m+2},$$

that is, $z f_m = f_{m+1} + c_{m,1} z^{m+2}$ for all $m \geq 0$. Then (4.3.1) yields

$$z f_m = f_{m+1} + c_{m,1}\sum_{t=0}^{\infty}(-1)^t\Big(\prod_{j=0}^{t-1} b_{m+2+j}\Big) f_{m+2+t} \qquad (m \geq 0). \tag{4.3.3}$$

Since $c_{m,1} = 0$ for all $m \geq n$, as pointed out earlier, it follows that $z f_m = f_{m+1}$ for all $m \geq n$. In particular, the matrix representation of M_z with respect to the orthonormal basis $\{f_m\}_{m\geq 0}$ is given by (also see [3, Page 729])

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots & 0 & 0 & \dots \\ c_{0,1} & 1 & 0 & \dots & 0 & 0 & \dots \\ -c_{0,1}b_2 & c_{1,1} & 1 & \dots & 0 & 0 & \dots \\ c_{0,1}b_2b_3 & -c_{1,1}b_3 & c_{2,1} & \ddots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_{n-1,1} & 1 & \ddots \\ 0 & 0 & 0 & \dots & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

We define the *canonical unitary map* $U : \mathcal{H}_k \longrightarrow H^2(\mathbb{D})$ by setting $Uf_m = z^m$, $m \geq 0$. It then follows that

$$UM_z = S_n U, \tag{4.3.4}$$

where $S_n := M_z + F$ is the n-shift corresponding to the n-perturbation F on $H^2(\mathbb{D})$ whose matrix representation with respect to the orthonormal basis $\{z^m\}_{m\geq 0}$ of $H^2(\mathbb{D})$ is given by

$$[F] = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots \\ c_{0,1} & 0 & 0 & \ldots & 0 & 0 & \ldots \\ -c_{0,1}b_2 & c_{1,1} & 0 & \ldots & 0 & 0 & \ldots \\ c_{0,1}b_2b_3 & -c_{1,1}b_3 & c_{2,1} & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & c_{n-1,1} & 0 & \cdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ldots & \ldots & \vdots & \vdots \end{bmatrix},$$

Definition 4.3.2. *We call S_n the n-shift corresponding to the truncated kernel k.*

Now we turn to the commutants of n-shifts corresponding to truncated kernels. Since M_z on $\mathcal{H}_k$ and S_n on $H^2(\mathbb{D})$ are unitarily equivalent, the problem of computing the commutant of S_n reduces to that of M_z.

Let $\mathcal{H}_k$ be a truncated space. Recall that a function $\varphi : \mathbb{D} \to \mathbb{C}$ is said to be a *multiplier* of $\mathcal{H}_k$ if $\varphi\mathcal{H}_k \subseteq \mathcal{H}_k$ [10]. We denote by $\mathcal{M}(\mathcal{H}_k)$ the set of all multipliers. By the closed graph theorem, a multiplier $\varphi \in \mathcal{M}(\mathcal{H}_k)$ defines a bounded linear operator M_φ on $\mathcal{H}_k$, where

$$M_\varphi f = \varphi f \qquad (f \in \mathcal{H}_k).$$

We call M_φ the *multiplication operator* corresponding to φ.

We will use the following notation: If $X \in \mathcal{B}(\mathcal{H})$, then the commutant of X, denoted by $\{X\}'$, is the algebra of all operators $T \in \mathcal{B}(\mathcal{H})$ such that $TX = XT$. In the following, we observe that $\{M_z\}' = \{M_\varphi : \varphi \in \mathcal{M}(\mathcal{H}_k)\}$. The proof is fairly standard:

Lemma 4.3.3. *Suppose $A \in \mathcal{B}(\mathcal{H}_k)$. Then $AM_z = M_zA$ if and only if there exists $\varphi \in \mathcal{M}(\mathcal{H}_k)$ such that $A = M_\varphi$.*

Proof. The "if" part is easy. To prove the "only if" part, suppose $AM_z = M_zA$ and let $A1 = \varphi$. Clearly, $\varphi \in \mathcal{H}_k$. Since $f_m = (1 + b_m z)z^m$, it follows that

$$Af_m = Az^m + b_m Az^{m+1} = (z^m + b_m z^{m+1})A1 = f_m\varphi = \varphi f_m,$$

for all $m \geq 0$. Since $\{f_m\}_{m\geq 0}$ is an orthonormal basis, we have $Af = \varphi f$ for all $f \in \mathcal{H}_k$, and hence, $\varphi\mathcal{H}_k \subseteq \mathcal{H}_k$. This proves that $A = M_\varphi$, and completes the proof of the lemma. ∎

Now we prove the main result of this section. It essentially says that $\mathcal{M}(\mathcal{H}_k) = H^\infty(\mathbb{D})$, that is, $\{M_z\}' = \{M_\varphi : \varphi \in H^\infty(\mathbb{D})\}$.

Theorem 4.3.4. *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be a function, and let $\mathcal{H}_k$ be a truncated space with $\{f_m\}_{m\geq 0}$ as an orthonormal basis, where $f_m(z) = (1 + b_m z)z^m$, $m \geq 0$, and $b_t = 0$ for all $t \geq n$. Then $\varphi \in \mathcal{M}(\mathcal{H}_k)$ if and only in $\varphi \in H^\infty(\mathbb{D})$.*

Proof. Recall from (4.3.3) that

$$zf_m = f_{m+1} + c_{m,1}\sum_{t=0}^{\infty}(-1)^t\Big(\prod_{j=0}^{t-1} b_{m+2+j}\Big)f_{m+2+t} \qquad (m \geq 0).$$

In general, for any $p \geq 1$, we have

$$z^p f_m = (1 + b_m z)z^{m+p} = f_{m+p} + (b_m - b_{m+p})z^{m+p+1}.$$

Since $c_{m,p} = b_m - b_{m+p}$ for all $m \geq 0$ and $p \geq 1$ (see (4.3.2)), it follows that

$$z^p f_m = f_{m+p} + c_{m,p}(f_{m+p+1} - b_{m+p+1}f_{m+p+2} + b_{m+p+1}b_{m+p+2}f_{m+p+3} - \cdots). \quad (4.3.5)$$

Note that $c_{m,p} = 0$ for all $m \geq n$. Let $\varphi \in \mathcal{H}_k$, and suppose $\varphi = \sum_{m=0}^{\infty} \alpha_m z^m$. Since $\varphi f_0 = \sum_{m=0}^{\infty}(\alpha_m z^m f_0)$ and $f_0 = 1 + b_0 z$, (4.3.5) implies

$$\varphi f_0 = \alpha_0 f_0 + \alpha_1 f_1 + (\alpha_2 + \beta_{1,0})f_2 + \cdots + (\alpha_n + \beta_{n-1,0})f_n + \sum_{t=n+1}^{\infty}(\alpha_t + c_{0,t-1}\alpha_{t-1})f_t,$$

where

$$\beta_{j,0} = \text{coefficient of } f_{j+1} - \alpha_{j+1} \qquad (j = 1, \ldots, n-1).$$

Observe that $\beta_{j,0}$ is a finite sum for each $j = 1, \ldots, n-1$. Similarly, for each $0 \leq m < n$, we have

$$\begin{aligned}\varphi f_m =& \alpha_0 f_m + \alpha_1 f_{m+1} + (\alpha_2 + \beta_{1,m})f_{m+2} + \cdots + (\alpha_{n-m} + \beta_{n-m-1,m})f_n \\ &+ \sum_{t=n+1}^{\infty}(\alpha_{t-m} + c_{m,t-m-1}\alpha_{t-m-1})f_t,\end{aligned}$$

where, as before, we let

$$\beta_{j,m} = \text{coefficient of } f_{j+m+1} - \alpha_{j+1} \qquad (j = 1, \ldots, n-m-1).$$

Finally, for each $m \geq n$, it is easy to see that

$$\varphi f_m = \sum_{j=0}^{\infty} \alpha_j f_{m+j}.$$

Therefore, the formal matrix representation of the linear operator M_φ (which is not necessarily bounded yet) is given by the formal sum of matrix operators

$$[M_\varphi] = [\tilde{T}_\varphi] + [N], \tag{4.3.6}$$

where

$$[\tilde{T}_\varphi] = \begin{bmatrix} \alpha_0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & \alpha_0 & 0 & 0 & \ddots \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.3.7}$$

and

$$[N] = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \ddots \\ \beta_{1,0} & 0 & 0 & \cdots & 0 & 0 & 0 & \ddots \\ \beta_{2,0} & \beta_{1,1} & 0 & \cdots & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \beta_{n-1,0} & \beta_{n-2,1} & \beta_{n-3,2} & \cdots & 0 & 0 & 0 & \ddots \\ c_{0,n}\alpha_n & c_{1,n-1}\alpha_{n-1} & c_{2,n-2}\alpha_{n-2} & \cdots & c_{n-1,1}\alpha_1 & 0 & 0 & \ddots \\ c_{0,n+1}\alpha_{n+1} & c_{1,n}\alpha_n & c_{2,n-1}\alpha_{n-1} & \cdots & c_{n-1,2}\alpha_2 & 0 & 0 & \ddots \\ c_{0,n+2}\alpha_{n+2} & c_{1,n+1}\alpha_{n+1} & c_{2,n}\alpha_n & \cdots & c_{n-1,3}\alpha_3 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.3.8}$$

Now assume that $\varphi \in \mathcal{M}(\mathcal{H}_k)$, that is, the multiplication operator M_φ is bounded on $\mathcal{H}_k$. Since

$$M_\varphi f_n = \varphi f_n = \sum_{j=0}^{\infty} \alpha_j f_{n+j},$$

it follows that $\{\alpha_m\}_{m\geq 0}$ is square summable, and hence $[\tilde{T}_\varphi]$ defines a linear (but not necessarily bounded yet) operator on $\mathcal{H}_k$. Since the matrix operator $[N]$ has at most n nonzero columns and

$$\sum_{m=0}^{\infty} |\alpha_m|^2 < \infty,$$

it follows that $[N]$ is bounded on $\mathcal{H}_k$. Therefore, by (4.3.6), $[\tilde{T}_\varphi]$ defines a bounded linear operator $\tilde{T}_\varphi$ on $\mathcal{H}_k$. Then we find that the canonical unitary map $U : \mathcal{H}_k \to H^2(\mathbb{D})$ defined by equation (4.3.4) satisfies

$$U\tilde{T}_\varphi = T_\varphi U,$$

where T_φ denote the (bounded) Toeplitz operator on $H^2(\mathbb{D})$ with symbol φ. In particular, $\varphi \in H^\infty(\mathbb{D})$.

For the converse, we assume that $\varphi = \sum_{m=0}^{\infty} \alpha_m z^m$ is in $H^\infty(\mathbb{D})$. If we set $\tilde{T}_\varphi = U^* T_\varphi U$, then $\tilde{T}_\varphi$ is a bounded linear operator on $\mathcal{H}_k$, and the matrix representation of $\tilde{T}_\varphi$ will be of the form (4.3.7). Finally, since $\sum_{m=0}^{\infty} |\alpha_m|^2 < \infty$, it follows that the matrix (4.3.8) defines a bounded linear operator on $\mathcal{H}_k$. Therefore, $M_\varphi = \tilde{T}_\varphi + N$ is bounded on $\mathcal{H}_k$, which completes the proof of the theorem. $\square$

Of course, the inclusion $\mathcal{M}(\mathcal{H}_k) \subseteq H^\infty(\mathbb{D})$ follows rather trivially from properties of kernel functions: Suppose $\varphi \in \mathcal{M}(\mathcal{H}_k)$. By the reproducing property of kernel functions, we have $M_\varphi^* k(\cdot, w) = \overline{\varphi(w)} k(\cdot, w)$, which implies

$$|\varphi(w)| = \frac{1}{\|k(\cdot, w)\|} \|M_\varphi^* k(\cdot, w)\| \leq \|M_\varphi\| \qquad (w \in \mathbb{D}).$$

In particular, $\varphi \in H^\infty(\mathbb{D})$ and $\|\varphi\|_\infty \leq \|M_\varphi\|$. Evidently, the content of the above theorem is different and proves much more than the standard inclusion $\mathcal{M}(\mathcal{H}_k) \subseteq H^\infty(\mathbb{D})$. Also, note that we have proved more than what has been explicitly stated in the above theorem:

Theorem 4.3.5. *Consider the n-shift S_n corresponding to the truncated space $\mathcal{H}_k$ defined as in Theorem 4.3.4, and let $X \in \mathcal{B}(H^2(\mathbb{D}))$. Then $X \in \{S_n\}'$ if and only if there exists $\varphi \in H^\infty(\mathbb{D})$ such that $X = T_\varphi + N$, where N is a matrix operator as in (4.3.8) with respect to $\{z^m\}_{m \geq 0}$.*

The proof follows easily, once one observe that

$$UM_\varphi = (T_\varphi + N)U, \tag{4.3.9}$$

for all $\varphi \in H^\infty(\mathbb{D}) = \mathcal{M}(\mathcal{H}_k)$, where $U : \mathcal{H}_k \to H^2(\mathbb{D})$ is the canonical unitary as in (4.3.4).

The following observation is now standard: The n-shift S_n as in Theorem 4.3.4 is irreducible. Indeed, if $\mathcal{M} \subseteq \mathcal{H}_k$ is a closed M_z-reducing subspace, then $P_\mathcal{M} M_z = M_z P_\mathcal{M}$ implies that $P_\mathcal{M} = M_\varphi$ for some $\varphi \in \mathcal{M}(\mathcal{H}_k)$. By Theorem 4.3.4, $\varphi \in H^\infty(\mathbb{D})$. Then $P_\mathcal{M}^2 = P_\mathcal{M}$ implies that $\varphi^2 = \varphi$ on $\mathbb{D}$, and we obtain $\varphi \equiv 0$ or 1. It now follows that $\mathcal{M} = \{0\}$ or $\mathcal{H}_k$.

Representations of commutants of n-shifts on even "simple" truncated spaces appear to be interesting and nontrivial. We will work out some concrete examples in Section 4.5.

4.4 Hyperinvariant subspaces

We continue from where we left in Section 4.3, and prove that invariant subspaces of n-shifts on truncated spaces are hyperinvariant. Recall that a closed subspace $\mathcal{M} \subseteq \mathcal{H}$

is called a *hyperinvariant subspace* for $T \in \mathcal{B}(\mathcal{H})$ if

$$X\mathcal{M} \subseteq \mathcal{M},$$

for all $X \in \{T\}'$. We assume that $\mathcal{H}_k$ is a truncated space corresponding to the orthonormal basis $\{f_m\}_{m\geq 0}$, where $f_m(z) = (1+b_m z)z^m$, $m \geq 0$, and $\{b_m\}_{m\geq 0}$ are scalars such that $b_t = 0$ for all $t \geq n$. In this case, recall that $\mathcal{M}(\mathcal{H}_k) = H^\infty(\mathbb{D})$ (see Theorem 4.3.4), and the canonical unitary $U : \mathcal{H}_k \to H^2(\mathbb{D})$ defined by equation (4.3.4) satisfies

$$UM_z = S_n U \text{ and } UM_\varphi = (T_\varphi + N)U,$$

for all $\varphi \in H^\infty(\mathbb{D})$, where N is the finite rank operator whose matrix representation with respect to the orthonormal basis $\{z^m\}_{m\geq 0}$ of $H^2(\mathbb{D})$ is given by (4.3.8).

We are now ready to solve the hyperinvariant subspace problem for n-shifts on truncated spaces.

Theorem 4.4.1. *Closed invariant subspaces of n-shifts on truncated spaces are hyperinvariant.*

Proof. Let M_z be an n-shift on a truncated space, and let S_n be the corresponding n-shift on $H^2(\mathbb{D})$. Suppose $\mathcal{M}$ is a nonzero closed S_n-invariant subspace of $H^2(\mathbb{D})$. By Theorem 4.2.2, there exist an inner function $\theta \in H^\infty(\mathbb{D})$ and polynomials $\{p_i, q_i\}_{i=0}^{n-1}$ such that

$$\mathcal{M} = (\mathbb{C}\varphi_0 \oplus \mathbb{C}\varphi_1 \oplus \cdots \oplus \mathbb{C}\varphi_{n-1}) \oplus z^n\theta H^2(\mathbb{D}),$$

where $\varphi_i = z^i p_i \theta - q_i$ for all $i = 0, \ldots, n-1$, and $S_n\varphi_j \in (\mathbb{C}\varphi_{j+1} \oplus \cdots \mathbb{C}\varphi_{n-1}) \oplus z^n\theta H^2(\mathbb{D})$ for all $j = 0, \ldots, n-2$, and $S_n\varphi_{n-1} = z^n p_{n-1}\theta$. In view of Theorem 4.3.5, we only need to prove that $(T_\varphi + N)\varphi_i \in \mathcal{M}$ for all $i = 0, 1, \ldots, n-1$, and $(T_\varphi + N)z^n\theta H^2(\mathbb{D}) \subseteq z^n\theta H^2(\mathbb{D})$ for all $\varphi \in H^\infty(\mathbb{D})$. To this end, let $\varphi \in \mathcal{M}(\mathcal{H}_k) = H^\infty(\mathbb{D})$, and suppose $\varphi(z) = \sum_{m=0}^\infty \alpha_m z^m$. Then for each $i = 0, 1, \ldots, n-1$, we have

$$(T_\varphi + N)\varphi_i = UM_\varphi U^*\varphi_i = U(\varphi U^*\varphi_i),$$

and hence

$$\begin{aligned}(T_\varphi + N)\varphi_i &= U(\sum_{m=0}^\infty \alpha_m z^m U^*\varphi_i) \\ &= U(\sum_{m=0}^\infty \alpha_m M_z^m U^*\varphi_i), \\ &= \sum_{m=0}^\infty \alpha_m S_n^m \varphi_i \in \mathcal{M}\end{aligned}$$

as $\varphi_i \in \mathcal{M}$ and $S_n\mathcal{M} \subseteq \mathcal{M}$. Finally, if $f \in H^2(\mathbb{D})$, then Lemma 4.1.3 implies

$$(T_\varphi + N)z^n\theta f = T_\varphi(z^n\theta f) + 0 = z^n\theta\varphi f \in z^n\theta H^2(\mathbb{D}),$$

and hence, $(T_\varphi + N)z^n\theta H^2(\mathbb{D}) \subseteq z^n\theta H^2(\mathbb{D})$, which completes the proof. □

Now let M_z be an n-shift, and let $\mathcal{M}(\mathcal{H}_k) = H^\infty(\mathbb{D})$. In particular, $\{M_z\}' = H^\infty(\mathbb{D})$. In this case, a similar argument as the above proof gives the same conclusion as Theorem 4.4.1. However, as is well known, explicit computation of $\mathcal{M}(\mathcal{H}_k)$ is a rather challenging problem.

4.5 Examples

In this section, we examine Theorem 4.2.2 from a more definite examples point of view. As we will see, these examples are instructive and bring out several analytic and geometric flavors, and points out additional complications to the theory of finite rank perturbations.

Fix scalars a_0 and b_0 such that $0 < |b_0| \leq |a_0|$, and consider the 1-shift $S_1 = M_z + F$ on $H^2(\mathbb{D})$ corresponding to the 1-perturbation

$$Fz^m = \begin{cases} ((a_0 - 1) + b_0 z)z & \text{if } m = 0 \\ 0 & \text{if } m \geq 1. \end{cases} \tag{4.5.1}$$

The fact that S_1 is a 1-shift follows from the inherited tridiagonal structure of S_1. Indeed, S_1 is unitarily equivalent to the shift M_z on the truncated space $\mathcal{H}_k$ with orthonormal basis $\{f_m\}_{m\geq 0}$, where $f_m = (a_m + b_m z)z^m$, $m \geq 0$, and $a_t = 1$ and $b_t = 0$ for all $t \geq 1$. Since

$$\left|\frac{a_m}{a_{m+1}}\right| \geq \min\{|a_0|, 1\} \qquad (m \geq 0),$$

the sequence $\{|\frac{a_m}{a_{m+1}}|\}_{m\geq 0}$ is bounded away from zero, and hence, M_z is left-invertible (see the discussion following Definition 4.3.1). Moreover, the canonical unitary $U : \mathcal{H}_k \to H^2(\mathbb{D})$ defined by equation (4.3.4) satisfies the required intertwining property $UM_z = S_1 U$. Therefore, it follows that $S_1 = M_z + F$ on $H^2(\mathbb{D})$ is indeed a 1-shift. We clearly have

$$Ff = f(0)((a_0 - 1) + b_0 z)z \qquad (f \in H^2(\mathbb{D})). \tag{4.5.2}$$

Now we observe three distinctive features of S_1: Note that the matrix representation of S_1 with respect to the orthonormal basis $\{z^m\}_{m\geq 0}$ of $H^2(\mathbb{D})$ is given by

$$[S_1] = [M_z + F] = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ a_0 & 0 & 0 & 0 & \ddots \\ b_0 & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Then, a simple computation yields that

$$[S_1^*, S_1] = \begin{bmatrix} |a_0|^2 + |b_0|^2 & \bar{b}_0 & 0 & 0 & \dots \\ b_0 & 1 - |a_0|^2 & -a_0\bar{b}_0 & 0 & \ddots \\ 0 & -\bar{a}_0 b_0 & -|b_0|^2 & 0 & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

is precisely a rank-3 operator. Indeed, the determinant of the 3×3 nonzero submatrix of $[S_1^*, S_1]$ is given by

$$(|a_0|^2 + |b_0|^2)\Big(- (1 - |a_0|^2)|b_0|^2 - |a_0|^2|b_0|^2\Big) - |b_0|^4 = -|a_0|^2|b_0|^2 < 0.$$

This also implies that $[S_1^*, S_1]$ is not a positive definite operator. Therefore:

1. S_1 is essentially normal, that is, $[S_1^*, S_1] = S_1^* S_1 - S_1 S_1^*$ is compact (in fact, here it is of finite rank).
2. S_1 is not hyponormal (and hence, not subnormal).
3. Invariant subspaces of S_1 are cyclic.

The proof of the final assertion is the main content of the following two theorems:

Theorem 4.5.1. *Let a_0 and b_0 be scalars such that $0 < |b_0| \leq |a_0|$. Suppose*

$$Fz^m = \begin{cases} ((a_0 - 1) + b_0 z)z & \textit{if } m = 0 \\ 0 & \textit{if } m \geq 1, \end{cases}$$

and consider the 1-shift $S_1 = M_z + F$ on $H^2(\mathbb{D})$. Then a nonzero closed subspace $\mathcal{M} \subseteq H^2(\mathbb{D})$ is invariant under S_1 if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that

$$\mathcal{M} = \mathbb{C}\varphi \oplus z\theta H^2(\mathbb{D}),$$

where

$$\varphi = \Big(1 + \frac{b_0}{a_0}|\theta(0)|^2 z\Big)\theta - \frac{\theta(0)}{a_0}\Big((a_0 - 1) + b_0 z\Big).$$

Moreover, if $\mathcal{M}$ is as above, then $\mathcal{M} = [\varphi]_{S_1}$.

Proof. In view of Theorem 4.2.2, we only have to prove the necessary part. Suppose $\mathcal{M}$ is a S_1-invariant closed subspace of $H^2(\mathbb{D})$. Again, by Theorem 4.2.2, there exists inner function $\theta \in H^\infty(\mathbb{D})$ such that $\mathcal{M} = \mathbb{C}\varphi \oplus z\theta H^2(\mathbb{D})$, where $S_1\varphi = zp\theta$ and

$$\varphi = q + p\theta \tag{4.5.3}$$

for some polynomials $p, q \in \mathbb{C}[z]$. Since $S_1\varphi = zp\theta$, we have $zp\theta = (M_z + F)\varphi$. Then (4.5.2) implies

$$zp\theta = (M_z + F)\varphi = z\varphi + \varphi(0)((a_0 - 1) + b_0 z)z,$$

that is, $p\theta = \varphi + \varphi(0)((a_0 - 1) + b_0 z)$. Therefore,

$$\varphi = p\theta - \varphi(0)((a_0 - 1) + b_0 z), \tag{4.5.4}$$

and by (4.5.3), it follows that $q = -\varphi(0)((a_0 - 1) + b_0 z)$. Now, if $m \geq 1$, then $\varphi \perp z^m\theta H^2(\mathbb{D})$ implies that $\langle \varphi, z^m\theta\rangle = 0$, and hence (4.5.4) yields

$$\langle p, z^m\rangle = \langle p\theta, z^m\theta\rangle = \varphi(0)\langle (a_0 - 1) + b_0 z, z^m\theta\rangle.$$

Since $\varphi(0) = \frac{p(0)\theta(0)}{a_0}$, by (4.5.4) again, it follows that

$$\langle p, z^m\rangle = \begin{cases} b_0\frac{p(0)|\theta(0)|^2}{a_0} & \text{if } m = 1\\ 0 & \text{if } m > 1.\end{cases}$$

Thus, we have

$$p = p(0)(1 + \frac{b_0}{a_0}|\theta(0)|^2 z),$$

which implies that (by recalling (4.5.4))

$$\begin{aligned}\varphi &= p\theta - \varphi(0)((a_0 - 1) + b_0 z)\\ &= p\theta - \frac{p(0)\theta(0)}{a_0}((a_0 - 1) + b_0 z)\\ &= p(0)\Big[(1 + \frac{b_0}{a_0}|\theta(0)|^2 z)\theta - \frac{\theta(0)}{a_0}((a_0 - 1) + b_0 z)\Big].\end{aligned}$$

Finally, since $\varphi \neq 0$, without loss of generality, we may assume that $p(0) = 1$. This completes the proof of the first part. We also have

$$p = 1 + \frac{b_0}{a_0}|\theta(0)|^2 z.$$

Since $0 < |b_0| \leq |a_0|$ and θ is inner, it follows that p is an outer polynomial. The remaining part of the statement is now a particular case of the following theorem. $\square$

In the level of S_1-invariant subspaces, we have the following general classification:

Theorem 4.5.2. *Let $\mathcal{M} \subseteq H^2(\mathbb{D})$ be a nonzero closed S_1-invariant subspace. Then*

$$\mathcal{M} = [\mathcal{M} \ominus S_1\mathcal{M}]_{S_1},$$

if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ and an outer polynomial $p \in \mathbb{C}[z]$ such that $\mathcal{M} \ominus S_1\mathcal{M} = \mathbb{C}\varphi$ and $S_1\varphi = zp\theta$.

Proof. Let $\mathcal{M} = \mathbb{C}\varphi \oplus z\theta H^2(\mathbb{D})$, where $\theta \in H^\infty(\mathbb{D})$ is an inner function, $\varphi = p\theta - q$, and $S_1\varphi = zp\theta$ for some $p, q \in \mathbb{C}[z]$ (see Theorem 4.2.2). Note that

$$\mathcal{M} \ominus S_1\mathcal{M} = \mathbb{C}\varphi.$$

Since $S_1\varphi = zp\theta$, by (4.1.2) we have

$$S_1^m\varphi = S_1^{m-1}(zp\theta) = M_z^{m-1}(zp\theta) = z^m p\theta,$$

for all $m \geq 2$. Therefore

$$S_1^m\varphi = z^m p\theta \qquad (m \geq 1). \tag{4.5.5}$$

Now suppose that $\mathcal{M} = [\varphi]_{S_1}$. The above equality then tells us that $[S_1\varphi]_{S_1} \subseteq z\theta H^2(\mathbb{D})$. Since $\varphi \perp z\theta H^2(\mathbb{D})$, we have

$$\mathcal{M} = [\varphi]_{S_1} = \mathbb{C}\varphi \oplus z\theta H^2(\mathbb{D}) = \mathbb{C}\varphi \oplus [S_1\varphi]_{S_1}.$$

Clearly, we have $[S_1\varphi]_{S_1} = z\theta H^2(\mathbb{D})$, where on the other hand

$$[S_1\varphi] = [zp\theta]_{M_z} = z\theta[p]_{M_z},$$

and hence $z\theta[p]_{M_z} = z\theta H^2(\mathbb{D})$. But since $z\theta$ is an inner function, we have $[p]_{M_z} = H^2(\mathbb{D})$, that is, p is an outer polynomial. In the converse direction, since p is outer, (4.5.5) implies that

$$z\theta H^2(\mathbb{D}) = z\theta[p]_{M_z} = [S_1\varphi]_{M_z} = [S_1\varphi]_{S_1}.$$

Therefore

$$\mathcal{M} = \mathbb{C}\varphi \oplus z\theta H^2(\mathbb{D}) = \mathbb{C}\varphi \oplus [S_1\varphi]_{S_1} = [\varphi]_{S_1},$$

which completes the proof of the theorem. $\square$

In the setting of Theorem 4.5.1, we now consider the particular case when $a_0 = b_0 = 1$. In this case

$$Fz^m = \begin{cases} z^2 & \text{if } m = 0 \\ 0 & \text{if } m \geq 1. \end{cases}$$

Then, by Theorem 4.5.1, we have:

Corollary 4.5.3. *Let $F1 = z^2$ and $Fz^m = 0$ for all $m \geq 1$. Suppose $\mathcal{M}$ is a nonzero closed subspace of $H^2(\mathbb{D})$. Then $\mathcal{M}$ is invariant under $S_1 = M_z + F$ if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that $\mathcal{M} = \mathbb{C}\varphi \oplus z\theta H^2(\mathbb{D})$, where*

$$\varphi = (1 + |\theta(0)|^2 z)\theta - \theta(0)z.$$

Moreover, if $\mathcal{M}$ is as above, then $\mathcal{M} = [\varphi]_{S_1}$.

Moreover, in the setting of Theorem 4.5.1, for $\mathcal{M} = \mathbb{C}\varphi \oplus z\theta H^2(\mathbb{D})$, we have the following curious observations:

1. $\mathcal{M}$ is of finite codimension if and only if θ is a finite Blaschke product (this is also true for general n-shift invariant subspaces in the setting of Theorem 4.2.2).
2. φ need not be an inner function. Indeed, in the setting of Corollary 4.5.3, consider the Blaschke factor $\theta(z) = \frac{\frac{1}{2}-z}{1-\frac{1}{2}z}$, and set $\varphi = (1 + |\theta(0)|^2 z)\theta - \theta(0)z$. Then $\varphi(z) = \frac{1}{2}\frac{1-\frac{11}{4}z}{1-\frac{1}{2}z}$ is a rational function with $z = 2$ as the only pole. Note that $\varphi(1) = -\frac{7}{4}$ and $\varphi(-1) = \frac{5}{4}$. Clearly, φ is not an inner function.
3. If $\theta(0) = 0$, then $\mathcal{M} = [\theta]_{M_z} = [\theta]_{S_1}$. Therefore, $S_1|_{\mathcal{M}}$ is an unilateral shift of multiplicity one. On the other hand, if $\tilde{\theta}$ is an inner function with $\tilde{\theta}(0) \neq 0$, then $S_1|_{\mathcal{M}}$ and $S_1|_{\tilde{\mathcal{M}}}$ are not unitarily equivalent, where $\tilde{\mathcal{M}} = \mathbb{C}\tilde{\varphi} \oplus z\tilde{\theta}H^2(\mathbb{D})$ and $\tilde{\varphi} = (1 + \frac{b_0}{a_0}|\tilde{\theta}(0)|^2 z)\tilde{\theta} - \frac{\tilde{\theta}(0)}{a_0}\Big((a_0 - 1) + b_0 z\Big)$.

The final observation is in sharp contrast with a well-known consequence of the Beurling theorem: If $\mathcal{M}_1$ and $\mathcal{M}_2$ are nonzero closed M_z-invariant subspaces of $H^2(\mathbb{D})$, then $M_z|_{\mathcal{M}_1}$ and $M_z|_{\mathcal{M}_2}$ are unitarily equivalent. In view of (3) above, this property fails to hold for invariant subspaces of n-shifts.

We still continue with the setting of Corollary 4.5.3, and examine Theorem 4.3.5 in the case of the commutators of S_1. In fact, we have the following observation: Let $X \in \mathcal{B}(H^2(\mathbb{D}))$. Then $X \in \{S_1\}'$ if and only if there exists $\varphi \in H^\infty(\mathbb{D})$ such that $X = T_\varphi + N$, where

$$Nz^m = \begin{cases} z(\varphi - \varphi(0)) & \text{if } m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, in this case, $f_0(z) = 1 + z$ and $f_m(z) = z^m$ for all $m \geq 1$. Let $X \in \mathcal{B}(H^2(\mathbb{D}))$, and let $XS_1 = S_1X$. Set $\tilde{X} = U^*XU$. Then, $\tilde{X} \in \mathcal{B}(\mathcal{H}_k) \cap \{M_z\}'$, and, as in the proof of Theorem 4.3.5, there exist $\varphi \in H^\infty(\mathbb{D})$ such that $\tilde{X} = M_\varphi$. Moreover, if $\varphi = \sum_{m=0}^\infty \alpha_m z^m$, then

$$M_\varphi f_0 = \alpha_0 f_0 + \alpha_1 f_1 + \sum_{j=2}^\infty (\alpha_j + \alpha_{j-1}) f_j,$$

and

$$M_\varphi f_m = \sum_{j=0}^\infty \alpha_j f_{m+j} \qquad (m \geq 1),$$

which implies that

$$[M_\varphi] = \begin{bmatrix} \alpha_0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & \alpha_0 & 0 & 0 & \ddots \\ \alpha_2 + \alpha_1 & \alpha_1 & \alpha_0 & 0 & \ddots \\ \alpha_3 + \alpha_2 & \alpha_2 & \alpha_1 & \alpha_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Therefore, $[M_\varphi] = [\tilde{T}_\varphi] + [N]$, where

$$[\tilde{T}_\varphi] = \begin{bmatrix} \alpha_0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & \alpha_0 & 0 & 0 & \ddots \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ddots \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \text{ and } [N] = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots \\ \alpha_1 & 0 & 0 & 0 & \ddots \\ \alpha_2 & 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

By the proof of Theorem 4.3.5, $X = U\tilde{X}U^* = T_\varphi + N$. Clearly, $N1 = \sum_{j=1}^\infty \alpha_j z^{j+1} = z(\varphi - \varphi(0))$, and $Nz^m = 0$ for all $m \geq 1$, which ends the proof of the claim.

In connection with Theorem 4.5.1, we now point out the other natural (but easier) example of 1-shift $S_1 = M_z + F$, where

$$Fz^m = \begin{cases} z & \text{if } m = 0 \\ 0 & \text{if } m \geq 1. \end{cases}$$

In this case, S_1 is a weighed shift with the weight sequence $\{2, 1, 1, \ldots\}$. Therefore, S_1 is similar to the unilateral shift M_z on $H^2(\mathbb{D})$ via an explicit similarity map. Using this, it is rather easy to deduce, by pulling back inner functions corresponding to M_z-invariant subspaces of $H^2(\mathbb{D})$, that S_1-invariant subspaces are cyclic and of the form $\mathbb{C}\varphi \oplus z\theta H^2(\mathbb{D})$, with $\theta \in H^\infty(\mathbb{D})$ inner and (after an appropriate scaling)

$$\varphi = \theta - \frac{1}{2}\theta(0).$$

We refer to [45] for the theory of invariant subspaces of weighted shifts.

Finally, as far as the results of this present chapter are concerned, n-shifts are more realistic shifts among shifts that are finite rank perturbations of the unilateral shift. However, a pressing question remains about the classification of invariant subspaces of general shifts that are finite rank perturbations of the unilateral shift.

Chapter 5

Tridiagonal shifts as compact + isometry

The starting point of our present chapter is the following classification of compact perturbations of isometries [33, page 191]:

Theorem 5.0.1 (Fillmore, Stampfli, and Williams)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then T = compact + isometry if and only if $I - T^*T$ is compact and T is semi-Fredholm with ind$(T) \leq 0$.*

In this chapter, we are interested in a quantitative version of the above theorem. For instance, consider a bounded sequence of non-zero scalars $\{w_n\}_{n\geq 0}$ and an infinite-dimensional Hilbert space $\mathcal{H}$ with an orthonormal basis $\{e_n\}_{n\geq 0}$. Then the *weighted shift* S_w defined by

$$S_w(e_n) = w_n e_{n+1} \qquad (n \geq 0),$$

is in $\mathcal{B}(\mathcal{H})$ with $\|S_w\| = \sup_n |w_n|$. We assume that the weight sequence $\{w_n\}$ is bounded away from zero. Since $\ker S_w = \{0\}$ and $\ker S_w^* = \{e_0\}$, it follows that S_w is semi-Fredholm and $\operatorname{ind}(S_w) = -1$. Moreover, using the fact that $S_w^* e_0 = 0$ and $S_w^* e_n = \bar{w}_{n-1} e_{n-1}$, $n \geq 1$, it follows that

$$I - S_w^* S_w = \operatorname{diag}(1 - |w_0|^2, 1 - |w_1|^2, \ldots).$$

Theorem 5.0.1 then readily implies that

$$\lim_{n\to\infty} |w_n| = 1 \text{ if and only if } S_w = \text{compact} + \text{isometry}. \tag{5.0.1}$$

In this case, since the weight sequence is bounded away from zero, S_w is necessarily left-invertible.

Also note that S_w is a concrete example of a left-invertible shift on an analytic Hilbert space.

A standard computation now reveals that S_w, under some appropriate assumption on the weight sequence $\{w_n\}_{n\geq 0}$ [54, proposition 7], is unitarily equivalent to M_z on a diagonal space. Therefore (5.0.1) yields a quantitative classification of shifts on diagonal spaces that are compact perturbations of isometries. This motivates the following natural question:

QUESTION 3. *Is it possible to find a quantitative classification of left-invertible shifts on analytic Hilbert spaces that are compact perturbations of isometries?*

The main purpose of this chapter is to provide an answer to the above question for the case of M_z on (tractable) tridiagonal spaces. Throughout the chapter, we fix sequences of scalars $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ with the assumption that $a_n \neq 0$, $n \geq 0$. We set

$$f_n(z) = (a_n + b_n z)z^n \qquad (n \geq 0),$$

and consider the Hilbert space $\mathcal{H}_k$ with $\{f_n\}_{n\geq 0}$ as an orthonormal basis. Then $\mathcal{H}_k$ is a tridiagonal space corresponding to the tridiagonal kernel

$$k(z, w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} \qquad (z, w \in \mathbb{D}).$$

We always assume that $\{|\frac{a_n}{a_{n+1}}|\}_{n\geq 0}$ is bounded away from zero and

$$\sup_{n\geq 0} |\frac{a_n}{a_{n+1}}| < \infty \text{ and } \limsup_{n\geq 0} |\frac{b_n}{a_{n+1}}| < 1.$$

The latter two assumptions ensure that M_z on $\mathcal{H}_k$ is bounded [3, Theorem 5], whereas the first assumption implies that M_z is left-invertible [24, Theorem 3.5]. In this case we also call M_z a *tridiagonal shift.*

The notion of tridiagonal shifts was introduced by Adams and McGuire [3]. A part of their motivation came from factorizations of positive operators on analytic Hilbert spaces [4] (also see [47]). Evidently, if $b_n = 0$, then k is a diagonal kernel and M_z is a weighted shift on $\mathcal{H}_k$. Therefore, in view of shifts on analytic Hilbert spaces, tridiagonal shifts are the "next best" concrete examples of shifts after weighted shifts. The following is the answer to Question 3 for tridiagonal shifts:

Theorem 5.0.2 (Main result)**.** *Let M_z be the tridiagonal shift on $\mathcal{H}_k$. Then M_z = compact + isometry if and only if $|\frac{a_n}{a_{n+1}}| \to 1$ and $|\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}| \to 0$.*

In Section 5.2, we present the proof of the above theorem. In Section 5.1 we prove a key proposition that says that if $T \in \mathcal{B}(\mathcal{H})$ is a left-invertible operator and if T is of finite index, then T = compact + isometry if and only if $L_T - T^*$ is compact, where $L_T = (T^*T)^{-1}T^*$. Section 5.3 concludes the chapter with some general remarks and additional observations.

5.1 Preparatory results

The aim of this section is to prove a key result of this chapter. We begin with some elementary properties of left-invertible operators. See [56] for more on this theme. Let $T \in \mathcal{B}(\mathcal{H})$ be a left-invertible operator. We use the fact that T^*T is invertible to see that

$$L_T = (T^*T)^{-1}T^*,$$

is a left inverse of T. Note that $(TL_T)^2 = TL_T = (TL_T)^*)$, that is, TL_T is an orthogonal projection. Moreover, if $T^*f = 0$ for some $f \in \mathcal{H}$, then $(I - TL_T)f = f$. On the other hand, if $(I - TL_T)f = f$ for some $f \in \mathcal{H}$, then $TL_Tf = 0$ and hence $T^*TL_Tf = 0$, which implies that $T^*f = 0$. Therefore, $I - TL_T$ is the orthogonal projection onto $\ker T^*$, that is

$$I - TL_T = P_{\ker T^*}.$$

Part of the following is a particular case of [33, Theorem 6.2]. However, part (3) appears to be new, which will be also a key to the proof of the main theorem of this chapter. For the sake of completeness, we present the argument with all details.

Proposition 5.1.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be left-invertible and of finite index. The following statements are equivalent:*

1. $T =$ *compact* $+$ *isometry.*
2. $I - T^*T$ *is compact.*
3. $L_T - T^*$ *is compact.*
4. $I - TT^*$ *is compact.*

Proof. Throughout the following, we will designate compact operators by letters such as K, K_1, K_2, etc.

(1) ⇒ (2): Suppose $T = S + K$ for some isometry S on $\mathcal{H}$. Then

$$T^*T = (S + K)^*(S + K) = S^*S + K_1 = I + K_1,$$

implies that $I - T^*T$ is compact.

(2) ⇒ (3): Since $I - TL_T = P_{\ker T^*}$ and $\dim \ker T^* < \infty$, we have $TL_T = I + K_1$. Now if $I - T^*T = K_2$, then $L_T - T^*TL_T = K_3$, and hence

$$K_3 = L_T - T^*TL_T = L_T - T^*(I + K_1) = L_T - T^* + K_4.$$

This gives us $L_T - T^* = K$.

To prove (3) $\Rightarrow$ (4), assume that $L_T - T^* = K$. Then $TL_T - TT^* = K_1$. Again, since $I - TL_T = P_{\ker T^*}$ and $\dim \ker T^* < \infty$, we have

$$I - TT^* = (I - TL_T) + (TL_T - TT^*) = P_{\ker T^*} + K_1 = K_2.$$

(4) $\Rightarrow$ (2): Let $K = I - TT^*$. Then $T^*K = T^* - T^*TT^* = (I - T^*T)T^*$ implies that $T(I - T^*T) = K_1$, and hence

$$I - T^*T = L_T T(I - T^*T) = L_T K_1 = K_2.$$

(2) $\Rightarrow$ (1) Suppose $I - T^*T = K$. Since $|T|$ is positive, we see that $(I + |T|)$ is invertible. Then $K = (I + |T|)(I - |T|)$ implies that $|T| = I + K_1$. Let $T = U|T|$ be the polar decomposition of T. Taking the injectivity property of T in account, we find that U is an isometry, which implies

$$T = U|T| = U(I + K_1) = U + K_2,$$

and completes the proof of the proposition. $\square$

Unlike the proof of [33], the above proof avoids employing the Calkin algebra method. Of course, as pointed out earlier, the result of [33] (modulo part (3)) holds without the left-invertibility assumption.

Now we turn to the tridiagonal shift M_z on $\mathcal{H}_k$, where

$$k(z, w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)} \qquad (z, w \in \mathbb{D}),$$

and $f_n(z) = (a_n + b_n z)z^n$, $a_n, b_n \in \mathbb{C}$, $n \geq 0$. Recall that $a_n \neq 0$ for all $n \geq 0$. Moreover, by assumption, $\{|\frac{a_n}{a_{n+1}}|\}_{n\geq 0}$ is bounded away from zero and $\sup_{n\geq 0} |\frac{a_n}{a_{n+1}}| < \infty$ and $\limsup_{n\geq 0} |\frac{b_n}{a_{n+1}}| < 1$, which ensures that M_z is bounded and left-invertible on $\mathcal{H}_k$. It will be convenient to work with the matrix representation of M_z with respect to the orthonormal basis $\{f_n\}_{n\geq 0}$. A standard computation reveals that [3, Section 3]

$$z^n = \frac{1}{a_n} \sum_{m=0}^{\infty} (-1)^m \Big(\frac{\prod_{j=0}^{m-1} b_{n+j}}{\prod_{j=0}^{m-1} a_{n+j+1}} \Big) f_{n+m} \qquad (n \geq 0),$$

where $\prod_{j=0}^{-1} x_{n+j} := 1$. A new round of computation then gives

$$M_z f_n = \Big(\frac{a_n}{a_{n+1}} \Big) f_{n+1} + c_n \sum_{m=0}^{\infty} (-1)^m \Big(\frac{\prod_{j=0}^{m-1} b_{n+2+j}}{\prod_{j=0}^{m-1} a_{n+3+j}} \Big) f_{n+2+m} \qquad (n \geq 0),$$

where

$$c_n = \frac{a_n}{a_{n+2}} \Big(\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \Big) \qquad (n \geq 0). \tag{5.1.1}$$

Therefore

$$[M_z] = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{a_0}{a_1} & 0 & 0 & 0 & \ddots \\ c_0 & \frac{a_1}{a_2} & 0 & 0 & \ddots \\ \frac{-c_0 b_2}{a_3} & c_1 & \frac{a_2}{a_3} & 0 & \ddots \\ \frac{c_0 b_2 b_3}{a_3 a_4} & \frac{-c_1 b_3}{a_4} & c_2 & \frac{a_3}{a_4} & \ddots \\ \frac{-c_0 b_2 b_3 b_4}{a_3 a_4 a_5} & \frac{c_1 b_3 b_4}{a_4 a_5} & \frac{-c_2 b_4}{a_5} & c_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \tag{5.1.2}$$

with respect to the orthonormal basis $\{f_n\}_{n\geq 0}$ [3, Page 729].

5.2 Proof of the main theorem

Now we are ready to prove the main result of this chapter. Throughout the proof, we will frequently use matrix representations of bounded linear operators on the tridiagonal space (as well as subspaces of) $\mathcal{H}_k$ as in (3.1.8).

Proof of Theorem 5.0.2. Since $\ker M_z^* = \mathbb{C}f_0$, we see that $\operatorname{ind}(M_z) = -1$. Using the left-invertibility of M_z applied to Proposition 5.1.1, we see that M_z = isometry + compact if and only if $L_{M_z} - M_z^*$ is compact. By (3.1.9), the matrix representation of M_z^* is given by

$$[M_z^*] = \begin{bmatrix} 0 & \frac{\bar{a}_0}{\bar{a}_1} & \bar{c}_0 & \frac{-\bar{c}_0 \bar{b}_2}{\bar{a}_3} & \frac{\bar{c}_0 \bar{b}_2 \bar{b}_3}{\bar{a}_3 \bar{a}_4} & \cdots \\ 0 & 0 & \frac{\bar{a}_1}{\bar{a}_2} & \bar{c}_1 & \frac{-\bar{c}_1 \bar{b}_3}{\bar{a}_4} & \ddots \\ 0 & 0 & 0 & \frac{\bar{a}_2}{\bar{a}_3} & \bar{c}_2 & \ddots \\ 0 & 0 & 0 & 0 & \frac{\bar{a}_3}{\bar{a}_4} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{5.2.1}$$

Recall that $L_{M_z} = (M_z^* M_z)^{-1} M_z^*$ is a left-inverse of M_z. It follows that the matrix representation of L_{M_z} with respect to the orthonormal basis $\{f_n\}_{n\geq 0}$ [24, Theorem 3.5] is given by

$$[L_{M_z}] = \begin{bmatrix} 0 & \frac{a_1}{a_0} & 0 & 0 & 0 & \cdots \\ 0 & d_1 & \frac{a_2}{a_1} & 0 & 0 & \ddots \\ 0 & \frac{-d_1 b_1}{a_2} & d_2 & \frac{a_3}{a_2} & 0 & \ddots \\ 0 & \frac{d_1 b_1 b_2}{a_2 a_3} & \frac{-d_2 b_2}{a_3} & d_3 & \frac{a_4}{a_3} & \ddots \\ 0 & \frac{-d_1 b_1 b_2 b_3}{a_2 a_3 a_4} & \frac{d_2 b_2 b_3}{a_3 a_4} & \frac{-d_3 b_3}{a_4} & d_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

where $d_n = \frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}$ for all $n \geq 1$. Therefore, we have the following matrix representation of $L_{M_z} - M_z^*$:

$$[L_{M_z} - M_z^*] = \begin{bmatrix} 0 & (\frac{a_1}{a_0} - \frac{\bar{a}_0}{\bar{a}_1}) & -\bar{c}_0 & \frac{\bar{c}_0\bar{b}_2}{\bar{a}_3} & -\frac{\bar{c}_0\bar{b}_2\bar{b}_3}{\bar{a}_3\bar{a}_4} & \cdots \\ 0 & d_1 & (\frac{a_2}{a_1} - \frac{\bar{a}_1}{\bar{a}_2}) & -\bar{c}_1 & \frac{\bar{c}_1\bar{b}_3}{\bar{a}_4} & \ddots \\ 0 & \frac{-d_1b_1}{a_2} & d_2 & (\frac{a_3}{a_2} - \frac{\bar{a}_2}{\bar{a}_3}) & -\bar{c}_2 & \ddots \\ 0 & \frac{d_1b_1b_2}{a_2a_3} & \frac{-d_2b_2}{a_3} & d_3 & (\frac{a_4}{a_3} - \frac{\bar{a}_3}{\bar{a}_4}) & \ddots \\ 0 & \frac{-d_1b_1b_2b_3}{a_2a_3a_4} & \frac{d_2b_2b_3}{a_3a_4} & \frac{-d_3b_3}{a_4} & d_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Finally, by (5.1.1), we see that $c_n = \frac{a_n}{a_{n+2}}(\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}})$ for all $n \geq 0$, and hence

$$d_{n+1} = -\frac{a_{n+2}}{a_n} c_n \qquad (n \geq 0). \tag{5.2.2}$$

Now suppose that $L_{M_z} - M_z^*$ is compact. Since $\{f_n\}_{n\geq 0}$ is an orthonormal basis of $\mathcal{H}_k$, a well-known property of compact operators on Hilbert spaces implies that

$$\|(L_{M_z} - M_z^*)f_n\| \to 0 \text{ as } n \to \infty.$$

For each $n \geq 1$, use the matrix representation of $L_{M_z} - M_z^*$ to see that

$$\begin{aligned}\|(L_{M_z} - M_z^*)f_{n+2}\|^2 =& \left|\frac{c_0b_2b_3\cdots b_{n+1}}{a_3a_4a_5\cdots a_{n+2}}\right|^2 + \cdots + \left|\frac{c_{n-1}b_{n+1}}{a_{n+2}}\right|^2 + |c_n|^2 \\ &+ \left|\frac{a_{n+2}}{a_{n+1}} - \frac{\bar{a}_{n+1}}{\bar{a}_{n+2}}\right|^2 + |d_{n+2}|^2 + \cdots .\end{aligned}$$

In particular

$$\|(L_{M_z} - M_z^*)f_{n+2}\|^2 \geq |c_n|^2 + \left|\frac{a_{n+2}}{a_{n+1}} - \frac{\bar{a}_{n+1}}{\bar{a}_{n+2}}\right|^2 \qquad (n \geq 1),$$

and hence, $|c_n| \to 0$ and $|\frac{a_{n+2}}{a_{n+1}} - \frac{\bar{a}_{n+1}}{\bar{a}_{n+2}}| \to 0$ as $n \to \infty$. Then we have

$$\left|\left|\frac{a_{n+1}}{a_{n+2}}\right|^2 - 1\right| = \left|\frac{a_{n+1}}{a_{n+2}}\right|\left|\frac{\bar{a}_{n+1}}{\bar{a}_{n+2}} - \frac{a_{n+2}}{a_{n+1}}\right| \leq \left|\frac{\bar{a}_{n+1}}{\bar{a}_{n+2}} - \frac{a_{n+2}}{a_{n+1}}\right|\Big(\sup_m \left|\frac{a_m}{a_{m+1}}\right|\Big),$$

and hence $|\frac{a_n}{a_{n+1}}| \to 1$. Finally, $|c_n| \to 0$ (see the definition of c_n in (5.1.1)) and the fact that $\{\frac{a_n}{a_{n+2}}\}_{n\geq 0}$ is bounded imply that $|\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}| \to 0$.

For the converse direction, we assume that $|\frac{a_n}{a_{n+1}}| \to 1$ and $|\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}| \to 0$. Taken together, these conditions mean that $|c_n| \to 0$ (see (5.1.1)). We claim that $L_{M_z} - M_z^*$ is compact. To prove this, we first let $(\mathbb{C}f_0)^\perp = \mathcal{H}$. Then, with respect to

$$\mathcal{H}_k = \mathbb{C}f_0 \oplus \mathcal{H},$$

the operator $L_{M_z} - M_z^*$ can be represented as

$$L_{M_z} - M_z^* = \begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix},$$

where $A = P_{\mathbb{C}f_0}(L_{M_z} - M_z^*)|_{\mathcal{H}}$ and $B = P_{\mathcal{H}}(L_{M_z} - M_z^*)|_{\mathcal{H}}$. Thus we only have to worry about the compactness of B. To this end, we consider the matrix representation of B with respect to the orthonormal basis $\{f_n\}_{n\geq 1}$ as

$$[B] = \begin{bmatrix} d_1 & (\frac{a_2}{a_1} - \frac{\bar{a}_1}{\bar{a}_2}) & -\bar{c}_1 & \frac{\bar{c}_1\bar{b}_3}{\bar{a}_4} & \ddots \\ \frac{-d_1b_1}{a_2} & d_2 & (\frac{a_3}{a_2} - \frac{\bar{a}_2}{\bar{a}_3}) & -\bar{c}_2 & \ddots \\ \frac{d_1b_1b_2}{a_2a_3} & \frac{-d_2b_2}{a_3} & d_3 & (\frac{a_4}{a_3} - \frac{\bar{a}_3}{\bar{a}_4}) & \ddots \\ \frac{-d_1b_1b_2b_3}{a_2a_3a_4} & \frac{d_2b_2b_3}{a_3a_4} & \frac{-d_3b_3}{a_4} & d_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In view of the above matrix representation, we define linear operators B_1, B_2 and B_3 on $\mathcal{H}$, which admit the following matrix representations:

$$[B_1] = \operatorname{diag}\Big(\frac{a_2}{a_1} - \frac{\bar{a}_1}{\bar{a}_2}, \frac{a_3}{a_2} - \frac{\bar{a}_2}{\bar{a}_3}, \dots\Big),$$

and

$$[B_2] = \begin{bmatrix} -c_1 & 0 & 0 & 0 & \ddots \\ \frac{c_1b_3}{a_4} & -c_2 & 0 & 0 & \ddots \\ \frac{-c_1b_3b_4}{a_4a_5} & \frac{c_2b_4}{a_5} & -c_3 & 0 & \ddots \\ \frac{c_1b_3b_4b_5}{a_4a_5a_6} & \frac{-c_2b_4b_5}{a_5a_6} & \frac{c_3b_5}{a_6} & -c_4 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \text{ and } [B_3] = \begin{bmatrix} d_1 & 0 & 0 & 0 & \ddots \\ \frac{-d_1b_1}{a_2} & d_2 & 0 & 0 & \ddots \\ \frac{d_1b_1b_2}{a_2a_3} & \frac{-d_2b_2}{a_3} & d_3 & 0 & \ddots \\ \frac{-d_1b_1b_2b_3}{a_2a_3a_4} & \frac{d_2b_2b_3}{a_3a_4} & \frac{-d_3b_3}{a_4} & d_4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Assume for a moment that B_1, B_2 and B_3 are compact. Denote by U the unilateral shift on $\mathcal{H}$ corresponding to the orthonormal basis $\{f_n\}_{n\geq 1}$. In other words, $Uf_n = f_{n+1}$ for all $n \geq 1$. Then

$$B = B_1U^* + B_2^*U^{*2} + B_3.$$

Clearly, this would imply that B is compact. Therefore, it suffices to show that B_1, B_2 and B_3 are compact operators. Note that there exist $\epsilon > 0$ and $M > 0$ such that

$$\epsilon < \Big|\frac{a_n}{a_{n+1}}\Big| < M. \tag{5.2.3}$$

Then

$$\Big|\frac{a_{n+1}}{a_n} - \frac{\bar{a}_n}{\bar{a}_{n+1}}\Big| = \Big|\frac{a_{n+1}}{a_n}\Big(1 - \Big|\frac{a_n}{a_{n+1}}\Big|^2\Big)\Big| < \frac{1}{\epsilon}\Big|1 - \Big|\frac{a_n}{a_{n+1}}\Big|^2\Big|,$$

implies that the sequence $\{|\frac{a_{n+1}}{a_n} - \frac{\bar{a}_n}{\bar{a}_{n+1}}|\}_{n\geq 0}$ converges to zero, which proves that B_1 is compact.

We now prove that B_2 is compact. Since $\limsup |\frac{b_n}{a_{n+1}}| < 1$, there exist $r \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$\Big|\frac{b_n}{a_{n+1}}\Big| < r \qquad (n \geq n_0).$$

Write

$$\mathcal{H} = (\bigoplus_{p=1}^{n_0-1} f_p) \oplus (\bigoplus_{q=0}^{\infty} f_{n_0+q}),$$

and, with respect to this orthogonal decomposition, we let

$$B_2 = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}.$$

It is now enough to prove that A_2 acting on the infinite dimensional space $\oplus_{q=0}^{\infty} f_{n_0+q}$ is compact. Note

$$[A_2] = \begin{bmatrix} -c_{n_0} & 0 & 0 & \ddots \\ \frac{c_{n_0} b_{n_0+2}}{a_{n_0+3}} & -c_{n_0+1} & 0 & \ddots \\ \frac{-c_{n_0} b_{n_0+2} b_{n_0+3}}{a_{n_0+3} a_{n_0+4}} & \frac{c_{n_0+1} b_{n_0+3}}{a_{n_0+4}} & -c_{n_0+2} & \ddots \\ \vdots & \vdots & \vdots & \ddots \\ (-1)^n \frac{c_{n_0} b_{n_0+2} \cdots b_{n_0+n}}{a_{n_0+3} \cdots a_{n_0+n+1}} & (-1)^{n-1} \frac{c_{n_0+1} b_{n_0+3} \cdots b_{n_0+n}}{a_{n_0+4} \cdots a_{n_0+n+1}} & (-1)^{n-2} \frac{c_{n_0+2} b_{n_0+4} \cdots b_{n_0+n}}{a_{n_0+5} \cdots a_{n_0+n+1}} & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Denote by W_{n_0} the bounded weighted shift on $\oplus_{q=0}^{\infty} f_{n_0+q}$ with weight sequence $\{\frac{b_{n_0+n}}{a_{n_0+n+1}}\}_{n \geq 2}$, that is

$$[W_{n_0}] = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \frac{b_{n_0+2}}{a_{n_0+3}} & 0 & 0 & 0 & \ddots \\ 0 & \frac{b_{n_0+3}}{a_{n_0+4}} & 0 & 0 & \ddots \\ 0 & 0 & \frac{b_{n_0+4}}{a_{n_0+5}} & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

and write

$$D_{n_0} = \operatorname{diag}(-c_{n_0}, -c_{n_0+1}, -c_{n_0+2}, \cdots).$$

Suppose $M_0 := \sup_{n \geq 0} |c_n|$. Then

$$\|D_{n_0}\| = \sup_{n \geq n_0} |c_n| \leq \sup_{n \geq 0} |c_n| = M_0,$$

and, by the fact that $c_n \to 0$, it follows that D_{n_0} is a compact operator. Moreover, A_2 can be rewritten as

$$A_2 = D_{n_0} - W_{n_0} D_{n_0} + W_{n_0}^2 D_{n_0} + \cdots = \sum_{n=0}^{\infty} (-1)^n W_{n_0}^n D_{n_0}.$$

Clearly, $W_{n_0}^n D_{n_0}$ is compact for all $n \geq 0$, and, for $m \geq 2$, we have

$$\begin{aligned}\|W_{n_0}^m\| &\leq \sup_{l\geq 0}\Big|\frac{b_{n_0+2+l}b_{n_0+3+l}\cdots b_{n_0+m+l+1}}{a_{n_0+3+l}a_{n_0+4+l}\cdots a_{n_0+m+l+2}}\Big| \\ &\leq r^m.\end{aligned}$$

Finally, consider the sequence $\{S_n\}_{n\geq 1}$ of partial sums of compact operators, where $S_n = \sum_{m=0}^n (-1)^m W_{n_0}^m D_{n_0}$ for all $n \geq 1$. Then

$$\begin{aligned}\|A_2 - S_n\| &= \|(-1)^{n+1}W_{n_0}^{n+1}D_{n_0} + (-1)^{n+2}W_{n_0}^{n+2}D_{n_0} + (-1)^{n+3}W_{n_0}^{n+3}D_{n_0} + \cdots\| \\ &\leq M_0 \sum_{m=1}^{\infty} r^{n+m} \\ &= M_1 r^n,\end{aligned}$$

for some $M_1 > 0$ (as $0 < r < 1$), and hence A_2 is the norm limit of a sequence of compact operators. This completes the proof of the fact that B_2 is compact.

It remains to prove that B_3 is compact. First note that $d_{n+1} = -\frac{a_{n+2}}{a_n}c_n$ for all $n \geq 0$ (see (5.2.2)). The estimate (5.2.3) then implies that $c_n \to 0$ if and only if $d_n \to 0$. In particular, we may assume that $d_n \to 0$. We are now in a similar situation as in the proof of the compactness of B_2. The proof of the fact that B_3 is compact now follows similarly as in the case of B_2. $\square$

Remark 5.2.1. *Note that if the sequence $\{\frac{b_n}{a_n}\}_{n\geq 0}$ is convergent, then $|\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}| \to 0$. But the converse, evidently, is not true.*

Note that if $b_n = 0$ for all $n \geq 0$, then $\mathcal{H}_k$ is a diagonal space and M_z on $\mathcal{H}_k$ is a weighted shift. So in this case, Theorem 5.0.2 recovers the classification of (the reproducing kernel version of) weighted shifts as obtained earlier in (5.0.1). We refer the reader to [54] for the transition between weighted shifts and shifts on reproducing kernel Hilbert spaces.

5.3 Concluding remarks

Let us now return to the general question (cf. Question 3) of quantitative classification of left-invertible shifts that are compact perturbations of isometries. Clearly, the equivalence in (5.0.1) and Theorem 5.0.2 yields a complete answer to this question for the case of weighted shifts and tridiagonal shifts, respectively. In particular, if M_z is the Bergman shift, or the weighted Bergman shift, or the Dirichlet shift, then (5.0.1) implies that M_z = compact + isometry.

However, unlike the diagonal case, it is not yet completely clear to us how to directly relate the kernel k of the tridiagonal space $\mathcal{H}_k$ to the conclusion of Theorem 5.0.2. In other words, our answer to Question 3 for the tridiagonal case does not seem to indicate a comprehensive understanding (if any) of the general question.

To conclude this chapter, we offer a general (but still abstract) classification of shifts that are compact perturbations of isometries. The proof is essentially a variant of Proposition 5.1.1.

Proposition 5.3.1. *Let $\mathcal{H}_k$ be an analytic Hilbert space. Suppose the shift M_z on $\mathcal{H}_k$ is left-invertible and of finite index. Define C on $\mathcal{H}_k$ by*

$$(Cf)(w) = \langle f, (1 - z\bar{w})k(\cdot, w)\rangle_{\mathcal{H}_k} \qquad (f \in \mathcal{H}_k, w \in \mathbb{D}).$$

Then $M_z =$ compact $+$ isometry if and only if C defines a compact operator on $\mathcal{H}_k$.

Proof. Since M_z is left-invertible, the index of M_z is negative. We know that $M_z =$ isometry $+$ compact if and only if $I - M_z M_z^*$ is compact (Proposition 5.1.1). A standard (and well known) computation shows that

$$M_z^* k(\cdot, w) = \bar{w} k(\cdot, w) \qquad (w \in \mathbb{D}).$$

Then

$$(I - M_z M_z^*)k(\cdot, w) = (1 - z\bar{w})k(\cdot, w) \qquad (w \in \mathbb{D}).$$

For each $f \in \mathcal{H}_k$ and $w \in \mathbb{D}$, we have $((I - M_z M_z^*)f)(w) = \langle (I - M_z M_z^*)f, k(\cdot, w)\rangle_{\mathcal{H}_k}$, and hence

$$((I - M_z M_z^*)f)(w) = \langle f, (I - M_z M_z^*)k(\cdot, w)\rangle_{\mathcal{H}_k} = \langle f, (1 - z\bar{w})k(\cdot, w)\rangle_{\mathcal{H}_k},$$

which implies that $(I - M_z M_z^*)f = Cf$. This completes the proof. $\square$

On one hand, the above proposition is an effective tool for weighted shifts (the easy case, cf. (5.0.1)). For example, if k is a diagonal kernel and

$$k(z, w) = \frac{1}{1 - z\bar{w}} \tilde{k}(z, w) \qquad (z, w \in \mathbb{D}),$$

for some diagonal kernel $\tilde{k}$, then Proposition 5.3.1 provides a definite criterion for answering Question 3. This is exactly the case with the Bergman and the weighted Bergman kernels. On the other hand, a quick inspection reveals that the (matrix) representation of $M_z M_z^*$ for a tridiagonal shift M_z is rather complicated and the above proposition is less effective in drawing the conclusion as we did in Theorem 5.0.2.

Finally, it is worth pointing out that often Berezin symbols play an important role in proving compactness of linear operators on analytic Hilbert spaces [46]. See [11, 59, 60] and also [19] for recent accounts on the theory Berezin symbols on analytic Hilbert spaces. However, in the present context, it is not clear what is the connection between Berezin symbols and compact perturbations of isometries.

Chapter 6

Left-invertibility of rank-one perturbations

Rank-one operators are the simplest as well as easy to spot among all bounded linear operators on Hilbert spaces. Indeed, for each pair of nonzero vectors f and g in a Hilbert space $\mathcal{H}$, one can associate a *rank-one* operator $f \otimes g \in \mathcal{B}(\mathcal{H})$ defined by

$$(f \otimes g)h = \langle h, g \rangle f \qquad (h \in \mathcal{H}).$$

These are the only operators whose range spaces are one-dimensional. Here $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. Note that finite-rank operators, that is, linear sums of rank-one operators are norm dense in the ideal of compact operators, where one of the most important and natural examples of a noncompact operator is an isometry: A linear operator V on $\mathcal{H}$ is an *isometry* if $\|Vh\| = \|h\|$ for all $h \in \mathcal{H}$, or equivalently

$$V^*V = I_{\mathcal{H}}.$$

Along this line, left-invertible operators (also known as, by a slight abuse of terminology, "operators close to an isometry" [56] also natural examples of noncompact operators: $T \in \mathcal{B}(\mathcal{H})$ is *left-invertible* if T is bounded below, that is, there exists $\epsilon > 0$ such that $\|Th\| \geq \epsilon\|h\|$ for all $h \in \mathcal{H}$, or equivalently, there exists $S \in \mathcal{B}(\mathcal{H})$ such that

$$ST = I_{\mathcal{H}}.$$

The intent of this chapter is to make a modest contribution to the delicate structure of rank-one perturbations of bounded linear operators [41]. More specifically, this chapter aims to introduce some methods for the left-invertibility of rank-one perturbations of isometries and, to some extent, diagonal operators. The following is the central question that interests us:

QUESTION 4. *Find necessary and sufficient conditions for left-invertibility of the rank-one perturbation $V + f \otimes g$, where $V \in \mathcal{B}(\mathcal{H})$ is an isometry or a diagonal operator and f and g are vectors in $\mathcal{H}$.*

The answer to this question is completely known for isometries. Given an isometry $V \in \mathcal{B}(\mathcal{H})$ and vectors $f, g \in \mathcal{H}$, the perturbation $X = V + f \otimes g$ is an isometry if and only if there exist a unit vector $h \in \mathcal{H}$ and a scalar α of modulus one such that $f = (\alpha - 1)h$ and $g = V^*h$. In other words, a rank-one perturbation X of the isometry V is an isometry if and only if there exists a unit vector $f \in \mathcal{H}$ and a scalar α of modulus one such that

$$X = V + (\alpha - 1) f \otimes V^* f. \tag{6.0.1}$$

This result is due to Nakamura [44, 43] (and also see [53]). For more on rank-one perturbations of isometries and related studies, we refer the reader to [13, 22, 21, 34] and also [39].

In this chapter, we extend the above idea to a more general setting of left-invertibility of rank-one perturbations of isometries. In this case, however, left-invertibility of rank-one perturbations of isometries completely relies on certain real numbers. More specifically, given an isometry $V \in \mathcal{B}(\mathcal{H})$ and a pair of vectors f and g in $\mathcal{H}$, we associate a real number $c(V; f, g)$ defined by

$$c(V; f, g) = (\|f\|^2 - \|V^* f\|^2)\|g\|^2 + |1 + \langle V^* f, g\rangle|^2. \tag{6.0.2}$$

This is the number which precisely determine the left-invertibility of $V + f \otimes g$:

Theorem 6.0.1. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry, and let f and g be vectors in $\mathcal{H}$. Then $V + f \otimes g$ is left-invertible if and only if*

$$c(V; f, g) \neq 0.$$

Note that since V is an isometry, we have $\|V^* f\| \leq \|f\|$, and hence, the quantity $c(V; f, g)$ is always nonnegative. Therefore, the condition $c(V; f, g) \neq 0$ in the above theorem can be rephrased as saying that $c(V; f, g) > 0$, or equivalently, $\|V^* f\| < \|f\|$ or $1 + \langle V^* f, g\rangle \neq 0$. However, in what follows, we will keep the constant $c(V; f, g)$ in our consideration. Not only $c(V; f, g)$ plays a direct role in the proof of the above theorem but, as we will see in Remark 6.1.1, this quantity also appears in the explicit representation of a left inverse of a left-invertible perturbation.

The following conclusion is now easy:

Corollary 6.0.2. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry, and let f and g be vectors in $\mathcal{H}$. Then $V + f \otimes g$ is not left-invertible if and only if*

$$\|V^* f\| = \|f\| \textit{ and } \langle V^* f, g\rangle = -1.$$

The above theorem also provides us with a rich source of natural examples of left-invertible operators. For instance, let us denote by $\mathbb{D}$ the open unit disc in $\mathbb{C}$. Consider the shift M_z on the $\mathcal{E}$-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ over $\mathbb{D}$, where $\mathcal{E}$ is a Hilbert space. Then for any

$$\eta \in \ker M_z^* = \mathcal{E} \subseteq H^2_{\mathcal{E}}(\mathbb{D}),$$

and nonzero vector $g \in H^2_{\mathcal{E}}(\mathbb{D})$, the rank-one perturbation $M_z + \eta \otimes g$ is left-invertible. A similar conclusion holds if $f, g \in H^2(\mathbb{D})$ and

$$\langle M_z^* f, g \rangle \neq -1.$$

Section 6.1 contains the proof of the above theorem. In Section 6.2, we discuss a follow-up question: Characterizations of shifts that are rank-one perturbations of isometries. Here a shift refers to the multiplication operator M_z on some Hilbert space of analytic functions (that is, a reproducing kernel Hilbert space) on a domain in $\mathbb{C}$. Note, however, that our analysis will be mostly limited to the level of elementary examples.

In Section 6.3, we study rank-one perturbations of diagonal operators. It is well known that the structure of rank-one perturbations of diagonal operators is also complicated (cf. [6, 32, 39]). Moreover, comparison between perturbations of diagonal operators and that of isometries is perhaps inevitable if one views diagonals as normal operators and isometries as one of the best tractable non-normal operators. Here we consider $D + f \otimes g$ on some Hilbert space $\mathcal{H}$, where D is a diagonal operator with nonzero diagonal entries with respect to an orthonormal basis $\{e_n\}_{n=0}^\infty$ of $\mathcal{H}$. We also assume that the Fourier coefficients of f and g with respect to $\{e_n\}_{n=0}^\infty$ are nonzero. In Theorem 6.3.6, we prove:

Theorem 6.0.3. *$D + f \otimes g$ is left-invertible if and only if $D + f \otimes g$ is invertible.*

In Section 6.4, we observe that the parameterized spaces considered in the work of Davidson, Paulsen, Raghupathi and Singh [25] is connected to rank-one perturbations of isometries. In the final section, Section 6.5, we compute $c(V; f, g)$ when $V + f \otimes g$ is an isometry and make some further comments on rank-one perturbations of diagonal operators.

Finally, we remark that the last two decades have witnessed more intense interest in the theory of left-invertible operators starting from the work of Shimorin [56]. For instance, see [48] and references therein. For a more recent account of Shimorin's approach in the context of analytic model theory, invariant subspaces, and wandering subspaces in several variables, we refer the reader to Eschmeier [27] (also see [16] as part of the motivation), Eschmeier and Langendörfer [28], and Eschmeier and Toth [30]. Also see the monograph by Eschmeier and Putinar [29] for the general framework and motivation.

6.1 Proof of Theorem 6.0.1

In this section, we present the proof of the left-invertibility criterion of rank-one perturbations of isometries. First note that by expanding the right-hand side of (6.0.2), we have

$$c(V; f, g) = 1 + \|f\|^2\|g\|^2 + 2\mathrm{Re}\langle V^*f, g\rangle + |\langle V^*f, g\rangle|^2 - \|V^*f\|^2\|g\|^2. \tag{6.1.1}$$

Next we make a list of the most commonly used rank-one operator arithmetic, which will be used several times in what follows. Let $f, g \in \mathcal{H}$ and let $T \in \mathcal{B}(\mathcal{H})$. The following holds true:

1. $(f \otimes g)^* = g \otimes f$.
2. $\alpha(f \otimes g) = (\alpha f) \otimes g = f \otimes (\bar{\alpha} g)$ for all $\alpha \in \mathbb{C}$.
3. $(f \otimes g)(f_1 \otimes g_1) = \langle f_1, g\rangle f \otimes g_1$ for all $f_1, g_1 \in \mathcal{H}$.
4. $T(f \otimes g) = (Tf) \otimes g$ and so $(f \otimes g)T = f \otimes (T^*g)$.
5. $\|f \otimes g\| = \|f\|\|g\|$.

Of course, part (2) is a particular case of part (4). We also note that $T \in \mathcal{B}(\mathcal{H})$ is left-invertible if and only if T^*T is invertible. Indeed, if T is left-invertible, then T^*T is an injective positive operator. Since T is bounded below, T^*T is also bounded below and hence of closed range. Therefore, T^*T is invertible. Conversely, suppose X is the inverse of T^*T. Then $(XT^*)T = I$ implies that T is left-invertible.

We are now ready for the proof of the theorem.

Proof of Theorem 6.0.1. The statement trivially holds for $f = 0$ or $g = 0$. So assume that both f and g are nonzero vectors. Suppose that $V + f \otimes g$ on $\mathcal{H}$ is left-invertible. Then $(V + f \otimes g)^*(V + f \otimes g)$ is invertible with the inverse, say L. We have

$$I = L(V + f \otimes g)^*(V + f \otimes g) = L(V^* + g \otimes f)(V + f \otimes g).$$

Since $V^*V = I$, it follows that

$$\begin{aligned} I &= L(V^* + g \otimes f)(V + f \otimes g) \\ &= L(I + V^*f \otimes g + g \otimes V^*f + \|f\|^2 g \otimes g) \\ &= L + (LV^*f) \otimes g + Lg \otimes V^*f + \|f\|^2 Lg \otimes g. \end{aligned}$$

In particular, evaluating both sides on the vector V^*f and g, respectively, we get

$$\begin{aligned} V^*f &= LV^*f + \langle V^*f, g\rangle LV^*f + \|V^*f\|^2 Lg + \|f\|^2\langle V^*f, g\rangle Lg \\ &= (\langle V^*f, g\rangle + 1)LV^*f + (\|V^*f\|^2 + \|f\|^2\langle V^*f, g\rangle)Lg, \end{aligned}$$

and

$$\begin{aligned} g &= Lg + \|g\|^2 LV^*f + \langle g, V^*f\rangle Lg + \|f\|^2\|g\|^2 Lg \\ &= \|g\|^2 LV^*f + (1 + \langle g, V^*f\rangle + \|f\|^2\|g\|^2)Lg \\ &= \|g\|^2 LV^*f + \alpha Lg, \end{aligned}$$

where $\alpha = 1 + \langle g, V^*f\rangle + \|f\|^2\|g\|^2$. The latter equality implies that

$$LV^*f = \frac{1}{\|g\|^2}(I - \alpha L)g.$$

Now plug the value for LV^*f into the expression for V^*f above to get

$$V^*f = \frac{1}{\|g\|^2}(1 + \langle V^*f, g\rangle)(I - \alpha L)g + (\|V^*f\|^2 + \|f\|^2\langle V^*f, g\rangle)Lg$$

A little rearrangement then shows that

$$V^*f = \frac{1}{\|g\|^2}\Big(1+\langle V^*f, g\rangle\Big)g + \Big(\|V^*f\|^2 + \|f\|^2\langle V^*f, g\rangle - \frac{\alpha}{\|g\|^2}(1+\langle V^*f, g\rangle)\Big)Lg. \quad (6.1.2)$$

We compute

$$\begin{aligned} \alpha(1 + \langle V^*f, g\rangle) &= (1 + \langle V^*f, g\rangle)(1 + \langle g, V^*f\rangle + \|f\|^2\|g\|^2) \\ &= \langle V^*f, g\rangle\|f\|^2\|g\|^2 + 2\mathrm{Re}\langle V^*f, g\rangle + |\langle V^*f, g\rangle|^2 + \|f\|^2\|g\|^2 + 1 \\ &= \langle V^*f, g\rangle\|f\|^2\|g\|^2 + \|V^*f\|^2\|g\|^2 + c(V; f, g), \end{aligned}$$

where the last equality follows from the definition of $c(V; f, g)$ as in (6.1.1). Now we simplify the coefficient of Lg, say a, in the right-hand side of (6.1.2) as follows:

$$\begin{aligned} a &= \|V^*f\|^2 + \|f\|^2\langle V^*f, g\rangle - \frac{1}{\|g\|^2}\Big(\langle V^*f, g\rangle\|f\|^2\|g\|^2 + \|V^*f\|^2\|g\|^2 + c(V; f, g)\Big) \\ &= -\frac{1}{\|g\|^2}c(V; f, g). \end{aligned}$$

Consequently, by (6.1.2), we have

$$V^*f = \frac{1}{\|g\|^2}(1 + \langle V^*f, g\rangle)g - c(V; f, g)\frac{1}{\|g\|^2}Lg.$$

Suppose if possible that $c(V; f, g) = 0$. Then $V^*f = \frac{1}{\|g\|^2}(1 + \langle V^*f, g\rangle)g$, and so

$$\langle V^*f, g\rangle = \frac{1}{\|g\|^2}\langle(1 + \langle V^*f, g\rangle)g, g\rangle = 1 + \langle V^*f, g\rangle,$$

which is absurd. This contradiction proves that $c(V; f, g) \neq 0$.

Conversely, suppose that $c := c(V; f, g) \neq 0$. Set $R = (1 + \langle g, V^* f\rangle)V^* f \otimes g$, and let

$$X = I + \frac{1}{c}\{\|g\|^2 V^* f \otimes V^* f + (\|V^* f\|^2 - \|f\|^2) g \otimes g - (R + R^*)\}. \tag{6.1.3}$$

We claim that $X(V + f \otimes g)^*$ is a left inverse of $V + f \otimes g$, that is

$$X(V + f \otimes g)^*(V + f \otimes g) = I.$$

Indeed, the left hand side of the above simplifies to

$$\begin{aligned}
X(V + f \otimes g)^*(V + f \otimes g) &= X(V^* + g \otimes f)(V + f \otimes g)\\
&= X(I + g \otimes V^* f + V^* f \otimes g + \|f\|^2 g \otimes g)\\
&= \Big(I + \frac{1}{c}\{\|g\|^2 V^* f \otimes V^* f + (\|V^* f\|^2 - \|f\|^2) g \otimes g\\
&\qquad - (R + R^*)\}\Big)\Big(I + g \otimes V^* f + V^* f \otimes g + \|f\|^2 g \otimes g\Big),
\end{aligned}$$

and hence, there exists scalars a_1, a_2, a_3, and a_4 such that

$$X(V + f \otimes g)^*(V + f \otimes g) = I + a_1 g \otimes g + a_2 V^* f \otimes g + a_3 g \otimes V^* f + a_4 V^* f \otimes V^* f.$$

It is now enough to show that $a_1 = a_2 = a_3 = a_4 = 0$. Before getting to the proof of this claim, let us observe that

$$R + R^* = (1 + \bar{\beta}) V^* f \otimes g + (1 + \beta) g \otimes V^* f,$$

where $\beta := \langle V^* f, g\rangle$. Now we prove that $a_1 = 0$:

$$\begin{aligned}
a_1 &= \text{coefficient of } g \otimes g\\
&= \|f\|^2 + \frac{1}{c}\Big\{ - (1 + \beta)(\|V^* f\|^2 + \bar{\beta}\|f\|^2) + (\|V^* f\|^2 - \|f\|^2)\Big((1 + \beta) + \|f\|^2\|g\|^2\Big)\Big\}\\
&= \|f\|^2 + \frac{1}{c}\Big\{ - \bar{\beta}(1 + \beta)\|f\|^2 + \|V^* f\|^2\|f\|^2\|g\|^2 - (1 + \beta)\|f\|^2 - \|f\|^4\|g\|^2\Big\}\\
&= \|f\|^2 + \frac{\|f\|^2}{c}\Big\{ - \bar{\beta}(1 + \beta) + \|V^* f\|^2\|g\|^2 - (1 + \beta) - \|f\|^2\|g\|^2\Big\}\\
&= \|f\|^2 + \frac{\|f\|^2}{c}(-c)\\
&= 0,
\end{aligned}$$

where the last but one equality follows from (6.1.1). Next we compute a_2:

$$\begin{aligned}
a_2 &= \text{coefficient of } V^* f \otimes g\\
&= 1 + \frac{1}{c}\Big\{\|g\|^2(\|V^* f\|^2 + \bar{\beta}\|f\|^2) - (1 + \bar{\beta})\Big((1 + \beta) + \|f\|^2\|g\|^2\Big)\Big\}\\
&= 1 + \frac{1}{c}\Big\{\|V^* f\|^2\|g\|^2 - |1 + \beta|^2 - \|f\|^2\|g\|^2\Big\}\\
&= 0,
\end{aligned}$$

as $\beta = \langle V^* f, g\rangle$. We turn now to compute a_3:

$$\begin{aligned}
a_3 &= \text{coefficient of } g \otimes V^* f \\
&= 1 + \frac{1}{c}\Big\{ - (1+\beta)\bar{\beta} - (1+\beta) + (\|V^* f\|^2 - \|f\|^2)\|g\|^2 \Big\} \\
&= 1 + \frac{1}{c}\Big\{ - |1+\beta|^2 + (\|V^* f\|^2 - \|f\|^2)\|g\|^2 \Big\} \\
&= 0,
\end{aligned}$$

and, finally

$$a_4 = \text{coefficient of } V^* f \otimes V^* f = -\frac{1}{c}\Big\{ \|g\|^2(1+\bar{\beta}) - (1+\bar{\beta})\|g\|^2 \Big\} = 0.$$

This completes the proof of the fact that $V + f \otimes g$ is left-invertible. $\square$

Remark 6.1.1. *From the definition of X in (6.1.3), it is clear that if $V + f\otimes g$ is left-invertible for some isometry $V \in \mathcal{B}(\mathcal{H})$ and vectors f and g in $\mathcal{H}$, then*

$$L = \Big(I + \frac{1}{c}\{\|g\|^2 V^* f \otimes V^* f + (\|V^* f\|^2 - \|f\|^2) g \otimes g - (R + R^*)\}\Big)\Big(V + f \otimes g\Big)^*,$$

is a left-inverse of $V + f \otimes g$, where $c = c(V; f, g)$ and $R = (1 + \langle g, V^* f\rangle) V^* f \otimes g$.

It is worthwhile to observe that for an isometry $V \in \mathcal{B}(\mathcal{H})$ and a vector $f \in \mathcal{H}$, we have $\|V^* f\| = \|f\|$ if and only if $f \in \operatorname{ran} V$. In particular, Theorem 6.0.1 yields the following:

Corollary 6.1.1. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry and let f and g are nonzero vectors in $\mathcal{H}$. If $f \notin \operatorname{ran} V$, then $V + f \otimes g$ is left-invertible.*

6.2 Analytic operators

Recall that an isometry $V \in \mathcal{B}(\mathcal{H})$ is called a *pure isometry* if $\bigcap_{n=0}^{\infty} V^n \mathcal{H} = \{0\}$. As we will see soon, this is also known as the *analytic property* of V. It is known that an isometry $V \in \mathcal{B}(\mathcal{H})$ is pure if and only if V is unitarily equivalent to M_z on the $\mathcal{W}$-valued Hardy space $H^2_{\mathcal{W}}(\mathbb{D})$, where $\mathcal{W} = \ker V^*$ is the *wandering subspace* corresponding to V. Here M_z denotes the multiplication operator by the coordinate function z on $H^2_{\mathcal{W}}(\mathbb{D})$ (see (6.2.1) below). Rank-one perturbations of isometries (or pure isometries) that are pure isometries form a rich class of operators and are fairly complex in nature [43]. The methods involve heavy machinery of $H^\infty(\mathbb{D})$-function theory, which is mostly unavailable for general function spaces (see [13, 34, 39, 53]). In this section we discuss some examples of rank-one perturbations of isometries that are shift or simply analytic.

We begin with a brief introduction to shift operators on reproducing kernel Hilbert spaces. Let $\mathcal{E}$ be a Hilbert space and Ω be a domain in $\mathbb{C}$. Let $\mathcal{H}$ be a Hilbert space of

$\mathcal{E}$-valued analytic functions on Ω. Suppose the *evaluation map*

$$ev_w(f) = f(w) \qquad (f \in \mathcal{H}),$$

defines a bounded linear operator $ev_w : \mathcal{H} \to \mathcal{E}$ for all $w \in \Omega$. Then the *kernel* function $k : \Omega \times \Omega \to \mathcal{B}(\mathcal{E})$ defined by $k(z, w) = ev_z \circ ev_w^*$, $z, w \in \Omega$, is *positive definite*, that is,

$$\sum_{i,j=1}^{n} \langle k(z_i, z_j)\eta_j, \eta_i\rangle_{\mathcal{E}} \geq 0,$$

for all $\{z_i\}_{i=1}^n \subseteq \Omega$, $\{\eta_i\}_{i=1}^n \subseteq \mathcal{E}$ and $n \geq 1$. Moreover, k is analytic in the first variable and satisfies the *reproducing property*

$$\langle ev_w(f), \eta\rangle_{\mathcal{E}} = \langle f(w), \eta\rangle_{\mathcal{E}} = \langle f, k(\cdot, w)\eta\rangle_{\mathcal{H}},$$

for all $f \in \mathcal{H}$, $w \in \Omega$ and $\eta \in \mathcal{E}$. We denote the space $\mathcal{H}$ by $\mathcal{H}_k$ and call it *analytic Hilbert space*. The *shift operator* M_z on $\mathcal{H}_k$ is defined by

$$(M_z f)(w) = wf(w) \qquad (f \in \mathcal{H}_k, w \in \Omega). \tag{6.2.1}$$

We always assume that M_z is a bounded linear operator on $\mathcal{H}_k$ (equivalently, $z\mathcal{H}_k \subseteq \mathcal{H}_k$). It is easy to see that if M_z is a shift on some $\mathcal{H}_k$, then

$$\bigcap_{n=0}^{\infty} M_z^n \mathcal{H}_k = \bigcap_{n=0}^{\infty} z^n \mathcal{H}_k = \{0\}.$$

This is the property which bridges the gap between left-invertible operators and left-invertible shifts. More precisely, following the ideas of Shimorin [56], a bounded linear operator T on $\mathcal{H}$ is called *analytic* if

$$\bigcap_{n=0}^{\infty} T^n \mathcal{H} = \{0\}.$$

If $T \in \mathcal{B}(\mathcal{H})$ is a left-invertible analytic operator, then there exists an analytic Hilbert space $\mathcal{H}_k$ such that T and the shift M_z on $\mathcal{H}_k$ are unitarily equivalent [56]. Therefore, up to unitary equivalence, analytic left-invertible operators are nothing but left-invertible shifts.

The following proposition collects some examples of analytic and shift operators.

Proposition 6.2.1. *Let $V \in \mathcal{B}(\mathcal{H})$ be a pure isometry, $m, n \in \mathbb{Z}_+$, and let $f_0 \in \ker V^*$. If $S = V + V^m f_0 \otimes V^n f_0$, then the following holds:*

1. *S is analytic whenever $m > n$.*
2. *S is a shift whenever $m > n + 1$.*

Proof. For simplicity, for each $t \in \mathbb{Z}$, we set

$$f_t = \begin{cases} V^t f_0 & \text{if } t \geq 0 \\ V^{*-t} f_0 & \text{if } t < 0. \end{cases}$$

Since $f_0 \in \ker V^*$, it follows that $f_t = 0$ for all $t < 0$. Suppose $m > n$. Observe that $\langle f_m, f_n \rangle = \langle V^m f_0, V^n f_0 \rangle = 0$, and hence

$$\begin{aligned} S^2 &= V^2 + f_{m+1} \otimes f_n + f_m \otimes f_{n-1} + \langle f_m, f_n \rangle f_m \otimes f_n \\ &= V^2 + f_m \otimes f_{n-1} + f_{m+1} \otimes f_n. \end{aligned}$$

Then, by induction, we have

$$S^{k+1} = V^{k+1} + f_m \otimes f_{n-k} + f_{m+1} \otimes f_{n-k+1} + \cdots + f_{m+k-1} \otimes f_{n-1} + f_{m+k} \otimes f_n,$$

that is

$$S^{k+1} = V^{k+1} + \sum_{j=0}^{k} f_{m+j} \otimes f_{n-k+j}, \tag{6.2.2}$$

for all $k \geq 1$. In particular, if $k = n + j$ and $j \geq 1$, then it follows that

$$S^{n+j+1} = V^{n+j+1} + f_m \otimes f_{-j} + f_{m+1} \otimes f_{-j+1} + \cdots + f_{m+n+j-1} \otimes f_{n-1} + f_{m+n+j} \otimes f_n.$$

At this point, we note that $f_{-p} = 0$ for all $p > 0$, and hence

$$\begin{aligned} S^{n+j+1} &= V^{n+j+1} + f_{m+j} \otimes f_0 + f_{m+j+1} \otimes f_1 + \cdots + f_{m+j+n-1} \otimes f_{n-1} + f_{m+n+j} \otimes f_n \\ &= V^{n+j+1}(I + \sum_{i=0}^{n} f_{m-n-1+i} \otimes f_i), \end{aligned}$$

as $m > n$. This implies that $S^{n+j+1}\mathcal{H} \subseteq V^{n+j+1}\mathcal{H}$, $j \geq 1$. From here we see that

$$\bigcap_{r \geq 0} S^r \mathcal{H} \subseteq \bigcap_{r \geq n+1} S^r \mathcal{H} \subseteq \bigcap_{r \geq n+1} V^r \mathcal{H} = \{0\},$$

where the last equality follows from the fact that V is pure. To prove (2), we compute the value of $c(V; f, g)$ with $f = V^m f_0$ and $g = V^n f_0$:

$$\begin{aligned} c(V; f, g) &= (\|f\|^2 - \|V^* f\|^2)\|g\|^2 + |1 + \langle V^* f, g \rangle|^2 \\ &= (\|V^m f_0\|^2 - \|V^{m-1} f_0\|^2)\|V^n f_0\|^2 + |1 + \langle V^{m-1} f_0, V^n f_0 \rangle|^2 \\ &= 0 \times \|f_0\|^2 + |1 + 0| \\ &= 1, \end{aligned}$$

where the last but one equality follows because $m - n - 1 > 0$ implies $\langle V^{*n} V^{m-1} f_0, f_0 \rangle = 0$. The first part and Theorem 6.0.1 then completes the proof of part (2). $\square$

The above observation is fairly elementary. The general classification of rank-one perturbations of isometries (or pure isometries) that are shift on some reproducing kernel Hilbert space is an open problem. However, see [44, Theorem 1] and [43] in the context of classifications of rank-one perturbations of isometries that are pure isometry.

The following is also a simple class of examples of analytic operators.

Proposition 6.2.2. *Let $V \in \mathcal{B}(\mathcal{H})$ be a pure isometry, f and g be vectors in $\mathcal{H}$, and suppose $V^*g + \langle g, f\rangle g = 0$. Then $V + f \otimes g$ is analytic.*

Proof. If we set $S := V + f \otimes g$, then

$$S^2 = V^2 + Vf \otimes g + f \otimes (V^*g + \langle g, f\rangle g) = VS.$$

Therefore, $S^{n+1} = V^n S$, $n \geq 1$, can be proved analogously by induction. In particular

$$S^{n+1}\mathcal{H} = V^n S\mathcal{H} \subseteq V^n\mathcal{H} \qquad (n \geq 0),$$

and hence, by using the fact that V is a pure isometry, it follows that

$$\bigcap_{n=0}^{\infty} (V + f \otimes g)^{n+1}\mathcal{H} \subseteq \bigcap_{n=0}^{\infty} V^n\mathcal{H} = \{0\},$$

that is, $V + f \otimes g$ is analytic. □

Note that $V^*g + \langle g, f\rangle g = 0$ is equivalent to the condition that $g \in \ker(V + f \otimes g)^*$.

Recall that the scalar-valued Hardy space $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space corresponding to the Szegö kernel $\mathbb{S} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$, where

$$\mathbb{S}(z, w) = (1 - z\bar{w})^{-1} \qquad (z, w \in \mathbb{D}).$$

For each $w \in \mathbb{D}$, consider the analytic function $\mathbb{S}(\cdot, w) : \mathbb{D} \to \mathbb{C}$ defined by (the kernel function, see the discussion at the beginning of this section) $(\mathbb{S}(\cdot, w))(z) = \mathbb{S}(z, w)$, $z \in \mathbb{D}$.

Example 6.2.3. *The following examples illustrate some direct application of the above propositions.*

1. *Fix $w \in \mathbb{D}$, and set $g = \mathbb{S}(\cdot, w)$. We know that $M_z^*\mathbb{S}(\cdot, w) = \bar{w}\mathbb{S}(\cdot, w)$. Choose $f \in H^2(\mathbb{D})$ such that $\langle g, f\rangle_{H^2(\mathbb{D})} = -\bar{w}$ (for instance, $f = \frac{-1}{\bar{w}^{n-1}} z^n$ for some $n \geq 1$). Evidently $M_z^*g + \langle g, f\rangle g = 0$, and hence, $M_z + f \otimes \mathbb{S}(\cdot, w)$ is an analytic operator.*

2. *Consider $f = z$ and $g = 1$ in $H^2(\mathbb{D})$. Then $c(M_z; f, g) = 2 \neq 0$, and hence $M_z + f \otimes g$ is a shift.*

3. *Consider $f = z$ and $g = -1$ in $H^2(\mathbb{D})$. Then $c(M_z; f, g) = 0$, and hence $M_z + f \otimes g$ not left-invertible, but analytic by Proposition 6.2.1.*

Note that the rank-one perturbation $M_z + z^2 \otimes z$ is similar to M_z on $H^2(\mathbb{D})$. Here the similarity follows easily from the fact that $M_z + z^2 \otimes z$ is a weighted shift with the weight sequence $\{1, 2, 1, 1, \ldots\}$. This implies that $M_z + z^2 \otimes z$ is analytic, where on the one hand

$$M_z^* z + \langle z, z^2 \rangle z = 1 \neq 0.$$

Therefore, $M_z + z^2 \otimes z$ is an example of an analytic rank-one perturbation of M_z which does not satisfy the hypothesis of Proposition 6.2.2.

6.3 Diagonal operators

In this section, we examine rank-one perturbations of diagonal operators. We prove that all the interesting left-invertible rank-one perturbations of diagonal operators are invertible.

Throughout this section, we fix a Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of $\mathcal{H}$. We also fix vectors $f = \sum_{n=0}^{\infty} a_n e_n$ and $g = \sum_{n=0}^{\infty} b_n e_n$ in $\mathcal{H}$ and diagonal operator $D \in \mathcal{B}(\mathcal{H})$ with diagonal entries $\{\alpha_n\}_{n\geq 0}$. Also, we set

$$T = D + f \otimes g.$$

We will assume throughout this section that

$$\alpha_n, a_n, b_n \neq 0 \qquad (n \geq 0),$$

as this is the class of perturbations we all are most interested in (cf. [39]). Also we denote

$$r := 1 + \sum_{n=0}^{\infty} \frac{a_n \bar{b}_n}{\alpha_n}.$$

The following result is from Ionascu [39, Proposition 2.4]:

Proposition 6.3.1. *T admits zero as an eigenvalue if and only if $r = 0$ and $\{\frac{a_n}{\alpha_n}\}_{n\geq 0}$ is square summable.*

The key to our analysis lies in the following observation which is also a result of independent interest.

Proposition 6.3.2. *$\{e_n\}_{n\geq 0} \subseteq ranT$ if and only if $r \neq 0$ and $\{\frac{a_n}{\alpha_n}\}_{n\geq 0}$ is square summable.*

Proof. Assume that $e_j \in \operatorname{ran} T$ for some arbitrary but fixed integer $j \geq 0$. Then there exists $x = \sum_{n=0}^{\infty} c_n e_n \in \mathcal{H}$ such that $Tx = (D + f \otimes g)x = e_j$. Therefore

$$e_j = \sum_{n=0}^{\infty} (c_n \alpha_n) e_n + \langle x, g \rangle \sum_{n=0}^{\infty} a_n e_n. \tag{6.3.1}$$

Note that $\langle x, g\rangle \neq 0$. Indeed, if $\langle x, g\rangle = 0$, then

$$c_n = \begin{cases} \frac{1}{\alpha_j} & \text{if } n = j \\ 0 & \text{otherwise,} \end{cases}$$

and hence $x = \frac{1}{\alpha_j} e_j$. Since $g = \sum_{n=0}^{\infty} b_n e_n$, using $\langle x, g\rangle = 0$, we have $b_j = 0$. This contradiction shows, as promised, that $\langle x, g\rangle \neq 0$. Now equating the coefficients of terms on either side of (6.3.1), we have $c_j = \frac{1}{\alpha_j}(1 - a_j\langle x, g\rangle)$, and $c_n = 0$ for all $n \neq j$. In particular, $\{\frac{a_n}{\alpha_n}\}_{n\geq 0}$ is a square summable sequence, and, as $\langle x, g\rangle = \sum_{n=0}^{\infty} c_n \bar{b}_n$, we have

$$\langle x, g\rangle = -\langle x, g\rangle \sum_{n=0}^{\infty} \frac{a_n \bar{b}_n}{\alpha_n} + \frac{\bar{b}_j}{\alpha_j},$$

which implies

$$\langle x, g\rangle \Big(1 + \sum_{n=0}^{\infty} \frac{a_n \bar{b}_n}{\alpha_n}\Big) = \langle x, g\rangle r = \frac{\bar{b}_j}{\alpha_j},$$

and hence $r \neq 0$. For the converse direction, fix an integer $j \geq 0$. Then

$$y = -\frac{\bar{b}_j}{r\alpha_j}\Big(\sum_{n=0}^{\infty} \frac{a_n}{\alpha_n} e_n\Big) + \frac{1}{\alpha_j} e_j,$$

is a vector in $\mathcal{H}$. Note that

$$\langle y, g\rangle = -\frac{\bar{b}_j}{r\alpha_j}(r-1) + \frac{\bar{b}_j}{\alpha_j} = \frac{\bar{b}_j}{r\alpha_j}.$$

Using the representation $f = \sum_{n=0}^{\infty} a_n e_n$, we deduce from the above that

$$Ty = (D + f \otimes g)y = -\frac{\bar{b}_j}{r\alpha_j} \sum_{n=0}^{\infty} a_n e_n + e_j + \langle y, g\rangle f = e_j.$$

This implies that $e_j \in \operatorname{ran} T$ for all $j \geq 0$ and completes the proof of the proposition. □

We also need the following lemma:

Lemma 6.3.3. *If T is bounded below, then D is invertible.*

Proof. Assume by contradiction that $\{\alpha_{n_k}\}$ is a subsequence of the sequence $\{\alpha_n\}$, which converges to zero. Now

$$Te_{n_k} = (D + f \otimes g)e_{n_k} = \alpha_{n_k} e_{n_k} + \langle e_{n_k}, g\rangle f = \alpha_{n_k} e_{n_k} + b_{n_k} f,$$

implies

$$\|Te_{n_k}\| \leq |\alpha_{n_k}| + |b_{n_k}| \|f\|.$$

This shows that $\{Te_{n_k}\}$ converges to zero for the sequence of unit vectors $\{e_{n_k}\}$. But this contradicts the fact that T is bounded below. Therefore the sequence $\{\alpha_n\}$ has no subsequence that converges to zero. Consequently, there exists $M > 0$ such that

$$|\alpha_n| > M \qquad (n \geq 0),$$

and hence $\{\frac{1}{\alpha_n}\}$ is a bounded sequence. We conclude that D is invertible. □

The converse is not true: Pick $f, g \in \mathcal{H}$ such that $\langle f, g\rangle = -1$. By Proposition 6.3.1, $I + f \otimes g$ is not injective, and hence not left-invertible. However, as a weak converse we have:

Proposition 6.3.4. *If D is bounded below and T is injective, then T is left-invertible.*

Proof. Assume by contradiction that $T = D + f \otimes g$ is not bounded below. Then there is a sequence $\{h_n\} \subseteq \mathcal{H}$ with $\|h_n\| = 1$ such that $Th_n \to 0$. By the compactness of $f \otimes g$, there exists a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ such that $(f \otimes g)h_{n_k}$ converges. Then, $Dh_{n_k} = (T - f \otimes g)h_{n_k}$ converges. But since D is bounded below, this gives us $h_{n_k} \to \tilde{h}$ for some $\tilde{h} \in \mathcal{H}$. In particular, we have $\|\tilde{h}\| = 1$. On the other hand, since T is a bounded linear operator, we have

$$T\tilde{h} = \lim_{k\to\infty} Th_{n_k} = 0,$$

that is, $\tilde{h} \in \ker T$. But, $\ker T = \{0\}$ by our assumption, and hence $\tilde{h} = 0$, which contradicts the fact that $\|\tilde{h}\| = 1$. Therefore, T is bounded below. □

Although Proposition 6.3.4 is not directly related to the main result of this section, but perhaps fits appropriately with our present context. We come now to the main result on left-invertibility of rank-one perturbations. The following result and its proof are also along the same line and perhaps of independent interest.

Proposition 6.3.5. *If D has a closed range, then T also has a closed range.*

Proof. Let $\mathcal{N} = \ker T$, and let $\mathrm{ran} D$ is closed. Then $T|_{\mathcal{N}^\perp}$ is injective. Assume by contradiction that $\mathrm{ran} T$ is not closed. Then $X := T|_{\mathcal{N}^\perp}$ is not left-invertible. Proceeding exactly as in the proof of Proposition 6.3.4 (by replacing the role of T by X), we will find a contradiction. □

We come now to the main result on left-invertibility of rank-one perturbations.

Theorem 6.3.6. *$D + f \otimes g$ is left-invertible if and only if $D + f \otimes g$ is invertible.*

Proof. For the nontrivial direction, assume that $T = D + f \otimes g$ is left-invertible. Assume by contradiction that T is not invertible. Since, in particular, $\mathrm{ran} T$ is closed, $\{e_n\}_{n\geq 0} \nsubseteq \mathrm{ran} T$. Now by Proposition 6.3.1, either $r \neq 0$ or the sequence $\{\frac{a_n}{\alpha_n}\}_{n\geq 0}$ is not square summable. On the other hand, we know from Lemma 6.3.3 that D is invertible, and hence

$$D^{-1}f = \sum_{n=0}^{\infty} \frac{a_n}{\alpha_n} e_n \in \mathcal{H}.$$

This implies, of course, that $\{\frac{a_n}{\alpha_n}\}_{n\geq 0}$ is a square summable sequence, and hence $r \neq 0$. As a consequence, we can apply Proposition 6.3.2 to T: the basis vectors $\{e_n\}_{n\geq 0} \subseteq$ ran$T$; which is a contradiction. This proves that T is invertible. □

If we know that D is invertible (which anyway follows from Lemma 6.3.3) and $r \neq 0$, then the surjectivity of $T = D + f \otimes g$ in the above proof also can be obtained as follows: Observe that

$$1 + \langle D^{-1}f, g\rangle = 1 + \sum_{n=0}^{\infty} \frac{a_n \bar{b}_n}{\alpha_n} = r.$$

Then for each $y \in \mathcal{H}$, we consider $x = D^{-1}y - \frac{1}{r}\langle D^{-1}y, g\rangle D^{-1}f$. We deduce easily that $Tx = y$, which completes the proof of the fact that T is onto.

6.4 An example

Let T be a bounded linear operator on $H^2(\mathbb{D})$. Suppose

$$[T] = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ a_{01} & 0 & 0 & \ddots \\ a_{02} & a_{12} & 0 & \ddots \\ a_{03} & a_{13} & a_{23} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

the matrix representation of T with respect to the standard orthonormal basis $\{z^n, n \geq 0\}$ of $H^2(\mathbb{D})$. Clearly, $T(z^n) \subseteq z^{n+1}H^2(\mathbb{D})$, and hence

$$T^n(H^2(\mathbb{D})) \subseteq z^n H^2(\mathbb{D}) \qquad (n \geq 0).$$

It follows that

$$\bigcap_{n=0}^{\infty} T^n H^2(\mathbb{D}) \subseteq \bigcap_{n=0}^{\infty} z^n H^2(\mathbb{D}) = \{0\},$$

that is, T is analytic. In particular, for each α and β in $\mathbb{C}$, the matrix operator

$$[T_{\alpha,\beta}] = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \alpha & 0 & 0 & 0 & \ddots \\ \beta & 0 & 0 & 0 & \ddots \\ 0 & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

defines an analytic operator $T_{\alpha,\beta}$ on $H^2(\mathbb{D})$. Moreover, one can show that

$$T_{\alpha,\beta} = M_z^2 + (\alpha z + (\beta - 1)z^2) \otimes 1,$$

that is, $T_{\alpha,\beta}$ is a rank-one perturbation of the shift M_z^2 on $H^2(\mathbb{D})$. Next, we compute $c(T_{\alpha,\beta}; f, g)$, where $f = \alpha z + (\beta - 1)z^2$ and $g = 1$. Since $\langle M_z^{*2} f, g\rangle_{H^2(\mathbb{D})} = \beta - 1$, and $\|M_z^{*2} f\|^2 = |\beta - 1|^2$, and $\|f\|^2 = |\alpha|^2 + |\beta - 1|^2$, it follows that

$$c(T_{\alpha,\beta}, \alpha z + (\beta - 1)z^2, 1) = |\alpha|^2 + |\beta|^2.$$

Thus we have proved:

Proposition 6.4.1. *Let $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. If $f = \alpha z + (\beta - 1)z^2$ and $g = 1$, then:*

1. *$T_{\alpha,\beta}$ is a shift on $H^2(\mathbb{D})$,*
2. *$T_{\alpha,\beta} = M_z^2 + f \otimes g$ and*
3. *$c(M_z^2; f, g) = |\alpha|^2 + |\beta|^2$.*

We recall in passing that $T_{\alpha,\beta}$ is a shift means the existence of an analytic Hilbert space $\mathcal{H}_k$ and a unitary $U : H^2(\mathbb{D}) \to \mathcal{H}_k$ such that $T_{\alpha,\beta} = U^* M_z U$.

We continue with the matrix representation $[T_{\alpha,\beta}]$. Clearly, $T_{\alpha,\beta}$ is an isometry if and only if

$$|\alpha|^2 + |\beta|^2 = 1.$$

Denote by $H^2_{\alpha,\beta}(\mathbb{D})$ the closed codimension one subspace of $H^2(\mathbb{D})$ with orthonormal basis $\{\alpha + \beta z, z^2, z^3, \dots\}$. Clearly, $H^2_{\alpha,\beta}(\mathbb{D})$ is an invariant subspace of M_z^2. One can verify straightforwardly that the map $U : H^2(\mathbb{D}) \to H^2_{\alpha,\beta}(\mathbb{D})$ defined by

$$U z^n = \begin{cases} \alpha + \beta z & \text{if } n = 0 \\ z^{n+1} & \text{otherwise} \end{cases}$$

is a unitary operator and

$$U T_{\alpha,\beta} = M_z^2 U,$$

that is, $T_{\alpha,\beta}$ on $H^2(\mathbb{D})$ and $M_z^2|_{H^2_{\alpha,\beta}(\mathbb{D})}$ on $H^2_{\alpha,\beta}(\mathbb{D})$ are unitarily equivalent. The operator $M_z^2|_{H^2_{\alpha,\beta}(\mathbb{D})}$, $(\alpha,\beta) \in \mathbb{C}^2$ with $|\alpha|^2+|\beta|^2 = 1$, has been considered in [25] in the context of invariant subspaces and a constrained Nevanlinna-Pick interpolation problem. Clearly, in the context of perturbation theory, it is worth exploring and explaining the results of [25].

6.5 Concluding remarks

We begin by computing $c(V; f, g)$ for rank-one perturbations that are isometries. Suppose $V \in \mathcal{B}(\mathcal{H})$ is an isometry and f and g are vectors in $\mathcal{H}$. It is curious to observe that

$$c(V; f, g) = 1,$$

whenever $V + f \otimes g$ is an isometry. Indeed, in the present case, by (6.0.1), there exist a unit vector $h \in \mathcal{H}$ and a scalar α of modulus one such that $f = (\alpha - 1)h$ and $g = V^*h$. Then (6.1.1) yields

$$\begin{aligned} c(V; f, g) - 1 &= |\alpha - 1|^2 \|V^*h\|^2 + 2(\mathrm{Re}(\alpha - 1))\|V^*h\|^2 + |\alpha - 1|^2 \|V^*h\|^4 (1 - 1) \\ &= (|\alpha - 1|^2 + 2\mathrm{Re}(\alpha - 1))\|V^*h\|^2, \end{aligned}$$

and hence $c(V; f, g) - 1 = 0$ as $|\alpha| = 1$. This completes the proof of the claim.

It would be interesting to investigate the nonnegative number $c(V; f, g)$ in terms of analytic and geometric invariants, if any, of rank-one perturbations of isometries. This is perhaps a puzzling question for which we do not have any meaningful answer or guess at this moment.

We conclude this chapter by making some additional comments on (non-analytic features of) perturbations of diagonal operators. The following easy-to-prove proposition says that rank-one perturbations of common diagonal operators do not fit well with shifts on reproducing kernel Hilbert spaces.

Proposition 6.5.1. *Let $D \in \mathcal{B}(\mathcal{H})$ be a Fredholm diagonal operator, and let $f, g \in \mathcal{H}$. Then $D + f \otimes g$ cannot be represented as shift.*

Proof. Suppose that $D + f \otimes g$ is unitarily equivalent to M_z on some reproducing kernel Hilbert space. Since D is Fredholm, and M_z and $D + f \otimes g$ are unitarily equivalent, we have $ind(M_z) = ind(D) = 0$. On the other hand, since M_z is injective, it follows that

$$ind(M_z) = \dim\ker M_z - \dim\ker M_z^* < 0,$$

which is a contradiction. $\square$

In the context of Theorem 6.3.6, we remark that rank-one perturbations of diagonal operators need not be left-invertible: Consider a compact diagonal operator D (for instance, consider D with diagonal entries $\{\frac{1}{n}\}$). Then a rank-one perturbation of D is also compact, and hence the perturbed operator cannot be left-invertible.

In Lemma 6.3.3, we prove that if $D + f \otimes g$ is bounded below, then D is invertible. This was one of the key tools in proving Theorem 6.3.6: $D + f \otimes g$ is left-invertible if and only if $D + f \otimes g$ is invertible. Of course, we assumed that the Fourier coefficients of f and g are nonzero. Here, we would like to point out that rank-one perturbation of an invertible operator need not be invertible. In fact, the invertibility property of rank-one perturbations of invertible operators can be completely classified (see [39, Lemma 2.7]): Let D be an invertible diagonal operator. Then $D + f \otimes g$ is invertible if and only if

$$1 + \langle D^{-1} f, g \rangle \neq 0.$$

Finally, in the context of left-invertibility, consider $D = I_{\mathcal{H}}$ and choose f and g from $\mathcal{H}$ such that $\langle f, g \rangle = -1$. Then $c(D; f, g) = 0$, and hence, $D + f \otimes g$ is not left-invertible.

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List of Publications

1. S. Das and J. Sarkar, *Tridiagonal kernels and left-invertible operators with applications to Aluthge transforms*, 2020, arXiv:2009.03410 (Communicated).

2. S. Das and J. Sarkar, *Invariant subspaces of analytic perturbations*, 2021, arXiv:2107.05052 (Communicated).

3. S. Das and J. Sarkar, *Tridiagonal shifts as compact + isometry*, 2021, arXiv:2111.04180 (Communicated).

4. S. Das and J. Sarkar, *Left-invertibility of rank-one perturbations*, 2022, arXiv:2201.10535 (Communicated).