Weighted inequalities for maximal operators and the Hardy space H^1 on LCA groups

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Dedicated to my Parents and Brothers

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Contents

Chapter 1

Introduction

The purpose of this dissertation is two fold: to study weighted norm inequalities for maximal type operators such as Hardy–Littlewood maximal operator associated with a family of general sets in a topological space and Fourier maximal operator in the context of the ring of integers of a local field, and to extend the classical theory of Hardy space and related topics such as the space BMO and the John–Nirenberg space in the setting of Locally Compact Abelian (LCA) groups having a covering family.

1.1 Maximal Operators Associated with General Sets

Let us start with recalling the necessary background and notation. The Hardy–Littlewood maximal function Mf of a locally integrable function f on $\mathbb{R}^d$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^d, \tag{1.1.1}$$

where $B(x,r)$ denotes the Euclidean ball of radius r centred at x and $|E|$ is the Lebesgue measure of a measurable set E. The operator M was shown to be strong type (p,p) for $1 < p \leq \infty$ and weak type $(1,1)$ by Hardy and Littlewood [49] for $d = 1$ and by Wiener [115] for general d. More precisely, M satisfies the strong type (p,p) inequality

$$\int_{\mathbb{R}^d} Mf(x)^p\, dx \leq C_p \int_{\mathbb{R}^d} |f(x)|^p\, dx, \quad 1 < p \leq \infty, \tag{1.1.2}$$

and the weak type $(1,1)$ inequality

$$|\{x \in \mathbb{R}^d : Mf(x) > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1 \quad \text{for all } \lambda > 0. \tag{1.1.3}$$

We are interested in certain mixed weighted versions of (1.1.2) and (1.1.3). By a weight w, we mean a nonnegative, integrable function on the underlying measure space. One of the principal problems concerning weighted inequalities is to identify criteria for w guaranteeing that a given bounded operator T on the Lebesgue space $L^p(\mathbb{R}^d)$ is bounded on the weighted space $L^p(\mathbb{R}^d, w)$, that is, to characterize the nonnegative functions w for which the inequality

$$\|Tf\|_{L^p(\mathbb{R}^d,w)} \leq C\|f\|_{L^p(\mathbb{R}^d,w)} \tag{1.1.4}$$

holds for all $f \in L^p(\mathbb{R}^d, w)$, where C is a positive constant that depends only on p and w. For $1 < p < \infty$, $L^p(\mathbb{R}^d, w)$ denotes the space of p-integrable functions on $\mathbb{R}^d$ with respect to the measure $w(x)\,dx$. Extensive study of such inequalities for some of the important operators in harmonic analysis began in the early 1970s with the seminal work of Muckenhoupt. In his work, he characterized all weights w so that (1.1.4) holds for the Hardy–Littlewood maximal operator M. Since then such classes of weights are termed as Muckenhoupt A_p weights in the literature and are defined as follows.

A weight w is said to be in the A_p class, $1 < p < \infty$, if

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)\Big(\frac{1}{|Q|)}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\Big)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes.

The weight w belongs to the weight class A_1 if there exists a constant $C > 0$ such that

$$Mw(x) \leq Cw(x) \quad \text{for a.e. } x \in \mathbb{R}^d.$$

Muckenhoupt [86] proved the following result.

Theorem 1.1.1. *(a) Let $1 < p < \infty$. Then $w \in A_p$ if and only if there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^d} Mf(x)^p w(x)\, dx \leq C \int_{\mathbb{R}^d} |f(x)|^p w(x)\, dx \quad \text{for all } f \in L^p(\mathbb{R}^d, w). \tag{1.1.5}$$

(b) The weight $w \in A_1$ if and only if there exists a constant $C > 0$ such that

$$w(\{x \in \mathbb{R}^d : Mf(x) > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} |f(x)| w(x)\, dx \quad \text{for all } f \in L(\mathbb{R}^d, w). \tag{1.1.6}$$

The quantity $[w]_{A_p}$ is called the A_p *constant* or the A_p *characteristic* of the weight w. In many applications it is of interest to have optimal or at least good bounds for the operator norm $\|M\|_{L^p(\mathbb{R}^d,w)\to L^p(\mathbb{R}^d,w)}$ in terms of the size of the constant $[w]_{A_p}$. Since $[w]_{A_p} \geq 1$, the problem is to find estimates of the form

$$\|M\|_{L^p(\mathbb{R}^d,w)\to L^p(\mathbb{R}^d,w)} \leq C[w]_{A_p}^{\alpha(p)}$$

with $\alpha(p)$ as small as possible, where C is a constant depending only on p and the dimension d.

We remark that the results of Muckenhoupt [86] are qualitative, in the sense that they do not provide any information about $\alpha(p)$. Buckley [9] proved the first quantitative result on the boundedness of M by providing the best possible power dependence on the A_p constant. He proved the following result.

Theorem 1.1.2. *Let M be the Hardy–Littlewood maximal operator defined in (1.1.1) and $1 < p < \infty$. Then there is a constant $C > 0$ such that*

$$\|M\|_{L^p(\mathbb{R}^d,w)\to L^p(\mathbb{R}^d,w)} \leq C[w]_{A_p}^{\frac{1}{p-1}}. \tag{1.1.7}$$

This estimate is sharp in the sense that the exponent $\frac{1}{p-1}$ cannot be replaced by any smaller quantity and hence $\alpha(p) = \frac{1}{p-1}$.

In a more general setting of the spaces of homogeneous type, Hytönen, Pérez and Rela [62] showed that Buckley's theorem can be improved further in terms of various mixed characteristics of the underlying weight w. Recall that a space of homogeneous type is a quasi-metric space $\mathcal{S}$ with quasi-metric d such that the d-balls $B(x,r) = \{y \in \mathcal{S} : d(x,y) < r\}, x \in \mathcal{S}, r > 0$, are open sets, and μ is a regular measure defined on the σ-algebra containing the d-balls that satisfies the "doubling condition", i.e., there is a constant A such that the measure of a ball of radius $2r$ is at most A times the measure of the ball of radius r with the same centre. Let $\mathcal{B} = \{B(x,r) : x \in \mathcal{S}, r > 0\}$. Analogous to the Euclidean case, the uncentred Hardy–Littlewood maximal operator $\mathcal{M}$ on $\mathcal{S}$ is obtained by replacing the cubes by balls in (1.1.1). More precisely,

$$\mathcal{M}f(x) = \sup_{x\in B\in\mathcal{B}} \frac{1}{\mu(B)} \int_B |f(x)|\, d\mu(x), \quad x \in \mathcal{S}.$$

In a similar way, we also define the A_p constants of a weight w. Furthermore, we define the A_∞ constant of w by

$$[w]_{A_\infty} = \sup_{B\in\mathcal{B}} \frac{1}{w(B)} \int_B \mathcal{M}(w\chi_B)(x)\, d\mu(x).$$

where $w(B) = \int_B w(x)\, d\mu(x)$. We also denote the space of p-integrable functions on $\mathcal{S}$ with respect to the measure $w(x)\, d\mu(x)$ as $L^p(\mathcal{S}, w)$. With these notation, one of the main results in [62] may be stated as follows.

Theorem 1.1.3. *Let $\mathcal{M}$ be the Hardy–Littlewood maximal operator on $\mathcal{S}$ and let $1 < p < \infty$. Then there is a constant $C > 0$ such that*

$$\|\mathcal{M}\|_{L^p(\mathcal{S},w)\to L^p(\mathcal{S},w)} \le C([w]_{A_p}[\sigma]_{A_\infty})^{\frac{1}{p}}, \tag{1.1.8}$$

where $\sigma = w^{-\frac{1}{p-1}}$ is the dual weight of w.

The mixed bound in (1.1.8) is sharper than the estimate involving only the A_p constant in (1.1.7) and improves Buckley's theorem since $[\sigma]_{A_\infty} \le C[\sigma]_{A_{p'}} = C[w]_{A_p}^{\frac{1}{p-1}}$, which yields (1.1.7).

The essence of Hytönen, Pérez and Rela's result is that the mixed type bounds can be obtained for the maximal operator and the weights associated with a family $\mathcal{B}$ of balls generated by a quasi-metric d of a space of homogeneous type. However, there are many examples of important families of measurable sets arising in harmonic analysis and PDE which cannot be generated by a quasi-metric d, and hence are not in the scope of spaces of homogeneous type. Let us illustrate this situation by exhibiting some concrete examples. Some more examples are given in [28].

Example 1.1.4. *A family of convex sets in* $\mathbb{R}^d$ *was considered by Caffarelli and Gutiérrez in [12] as follows. Let* ϕ *be a convex smooth function on* $\mathbb{R}^d$*. For* $x \in \mathbb{R}^d$*, let* $\ell(y)$ *be a supporting hyperplane of* ϕ *at the point* $(x, \phi(x))$*. For* $r > 0$*, define the set* $S_\phi(x,r) = \{y \in \mathbb{R}^d : \phi(y) < l(y) + r\}$*. These sets are called sections and are obtained by projecting on* $\mathbb{R}^d$ *the points on the graph of* ϕ *that are below a supporting hyperplane lifted in* r*. Let* $\mu = \det D^2\phi$ *be the Monge–Ampère measure. The family* $\mathcal{F} = \{S_\phi(x,r) : x \in \mathbb{R}^d, r > 0\}$ *is related to the convex solution of Monge–Ampère equation.*

Example 1.1.5. *Let* $\gamma : [0,\infty) \to \mathbb{R}$ *be a convex function on* $[0,\infty)$ *and in* C^2 *on* $(0,\infty)$ *and satisfies* $\gamma(0) = \gamma'(0) = 0$*. For* $r > 0$*, set* $h(r) = r\gamma'(r) - \gamma(r)$*. Also suppose that there exists a constant* $C > 0$ *such that* $\frac{h(2r)}{h(r)} \leq C$ *for all* $r > 0$*. Moreover, let* $Q_0 = \{(x,y) \in \mathbb{R}^2 : |x|, |y| < 1\}$ *and for any* $r > 0$*, set* $P_r(0) = A_r Q_0$*, where*

$$A_r := \begin{pmatrix} r & 0 \\ \gamma(r) + h(r) & h(r) \end{pmatrix}.$$

Let $\mathcal{P} := \{P_r(x) = x + P_r(0) : x \in \mathbb{R}^2, r > 0\}$ *and* μ *be the Lebesgue measure. The family* $\mathcal{P}$ *arises in studying the* L^p *estimates of the maximal operator and Hilbert transform associated convex curve by Carbery et al. [15].*

Observe that we can define an analogue of the maximal operator and the A_p weights corresponding to the families $\mathcal{F}$ and $\mathcal{P}$ in the above examples. Thus, the following question arises naturally.

Let $\mathcal{F}$ *be a family of measurable sets which are not associated with any quasi-metric. Consider the maximal operator and the weights associated with the family* $\mathcal{F}$ *in the usual way. Can we extend Theorem 1.1.3 for such families?*

One of the objectives of this thesis is to give an affirmative answer to this question. We consider this question in a more general framework: given a very general basis $\mathbb{E}$ of open sets on a measure space X, we prove quantitative estimates in terms of different A_p characteristics for the maximal operator $M_{\mathbb{E}}$ defined with respect to this basis $\mathbb{E}$. For the precise statement of our result, see Theorem 2.2.2. Let us mention that our results will not only cover all these families, but also give a mixed type bound for Stein's maximal operator (see Theorem 2.1.1 of Chapter 2) and also present an improvement of Theorem 1.1.3.

Another fundamental generalization of the maximal inequality (1.1.3) is due to Fefferman and Stein [36]. In their pioneering work, they proved that the following two-weights inequality

$$w(\{x \in \mathbb{R}^d : Mf(x) > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} |f(x)| Mw(x)\, dx \tag{1.1.9}$$

holds for all nonnegative functions f and w.

In the literature, this inequality is popularly known as the endpoint Fefferman–Stein weighted inequality and is interesting for several reasons. The first of them is that it was a precursor of the weighted theory of Muckenhoupt and gives an improvement of the inequality (1.1.6). Note that if $w \in A_1$, then inequality (1.1.6) readily follows from (1.1.9). In [36], Fefferman and Stein exploited this inequality to derive the vector-valued analogue of maximal inequalities (1.1.2) and (1.1.3) and applied these to obtain certain estimatess for Marcinkiewicz integral. If $f = (f_1, f_2, \dots)$ is a sequence of functions on $\mathbb{R}^d$, $Mf = (Mf_1, Mf_2, \dots)$ and $\|f(x)\|_{\ell^r} = (\sum_{k=1}^{\infty} |f_k(x)|^r)^{\frac{1}{r}}$, $1 < r < \infty$, then their result is the following.

Theorem 1.1.6. *Let $1 < p < \infty$. If $1 < r < \infty$, then there exists a constant $C_{r,p} > 0$ such that*

$$\int_{\mathbb{R}^d} \|Mf(x)\|_{\ell^r}^p\, dx \leq C_{r,p} \int_{\mathbb{R}^d} \|f(x)\|_{\ell^r}^p\, dx.$$

Moreover, the following weak type $(1,1)$ estimate holds: there exists a constant $C > 0$ such that

$$|\{x \in \mathbb{R}^d : \|Mf(x)\|_{\ell^r} > \lambda\}| \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} \|f(x)\|_{\ell^r}\, dx \quad \textit{for all } \lambda > 0.$$

Note that if we put $f_1 = f, f_2 = f_3 = \cdots = 0$, then we obtain the inequalities (1.1.2) and (1.1.3).

This is a very deep theorem and has been generalized and used in many different contexts in modern harmonic analysis explaining the central role of the inequality (1.1.9). Our purpose here is to extend the endpoint Fefferman–Stein weighted inequality (1.1.9) and Theorem 1.1.6 for the maximal operator $M_{\mathbb{E}}$ associated with a family of general sets $\mathbb{E}$. This task has been addressed in Chapter 2.

At this point, we take the opportunity to mention some related works on endpoint Fefferman–Stein inequalities in a variety of contexts. In the setting of the spaces of homogeneous type, Aimar, Bernardis and Nowak [2] proved a dyadic version of (1.1.9) following the proof given in [36]. Luque and Parissis [76] derived similar inequalities for the strong maximal function on the Euclidean spaces. In a very recent work, Ombrosi, Rivera-Ríos and Safe [92] proved an analogue of (1.1.9) on the infinite rooted k-ary tree.

1.2 Fourier Partial Sum Operators on Local Fields

Another important operator in harmonic analysis is the Fourier maximal operator acting on integrable functions on the circle $\mathbb{T}$ defined by

$$\mathsf{M}f(x) = \sup_{n\in\mathbb{N}} |S_n f(x)|, \tag{1.2.1}$$

where $S_n f$ denotes the nth partial sum of the Fourier series of a function $f \in L^1(\mathbb{T})$. The operator M was shown to be of strong type (p,p) by Carleson [17] for $p = 2$ and by Hunt [57] for the case $1 < p < \infty$. More precisely, Carleson and Hunt proved that M satisfies the strong type (p,p) inequality

$$\int_{\mathbb{T}} \mathsf{M}f(x)^p \, dx \le C_p \int_{\mathbb{T}} |f(x)|^p \, dx, \quad 1 < p < \infty. \tag{1.2.2}$$

These inequalities are fundamental in harmonic analysis establishing the pointwise almost everywhere convergence of Fourier series of L^p functions. Hunt and Young [60] extended these inequalities for functions in the weighted spaces $L^p(\mathbb{T}, w)$ for the weights satisfying the Muckenhoupt A_p condition. For $1 < p < \infty$, $L^p(\mathbb{T}, w)$ denotes the space of p-integrable functions on $\mathbb{T}$ with respect to the measure $w(x)\,dx$.

Hunt and Young [60] proved the following weighted variant of the inequality (1.2.2).

Theorem 1.2.1. *Let $1 < p < \infty$ and $w \in A_p$. Then there exists a constant $C_p > 0$, depending only on w and p, such that*

$$\int_{\mathbb{T}} \mathsf{M}f(x)^p w(x)\,dx \leq C_p \int_{\mathbb{T}} |f(x)|^p w(x)\,dx \quad \textit{for all } f \in L^p(\mathbb{T}, w). \tag{1.2.3}$$

Another purpose of this thesis is to generalize the inequality (1.2.3) for the maximal operator of Fourier series in the context of the ring of integers of a locally compact, totally disconnected and non-discrete field. This problem is considered in Chapter 3.

We now turn our attention to applications of Muckenhoupt weights. In the study of Gabor theory, A_2 weights play an important role. To facilitate our discussion further we first recall some relevant concepts. Let $\psi \in L^2(\mathbb{R})$. The Gabor system associated with ψ is the collection $\{e^{2\pi ibnx}\psi(x - ak) : k, n \in \mathbb{Z}\}$ of simple time-frequency shifts, where $a, b > 0$ are fixed. The question of when a Gabor system forms an orthonormal basis, a frame or a Riesz basis for $L^2(\mathbb{R})$ has been a matter of great interest not only mathematically but for applications as well. In this context the Zak transform has proved to be an important tool. For $\psi \in L^2(\mathbb{R})$, the Zak transform $Z\psi$ is the function on $\mathbb{R} \times \mathbb{R}$ defined by

$$Z\psi(x, y) = \sum_{k \in \mathbb{Z}} \psi(x + k) \exp(2\pi iyk).$$

For instance, $\{e^{2\pi inx}\psi(x - k) : k, n \in \mathbb{Z}\}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$ precisely when $|Z\psi| = 1$ a.e. It is complete if and only if $Z\psi \neq 0$ a.e. In [50], Heil and Powell showed that the Gabor systems which form Schauder bases for $L^2(\mathbb{R})$ also admit a simple characterization in the Zak transform domain and manifested the importance of Muckenhoupt A_2 weights in time-frequency analysis. Recall that a Schauder basis in $L^2(\mathbb{R})$ is a system of functions $\{f_k\}$ such that for every $\psi \in L^2(\mathbb{R})$ there exists a unique sequence $\{\alpha_k\}$ of scalars with $\psi = \sum_k \alpha_k f_k$, where the series converges in the L^2-norm with respect to a fixed order.

Since Schauder basis expansions may converge conditionally, the order of summation is important. Heil and Powell constructed a suitable family Γ of enumerations of $\mathbb{Z} \times \mathbb{Z}$ and with respect to each of these enumerations, they proved that the Gabor system generated by ψ is a Schauder basis for $L^2(\mathbb{R})$ if and only if $|Z\psi|^2$ is an A_2 weight on

$\mathbb{T} \times \mathbb{T}$. Here v is an A_2 weight on $\mathbb{T} \times \mathbb{T}$ means that for a.e. $x, y \in \mathbb{T}$, the functions $v(x, .)$ and $v(., y)$ are A_2 weights on $\mathbb{T}$.

The notion of Zak transform admits a natural generalization to locally compact abelian groups (see [113]) and since the Gabor theory rests mainly on the structure of translations and modulations, it is natural to ask whether the above mentioned characterization of Gabor Schauder bases is also valid in the general context. Such a development is quite useful as it includes all other examples which are important for applications and it emphasizes the basic features of time-frequency analysis in a comprehensive way. Even though the generalization of many aspects of Gabor theory to locally compact abelian groups are easily carried out and is based on standard harmonic analysis on such groups (see, for example, [46], [53] and [72]), characterization of Gabor Schauder bases to a general locally compact abelian group is much harder. Here our aim is to extend the Heil–Powell theorem in the setting of local fields. Furthermore, in a local field K of positive characteristic, we provide a necessary and sufficient condition on a function $\varphi \in L^2(K)$ for which the collection of translates of φ forms a Schauder basis for its closed linear span.

1.3 H^1 and BMO on LCA Groups

Let us move on to the theory of Hardy spaces, which is a central analytic tool in the study of various aspects of harmonic analysis. The scope of its applications and connections is much wider, including complex analysis, partial differential equations (PDEs), functional analysis and geometric analysis. Let us first recall the definition of the classical real-variable Hardy space $H^1(\mathbb{R}^d)$ in the d-dimensional Euclidean space $\mathbb{R}^d$. Let $f \in L^1(\mathbb{R}^d)$. We say that f belongs to $H^1(\mathbb{R}^d)$ if the Riesz transform $\Delta\nabla^{\frac{-1}{2}} f \in L^1(\mathbb{R}^d)$. The origin of real-variable theory of Hardy space $H^1(\mathbb{R}^d)$ goes back to the fundamental work of Stein and Weiss [105] in the early 60's. Since its introduction, the theory and applications of Hardy space have been under intensive study. There exists an abundance of equivalent characterizations for $H^1(\mathbb{R}^d)$, among which one of the most useful is the characterization by atomic decompositions, that is, the decomposition of functions in $H^1(\mathbb{R}^d)$ into simple building blocks, atoms, originally proved by Coifman [23] for dimension $d = 1$ and by Latter [74] in higher dimensions. This remarkable feature freed the theory of $H^1(\mathbb{R}^d)$ from the rigidity of differentiable structure and makes possible its extension in a variety of contexts. In the abstract spaces of homogeneous type X, Coifman and Weiss [24] introduced the atomic Hardy spaces $H^p(X)$, $0 < p \le 1$. Tolsa [112] contributed to the theory by considering a nonnegative Radon measure μ on $\mathbb{R}^d$ which only satisfies the polynomial growth condition that there exist positive constants C and $\kappa \in (0, n]$ such that for all $x \in \mathbb{R}^d$ and $r \in (0, \infty)$,

$$\mu\Big(\{y \in \mathbb{R}^d : |x - y| < r\}\Big) \le C r^\kappa.$$

Such a measure need not satisfy the doubling condition (see also [118]). Hytönen et. al. [63] defined and investigated the H^1 theory on non-homogeneous spaces, that include both the spaces of homogeneous type and metric spaces with polynomial growth measures as special cases. The literature of Hardy spaces is very rich and still continuing to flourish. Interested reader may consult some recent articles on this topic [16, 28, 75, 64, 67].

Another goal of this thesis is to conduct an extensive study on the Hardy space in the context of Locally Compact Abelian (LCA) groups. More specifically, the general assumption on groups will be that any point admits a sequence of neighbourhoods of decreasing base sets shrinking to it and, in addition, the whole space can be covered by the increasing union of such sets. This is discussed in the first part of Chapter 4.

We now turn our attention to the well-known class of functions of bounded mean oscillation (BMO). The space BMO, originally introduced by John and Nirenberg [65] in the context of partial differential equations (PDEs), consists of functions whose mean oscillations over Euclidean balls are uniformly bounded. Since then it has played a central role in the regularity theory for nonlinear PDEs. Our interest is in its connection with $H^1(\mathbb{R}^d)$. In [37], Fefferman showed that BMO can be identified as the dual of $H^1(\mathbb{R}^d)$, i.e., $[H^1(\mathbb{R}^d)]^* = BMO$. Our second goal in Chapter 4 is to extend the classical notion of BMO to the framework of LCA groups and relate it with the Hardy space, we aim to construct, via duality.

Another fundamental property of BMO functions is the John–Nirenberg inequality which encodes self-improving properties of the oscillations of the functions involved. To be more precise, if there is a constant $C > 0$ such that for every $f \in L^1_{\rm loc}(\mathbb{R}^d)$ and for every cube Q in $\mathbb{R}^d$,

$$\fint_Q |f - f_Q|\, dx \leq C,$$

where

$$f_Q = \fint_Q f(x)\, dx = \frac{1}{|Q|} \int_Q f(x)\, dx,$$

then the oscillations $f - f_Q$ are exponentially integrable. Our another main result of this work is an extension of this quintessential feature of BMO in the context of LCA groups. Moreover, we also present an application of these results to the theory of convolution operators (see Theorem 4.5.2).

1.4 John–Nirenberg Spaces on LCA Groups

In the same paper [65], John and Nirenberg also discussed a larger BMO-type space which has since become known as the John–Nirenberg space, or JN_p, for some parameter p. The space JN_p is defined as follows. Suppose $1 < p < \infty$ and Q_0 is a cube in $\mathbb{R}^d$ with sides parallel to the coordinate axes. A function $f \in L^1(Q_0)$ is said to be in the space $JN_p(Q_0)$ provided

$$\|f\|_{JN_p(Q_0)} := \sup \Big(\frac{1}{|Q_0|} \sum_i \Big(\fint_{Q_i} |f - f_{Q_i}|\, dx \Big)^p |Q_i| \Big)^{\frac{1}{p}} < \infty, \tag{1.4.1}$$

where the supremum is taken over all possible countable collections $\{Q_i\}_{i\in\mathbb{N}}$ of pairwise disjoint subcubes of Q_0.

In the last ten years or so, there has been substantial interest in studying the John–Nirenberg spaces JN_p [65, 1, 26, 7, 110] and its several variants such as its dyadic version JN_p [71], the John–Nirenberg-Campanato spaces [111], their localized versions [107] and so on because of its connection with the space BMO and the self-improving phenomenon. The space JN_p retains some of the fundamental features of BMO spaces such as the John–Nirenberg inequality. The corresponding inequality for $JN_p(Q_0)$ spaces reveals that $JN_p(Q_0)$ can be embedded into $L^{p,\infty}(Q_0)$. Therefore, the function f, which is a priori in $L^1(Q_0)$, turns out to be in the space $L^{p,\infty}(Q_0)$ and this may be regarded as a self-improving property for the space $JN_p(Q_0)$.

In the present work we generalize the notion of John–Nirenberg spaces JN_p to LCA groups and study their properties. Our result (Theorem 5.2.1) provides an extension of the John–Nirenberg inequality for JN_p spaces in this context. In particular, we exhibit self-improving phenomenon in this framework. The key idea for the proof of John–Nirenberg inequality is inspired from [7] and relies on two main ingredients: a local Calderón–Zygmund decomposition in this setting (Lemma 5.2.4) and a certain relative distributional inequality, also referred to as the "good-λ inequality" in the literature (Proposition 5.2.2).

Chapter 2

Inequalities for the Maximal Operator Associated with General Sets

In this chapter we study norm inequalities for the maximal operator $M_{\mathbb{E}}$ associated with a family $\mathbb{E}$ of general sets from various points of view. Our first main result is the mixed $A_p - A_\infty$ weighted estimates for the operator $M_{\mathbb{E}}$. The main ingredient to prove this result is a sharp form of a weak reverse Hölder inequality for the $A_{\infty,\mathbb{E}}$ weights. As an application of this inequality, we also provide a quantitative version of the open property for $A_{p,\mathbb{E}}$ weights. Our second main result in this setting is the establishment of the endpoint Fefferman–Stein weighted inequalities for the operator $M_{\mathbb{E}}$. Furthermore, vector-valued extensions for maximal inequalities are also obtained in this context.

2.1 Introduction

Recall that the Hardy–Littlewood maximal function Mf of a locally integrable function f on $\mathbb{R}^d$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^d, \tag{2.1.1}$$

where $B(x,r)$ denotes the Euclidean ball of radius r centred at x and $|E|$ is the Lebesgue measure of a measurable set E. In [106], Stein generalized the results of Hardy–Littlewood [49] and Wiener [115] on L^p-boundedness of the maximal operator to more general situations by replacing the balls in (2.1.1) by a suitable collection of sets in $\mathbb{R}^d$. Stein observed that the Euclidean balls and the translation invariant Lebesgue measure in $\mathbb{R}^d$ can be replaced by more general sets and a nonnegative Borel measure respectively to pose and answer similar questions. More specifically, we have the following.

For each $x \in \mathbb{R}^d$, let $\{E_r(x) : 0 < r < \infty\}$ be a collection of nonempty, bounded open subsets of $\mathbb{R}^d$ containing x and let $\mathcal{E} = \{E_r(x) : 0 < r < \infty, x \in \mathbb{R}^d\}$. Assume that the sets in $\mathcal{E}$ are monotonic in r in the sense that $E_r(x) \subset E_s(x)$ if $0 < r \leq s$. Let μ be a nonnegative Borel measure with $\mu(\mathbb{R}^d) > 0$. Further, assume that the sets in the family $\mathcal{E}$ and the measure μ satisfy the following properties:

(i) there exists a constant $\theta > 1$ such that for all x, y and r, $E_r(x) \cap E_r(y) \neq \emptyset$ implies $E_r(y) \subset E_{\theta r}(x)$. Here, we call $E_{\theta r}(x)$ as the θ dilation of $E_r(x)$;

(ii) there exists a constant $C_\mu > 1$ such that

$$\mu(E_{2r}(x)) \leq C_\mu \, \mu(E_r(x)) \text{ for all } x \in \mathbb{R}^d \text{ and } 0 < r < \infty;$$

(iii) $\bigcap_{r>0} \overline{E_r(x)} = \{x\}$ and $\bigcup_{r>0} E_r(x) = \mathbb{R}^d$;

(iv) for each open set U and $r > 0$, the function $x \to \mu(E_r(x) \cap U)$ is continuous.

Define the associated maximal operator $M_{\mathcal{E}}$ by

$$M_{\mathcal{E}} f(x) = \sup_{r>0} \frac{1}{|E_r(x)|} \int_{E_r(x)} |f(y)| \, dy, \quad x \in \mathbb{R}^d.$$

Stein proved the following result. We refer to Chapter I of [106] for the proof of the following theorem and several examples of families $\mathcal{E}$ in $\mathbb{R}^d$ satisfying these properties.

Theorem 2.1.1 (Stein)**.** *Let f be a function defined on $\mathbb{R}^d$ and $\mathcal{E} = \{E_r(x) : 0 < r < \infty, x \in \mathbb{R}^d\}$ be as above. If $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, then $M_{\mathcal{E}} f$ is defined almost everywhere. Moreover, the operator $M_{\mathcal{E}}$ is of weak type $(1,1)$ and strong type (p,p) for $1 < p \leq \infty$.*

This result can further be extended to other settings. For example, the underlying space $\mathbb{R}^d$ can be replaced by a finitely generated discrete group of polynomial growth, or a smooth compact Riemannian manifold. See p. 37 in [106] for the details.

If $\mathcal{E}$ consists of all Euclidean balls in $\mathbb{R}^d$, then we get the usual Hardy–Littlewood maximal operator defined in (2.1.1) which has the weak type $(1,1)$ and strong type (p,p) properties, as mentioned earlier. However, if $\mathcal{E}$ is the family of all rectangles in $\mathbb{R}^d$, then the corresponding maximal operator is not of weak type $(1,1)$ and strong type (p,p) (see [48]). If the rectangles have sides parallel to the coordinate axes, then $M_{\mathcal{E}}$ is not of weak type $(1,1)$, even though it is of strong type (p,p) for $p>1$, see [100]. The situation does not improve even if the sets in $\mathcal{E}$ satisfy a monotone condition. For instance, Hunt provided the following example on the real line. Let $E_N(x) = x + S_N$, where $S_N = \bigcup_{k=N}^{\infty}(2^{-k}, 2^{-k}+2^{-2k})$. Then $S_{N+1} \subset S_N$ and the associated maximal operator is still not of weak type $(1,1)$, see [88] for the details of this construction.

Observe that it makes sense to define the sets $E_r(x)$ of Theorem 2.1.1 in topological spaces. In [28], Ding, Lee and Lin considered the following more general setting. This is the context in which we are going to prove the main results of this chapter.

Let X be a topological space equipped with a nonnegative Borel measure μ. Let $\mathbb{E} = \{E_r(x) : r>0, x\in X\}$ be a family of open subsets of X, where x is an interior point of $E_r(x)$. We assume that the family $\mathbb{E}$ and μ satisfy the following conditions:

(A) $\bigcup_{r>0} E_r(x) = X$;

(B) $\bigcap_{r>0} E_r(x) = \{x\}$;

(C) $E_r(x) \subset E_s(x)$ if $0<r\le s$;

(D) for all $x\in X$ and $r>0$, we have $0<\mu(E_r(x))<\infty$, and μ satisfies a *doubling condition*, i.e., there exists a constant $C_\mu > 1$ such that

$$\mu(E_{2r}(x)) \le C_\mu\, \mu(E_r(x)) \qquad \text{for all } x\in X \text{ and } E_r(x)\in\mathbb{E}; \tag{2.1.2}$$

(E) for each open set U and $r>0$, the function $x\to\mu(E_r(x)\cap U)$ is continuous;

(F) there exists a constant $\theta>1$ such that for all $E_r(x)\in\mathbb{E}$, $y\in E_r(x)$ implies $E_r(x)\subset E_{\theta r}(y)$ and $E_r(y)\subset E_{\theta r}(x)$;

(G) the mapping $r \to \mu(E_r(x))$ is continuous for each $x \in X$.

It is easy to verify that (F) is equivalent to the following condition:

(F′) there exists a constant $\theta > 1$ such that for all $x, y \in X$ and $r > 0$, $E_r(x) \cap E_r(y) \neq \emptyset$ implies $E_r(y) \subset E_{\theta r}(x)$.

It is not difficult to verify that the families of sets in Examples 1.1.4, 1.1.5 and the family of balls $\mathcal{B}$ generated by a quasi-metric in a space of homogeneous type satisfy the conditions (A)–(G). We now provide some more examples of such families.

Example 2.1.2. *Let Ω be an open proper subset of $\mathbb{R}^d$. For $r > 0$ and $x \in \Omega$, let $\widetilde{B}_r(x) = B_r(x) \cap \Omega$, where $B_r(x)$ denotes the Euclidean open ball centred at x with radius r. Let $\widetilde{\mathcal{B}} = \{\widetilde{B}_r(x) : x \in \Omega, r > 0\}$ and μ be the Lebesgue measure. This family $\widetilde{\mathcal{B}}$ arises in the study of Hardy spaces on open sets in $\mathbb{R}^d$ (see [18, 19]).*

Example 2.1.3. *Another important example of such a family arises naturally in the Dirichlet and Neumann problems for the Laplacian on Lipschitz domains [68, 69]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $\partial\Omega$ be its boundary. Define the family of sets $\mathcal{K} = \{B_r(x) \bigcap \partial\Omega : x \in \partial\Omega, r > 0\}$ and let μ be the Lebesgue measure $d\sigma$ on $\partial\Omega$.*

2.2 Results on the Maximal Operator $M_{\mathbb{E}}$

2.2.1 Mixed $A_p - A_\infty$ Bound for $M_{\mathbb{E}}$

In this section first we recall some basic notions related to the maximal function $M_{\mathbb{E}}$ and the theory of Muckenhoupt weights associated with the family $\mathbb{E}$ of general sets developed in [28]. Then we state our result on two-weight estimate for the operator $M_{\mathbb{E}}$ that generalizes Theorem 1.1.3 on spaces of homogeneous type.

The maximal operator $M_{\mathbb{E}}$ associated with the family $\mathbb{E}$ on X is defined by

$$M_{\mathbb{E}} f(x) = \sup_{E_r(x) \in \mathbb{E}} \frac{1}{\mu(E_r(x))} \int_{E_r(x)} |f(y)| \, d\mu(y), \quad x \in X. \tag{2.2.1}$$

The Muckenhoupt weights can also be defined analogously in this setting. Let $1 < p < \infty$. A nonnegative locally integrable function w on X is said to be in the weight class

$A_{p,\mathbb{E}}$ if

$$[w]_{A_{p,\mathbb{E}}} = \sup_{E\in\mathbb{E}} \Big(\frac{1}{\mu(E)}\int_E w(x)\,d\mu(x)\Big)\Big(\frac{1}{\mu(E)}\int_E w(x)^{-\frac{1}{p-1}}\,d\mu(x)\Big)^{p-1} < \infty.$$

The $A_{\infty,\mathbb{E}}$ constant of w is defined by

$$[w]_{A_{\infty,\mathbb{E}}} = \sup_{E\in\mathbb{E}} \frac{1}{w(E)}\int_E M_{\mathbb{E}}(w\chi_E)(x)\,d\mu(x),$$

where $w(E) = \int_E w(x)\,d\mu(x)$.

Observe that if $\mathbb{E}$ is the set of all balls (or cubes with sides parallel to the coordinate axes) and μ is the Lebesgue measure on $\mathbb{R}^d$, then the $A_{p,\mathbb{E}}$ weights are the usual Muckenhoupt A_p weights.

In [28], the authors obtained the following characterization of weighted strong type (p,p) properties for the maximal operator $M_{\mathbb{E}}$.

Theorem 2.2.1. *Let $1 < p < \infty$. Then $w \in A_{p,\mathbb{E}}$ if and only if the maximal operator $M_{\mathbb{E}}$ satisfies the strong type (p,p) inequality*

$$\|M_{\mathbb{E}}f\|_{L^p(X,w)} \le C\|f\|_{L^p(X,w)}, \quad f \in L^p(X,w),$$

for some constant $C > 0$.

Here $L^p(X,w)$ denotes the space of p-integrable functions on X with respect to the measure $w(x)\,d\mu(x)$.

Inspired by the work of Hytönen, Pérez and Rela, as mentioned in Chapter 1, we study the quantitative aspect of the constant C, i.e., the dependence of the operator norm of the operator $M_{\mathbb{E}}$ in terms of mixed characteristics of the underlying weight $w \in A_{p,\mathbb{E}}$. We obtain sharp quantitative norm estimates of $M_{\mathbb{E}}$ in the spirit of Theorem 1.1.3, and our result is the following.

Theorem 2.2.2. *Let $M_{\mathbb{E}}$ be the maximal operator as defined in (2.2.1) and $w \in A_{p,\mathbb{E}}$, $1 < p < \infty$. Then there exists a constant $C > 0$ such that*

$$\|M_{\mathbb{E}}f\|_{L^p(X,w)} \le C\Big([w]_{A_{p,\mathbb{E}}}[\sigma]_{A_{\infty,\mathbb{E}}}\Big)^{\frac{1}{p}}\|f\|_{L^p(X,w)} \quad \text{for all } f \in L^p(X,w). \tag{2.2.2}$$

Here we denote by $\sigma := w^{-\frac{1}{p-1}}$ the dual weight of w.

Some words are in order regarding the proof of this result.

We will exploit the flexibility in the approach provided by Hytönen, Pérez and Rela in [62]. However, we would like to remark that one of the main difficulties arising in our setup is the lack of geometry. We work directly with our basic assumptions (A)–(G). We also refer the reader to [94] for a variant of Theorem 1.1.3 on a locally compact abelian group having a covering family as defined in [32].

In [28], the authors have proved the well-known open property of A_p weights: if $w \in A_p$ for some $p > 1$, then w also belongs to $A_{p-\delta}$ for some $\delta > 0$. In order to prove Theorem 2.2.2, we need some quantitative information about δ. To achieve this, we prove the following sharp version of reverse Hölder inequality for weights in $A_{\infty,\mathbb{E}}$ class with a precise quantitative expression for the exponent. The price we pay for this is that the inequality is in a weak form.

Theorem 2.2.3 (Sharp weak reverse Hölder inequality). *If $w \in A_{\infty,\mathbb{E}}$, then*

$$\frac{1}{\mu(E)} \int_E w^{1+\epsilon}\, d\mu \leq 4C_\mu \theta^{2\alpha} \Big(\frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} w\, d\mu \Big)^{1+\epsilon} \quad \textit{for all } E \in \mathbb{E},$$

where $\epsilon = \frac{1}{2[w]_{A_{\infty,\mathbb{E}}} C - 1}$ *and* $C = 2(2C_\mu)^4 4^\alpha \theta^{8\alpha}$.

The constant α that appears above is defined in (2.4.1). The above estimate is a weaker version of reverse Hölder inequality in the sense that the set on the right of the inequality is an enlargement of the set on the left. The set $\widehat{E}$ is defined in (2.4.6). In case of spaces of homogeneous type, it is a dilation of E. As an application of Theorem 2.2.3, we appropriately quantify the δ associated with the open property of A_p weights. Finally, using this precise open property of A_p weights, together with an interpolation type argument, we obtain the desired mixed bound in (2.2.2).

In Section 2.5, we give the proofs of the results described in this section.

2.2.2 Endpoint Fefferman–Stein Inequality for $M_{\mathbb{E}}$

In this section we state our result on the generalization of the endpoint Fefferman–Stein weighted inequality (1.1.9) to the present setting. Prior to stating our results, we indicate below some difficulties we encounter in our setting.

There are many different proofs of Fefferman–Stein weighted inequality available in the literature in the standard case of $\mathbb{R}^d$. In the original paper [36], the authors first proved the inequality (1.1.9) for the dyadic maximal operator M_d. Then a pointwise inequality connecting M_d with the truncated maximal operators allows them to obtain the inequality (1.1.9) for the truncated maximal operators by a simple application of Minkowski's integral inequality. Finally, using the monotone convergence theorem, they extended those inequalities for the ordinary maximal operator.

There is another proof in [43]. The essence of their proof is based on the following observation. At each level $\lambda > 0$, the set $\{x \in \mathbb{R}^d : Mf(x) > \lambda\}$ can be covered by a countable union of dilated dyadic cubes with some control on the average of f on those dyadic cubes. By reprising the classical argument for proving the inequality (1.1.3) as mentioned in [106], this proof avoids the use of dyadic cubes. However, it relies on the Vitali covering lemma and regularity of the Lebesgue measure.

In our situation, we lack the concept of dyadic cubes and we do not assume any regularity on the measure μ, although we have Vitali type covering lemma at our disposal in our setup. Consequently, it seems difficult to adapt the above approaches to our setting to obtain an analogue of (1.1.9). To overcome this problem, we prove the following covering lemma in our setup.

Lemma 2.2.4. *Let $\mathcal{F} = \{E_{r_\alpha}(x_\alpha) : x_\alpha \in X, \alpha \in \Lambda\}$ be a family in $\mathbb{E}$ and $\Sigma = \bigcup_{\alpha\in\Lambda} E_{r_\alpha}(x_\alpha)$ such that $\mu(\Sigma) < \infty$. Then there exists a disjoint countable subfamily $\{E_{r_i}(x_i)\} \subset \mathcal{F}$ satisfying the property: for any $E_{r_\alpha}(x_\alpha) \in \mathcal{F}$, there is an $E_{r_i}(x_i)$ such that $E_{r_\alpha}(x_\alpha) \subset E_{\theta^3 r_i}(x_i)$, where θ is the constant given in condition* (F)*.*

As an application of this lemma, we shall derive an analogue of (1.1.9) in our setting as follows.

Theorem 2.2.5. *There exists a constant $C > 0$ such that*

$$w(\{x \in X : M_{\mathbb{E}}f(x) > \lambda\}) \leq \frac{C}{\lambda}\int_X |f(x)|Mw(x)\,d\mu(x) \tag{2.2.3}$$

for all measurable functions f and w on X.

The L^p version of the endpoint Fefferman–Stein estimate (2.2.3) is the following inequality. Let $1 < p < \infty$. Then

$$\int_X M_{\mathbb{E}} f(x)^p w(x)\, d\mu(x) \le C_p \int_X |f(x)|^p M_{\mathbb{E}} w(x)\, d\mu(x) \tag{2.2.4}$$

for all measurable functions f and w on X. This estimate is a simple consequence of the fact

$$\|M_{\mathbb{E}} f\|_{L^\infty(X,w)} \le C \|f\|_{L^\infty(X, M_{\mathbb{E}} w)}$$

combined with (2.2.3) and Marcinkiewicz interpolation theorem.

We shall prove all the results that we have presented in this section in Section 2.5.

2.2.3 Vector-Valued Inequalities for $M_{\mathbb{E}}$

Our result on the generalization of Theorem 1.1.6 to the present setting is the following. We will prove this result in Section 2.5.

Theorem 2.2.6. *Suppose $1 < p, q < \infty$. Let $f = \{f_i\}_i$ be a sequence of measurable functions on X. Then, we have the strong type (p,p) inequality*

$$\int_X \Big(\sum_i M_{\mathbb{E}} f_i(x)^q \Big)^{p/q} d\mu(x) \le C \int_X \Big(\sum_i |f_i(x)|^q \Big)^{p/q} d\mu(x).$$

Moreover, we have the weak type $(1,1)$ inequality

$$\mu\Big(\Big\{ x \in X : \Big(\sum_i M_{\mathbb{E}} f_i(x)^q \Big)^{1/q} > \lambda \Big\} \Big) \le \frac{C}{\lambda} \int_X \Big(\sum_i |f_i(x)|^q \Big)^{1/q} d\mu(x).$$

2.3 Notation

Throughout this chapter, we use the following notation. For $1 \le p \le \infty$, p' denotes the conjugate exponent of p defined by the condition $\frac{1}{p} + \frac{1}{p'} = 1$. For a measurable subset S of X and a measurable function f on X, we will use the notation $f(S) = \int_S f\, d\mu$. We will also write f_S to denote the μ-average of f over S, i.e.,

$$f_S = \frac{1}{\mu(S)} \int_S f(x)\, d\mu(x).$$

For $1 \le p < \infty$, the spaces $L^p(X)$ and $L^p(X, w)$ denote the p-integrable functions on X with respect to the measures μ and $w(x)\,d\mu(x)$, respectively.

For a measurable function f on a measure space (X, μ), the *distribution function* of f is the function d_f defined on $[0, \infty)$ as

$$d_f(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\}).$$

The set $\{x \in X : |f(x)| > \lambda\}$ will be called the *distribution set* of f and will be denoted by $\mathcal{D}_f(\lambda)$. That is,

$$\mathcal{D}_f(\lambda) = \{x \in X : |f(x)| > \lambda\}.$$

For any $p > 0$, the weak L^p space $L^{p,\infty}(X, \mu)$ is defined as the set of all μ-measurable functions f on X such that

$$\|f\|_{L^{p,\infty}} = \inf_{\lambda > 0} \{\lambda\, d_f(\lambda)^{\frac{1}{p}}\} < \infty. \tag{2.3.1}$$

In many occasions, we will need to evaluate the L^p-norm of a function precisely. The following formula, which computes the L^p-norm of f in terms of its distribution set $\mathcal{D}_f(\cdot)$, is very helpful and will be used several times.

Let (X, μ) be a σ-finite measure space. Then for all $p > 0$, we have

$$\begin{aligned}
\int_X |f(x)|^p \, d\mu(x) &= \int_0^\infty p\lambda^{p-1} \mu(\mathcal{D}_f(\lambda))\, d\lambda \\
&= \int_0^\infty p\lambda^{p-1} \mu(\{x \in X : |f(x)| > \lambda\})\, d\lambda. \qquad (2.3.2)
\end{aligned}$$

2.4 Basics on the Family $\mathbb{E}$ of General Sets

Let X be a topological space equipped with a nonnegative Borel measure μ. Let $\mathbb{E} = \{E_r(x) : r > 0, x \in X\}$ be a family of open subsets of X, where x is an interior point of $E_r(x)$. We assume that $\mathbb{E}$ and μ satisfy the conditions (A)–(G) described in Section 2.1.

It is easy to verify that there exists $\alpha > 1$ such that for all $a > 1$, we have

$$\mu(E_{ar}(x)) \leq 2C_\mu a^\alpha \mu(E_r(x)), \quad x \in X, r > 0. \tag{2.4.1}$$

Indeed, since $a > 1$ and $C_\mu > 1$, we can find nonnegative integers j and k such that $2^j < a \leq 2^{j+1}$ and $2^k < C_\mu \leq 2^{k+1}$. Using property (C) and the doubling condition (2.1.2), we get

$$\begin{aligned} \mu(E_{ar}(x)) &\leq \mu(E_{2^{j+1}r}(x)) \leq C_\mu^{j+1}\mu(E_r(x)) \\ &\leq 2^{(j+1)(k+1)}\mu(E_r(x)) \leq 2C_\mu a^\alpha \mu(E_r(x)), \end{aligned}$$

where $\alpha = \log_2 C_\mu + 1 > 1$ is independent of a.

We shall make use of the following Vitali type covering lemma several times in this chapter.

Lemma 2.4.1 (Lemma 2.2, [28]). *For $0 < r_0 < \infty$, $X_0 \subset X$, let*

$$\mathcal{F} = \{E_r(x) \in \mathbb{E} : 0 < r \leq r_0, x \in X_0\}.$$

Then there exists a disjoint countable subfamily $\{E_{r_i}(x_i) : i \in J\} \subset \mathcal{F}$ satisfying the following property: for any $E_r(x) \in \mathcal{F}$, there exists an $E_{r_i}(x_i)$ such that $E_r(x) \subset E_{4\theta^4 r_i}(x_i)$.

Remark 2.4.2. *Note that in Lemma 2.4.1, there is a restriction on the choices of $E_r(x)$ in terms of r. However, in Lemma 2.2.4 there is no restriction on r but on the measure of the union of sets $E_r(x)$. This assumption enables us to get better covering of sets $E_r(x)$ in terms of the dilations of $E_{r_i}(x_i)$s.*

An immediate consequence of this lemma is the L^p-mapping properties of $M_{\mathbb{E}}$ which are also extensions of the inequalities (1.1.2) and (1.1.3) in this setup. More precisely, we have the following theorem.

Theorem 2.4.3 (Theorem 2.4, [28]). *Let $M_{\mathbb{E}}$ be the maximal operator on X associated with the family $\mathbb{E}$ defined in (2.2.1). Then*

(a) $M_{\mathbb{E}}$ is bounded from $L^1(X)$ to $L^{1,\infty}(X)$;

(b) $M_{\mathbb{E}}$ *is bounded on* $L^p(X)$ *for* $1 < p \leq \infty$.

As a corollary of the above maximal theorem, we have the Lebesgue differentiation theorem. We will need this important theorem later, so we state here for easy reference.

Corollary 2.4.4. *Let* f *be a locally integrable function on* X*. Then*

$$\lim_{r\to 0} \frac{1}{\mu(E_r(x))} \int_{E_r(x)} f(y)\, d\mu(y) = f(x) \quad \text{for a.e.}\, x \in X,$$

where $E_r(x) \in \mathbb{E}$.

In [28], the authors developed an analogue of the classical theory of A_p weights for the family $\mathbb{E}$ of open subsets in X with the underlying measure μ as above. In particular, they proved that the class of weights for which the operator $M_{\mathbb{E}}$ acts boundedly from $L^p(X, w)$ to weak $L^p(X, w)$ is precisely the Muckennhoupt $A_{p,\mathbb{E}}$ class, $1 \leq p < \infty$. However, no information was given regarding the weak norm of $M_{\mathbb{E}}$ in terms of the $A_{p,\mathbb{E}}$ characteristic of the weight w. In the sequel, we will need a quantitative estimate of the operator norm of $M_{\mathbb{E}}$. Therefore, we now state this requirement in a precise form and prove it by modifying the argument given in [28]. The estimate we need is the following.

Lemma 2.4.5. *Let* $1 < p < \infty$ *and* $w \in A_{p,\mathbb{E}}$*. Then the maximal operator* $M_{\mathbb{E}}$ *satisfies the following weak type* (p, p) *inequality*

$$w(\{x \in X : M_{\mathbb{E}}f(x) > \lambda\}) \leq \frac{(2C_\mu(4\theta^4)^\alpha)^p}{\lambda^p} [w]_{A_{p,\mathbb{E}}} \|f\|^p_{L^p(X,w)}, \quad \lambda > 0. \tag{2.4.2}$$

Proof. Let $\lambda > 0$ and $f \in L^p(X, w)$ be given. By a simple application of Hölder's inequality, it is easy to check that f is locally integrable and therefore $M_{\mathbb{E}}f$ makes sense. Let $r_0 > 0$ be fixed and define the following truncated (uncentred) maximal operator $\widetilde{M}_{\mathbb{E},r_0}$ by

$$\widetilde{M}_{\mathbb{E},r_0} f(x) = \sup_{\substack{x\in E_r(z)\in\mathbb{E} \\ r\leq r_0}} \frac{1}{\mu(E_r(z))} \int_{E_r(z)} |f(y)|\, d\mu(y). \tag{2.4.3}$$

We consider the corresponding distribution set

$$\mathcal{D}_{\widetilde{M}_{\mathbb{E},r_0} f}(\lambda) = \{x \in X : \widetilde{M}_{\mathbb{E},r_0} f(x) > \lambda\}.$$

Then it is easy to see that the family $\{\mathcal{D}_{\widetilde{M}_{\mathbb{E},r_0}f}(\lambda) : r_0 > 0\}$ is increasing in r_0 and its limit is $\mathcal{D}_{M_{\mathbb{E}}f}(\lambda) = \{x \in X : M_{\mathbb{E}}f(x) > \lambda\}$. Therefore, in order to establish (2.4.2), it will be enough to prove that the inequality

$$w(\mathcal{D}_{\widetilde{M}_{\mathbb{E},r_0}f}(\lambda)) \leq \frac{C}{\lambda^p}[w]_{A_{p,\mathbb{E}}}\|f\|^p_{L^p(X,w)} \tag{2.4.4}$$

holds, where the constant C is independent of r_0 and λ.

First of all, we note that

$$\mathcal{D}_{\widetilde{M}_{\mathbb{E},r_0}f}(\lambda) \subseteq \bigcup_{E_r(z)\in\mathcal{F}} E_r(z),$$

where the family $\mathcal{F} \subset \mathbb{E}$ and for any $E_r(z) \in \mathcal{F}$ and $r \leq r_0$, we have $\frac{1}{\mu(E_r(z))}\int_{E_r(z)} |f|\,d\mu > \lambda$. Thus, by Lemma 2.4.1, there exists a countable disjoint family $\{E_{r_i}(x_i)\}_i$ in $\mathcal{F}$ such that $\mathcal{D}_{\widetilde{M}_{\mathbb{E},r_0}f}(\lambda) \subseteq \bigcup_i E_{4\theta^4 r_i}(x_i)$ and

$$\frac{1}{\mu(E_{r_i}(x_i))}\int_{E_{r_i}(x_i)} |f|\,d\mu > \lambda \quad \text{for all } i.$$

Using these facts along with (2.4.1), we obtain

$$\begin{aligned}
&\lambda^p\, w(\mathcal{D}_{\widetilde{M}_{\mathbb{E},r_0}f}(\lambda)) \\
&\quad\leq \sum_i \lambda^p w(E_{4\theta^4 r_i}(x_i)) \\
&\quad\leq \sum_i w(E_{4\theta^4 r_i}(x_i))\Big(\frac{2C_\mu(4\theta^4)^\alpha}{\mu(E_{4\theta^4 r_i}(x_i))}\int_{E_{r_i}(x_i)} |f(y)|\,d\mu(y)\Big)^p.
\end{aligned} \tag{2.4.5}$$

Now we estimate each term in this sum. By Hölder's inequality and the $A_{p,\mathbb{E}}$ condition, we get

$$\begin{aligned}
&\frac{w(E_{4\theta^4 r_i}(x_i))}{\mu(E_{4\theta^4 r_i}(x_i))^p}\Big(\int_{E_{r_i}(x_i)} |f(y)|\,d\mu(y)\Big)^p \\
&\quad\leq \frac{w(E_{4\theta^4 r_i}(x_i))}{\mu(E_{4\theta^4 r_i}(x_i))^p}\Big(\int_{E_{r_i}(x_i)} |f|^p w\,d\mu\Big)\Big(\int_{E_{4\theta^4 r_i}(x_i)} w^{1-p'}\,d\mu\Big)^{p-1} \\
&\quad= \Big(\frac{1}{\mu(E_{4\theta^4 r_i}(x_i))}\int_{E_{4\theta^4 r_i}(x_i)} w\,d\mu\Big) \\
&\qquad\times\Big(\frac{1}{\mu(E_{4\theta^4 r_i}(x_i))}\int_{E_{4\theta^4 r_i}(x_i)} w^{1-p'}\,d\mu\Big)^{p-1}\int_{E_{r_i}(x_i)} |f|^p w\,d\mu
\end{aligned}$$

$$\leq \quad [w]_{A_{p,\mathbb{E}}} \int_{E_{r_i}(x_i)} |f|^p w \, d\mu.$$

Substituting in (2.4.5) and using the fact that $\{E_{r_i}(x_i)\}_i$ is a disjoint family, we get our desired inequality

$$w(\mathcal{D}_{\widetilde{M}_{\mathbb{E},r_0} f}(\lambda)) \leq \frac{(2C_\mu(4\theta^4)^\alpha)^p}{\lambda^p} [w]_{A_{p,\mathbb{E}}} \|f\|^p_{L^p(X,w)}.$$

This proves (2.4.4) and the proof of the lemma is complete. □

We will also need the following Calderón–Zygmund decomposition of an integrable function. For a proof, we refer to [29].

Lemma 2.4.6 (Calderón–Zygmund decomposition)**.** *Let $f \in L^1(X)$ and $\lambda > 0$. There exists a countable collection of pairwise disjoint open sets $\{E_{r_i}(x_i) : i \in J\}$ such that*

(a) $\lambda < \frac{1}{\mu(E_{r_i}(x_i))} \int_{E_{r_i}(x_i)} |f| \, d\mu \leq 2C_\mu \theta^\alpha \lambda$ *for all* $i \in J$*,*

(b) $\sum_{i \in J} \mu(E_{r_i}(x_i)) \leq \frac{\|f\|_1}{\lambda}$*,*

(c) $|f(x)| \leq \lambda$ *if* $x \notin \bigcup_{i \in J} E_{r_i}(x_i)$*.*

To prove the sharp reverse Hölder inequality, we need to enlarge the sets $E_r(x)$. We first define this concept of enlargement and study some basic properties.

Let $E = E_{r_0}(x_0)$ be a fixed open set in $\mathbb{E}$. Define

$$\mathcal{B}_E = \{E_r(y) : y \in E, r \leq r_0\}$$

and

$$\widehat{E} = \bigcup_{E_r(y) \in \mathcal{B}_E} E_r(y). \tag{2.4.6}$$

Lemma 2.4.7. *Let $E = E_{r_0}(x_0)$ be a fixed open set in $\mathbb{E}$.*

(a) If $F \in \mathcal{B}_E$, then $F \subset E_{\theta r_0}(x_0)$.

(b) For any $z \in E$, we have $\widehat{E} \subset E_{\theta^2 r_0}(z)$. This implies

$$\mu(\widehat{E}) \leq \mu(E_{\theta^2 r_0}(z)) \leq 2C_\mu \theta^{2\alpha} \mu(E_{r_0}(z)).$$

Proof. (a) If $F \in \mathcal{B}_E$, then $F = E_r(x)$ for some $x \in E_{r_0}(x_0)$ and $r \leq r_0$. Hence, $x \in E_{r_0}(x) \cap E_{r_0}(x_0)$ so that $E_{r_0}(x) \subset E_{\theta r_0}(x_0)$, by property (F′). Therefore, $F = E_r(x) \subset E_{r_0}(x) \subset E_{\theta r_0}(x_0)$.

(b) Let $F \in \mathcal{B}_E$ so that $F = E_r(x)$, $x \in E_{r_0}(x_0)$, $r \leq r_0$. By (a), $F = E_r(x) \subset E_{\theta r_0}(x_0)$. Now, if $z \in E_{r_0}(x_0)$, then $z \in E_{\theta r_0}(x_0)$. Also, $z \in E_{\theta r_0}(z)$. Therefore, $z \in E_{\theta r_0}(x_0) \cap E_{\theta r_0}(z)$. Hence, $E_{\theta r_0}(x_0) \subset E_{\theta^2 r_0}(z)$. Thus, $F = E_r(x) \subset E_{\theta r_0}(x_0) \subset E_{\theta^2 r_0}(z)$. This shows that $\widehat{E} \subset E_{\theta^2 r_0}(z)$. The last assertion follows from (2.4.1). □

Given a fixed open set $E = E_{r_0}(x_0)$ in $\mathbb{E}$, we define the *local maximal function* relative to E as follows:

$$M_E f(y) = \begin{cases} \sup\limits_{F \in \widehat{E}, y \in F} \frac{1}{\mu(F)} \int_F |f(z)|\, d\mu(z), & \text{if } y \in \widehat{E}, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Note that by Lebesgue differentiation theorem (Corollary 2.4.4), it follows that

$$|f(y)| \leq M_E f(y) \quad \text{for a.e. } y \in E. \tag{2.4.7}$$

We consider the distribution set at level $\lambda > 0$ for the local maximal function of a weight w:

$$\mathcal{D}_{M_E w}(\lambda) = \{x \in \widehat{E} : M_E w(x) > \lambda\}. \tag{2.4.8}$$

The next result contains a key decomposition of $\mathcal{D}_{M_E w}(\lambda)$ of Calderón–Zygmund type. We shall make use of it several times in the sequel.

Lemma 2.4.8. *Let $E = E_{r_0}(x_0)$ be a fixed open set in $\mathbb{E}$ and w be a nonnegative integrable function with support in $\widehat{E}$. For $\lambda > w_{\widehat{E}}$, define $\mathcal{D}_{M_E w}(\lambda)$ as above. Then, there exists a countable family of pairwise disjoint open sets $\{E_{r_i}(x_i) : i \in J\}$ from $\mathcal{B}_E$ such that*

(i) $\bigcup_{i \in J} E_{r_i}(x_i) \subset \mathcal{D}_{M_E w}(\lambda) \subset \bigcup_{i \in J} E_{4\theta^4 r_i}(x_i)$,

(ii) $\lambda < \frac{1}{\mu(E_{r_i}(x_i))} \int_{E_{r_i}(x_i)} w\, d\mu \leq 2C_\mu \theta^\alpha \lambda$ *for all* $i \in J$,

(iii) $r_i \leq r_0$ *for all* $i \in J$,

(iv) if $r > r_i$ for some i, then $\frac{1}{\mu(E_r(x_i))}\int_{E_r(x_i)} w\,d\mu \le 2C_\mu\theta^\alpha\lambda$.

Proof. The proof is based on Lemma 2.4.1 and inequality (2.4.1), the doubling property of μ. For $x \in \mathcal{D}_{M_E w}(\lambda)$, define

$$R_x = \sup\{r : \text{there exists } F = E_r(y) \text{ with } x \in F \text{ and } \frac{1}{\mu(F)}\int_F w\,d\mu > \lambda\}.$$

Note that $R_x < \infty$ as $r \le r_0$. So, for each $x \in \mathcal{D}_{M_E w}(\lambda)$, there exists $r_x > 0$ and $y_x \in E_{r_0}(x_0)$ such that

$$\frac{1}{\mu(E_{r_x}(y_x))}\int_{E_{r_x}(y_x)} w\,d\mu > \lambda \quad \text{and} \quad r_x \le R_x < \theta r_x.$$

Observe that

$$\mathcal{D}_{M_E w}(\lambda) \subseteq \bigcup_{x\in\mathcal{D}_{M_E w}(\lambda)} E_{r_x}(y_x) \quad \text{and} \quad r_x \le r_0 \text{ for all } x.$$

Therefore, by Lemma 2.4.1, there exists a countable set J and a collection of pairwise disjoint open sets $\{E_{r_{x_i}}(y_{x_i}) : i \in J\}$ such that $\mathcal{D}_{M_E w}(\lambda) \subseteq \bigcup_{i\in J} E_{4\theta^4 r_{x_i}}(y_{x_i})$. To simplify notation, we write r_{x_i} by r_i, R_{x_i} by R_i, and y_{x_i} by x_i.

Note that (i) is trivial since each $E_{r_i}(x_i)$ is contained in $\mathcal{D}_{M_E w}(\lambda)$. To prove (ii), observe that by (2.4.1), we get

$$\begin{aligned}\lambda &< \frac{1}{\mu(E_{r_i}(x_i))}\int_{E_{r_i}(x_i)} w\,d\mu \\ &\le \frac{\mu(E_{\theta r_i}(x_i))}{\mu(E_{r_i}(x_i))}\cdot\frac{1}{\mu(E_{\theta r_i}(x_i))}\int_{E_{\theta r_i}(x_i)} w\,d\mu \le 2C_\mu\theta^\alpha\lambda.\end{aligned}$$

Part (iii) is obvious. Finally, if $r > r_i$ for some i, then $\theta r > \theta r_i > R_i$. Hence, again by (2.4.1),

$$\begin{aligned}\frac{1}{\mu(E_r(x_i))}\int_{E_r(x_i)} w\,d\mu &= \frac{\mu(E_{\theta r}(x_i))}{\mu(E_r(x_i))}\cdot\frac{1}{\mu(E_{\theta r}(x_i))}\int_{E_{r_i}(x_i)} w\,d\mu \\ &\le 2C_\mu\theta^\alpha\frac{1}{\mu(E_{\theta r}(x_i))}\int_{E_{\theta r}(x_i)} w\,d\mu \\ &\le 2C_\mu\theta^\alpha\lambda.\end{aligned}$$

This proves (iv). □

The next result is a continuation of the above lemma where we present a localization argument for the local maximal function. The corresponding result for the dyadic case in $\mathbb{R}^d$ is a direct consequence of the maximality of the cubes in the Calderón–Zygmund decomposition. Preserving the notation of Lemma 2.4.8, we have the following.

Lemma 2.4.9. *Let $E = E_{r_0}(x_0)$ be a fixed open set in X and $L = (2C_\mu\theta^\alpha)^3$. If $x \in E_{4\theta^4 r_i}(x_i) \cap \mathcal{D}_{M_E w}(L\lambda)$, then $M_E w(x) \le M_E(w\chi_{E_{4\theta^5 r_i}(x_i)})(x)$.*

Proof. Let $x \in E_{4\theta^4 r_i}(x_i) \cap \mathcal{D}_{M_E w}(L\lambda)$. Then, there exists $F = E_r(y)$, $y \in E$ and $x \in E_r(y)$ with $r \le r_0$ such that $\frac{1}{\mu(F)}\int_F w\,d\mu > L\lambda$. Thus, $x \in E_r(y) \cap E_{4\theta^4 r_i}(x_i)$.

We claim that $r \le 4\theta^4 r_i$. Once the claim is proved, we have $x \in E_{4\theta^4 r_i}(y) \cap E_{4\theta^4 r_i}(x_i)$. Hence, $F = E_r(y) \subset E_{4\theta^4 r_i}(y) \subset E_{4\theta^5 r_i}(x_i)$, by property (F′). Then, by the definition of M_E, we have

$$\frac{1}{\mu(F)}\int_F w\,d\mu = \frac{1}{\mu(F)}\int_F w\chi_{E_{4\theta^5 r_i}(x_i)}\,d\mu \le M_E\big(w\chi_{E_{4\theta^5 r_i}(x_i)}\big)(x).$$

Therefore, $M_E w(x) \le M_E\big(w\chi_{E_{4\theta^5 r_i}(x_i)}\big)(x)$.

We now prove the claim. If possible, let $r > 4\theta^4 r_i$. Since $x \in E_r(x_i) \cap E_r(y)$, using property (F′), we have $E_r(x_i) \subset E_{\theta r}(y)$ and $E_r(y) \subset E_{\theta r}(x_i)$. Hence,

$$\frac{\mu(E_{\theta r}(x_i))}{\mu(E_r(y))} \le 2C_\mu\theta^\alpha \frac{\mu(E_r(x_i))}{\mu(E_r(y))} \le 2C_\mu\theta^\alpha \frac{\mu(E_{\theta r}(y))}{\mu(E_r(y))} \le (2C_\mu\theta^\alpha)^2.$$

Furthermore, by Lemma 2.4.8 (iv), we have

$$\frac{1}{\mu(E_{\theta r}(x_i))}\int_{E_{\theta r}(x_i)} w\,d\mu \le 2C_\mu\theta^\alpha\lambda,$$

since $\theta r > r > 4\theta^4 r_i > r_i$. So we conclude that

$$\begin{aligned}
\frac{1}{\mu(F)}\int_F w\,d\mu &= \frac{1}{\mu(E_r(y))}\int_{E_r(y)} w\,d\mu \\
&\le \frac{\mu(E_{\theta r}(x_i))}{\mu(E_r(y))}\cdot\frac{1}{\mu(E_{\theta r}(x_i))}\int_{E_{\theta r}(x_i)} w\,d\mu \\
&\le (2C_\mu\theta^\alpha)^2 \frac{1}{\mu(E_{\theta r}(x_i))}\int_{E_{\theta r}(x_i)} w\,d\mu \\
&\le (2C_\mu\theta^\alpha)^2(2C_\mu\theta^\alpha)\lambda \\
&= L\lambda,
\end{aligned}$$

which is a contradiction. This finishes the proof of the lemma. □

2.5 Proofs of the Main Results

2.5.1 Sharp Weak Reverse Hölder Inequality

Proof of Theorem 2.2.3. We want to prove that if $w \in A_{\infty,\mathbb{E}}$, then for all $E \in \mathbb{E}$, we have

$$\frac{1}{\mu(E)} \int_E w^{1+\epsilon}\, d\mu \leq 4C_\mu \theta^{2\alpha} \Big(\frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} w\, d\mu \Big)^{1+\epsilon}, \tag{2.5.1}$$

where $\epsilon = \frac{1}{2[w]_{A_{\infty,\mathbb{E}}} C - 1}$ and $C = 2(2C_\mu)^4 \theta^{8\alpha} 4^\alpha$.

Let $E = E_{r_0}(x_0) \in \mathbb{E}$ be a fixed open set. We begin by reducing the above inequality to a self-improving property of the maximal operator M_E when restricted to $A_{\infty,\mathbb{E}}$ weights. Using (2.4.7), we obtain

$$\int_E w^{1+\epsilon}\, d\mu \leq \int_E (M_E w)^\epsilon w\, d\mu \leq \int_{\widehat{E}} (M_E w)^\epsilon w\, d\mu.$$

Let $\mathcal{D}_{M_E w}(\lambda)$ be defined as in (2.4.8). Using (2.3.2), we write the last integral as

$$\begin{aligned} \int_{\widehat{E}} (M_E w)^\epsilon w\, d\mu &= \int_0^\infty \epsilon \lambda^{\epsilon-1} w(\mathcal{D}_{M_E w}(\lambda))\, d\lambda \\ &= \int_0^{w_{\widehat{E}}} \epsilon \lambda^{\epsilon-1} w(\mathcal{D}_{M_E w}(\lambda))\, d\lambda + \int_{w_{\widehat{E}}}^\infty \epsilon \lambda^{\epsilon-1} w(\mathcal{D}_{M_E w}(\lambda))\, d\lambda \\ &\leq w(\widehat{E})(w_{\widehat{E}})^\epsilon + \int_{w_{\widehat{E}}}^\infty \epsilon \lambda^{\epsilon-1} w(\mathcal{D}_{M_E w}(\lambda))\, d\lambda. \end{aligned} \tag{2.5.2}$$

Now, in order to estimate the second term of (2.5.2), we apply Calderón–Zygmund decomposition of $\mathcal{D}_{M_E w}(\lambda)$ for each $\lambda > w_{\widehat{E}}$ (see Lemma 2.4.8) and obtain a countable family of pairwise disjoint open sets $\{E_{r_i}(x_i)\}_i$ such that $\mathcal{D}_{M_E w}(\lambda) \subseteq \bigcup_i E_{4\theta^4 r_i}(x_i)$. Moreover,

$$\frac{1}{\mu(E_{4\theta^4 r_i}(x_i))} \int_{E_{4\theta^4 r_i}(x_i)} w(y)\, d\mu(y) \leq 2C_\mu \theta^\alpha \lambda \quad \text{for all } i.$$

That is,

$$w(E_{4\theta^4 r_i}(x_i)) \leq 2C_\mu \theta^\alpha \lambda \mu(E_{4\theta^4 r_i}(x_i)).$$

Therefore, using the doubling property (2.4.1), we have

$$\begin{aligned}
\int_{w_{\widehat{E}}}^{\infty} \epsilon\lambda^{\epsilon-1} w(\mathcal{D}_{M_E w}(\lambda))\,d\lambda &\leq \int_{w_{\widehat{E}}}^{\infty} \epsilon\lambda^{\epsilon-1} \sum_i w(E_{4\theta^4 r_i}(x_i))\,d\lambda \\
&\leq 2C_\mu\theta^\alpha \int_{w_{\widehat{E}}}^{\infty} \epsilon\lambda^{\epsilon} \sum_i \mu(E_{4\theta^4 r_i}(x_i))\,d\lambda \\
&\leq 2C_\mu\theta^\alpha \cdot 2C_\mu(4\theta^4)^\alpha \int_{w_{\widehat{E}}}^{\infty} \epsilon\lambda^{\epsilon} \sum_i \mu(E_{r_i}(x_i))\,d\lambda \\
&\leq (2C_\mu)^2 4^\alpha \theta^{5\alpha} \int_0^\infty \epsilon\lambda^\epsilon \mu(\mathcal{D}_{M_E w}(\lambda))\,d\lambda \\
&= (2C_\mu)^2 4^\alpha \theta^{5\alpha} \frac{\epsilon}{\epsilon+1} \int_{\widehat{E}} (M_E w)^{1+\epsilon}\,d\mu.
\end{aligned}$$

The last line follows since

$$\int_{\widehat{E}} (M_E w)^{\epsilon+1}\,d\mu = \int_0^\infty (\epsilon+1)\lambda^\epsilon \mu(\mathcal{D}_{M_E w}(\lambda))\,d\lambda.$$

Substituting this estimate in (2.5.2) and taking average over E, we get

$$\begin{aligned}
&\frac{1}{\mu(E)} \int_E w^{1+\epsilon}\,d\mu \\
&\leq \frac{1}{\mu(E)} w(\widehat{E})(w_{\widehat{E}})^\epsilon + (2C_\mu)^2 4^\alpha \theta^{5\alpha} \frac{\epsilon}{\epsilon+1} \cdot \frac{1}{\mu(E)} \int_{\widehat{E}} (M_E w)^{1+\epsilon}\,d\mu \\
&\leq (w_{\widehat{E}})^{1+\epsilon} 2C_\mu \theta^{2\alpha} + (2C_\mu)^2 4^\alpha \theta^{5\alpha} \frac{\epsilon}{\epsilon+1} \cdot 2C_\mu\theta^{2\alpha} \frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} (M_E w)^{1+\epsilon}\,d\mu,
\end{aligned}$$

where in the last inequality, we have used Lemma 2.4.7 (b).

We turn now to the estimation of the integral in the second term of the last inequality. We intend to show that

$$\frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} (M_E w)^{1+\epsilon}\,d\mu \leq 2[w]_{A_{\infty,\mathbb{E}}} \Big(\frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} w\,d\mu \Big)^{1+\epsilon} \tag{2.5.3}$$

for any $\epsilon \leq \frac{1}{2C[w]_{A_{\infty,\mathbb{E}}}-1}$, where $C = 2(2C_\mu)^4 4^\alpha \theta^{8\alpha}$.

If this inequality is proved, then by a simple computation, we obtain

$$\frac{1}{\mu(E)} \int_E w^{1+\epsilon}\,d\mu \leq 4C_\mu\theta^{2\alpha} \Big(\frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} w\,d\mu \Big)^{1+\epsilon},$$

which is precisely (2.5.1). So the proof of the theorem will be complete once we establish (2.5.3) and we now proceed to do so.

By the definition of the local maximal function $M_E w$ of w, we may assume that the weight w is supported on $\widehat{E}$. Again, we consider the distribution set $\mathcal{D}_{M_E w}(\lambda)$ as above and using (2.3.2), we write

$$\begin{aligned}
&\int_{\widehat{E}} (M_E w)^{1+\epsilon}\, d\mu \\
&= \int_0^\infty \epsilon\lambda^{\epsilon-1} M_E w(\mathcal{D}_{M_E w}(\lambda))\, d\lambda \\
&= \int_0^{w_{\widehat{E}}} \epsilon\lambda^{\epsilon-1} M_E w(\mathcal{D}_{M_E w}(\lambda))\, d\lambda + \int_{w_{\widehat{E}}}^\infty \epsilon\lambda^{\epsilon-1} M_E w(\mathcal{D}_{M_E w}(\lambda))\, d\lambda. \quad (2.5.4)
\end{aligned}$$

Now, we estimate each integral separately. The first integral is relatively easier to handle. Indeed, an application of Lemma 2.4.7, combined with the definition of $A_{\infty,\mathbb{E}}$ constant, yields

$$\begin{aligned}
\int_0^{w_{\widehat{E}}} \epsilon\lambda^{\epsilon-1} M_E w(\mathcal{D}_{M_E w}(\lambda))\, d\lambda &\leq \Big(\int_0^{w_{\widehat{E}}} \epsilon\lambda^{\epsilon-1}\, d\lambda\Big) M_E w(\widehat{E}) \\
&= (w_{\widehat{E}})^\epsilon \int_{\widehat{E}} M_E w(x)\, d\mu(x) \\
&\leq (w_{\widehat{E}})^\epsilon \int_{E_{\theta^2 r_0}(y)} M_E(w\chi_{E_{\theta^2 r_0}(y)})(x)\, d\mu(x) \\
&\leq (w_{\widehat{E}})^\epsilon [w]_{A_{\infty,\mathbb{E}}} w(\widehat{E}), \quad (2.5.5)
\end{aligned}$$

where in the second last inequality $y \in E$.

We now estimate the second integral. Apply the Calderón–Zygmund decomposition of $\mathcal{D}_{M_E w}(\lambda)$ for each $\lambda > w_{\widehat{E}}$ to obtain a collection of pairwise disjoint open sets $\{E_{r_i}(x_i)\}$ satisfying conditions (i) to (iv) of Lemma 2.4.8. In order to simplify notation, we write $E_i = E_{4\theta^4 r_i}(x_i)$ and decompose E_i as $E_i = E_i^1 \cup E_i^2$, where

$$E_i^1 = E_i \cap \mathcal{D}_{M_E w}(L\lambda) \text{ and } E_i^2 = E_i \setminus \mathcal{D}_{M_E w}(L\lambda).$$

Recall that $L = (2C_\mu \theta^\alpha)^3$. So, in our new notation, $\mathcal{D}_{M_E w}(\lambda) \subseteq \bigcup_i E_i$ and thus we have

$$M_E w(\mathcal{D}_{M_E w}(\lambda)) \leq \sum_i M_E w(E_i).$$

Now, we proceed to estimate each $M_E w(E_i)$. We have

$$M_E w(E_i) \;=\; \int_{E_i^1} M_E w\, d\mu + \int_{E_i^2} M_E w\, d\mu$$

$$\begin{aligned} &\le \int_{E_i^1} M_E(w\chi_{E_{4\theta^5 r_i}(x_i)})\,d\mu + L\lambda\mu(E_i^2) \\ &\le [w]_{A_{\infty,\mathbb{E}}} w(E_{4\theta^5 r_i}(x_i)) + L\lambda\mu(E_{4\theta^5 r_i}(x_i)) \\ &= \Big([w]_{A_{\infty,\mathbb{E}}} w_{E_{4\theta^5 r_i}(x_i)} + L\lambda\Big)\mu(E_{4\theta^5 r_i}(x_i)), \end{aligned}$$

where we have used Lemma 2.4.9 in the first inequality and the definition of $A_{\infty,\mathbb{E}}$ constant in the second inequality.

Now if $r > r_i$, then by Lemma 2.4.8, $w_{E_r(x_i)} = \frac{1}{\mu(E_r(x_i))}\int_{E_r(x_i)} w\,d\mu \le 2C_\mu\theta^\alpha\lambda$. Taking this fact into account and using the doubling condition (2.4.1), we obtain

$$\begin{aligned} M_E w(E_i) &\le \Big([w]_{A_{\infty,\mathbb{E}}} 2C_\mu\theta^\alpha\lambda + L\lambda\Big)2C_\mu(4\theta^5)^\alpha\mu(E_{r_i}(x_i)) \\ &\le \Big([w]_{A_{\infty,\mathbb{E}}}(2C_\mu\theta^\alpha)^3 + (2C_\mu\theta^\alpha)^3\Big)\lambda 2C_\mu(4\theta^5)^\alpha\mu(E_{r_i}(x_i)) \\ &\le [w]_{A_{\infty,\mathbb{E}}}\lambda C\mu(E_{r_i}(x_i)), \end{aligned}$$

where $C = 2(2C_\mu)^4 4^\alpha\theta^{8\alpha}$. Adding these inequalities, we get

$$\begin{aligned} M_E w(\mathcal{D}_{M_E w}(\lambda)) &\le \sum_i M_E w(E_i) \\ &\le [w]_{A_{\infty,\mathbb{E}}}\lambda C\sum_i \mu(E_{r_i}(x_i)) \\ &\le \lambda C[w]_{A_{\infty,\mathbb{E}}}\mu(\mathcal{D}_{M_E w}(\lambda)). \end{aligned}$$

Thus, the second integral in (2.5.4) is bounded by

$$\begin{aligned} \int_{w_{\widehat{E}}}^\infty \epsilon\lambda^{\epsilon-1} M_E w(\mathcal{D}_{M_E w}(\lambda))\,d\lambda &\le C[w]_{A_{\infty,\mathbb{E}}}\int_{w_{\widehat{E}}}^\infty \epsilon\lambda^\epsilon\mu(\mathcal{D}_{M_E w}(\lambda))\,d\lambda \\ &\le C[w]_{A_{\infty,\mathbb{E}}}\int_0^\infty \epsilon\lambda^\epsilon\mu(\mathcal{D}_{M_E w}(\lambda))\,d\lambda \\ &= C[w]_{A_{\infty,\mathbb{E}}}\frac{\epsilon}{\epsilon+1}\int_{\widehat{E}} (M_E w)^{1+\epsilon}\,d\mu. \end{aligned} \tag{2.5.6}$$

Now, we put together the estimates (2.5.5) and (2.5.6) in (2.5.4) and get

$$\int_{\widehat{E}} (M_E w)^{1+\epsilon}\,d\mu \le (w_{\widehat{E}})^\epsilon [w]_{A_{\infty,\mathbb{E}}} w(\widehat{E}) + C[w]_{A_{\infty,\mathbb{E}}}\frac{\epsilon}{\epsilon+1}\int_{\widehat{E}} (M_E w)^{1+\epsilon}\,d\mu.$$

Finally, taking average over $\widehat{E}$, we have

$$\Big(1 - [w]_{A_{\infty,\mathbb{E}}} C\frac{\epsilon}{\epsilon+1}\Big)\frac{1}{\mu(\widehat{E})}\int_{\widehat{E}} (M_E w)^{1+\epsilon}\,d\mu \le (w_{\widehat{E}})^{1+\epsilon}[w]_{A_{\infty,\mathbb{E}}}.$$

Note that $1 - C[w]_{A_{\infty,\mathbb{E}}} \frac{\epsilon}{\epsilon+1} \geq \frac{1}{2}$ if and only if $\epsilon \leq \frac{1}{2[w]_{A_{\infty,\mathbb{E}}} C - 1}$. In particular, if we choose $\epsilon = \frac{1}{2[w]_{A_{\infty,\mathbb{E}}} C - 1}$, where $C = 2(2C_\mu)^4 4^\alpha \theta^{8\alpha}$, then

$$\frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} (M_E w)^{1+\epsilon} \, d\mu \leq 2[w]_{A_{\infty,\mathbb{E}}} \Big(\frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} w \, d\mu \Big)^{1+\epsilon}.$$

Thus, (2.5.3) holds and the proof of Theorem 2.2.3 is complete. □

2.5.2 Open Property of $A_{p,\mathbb{E}}$ Weights

We denote $r(w) = 1 + \epsilon$, where ϵ is as above. Explicitly,

$$r(w) = 1 + \frac{1}{2[w]_{A_{\infty,\mathbb{E}}} C - 1}, \quad \text{where } C = 2(2C_\mu)^4 4^\alpha \theta^{8\alpha}.$$

An immediate consequence of Theorem 2.2.3 is the following quantitative open property of the $A_{p,\mathbb{E}}$ weights.

Corollary 2.5.1 (Open property of $A_{p,\mathbb{E}}$ weights)**.** *Let $1 < p < \infty$ and $w \in A_{p,\mathbb{E}}$. Let $\sigma = w^{1-p'}$ be the dual weight of w. Put $\delta = \frac{p-1}{(r(\sigma))'}$. Then, $w \in A_{p-\delta,\mathbb{E}}$ and*

$$[w]_{A_{p-\delta,\mathbb{E}}} \leq (2C_\mu \theta^{2\alpha})^p (4C_\mu \theta^{2\alpha})^{\frac{p-1}{r(\sigma)}} [w]_{A_{p,\mathbb{E}}}. \tag{2.5.7}$$

Proof. We use the same classical ideas as in the Euclidean case. For the above choice of $\delta = \frac{p-1}{(r(\sigma))'}$, we have $1 - (p-\delta)' = (1-p')r(\sigma)$ and $r(\sigma) = \frac{p-1}{p-\delta-1}$. Applying the sharp weak reverse Hölder inequality for the weight σ, we have

$$\begin{aligned} \Big(\frac{1}{\mu(E)} \int_E w^{1-(p-\delta)'} \, d\mu \Big)^{p-\delta-1} &= \Big(\frac{1}{\mu(E)} \int_E w^{(1-p')r(\sigma)} \, d\mu \Big)^{\frac{p-1}{r(\sigma)}} \\ &= \Big(\frac{1}{\mu(E)} \int_E \sigma^{r(\sigma)} \, d\mu \Big)^{\frac{p-1}{r(\sigma)}} \\ &\leq \Big((4C_\mu \theta^{2\alpha})^{\frac{1}{r(\sigma)}} \frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} \sigma \, d\mu \Big)^{p-1} \end{aligned}$$

for any $E \in \mathbb{E}$.

Let $E = E_{r_0}(x_0)$ be a fixed open set in $\mathbb{E}$. By Lemma 2.4.7, we have $\widehat{E} \subset E_{\theta^2 r_0}(z)$. Hence, by the doubling property of μ, we have

$$\begin{aligned}
&\Big(\frac{1}{\mu(E)}\int_E w\,d\mu\Big)\cdot\Big(\frac{1}{\mu(E)}\int_E w^{1-(p-\delta)'}\,d\mu\Big)^{p-\delta-1}\\
\leq\ & \Big(\frac{1}{\mu(E)}\int_E w\,d\mu\Big)\cdot\Big((4C_\mu\theta^{2\alpha})^{\frac{1}{r(\sigma)}}\frac{1}{\mu(\widehat{E})}\int_{\widehat{E}}\sigma\,d\mu\Big)^{p-1}\\
\leq\ & 2C_\mu\theta^{2\alpha}\Big(\frac{1}{\mu(E_{\theta^2 r_0}(x_0))}\int_{E_{\theta^2 r_0}(x_0)} w\,d\mu\Big)\\
&\quad\times\Big((4C_\mu\theta^{2\alpha})^{\frac{1}{r(\sigma)}}2C_\mu\theta^{2\alpha}\frac{1}{\mu(E_{\theta^2 r_0}(x_0))}\int_{E_{\theta^2 r_0}(x_0)} w^{1-p'}\,d\mu\Big)^{p-1}\\
\leq\ & (2C_\mu\theta^{2\alpha})^p(4C_\mu\theta^{2\alpha})^{\frac{p-1}{r(\sigma)}}[w]_{A_{p,\mathbb{E}}}.
\end{aligned}$$

In other words, $[w]_{A_{p-\delta,\mathbb{E}}} \leq (2C_\mu\theta^{2\alpha})^p(4C_\mu\theta^{2\alpha})^{\frac{p-1}{r(\sigma)}}[w]_{A_{p,\mathbb{E}}}$. □

2.5.3 Sharp Mixed Bound for the Maximal Operator

Proof of Theorem 2.2.2. Let $f \in L^p(X,w)$ be given. As we mentioned earlier, the idea behind the proof of (2.2.2) is to use an interpolation type argument. To do so, we need a suitable truncation of f, namely $f_t = f\chi_{|f|>t}$, $t > 0$. Then, a simple computation shows that

$$\{x \in X : M_{\mathbb{E}}f(x) > 2t\} \subset \{x \in X : M_{\mathbb{E}}f_t(x) > t\}. \tag{2.5.8}$$

Let us estimate the $L^p(X,w)$-norm of $M_{\mathbb{E}}f$. First of all, by (2.5.8), we write

$$\begin{aligned}
\|M_{\mathbb{E}}f\|^p_{L^p(X,w)} &= \int_0^\infty pt^{p-1}w(\{x\in X : M_{\mathbb{E}}f(x) > t\})\,dt\\
&= 2^p\int_0^\infty pt^{p-1}w(\{x\in X : M_{\mathbb{E}}f(x) > 2t\})\,dt\\
&\leq 2^p\int_0^\infty pt^{p-1}w(\{x\in X : M_{\mathbb{E}}f_t(x) > t\})\,dt.
\end{aligned}$$

We will now use the tools that we have developed earlier. Let $w \in A_{p,\mathbb{E}}$. By Corollary 2.5.1, we have $w \in A_{p-\delta,\mathbb{E}}$, where $\delta = \frac{p-1}{(r(\sigma))'}$. Therefore, by Lemma 2.4.5 and (2.5.7), we obtain

$$\begin{aligned}
\|M_{\mathbb{E}}f\|^p_{L^p(X,w)} &\leq 2^p p\int_0^\infty t^{p-1}\frac{[w]_{A_{p-\delta,\mathbb{E}}}}{t^{p-\delta}}(2C_\mu(4\theta^4)^\alpha)^{p-\delta}\int_X |f_t|^{p-\delta}w\,d\mu\,dt\\
&\leq 2^p p\int_0^\infty t^{\delta-1}(2C_\mu\theta^{2\alpha})^p(4C_\mu\theta^{2\alpha})^{\frac{p-1}{r(\sigma)}}[w]_{A_{p,\mathbb{E}}}
\end{aligned}$$

$$\begin{aligned}
&\quad \times(2C_\mu(4\theta^4)^\alpha)^{p-\delta}\int_X |f_t|^{p-\delta} w\, d\mu\, dt\\
&\le 2^p p(2C_\mu\theta^{2\alpha})^p(4C_\mu\theta^{2\alpha})^{\frac{p-1}{r(\sigma)}}[w]_{A_{p,\mathbb{E}}}\\
&\quad \times(2C_\mu(4\theta^4)^\alpha)^p\int_0^\infty t^{\delta-1}\int_X |f_t|^{p-\delta} w\, d\mu\, dt\\
&= A[w]_{A_{p,\mathbb{E}}}\int_0^\infty t^{\delta-1}\int_X |f_t|^{p-\delta} w\, d\mu\, dt\\
&\le \frac{A[w]_{A_{p,\mathbb{E}}}}{\delta}\int_X |f|^p w\, d\mu,
\end{aligned}$$

where $A = 2^p p(2C_\mu\theta^{2\alpha})^p(4C_\mu\theta^{2\alpha})^{\frac{p-1}{r(\sigma)}}(2C_\mu(4\theta^4)^\alpha)^p$. Also note that $r(\sigma) = 1+\frac{1}{2[\sigma]_{A_{\infty,\mathbb{E}}}C-1}$, where $C = 2(2C_\mu)^4 4^\alpha\theta^{8\alpha}$. Hence,

$$\delta = \frac{p-1}{(r(\sigma))'} = \frac{p-1}{2[\sigma]_{A_{\infty,\mathbb{E}}}C}.$$

With δ as above, we obtain

$$\|M_{\mathbb{E}}f\|^p_{L^p(X,w)} \le 2AC\frac{[\sigma]_{A_{\infty,\mathbb{E}}}[w]_{A_{p,\mathbb{E}}}}{p-1}\int_X |f|^p w\, d\mu.$$

From this we conclude that

$$\|M_{\mathbb{E}}f\|_{L^p(X,w)} \le C\Big([w]_{A_{p,\mathbb{E}}}[\sigma]_{A_{\infty,\mathbb{E}}}\Big)^{\frac{1}{p}}\|f\|_{L^p(X,w)}.$$

This completes the proof of the theorem. □

2.5.4 Endpoint Fefferman–Stein Weighted Inequality

A crucial ingredient to prove the endpoint Fefferman–Stein weighted inequality is the covering lemma outlined in Lemma 2.2.4. We first prove this result.

Proof of Lemma 2.2.4. The idea is to use a sort of inductive type argument.

Step I. We start by picking an open set $E_{r_{0,1}}(x_{0,1}) \in \mathcal{F}$ such that

$$2\mu(E_{r_{0,1}}(x_{0,1})) > \sup_{\alpha\in\Lambda}\mu(E_{r_\alpha}(x_\alpha)).$$

Define the sets $\mathcal{F}_1$ and $\mathcal{R}_1$ as follows:

$$\mathcal{F}_1 = \{E_r(x) \in \mathcal{F} : E_{\theta^2 r}(x) \cap E_{r_{0,1}}(x_{0,1}) \neq \emptyset\}$$

and

$$\mathcal{R}_1 = \{r : E_r(x) \in \mathcal{F}_1\}.$$

It is obvious that $E_{r_{0,1}}(x_{0,1}) \in \mathcal{F}_1$. We claim that $\sup \mathcal{R}_1 < \infty$. Suppose on the contrary $\sup \mathcal{R}_1 = \infty$. Then, we can find a sequence $\{r_j\}$ such that $r_j \to \infty$ as $j \to \infty$ with the property that $E_{r_j}(x_j) \in \mathcal{F}_1$ and $E_{\theta^2 r_j}(x_j) \cap E_{r_{0,1}}(x_{0,1}) \neq \emptyset$. Without loss, we can assume that $\theta^2 r_j > r_{0,1}$ for all j. Since $E_{r_{0,1}}(x_{0,1}) \subset E_{\theta^2 r_j}(x_{0,1})$, we have $E_{\theta^2 r_j}(x_j) \cap E_{\theta^2 r_j}(x_{0,1}) \neq \emptyset$. By property (F′), $E_{\theta^2 r_j}(x_{0,1}) \subset E_{\theta^3 r_j}(x_j)$ for all j. Applying (2.4.1), we see that

$$\mu(E_{\theta^2 r_j}(x_{0,1})) \leq \mu(E_{\theta^3 r_j}(x_j)) \leq 2C_\mu \theta^{3\alpha} \mu(E_{r_j}(x_j)) \leq 2C_\mu \theta^{3\alpha} \mu(\Sigma).$$

Since $\mu(E_{\theta^2 r_j}(x_{0,1})) \to \infty$ as $j \to \infty$, this would mean that $\mu(\Sigma) = \infty$. This is a contradiction and hence the claim is proved. Therefore, we can choose $E_{r_1}(x_1) \in \mathcal{F}_1$ such that $\theta r_1 > \sup \mathcal{R}_1$.

We further claim that if $E_r(x) \in \mathcal{F}_1$ such that $E_r(x) \cap E_{r_1}(x_1) \neq \emptyset$, then $E_r(x) \subset E_{\theta^3 r_1}(x_1)$. To prove this claim, we first observe that for any such $E_r(x)$, we have $r \leq \theta^2 r_1$. This is for the following reason. Suppose there is an r such that $r > \theta^2 r_1$. Since $E_{r_{0,1}}(x_{0,1}) \in \mathcal{F}_1$, by the definition of $\mathcal{R}_1$, we have $\theta r_1 > r_{0,1}$ so that $r > \theta^2 r_1 > \theta r_1 > r_{0,1}$. Also, by the choice of $E_{r_1}(x_1)$, we have $E_{r_{0,1}}(x_{0,1}) \cap E_{\theta^2 r_1}(x_1) \neq \emptyset$. Let y be a point in their intersection. Then, by property (F′),

$$E_{r_{0,1}}(x_{0,1}) \subset E_{\theta r_{0,1}}(y) \subset E_{\theta^2 r_1}(y) \subset E_{\theta^3 r_1}(x_1) \subset E_{\theta r}(x_1).$$

On the other hand, by our hypothesis $E_r(x) \cap E_{r_1}(x_1) \neq \emptyset$. This implies $E_{\theta r}(x) \cap E_{\theta r}(x_1) \neq \emptyset$. Hence, $E_{\theta r}(x_1) \subset E_{\theta^2 r}(x)$ by property (F′). Thus, $E_{r_{0,1}}(x_{0,1}) \subset E_{\theta^2 r}(x)$. In particular, $E_{r_{0,1}}(x_{0,1}) \cap E_{\theta^2 r}(x) \neq \emptyset$. This implies that $r \in \mathcal{R}_1$ so that $r \leq \sup \mathcal{R}_1 < \theta r_1 < \theta^2 r_1$. This is a contradiction.

So we have proved that if $E_r(x) \in \mathcal{F}_1$ such that $E_r(x) \cap E_{r_1}(x_1) \neq \emptyset$, then $r \leq \theta^2 r_1$. Consequently $E_{\theta^2 r_1}(x) \cap E_{\theta^2 r_1}(x_1) \neq \emptyset$. Therefore, by property (F′), $E_r(x) \subset$

$E_{\theta^2 r_1}(x) \subset E_{\theta^3 r_1}(x_1)$. Thus, the second claim is also proved.

Step II. Let

$$\widetilde{\mathcal{F}_2} = \{E_r(x) \in \mathcal{F} : E_r(x) \cap E_{r_1}(x_1) = \emptyset\}.$$

We choose $E_{r_{0,2}}(x_{0,2}) \in \widetilde{\mathcal{F}_2}$ such that

$$2\mu(E_{r_{0,2}}(x_{0,2})) > \sup_{E_r(x) \in \widetilde{\mathcal{F}_2}} \mu(E_r(x)).$$

Let

$$\mathcal{F}_2 = \{E_r(x) \in \widetilde{\mathcal{F}_2} : E_{\theta^2 r}(x) \cap E_{r_{0,2}}(x_{0,2}) \neq \emptyset\}$$

and

$$\mathcal{R}_2 = \{r : E_r(x) \in \mathcal{F}_2\}.$$

Proceeding as in Step I, we have $\sup \mathcal{R}_2 < \infty$ and hence we can choose $E_{r_2}(x_2) \in \mathcal{F}_2$ such that $\theta r_2 > \sup \mathcal{R}_2$. Furthermore, if $E_r(x) \in \mathcal{F}_2$ such that $E_r(x) \cap E_{r_2}(x_2) \neq \emptyset$, then $E_r(x) \subset E_{\theta^3 r_2}(x_2)$.

Step III. We continue as above. If the process stops after J steps, then $\{E_{r_i}(x_i) : i = 1, 2, \ldots, J\}$ is the required collection. Otherwise, this process generates a countable collection of disjoint open sets $\{E_{r_i}(x_i) : i = 1, 2, \ldots\}$. Now, we show that if $E_r(x) \in \mathcal{F}$, then it intersects with at least one $E_{r_i}(x_i)$. This will complete the proof of the lemma. If this is not the case, then $E_r(x) \cap E_{r_i}(x_i) = \emptyset$ for all i. This means that $E_r(x) \in \widetilde{\mathcal{F}_i}$ for every i. Therefore,

$$2\mu(E_{r_{0,i}}(x_{0,i})) > \mu(E_r(x)) \quad \text{for all } i.$$

Also, since $E_{\theta^2 r_{0,i}}(x_{0,i}) \cap E_{r_{0,i}}(x_{0,i}) \neq \emptyset$, by the definition of $\sup \mathcal{R}_i$, it follows that $r_{0,i} < \theta r_i$.

On the other hand, by the choice of $E_{r_i}(x_i)$, the condition $E_{\theta^2 r_i}(x_i) \cap E_{r_{0,i}}(x_{0,i}) \neq \emptyset$ implies that $E_{\theta^2 r_i}(x_i) \cap E_{\theta^2 r_i}(x_{0,i}) \neq \emptyset$. Then it follows that $E_{\theta^2 r_i}(x_{0,i}) \subset E_{\theta^3 r_i}(x_i)$. Therefore,

$$E_{r_{0,i}}(x_{0,i}) \subset E_{\theta r_i}(x_{0,i}) \subset E_{\theta^2 r_i}(x_{0,i}) \subset E_{\theta^3 r_i}(x_i).$$

Summarizing these, we obtain

$$0 < \mu\big(E_r(x)\big) < 2\mu\big(E_{r_{0,i}}(x_{0,i})\big) \le 2\mu\big(E_{\theta^3 r_i}(x_i)\big) \le 2\cdot 2C_\mu \theta^{3\alpha}\mu\big(E_{r_i}(x_i)\big).$$

Since $\{E_{r_i}(x_i)\}_i$ is a pairwise disjoint collection, this will imply that $\mu(\Sigma) = \infty$, a contradiction. Hence, we conclude that every $E_r(x)$ in $\mathcal{F}$ intersects at least one $E_{r_i}(x_i)$ and in that case $E_r(x) \subseteq E_{\theta^3 r_i}(x_i)$. This completes the proof of the lemma. □

We are now ready to prove the endpoint Fefferman–Stein weighted inequality.

Proof of Theorem 2.2.5. Let f be a locally integrable function on X and $\lambda > 0$ be given. We want to show that

$$w(\{x \in X : M_{\mathbb{E}} f(x) > \lambda\}) \le C \int_X |f(x)| M_{\mathbb{E}} w(x)\, d\mu(x), \tag{2.5.9}$$

where C is independent of f and λ.

Let $\mathcal{D}_{M_{\mathbb{E}}f}(\lambda) = \{x \in X : M_{\mathbb{E}} f(x) > \lambda\}$ be the distribution set of $M_{\mathbb{E}} f$ at the level λ. Fix an arbitrary element x_0 in X. For $n \in \mathbb{N}$, let f_n be the function obtained by restricting f to the set $E_n(x_0)$, i.e., $f_n = f \cdot \chi_{E_n(x_0)}$. We also consider the corresponding distribution set $\mathcal{D}_{M_{\mathbb{E}}f_n}(\lambda) = \{x \in X : M_{\mathbb{E}} f_n(x) > \lambda\}$. We clearly have that the family $\mathcal{D}_{M_{\mathbb{E}}f_n}(\lambda)$ is increasing in n and moreover, $\mathcal{D}_{M_{\mathbb{E}}f}(\lambda) = \bigcup_n \mathcal{D}_{M_{\mathbb{E}}f_n}(\lambda)$. This suggests that we may compute the value of $w(\mathcal{D}_{M_{\mathbb{E}}f}(\lambda))$ as limit of $w(\mathcal{D}_{M_{\mathbb{E}}f_n}(\lambda))$ as $n \to \infty$.

So let n be fixed and $x \in \mathcal{D}_{M_{\mathbb{E}}f_n}(\lambda)$. Then there exists an open set $E_s(z) \in \mathbb{E}$ containing x such that

$$\frac{1}{\mu\big(E_s(z)\big)} \int_{E_s(z)} |f_n(y)|\, d\mu(y) > \lambda.$$

The last inequality also shows that $E_s(z) \subseteq \mathcal{D}_{M_{\mathbb{E}}f_n}(\lambda)$ and thus $\mathcal{D}_{M_{\mathbb{E}}f_n}(\lambda)$ may be written as a union of open sets from $\mathbb{E}$. Further, by the weak $(1,1)$ property of $M_{\mathbb{E}}$ (Theorem 2.4.3), we have $\mu(\mathcal{D}_{M_{\mathbb{E}}f_n}(\lambda)) < \infty$. Hence, by Lemma 2.2.4, there exists a countable family of pairwise disjoint open sets $\{E_{r_i}(x_i)\}_i$ such that

$$\mathcal{D}_{M_{\mathbb{E}}f_n}(\lambda) \subseteq \bigcup_i E_{\theta^3 r_i}(x_i). \tag{2.5.10}$$

Moreover,

$$\frac{1}{\mu\big(E_{r_i}(x_i)\big)}\int_{E_{r_i}(x_i)}|f_n(x)|\,d\mu(x)>\lambda \quad \text{for all } i. \tag{2.5.11}$$

Therefore, using (2.5.10), (2.5.11) and (2.4.1), we obtain

$$\begin{aligned}
w\big(\mathcal{D}_{M_{\mathbb{E}}f_n}(\lambda)\big) &\leq \sum_i w\big(E_{\theta^3 r_i}(x_i)\big)\\
&= \sum_i \mu\big(E_{\theta^3 r_i}(x_i)\big)\cdot\frac{1}{\mu\big(E_{\theta^3 r_i}(x_i)\big)}\int_{E_{\theta^3 r_i}(x_i)} w\,d\mu\\
&\leq 2C_\mu\theta^{3\alpha}\sum_i\Big(\frac{1}{\lambda}\int_{E_{r_i}(x_i)}|f_n|\,d\mu\Big)\frac{1}{\mu\big(E_{\theta^3 r_i}(x_i)\big)}\int_{E_{\theta^3 r_i}(x_i)} w\,d\mu\\
&= \frac{2C_\mu\theta^{3\alpha}}{\lambda}\sum_i\Big\{\int_{E_{r_i}(x_i)}|f_n|\Big(\frac{1}{\mu\big(E_{\theta^3 r_i}(x_i)\big)}\int_{E_{\theta^3 r_i}(x_i)} w\,d\mu\Big)\,d\mu\Big\}\\
&\leq \frac{2C_\mu\theta^{3\alpha}}{\lambda}\sum_i\int_{E_{r_i}(x_i)}|f_n(x)|M_{\mathbb{E}}w(x)\,d\mu(x)\\
&\leq \frac{2C_\mu\theta^{3\alpha}}{\lambda}\int_X|f_n(x)|M_{\mathbb{E}}w(x)\,d\mu(x)\\
&\leq \frac{2C_\mu\theta^{3\alpha}}{\lambda}\int_X|f(x)|M_{\mathbb{E}}w(x)\,d\mu(x).
\end{aligned}$$

We observe that the last expression is independent of x_0 and n. Hence, we conclude that

$$w\big(\mathcal{D}_{M_{\mathbb{E}}f}(\lambda)\big)\leq\frac{2C_\mu\theta^{3\alpha}}{\lambda}\int_X|f(x)|M_{\mathbb{E}}w(x)\,d\mu(x)$$

for all $\lambda>0$. □

Remark 2.5.2. *Strictly speaking, we have proved Fefferman–Stein weighted inequality for the case* $\mu(X)=\infty$ *since the covering Lemma 2.2.4 is valid only under this assumption. However, the assertion of Theorem 2.2.5 is still true for the case* $\mu(X)<\infty$*. In this case, we use the covering Lemma 2.4.1 instead of Lemma 2.2.4 and replace the assumptions* $0<r<\infty$ *and* $0<\mu(E_r(x))<\infty$ *by* $0<r<\mu(X)$ *and* $0<\mu(E_r(x))<\mu(X)$ *respectively. Then, the argument presented here for the proof of* (2.5.9) *will go through for this case as well.*

2.5.5 Vector-Valued Inequalities

Proof of Theorem 2.2.6. We use the following notation. If $f = (f_1, f_2, \dots)$ is a sequence of functions on X, then $M_{\mathbb{E}} f = (M_{\mathbb{E}} f_1, M_{\mathbb{E}} f_2, \dots)$ and

$$\|f(x)\|_{\ell^q} = \big(\sum_i |f_i(x)|^q\big)^{\frac{1}{q}}, \quad 1 < q < \infty.$$

We shall follow the same basic ideas as the arguments presented in [36]. If $p = q > 1$, then the proof is straightforward and therefore we omit the details. We will need this result for the proof of the case $p = 1$. Thus, for all $q > 1$, there is a constant A_q such that

$$\int_X \|M_{\mathbb{E}} f(x)\|_{\ell^q}^q \, d\mu(x) \le A_q \int_X \|f(x)\|_{\ell^q}^q \, d\mu(x). \tag{2.5.12}$$

For the case $p = 1$, we proceed as follows. Let $f = (f_1, f_2, \dots)$ be such that the integral of $\|f(x)\|_{\ell^q}$ is finite and $\lambda > 0$ be given. We want to prove

$$\mu\big(\{x \in X : \|M_{\mathbb{E}} f(x)\|_{\ell^q} > \lambda\}\big) \le \frac{C_q}{\lambda} \int_X \|f(x)\|_{\ell^q} \, d\mu(x), \tag{2.5.13}$$

where C_q is independent of f and λ.

We apply Calderón–Zygmund decomposition (Lemma 2.4.8) to the function $(\sum_i |f_i|^q)^{1/q}$ to obtain a countable collection of pairwise disjoint open sets $\{E_{r_j}(x_j) : j \in J\}$ such that

(a) $\lambda < \frac{1}{\mu(E_{r_j}(x_j))} \int_{E_{r_j}(x_j)} \|f(x)\|_{\ell^q} \, d\mu(x) \le 2C_\mu \theta^\alpha \lambda$ for all $j \in J$,

(b) $\sum_{j \in J} \mu(E_{r_j}(x_j)) \le \frac{1}{\lambda} \int_X \|f(x)\|_{\ell^q} \, d\mu(x)$,

(c) $\|f(x)\|_{\ell^q} \le \lambda$, if $x \notin \bigcup_{j \in J} E_{r_j}(x_j)$.

Let $\Omega = \bigcup_j E_{r_j}(x_j)$. We shall use the open sets $E_{r_j}(x_j)$ to decompose the function f into two parts as follows. For each i, define $f_{1,i} = f_i \cdot \chi_\Omega$ on X. Setting $f_{2,i} = f_i - f_{1,i}$, we obtain a decomposition $f = f^1 + f^2$, where

$$f^1 = (f_{1,1}, f_{1,2}, f_{1,3}, \dots) \quad \text{and} \quad f^2 = (f_{2,1}, f_{2,2}, f_{2,3}, \dots).$$

Observe that f^2 is supported outside of Ω, $\|f^2(x)\|_{\ell^q} \leq \lambda$ for a.e. x, and $\|f^2(x)\|_{\ell^q} \leq \|f(x)\|_{\ell^q}$. Using these facts and (2.5.12), we obtain

$$\begin{aligned}
&\lambda^q \mu\big(\{x \in X : \|M_{\mathbb{E}} f^2(x)\|_{\ell^q} > \lambda\}\big) \\
&\leq q\int_0^\lambda \mu\big(\{x \in X : \|M_{\mathbb{E}} f^2(x)\|_{\ell^q} > t\}\big) t^{q-1}\, dt \\
&\leq \int_0^\infty \mu\big(\{x \in X : \|M_{\mathbb{E}} f^2(x)\|_{\ell^q}^q > s\}\big)\, ds \\
&= \int_X \|M_{\mathbb{E}} f^2(x)\|_{\ell^q}^q \, d\mu(x) \\
&\leq A_q \int_X \|f^2(x)\|_{\ell^q}^q \, d\mu(x) \\
&\leq \lambda^{q-1} A_q \int_X \Big(\sum_i |f_{2,i}(x)|^q\Big)^{\frac{1}{q}} d\mu(x) \\
&\leq \lambda^{q-1} A_q \int_X \|f(x)\|_{\ell^q}\, d\mu(x).
\end{aligned}$$

Hence,

$$\mu\big(\{x \in X : \|M_{\mathbb{E}} f^2(x)\|_{\ell^q} > \lambda\}\big) \leq \frac{A_q}{\lambda} \int_X \|f(x)\|_{\ell^q}\, d\mu(x). \tag{2.5.14}$$

Also, by Minkowski's inequality, we have

$$\|M_{\mathbb{E}} f(x)\|_{\ell^q} \leq \|M_{\mathbb{E}} f^1(x)\|_{\ell^q} + \|M_{\mathbb{E}} f^2(x)\|_{\ell^q}.$$

So, in view of the estimate (2.5.14), it follows that inequality (2.5.13) will be proved once we prove that

$$\mu\big(\{x \in X : \|M_{\mathbb{E}} f^1(x)\|_{\ell^q} > \lambda\}\big) \leq \frac{A_q}{\lambda} \int_X \|f(x)\|_{\ell^q}\, d\mu(x). \tag{2.5.15}$$

Now we further reduce the above inequality to simpler inequalities. We split the distribution set of $\|M_{\mathbb{E}} f^1(\cdot)\|_{\ell^q}$ as follows. Set $\widetilde{\Omega} = \bigcup_{j\in J} \widetilde{E}_{r_j}(x_j)$, where $\widetilde{E}_{r_j}(x_j) = E_{\theta r_j}(x_j)$. Then

$$\begin{aligned}
&\mu\big(\{x \in X : \|M_{\mathbb{E}} f^1(x)\|_{\ell^q} > \lambda\}\big) \\
&\leq \mu(\widetilde{\Omega}) + \mu\big(\{x \in X \setminus \widetilde{\Omega} : \|M_{\mathbb{E}} f^1(x)\|_{\ell^q} > \lambda\}\big).
\end{aligned}$$

The first term is easily controlled by (2.4.1) and (b):

$$\begin{aligned}
\mu(\widetilde{\Omega}) &\leq \sum_{j\in J} \mu(\widetilde{E}_{r_j}(x_j)) \\
&\leq 2C_\mu \theta^\alpha \sum_{j\in J} \mu(E_{r_j}(x_j)) \\
&\leq \frac{2C_\mu\theta^\alpha}{\lambda} \int_X \|f(x)\|_{\ell^q}\, d\mu(x).
\end{aligned}$$

Taking the above estimate into account, we see that proving (2.5.15) reduces to showing that

$$\mu\big(\big\{x \in X \setminus \widetilde{\Omega} : \|M_{\mathbb{E}} f^1(x)\|_{\ell^q} > \lambda\big\}\big) \leq \frac{C}{\lambda} \int_X \|f(x)\|_{\ell^q}\, d\mu(x). \tag{2.5.16}$$

Let us define

$$\widetilde{f}_i(x) = \begin{cases} \frac{1}{\mu\left(E_{r_j}(x_j)\right)} \int_{E_{r_j}(x_j)} |f_i(y)|\, d\mu(y), & \text{if } x \in E_{r_j}(x_j), \\ \\ 0, & \text{otherwise.} \end{cases}$$

Then, the function $\|\widetilde{f}(\cdot)\|_{\ell^q} = \left(\sum_i |\widetilde{f}_i(\cdot)|^q\right)^{1/q}$ is supported on Ω and is bounded by $2C_\mu\theta^\alpha$. By a similar computation as above, we obtain

$$\mu\big(\big\{x \in X : \|M_{\mathbb{E}} \widetilde{f}(x)\|_{\ell^q} > \lambda\big\}\big) \leq \frac{A_q(2C_\mu\theta^\alpha)^q}{\lambda} \int_X \|f(x)\|_{\ell^q}\, d\mu(x).$$

Therefore, in order to prove (2.5.16), it is enough to show that

$$M_{\mathbb{E}} f_{1,i}(x) \leq C M_{\mathbb{E}} \widetilde{f}_i(x) \quad \text{for a.e. } x \in X \setminus \widetilde{\Omega} \text{ and for all } i. \tag{2.5.17}$$

This is not hard to prove. Indeed, let $x \in X \setminus \widetilde{\Omega}$. Consider an open set $E_r(y) \in \mathbb{E}$ such that $x \in E_r(y)$ and $\frac{1}{\mu\left(E_r(y)\right)} \int_{E_r(y)} |f_{1,i}(z)|\, d\mu(z) > 0$. Now, we compute the average of $f_{1,i}$ over $E_r(y)$. We have

$$\begin{aligned}
&\frac{1}{\mu\big(E_r(y)\big)} \int_{E_r(y)} |f_{1,i}(z)|\, d\mu(z) \\
&\quad = \frac{1}{\mu\big(E_r(y)\big)} \sum_{\substack{j\in J \\ E_r(y)\cap E_{r_j}(x_j)\neq\emptyset}} \int_{E_r(y)\cap E_{r_j}(x_j)} |f_{1,i}(z)|\, d\mu(z)
\end{aligned}$$

$$\leq \frac{1}{\mu(E_r(y))} \sum_{\substack{j\in J \\ E_r(y)\cap E_{r_j}(x_j)\neq\emptyset}} \int_{E_{r_j}(x_j)} |f_{1,i}(z)|\, d\mu(z)$$
$$= \frac{1}{\mu(E_r(y))} \sum_{\substack{j\in J \\ E_r(y)\cap E_{r_j}(x_j)\neq\emptyset}} \int_{E_{r_j}(x_j)} |\widetilde{f}_i(z)|\, d\mu(z).$$

If $j \in J$ is such that $E_r(y) \cap E_{r_j}(x_j) \neq \emptyset$, then we claim that $r_j \leq r$. Suppose $r < r_j$. Then, $E_{r_j}(y) \cap E_{r_j}(x_j) \neq \emptyset$ so that $E_{r_j}(y) \subset E_{\theta r_j}(x_j)$ by property (F′). This would mean that $x \in E_r(y) \subset E_{r_j}(y) \subset E_{\theta r_j}(x_j) \subset \widetilde{\Omega}$, a contradiction. Therefore, $E_r(y) \cap E_{r_j}(x_j) \neq \emptyset$ implies that $E_{r_j}(x_j) \subset E_r(x_j) \subset E_{\theta r}(y)$. Using this fact in the last integral, we get

$$\begin{aligned} \frac{1}{\mu(E_r(y))} \int_{E_r(y)} |f_{1,i}(z)|\, d\mu(z) &\leq \frac{1}{\mu(E_r(y))} \int_{E_{\theta r}(y)} |\widetilde{f}_i(z)|\, d\mu(z) \\ &\leq \frac{2C_\mu \theta^\alpha}{\mu(E_{\theta r}(y))} \int_{E_{\theta r}(y)} |\widetilde{f}_i(z)|\, d\mu(z) \\ &\leq 2C_\mu \theta^\alpha M_{\mathbb{E}} \widetilde{f}_i(x). \end{aligned}$$

Taking supremum over all such $E_r(y)$, we see that (2.5.17) holds. But the proof of (2.5.13) was reduced to this inequality. Hence, we have proved the theorem for the case $p = 1$.

For the exponent $1 < p < q$, we use Marcinkiewicz interpolation theorem. We interpolate between the estimates in $p = 1$ and $p = q$ to conclude that

$$\int_X \|M_{\mathbb{E}} f(x)\|_{\ell^q}^p \, d\mu(x) \leq A_q \int_X \|f(x)\|_{\ell^q}^p \, d\mu(x), \quad 1 < p < q.$$

For the remaining values of the exponent p, namely for $1 < q < p < \infty$, we use the duality argument presented in [36]. A careful reading of the proof reveals that the arguments given there only use Hölder's inequality, Hahn–Banach theorem and an analogue of the inequality (2.2.4) as the main ingredients. Therefore, a similar argument gives us the strong type (p, p) inequality for this case as well. This completes the proof of the theorem. □

Chapter 3

Weighted Norm Inequalities for Fourier Series and Applications

In this chapter we focus on the weighted norm inequalities for Fourier series in the context of the ring of integers $\mathfrak{D}$ of a local field K and some important applications. We establish weighted estimates for the maximal partial sum operator $\mathfrak{M}$ of Fourier series on the weighted spaces $L^p(\mathfrak{D}, w)$, $1 < p < \infty$, where w is a Muckenhoupt A_p weight. As a consequence of this result, we obtain the uniform boundedness of the Fourier partial sum operators $S_n, n \in \mathbb{N}$, on $L^p(\mathfrak{D}, w)$. Both these results include the cases when $\mathfrak{D}$ is the ring of integers of the p-adic field $\mathbb{Q}_p$ and the field $\mathbb{F}_q((X))$ of formal Laurent series over a finite field $\mathbb{F}_q$, and in particular, when $\mathfrak{D}$ is the Walsh–Paley or dyadic group 2^ω.

We present several applications of these results. In a local field K of positive characteristic, we show that for a function $\varphi \in L^2(K)$, its collection of discrete translates forms a Schauder basis for its closed linear span if and only if the periodization of $|\hat{\varphi}|^2$ is an A_2 weight on $\mathfrak{D}$. Using the uniform boundedness of $\{S_n : n \in \mathbb{N}\}$, we also characterize the Schauder basis property of the Gabor systems in a local field K of positive characteristic K in terms of A_2 weights on $\mathfrak{D} \times \mathfrak{D}$ and the Zak transform Zg of the window function g that generates the Gabor system. More precisely, we show that the Gabor system generated by g is a Schauder basis for $L^2(K)$ if and only if $|Zg|^2$ is an A_2 weight on $\mathfrak{D} \times \mathfrak{D}$. Some examples are given to illustrate this result.

Furthermore, we construct a Gabor system which is complete and minimal, but fails to be a Schauder basis for $L^2(K)$.

3.1 Basics on Fourier Analysis on Local Fields

Here we first present some basic facts about local fields K and then recall some results on Fourier series of functions defined on the ring of integers $\mathfrak{D}$ of K. Many of these facts are well-known and can be found, for example, in the books [109] and [96]. Moreover, we describe some results from the weighted theory of maximal functions on local fields [20, 94] which are needed to prove our results.

3.1.1 Local Fields

Let K be a field and a topological space. Then K is called a locally compact field or a *local field* if both K^+ and K^* are locally compact abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K respectively.

If K is any field and is endowed with the discrete topology, then K is a locally compact field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. So, by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. We use the notation of the book by Taibleson [109].

Let K be a local field. Since K^+ is a locally compact abelian group, we choose a Haar measure dx for K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. By the uniqueness of the Haar measure, $d(\alpha x) = c\,dx$ for some $c > 0$. Let $c = |\alpha|$. We call $|\alpha|$ the *absolute value* or the *valuation* of α. We also let $|0| = 0$.

The map $x \to |x|$ has the following properties:

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|xy| = |x||y|$ for all $x, y \in K$;

(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the *ultrametric inequality*. It follows that

$$|x+y| = \max\{|x|, |y|\} \text{ if } |x| \neq |y|. \tag{3.1.1}$$

The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. It is the unique maximal compact subring of K. Define $\mathfrak{P} = \{x \in K : |x| < 1\}$. The set $\mathfrak{P}$ is called the *prime ideal* in K and it is the unique maximal ideal in $\mathfrak{D}$.

Since K is totally disconnected, the set of values $|x|$, as x varies over K, is a discrete set of the form $\{s^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $s > 0$. Hence, there is an element of $\mathfrak{P}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{P}$. Such an element is called a *prime element* of K. It can be proved that $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{P}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{P}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. For a proof of this fact, we refer to [109].

For a measurable subset E of K, let $|E| = \int_K \mathbf{1}_E(x)\,dx$, where $\mathbf{1}_E$ is the characteristic function of E and dx is the Haar measure of K normalized so that $|\mathfrak{D}| = 1$. Then, it is easy to see that $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{p}| = q^{-1}$ (see [109]). It follows that if $x \neq 0$, and $x \in K$, then $|x| = q^k$ for some $k \in \mathbb{Z}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{P} = \{x \in K : |x| = 1\}$. It is the *group of units* in K^*. Let $\mathfrak{P}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| \leq q^{-k}\}$, $k \in \mathbb{Z}$. These are called *fractional ideals*. Each $\mathfrak{P}^k$ is compact and open and is a subgroup of K^+ (see [96]). It follows that $|\mathfrak{P}^k| = q^{-k}$ for $k \in \mathbb{Z}$.

A character on locally compact abelian group G is a group homomorphism from G into the circle group $\mathbb{T}$. If K is a local field, then there is a nontrivial, unitary, continuous character χ on K^+. It can be proved that K^+ is self dual (see [109]).

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{P}^{-1}$. We can find such a character by starting with any nontrivial character and rescaling. For $y \in K$, we define $\chi_y(x) = \chi(yx)$, $x \in K$.

Let $h \in K$ and $k \in \mathbb{Z}$. A set B of the form $h + \mathfrak{P}^k$ will be called a *ball with centre h and radius q^{-k}*. It is easy to verify the following facts. For a proof, we refer to [109].

Proposition 3.1.1. *(a) Every point of a ball is its centre.*

(b) If two balls intersect, then one contains the other.

(c) The number of balls in K is countable.

(d) If $k < l$, then $\mathfrak{P}^k$ is a disjoint union of q^{l-k} cosets of $\mathfrak{P}^l$.

For $h \in K$, the *translation operator* τ_h is defined by $\tau_h f(x) = f(x-h)$, whenever f is a function on K. For $k \in \mathbb{Z}$, let $\Phi_k = \mathbf{1}_{\mathfrak{P}^k}$. Note that the characteristic function of the ball $h + \mathfrak{P}^k$ is $\tau_h \Phi_k = \Phi_k(\cdot - h)$. It follows from Proposition 3.1.1 (b) that $\tau_h \Phi_k$ is constant on cosets of $\mathfrak{P}^k$.

The space of all finite linear combinations of functions of the form $\tau_h \Phi_k$, $h \in K$, $k \in \mathbb{Z}$, will be denoted by $\mathcal{S}$. The following two theorems describe the properties of functions in $\mathcal{S}$ and their Fourier transforms. For proofs of these statements, see Theorem 3.1 and Theorem 3.2, Chapter II in [109].

Theorem 3.1.2. *The function $g \in \mathcal{S}$ if and only if there exist integers k and l such that g is constant on cosets of $\mathfrak{P}^k$ and is supported on $\mathfrak{P}^l$.*

Theorem 3.1.3. *If $g \in \mathcal{S}$ is constant on cosets of $\mathfrak{P}^k$ and is supported on $\mathfrak{P}^l$, then $\hat{g} \in \mathcal{S}$ is constant on cosets of $\mathfrak{P}^{-l}$ and is supported on $\mathfrak{P}^{-k}$.*

3.1.2 Fourier Series on the Compact Abelian Group $\mathfrak{D}$

Let χ_u be any character on K^+. Since $\mathfrak{D}$ is a subgroup of K^+, the restriction $\chi_u|_{\mathfrak{D}}$ is a character on $\mathfrak{D}$. Also, as characters on $\mathfrak{D}$, $\chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. That is, $\chi_u = \chi_v$ if $u + \mathfrak{D} = v + \mathfrak{D}$ and $\chi_u \neq \chi_v$ if $(u+\mathfrak{D}) \cap (v + \mathfrak{D}) = \emptyset$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ is a list of distinct characters on $\mathfrak{D}$. It is proved in [109] that this list is complete.

Proposition 3.1.4. *Let $\{u(n) : n \in \mathbb{N}_0\}$ be a complete list of coset representatives of $\mathfrak{D}$ in K^+. Then $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ is a complete list of characters on $\mathfrak{D}$. Moreover, it is an orthonormal basis for $L^2(\mathfrak{D})$.*

Given such a list of characters $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$, we define the *Fourier coefficients* of $f \in L^1(\mathfrak{D})$ as

$$\hat{f}\big(u(n)\big) = \langle f, \chi_{u(n)} \rangle = \int_{\mathfrak{D}} f(x) \overline{\chi_{u(n)}(x)} dx, \quad n \in \mathbb{N}_0.$$

For $N \in \mathbb{N}$, let $S_N f$ be the Nth partial sum of the Fourier series of f:

$$S_N f(x) = \sum_{n=0}^{N-1} \langle f, \chi_{u(n)} \rangle \chi_{u(n)}(x).$$

The series $\sum_{n=0}^{\infty} \langle f, \chi_{u(n)} \rangle \chi_{u(n)}(x)$ is called the *Fourier series* of f at x.

For brevity, we will now write $\chi_n = \chi_{u(n)}|_{\mathfrak{D}}$ for $n \in \mathbb{N}_0$. With this notation, we have

$$S_N f(x) = \sum_{n=0}^{N-1} \langle f, \chi_n \rangle \chi_n(x). \tag{3.1.2}$$

From the standard L^2-theory for compact abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and *Parseval's identity* holds:

$$\int_{\mathfrak{D}} |f(x)|^2 \, dx = \sum_{n \in \mathbb{N}_0} |\hat{f}\big(u(n)\big)|^2.$$

Also, we have the *uniqueness of the Fourier coefficients*. That is, if $f \in L^1(\mathfrak{D})$ and $\hat{f}\big(u(n)\big) = 0$ for all $n \in \mathbb{N}_0$, then $f = 0$ a.e.

We now proceed to construct a sequence $\{u(n) : n \in \mathbb{N}_0\}$ which forms a complete list of coset representatives of $\mathfrak{D}$ in K^+. Note that $\Gamma = \mathfrak{D}/\mathfrak{P}$ is isomorphic to the finite field $GF(q)$ and $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{c-1}\} \subset \mathfrak{D}^*$ such that $\operatorname{span}\{\epsilon_j : j = 0, 1, \ldots, c-1\} \cong GF(q)$. For $n \in \mathbb{N}_0$ such that $0 \le n < q$, we have

$$n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \quad 0 \le a_k < p, \quad k = 0, 1, \ldots, c-1.$$

Define

$$u(n) = (a_0 + a_1 \epsilon_1 + \cdots + a_{c-1} \epsilon_{c-1}) \mathfrak{p}^{-1}. \tag{3.1.3}$$

Note that $\{u(n) : n = 0, 1, \ldots, q-1\}$ is a complete set of coset representatives of $\mathfrak{D}$ in $\mathfrak{P}^{-1}$ so that we can write

$$\mathfrak{P}^{-1} = \bigcup_{l=0}^{q-1} \big(u(l) + \mathfrak{D}\big). \tag{3.1.4}$$

Now, for $n \geq 0$, write

$$n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s, \quad 0 \leq b_k < q, \quad k = 0, 1, 2, \ldots, s,$$

and define

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{3.1.5}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But it follows that

$$u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s) \quad \text{if } r \geq 0, k \geq 0 \text{ and } 0 \leq s < q^k. \tag{3.1.6}$$

In the following proposition we list some properties of $\Lambda = \{u(n) : n \in \mathbb{N}_0\}$ which will be used later. We refer to [4] for a proof.

Proposition 3.1.5. *For $n \in \mathbb{N}_0$, let $u(n)$ be defined as in (3.1.3) and (3.1.5). Then*

(a) $u(n) = 0$ if and only if $n = 0$. If $k \geq 1$, then $|u(n)| = q^k$ if and only if $q^{k-1} \leq n < q^k$.

Moreover, if K is a local field of positive characteristic, then

(b) $\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\}$;

(c) for a fixed $l \in \mathbb{N}_0$, we have $\{u(l) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$.

In particular, $\Lambda = \{u(n) : n \in \mathbb{N}_0\}$ is a subgroup of K^+.

The *Dirichlet kernels* $D_n, n \in \mathbb{N}$, are defined by

$$D_n(x) = \sum_{k=0}^{n-1} \chi_k(x). \tag{3.1.7}$$

We will need the following property of the Dirichlet kernels for some specific values of n.

Lemma 3.1.6. *For $k \in \mathbb{N}_0$, we have $D_{q^k} = q^k \mathbf{1}_{\mathfrak{P}^k}$.*

Proof. See Page 86 in [109]. □

Let f be a locally integrable function in K. We say that $x \in K$ is a *regular point* of f if

$$q^k \int_{\{y:|x-y|\leq q^{-k}\}} f(y)\, dy = \frac{1}{|x+\mathfrak{P}^k|} \int_{x+\mathfrak{P}^k} f(y)\, dy \to f(x) \quad \text{as } k \to \infty.$$

The following result, called Lebesgue differentiation theorem, asserts that almost every point of a locally integrable function is a regular point. We refer to [109] for a proof of this theorem.

Theorem 3.1.7. *If f is a locally integrable function in K, then almost every $x \in K$ is a regular point of f.*

For $f \in L^1_{\text{loc}}(K)$, the *Hardy–Littlewood maximal function* Mf is defined by

$$Mf(x) = \sup_{k\in\mathbb{Z}} \frac{1}{|x+\mathfrak{P}^k|} \int_{x+\mathfrak{P}^k} |f(y)|\, dy, \quad x \in K. \tag{3.1.8}$$

For $1 < p < \infty$, let $L^p(K, w)$ denote the space of p-integrable functions on K with respect to the measure $w(x)\, dx$. Analogous to the Euclidean case, a weight w on K is said to be a Muckenhoupt A_p weight if

$$\sup_B \Big(\frac{1}{|B|} \int_B w(x)\, dx\Big) \Big(\frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}}\, dx\Big)^{p-1} < \infty. \tag{3.1.9}$$

Here the supremum is taken over all balls B in K. In this case we say that $w \in A_p$. We also say that $w \in A_1$ if there exists a constant $C > 0$ such that

$$Mw(x) \leq Cw(x) \quad \text{for a.e.}\, x \in K,$$

where M is the Hardy–Littlewood maximal operator defined in (3.1.8).

The weighted L^p-norm of any function $f \in L^p(K, w)$ and the corresponding maximal function Mf are comparable. For a proof of the following theorem, see [20] or [94].

Theorem 3.1.8. *Let M be the Hardy–Littlewood maximal operator on K and $w \in A_p$, $1 < p < \infty$. Then there exists a constant $C_p > 0$ such that*

$$\int_K Mf(x)^p w(x)\,dx \leq C_p \int_K |f(x)|^p w(x)\,dx \quad \text{for all } f \in L^P(K,w).$$

We define a variant of the maximal operator by putting

$$M_s f(x) = \big(M|f|^s(x)\big)^{\frac{1}{s}}, \quad 1 < s < \infty. \tag{3.1.10}$$

Since $|f|^s \leq M|f|^s$, if follows that

$$|f| \leq (M|f|^s)^{\frac{1}{s}} = M_s f. \tag{3.1.11}$$

Finally, for $f \in L^1_{\text{loc}}(K)$, we introduce the *Fefferman–Stein maximal function* $f^\sharp$ analogously as in $\mathbb{R}^n$ by setting

$$f^\sharp(x) = \sup_{k\in\mathbb{Z}} \frac{1}{|x+\mathfrak{P}^k|} \int_{x+\mathfrak{P}^k} |f(y) - f_{x+\mathfrak{P}^k}|\,dy, \quad x \in K,$$

where $f_B = \frac{1}{|B|}\int_B f(x)\,dx$ is the average of f over the ball B. We will also need the following result which relates the weighted L^p-norm of the Hardy–Littlewood maximal function Mf and the Fefferman–Stein maximal function $f^\sharp$.

Theorem 3.1.9. *Let $1 < p < \infty$ and $w \in A_p$. Then there exists a constant $C_p > 0$, depending only on p and w, such that*

$$\int_K Mf(x)^p w(x)\,dx \leq C_p \int_K f^\sharp(x)^p w(x)\,dx$$

for every $f \in L^p(K,w)$.

We skip the proof of the above result as it can be proved by following the same ideas as in the case of Euclidean spaces (see [43]).

3.2 Weighted Norm Inequalities for Fourier Series

In this section we extend Theorem 1.2.1 to the present setting of the ring of integers $\mathfrak{D}$ of a general local field K. Consider the maximal operator $\mathfrak{M}$ corresponding to the Fourier partial sum operators $\{S_n : n \in \mathbb{N}\}$:

$$\mathfrak{M}f(x) = \sup_n |S_n f(x)|, \quad f \in L^1(\mathfrak{D}).$$

Hunt and Taibleson [58] showed that the operator $\mathfrak{M}$ is bounded on $L^p(\mathfrak{D})$ and obtained the almost everywhere convergence of the Fourier series of functions in $L^p(\mathfrak{D})$ for $1 < p < \infty$. Here we are interested in a weighted version of their result. For $1 < p < \infty$, let $L^p(\mathfrak{D}, w)$ denote the space of p-integrable functions on $\mathfrak{D}$ with respect to the measure $w(x)\,dx$.

Recall that $\Lambda = \{u(n) : n \in \mathbb{N}_0\}$. A function f on K is said to be Λ-periodic if $f(x + u(n)) = f(x)$ for all $n \in \mathbb{N}_0$ and *a.e.* $x \in K$. The main result of this section is the following theorem.

Theorem 3.2.1. *Let K be a local field, w be a Λ-periodic weight on K and $1 < p < \infty$. If $w \in A_p$, then there exists a constant $C_p > 0$, depending only on w and p, such that*

$$\int_{\mathfrak{D}} \mathfrak{M}f(x)^p w(x)\,dx \le C_p \int_{\mathfrak{D}} |f(x)|^p w(x)\,dx \quad \textit{for all } f \in L^p(\mathfrak{D}, w). \tag{3.2.1}$$

Proof. Recall that the Dirichlet kernels D_n, $n \in \mathbb{N}$, are defined by

$$D_n(x) = \sum_{k=0}^{n-1} \chi_k(x).$$

Let Φ_0 be the characteristic function of $\mathfrak{D}$. For functions on $\mathfrak{D}$, we treat them as functions defined on K but supported on $\mathfrak{D}$. By this convention, we have $D_n = \Phi_0 D_n$ and $S_n f = f * D_n$, where the integration that defines the convolution is over all of K. In case of the Euclidean Fourier series, ordinary partial sums are studied in terms of the modified partial sums after replacing the Dirichlet kernel by modified Dirichlet kernel. The same program was also carried out by Taibleson's development of Fourier series in local fields (see [58, 108]). Following their footsteps, we define

$$\widetilde{D}_n = \overline{\chi}_n D_n \quad \text{and} \quad \widetilde{S}_n f = \widetilde{D}_n * f.$$

The functions $\widetilde{D}_n$, $n \in \mathbb{N}$, are called the *modified Dirichlet kernels.* If $x \in \mathfrak{D}$, then

$$\begin{aligned} S_n f(x) &= \int_{\mathfrak{D}} f(y) D_n(x-y)\, dy \\ &= \chi_n(x) \int_{\mathfrak{D}} f(y)\overline{\chi}_n(y)\overline{\chi}_n(x-y) D_n(x-y)\, dy \\ &= \chi_n(x) \int_{\mathfrak{D}} \big(\overline{\chi}_n f\big)(y)\widetilde{D}_n(x-y)\, dy. \end{aligned}$$

Therefore,

$$S_n f = \chi_n \cdot \widetilde{S}_n(\overline{\chi}_n f). \tag{3.2.2}$$

By our convention, $\widetilde{D}_n = \Phi_0 \widetilde{D}_n$ is the kernel of the operator $\widetilde{S}_n$. We denote this kernel by k_n. Note that k_n is supported on $\mathfrak{D}$.

Define the associated maximal operator $\mathfrak{M}^*$ by

$$\mathfrak{M}^* f(x) = \sup_n |\widetilde{S}_n f(x)|, \quad x \in \mathfrak{D}. \tag{3.2.3}$$

Therefore, in order to prove Theorem 3.2.1, it suffices to show that there exists $C_p > 0$ such that

$$\int_{\mathfrak{D}} \mathfrak{M}^* f(x)^p w(x)\, dx \le C_p \int_{\mathfrak{D}} |f(x)|^p w(x)\, dx \tag{3.2.4}$$

for all $f \in L^p(\mathfrak{D}, w)$.

It is proved in [58] (see also [109]) that $\mathfrak{M}^*$ is bounded on $L^p(\mathfrak{D})$. Using this fact and the reverse Hölder inequality for A_p weights (see [20]), it is easy to see that $\mathfrak{M}^* f \in L^p(\mathfrak{D}, w)$. Since each $\widetilde{S}_n f$ is supported on $\mathfrak{D}$, so is $\mathfrak{M}^* f$ and hence $\mathfrak{M}^* f$ also lies in $L^p(K, w)$.

We apply Theorem 3.1.8 and Lemma 3.1.9 to $\mathfrak{M}^* f$ and obtain

$$\|M(\mathfrak{M}^* f)\|_{L^p(K,w)} \le C \|\mathfrak{M}^* f\|_{L^p(K,w)}$$

and

$$\|M(\mathfrak{M}^* f)\|_{L^p(K,w)} \le C \|(\mathfrak{M}^* f)^\sharp\|_{L^p(K,w)}.$$

These two inequalities, along with the fact that $\mathfrak{M}^* f(x) \le M(\mathfrak{M}^* f)(x)$ for almost every x, yield

$$\int_{\mathfrak{D}} \mathfrak{M}^* f(x)^p w(x)\,dx \le C \int_K \big((\mathfrak{M}^* f)^\sharp(x)\big)^p w(x)\,dx.$$

Finally, in order to estimate the last integral, we dominate $(\mathfrak{M}^* f)^\sharp$ pointwise a.e. by the function $M_r f$ as follows.

Proposition 3.2.2. *Let $1 < r < \infty$. There is a constant $C > 0$, depending only on r, such that for any $f \in L^r(\mathfrak{D}, w)$, we have*

$$(\mathfrak{M}^* f)^\sharp(x) \le C M_r f(x) \quad \textit{for a.e. } x.$$

We postpone the proof of this proposition and continue with the proof of inequality (3.2.4). We recall an important property of the A_p weights (see [20]).

Lemma 3.2.3. *Suppose $1 < p < \infty$ and $w \in A_p$. Then there exists s with $1 < s < p$ such that $w \in A_s$.*

Taking the s obtained from Lemma 3.2.3, we choose $r = \frac{p}{s}$. Then $\frac{p}{r} = s > 1$ and $w \in A_{\frac{p}{r}}$. Hence, by Proposition 3.2.2, we have

$$\begin{aligned}
\int_{\mathfrak{D}} \mathfrak{M}^* f(x)^p w(x)\,dx &\le C \int_K \big((\mathfrak{M}^* f)^\sharp(x)\big)^p w(x)\,dx \\
&\le C_{r,p} \int_K M_r f(x)^p w(x)\,dx \\
&= C_{r,p} \int_K (M|f|^r(x))^{\frac{p}{r}} w(x)\,dx.
\end{aligned}$$

Since $|f|^r \in L^{\frac{p}{r}}(K, w)$ and $w \in A_{\frac{p}{r}}$, we apply Theorem 3.1.8 to obtain

$$\int_{\mathfrak{D}} \mathfrak{M}^* f(x)^p w(x)\,dx \le C \int_{\mathfrak{D}} |f(x)|^p w(x)\,dx.$$

Hence, inequality (3.2.4) is proved.

Therefore, to complete the proof of Theorem 3.2.1, all that remains is to prove Proposition 3.2.2.

We need the following lemmas. Recall that $\mathcal{S}$ is the space of all finite linear combinations of functions of the form $\tau_h \Phi_k$, $h \in K$, $k \in \mathbb{Z}$.

Lemma 3.2.4. *Let φ be a function in $\mathcal{S}$. If φ is constant on cosets of $\mathfrak{P}^{k+1}$ in $\mathfrak{P}^k \setminus \mathfrak{P}^{k+1}$ for all $k \in \mathbb{Z}$, then $\hat{\varphi}$ is constant on cosets of $\mathfrak{P}^{k+1}$ in $\mathfrak{P}^k \setminus \mathfrak{P}^{k+1}$ for all $k \in \mathbb{Z}$.*

Proof. Observe that the property φ is constant on cosets of $\mathfrak{P}^{k+1}$ in $\mathfrak{P}^k \setminus \mathfrak{P}^{k+1}$ for all $k \in \mathbb{Z}$ is equivalent to the statement that $\varphi(x+y) = \varphi(x)$ whenever $|x| > |y|$.

Since $\varphi \in \mathcal{S}$, there exists $N \in \mathbb{N}$ such that supp $\varphi \subseteq \mathfrak{P}^{-N}$. By Theorem 3.1.3, $\hat{\varphi}$ is constant on cosets of $\mathfrak{P}^N$. Hence, it is enough to prove the result for $\hat{\varphi}_l$, where $\varphi_l = \varphi \cdot \mathbf{1}_{\mathfrak{P}^{-l}\setminus\mathfrak{P}^{-l+1}}$ for all $l \in \mathbb{Z}$. Fix $l \in \mathbb{Z}$. Since φ_l is supported on $\mathfrak{P}^{-l}$ and constant on cosets of $\mathfrak{P}^{-l+1}$, by Theorem 3.1.3, $\hat{\varphi}_l$ is supported on $\mathfrak{P}^{l-1}$ and constant on cosets of $\mathfrak{P}^l$.

If $|x| > q^{-l+1}$, then $\hat{\varphi}_l(x) = 0$. Then for each y with $|y| < |x|$, we have $|x+y| = |x|$ so that $\hat{\varphi}_l(x+y) = 0$. Now, if $|x| \le q^{-l+1}$ and $|y| < |x|$, then $|y| \le q^{-l}$. Since $x \in \mathfrak{P}^{l-1}$, we have $x \in \mathfrak{P}^l + a$ for some $a \in K$. So $\mathfrak{P}^l + a = B(x, q^{-l})$, by Proposition 3.1.1 (a). Hence, $|x+y-x| = |y| \le q^{-l}$ so that $x+y \in B(x, q^{-l}) = \mathfrak{P}^l + a$. This shows that $x+y$ and x are in the same coset of $\mathfrak{P}^l$. Hence, $\hat{\varphi}_l(x+y) = \hat{\varphi}_l(x)$. □

We derive an important property of the kernel k_n which will be needed in the proof of Proposition 3.2.2.

Lemma 3.2.5. *Let $y \in \mathfrak{D}$ such that $|y| < 1$. Then the functions $\tau_y k_n$ and k_n agree at $x \in \mathfrak{D}$ if $|y| < |x|$.*

Proof. We first write k_n as follows:

$$k_n = \Phi_0 \overline{\chi}_n D_n = \Phi_0 \overline{\chi}_n \sum_{m=0}^{n-1} \chi_m = \sum_{m=0}^{n-1} \Phi_0 \chi_{u(m)-u(n)}.$$

Taking Fourier transform and observing that $(\Phi_0 \chi_y)^\wedge = \tau_y \Phi_0$, we get

$$\widehat{k}_n = \sum_{m=0}^{n-1} \tau_{u(m)-u(n)} \Phi_0 = \sum_{m=0}^{n-1} \mathbf{1}_{\mathfrak{D}+u(m)-u(n)}.$$

That is, $\widehat{k}_n$ is the characteristic function of the union of n disjoint cosets

$$\{\mathfrak{D} + u(m) - u(n) : m = 0, 1, \ldots, n-1\} \tag{3.2.5}$$

of $\mathfrak{D}$. Let $x, y \in \mathfrak{D}$ be such that $|y| < |x|$. Then from (3.1.1), we have $|x + y| = |x|$.

We claim that $\widehat{k}_n(x + y) = \widehat{k}_n(x)$. Once we prove this, it is easy to see that the functions $\tau_y k_n$ and k_n agree at x. Indeed, applying Lemma 3.2.4 to $\widehat{k}_n$ and observing that $\widehat{\widehat{k}}_n$ is the reflection of k_n, we get $k_n(x + y) = k_n(x)$ whenever $|y| < |x|$. Hence, $\tau_y k_n(x) = k_n(x)$ whenever $|y| < |x|$.

Let us note that $q^k \leq n \leq q^{k+1} - 1$ for some $k \in \mathbb{N}_0$. Now we prove the claim. This is equivalent to show that the union of the cosets in (3.2.5) either contains both x and $x + y$ or neither. We will prove this by induction on k.

If $k = 0$, then $1 \leq n \leq q - 1$. Hence, $\{\mathfrak{D} + u(m) - u(n) : m = 0, 1, \ldots, n - 1\}$ consists of n distinct cosets of $\mathfrak{D}$ in $\mathfrak{P}^{-1}$, since $u(m) \neq u(n)$ and $|u(m) - u(n)| = q$ for $m = 0, 1, \ldots, n - 1$. If $|x| \leq 1$, then $|x + y| \leq 1$ so that both $x + y$ and x are in $\mathfrak{D}$ and hence $\widehat{k}_n(x + y) = \widehat{k}_n(x) = 0$. Similarly, if $|x| > q$, then $|x + y| > q$ and again $\widehat{k}_n(x + y) = \widehat{k}_n(x) = 0$. If $|x| = q$, then $|y| \leq 1$. Suppose $x \in \mathfrak{D} + u(m) - u(n)$ for some $m = 0, 1, \ldots, n - 1$. Then $\mathfrak{D} + u(m) - u(n) = B(x, 1)$. Now, $|x + y - x| = |y| \leq 1$ so that $x + y \in B(x, 1)$. Hence, both x and $x + y$ are in the same coset of $\mathfrak{D}$. So the induction hypothesis is true for $k = 0$.

Now, assume that the assertion holds for $n < q^k$. We will prove it for all n such that $q^k \leq n \leq q^{k+1} - 1$. We write

$$n = rq^k + s, \quad 1 \leq r \leq q - 1, \quad 0 \leq s \leq q^k - 1.$$

Then $u(n) = u(rq^k) + u(s)$, by (3.1.6). If $0 \leq m \leq rq^k - 1$, then

$$m = lq^k + t, \quad 0 \leq l \leq r - 1, \quad 0 \leq t \leq q^k - 1$$

so that $u(m) = u(lq^k) + u(t)$. Hence,

$$u(m) - u(n) = \Big(u(lq^k) - u(rq^k)\Big) + \Big(u(t) - u(s)\Big).$$

If $rq^k \leq m \leq rq^k + s - 1$, then $m = rq^k + \nu$, $0 \leq \nu \leq s - 1$ and $u(m) - u(n) = u(rq^k) + u(\nu) - u(rq^k) - u(s) = u(\nu) - u(s)$. Therefore, the union of the cosets in (3.2.5) is the union of

$$\bigcup_{l=0}^{r-1}\bigcup_{t=0}^{q^k-1}\Big(\mathfrak{D}+u(lq^k)-u(rq^k)+u(t)-u(s)\Big) \tag{3.2.6}$$

and

$$\bigcup_{\nu=0}^{s-1}\big(\mathfrak{D}+u(\nu)-u(s)\big). \tag{3.2.7}$$

Since $s<q^k$, the cosets in (3.2.7) satisfy the induction hypothesis. Hence, both x and $x+y$ belong to this union or neither does. For the cosets in (3.2.6), we observe that $|u(t)-u(s)|\le q^k$ for $t=0,1,\dots,q^k-1$ so that $\bigcup_{t=0}^{q^k-1}\big(\mathfrak{D}+u(t)-u(s)\big)=\mathfrak{P}^{-k}$. Hence, the union in (3.2.6) is

$$\bigcup_{l=0}^{r-1}\Big(\mathfrak{P}^{-k}+u(lq^k)-u(rq^k)\Big).$$

This is a union of r cosets of $\mathfrak{P}^{-k}$ in $\mathfrak{P}^{-k-1}\setminus\mathfrak{P}^{-k}$. If x is in any of these r cosets, then $|x|=q^{k+1}$ and if $|y|<|x|$, then $|y|\le q^k$. As in the case of $k=0$, it follows that x and $x+y$ are in the same coset of $\mathfrak{P}^{-k}$. The induction is complete. □

We are now ready to prove Proposition 3.2.2. We begin with the observation that

$$\frac{1}{2}\|f^\sharp\|_\infty\le\sup_B\inf_{\alpha\in\mathbb{C}}\frac{1}{|B|}\int_B|f(x)-\alpha|\,dx. \tag{3.2.8}$$

Indeed, first we note that, for all $\alpha\in\mathbb{C}$,

$$\int_B|f(x)-f_B|\,dx\le\int_B|f(x)-\alpha|\,dx+\int_B|\alpha-f_B|\,dx\le 2\int_B|f(x)-\alpha|\,dx.$$

Dividing both sides by $|B|$, taking the infimum over $\alpha\in\mathbb{C}$ and then supremum over all balls B, we get (3.2.8).

Now, fix $r>1$ and assume $f\in L^r(\mathfrak{D},w)$. Let $x\in K$ and B be any ball containing x. In view of (3.2.8), it is enough to show that there exists a constant α depending on the ball B such that

$$\frac{1}{|B|}\int_B|\mathfrak{M}^*f(y)-\alpha|\,dy\le CM_rf(x). \tag{3.2.9}$$

By Proposition 3.1.1 (a), $B = \{y \in K : |y - x| < q^l\}$ for some $l \in \mathbb{Z}$. We write $f_1 = f \cdot \mathbf{1}_B$ and $f_2 = f - f_1$ and set $\alpha = \mathfrak{M}^* f_2(x)$. Then for any $y \in B$,

$$\begin{aligned} |\mathfrak{M}^* f(y) - \mathfrak{M}^* f_2(x)| &= \Big|\sup_n |\widetilde{S}_n f(y)| - \sup_n |\widetilde{S}_n f_2(x)|\Big| \\ &\leq \sup_n |\widetilde{S}_n f(y) - \widetilde{S}_n f_2(x)| \\ &\leq \sup_n |\widetilde{S}_n f_1(y)| + \sup_n |\widetilde{S}_n f_2(y) - \widetilde{S}_n f_2(x)| \\ &= \mathfrak{M}^* f_1(y) + \sup_n |\widetilde{S}_n f_2(y) - \widetilde{S}_n f_2(x)|. \end{aligned} \tag{3.2.10}$$

For the first term of (3.2.10), we use Hölder's inequality and the boundedness of $\mathfrak{M}^*$ on $L^r(\mathfrak{D})$ to get

$$\begin{aligned} \frac{1}{|B|}\int_B \mathfrak{M}^* f_1(y)\,dy &\leq \frac{1}{q^{l-1}}\Big(\int_{|y-x|<q^l} \mathfrak{M}^* f_1(y)^r\,dy\Big)^{\frac{1}{r}} \cdot (q^{l-1})^{\frac{1}{r'}} \\ &\leq \frac{C}{q^{l-1}}\Big(\int_{|y-x|<q^l} |f(y)|^r\,dy\Big)^{\frac{1}{r}} \cdot (q^{l-1})^{\frac{1}{r'}} \\ &\leq CM_r f(x). \end{aligned}$$

Now, integrating the second term of (3.2.10) over the ball B, we get

$$\begin{aligned} &\frac{1}{|B|}\int_B \sup_n |\widetilde{S}_n f_2(y) - \widetilde{S}_n f_2(x)|\,dy \\ &= \frac{1}{q^{l-1}}\int_{|x-y|<q^l} \sup_n \Big|\int_K \big(k_n(y-z) - k_n(x-z)\big) f_2(z)\,dz\Big|\,dy \\ &\leq \frac{1}{q^{l-1}}\int_{|x-y|\leq q^{l-1}} \sup_n \int_{|x-z|\geq q^l} \big|\tau_{x-y}k_n(x-z) - k_n(x-z)\big||f_2(z)|\,dz\,dy. \end{aligned}$$

Since $|x - z| \geq q^l > q^{l-1} \geq |x - y|$ and k_n is supported on $\mathfrak{D}$, by Lemma 3.2.5, we have $\tau_{x-y}k_n(x - z) = k_n(x - z)$. From this we conclude that the right hand side of the above inequality is zero. Hence, (3.2.9) holds with $\alpha = \mathfrak{M}^* f_2(x)$. This completes the proof of Proposition 3.2.2, and hence, we have also proved Theorem 3.2.1. □

Since $|S_n f(x)|$ is dominated by $\mathfrak{M}f(x)$ for all $n \in \mathbb{N}$, an immediate consequence of Theorem 3.2.1 is the uniform boundedness of the Fourier partial sum operators $S_n, n \in \mathbb{N}$, in the weighted spaces $L^p(\mathfrak{D}, w)$.

Theorem 3.2.6. *Let K be a local field, w be a Λ-periodic weight function on K and $1 < p < \infty$. Then $w \in A_p$ if and only if there is a positive constant $C_{p,w}$, depending* *only on w and p, such that for every $f \in L^p(\mathfrak{D}, w)$,*

$$\int_{\mathfrak{D}} |S_n f(x)|^p w(x)\, dx \leq C_{p,w} \int_{\mathfrak{D}} |f(x)|^p w(x)\, dx \tag{3.2.11}$$

for all $n \in \mathbb{N}$. In other words, $w \in A_p$ if and only if the Fourier partial sum operators S_n, $n \in \mathbb{N}$, are uniformly bounded on the weighted space $L^p(\mathfrak{D}, w)$.

Proof. The proof of sufficiency part follows from Theorem 3.2.1. So, we only prove the necessity part. First we observe that, it is enough to show that inequality (3.1.9) holds for all balls B with $|B| \leq 1$. In fact, let B be any ball with $|B| > 1$, then $|B| = q^k$ for some $k \geq 1$. Hence, by Proposition 3.1.1 (d), B can be written as a disjoint union of q^k cosets of $\mathfrak{D}$ as $B = \bigcup_{i=1}^{q^k} (u(l_i) + \mathfrak{D})$, where $l_i \in \mathbb{N}_0$. We observe that

$$\frac{1}{|B|} \int_B w(x)\, dx = \frac{1}{|B|} \sum_{i=1}^{q^k} \int_{\mathfrak{D}} w(x)\, dx = \int_{\mathfrak{D}} w(x)\, dx,$$

since w is Λ-periodic. This reduces to the case when $|B| = 1$. Therefore, we assume that $|B| \leq 1$. Then, $|B| = q^{-r}$ for some $r \in \mathbb{N}_0$ and $B \subset u(l) + \mathfrak{D}$ for some $l \in \mathbb{N}_0$. Let f be a nonnegative function on B and 0 on $\big(u(l) + \mathfrak{D}\big) \setminus B$. Extend f to K Λ-periodically. Now, for any $x \in B$, we have $B = x + \mathfrak{P}^r$. If $y \in B$, then $y - x \in \mathfrak{P}^r$. Using the fact that $D_{q^r} = q^r \mathbf{1}_{\mathfrak{P}^r}$, $r \geq 0$ (see Lemma 3.1.6), we get

$$S_{q^r} f(x) = \int_B f(y) D_{q^r}(x - y)\, dy = \int_B f(y) q^r\, dy = \frac{1}{|B|} \int_B f(y)\, dy.$$

Hence, by (3.2.11), we get

$$\Big(\int_B w(x)\, dx\Big) \Big(\frac{1}{|B|} \int_B f(y)\, dy\Big)^p \leq C \int_B |f(x)|^p w(x)\, dx.$$

Then, by a standard argument as in page 247 in [59], it can be shown that $w \in A_p$. We omit the details. □

Remark 3.2.7. *Few comments are in order regarding the method we have used to prove the above results. Hunt and Young [60] proved the boundedness of the Fourier maximal operator M on $L^p(\mathbb{T}, w)$ by verifying a Burkholder–Gundy [10] type distribution function inequality. This inequality relates the weighted distribution functions of modified versions of $\mathsf{M}f$ and the Hardy–Littlewood maximal function of f. Following* *their method, Gosselin obtained a similar result for Vilenkin–Fourier series in [45]. In addition, the proof in [45] is based on a joint distribution inequality similar to those in [11] and [21]. A Vilenkin group G is a direct product of cyclic groups of order p_i, where each p_i is an integer greater than or equal to 2. In particular, if we take each p_i to be equal to a prime p, then G becomes the ring of integers of the p-series field $\mathbb{F}_p((X))$ which is a special case of a local field of positive characteristic. On the other hand, the ring of integers of a p-adic field $\mathbb{Q}_p$ is not a Vilenkin group and the fields $\mathbb{Q}_p$ are example of local fields of characteristic zero. It is therefore natural to look for an analogue of the Hunt–Young result on the ring of integers of a local field which are not included in Gosselin's result. Our result (Theorem 3.2.1) is on the ring of integers $\mathfrak{D}$ of a general local field which, of course, includes $\mathbb{Q}_p$ and $\mathbb{F}_p((X))$ as special cases. The Walsh–Paley or dyadic group 2^ω can be identified with the additive group of the ring of integers of the field $\mathbb{F}_2((X))$. Hence, our result is also valid for the Walsh–Paley group. It is also applicable to the p-adic fields $\mathbb{Q}_p$ which are not included in Gosselin's result. We would also like to remark that our approach is different from those of [60] and [45] and hence it also gives another proof of the boundedness of $\mathfrak{M}$ for the case of Vilenkin groups which are products of cyclic groups of the same order p.*

3.3 Applications

The purpose of this section is to exhibit several applications of Theorem 3.2.6 to the theory of Schauder bases on shift-invariant spaces and to Gabor theory. We begin by recalling the definition of Schauder bases and some other relevant notions.

Definition 3.3.1. *A sequence $\{x_k : k \in \mathbb{N}_0\}$ of elements of a Banach space $\mathbb{B}$ is called a* Schauder basis *for $\mathbb{B}$ if for every $x \in \mathbb{B}$ there exists a unique sequence $\{\alpha_k : k \in \mathbb{N}_0\}$ of scalars such that*

$$x = \sum_{k\in\mathbb{N}_0} \alpha_k x_k,$$

where the partial sums of the series converge in the norm of $\mathbb{B}$, that is,

$$\lim_{N\to\infty} \Big\| x - \sum_{k=0}^{N} \alpha_k x_k \Big\| = 0.$$

Let $\{x_n : n \in \mathbb{N}_0\}$ be a sequence in a Hilbert space $\mathbb{H}$. A sequence $\{\tilde{x}_n : n \in \mathbb{N}_0\}$ in $\mathbb{H}$ is said to be *biorthogonal* to $\{x_n : n \in \mathbb{N}_0\}$ if $\langle x_k, \tilde{x}_l \rangle = \delta_{k,l}$ for all $k, l \in \mathbb{N}_0$. It is easy to verify that if $\{x_n : n \in \mathbb{N}_0\}$ is complete in $\mathbb{H}$, that is, if $\overline{\text{span}}\{x_n : n \in \mathbb{N}_0\} = \mathbb{H}$, then there is a unique sequence $\{\tilde{x}_n : n \in \mathbb{N}_0\} \subset \mathbb{H}$ which is biorthogonal to $\{x_n : n \in \mathbb{N}_0\}$. Such a sequence is called the *biorthogonal dual* of $\{x_n : n \in \mathbb{N}_0\}$. Every Schauder basis has a unique biorthogonal dual.

3.3.1 Schauder Bases on Shift-Invariant Spaces

Let K be a local field, Γ be a countable set in K, and $\varphi \in L^2(K)$. Define

$$V(\varphi, \Gamma) = \overline{\text{span}}\{\varphi(\cdot - \gamma) : \gamma \in \Gamma\},$$

the closure in $L^2(K)$ of the finite linear combinations of translates of φ by elements of Γ.

Let us consider the following problem. **When does the system of translates $\{\varphi(\cdot - \gamma) : \gamma \in \Gamma\}$ form an orthonormal basis/Schauder basis for the space $V(\varphi, \Gamma)$?**

If $K = \mathbb{Q}_p$, the field of p-adic numbers, then we can answer this question using Fuglede's conjecture on $\mathbb{Q}_p$.

Theorem 3.3.2. *A Borel set Ω of positive and finite Haar measure in $\mathbb{Q}_p$ is a spectral set if and only if it tiles $\mathbb{Q}_p$ by translations.*

We say that Ω tiles $\mathbb{Q}_p$ by translations if there exists a set $T \subset \mathbb{Q}_p$ such that $\sum_{t \in T} \mathbf{1}_\Omega(x - t) = 1$ for a.e. $x \in \mathbb{Q}_p$, where $\mathbf{1}_\Omega$ is the characteristic function of Ω. The set Ω is said to be a spectral set if there exists a set $S \subset \mathbb{Q}_p$ such that $\{\chi_s : s \in S\}$ is an orthonormal basis for $L^2(\Omega)$. Theorem 3.3.2 was recently proved by Fan, Fan, Liao, and Shi [34]. Using this theorem we can prove the following result.

Theorem 3.3.3. *Let Ω be a Borel set of positive and finite Haar measure in $\mathbb{Q}_p$. Let $\varphi \in L^2(\mathbb{Q}_p)$ with $\operatorname{supp} \hat{\varphi} \subseteq \Omega$ and $|\hat{\varphi}| = 1$ on Ω a.e. If Ω tiles $\mathbb{Q}_p$ by translations, then there exists $\Gamma \subset \mathbb{Q}_p$ such that $\{\varphi(\cdot - \gamma) : \gamma \in \Gamma\}$ forms an orthonormal basis for $V(\varphi, \Gamma)$. The converse is also true.*

Fuglede's conjecture is still open for general local fields, in particular for local fields of positive characteristic. Nevertheless, for the set $\Gamma = \{u(k) : k \in \mathbb{N}_0\}$, we characterize the Schauder basis property of $V(\varphi, \Gamma)$ in terms of A_2 weights in $\mathfrak{D}$. This extends the result from the real line obtained in [90].

Theorem 3.3.4. *Let K be a local field of positive characteristic, $\Gamma = \{u(k) : k \in \mathbb{N}_0\}$, and $\varphi \in L^2(K)$. Then, the family $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is a Schauder basis for $V(\varphi, \Gamma)$ if and only if $w_\varphi \in A_2(\mathfrak{D})$, where $w_\varphi(\xi) = \sum_{n=0}^{\infty} |\hat{\varphi}(\xi + u(n))|^2$.*

Note that w_φ is Λ-periodic. Indeed, by Proposition 3.1.5 (c), for $l \in \mathbb{N}_0$, we have

$$w_\varphi(\xi + u(l)) = \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(l) + u(k))|^2 = \sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = w_\varphi(\xi)$$

for a.e. $\xi \in K$. In order to prove this theorem, we need some preliminary results. For $\Gamma = \{u(k) : k \in \mathbb{N}_0\}$, we write $V(\varphi, \Gamma)$ as V_φ. That is $V_\varphi = \overline{\operatorname{span}}\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$.

Consider the map $J_\varphi : L^2(\mathfrak{D}, w_\varphi) \to V_\varphi$ defined by $J_\varphi f = (f\hat{\varphi})^\vee$, where $f^\vee$ is the inverse Fourier transform of f. It can be shown that the map J_φ is an isometry. For a proof of this fact, we refer to [3]. Note that $(J_\varphi \overline{\chi}_k)^\wedge = \overline{\chi}_k \hat{\varphi} = [\varphi(\cdot - u(k))]^\wedge$ so that J_φ maps $\overline{\chi}_k$ to $\varphi(\cdot - u(k))$. Thus, various properties of $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ on V_φ correspond to similar properties of the system $\{\chi_k : k \in \mathbb{N}_0\}$ on $L^2(\mathfrak{D}, w_\varphi)$. Therefore, our original problem is equivalent to the problem of finding conditions on w_φ so that $\{\chi_k : k \in \mathbb{N}_0\}$ forms a Schauder basis for $L^2(\mathfrak{D}, w_\varphi)$.

Let $\mathbb{H} = V_\varphi$. Suppose there exists $\tilde{\varphi} \in V_\varphi$ such that $\langle \varphi(\cdot - u(k)), \tilde{\varphi} \rangle = \delta_{k,0}$ for all $k \in \mathbb{N}_0$. Now,

$$\langle \varphi(\cdot - u(k)), \tilde{\varphi}(\cdot - u(l)) \rangle = \langle \varphi(\cdot - (u(k) - u(l))), \tilde{\varphi} \rangle. \tag{3.3.1}$$

If $k = l$, then the above inner product is equal to $\langle \varphi, \tilde{\varphi} \rangle = 1$. If $k \neq l$, then $0 \neq u(k) - u(l) = u(m)$ for some $m \in \mathbb{N}_0$, by Proposition 3.1.5. Hence, $m \neq 0$. So, the inner product is equal to $\delta_{m,0} = 0$. Thus, $\langle \varphi(\cdot - u(k)), \tilde{\varphi}(\cdot - u(l)) \rangle = \delta_{k,l}$. That is, if there exists $\tilde{\varphi} \in V_\varphi$ such that $\langle \varphi(\cdot - u(k)), \tilde{\varphi} \rangle = \delta_{k,0}$ for all $k \in \mathbb{N}_0$, then $\{\tilde{\varphi}(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is a biorthogonal dual of $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$. The function $\tilde{\varphi}$ will then be called the *canonical dual function* to φ. As above, if it exists, then it is unique. We will need the following result. For a proof, we refer to [104] (see Theorem 4.1, Chapter 1).

Lemma 3.3.5. *A complete sequence $\{x_n : n \in \mathbb{N}_0\}$ with biorthogonal dual $\{\tilde{x}_n : n \in \mathbb{N}_0\}$ is a Schauder basis for $\mathbb{H}$ if and only if the partial sum operators*

$$s_n(x) = \sum_{k=0}^{n-1} \langle x, \tilde{x_k} \rangle x_k, \quad n \in \mathbb{N},$$

are uniformly bounded in $\mathbb{H}$.

The following result provides a necessary and sufficient condition for the existence of a canonical dual.

Proposition 3.3.6. *Let $\varphi \in L^2(K)$. There exists a canonical dual $\tilde{\varphi}$ of φ in V_φ if and only if $\frac{1}{w_\varphi} \in L^1(\mathfrak{D})$. In this case, $\tilde{\varphi} = (\frac{1}{w_\varphi}\hat{\varphi})^\vee$.*

Proof. Since the map $J_\varphi : L^2(\mathfrak{D}, w_\varphi) \to V_\varphi$ is an isometry, $\tilde{\varphi} \in V_\varphi$ if and only if there exists a unique m in $L^2(\mathfrak{D}, w_\varphi)$ such that $\widehat{\tilde{\varphi}} = m\hat{\varphi}$. Moreover, to be a canonical dual, $\tilde{\varphi}$ must satisfy $\langle \varphi(\cdot - u(k)), \tilde{\varphi} \rangle = \delta_{k,0}$ for all $k \in \mathbb{N}_0$. But

$$\begin{aligned}
\langle \varphi(\cdot - u(k)), \tilde{\varphi} \rangle &= \int_K \hat{\varphi}(\xi)\overline{\chi_k(\xi)}\overline{\widehat{\tilde{\varphi}}(\xi)}\, d\xi \\
&= \int_K \overline{m(\xi)}|\hat{\varphi}(\xi)|^2\overline{\chi_k(\xi)}\, d\xi \\
&= \int_{\mathfrak{D}} \overline{m(\xi)}w_\varphi(\xi)\overline{\chi_k(\xi)}\, d\xi \\
&= (\overline{m}w_\varphi)^\wedge(u(k)).
\end{aligned}$$

Thus, the kth Fourier coefficient of $\overline{m}w_\varphi$ is equal to $\delta_{k,0}$ for all $k \in \mathbb{N}_0$. This will happen if and only if $\overline{m}w_\varphi = 1$ for a.e. $\xi \in \mathfrak{D}$. Since w_φ is real-valued, we have $\overline{m} = m = \frac{1}{w_\varphi}$. Finally, $m = \frac{1}{w_\varphi} \in L^2(\mathfrak{D}, w_\varphi)$ if and only if $\int_{\mathfrak{D}} \frac{1}{w_\varphi^2(\xi)} w_\varphi(\xi)\, d\xi = \int_{\mathfrak{D}} \frac{1}{w_\varphi(\xi)}\, d\xi < \infty$ if and only if $\frac{1}{w_\varphi} \in L^1(\mathfrak{D})$. □

We are now ready to prove Theorem 3.3.4.

Proof of Theorem 3.3.4. Let $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ be a Schauder basis for V_φ. Then, since J_φ is an isometry, $\{\chi_k : k \in \mathbb{N}_0\}$ is a Schauder basis for $L^2(\mathfrak{D}, w_\varphi)$. Let $\{z_k : k \in \mathbb{N}_0\}$ be the biorthogonal dual of $\{\chi_k : k \in \mathbb{N}_0\}$ in $L^2(\mathfrak{D}, w_\varphi)$. By Proposition 3.3.6, $\frac{1}{w_\varphi} \in L^1(\mathfrak{D})$. In particular, $w_\varphi > 0$ a.e. We have,

$$\delta_{k,l} = \langle \chi_k, z_l \rangle_{L^2(\mathfrak{D}, w_\varphi)} = \int_{\mathfrak{D}} \chi_k(\xi)\overline{z_l}(\xi)w_\varphi(\xi)\, d\xi = \int_{\mathfrak{D}} z_l(\xi)w_\varphi(\xi)\overline{\chi_k}(\xi)\, d\xi.$$

Hence, the function $z_l w_\varphi$ has all but the lth Fourier coefficient are zero. By the uniqueness of Fourier coefficients, we have $z_l w_\varphi = \chi_l$, $l \in \mathbb{N}_0$. An easy computation shows that for any $f \in L^2(\mathfrak{D}, w_\varphi)$, we have

$$\langle f, z_k \rangle_{L^2(\mathfrak{D}, w_\varphi)} = \langle f, \chi_k \rangle_{L^2(\mathfrak{D})}. \tag{3.3.2}$$

Now, we define

$$s_n f = \sum_{k=0}^{n-1} \langle f, z_k \rangle_{L^2(\mathfrak{D}, w_\varphi)} \chi_k.$$

Since $\{\chi_k : k \in \mathbb{N}_0\}$ is a Schauder basis for $L^2(\mathfrak{D}, w_\varphi)$ with biorthogonal dual $\{z_k : k \in \mathbb{N}_0\}$, by Lemma 3.3.5, the partial sum operators $s_n : L^2(\mathfrak{D}, w_\varphi) \to L^2(\mathfrak{D}, w_\varphi), n \in \mathbb{N}$, are uniformly bounded. But from (3.3.2), we that $s_n f = S_n f$, the usual partial sums of the Fourier series of f. Hence, $S_n : L^2(\mathfrak{D}, w_\varphi) \to L^2(\mathfrak{D}, w_\varphi)$ are uniformly bounded. Therefore, by Theorem 3.2.6, it follows that $w_\varphi \in A_2(\mathfrak{D})$.

Conversely, suppose that $w_\varphi \in A_2(\mathfrak{D})$. Then $w_\varphi > 0$ a.e. and $\frac{1}{w_\varphi} \in L^1(\mathfrak{D})$. Hence, $\{z_k = \frac{\chi_k}{w_\varphi} : k \in \mathbb{N}_0\}$ is the biorthogonal dual of the complete system $\{\chi_k : k \in \mathbb{N}_0\}$. Again, since $\langle f, z_k \rangle_{L^2(\mathfrak{D}, w_\varphi)} = \langle f, \chi_k \rangle_{L^2(\mathfrak{D})}$, we see that the operator s_n coincides with the Fourier partial sum operator S_n. By Theorem 3.2.6, $S_n : L^2(\mathfrak{D}, w_\varphi) \to L^2(\mathfrak{D}, w_\varphi)$ are uniformly bounded. Hence, $s_n : L^2(\mathfrak{D}, w_\varphi) \to L^2(\mathfrak{D}, w_\varphi)$ are also uniformly bounded. Again, by Lemma 3.3.5, $\{\chi_k : k \in \mathbb{N}_0\}$ is a Schauder basis for $L^2(\mathfrak{D}, w_\varphi)$ which, in turn, shows that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is a Schauder basis for V_φ. □

We conclude this section with a proof of Theorem 3.3.3.

Proof of Theorem 3.3.3. Suppose Ω tiles $\mathbb{Q}_p$ by translations. Then, by Theorem 3.3.2, Ω is a spectral set. That is, there is a subset Γ of $\mathbb{Q}_p$ such that $\{\chi_\gamma : \gamma \in \Gamma\}$ forms an orthonormal basis for $L^2(\Omega)$. We have to show that the system of translates $\{\varphi(\cdot - \gamma) : \gamma \in \Gamma\}$ forms an orthonormal basis for $V(\varphi, \Gamma)$.

Let $f \in V(\varphi, \Gamma)$. Observe that the Fourier transform $\hat{f}$ of f is supported on Ω and hence $\hat{f} \in L^2(\Omega)$. Since $|\hat{\varphi}| = 1$ on Ω, $\hat{f}/\hat{\varphi} \in L^2(\Omega)$. Now, since $\{\chi_\gamma : \gamma \in \Gamma\}$ is an orthonormal basis for $L^2(\Omega)$, we have $\frac{\hat{f}}{\hat{\varphi}} = \sum_{\gamma \in \Gamma} \langle \frac{\hat{f}}{\hat{\varphi}}, \overline{\chi}_\gamma \rangle \overline{\chi}_\gamma$. Using $|\hat{\varphi}|^2 = 1$ and

$[\varphi(\cdot-\gamma)]^\wedge = \hat{\varphi}\overline{\chi}_\gamma$, we get

$$\frac{\hat{f}}{\hat{\varphi}} = \sum_{\gamma\in\Gamma}\langle \hat{f}, \tfrac{1}{\hat{\varphi}}\overline{\chi}_\gamma\rangle \overline{\chi}_\gamma = \sum_{\gamma\in\Gamma}\langle \hat{f}, \hat{\varphi}\overline{\chi}_\gamma\rangle \overline{\chi}_\gamma = \sum_{\gamma\in\Gamma}\langle f, \varphi(\cdot-\gamma)\rangle \overline{\chi}_\gamma.$$

Hence,

$$\hat{f} = \sum_{\gamma\in\Gamma}\langle f, \varphi(\cdot-\gamma)\rangle \hat{\varphi}\overline{\chi}_\gamma = \sum_{\gamma\in\Gamma}\langle f, \varphi(\cdot-\gamma)\rangle [\varphi(\cdot-\gamma)]^\wedge.$$

Therefore,

$$f = \sum_{\gamma\in\Gamma}\langle f, \varphi(\cdot-\gamma)\rangle \varphi(\cdot-\gamma).$$

To see the orthonormality of $\{\varphi(\cdot-\gamma) : \gamma\in\Gamma\}$, we observe that, for $\gamma,\lambda\in\Gamma$, we have

$$\langle \varphi(\cdot-\gamma), \varphi(\cdot-\lambda)\rangle = \langle \hat{\varphi}\overline{\chi}_\gamma, \hat{\varphi}\overline{\chi}_\lambda\rangle = \langle \overline{\chi}_\gamma, \overline{\chi}_\lambda\rangle = \delta_{\gamma,\lambda}.$$

Here, we have used Parseval's identity and the fact that $|\hat{\varphi}|^2 = 1$ on Ω.

Conversely, suppose there is a countable set Γ for which $\{\varphi(\cdot-\gamma) : \gamma\in\Gamma\}$ forms an orthonormal basis for $V(\varphi,\Gamma)$. Then, by a similar argument as above, $\{\chi_\gamma : \gamma\in\Gamma\}$ forms an orthonormal basis for $L^2(\Omega)$ and then by Theorem 3.3.2, we conclude that Ω tiles $\mathbb{Q}_p$ by translations. □

3.3.2 Characterization of Schauder Basis Property of Gabor Systems

Let K be a local field of positive characteristic. Fix $a,b\in K$ and $g\in L^2(K)$. The *Gabor system* $\mathcal{G}(g,a,b)$ is the collection of functions

$$\mathcal{G}(g,a,b) = \{M_{bu(n)}T_{au(k)}g : n,k\in\mathbb{N}_0\}$$

where $T_y f(x) = f(x-y)$ and $M_\xi f(x) = \chi_\xi(x)f(x)$, $y,\xi\in K$, are the usual *translation* and *modulation* operators, respectively. The operators T_y and M_ξ are also called the time shifts and frequency shifts. Their compositions T_yM_ξ and $M_\xi T_y$ are called the *time-frequency shift* operators. The function g is called a *window function* or an *atom*. We are concerned about the characterization of Schauder basis property of the system

$$\mathcal{G}(g) = \mathcal{G}(g,1,1) = \{M_{u(n)}T_{u(k)}g : n,k\in\mathbb{N}_0\}$$

for $L^2(K)$ in terms of Zak transform.

The Zak transform was first introduced by Gelfand [44] and it was later rediscovered by Zak [121] and Brezin [8]. For more details about the history of the Zak transform, we refer to [47] and [52]. Weil [113] introduced the concept of Zak transform on locally compact abelian groups and formulated its basic properties. For the definition of Zak transform and derivation of its properties on certain locally compact nonabelian groups, we refer to [72]. In order to define Zak transform on a locally compact abelian group G, we need a lattice. Recall that a discrete subgroup D of G is called a *lattice* if the quotient G/D is a compact group. Since the characteristic of K is positive, it follows that $\Lambda = \{u(k) : k \in \mathbb{N}_0\}$ is a discrete subgroup of K^+ (see Proposition 3.1.5) and that $K^+/\Lambda = \mathfrak{D}$ is compact, and hence Λ is a lattice in K^+. With respect this lattice, we make the following definition.

Definition 3.3.7. *Let K be a local field of positive characteristic. The* Zak transform *of a function $f \in L^2(K)$ is the function of two variables defined by*

$$Zf(x,\xi) = \sum_{k\in\mathbb{N}_0} T_{u(k)}f(x)\chi_k(\xi) = \sum_{k\in\mathbb{N}_0} f\big(x-u(k)\big)\chi\big(u(k)\xi\big), \quad x,\xi \in K.$$

Using the Λ-periodicity of the characters χ_k, we can show that $|Zf|$ is Λ-periodic in both the variables. It turns out that Z maps $L^2(K)$ isometrically onto $L^2(\mathfrak{D}\times\mathfrak{D})$. The proof of this fact can be obtained from the corresponding result on Euclidean spaces with necessary modifications, see e. g. [52].

For $n, k \in \mathbb{N}_0$, define

$$E_{n,k}(x,\xi) = \chi_n(x)\overline{\chi_k(\xi)} = \chi\Big(u(n)x - u(k)\xi\Big). \tag{3.3.3}$$

The following theorem shows that the Zak transform diagonalizes the time-frequency shifts.

Theorem 3.3.8. *Let $g \in L^2(K)$. Then*

$$Z(M_{u(n)}T_{u(k)}g)(x,\xi) = (E_{n,k}\cdot Zg)(x,\xi) = E_{n,k}(x,\xi)\cdot Zg(x,\xi).$$

Proof. This is a straightforward verification. □

If $\mathcal{G}(g)$ is a Schauder basis for $L^2(K)$, then using Theorem 3.3.8, we can show that the biorthogonal system is of the form $\mathcal{G}(\tilde{g})$, where the dual window $\tilde{g} \in L^2(K)$ is defined by the condition $Z\tilde{g} = 1/\overline{Zg}$. Indeed, since Z is an isometry, for $k, l, m, n \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\langle M_{u(n)}T_{u(k)}g, M_{u(m)}T_{u(l)}\tilde{g}\rangle \\
&= \langle Z(M_{u(n)}T_{u(k)}g), Z(M_{u(m)}T_{u(l)}\tilde{g})\rangle \\
&= \langle E_{n,k} \cdot Zg, E_{m,l} \cdot Z\tilde{g}\rangle \\
&= \int_{\mathfrak{D}}\int_{\mathfrak{D}} \chi_n(x)\overline{\chi_k(\xi)}Zg(x,\xi)\overline{\chi_m(x)}\chi_l(\xi)\overline{Z\tilde{g}(x,\xi)}\,dx\,d\xi \\
&= \langle \chi_n, \chi_m\rangle\langle \chi_l, \chi_k\rangle \\
&= \delta_{n,m}\delta_{l,k}.
\end{aligned}$$

Therefore, $\mathcal{G}(\tilde{g})$ is biorthogonal to $\mathcal{G}(g)$. Since a Schauder basis has a unique biorthogonal dual, it follows that $\mathcal{G}(\tilde{g})$ is the biorthogonal dual of $\mathcal{G}(g)$.

3.3.3 A_p Weights on the Product Space $\mathfrak{D} \times \mathfrak{D}$

Let w be a nonnegative function on $\mathfrak{D} \times \mathfrak{D}$. Following Fefferman and Stein [39], we say that $w \in A_p(\mathfrak{D} \times \mathfrak{D})$ if for a.e. $y \in \mathfrak{D}$, the function $x \to w(x,y)$ is an $A_p(\mathfrak{D})$ weight and the A_p characteristic $[w(\cdot,y)]_{A_p}$ is independent of y, and a similar condition holds for the function $y \to w(x,y)$ for a.e. $x \in \mathfrak{D}$.

That is, $w \in A_p(\mathfrak{D} \times \mathfrak{D})$ if there exists a constant $C > 0$ such that for a.e. $y \in \mathfrak{D}$ and all balls $B \subset \mathfrak{D}$

$$\Big(\frac{1}{|B|}\int_B w(x,y)\,dx\Big)\Big(\frac{1}{|B|}\int_B w(x,y)^{-\frac{1}{p-1}}\,dx\Big)^{p-1} \le C, \tag{3.3.4}$$

and for a.e. $x \in \mathfrak{D}$ and balls $B \subset \mathfrak{D}$

$$\Big(\frac{1}{|B|}\int_B w(x,y)\,dy\Big)\Big(\frac{1}{|B|}\int_B w(x,y)^{-\frac{1}{p-1}}\,dy\Big)^{p-1} \le C. \tag{3.3.5}$$

We will show that the above definition is equivalent to the existence of a constant $C > 0$ such that for all balls $B_1, B_2 \subset \mathfrak{D}$, we have

$$\Big(\frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2} w(x,y)\,dx\,dy\Big) \times \Big(\frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2} w(x,y)^{-\frac{1}{p-1}}\,dx\,dy\Big)^{p-1} \le C. \tag{3.3.6}$$

Let us temporarily write $w \in A_{p,*}(\mathfrak{D}\times\mathfrak{D})$ if the weight w satisfies (3.3.6) for all balls $B_1, B_2 \subset \mathfrak{D}$. Also, let us denote the supremum of the left hand side in (3.3.6) taken over all balls $B_1, B_2 \subset \mathfrak{D}$ by $[w]_{A_{p,*}}$. We will now show that both these definitions of weights on $\mathfrak{D}\times\mathfrak{D}$ are equivalent to the boundedness on $L^p(\mathfrak{D}\times\mathfrak{D}, w)$ of the Hardy–Littlewood maximal operator M_* adapted to this definition.

For a function f on $\mathfrak{D}$, we define the maximal function M_*f as follows:

$$M_*f(x,y) = \sup_{x\in B_1, y\in B_2} \frac{1}{|B_1|}\frac{1}{|B_2|}\int_{B_1}\int_{B_2} |f(u,v)|\,du\,dv,$$

where the supremum is taken over all balls B_1 in $\mathfrak{D}$ containing x and B_2 in $\mathfrak{D}$ containing y.

Theorem 3.3.9. *Let w be a weight on $\mathfrak{D}\times\mathfrak{D}$ and $1 < p < \infty$. Then the following are equivalent.*

(a) $w \in A_{p,*}(\mathfrak{D}\times\mathfrak{D})$.

(b) $w \in A_p(\mathfrak{D}\times\mathfrak{D})$.

(c) M_* *is a bounded operator on* $L^p(\mathfrak{D}\times\mathfrak{D}, w)$.

Proof. Suppose (a) holds. Then for all balls $B_1, B_2 \subset \mathfrak{D}$, we have

$$\Big(\frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2} w(x,y)\,dx\,dy\Big) \times \Big(\frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2} w(x,y)^{-\frac{1}{p-1}}\,dx\,dy\Big)^{p-1} \le [w]_{A_{p,*}}. \tag{3.3.7}$$

Fix a ball $B_1 \subset \mathfrak{D}$. By Lebesgue differentiation theorem (Theorem 3.1.7), for a.e. y, we have

$$\frac{1}{|x+\mathfrak{P}^k|}\int_{x+\mathfrak{P}^k}\Big(\frac{1}{|B_1|}\int_{B_1} w(x,z)\,dx\Big)\,dz \longrightarrow \frac{1}{|B_1|}\int_{B_1} w(x,y)\,dx$$

as $k \to \infty$. Similarly,

$$\Big(\frac{1}{|x+\mathfrak{P}^k|}\int_{x+\mathfrak{P}^k}\Big(\frac{1}{|B_1|}\int_{B_1} w(x,z)^{-\frac{1}{p-1}}\,dx\Big)\,dz\Big)^{p-1}$$

$$\longrightarrow \Big(\frac{1}{|B_1|}\int_{B_1} w(x,y)^{-\frac{1}{p-1}}\,dx\Big)^{p-1}$$

as $k \to \infty$. Therefore, the product of the left sides in the last two equations converge to the product of the right sides. By (3.3.7) the product of the left sides is at most $[w]_{A_{p,*}}$. Hence,

$$\Big(\frac{1}{|B_1|}\int_{B_1} w(x,y)\,dx\Big)\Big(\frac{1}{|B_1|}\int_{B_1} w(x,y)^{-\frac{1}{p-1}}\,dx\Big)^{p-1} \le [w]_{A_{p,*}}.$$

This is true for every ball $B_1 \in \mathfrak{D}$ and a.e. $y \in \mathfrak{D}$. Hence, $w(\cdot,y) \in A_p(\mathfrak{D})$ and the A_p characteristic $[w(\cdot,y)]_{A_p}$ is independent of y. Similarly, $w(x,\cdot) \in A_p(\mathfrak{D})$ and $[w(x,\cdot)]_{A_p}$ is independent of x. Therefore, $w \in A_p(\mathfrak{D}\times\mathfrak{D})$. So (b) is proved.

Assume (b). Define the Hardy–Littlewood maximal operators M_1 and M_2 corresponding to the first and second variables:

$$M_1 f(x,y) = \sup_B \frac{1}{|B|}\int_B |f(s,y)|\,ds$$

and

$$M_2 f(x,y) = \sup_B \frac{1}{|B|}\int_B |f(x,t)|\,dt.$$

Since $w(x,\cdot)$ and $w(\cdot,y)$ are in $A_p(\mathfrak{D})$, the operators M_1 and M_2 are bounded on $L^p(\mathfrak{D}, w(\cdot,y))$ and $L^p(\mathfrak{D}, w(x,\cdot))$ respectively, by Theorem 3.1.8. Clearly, $M_* f(x,y) \le M_1 \circ M_2 f(x,y)$. Since the A_p characteristic of $w(x,\cdot)$ and $w(\cdot,y)$ have uniform bound, by applying Fubini's theorem, we get

$$\begin{aligned}\int_{\mathfrak{D}}\int_{\mathfrak{D}} M_* f(x,y)^p w(x,y)\,dx\,dy &\le \int_{\mathfrak{D}}\int_{\mathfrak{D}} M_1\circ M_2 f(x,y)^p w(x,y)\,dx\,dy \\ &\le C\int_{\mathfrak{D}}\int_{\mathfrak{D}} M_2 f(x,y)^p w(x,y)\,dx\,dy\end{aligned}$$

$$\leq \quad C^2 \int_{\mathfrak{D}} \int_{\mathfrak{D}} |f(x,y)|^p w(x,y)\, dx\, dy.$$

This proves (c).

Now, assume that (c) is true. Let B_1 and B_2 be balls in $\mathfrak{D}$. Then for each $(x,y) \in B_1 \times B_2$, we have

$$M_*(f \cdot \mathbf{1}_{B_1\times B_2})(x,y) \geq \frac{1}{|B_1|}\frac{1}{|B_2|}\int_{B_1}\int_{B_2} |f(u,v)|\, du\, dv.$$

Hence,

$$\begin{aligned}
&\Big(\frac{1}{|B_1|}\frac{1}{|B_2|}\int_{B_1}\int_{B_2} |f(u,v)|\, du\, dv\Big)^p \int_{B_1}\int_{B_2} w(x,y)\, dx\, dy \\
&\leq \quad \int_{B_1}\int_{B_2} [M_*(f\cdot \mathbf{1}_{B_1\times B_2})(x,y)]^p w(x,y)\, dx\, dy \\
&\leq \quad C\int_{B_1}\int_{B_2} |f(x,y)|^p w(x,y)\, dx\, dy.
\end{aligned}$$

Taking $f = w^{-\frac{1}{p-1}}$, we see that (3.3.6) is satisfied so that we obtain (a). This completes the proof of the theorem. □

Remark 3.3.10. *(a) In view of Theorem 3.3.9, we say that $w \in A_p(\mathfrak{D}\times\mathfrak{D})$ if w satisfies either (3.3.4) and (3.3.5) or (3.3.6).*

(b) In [80], several characterizations are provided for the A_p weights on $\mathbb{T}^n$ in terms of strong maximal functions, rectangular conjugate functions, and rectangular partial sums.

For $M, N \in \mathbb{N}$, define

$$S_{M,N}F = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\langle F, E_{m,n}\rangle E_{m,n},$$

where $E_{n,k}(x,\xi) = \chi_n(x)\overline{\chi_k(\xi)}$ and the inner product is in $L^2(\mathfrak{D}\times\mathfrak{D})$.

Let $T : L^p(\mathfrak{D}\times\mathfrak{D}, w) \to L^p(\mathfrak{D}\times\mathfrak{D}, w)$ be a bounded operator. The norm of this operator is denoted by $\|T\|_{p,w}$. Also, we denote $\|T\|_p$ to be the norm of the bounded operator $T : L^p(\mathfrak{D}\times\mathfrak{D}) \to L^p(\mathfrak{D}\times\mathfrak{D})$.

The following theorem characterizes the uniform boundedness of the operators $\{S_{M,N} : M, N \in \mathbb{N}\}$ on the weighted space $L^2(\mathfrak{D} \times \mathfrak{D}, w)$.

Theorem 3.3.11. *Let w be a nonnegative function in $L^2(\mathfrak{D}\times\mathfrak{D})$. Then $\sup_{M,N} \|S_{M,N}\|_{2,w} < \infty$ if and only if $w \in A_2(\mathfrak{D} \times \mathfrak{D})$.*

Proof. Suppose $C = \sup_{M,N} \|S_{M,N}\|_{2,w} < \infty$. In order to show that $w \in A_2(\mathfrak{D} \times \mathfrak{D})$, we will prove that

$$\Big(\frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2} w\Big)\Big(\frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2} \frac{1}{w}\Big) \le C$$

for all balls $B_1, B_2 \subset \mathfrak{D}$. Suppose $|B_1| = q^{-r}$ and $|B_2| = q^{-s}$, where $r, s \geq 0$. Choose F to be nonnegative on $B_1 \times B_2$ and 0 on $(\mathfrak{D} \times \mathfrak{D}) \setminus (B_1 \times B_2)$. Then extend F $\Lambda \times \Lambda$-periodically. Now, for $(x, \xi) \in B_1 \times B_2$, we have

$$\begin{aligned}
S_{M,N}F(x,\xi) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \langle F, E_{m,n}\rangle E_{m,n}(x,\xi) \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \Big[\int_{B_1}\int_{B_2} F(y,\eta)\overline{\chi_m(y)}\chi_n(\eta)\,dy\,d\eta\Big]\chi_m(x)\overline{\chi_n(\xi)} \\
&= \int_{B_1}\int_{B_2} F(y,\eta)\Big[\sum_{m=0}^{M-1}\chi_m(x-y)\sum_{n=0}^{N-1}\chi_n(\eta-\xi)\Big]\,dy\,d\eta \\
&= \int_{B_1}\int_{B_2} F(y,\eta) D_M(x-y) D_N(\eta-\xi)\,dy\,d\eta,
\end{aligned}$$

where D_n is the Dirichlet kernel defined in (3.1.7). Now, B_1 is a ball in $\mathfrak{D}$ of measure q^{-r}. Since $\mathfrak{D} = \bigcup_{l=0}^{q^r-1}(a_l + \mathfrak{P}^r)$, we have $B_1 = a_l + \mathfrak{P}^r$ for some a_l, by Proposition 3.1.1. Since $x \in B_1$, we have $B_1 = x + \mathfrak{P}^r$ since every point of a ball is its centre (see Proposition 3.1.1). Also, we have chosen F to be supported on $B_1 \times B_2$ so that $y \in B_1$ in the above integral, that is, $y \in x + \mathfrak{P}^r$. Hence, $x - y \in \mathfrak{P}^r$. Similarly, $\eta - \xi \in \mathfrak{P}^s$. Now, we choose $M = q^r$ and $N = q^s$. By Lemma 3.1.6, $D_{q^k} = q^k \mathbf{1}_{\mathfrak{P}^k}$. Hence, we obtain

$$S_{q^r,q^s}F(x,\xi) = \int_{B_1}\int_{B_2} F(y,\eta) q^r q^s\,dy\,d\eta = \frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2} F(y,\eta)\,dy\,d\eta.$$

Therefore,

$$\frac{1}{|B_1|^2|B_2|^2}\Big(\int_{B_1}\int_{B_2} F(y,\eta)\,dy\,d\eta\Big)^2\Big(\int_{B_1}\int_{B_2} w(x,\xi)\,dx\,d\xi\Big)$$

$$\begin{aligned} &= \int_{B_1}\int_{B_2} |S_{q^r,q^s}F(x,\xi)|^2 w(x,\xi)\,dx\,d\xi \\ &= \|S_{q^r,q^s}F\|_{2,w}^2 \le C^2\|F\|_{2,w}^2. \end{aligned} \tag{3.3.8}$$

In particular, if F is the $\Lambda\times\Lambda$-periodic extension of $\frac{1}{w}\mathbf{1}_{B_1\times B_2}$, then

$$\|F\|_{2,w}^2 = \int_{B_1}\int_{B_2}\frac{1}{w^2}w = \int_{B_1}\int_{B_2}\frac{1}{w}.$$

Hence,

$$\Big(\frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2}\frac{1}{w}\Big)\Big(\frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2}w\Big)\Big(\int_{B_1}\int_{B_2}\frac{1}{w}\Big) \le C^2\int_{B_1}\int_{B_2}\frac{1}{w}.$$

From this it follows that if $\int_{B_1}\int_{B_2}\frac{1}{w}<\infty$, then

$$\Big(\frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2}\frac{1}{w}\Big)\Big(\frac{1}{|B_1||B_2|}\int_{B_1}\int_{B_2}w\Big) \le C^2. \tag{3.3.9}$$

If $\int_{B_1}\int_{B_2}\frac{1}{w}=0$, then (3.3.9) holds trivially. Finally, if $\int_{B_1}\int_{B_2}\frac{1}{w}=\infty$, then there exists $G\in L^2(B_1\times B_2)$ such that $\frac{G}{w^{1/2}}\notin L^1(B_1\times B_2)$. Let $F=\frac{|G|}{w^{1/2}}$. Then $\int_{B_1}\int_{B_2}F=\infty$ but $\|F\|_{2,w}=\|G\|_2<\infty$. So, from (3.3.8), we get $\int_{B_1}\int_{B_2}w=0$. Hence, (3.3.9) holds in this case also. Therefore, $w\in A_2(\mathfrak{D}\times\mathfrak{D})$.

We will now prove the converse. Suppose $w\in A_2(\mathfrak{D}\times\mathfrak{D})$. Let $w_x=w(x,\cdot)$ and $w^\xi=w(\cdot,\xi)$. Since the A_2 characteristics of w_x and w^ξ are uniformly bounded, by Theorem 3.2.6, there exists $C>0$ such that for all $n\in\mathbb{N}$ and for a.e. x,ξ, we have

$$\int_{\mathfrak{D}}|S_Nf(x)|^2w^\xi(x)\,dx \le C\int_{\mathfrak{D}}|f(x)|^2w^\xi(x)\,dx, \quad f\in L^2(\mathfrak{D},w^\xi)$$

and

$$\int_{\mathfrak{D}}|S_Nf(\xi)|^2w_x(\xi)\,d\xi \le C\int_{\mathfrak{D}}|f(\xi)|^2w_x(\xi)\,d\xi, \quad f\in L^2(\mathfrak{D},w_x).$$

Now, let $F\in L^2(\mathfrak{D}\times\mathfrak{D},w)$. By Fubini's theorem, $F_x=F(x,\cdot)\in L^2(\mathfrak{D},w_x)$ for a.e. x and $F^\xi=F(\cdot,\xi)\in L^2(\mathfrak{D},w^\xi)$ for a.e. ξ. For $M,N\in\mathbb{N}$, let

$$S_N^1F(x,\xi)=S_NF^\xi(x)=\sum_{n=0}^{N-1}\langle F^\xi,\chi_n\rangle\chi_n(x)$$

and

$$S_M^2 F(x,\xi) = S_M F_x(\xi) = \sum_{m=0}^{M-1} \langle F_x, \chi_m \rangle \chi_m(\xi).$$

Then $S_N^1 S_M^2 F = S_{M,N} F$. Hence,

$$\begin{aligned}
\|S_{M,N}F\|_{2,w}^2 &= \int_{\mathfrak{D}} \int_{\mathfrak{D}} |S_N^1 S_M^2 F(x,\xi)|^2 w(x,\xi)\,dx\,d\xi \\
&\leq C \int_{\mathfrak{D}} \int_{\mathfrak{D}} |S_M^2 F(x,\xi)|^2 w(x,\xi)\,dx\,d\xi \\
&\leq C^2 \int_{\mathfrak{D}} \int_{\mathfrak{D}} |F(x,\xi)|^2 w(x,\xi)\,dx\,d\xi \\
&\leq C^2 \|F\|_{2,w}^2.
\end{aligned}$$

This completes the proof of the theorem. □

We rewrite the above result in terms of the partial sum operators involving the windowed system $E_{m,n} \cdot (1/\overline{W})$.

Corollary 3.3.12. *Let W be a nonnegative function in $L^2(\mathfrak{D} \times \mathfrak{D})$ and for $M, N \in \mathbb{N}$, let $T_{M,N} : L^2(\mathfrak{D} \times \mathfrak{D}) \to L^2(\mathfrak{D} \times \mathfrak{D})$ be the operator*

$$T_{M,N} F = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \Big\langle F, E_{m,n} \cdot \frac{1}{\overline{W}} \Big\rangle E_{m,n} \cdot W.$$

Then $\sup_{M,N} \|T_{M,N}\|_2 < \infty$ if and only if $|W|^2 \in A_2(\mathfrak{D} \times \mathfrak{D})$.

Proof. This is a reformulation of Theorem 3.3.11, we omit the proof since it is similar to that of Corollary 5.7 in [50]. □

3.3.4 Characterization of Gabor Systems that are Schauder Bases

We consider the problem of determining whether the Gabor system $\mathcal{G}(g) = \{M_{u(n)} T_{u(k)} g : n, k \in \mathbb{N}_0\}$ is a Schauder basis for $L^2(K)$. Since Schauder basis expansions may converge conditionally, the order of summation is important. The Gabor system $\mathcal{G}(g)$ involves two indices each in $\mathbb{N}_0$ so that we have to consider permutations of $\mathbb{N}_0 \times \mathbb{N}_0$. We define a family of permutationsof $\mathbb{N}_0 \times \mathbb{N}_0$. which are compatible with the partial sum operators. Following Heil and Powell [50], we define the following enumerations.

Definition 3.3.13. *Let Γ be the set of all enumerations $\{(k_j,n_j)\}_{j=1}^{\infty}$ of $\mathbb{N}_0\times\mathbb{N}_0$ defined in the following recursive manner.*

(a) The initial terms $(k_1,n_1),(k_2,n_2),\ldots,(k_{J_1},n_{J_1})$ are either

$$(0,0),(0,1),\ldots,(0,M_1) \quad or \quad (0,0),(1,0),\ldots,(N_1,0)$$

for some positive integers M_1 and N_1.

(b) If $\{(k_j,n_j)\}_{j=1}^{J_k}$ has been constructed to be of the form

$$\{0,1,\ldots,M_k\}\times\{0,1,\ldots,N_k\}$$

for some positive integers M_k and N_k, then terms are added to the top or the right side to obtain either the rectangle

$$\{0,1,\ldots,M_k\}\times\{0,1,\ldots,N_k+1\}$$

or

$$\{0,1,\ldots,M_k+1\}\times\{0,1,\ldots,N_k\}.$$

With respect to the above enumerations of $\mathbb{N}_0\times\mathbb{N}_0$, we consider the problem of characterization of Schauder basis property of the Gabor system $\mathcal{G}(g)$ in terms of the Zak transform. The main result of this section is the following.

Theorem 3.3.14. *Let K be a local field of positive characteristic and $g\in L^2(K)$. A necessary and sufficient condition for the Gabor system $\mathcal{G}(g)$ to be a Schauder basis for $L^2(K)$ with respect to every enumeration $\sigma\in\Gamma$ is that $|Zg|^2\in A_2(\mathfrak{D}\times\mathfrak{D})$.*

Proof. We first prove the necessity of the condition. Let $\sigma=\{(k_j,n_j)\}_{j=1}^{\infty}\in\Gamma$ be an enumeration of $\mathbb{N}_0\times\mathbb{N}_0$ and $\mathcal{G}(g)$ be a Schauder basis of $L^2(K)$ with respect to σ. Consider the partial sum operators $T_N^{\sigma}: L^2(\mathfrak{D}\times\mathfrak{D})\to L^2(\mathfrak{D}\times\mathfrak{D})$ corresponding to σ, defined by

$$T_N^{\sigma}F=\sum_{j=1}^{N}\Big\langle F, E_{n_j,k_j}\cdot\frac{1}{\overline{Zg}}\Big\rangle E_{n_j,k_j}\cdot Zg,$$

where $E_{n,k}(x,\xi) = \chi_n(x)\overline{\chi_k(\xi)}$ (see (3.3.3)). Then by Lemma 3.3.5 and Theorem 3.3.8, $\frac{1}{|Zg|} \in L^2(\mathfrak{D} \times \mathfrak{D})$ and the operators T_N^σ are uniformly bounded. Hence, by Corollary 3.3.12, $|Zg|^2 \in A_2(\mathfrak{D} \times \mathfrak{D})$.

We will now show the sufficiency. Let $|Zg|^2 \in A_2(\mathfrak{D} \times \mathfrak{D})$ and $\sigma \in \Gamma$ be an enumeration of $\mathbb{N}_0 \times \mathbb{N}_0$. Note that by the definition of the A_2 condition, it follows that $\frac{1}{Zg} \in L^2(\mathfrak{D} \times \mathfrak{D})$.

We again consider the operators T_N^σ defined above and claim that they are uniformly bounded i.e., $\sup_{N,\sigma} \|T_N^\sigma\|_2 < \infty$. Once we have this, then again by Lemma 3.3.5 and Theorem 3.3.8, $\mathcal{G}(g)$ will be a Schauder basis of $L^2(K)$ with respect to every enumeration $\sigma \in \Gamma$.

By Corollary 3.3.12, $\sup_{M,N} \|T_{M,N}\|_2 = C < \infty$. Choose an enumeration $\sigma \in \Gamma$ and $N \in \mathbb{N}$. Let M_N be the largest integer $M_N < N$ such that $T_{M_N}^\sigma F = T_{J,K}F$ for some integers J, K. Observe that

$$\begin{aligned} \|T_N^\sigma F\|_2 &= \|T_{M_N}^\sigma F + T_N^\sigma F - T_{M_N}^\sigma F\|_2 \\ &\leq \|T_{J,K}F\|_2 + \|(T_N^\sigma - T_{M_N}^\sigma)F\|_2. \end{aligned}$$

We now estimate the second term. We have

$$(T_N^\sigma - T_{M_N}^\sigma)F = \sum_{j=M_N+1}^{N} \Big\langle F, E_{n_j,k_j} \cdot \frac{1}{\overline{Zg}} \Big\rangle E_{n_j,k_j} \cdot Zg. \tag{3.3.10}$$

According to the specific nature of the enumerations σ and the definition of M_N, it follows that the terms in the above sum correspond to terms that have been added to a rectangle on top or on right. That is, the sum is equal to either of the following two sums:

$$\sum_{n=0}^{L} \Big\langle F, E_{n,K+1} \cdot \frac{1}{\overline{Zg}} \Big\rangle E_{n,K+1} \cdot Zg, \quad L \leq J, \tag{3.3.11}$$

or

$$\sum_{k=0}^{R} \Big\langle F, E_{J+1,k} \cdot \frac{1}{\overline{Zg}} \Big\rangle E_{J+1,k} \cdot Zg, \quad R \leq K. \tag{3.3.12}$$

Note that by Proposition 3.1.5 (b), $-u(K+1) = u(K')$ for some $K' \in \mathbb{N}$. Hence, $\chi_{-u(K+1)}(\xi) = \chi_{u(K')}(\xi)$ so that

$$E_{0,K'}(x,\xi) = \chi_0(x)\overline{\chi_{u(K')}(\xi)} = \chi_{-u(K')}(\xi) = \chi_{u(K+1)}(\xi).$$

Also, $E_{n,0}(x,\xi) = \chi_{u(n)}(x)$. Therefore, we have

$$\begin{aligned}
\Big\langle F, E_{n,K+1}\cdot\frac{1}{Zg}\Big\rangle &= \int_{\mathfrak{D}}\int_{\mathfrak{D}} F(x,\xi)\overline{\chi_{u(n)}(x)}\chi_{u(K+1)}(\xi)\frac{1}{Zg(x,\xi)}\,dx\,d\xi \\
&= \int_{\mathfrak{D}}\int_{\mathfrak{D}} F(x,\xi)\chi_{u(K+1)}(\xi)\overline{\chi_{u(n)}(x)}\frac{1}{Zg(x,\xi)}\,dx\,d\xi \\
&= \int_{\mathfrak{D}}\int_{\mathfrak{D}} (F\cdot E_{0,K'})(x,\xi)\overline{E_{n,0}(x,\xi)}\frac{1}{Zg(x,\xi)}\,dx\,d\xi \\
&= \Big\langle F\cdot E_{0,K'}, E_{n,0}\cdot\frac{1}{Zg}\Big\rangle.
\end{aligned}$$

Hence, the first sum (3.3.11) is bounded by

$$\begin{aligned}
&\Big\|\sum_{n=0}^{L}\Big\langle F, E_{n,K+1}\cdot\frac{1}{Zg}\Big\rangle E_{n,K+1}\cdot Zg\Big\|_2 \\
&= \Big\|\sum_{n=0}^{L}\Big\langle F\cdot E_{0,K'}, E_{n,0}\cdot\frac{1}{Zg}\Big\rangle E_{n,K+1}\cdot\frac{1}{Zg}\Big\|_2 \\
&= \Big\|\sum_{n=0}^{L}\Big\langle F\cdot E_{0,K'}, E_{n,0}\frac{1}{Zg}\Big\rangle E_{n,K+1}\cdot\frac{1}{Zg}\Big\|_2 \\
&= \Big\|T_{L,0}(F\cdot E_{0,K'})\Big\|_2 \\
&\le C\Big\|F\cdot E_{0,K'}\Big\|_2 \\
&= C\|F\|_2.
\end{aligned}$$

For the second sum we observe that

$$\begin{aligned}
\Big\langle F, E_{J+1,k}\cdot\frac{1}{Zg}\Big\rangle &= \int_{\mathfrak{D}}\int_{\mathfrak{D}} F(x,\xi)\overline{\chi_{u(J+1)}(x)}\chi_{u(k)}(\xi)\frac{1}{Zg(x,\xi)}\,dx\,d\xi \\
&= \int_{\mathfrak{D}}\int_{\mathfrak{D}} F(x,\xi)\overline{E_{J+1,0}(x,\xi)}\,\overline{E_{0,k}(x,\xi)}\frac{1}{Zg(x,\xi)}\,dx\,d\xi \\
&= \Big\langle F\cdot\overline{E_{J+1,0}}, E_{0,k}\cdot\frac{1}{Zg}\Big\rangle.
\end{aligned}$$

Hence, the second sum (3.3.12) is bounded by

$$\begin{aligned}
&\Big\|\sum_{k=0}^{R}\Big\langle F, E_{J+1,k}\cdot\frac{1}{\overline{Zg}}\Big\rangle E_{J+1,k}\cdot Zg\Big\|_2 \\
&= \Big\|\sum_{k=0}^{R}\Big\langle F\cdot\overline{E_{J+1,0}}, E_{0,k}\cdot\frac{1}{\overline{Zg}}\Big\rangle E_{J+1,k}\cdot Zg\Big\|_2 \\
&= \Big\|\sum_{k=0}^{R}\Big\langle F\cdot\overline{E_{J+1,0}}, E_{0,k}\cdot\frac{1}{\overline{Zg}}\Big\rangle E_{0,k}\cdot Zg\Big\|_2 \\
&= \Big\|T_{0,R}(F\cdot\overline{E_{J+1,0}})\Big\|_2 \\
&\le C\Big\|F\cdot\overline{E_{J+1,0}}\Big\|_2 \\
&= C\|F\|_2.
\end{aligned}$$

Substituting these estimates in (3.3.10), we obtain $\|T_N^\sigma F\|_2 \le 2C\|F\|_2$ for all enumeration $\sigma\in\Gamma$ and all $N\in\mathbb{N}$. Therefore, $\sup_{N,\sigma}\|T_N^\sigma\|_2 \le 2C < \infty$. Hence, the claim is proved and this completes the proof of the theorem. □

For applications, it is important to know whether a Gabor system is complete, minimal, a frame, a Riesz basis or an orthonormal basis. We mention some results in Gabor theory on the characterizations of such systems in terms of the Zak transform. We refer to [72] for a proof of the following theorem. For an introduction to frame theory, and definitions of frames and Riesz bases, see [51].

Theorem 3.3.15. *Let K be a local field of positive characteristic, $g\in L^2(K)$ and $\mathcal{G}(g)$ the Gabor system generated by g. Then*

(a) $\mathcal{G}(g)$ is complete in $L^2(K)$ if and only if $Zg \neq 0$ a.e.

(b) $\mathcal{G}(g)$ is minimal and complete in $L^2(K)$ if and only if $\frac{1}{Zg}\in L^2(\mathfrak{D}\times\mathfrak{D})$.

(c) $\mathcal{G}(g)$ is a frame for $L^2(K)$ with bounds A and B if and only if $A\le |Zg|^2\le B$ a.e. In this case $\mathcal{G}(g)$ is a Riesz basis for $L^2(K)$ with bounds A and B.

(d) $\mathcal{G}(g)$ is an orthonormal basis for $L^2(K)$ if and only if $|Zg| = 1$ a.e.

Now, to illustrate Theorem 3.3.14, we present some examples of Gabor systems which form Schauder bases for $L^2(K)$.

Example 3.3.16. *Let $-\frac{1}{2} < \alpha < \frac{1}{2}$. Consider the function g which is supported on $\mathfrak{D}$ and $g(x) = |x|^\alpha$, $x \in \mathfrak{D}$. Observe that $g \in L^2(K)$ since $\alpha > -\frac{1}{2}$. Since g is supported on $\mathfrak{D}$, the only term which contributes to the sum in the definition of Zg (see (3.3.7)) corresponds to $k = 0$. Hence, $Zg(x, \xi) = g(x)$. It follows from the definition that for a function of the form $w(x, y) = v(x)$, $w \in A_p(\mathfrak{D} \times \mathfrak{D})$ if $v \in A_p(\mathfrak{D})$. The function $g(x) = |x|^\alpha$ is an $A_p(\mathfrak{D})$ weight if and only if $-1 < \alpha < p-1$. Hence, $|g|^2$ is an $A_2(\mathfrak{D})$ weight if and only if $-\frac{1}{2} < \alpha < \frac{1}{2}$. This shows that $|Zg|^2 \in A_2(\mathfrak{D} \times \mathfrak{D})$. Therefore, by Theorem 3.3.14, it follows that the Gabor system $\mathcal{G}(g)$ is a Schauder basis for $L^2(K)$ with respect to every enumeration $\sigma \in \Gamma$.*

Also, observe that $|Zg|$ is not bounded away from zero. Therefore, by Theorem 3.3.15, it follows that $\mathcal{G}(g)$ is not a Riesz basis for $L^2(K)$.

Example 3.3.17. *Let g be a function constructed in Example 3.3.16 and h be any function supported in $\mathfrak{D}$ such that $h \in A_2(\mathfrak{D})$. For example, we can take $h(\xi) = |\xi|^\alpha$ with $-\frac{1}{2} < \alpha < \frac{1}{2}$. Let $G(x, \xi) = g(x)h(\xi)$. Then $|G|^2 \in A_2(\mathfrak{D} \times \mathfrak{D})$ so that $f = Z^{-1}G \in L^2(K)$. By Theorem 3.3.14, $\mathcal{G}(f)$ is a Schauder basis for $L^2(K)$ with respect to every enumeration $\sigma \in \Gamma$.*

Note that every Schauder basis is complete and minimal, but the converse need not be true in general. We now construct an example of a Gabor system which is complete and minimal, and using Theorem 3.3.14 we will show that this system cannot be a Schauder basis for some permutation of $\mathbb{N}_0 \times \mathbb{N}_0$.

Example 3.3.18. *Let $A_1 = \mathfrak{P}$ and*

$$A_n = \mathfrak{p}u(1) + \mathfrak{p}^2 u(1) + \cdots + \mathfrak{p}^{n-1} u(1) + \mathfrak{P}^n, \quad n \geq 2.$$

We first show that the balls A_n, $n \geq 1$, are pairwise disjoint. Note that $|A_n| = q^{-n}$. Let $k, l \in \mathbb{N}$ with $k < l$. Suppose A_k and A_l are not disjoint. By Proposition 3.1.1 (b), $A_l \subset A_k$. Let $y = \mathfrak{p}u(1) + \mathfrak{p}^2 u(1) + \cdots + \mathfrak{p}^{k-1} u(1)$. Then $A_k = y + \mathfrak{P}^k$ and $A_l = y + \mathfrak{p}^k u(1) + \cdots + \mathfrak{p}^{l-1} u(1) + \mathfrak{P}^l$. Now, if $A_l \subset A_k$, then $(A_l - y) \subset (A_k - y)$. But, this is not possible since $\mathfrak{p}^k u(1) + \cdots + \mathfrak{p}^{l-1} u(1) \in A_l - y$ and $|\mathfrak{p}^k u(1) + \cdots + \mathfrak{p}^{l-1} u(1)| = q^{-k+1}$ whereas $A_k - y = \mathfrak{P}^k$ is the ball of radius q^{-k} centred at 0. Hence, A_n, $n \geq 1$, are disjoint.

By Proposition 3.1.1 (d), each A_n is a union of q balls of radius q^{-n-1}. Choose any two such balls and call them E_n and F_n. Then $|E_n| = |F_n| = q^{-n-1}$. Let $S = \cup_{n=1}^{\infty}(E_n \cup F_n)$ and α be a real number such that $1 < \alpha < q$. Define the function g which is supported on $\mathfrak{D}$ and

$$g(x) = \begin{cases} \alpha^{n/2}, & x \in E_n, n \geq 1, \\ \alpha^{-n/2}, & x \in F_n, n \geq 1, \\ 1, & x \in \mathfrak{D} \setminus S. \end{cases}$$

We have

$$\begin{aligned} \int_{\mathfrak{D}} |g(x)|^2\, dx &= \sum_{n=1}^{\infty}\Big(\alpha^n |E_n| + \alpha^{-n}|F_n|\Big) + |\mathfrak{D} \setminus S| \\ &\leq \sum_{n=1}^{\infty}(\alpha^n + \alpha^{-n})q^{-n-1} + 1 \\ &= \frac{1}{q}\sum_{n=1}^{\infty}\Big[\Big(\frac{\alpha}{q}\Big)^n + \Big(\frac{1}{\alpha q}\Big)^n\Big] + 1 < \infty, \end{aligned}$$

since $\frac{\alpha}{q}, \frac{1}{\alpha q} < 1$. Similarly,

$$\int_{\mathfrak{D}} \frac{1}{|g(x)|^2}\, dx = \sum_{n=1}^{\infty}\Big(\alpha^{-n}|E_n| + \alpha^n |F_n|\Big) + |\mathfrak{D} \setminus S| < \infty.$$

Since $Zg(x, \xi) = g(x)$, it follows that $Zg, \frac{1}{Zg} \in L^2(\mathfrak{D} \times \mathfrak{D})$. Hence, by Theorem 3.3.15, $\mathcal{G}(g)$ is minimal and complete in $L^2(K)$.

We now compute the average of $|Zg(x, \cdot)|^2$ over the ball A_n. Note that $|A_n \setminus (E_n \cup F_n)| = \frac{q-2}{q^{n+1}}$. Hence,

$$\begin{aligned} \frac{1}{|A_n|}\int_{A_n} |Zg(x, \xi)|^2\, dx &= \frac{1}{|A_n|}\int_{A_n} |g(x)|^2\, dx \\ &= q^n\Big[(\alpha^n + \alpha^{-n})q^{-n-1} + \frac{q-2}{q^{n+1}}\Big] \\ &= \frac{1}{q}\Big(\alpha^n + \alpha^{-n} + q - 2\Big) \longrightarrow \infty \quad \text{as } n \to \infty. \end{aligned}$$

Similarly,

$$\frac{1}{|A_n|}\int_{A_n} \frac{1}{|Zg(x, \xi)|^2}\, dx \longrightarrow \infty \quad \text{as } n \to \infty.$$

Therefore, (3.3.4) does not hold for $w = |Zg|^2$ and $p = 2$. Hence, $|Zg|^2 \notin A_2(\mathfrak{D} \times \mathfrak{D})$. By Theorem 3.3.14, there exists an enumeration $\sigma \in \Gamma$ such that $\mathcal{G}(g)$ is not a Schauder *basis with respect to* σ.

Chapter 4

H^1 and BMO on LCA Groups Having a Covering Family

The aim of this chapter is to extend the classical theory of the Hardy space H^1 and its dual space of BMO functions with "bounded mean oscillation" to the setting of Locally Compact Abelian (LCA) groups G having covering families. First, we discuss in details the setting of LCA groups where our work is developed. Next, we introduce the notion of atomic Hardy spaces $H^{1,q}(G)$ with atom parameter $1 < q \leq \infty$ and the notion of the space $BMO(G)$ in this setting. After presenting some basic properties of these spaces, we then establish the main feature for functions in $BMO(G)$, namely the John–Nirenberg inequality. Moreover, we show that the atomic Hardy spaces $H^{1,q}(G)$ are independent of the choice of the parameter q. Finally, we relate $H^{1,q}(G)$ with $BMO(G)$ via duality in this setting.

As an application of our results, we obtain the boundedness of certain multiplier operators from the Hardy space to the Lebesgue spaces. In addition, we estimate the norms of these multiplier operators.

4.1 Locally Compact Abelian Groups with Covering Families

In this section we first review the notion of a covering family in an LCA group and then we record several preliminary results which will be required for establishing the results that we have in mind.

Let G be an LCA group with a measure μ that is inner regular and such that $\mu(K) < \infty$ for every compact set $K \subset G$. Notice that μ does not need to be the Haar measure because we do not assume μ to be translation invariant. Furthermore, suppose that the group possesses a local base of $0 \in G$ consisting of relatively compact neighbourhoods U_i, $i \in \mathbb{Z}$, satisfying the following basic conditions:

(a) $\{U_i\}_{i\in\mathbb{Z}}$ is monotonic in $i \in \mathbb{Z}$ in the sense that $U_i \subseteq U_j$ if $i \leq j$, moreover, $\bigcup_{i\in\mathbb{Z}} U_i = G$ and $\bigcap_{i\in\mathbb{Z}} U_i = \{0\}$;

(b) there exists an increasing function $\theta : \mathbb{Z} \to \mathbb{Z}$ such that

- $i \leq \theta(i)$,
- $U_i - U_i \subseteq U_{\theta(i)}$;

(c) the measure μ satisfies a doubling condition, i.e., there exists a constant $D > 1$ such that

$$\mu(x + U_{\theta(i)}) \leq D\,\mu(x + U_i) \quad \text{for all } x \in G \text{ and } i \in \mathbb{Z}.$$

Any group G admitting a sequence $\{U_i\}_{i\in\mathbb{Z}}$ of neighbourhoods of 0 and satisfying the above postulates (a)–(c) is said to have a covering family. This concept was introduced in [32] by Edwards and Gaudry. For each $x \in G$, the set $x + U_i$ will be called a *base set* and the collection of all base sets will be denoted by $\mathcal{B} = \{x + U_i : x \in G, i \in \mathbb{Z}\}$. A detailed exposition of harmonic analysis on LCA groups can be found in the monographs [32, 53, 54]. Some concrete examples of groups possessing a covering family are given below.

Example 4.1.1. *The simplest examples are $\mathbb{R}^n$, $\mathbb{Z}$ and the circle group $\mathbb{T}$. In the case of $\mathbb{R}^n$ equipped with the natural metric and the Lebesgue measure, we may take U_i to* *be the Euclidean ball $B(0,2^i)$ centred at 0 with radius 2^i, $i \in \mathbb{Z}$. Then the collection $\{B(0,2^i) : i \in \mathbb{Z}\}$ forms a covering family for $\mathbb{R}^n$ with the doubling constant $D = 2^n$ and $\theta(i) = i+1$.*

For the circle group $\mathbb{T} = \{\exp(2\pi it) : t \in [-\frac{1}{2}, \frac{1}{2})\}$ with the Haar measure, take $U_0 = \mathbb{T}$ and for $i \in \mathbb{N}, U_i = \{0\}$ and $U_{-i} = \{\exp(2\pi it) : |t| < \frac{1}{2i+1}\}$. Then, $\{U_i\}_{i\in\mathbb{Z}}$ is a covering family for $\mathbb{T}$ with $\theta(i) = i+1$ and $D = 2$.

When $G = \mathbb{Z}$, equipped with the counting measure, we may take $U_i = \{k \in \mathbb{Z} : |k| \leq 2^{i-1}\}$ for $i \geq 1$ and $U_i = 0$ otherwise; $\theta(i) = i+1$, and $D = 2$.

Example 4.1.2. *Let G be an LCA group with Haar measure μ and let H be a compact and open subgroup of G with $\mu(H) = 1$. Let A be an automorphism on G such that $H \subsetneq AH$ and $\bigcap_{i<0} A^i H = \{0\}$. In addition, suppose $G = \bigcup_{i\in\mathbb{Z}} A^i H$. Then one can check that $\{A^i H\}_{i\in\mathbb{Z}}$ satisfies the required properties to be a covering family. A structure of this type is considered in [6] for constructing wavelets on LCA groups with open and compact subgroups. The p-adic group $\mathbb{Q}_p$, where p is a prime number, or more generally, the additive groups of local fields, considered in Chapter 3, are some important examples of this situation.*

Now we collect some basic results which will be used in the rest of this chapter. In several occasions, we will make use of the following engulfing property of base sets, proved in [94, Lemma 2.2].

Lemma 4.1.3. *Let U and V be two base sets such that $U = x + U_i$ and $V = y + U_j$ with $i \leq j$ and $x, y \in G$. If $U \cap V \neq \emptyset$, then $x + U_i \subseteq y + U_{\theta^2(j)}$.*

If we assume μ to be translation invariant, then Lebesgue differentiation theorem (LDT) holds in our setting, see the remarks after Lemma 2.2.1 of [32]. However, since we do not confine ourselves to translation invariant measures, we need LDT to hold in this case also. In [94], Paternostro and Rela pointed out that the LDT still holds in this case as well. To be rigorous, there is a version of LDT in [54, Theorem 44.18], originally proved for the Haar measure on LCA groups having a D'-sequence (see [54] for the definition of a D'-sequence). A careful reading of the proof of Theorem 44.18 of [54] reveals that the result still holds with appropriate changes for measures which need not be translation invariant. Since a covering family is in particular a D'-sequence, therefore we have LDT in the present situation as well.

We remark that for some technical reasons, we may and do assume that the base sets U_i, $i \in \mathbb{Z}$, are symmetric. This is possible since one may always replace the base set U_i by the symmetric set $V_i = U_i - U_i$ and can verify that $\{V_i\}_{i\in\mathbb{Z}}$ still forms a local base of the identity 0 (see e.g. Appendix B.4 of [99]), and satisfies (a)–(c).

Throughout the chapter, we will use letters like C, C_1, C_2, etc. to denote positive constants independent of the main parameters, but may vary from line to line. While writing estimates, we shall use the notation $f \lesssim g$ to indicate $f \le Cg$ for some $C > 0$, and whenever $f \lesssim g \lesssim f$, we shall write $f \sim g$. For a base set of the form $V = x + U_i$, its θ dilation will be denoted by $V^* = x + U_{\theta(i)}$. Further iterations of this operation are defined recursively, that is, $V^{**} = (V^*)^*$ and V^{n*} for n iterations of the dilation operation. Let $1 \le q \le \infty$. For any μ-measurable set E in G, we denote by $L^q(E, \mu)$ the subspace of functions in $L^q(G)$ supported in E.

4.2 Atomic H^p Spaces on LCA Groups

In this section we introduce the notion of an atom in the present context and define atomic Hardy spaces. We then discuss several basic results regarding the nature of these spaces.

The standard way to define atoms is to use balls associated with some specific metric for the corresponding space. However, we lack such a concept in our case. Here, base sets take the role played by the balls.

Definition 4.2.1. *Let $1 < q \le \infty$. A function $b \in L^q(G,\mu)$ is called a $(1,q)$-atom if there exists a base set $V \in \mathcal{B}$ such that*

(i) $\operatorname{supp} b \subseteq V$*;*

(ii) $\int_V b(x)\, d\mu(x) = 0$*;*

(iii) $\|b\|_{L^q(G,\mu)} \le \left[\mu(V)\right]^{\frac{-1}{q'}}$, *where* $\frac{1}{q} + \frac{1}{q'} = 1$.

The corresponding atomic Hardy space $H^{1,q}(G)$ is defined by

$$H^{1,q}(G) = \left\{ \sum_{j=0}^{\infty} \lambda_j b_j : \{\lambda_j\}_{j=0}^{\infty} \in \ell^1 \text{ and } b_j' s \text{ are } (1,\ q)\text{-atoms} \right\}$$

with the norm given by

$$\|f\|_{H^{1,q}(G)} = \inf\Big\{\sum_{j=0}^{\infty}|\lambda_j| : f = \sum_{j=0}^{\infty}\lambda_j b_j, \{\lambda_j\}_{j=0}^{\infty} \in \ell^1 \text{ and } b_j' s \text{ are } (1,q)\text{-atoms}\Big\}.$$

4.2.1 Basic Properties

It is clear that the spaces $H^{1,q}(G)$ are vector spaces over $\mathbb{C}$. We now highlight some important properties of $H^{1,q}(G)$.

Denote by $H^{1,q}_{fin}(G)$ the subspace of $H^{1,q}(G)$ consisting of all finite linear combinations of $(1,q)$-atoms. We observe that $H^{1,q}_{fin}(G)$ is dense in $H^{1,q}(G)$.

Proposition 4.2.2. *Let $1 < q \leq \infty$.*

(i) The space $H^{1,q}(G)$ is a Banach space.

(ii) We have $H^{1,\infty}(G) \subset H^{1,q}(G) \subset L^1(G)$. Moreover, $H^{1,\infty}(G)$ is dense $H^{1,q}(G)$.

Proof. The proof of (i) is standard and hence omitted.

Let us proof (ii). The inclusions $H^{1,\infty}(G) \subset H^{1,q}(G) \subset L^1(G)$ are obvious. So, we only prove that $H^{1,\infty}(G)$ is dense in $H^{1,q}(G)$.

Since $H^{1,q}_{fin}(G)$ is dense in $H^{1,q}(G)$, it suffices to approximate $H^{1,q}_{fin}(G)$ functions in the $H^{1,q}(G)$-norm by $H^{1,\infty}(G)$ functions.

Let $\epsilon > 0$ be given and $g \in H^{1,q}_{fin}(G)$ be such that $g = \sum_{j=1}^{M} b_j$, where each b_j is a $(1,q)$-atom and supported on some base set, say, $V_j \in \mathcal{B}$.

First, using the fact that compactly supported bounded functions are dense in $L^q(G)$, we obtain $\varphi_j \in L^\infty(V_j, \mu)$ satisfying the condition

$$\big\|b_j - \varphi_j\big\|_{L^q(V_j,\mu)} \leq \frac{\epsilon}{2M}\frac{1}{\big(\mu(V_j)\big)^{1-\frac{1}{q}}}, \quad 1 \leq j \leq M. \tag{4.2.1}$$

Next, for each j, we define

$$\widetilde{b_j} = \varphi_j - \widetilde{\varphi_j} \quad \text{and} \quad \widetilde{\varphi_j} = \frac{\Delta_j}{\mu(V_j)}\chi_{V_j},$$

where Δ_j is a constant defined by the formula $\Delta_j = \int_{V_j} \varphi_j \, d\mu$. Then we note that

$$\widetilde{b_j} \in L^\infty(G,\mu), \text{ supp}\,(\widetilde{b_j}) \subset V_j, \text{ and } \int_{V_j} \widetilde{b_j} = 0,$$

and hence $\widetilde{b_j} \in H^{1,\infty}(G)$. Using the fact that the integral of b_j is zero, the Hölder's inequality, and equation (4.2.1), we get

$$\begin{aligned} \|\widetilde{\varphi_j}\|_{L^q(G,\mu)} &= |\Delta_j|\big(\mu(V_j)\big)^{\frac{1}{q}-1} \le \frac{1}{\big(\mu(V_j)\big)^{1-\frac{1}{q}}}\Big(\int_{V_j} |\varphi_j - b_j|\, d\mu\Big) \\ &\le \|\varphi_j - b_j\|_{L^q(V_j,\mu)} \le \frac{\epsilon}{2M}\frac{1}{\big(\mu(V_j)\big)^{1-\frac{1}{q}}}. \end{aligned}$$

Thus, we have

$$\|b_j - \widetilde{b_j}\|_{L^q(G,\mu)} \le \|b_j - \varphi_j\|_{L^q(G,\mu)} + \|\varphi_j - \widetilde{b_j}\|_{L^q(G,\mu)} \le \frac{\epsilon}{M}\frac{1}{\big(\mu(V_j)\big)^{1-\frac{1}{q}}}.$$

This means that

$$\big(b_j - \widetilde{b_j}\big) \in H^{1,q}(G) \quad \text{and} \quad \|b_j - \widetilde{b_j}\|_{H^{1,q}(G)} \le \frac{\epsilon}{M}.$$

Finally, setting $\widetilde{g} = \sum_{j=1}^{M} \widetilde{b_j}$, we see that

$$\widetilde{g} \in H^{1,\infty}(G) \quad \text{and} \quad \|g - \widetilde{g}\|_{H^{1,q}(G)} < \epsilon.$$

Hence, the desired conclusion follows. This completes the proof of the proposition. □

4.3 BMO functions on LCA Groups

In this section, we focus our attention to the space BMO. The concept of base sets allows us to extend the classical notion of BMO in the present framework naturally as follows.

Definition 4.3.1. *A function $f \in L^1_{\text{loc}}(G)$ is said to be in the space $BMO(G)$ if*

$$\|f\|_{BMO(G)} := \sup_{V \in \mathcal{B}} \frac{1}{\mu(V)} \int_V |f(x) - m_V(f)|\, d\mu(x) < \infty, \tag{4.3.1}$$

where $m_V(f)$ denotes the average of f over the base set V defined by

$$m_V(f) := \frac{1}{\mu(V)} \int_V f(y)\, d\mu(y).$$

It is easy to check that if $\|f\|_{BMO(G)} = 0$, then the function f is constant almost everywhere. Conversely, if f is constant almost everywhere, then $\|f\|_{BMO(G)} = 0$. Thus, $\|\cdot\|_{BMO(G)}$ is only a semi-norm. Therefore, it will be convenient to consider the quotient of $BMO(G)$ by the space of almost everywhere constant functions. Abusing notation, we still denote the resulting space by $BMO(G)$. Hence, f and $f + c$ have the same $BMO(G)$-norm and they are always identified in the same class.

4.3.1 Some Characterizations of $BMO(G)$

We now present some equivalent characterizations of $BMO(G)$ which will be crucial for the proofs of some subsequent results. First we introduce some notation. Let $f \in L^1_{\text{loc}}(G)$. Suppose that for every base set $V \in \mathcal{B}$, there is a constant f_V, which may depend on V, such that

$$\frac{1}{\mu(V)} \int_V |f(x) - f_V|\, d\mu(x) \leq M, \tag{4.3.2}$$

where $M < \infty$ is a constant that does not depend on V. We then define the *norm* $\|f\|_* := \inf\{M\}$, where the infimum runs over all constants M as above and all the numbers $\{f_V\}_{V \in \mathcal{B}}$ satisfying (4.3.2). The following proposition contains a useful characterization of the space $BMO(G)$.

Proposition 4.3.2. *The quantities $\|\cdot\|_*$ and $\|\cdot\|_{BMO(G)}$ are equivalent. That is, $\|\cdot\|_* \sim \|\cdot\|_{BMO(G)}$.*

Proof. This result can be proved in a straightforward manner by using only triangle inequality and hence we omit the details. □

We now present another characterization for a suitable subspace of $BMO(G)$. Fix a real-valued function f in $BMO(G)$. Let

$$M_{f,V} = \inf_{\alpha \in \mathbb{R}} m_V(|f - \alpha|)$$

and

$$\mathcal{M}_{f,V} = \{\hat{\alpha} : m_V(|f - \hat{\alpha}|) = M_{f,V}\}.$$

It can be shown that the set $\mathcal{M}_{f,V}$ is nonempty. We call $\mathcal{M}_{f,V}$ the set of all medians of f on V. Now from each set $\mathcal{M}_{f,V}$, we choose an element $\alpha_f(V)$, and we call it the median of f on V. Notice that for any real number c, $c + \alpha_f(V) \in \mathcal{M}_{f+c,V}$. Now, we make an agreement on $\alpha_{f+c}(V)$ and $c + \alpha_f(V)$ that $\alpha_{f+c}(V) = c + \alpha_f(V)$.

It can be shown that $\alpha_f(V)$ satisfies

$$\mu\big(\{x \in V : f(x) > \alpha_f(V)\}\big) \leq \frac{\mu(V)}{2} \tag{4.3.3}$$

and

$$\mu\big(\{x \in V : f(x) < \alpha_f(V)\}\big) \leq \frac{\mu(V)}{2}; \tag{4.3.4}$$

see [66, page 30].

Furthermore, we denote $\|f\|_\circ$ to be the minimal nonnegative constant C such that for any base set V,

$$\frac{1}{\mu(V)} \int_V |f(x) - \alpha_f(V)| \, d\mu(x) \leq C$$

holds. Notice that $\|f\|_\circ = \|f + c\|_\circ$ for any real number c.

Proposition 4.3.3. *If f is real-valued, then the quantities $\|f\|_\circ$ and $\|f\|_*$ are equivalent, i.e., $\|f\|_\circ \sim \|f\|_*$.*

Proof. The proof of $\|f\|_* \leq \|f\|_\circ$ is clear. So we only prove the reverse inequality $\|f\|_\circ \lesssim \|f\|_*$. To this end, suppose $f \in L^1_{\text{loc}}(G)$ be such that $\|f\|_* < \infty$.

Let $\epsilon > 0$ be arbitrary. Then from the definition of $\|f\|_\star$, there exists a collection of numbers $\{f_V\}_{V \in \mathcal{B}}$ such that

$$\frac{1}{\mu(V)} \int_V |f(x) - f_V| \, d\mu \leq \|f\|_\star + \epsilon. \tag{4.3.5}$$

Using (4.3.5), we find that for every base set $V \in \mathcal{B}$,

$$\begin{aligned} &\frac{1}{\mu(V)} \int_V \big|f(x) - \alpha_f(V)\big| \, d\mu \\ &\leq \Big(\frac{1}{\mu(V)} \int_V \big|f(x) - f_V\big| \, d\mu\Big) + \big|f_V - \alpha_f(V)\big| \end{aligned}$$

$$\leq \quad \|f\|_* + \epsilon + \big|f_V - \alpha_f(V)\big|.$$

Now using the definition of $\alpha_f(V)$, we further obtain

$$\begin{aligned}\big|f_V - \alpha_f(V)\big| &\leq \frac{1}{\mu(V)}\int_V \Big[\big|f(x) - f_V\big| + \big|f(x) - \alpha_f(V)\big|\Big]\, d\mu \\ &\leq \frac{2}{\mu(V)}\int_V \big|f(x) - f_V\big|\, d\mu \leq 2\,\big(\, \|f\|_* + \epsilon\,\big).\end{aligned}$$

Thus, we have

$$\frac{1}{\mu(V)}\int_V \big|f(x) - \alpha_f(V)\big|\, d\mu \leq 3\,\big(\, \|f\|_* + \epsilon\,\big)\,.$$

Since ϵ is arbitrary, it follows from the definition of $\|\cdot\|_\circ$ that $\|f\|_\circ \lesssim \|f\|_*$. □

Corollary 4.3.4. *For real-valued functions, the quantities $\|\cdot\|_{BMO(G)}$ and $\|\cdot\|_\circ$ are equivalent.*

4.3.2 The Space $BMO(G)$ and the Inequality of John–Nirenberg

At this stage, we are in a position to describe the most important behaviour of functions with bounded mean oscillation. It is well-known in the Euclidean setting that the logarithmic blowup is the worst possible behaviour for a BMO function. Our goal here is to prove an analogous version of this result in the present setting of LCA groups. The following theorem is the main result of this section.

Theorem 4.3.5. *Let $f \in BMO(G)$. Then there exist constants C_1 and C_2, independent of f, such that for every $\lambda > 0$ and for every $U \in \mathcal{B}$, one has*

$$\mu\Big(\big\{x \in U : |f(x) - m_U(f)| > \lambda\big\}\Big) \leq C_1 \exp\Big(\frac{-C_2\lambda}{\|f\|_{BMO(G)}}\Big)\mu(U)\,. \qquad (4.3.6)$$

Proof. The main idea is borrowed from [77, Theorem 1.4]. Let $f \in BMO(G)$ and $\lambda > 0$ be given. For any base set $U \in \mathcal{B}$, let $\mathcal{D}_\lambda(U)$ denote the distribution set of the function $f - m_U(f)$ on U at scale $\lambda > 0$, that is,

$$\mathcal{D}_\lambda(U) = \{x \in U : |f(x) - m_U(f)| > \lambda\}.$$

Next, we define a function Θ by

$$\Theta(\lambda) = \sup_{U \in \mathcal{B}} \frac{\mu(\mathcal{D}_\lambda(U))}{\mu(U)}, \quad \lambda > 0.$$

So, in our new notation, (4.3.6) is equivalent to

$$\Theta(\lambda) \leq C_1 \exp\left(\frac{-C_2\lambda}{\|f\|_{BMO(G)}}\right) \quad \text{for all } \lambda > 0 \tag{4.3.7}$$

for some constants $C_1, C_2 > 0$.

We may assume that $\|f\|_{BMO(G)} = 1$, because the above inequality remains unaffected if we replace f and λ by their same constant multiples. Notice that we don't need to worry about the validity of the above inequality for λ not too large as the function Θ is always bounded by 1. On the other hand, if we can manage to show that there exists a constant $\lambda_0 > 0$, independent of every base set U, such that for all $\lambda > \lambda_0$,

$$\frac{\mu(\mathcal{D}_\lambda(U))}{\mu(U)} \leq \frac{1}{2}\Theta(\lambda - \lambda_0) \tag{4.3.8}$$

holds, then iteration of this inequality yields (4.3.7) with $\|f\|_{BMO(G)} = 1$. Thus, Theorem 4.3.5 is reduced to proving the inequality (4.3.8). What follows is a proof this inequality.

Fix a base set $U = x_0 + U_k$. Observe that we may assume that $m_U(f) = 0$. Combining (4.3.1) with the fact that $\|f\|_{BMO(G)} = 1$, we obtain

$$m_{U^*}(|f|) \leq D, \tag{4.3.9}$$

where $U^* = x_0 + U_{\theta(k)}$. A key object in our proof is the family of base sets associated with U,

$$\mathcal{B}_U = \{y + U_i : y \in U, i \leq k\}.$$

Corresponding to this family of base sets, we define $\widetilde{U} = \bigcup_{V \in \mathcal{B}_U} V$. Then each member of $\widetilde{U}$ is contained in U^*. Indeed, if $V = y + U_i$ with $y \in U$ and $i \leq k$, take any $z \in V$. Then $z = y + u$ with $u \in U_i \subseteq U_k$. Since $y \in U$, we can write $y = x_0 + v$, $v \in U_k$. Thus,

$$z = x_0 + u + v \in x_0 + U_k + U_k \subseteq x_0 + U_{\theta(k)} = U^*.$$

As a consequence, we have $\widetilde{U} \subseteq U^*$.

Now we consider the Hardy–Littlewood maximal operator M_U restricted to the family $\mathcal{B}_U$ (if we replace U by another base set, then the maximal operator changes), that is,

$$M_U f(x) = \sup_{x \in V \in \mathcal{B}_U} \frac{1}{\mu(V)} \int_V |f(y)| \, d\mu(y),$$

with the convention that $M_U f(x) = 0$ if there is no base set in $\mathcal{B}_U$ containing x. In particular, $M_U f$ vanishes outside of U^*. By the Lebesgue differentiation theorem, we have

$$|f(x)| \leq M_U f(x) \quad \text{for a.e.}\, x \in U. \tag{4.3.10}$$

We now consider the distribution set Ω_λ of $M_U f$ at scale $\lambda > 0$:

$$\Omega_\lambda = \{x \in G : M_U f(x) > \lambda\}.$$

Clearly, this set is contained in U^*. In the following lemma, we present a decomposition of Calderón–Zygmund type for the set Ω_λ.

Lemma 4.3.6. *Let $\lambda > D^8$ and $\Omega_\lambda \neq \emptyset$. Then there exists a pairwise disjoint sequence $\{V_i\}_{i \in I}$ in $\mathcal{B}_U$ such that*

(i) $\bigcup_{i \in I} V_i \subseteq \Omega_\lambda \subseteq \bigcup_{i \in I} V_i^{4*}$;

(ii) $m_{V_i}(|f|) > \lambda$ *for all* $i \in I$;

(iii) $m_{V_i^{4*}}(|f|) \leq \lambda$ *for all* $i \in I$.

Proof. Define a function $\alpha : \Omega_\lambda \to \mathbb{Z}$ by

$$\alpha(x) = \max\{j \in \mathbb{Z} : \exists\, V = y + U_j \in \mathcal{B}_U, x \in V, m_V(|f|) > \lambda\}.$$

This mapping is well-defined since $V = y + U_j \in \mathcal{B}_U$ implies $j \leq k$. So, for each $x \in \Omega_\lambda$, we choose a base set $V_x \in \mathcal{B}_U$ so that $x \in V_x = y_x + U_{\alpha(x)}$ for some $y_x \in U$. Set $\Sigma = \bigcup_{x \in \Omega_\lambda} V_x$. Since $V_x \subseteq U^*$, it is obvious that $\mu(\Sigma) < \infty$. Now we claim that, for any $x \in \Omega_\lambda$, $V_x^{4*} \in \mathcal{B}_U$. For this, it suffices to show that $\theta^4(\alpha(x)) \leq k$. If not, then $\theta^4(\alpha(x)) > k$. This implies that $\theta^6(\alpha(x)) > \theta^2(k)$. Also, since $y_x \in V_x$, we have

$V_x^{6*} \cap U^{**} \neq \emptyset$. Hence, by the engulfing property (Lemma 4.1.3), we obtain $U^{**} \subseteq V_x^{8*}$. This fact, together with the doubling property of μ, gives us

$$\frac{\mu(U^*)}{\mu(V_x)} \leq D\frac{\mu(U^{**})}{\mu(V_x^*)} \leq D\frac{\mu(V_x^{8*})}{\mu(V_x^*)} \leq D^8.$$

Taking this into account and since $V_x \subseteq U^*$, we deduce that

$$\frac{1}{\mu(V_x)} \int_{V_x} |f|\, d\mu \leq \frac{\mu(U^*)}{\mu(V_x)} \frac{1}{\mu(U^*)} \int_{U^*} |f|\, d\mu \leq D^8 < \lambda,$$

which is a contradiction to the fact $m_{V_x}(|f|) > \lambda$. Thus, the claim holds true.

Let $\mathcal{V}_1 = \{V_x = y_x + U_{\alpha(x)} : x \in \Omega_\lambda\}$. The selection process of the desired V_i's from the collection $\mathcal{V}_1$ is based on an iteration process and is divided into the following three steps.

Step I. We start by picking a base set $V_{0,1} = x_{0,1} + U_{\alpha(x_{0,1})} \in \mathcal{V}_1$ such that

$$2\mu(V_{0,1}) > \sup_{V_x \in \mathcal{V}_1} \mu(V_x).$$

Define the sets $\widetilde{\mathcal{V}_1}$ and $\mathcal{I}_1$ as follows:

$$\widetilde{\mathcal{V}_1} = \{V_x \in \mathcal{V}_1 : V_x^{**} \cap V_{0,1} \neq \emptyset\}$$

and

$$\mathcal{I}_1 = \{\alpha(x) : V_x \in \widetilde{\mathcal{V}_1}\}.$$

It is obvious that $\widetilde{\mathcal{V}_1} \neq \emptyset$ and that $\max \mathcal{I}_1 < \infty$ is attained. We then choose a base set $x_1 + U_{\alpha(x_1)} \in \widetilde{\mathcal{V}_1}$ such that $\alpha(x_1) = \max \mathcal{I}_1$. Call $V_1 = x_1 + U_{\alpha(x_1)}$. Now we claim that if $V_x = y_x + U_{\alpha(x)} \in \mathcal{V}_1$ and $V_x \cap V_1 \neq \emptyset$, then $V_x \subseteq V_1^{4*}$. To see this, we first observe that $\alpha(x) \leq \theta^2(\alpha(x_1))$ must hold. Otherwise, if $\alpha(x) > \theta^2(\alpha(x_1))$, then $\alpha(x_1) < \alpha(x)$. Now by the choice of $V_1 \in \widetilde{\mathcal{V}_1}$, we know that $V_1^{**} \cap V_{0,1} \neq \emptyset$. Hence, there exists some $u \in U_{\theta^2(\alpha(x_1))}$ and $v \in V_{0,1}$ such that $x_1 + u = x_{0,1} + v$. Furthermore, $V_x \cap V_1 \neq \emptyset$ implies that $x + a = x_1 + b$, for some $a \in U_{\alpha(x)}$ and $b \in U_{\alpha(x_1)}$. Thus, we get $x + a - b + u = x_{0,1} + v$. Now, we see that

$$a - b + u \quad \in \quad U_{\alpha(x)} - U_{\alpha(x_1)} + U_{\theta^2(\alpha(x_1))} \subseteq U_{\alpha(x)} - U_{\alpha(x_1)} + U_{\alpha(x)}$$

$$\subseteq \quad U_{\alpha(x)} - U_{\alpha(x)} + U_{\theta(\alpha(x))} \subseteq U_{\theta(\alpha(x))} + U_{\theta(\alpha(x))} \subseteq U_{\theta^2(\alpha(x))}.$$

Therefore,

$$x_{0,1} + v = x + a - b + u \in x + U_{\theta^2(\alpha(x))}.$$

But this means actually $\big(x + U_{\theta^2(\alpha(x))}\big) \bigcap V_{0,1} \neq \emptyset$. Hence, it follows from the definition of $\widetilde{\mathcal{V}_1}$ that $V_x \in \widetilde{\mathcal{V}_1}$ and $\alpha(x) \leq \alpha(x_1) \leq \theta^2(\alpha(x_1))$. This is a contradiction. So, we have $\big(x + U_{\alpha(x)}\big) \bigcap \big(x_1 + U_{\theta^2(\alpha(x_1))}\big) \neq \emptyset$ with the condition that $\alpha(x) \leq \theta^2(\alpha(x_1))$. From the engulfing property, we conclude that $\big(x + V_{\alpha(x)}\big) \subseteq \big(x_1 + U_{\theta^4(\alpha(x_1))}\big)$.

Step II. Next, let $\mathcal{V}_2 = \{V_x \in \mathcal{V}_1 : V_x \cap V_1 = \emptyset\}$. We choose a base set $V_{0,2} = x_{0,2} + U_{\alpha(x_{0,2})} \in \mathcal{V}_2$ such that

$$2\mu(V_{0,2}) > \sup_{V_x \in \mathcal{V}_2} \mu(V_x).$$

Define the sets $\widetilde{\mathcal{V}_2}$ and $\mathcal{I}_2$ as follows:

$$\widetilde{\mathcal{V}_2} = \{V_x \in \mathcal{V}_2 : V_x^{**} \cap V_{0,2} \neq \emptyset\}$$

and

$$\mathcal{I}_2 = \{\alpha(x) : V_x \in \widetilde{\mathcal{V}_2}\}.$$

Proceeding as in Step I, we can get $V_2 = x_2 + U_{\alpha(x_2)} \in \widetilde{\mathcal{V}_2}$ such that if $V_x \in \mathcal{V}_2$ and $V_x \cap V_2 \neq \emptyset$, then $V_x \subseteq V_2^{4*}$.

Step III. We continue as above. If the process stops after N steps, then $\{V_i = x_i + U_{\alpha(x_i)} : i = 1, 2, \dots, N\}$ is the required collection. Otherwise, this process generates a countable collection of disjoint base sets $\{V_i = x_i + U_{\alpha(x_i)} : i = 1, 2, \dots\}$. Now, we show that if $V_x \in \mathcal{V}_1$, then it intersects with at least one $V_i = x_i + U_{\alpha(x_i)}$. If this is not the case, then we would have $V_x \bigcap V_i = \emptyset$ for all i. This means that $V_x \in \mathcal{V}_i$, and hence $2\mu(V_{0,i}) > \mu(V_x)$, for every i.

On the other hand, by the choice of V_i, $V_i^{**} \bigcap V_{0,i} \neq \emptyset$ holds. That is, $\big(x_i + U_{\theta^2(\alpha(x_i))}\big) \bigcap \big(x_{0,i} + U_{\alpha(x_{0,i})}\big) \neq \emptyset$ holds. Also, since $V_{0,i}^{**} \bigcap V_{0,i} \neq \emptyset$, it follows, from the definition of $\max \mathcal{I}_i$ that $\alpha(x_{0,i}) \leq \alpha(x_i) \leq \theta^2(\alpha(x_i))$. Therefore, by the engulfing property, we obtain $V_{0,i} \subseteq V_i^{4*}$.

Summarizing these, we have

$$0 < \mu(V_x) < 2\mu(V_{0,i}) \leq 2\mu(V_i^{4*}) \leq 2D^4\mu(V_i).$$

Since $\{V_i\}_{i\in I}$ is a pairwise disjoint infinite collection of base sets, this will imply that $\mu(\Sigma) = \infty$, a contradiction. Hence, we conclude that every V_x in $\mathcal{V}_1$ intersects at least one V_i and in that case $V_x \subseteq V_i^{4*}$. Clearly, the sequence $\{V_i\}_{i\in I}$ satisfies the desired conditions (i) and (ii). Since $V_i^{4*} = x_i + U_{\theta^4(\alpha(x_i))} \in \mathcal{B}_U$, item (iii) follows from the definition of $\alpha(x_i)$. This completes the proof of the lemma. □

Let us return to the proof of the inequality (4.3.8). Set $\lambda_0 = \max\{2D^5, D^8\} + 1$. Then for all $\lambda > \lambda_0$, we have $\mathcal{D}_\lambda(U) \subseteq \mathcal{D}_{\lambda_0}(U)$. Moreover, combining (4.3.10) with the fact that $m_U f = 0$, we see that $\mathcal{D}_{\lambda_0}(U) \subseteq \Omega_{\lambda_0}$. Thus, we write

$$\mathcal{D}_\lambda(U) = \mathcal{D}_\lambda(U) \cap \mathcal{D}_{\lambda_0}(U) \subseteq \mathcal{D}_\lambda(U) \cap \Omega_{\lambda_0}.$$

We now apply Lemma 4.3.6 at the value λ_0 to obtain a pairwise disjoint sequence of base sets $\{V_i\}_{i\in I}$ satisfying the conditions (i)–(iii). Therefore, we have

$$\mathcal{D}_\lambda(U) \cap \Omega_{\lambda_0} \subseteq \bigcup_{i\in I} \big(V_i^{4*} \cap \mathcal{D}_\lambda(U)\big).$$

Now, for each $i \in I$, $V_i^{4*} \cap \mathcal{D}_\lambda(U) \subseteq \mathcal{D}_{\lambda-\lambda_0}(V_i^{4*})$ holds. To see this, notice that $m_{V_i^{4*}}(|f|) \leq \lambda_0$. These facts imply that, for all $x \in V_i^{4*} \cap \mathcal{D}_\lambda(U)$,

$$\begin{aligned} \lambda < |f(x)| &\leq |f(x) - m_{V_i^{4*}}(f)| + |m_{V_i^{4*}}(f)| \\ &\leq |f(x) - m_{V_i^{4*}}(f)| + m_{V_i^{4*}}(|f|) \\ &\leq |f(x) - m_{V_i^{4*}}(f)| + \lambda_0. \end{aligned}$$

That is, $|f(x) - m_{V_i^{4*}}(f)| \geq \lambda - \lambda_0$. Hence, from the definition, the desire inclusion follows. So, summarizing the above discussions, we find that $\mathcal{D}_\lambda(U) \subseteq \bigcup_i \big(\mathcal{D}_{\lambda-\lambda_0}(V_i^{4*})\big)$. Therefore, we conclude that

$$\begin{aligned} \mu(\mathcal{D}_\lambda(U)) &\leq \sum_i \mu(\mathcal{D}_{\lambda-\lambda_0}(V_i^{4*})) = \sum_i \mu(V_i^{4*}) \frac{\mu(\mathcal{D}_{\lambda-\lambda_0}(V_i^{4*}))}{\mu(V_i^{4*})} \\ &\leq D^4\,\Theta(\lambda-\lambda_0) \sum_i \mu(V_i) \leq D^4 \frac{\Theta(\lambda-\lambda_0)}{\lambda_0} \sum_i \int_{V_i} |f|\, d\mu \end{aligned}$$

$$\begin{aligned} &\leq D^4 \frac{\Theta(\lambda-\lambda_0)}{\lambda_0} \int_{U*} |f|\, d\mu \leq D^4 \frac{\Theta(\lambda-\lambda_0)}{\lambda_0} \mu(U^*) \\ &\leq D^5 \frac{\Theta(\lambda-\lambda_0)}{\lambda_0}\, \mu(U) \leq \frac{1}{2}\Theta(\lambda-\lambda_0)\, \mu(U). \end{aligned}$$

In the above chain of inequalities, we have used the doubling property of μ, the definition of Θ, Lemma 4.3.6(ii), and the inequality (4.3.9). Thus, inequality (4.3.8), to which we reduced the proof of Theorem 4.3.5, holds. The theorem is now completely proven. □

We conclude this section with the following consequence of Theorem 4.3.5.

Corollary 4.3.7. *Let G be an LCA group and $1 \leq p < \infty$. There is a constant $C > 0$ such that for every $f \in BMO(G)$ and every base set $V \in \mathcal{B}$,*

$$\Big(\frac{1}{\mu(V)} \int_V |f - m_V(f)|^p\, d\mu\Big)^{\frac{1}{p}} \leq C\|f\|_{BMO(G)}.$$

4.4 Duality

This section is devoted to prove that the spaces $H^{1,q}(G)$, $1 < q \leq \infty$, coincide and that the dual of $H^{1,\infty}(G)$ may be identified with the space $BMO(G)$, simultaneously. Following the scheme provided by Journé [66], we first prepare several auxiliary lemmas.

Let $g \in BMO(G)$. We define a linear functional $\mathcal{L}_g$ on $H^{1,q}_{fin}(G)$ by setting

$$\mathcal{L}_g(f) = \int_G f(x)g(x)\, d\mu(x), \quad f \in H^{1,q}_{fin}(G). \tag{4.4.1}$$

From Corollary 4.3.7, it follows that $g \in L^{q'}_{\text{loc}}(G)$, and hence the action of the functional $\mathcal{L}_g$, defined in (4.4.1), makes sense on $H^{1,q}_{fin}(G)$. In fact, $\mathcal{L}_g$ extends to a bounded linear functional on $H^{1,q}(G)$. This can be proved by following the line of the proof in [24] which is based on the classical result of Fefferman and Stein [37, 38], and hence we omit the details and record this result as a lemma for later purposes.

Lemma 4.4.1. *For $1 < q \leq \infty$, $BMO(G) \subseteq [H^{1,q}(G)]^*$. More precisely, for any $g \in BMO(G)$, the linear functional $\mathcal{L}_g$ defined in (4.4.1) for functions in $H^{1,q}_{fin}(G)$,* *can be extended to a bounded linear functional* $\mathcal{L}_g$ *over* $H^{1,q}(G)$ *with*

$$\|\mathcal{L}_g\|_{[H^{1,q}(G)]^*} \lesssim \|g\|_{BMO(G)}. \tag{4.4.2}$$

If we take $q = \infty$ in the above lemma, we have slightly more information: the reverse of inequality (4.4.2) can also be achieved. To be more precise, we have the following result.

Lemma 4.4.2. *If* $g \in BMO(G)$, *we have*

$$\|\mathcal{L}_g\|_{[H^{1,\infty}(G)]^*} \sim \|g\|_{BMO(G)}.$$

Proof. Let $g \in BMO(G)$. Without loss of generality, we may assume that g is real-valued. In view of Lemma 4.4.1, we only need to prove that

$$\|\mathcal{L}_g\|_{[H^{1,\infty}]^*} \gtrsim \|g\|_{BMO(G)}.$$

In addition, with the aid of Corollary 4.3.4, we only have to construct a $(1,\infty)$-atom b such that

$$|\mathcal{L}_g(b)| \gtrsim \|g\|_\circ \|b\|_{H^{1,\infty}(G)}. \tag{4.4.3}$$

First of all, by the definition of $\|g\|_\circ$, there exists a base set $V \in \mathcal{B}$ satisfying the condition

$$\frac{\|g\|_\circ}{2} \leq \frac{1}{\mu(V)} \int_V |g(x) - \alpha_g(V)|\, d\mu(x).$$

Decompose the base set $V = V_1 \cup V_2 \cup V_3$, where $V_1 = \{x \in V : g(x) > \alpha_g(V)\}$, $V_2 = \{x \in V : g(x) < \alpha_g(V)\}$ and $V_3 = \{x \in V : g(x) = \alpha_g(V)\}$. Now define a function b by

$$b(x) = \begin{cases} 0 & \text{if} \quad x \notin V, \\ 1 & \text{if} \quad x \in V_1, \\ -1 & \text{if} \quad x \in V_2. \end{cases}$$

We are going to show that b is a $(1,q)$-atom and satisfies (4.4.3). By (4.3.3) and (4.3.4), $\mu(V_1) \leq \frac{\mu(V)}{2}$ and $\mu(V_2) \leq \frac{\mu(V)}{2}$. Using these facts, we can assign values ± 1 to b on V_3 suitably to make $\int_G b(x)\, d\mu(x) = 0$. From this, together with the boundedness of b, we conclude that b is a $(1,q)$ atom. Furthermore, the definition of b

implies that

$$\|b\|_{H^{1,\infty}(G)} \lesssim \mu(V).$$

We now estimate the quantity $\mathcal{L}_g(b)$ as follows:

$$\begin{aligned}
\left|\mathcal{L}_g(b)\right| &= \Big|\int_G g(x)b(x)\, d\mu(x)\Big| \\
&= \Big|\int_G \left[g(x) - \alpha_g(V)\right] b(x)\, d\mu(x)\Big| \\
&= \int_G \left|g(x) - \alpha_g(V)\right| d\mu(x) \\
&\geq \|g\|_\circ \frac{\mu(V)}{2} \gtrsim \|g\|_\circ \|b\|_{H^{1,\infty}(G)}.
\end{aligned}$$

That is,

$$\left|\mathcal{L}_g(b)\right| \gtrsim \|g\|_\circ \|b\|_{H^{1,\infty}(G)}.$$

Hence, $\|\mathcal{L}_g\|_{[H^{1,\infty}]^*} \gtrsim \|g\|_{BMO(G)}$. This finishes the proof of the lemma. □

If we ignore the case when $q = \infty$ in Lemma 4.4.1, then we have the converse conclusion as well. That is, for $1 < q < \infty$, every $\mathcal{L} \in [H^{1,q}(G)]^*$ is essentially $\mathcal{L}_g$ for some $g \in BMO(G)$. To see this, we need some notation and preliminary observations. So let $\mathcal{L} \in \left[H^{1,q}(G)\right]^*$. For any base set $V \in \mathcal{B}$, let $L_0^q(V, \mu)$ denote the space of all functions $f \in L^q(V, \mu)$ such that $\int_V f(x)\, d\mu(x) = 0$. If $f \in L_0^q(V, \mu)$, then $f \in H^{1,q}(G)$ and

$$\|f\|_{H^{1,q}(G)} \leq \left(\mu(V)\right)^{\frac{1}{q'}} \|f\|_{L^q(V,\mu)}.$$

Consider the restriction of $\mathcal{L}$ to $L_0^q(V, \mu)$. For all $f \in L_0^q(V, \mu)$, we have

$$\left|\mathcal{L}(f)\right| \leq \|\mathcal{L}\|_{[H^{1,q}(G)]^*} \left(\mu(V)\right)^{\frac{1}{q'}} \|f\|_{L^q(V,\mu)}.$$

Therefore, $\mathcal{L}$ defines a bounded linear functional on $L_0^q(V, \mu)$. By Hahn–Banach extension theorem, $\mathcal{L}$ has a unique bounded extension on $L^q(V, \mu)$. Since $1 < q < \infty$, by the Riesz representation theorem, there exists a unique function $h \in L^{q'}(V, \mu)$ which represents the restriction of $\mathcal{L}$ on V:

$$\mathcal{L}(f) = \int_V f(x)\, h(x)\, d\mu(x) \quad \text{for all } f \in L_0^q(V, \mu).$$

Lemma 4.4.3. *If $1 < q < \infty$, then $\left[H^{1,q}(G)\right]^* \subseteq BMO(G)$; that is, for any bounded linear functional $\mathcal{L}$ on $H^{1,q}(G)$, there exists $g \in BMO(G)$ such that*

$$\mathcal{L}(f) = \int_G f(x)g(x)\,d\mu(x) \quad \text{for all } f \in H^{1,q}(G). \tag{4.4.4}$$

Proof. Suppose $\mathcal{L} \in \left[H^{1,q}(G)\right]^*$. We begin by constructing such a function g by a labourious but conceptually simple argument as follows.

Fix a base set $V = x + U_i$ and let $V_n = V^{n*}$, $n \in \mathbb{N}$. By the above observation, for each $n \in \mathbb{N}$, we get $g_n \in L^{q'}(V_n, \mu)$ such that

$$\mathcal{L}(f) = \int_{V_n} f(x)g_n(x)\,d\mu(x), \quad f \in L_0^q(V_n, \mu)\,.$$

Moreover, we see that for all $j, l \in \mathbb{N}$ with $j \le l$ and μ-almost every $x \in V_j$, $g_j(x) - m_{V_j}(g_j) = g_l(x) - m_{V_l}(g_l)$. Indeed, for any $f \in L_0^q(V_j, \mu) \subseteq L_0^q(V_l, \mu)$, we have

$$\mathcal{L}(f) = \int_{V_j} f(x)g_j(x)\,d\mu(x) = \int_{V_l} f(x)g_l(x)\,d\mu(x) = \int_{V_j} f(x)g_l(x)\,d\mu(x)\,.$$

This implies that for all $f \in L_0^q(V_j, \mu)$,

$$\int_{V_j} f(x)\Big[g_j(x) - g_l(x)\Big]\,d\mu(x) = 0.$$

Now, notice that, for any $h \in L^q(V_j, \mu)$, we have $h - m_{V_j}(h) \in L_0^q(V_j, \mu)$. Therefore,

$$\begin{aligned} 0 &= \int_{V_j} \Big(g_j(x) - g_l(x)\Big)\Big[h(x) - m_{V_j}(h)\Big]\,d\mu(x) \\ &= \int_{V_j} h(x)\Big[g_j(x) - g_l(x) - m_{V_j}(g_j) + m_{V_l}(g_l)\Big]\,d\mu(x) \end{aligned}$$

for all $h \in L^q(V_j, \mu)$. Hence, $g_j(x) - m_{V_j}(g_j) = g_l(x) - m_{V_l}(g_l)$ for almost every $x \in V_j$.

Define

$$g = g_j - m_{V_j}(g_j) \quad \text{on } V_j,\ j \in \mathbb{N}.$$

Then g is well-defined and we have

$$\mathcal{L}(f) = \int_{V_j} f(x)\,g(x)\,d\mu(x) \quad \text{for all } f \in L_0^q(V_j, \mu), j \in \mathbb{N}. \tag{4.4.5}$$

Now we claim that $g \in BMO(G)$ and satisfies (4.4.4).

First we need the following fact: Let $V = y + U_j$ be an arbitrary base set from $\mathcal{B}$. Then there exists $k \in \mathbb{N}$ such that $V \subseteq V_k$. Recall that $\{V_n\}_{n\in\mathbb{N}}$ is an increasing family and $\bigcup_{n\in\mathbb{N}} V_n = G$, where $V_n = V^{n*} = x + U_{\theta^n(i)}$. Hence, we can find $l \in \mathbb{N}$ such that $j \le \theta^l(i)$ and $\big(y + U_j\big) \cap \big(x + U_{\theta^l(i)}\big) \neq \emptyset$. Then from the engulfing property of base sets, we have $y + U_j \subset x + U_{\theta^{l+2}(i)}$. That is, $V \subseteq V_{l+2}$ with $k = l + 2$.

Now, let us return to the proof of (4.4.4). For any $f \in H^{1,q}(G)$, we may write $f(x) = \sum_{k=0}^{\infty} \lambda_k\, b_k(x)$, where b_k is a $(1, q)$-atom supported on the base set $F_k \in \mathcal{B}$. By the above fact, for each k there exists j_k such that $F_k \subset V_{j_k}$. Therefore, from (4.4.5) and the linearity of $\mathcal{L}$, we have

$$\mathcal{L}(f) = \sum_{k=0}^{\infty} \lambda_k\, \mathcal{L}\,(b_k) = \sum_{k=0}^{\infty} \lambda_k \int_{V_{j_k}} b_k(x)\, g(x)\, d\mu(x) = \int_G f(x)\, g(x)\, d\mu(x).$$

To complete the proof of Lemma 4.4.3, we are thus left with the task of proving that $g \in BMO(G)$. For any fixed base set $V \in \mathcal{B}$, we notice that

$$\begin{aligned}
&\frac{1}{\mu(V)} \int_V \big|g(x) - m_V(g)\big|\, d\mu(x)\\
&\le \big(\mu(V)\big)^{\frac{-1}{q'}} \big\|g - m_V(g)\big\|_{L^{q'}(V,\,\mu)}\\
&= \big(\mu(V)\big)^{\frac{-1}{q'}} \sup_{\|h\|_{L^q(V,\,\mu)}\le 1} \Big| \int_V h(x)\big[g(x) - m_V(g)\big]\, d\mu(x)\Big|\\
&= \big(\mu(V)\big)^{\frac{-1}{q'}} \sup_{\|h\|_{L^q(V,\,\mu)}\le 1} \Big| \int_V g(x)\big[h(x) - m_V(h)\big]\, d\mu(x)\Big|.
\end{aligned}$$

Now suppose that $h \in L^q(V, \mu)$ and $\|h\|_{L^q(V,\mu)} \le 1$. Set $b(x) = \frac{1}{2}\big(\mu(V)\big)^{\frac{-1}{q'}}\big[h(x) - m_V(h)\big]\chi_V$. Then b is a $(1, q)$-atom supported in V and $\|b\|_{L^q(V,\mu)} \le 1$. This, together with (4.4.5), implies that

$$\big(\mu(V)\big)^{\frac{-1}{q'}} \Big| \int_V \big[h(x) - m_V(h)\big] g(x)\, d\mu(x)\Big| \le 2\big\|\mathcal{L}\big\|.$$

Hence,

$$\frac{1}{\mu(V)} \int_V \big|g(x) - m_V(g)\big|\, d\mu(x) \le 2\big\|\mathcal{L}\big\|.$$

Since the base set $V \in \mathcal{B}$ is arbitrary, we deduce that $g \in BMO(G)$ and that $\big\|g\big\|_{BMO(G)} \lesssim \big\|\mathcal{L}\big\|$. This completes the proof of the lemma. □

The time has come now to apply everything that we have prepared for the proof of results that we have mentioned in the beginning of this section. We state them together and present the proof.

Theorem 4.4.4. *For* $1 < q < \infty$, $H^{1,q}(G) = H^{1,\infty}(G)$. *Also* $[H^{1,\infty}(G)]^* = BMO(G)$.

Proof. First of all, we observe that by Lemmas 4.4.1 and 4.4.3, $[H^{1,q}(G)]^* = BMO(G)$ for $1 < q < \infty$. Next, by Proposition 4.2.2(ii), we see that $\big[H^{1,q}(G)\big]^* \subseteq \big[H^{1,\infty}(G)\big]^*$. We consider the maps

$$i : H^{1,\infty}(G) \to H^{1,q}(G)$$

and

$$i^* : BMO(G) = \big[H^{1,q}(G)\big]^* \to \big[H^{1,\infty}(G)\big]^*,$$

where i is the inclusion map and i^* is the canonical injection of $BMO(G)$ in $\big[H^{1,\infty}(G)\big]^*$. Here we identify g with $\mathcal{L}_g$ for $g \in BMO(G)$. We now claim that i maps $H^{1,\infty}(G)$ onto $H^{1,q}(G)$. To see this, first we notice that Lemma 4.4.2 implies $i^*\big(BMO(G)\big)$ is closed in $\big[H^{1,\infty}(G)\big]^*$. Now applying the Banach closed range theorem, we see that $H^{1,\infty}(G)$ is closed in $[H^{1,q}(G)]$. This, combined with Proposition 4.2.2(ii), implies that

$$H^{1,\infty}(G) = H^{1,q}(G)$$

as a set. We now recall that both $H^{1,\infty}(G)$ and $H^{1,q}(G)$ are Banach spaces. Therefore, by a corollary of open mapping theorem (see, e.g., [119, page 77]), we conclude that

$$H^{1,\infty}(G) = H^{1,q}(G)$$

with equivalent norms. As a consequence, we have that $[H^{1,\infty}(G)]^* = BMO(G)$. This completes the proof of the theorem. □

Thus, we have the liberty to define the Hardy space $H^1(G)$ by choosing any value of q. We choose the special value $q = \infty$:

$$H^1(G) := H^{1,\infty}(G) \quad \text{and} \quad \|\cdot\|_{H^1(G)} := \|\cdot\|_{H^{1,\infty}(G)}.$$

Remark 4.4.5. *Our another motivation for studying the Hardy space on LCA groups is a result of Kim [70]. In the special case* $G = \mathbb{Q}_p$, *the group of p-adic numbers,* *Kim [70] defined the space $BMO(\mathbb{Q}_p)$ associated with family of balls and the Haar measure that naturally arise on $\mathbb{Q}_p$ and proved a version of John–Nirenberg type inequality similar to the classical one. Hence, a natural and important question arises whether it is possible to set up a Hardy space on $\mathbb{Q}_p$, say $H^1(\mathbb{Q}_p)$, and characterize $BMO(\mathbb{Q}_p)$ as the dual of the space $H^1(\mathbb{Q}_p)$. Our Theorem 4.4.4 answers this question in an affirmative way in the more general setting of LCA groups.*

4.5 Application to Convolution Operators

We present an application of Theorem 4.4.4 to the theory of convolution operators. In order to bound the norms of certain multipliers on G, Edwards and Gaudry [32] introduced some specific convolution operators and estimated their norms as operators from $L^p(G)$ to $L^p(G)$. Our purpose here is to study their boundedness property from $H^1(G)$ to $L^1(G)$ and bound their $H^1(G) \to L^1(G)$-norms. To this end, let us first recall the relevant definitions and results. Let k be an integrable function on G. For each $l \in \mathbb{Z}$, define

$$J_l(k) = \sup_{y \in U_l} \int_{G \setminus U_{\theta(l)}} |k(x-y) - k(x)| \, d\mu(x)$$

and let $J(k) = \sup_{l \in \mathbb{Z}} J_l(k)$. With respect to such a kernel k, Edwards and Gaudry considered the usual convolution operator, that is,

$$L_k : f \to k \star f$$

and proved the following result.

Theorem 4.5.1. *Suppose $k \in L^1(G)$. Then*

(i) the weak $(1,1)$ norm of L_k on L^1 is at most

$$B = D^2 + 4D\|\hat{k}\|_\infty^2 + 4\,J(k),$$

(ii) for $p \in (1,\infty)$,

$$\|L_k\|_{L^p(G) \to L^p(G)} \leq D^{\frac{2}{p^*}} B_p \max\{J(k), \|\hat{k}\|_\infty\},$$

where $p^* = \min\{p, p'\}$ *and* B_p *depends solely on* p.

As an application of the atomic decomposition of $H^1(G)$, we prove the following result for the operator L_k.

Theorem 4.5.2. *The operator* L_k *is bounded from* $H^1(G)$ *to* $L^1(G)$. *Moreover,*

$$\|L_k\|_{H^1(G)\to L^1(G)} \le D^{\frac{3}{2}} B_2 \max\{J(k), \|\hat{k}\|_\infty\} + J(k).$$

For the proof of this theorem, we will require the following lemma, proved in [32].

Lemma 4.5.3. *Suppose that* $l \in \mathbb{Z}$, $x_0 \in G$, k *is an integrable function on* G, *and that* u *is an integrable function on* G *which vanishes off* $x_0 + U_l$, *and has zero integral. Then*

$$\int_{G\setminus(x_0+U_{\theta(l)})} |k \star u(x)|\, d\mu(x) \le \|u\|_1 J_l(k) \le \|u\|_1 J(k).$$

Proof of Theorem 4.5.2. Applying Theorem 4.4.4, it is enough to show that L_k is bounded from $H^{1,2}(G)$ to $L^1(G)$. Further, we observe that it suffices to show that there exists a constant $C > 0$ such that

$$\|L_k(b)\|_{L^1(G)} \le C$$

for all $(1, 2)$-atoms. Indeed, if $f = \sum_{j=0}^{\infty} \lambda_j\, b_j \in H^{1,2}(G)$, then from the linearity of L_k, we find that

$$\int_G |L_k\, f(x)|\, d\mu(x) \le \sum_{j=0}^{\infty} |\lambda_j| \int_G |L_k\, b_j(x)|\, d\mu(x) \le C \sum_{j=0}^{\infty} |\lambda_j|.$$

Taking infimum over all possible decompositions of f gives

$$\int_G |L_k\, f(x)|\, d\mu(x) \le C\|f\|_{H^{1,2}(G)}.$$

So let b be a $(1, 2)$-atom. Then from its definition, there exists a base set $V = x_0 + U_l \in \mathcal{B}$ such that $\operatorname{supp} b \subseteq V$, $\int_V b(x)\, d\mu(x) = 0$ and $\|b\|_{L^2(G)} \le \left[\mu(V)\right]^{\frac{-1}{2}}$. We write

$$\begin{aligned}\int_G |L_k\, b(x)|\, d\mu(x) &= \int_{V^*} |L_k\, b(x)|\, d\mu(x) + \int_{G\setminus V^*} |L_k\, b(x)|\, d\mu(x)\\ &= I_1 + I_2, \quad \text{say.}\end{aligned}$$

Applying Hölder's inequality, the doubling property of μ and Theorem 4.5.1 for the case $p = 2$, we obtain

$$\begin{aligned} I_1 = \int_{V^*} |L_k\, b(x)|\, d\mu(x) &\leq \big[\mu(V^*)\big]^{\frac{1}{2}} \cdot \Big[\int_{V^*} |L_k\, b(x)|^2\, d\mu(x)\Big]^{\frac{1}{2}} \\ &\leq D^{\frac{1}{2}} \cdot \big(\mu(V)\big)^{\frac{1}{2}}\, \|L_k\|_{L^2(G)\to L^2(G)}\, \|b\|_{L^2(G)} \\ &\leq D^{\frac{3}{2}}\, B_2\, \max\{J(k), \|\hat{k}\|_\infty\}\,. \end{aligned} \tag{4.5.1}$$

To estimate the second integral, we use Lemma 4.5.3 and Hölder's inequality. We have

$$\begin{aligned} I_2 &= \int_{G\setminus V^*} |k \star b(x)|\, d\mu(x) \\ &\leq \|b\|_{L^1(G)} \sup_{y\in V} \int_{G\setminus U_{\theta(l)}} |k(x-y) - k(x)|\, d\mu(x) \\ &\leq \big(\mu(V)\big)^{\frac{1}{2}} \|b\|_{L^2(G)} \cdot J(k) \leq J(k), \end{aligned} \tag{4.5.2}$$

which combined with (4.5.1), implies

$$\|L_k(b)\|_{L^1(G)} \leq D^{\frac{3}{2}} B_2 \max\{J(k), \|\hat{k}\|_\infty\} + J(k).$$

Hence, we conclude that

$$\int_G |L_k\, f(x)|\, d\mu(x) \leq \Big[D^{\frac{3}{2}} B_2 \max\{J(k), \|\hat{k}\|_\infty\} + J(k)\Big] \|f\|_{H^{1,2}(G)}.$$

□

Chapter 5

John–Nirenberg Spaces on LCA Groups

In the final chapter of this thesis, we explore the theory of John–Nirenberg spaces JN_p in the setting of LCA groups having covering families. The main result of this chapter is the John–Nirenberg inequality for functions in JN_p spaces which describes, as it happens in Euclidean setting, that JN_p can be embedded into weak L^p spaces.

5.1 The John–Nirenberg Space JN_p

To facilitate our discussion, we begin by recalling the space $JN_p(Q_0)$ that was introduced in Chapter 1. Let Q_0 be a cube in $\mathbb{R}^d$ with sides parallel to coordinate axes and $1 < p < \infty$. A function $f \in L^1(Q_0)$ is said to be in the space $JN_p(Q_0)$ if there exists a positive constant C such that

$$\|f\|_{JN_p(Q_0)} := \sup \Big(\frac{1}{|Q_0|} \sum_i \Big(\fint_{Q_i} |f - f_{Q_i}| \, dx \Big)^p |Q_i| \Big)^{\frac{1}{p}} \le C, \tag{5.1.1}$$

where the supremum is taken over all possible countable collections $\{Q_i\}_{i \in \mathbb{N}}$ of pairwise disjoint subcubes of Q_0 and

$$f_{Q_i} = \fint_{Q_i} f \, dx = \frac{1}{|Q_i|} \int_{Q_i} f \, dx.$$

There are other definitions of John–Nirenberg spaces in the literature depending on whether we require the cubes $\{Q_i\}_i$ in equation (5.1.1) to be mutually disjoint (see [7, 78]) or allow them to overlap in some nice and natural way (see [1]). In some cases, one obtains different spaces which are not equivalent. However, for Boman sets, all the definitions coincide and one ends up with a single space. For more details, the reader is referred to [78]. In case of spaces of homogeneous type, Berkovits et al. [7] defined a version of John–Nirenberg spaces by allowing the balls associated with the pseudo-metric to be pairwise disjoint. In the present context of LCA groups, we choose to adopt their approach. The concept of base sets allows us to extend the notion of John–Nirenberg spaces in the present framework as follows.

Definition 5.1.1. *Let G be an LCA group with a covering family and $1 < p < \infty$. Given a base set V and $f \in L^1(V)$, we say that f belongs to the John–Nirenberg space $JN_p(V)$ with exponent p if*

$$\|f\|_{JN_p(V)} := \sup \Big(\sum_i \big(\inf_{c_i \in \mathbb{R}} ⨍_{V_i} |f - c_i| \big)^p \mu(V_i) \Big)^{\frac{1}{p}} < \infty, \tag{5.1.2}$$

where the supremum runs over all collections $\{V_i\}$ of pairwise disjoint base sets in V.

Remark 5.1.2. *Likewise in the Euclidean spaces, we may also define the John–Nirenberg space via the integral averages f_{V_i} by replacing c_i in (5.1.2). To be specific, we may define*

$$\|f\|_{JN_p[V]} := \sup \Big(\sum_i \big(⨍_{V_i} |f - f_{V_i}| \, d\mu \big)^p \mu(V_i) \Big)^{\frac{1}{p}} < \infty.$$

However, both the definitions lead to the same space. Indeed, since

$$\|f\|_{JN_p(V)} \leq \Big(\sum_i \big(⨍_{V_i} |f - f_{V_i}| \, d\mu \big)^p \mu(V_i) \Big)^{\frac{1}{p}},$$

it follows that $\|f\|_{JN_p(V)} \lesssim \|f\|_{JN_p[V]}$. On the other hand, the reverse inequality

$$\Big(\sum_i \big(⨍_{V_i} |f - f_{V_i}| \, d\mu \big)^p \mu(V_i) \Big)^{\frac{1}{p}} \lesssim \|f\|_{JN_p(V)}$$

follows from the fact that $⨍_{V_i} |f - f_{V_i}| \, d\mu \leq 2 \inf_{c_i \in \mathbb{R}} ⨍_{V_i} |f - c_i| \, d\mu$. Hence, we have $\|f\|_{JN_p(V)} \sim \|f\|_{JN_p[V]}$. At this point, we would like to remark that in practice the *norm given in (5.1.2) is more flexible to establish some crucial results such as the John–Nirenberg inequality for functions in JN_p, as we shall see in the next section.*

5.1.1 Properties of the Spaces JN_p

The spaces JN_p and BMO are intimately related which we exhibit below.

Proposition 5.1.3. *If the group G itself is a base set, then $BMO(G) \subseteq JN_p(G)$. Moreover, the space $BMO(G)$ can be obtained as the limit of the John–Nirenberg spaces $JN_p(G)$ as $p \to \infty$ in the sense that*

$$\lim_{p\to\infty} \|f\|_{JN_p(G)} = \|f\|_{BMO(G)}.$$

Proof. For any sequence of pairwise disjoint base sets $\{V_i\}_i$ in G, we observe that

$$\begin{aligned}
&\Big(\sum_i \Big(\inf_{c_i\in\mathbb{R}} \fint_{V_i} |f - c_i|\, d\mu\Big)^p \mu(V_i)\Big)^{\frac{1}{p}} \\
&= \Big(\sum_i \int_{V_i} \Big(\inf_{c_i\in\mathbb{R}} \fint_{V_i} |f - c_i|\, d\mu\Big)^p d\mu\Big)^{\frac{1}{p}} \\
&= \Big(\sum_i \int_{V_i} \Big(\sum_j \chi_{V_j}\Big(\inf_{c_j\in\mathbb{R}} \fint_{V_j} |f - c_j|\, d\mu\Big)\Big)^p d\mu\Big)^{\frac{1}{p}} \\
&= \Big(\int_G \Big(\sum_j \chi_{V_j}\Big(\inf_{c_j\in\mathbb{R}} \fint_{V_j} |f - c_j|\, d\mu\Big)\Big)^p d\mu\Big)^{\frac{1}{p}}.
\end{aligned}$$

Now, in the last equality, invoking a standard measure theory fact that $\|g\|_{L^p(G)} \to \|g\|_{L^\infty(G)}$ as $p \to \infty$, leads to

$$\begin{aligned}
&\lim_{p\to\infty} \Big(\sum_i \Big(\inf_{c_i\in\mathbb{R}} \fint_{V_i} |f - c_i|\, d\mu\Big)^p \mu(V_i)\Big)^{\frac{1}{p}} \\
&= \sup_{x\in G} \sum_i \chi_{V_i}\Big(\fint_{V_i} |f - c_i|\, d\mu\Big) \\
&= \sup_i \Big(\inf_{c_i\in\mathbb{R}} \fint_{V_i} |f - c_i|\, d\mu\Big).
\end{aligned}$$

Then taking supremum over all possible sequences of pairwise disjoint base sets $\{V_i\}_i$ in G in the above equation, we further deduce that

$$\sup_{\{V_i\}_i} \lim_{p\to\infty} \Big(\sum_i \mu(V_i)\big(\inf_{c_i\in\mathbb{R}} \fint_{V_i} |f - c_i|\, d\mu)^p\Big)^{\frac{1}{p}}$$

$$
\begin{aligned}
&= \sup_{\{V_i\}_i} \sup_i \big(\inf_{c_i \in \mathbb{R}} \fint_{V_i} |f - c_i| \, d\mu \big) \\
&= \sup_{V \in \mathcal{B}} \Big(\inf_{c \in \mathbb{R}} \fint_{V_i} |f - c| \, d\mu \Big) \\
&= \|f\|_{BMO(G)}.
\end{aligned}
$$

In other words,

$$\lim_{p \to \infty} \|f\|_{JN_p(G)} = BMO(G).$$

This completes the proof of Proposition 5.1.3. □

Taking into consideration the above proposition, the space JN_p may be viewed as a generalization of the space BMO to some extent, and the BMO norm of a function can be interpreted as the limit of its JN_p norms when p tends to ∞.

Remark 5.1.4. *The space $JN_p(V)$ contains $L^p(V)$ and hence JN_p contains plenty of nontrivial functions. Indeed, for any countable collection of pairwise disjoint base sets $\{V_i\}_i$, we find that*

$$
\begin{aligned}
\sum_i \mu(V_i) \Big(\inf_{c_i \in \mathbb{R}} \fint_{V_i} |f - c_i| \, d\mu \Big)^p &\leq \sum_i \mu(V_i) \Big(\fint_{V_i} |f| \, d\mu \Big)^p \\
&\leq \sum_i \mu(V_i) \Big(\fint_{V_i} |f|^p \, d\mu \Big) \\
&\leq \int_V |f|^p \, d\mu.
\end{aligned}
$$

Then by taking the supremum over all such $\{V_i\}_i$, we get the desired conclusion.

5.2 John–Nirenberg Inequality for JN_p

In light of Remark 5.1.4, we know that $L^p(V) \subseteq JN_p(V)$. Now we show further containment relations for the spaces $JN_p(V)$. With this in mind we present the main result of this chapter. Recall that if $V = x + U_i$ is a base set, then $V^* = x + U_{\theta(i)}$ (see section 4.1).

Theorem 5.2.1 (John–Nirenberg inequality)**.** *Let $1 < p < \infty$. For every base set V and for every $f \in L^1(V^*)$, we have*

$$\|f - f_V\|_{L^{p,\infty}(V)} \leq C_{p,\mu} \|f\|_{JN_p(V^*)}, \tag{5.2.1}$$

where $C_{p,\mu}$ is a constant independent of f and V.

Proof. Fix an arbitrary base set $V_0 = x_0 + U_{i_0}$, where $x_0 \in G$ and $i_0 \in \mathbb{Z}$. Suppose that $f \in L^1(V_0^*)$ is given. To begin with the proof of (5.2.1), it will be convenient to establish certain notation and terminologies first that will be used frequently throughout the proof.

Let $C_{V_0^*}$ be a real number for which

$$\inf_{c\in\mathbb{R}} \fint_{V_0^*} |f - c|\, d\mu = \fint_{V_0^*} |f - C_{V_0^*}|\, d\mu \tag{5.2.2}$$

is attained. Let us first assert the existence of such a $C_{V_0^*}$. Choose an approximating sequence $\{\lambda_n\}_n$ so that

$$\fint_{V_0^*} |f - \lambda_n|\, d\mu < \inf_{c\in\mathbb{R}} \fint_{V_0^*} |f - c|\, d\mu + \frac{1}{n}.$$

From this, it follows that, for all n,

$$\begin{aligned}
|\lambda_n| &\le \left| \fint_{V_0^*} |f|\, d\mu - |\lambda_n| \right| + \fint_{V_0^*} |f|\, d\mu \\
&\le \fint_{V_0^*} |f - \lambda_n|\, d\mu + \fint_{V_0^*} |f|\, d\mu \\
&\le \fint_{V_0^*} |f|\, d\mu + 1 + \fint_{V_0^*} |f|\, d\mu \\
&= 2\fint_{V_0^*} |f|\, d\mu + 1.
\end{aligned}$$

This shows that the sequence $\{\lambda_n\}_n$ is bounded from above. So by Bolzano–Weierstrass theorem, there exists a subsequence $\{\lambda_{n_k}\}_k$ of $\{\lambda_n\}_n$ converging to $C_{V_0^*}$, say. Now using the dominated convergence theorem, we find that

$$\inf_{c\in\mathbb{R}} \fint_{V_0^*} |f - c|\, d\mu = \lim_{k\to\infty} \fint_{V_0^*} |f - \lambda_{n_k}|\, d\mu = \fint_{V_0^*} |f - C_{V_0^*}|\, d\mu,$$

which is (5.2.2).

Associated to the fixed base set V_0, we consider the following family of base sets which is the key object in our proof of (5.2.1):

$$\mathcal{F} = \{x + U_i : x \in V_0, i \le i_0\}.$$

With respect to the above basis $\mathcal{F}$, we consider the Hardy–Littlewood maximal function of $f - C_{V_0^*}$:

$$M_{\mathcal{F}}(f - C_{V_0^*})(x) = \sup_{x \in V \in \mathcal{F}} \fint_V |f - C_{V_0^*}| \, d\mu, \quad x \in G.$$

Also, we consider the distribution set of the maximal function $M_{\mathcal{F}}(f - C_{V_0^*})$ at scale $\lambda > 0$,

$$\Omega_\lambda = \{x \in V_0^* : M_{\mathcal{F}}(f - C_{V_0^*})(x) > \lambda\}. \tag{5.2.3}$$

We claim that $M_{\mathcal{F}}(f - C_{V_0^*})$ vanishes outside of V_0^*. Observe that it suffices to show that each member of $\mathcal{F}$ is contained in V_0^*. For this purpose let $V = y + U_i \in \mathcal{F}$, where $y \in V_0$ and $i \leq i_0$. Choose any point $z \in V$. Then, $z = y + u$ for some $u \in U_i \subseteq U_{i_0}$. Note also that $y = x_0 + v \in V_0$ with $v \in U_{i_0}$. Now, by Lemma 4.1.3, we have

$$z = x_0 + u + v \in x_0 + U_{i_0} + U_{i_0} \subseteq x_0 + U_{\theta(i_0)} = V_0^*.$$

Hence, the claim follows.

Let us return to the proof of (5.2.1). By Lebesgue differentiation theorem, we have for a.e. $x \in V_0$,

$$M_{\mathcal{F}}(f - C_{V_0^*})(x) \geq |f - C_{V_0^*}|(x)$$

which, together with the fact that $\|f - f_{V_0}\|_{L^{p,\infty}(V_0)} \leq 2\|f - C_{V_0^*}\|_{L^{p,\infty}(V_0)}$, implies

$$\|f - f_{V_0}\|_{L^{p,\infty}} \lesssim \|M_{\mathcal{F}}(f - C_{V_0^*})\|_{L^{p,\infty}(V_0^*)}.$$

So in order to prove (5.2.1), it will be enough to prove that

$$\|M_{\mathcal{F}}(f - C_{V_0^*})\|_{L^{p,\infty}(V_0^*)} \lesssim \|f\|_{JN_p(V_0^*)}. \tag{5.2.4}$$

The proof of this estimate is based on a Burkholder–Gundy [10] type good-λ inequality (See Proposition 5.2.2 below) which relates the local maximal function $M_{\mathcal{F}}(f - C_{V_0^*})$ and a variant of the sharp maximal function $M_{\mathcal{F}}^\sharp f$ associated with the family $\mathcal{F}$ defined by

$$M_{\mathcal{F}}^\sharp f(x) = \sup_{x \in V \in \mathcal{F}} \fint_V |f - C_V| \, d\mu,$$

where C_V is a constant that satisfies

$$\inf_{c\in\mathbb{R}} \fint_V |f-c|\,d\mu = \fint_V |f-C_V|\,d\mu.$$

The proof of the existence of such a C_V is identical to that of $C^*_{V_0}$ in (5.2.2), the details being omitted.

We consider the set

$$\{x\in V_0^*: M_{\mathcal{F}}(f-C_{V_0^*})(x)>k\lambda, M^\sharp_{\mathcal{F}}f(x)\le\gamma\lambda\},$$

where $\gamma\in(0,1)$. For convenience, in what follows, we denote this set by Σ_λ.

Proposition 5.2.2. *Let* $K=\max\{3,D^6\}$. *If*

$$\lambda \ge D^{11}\fint_{V_0^*}|f-C^*_{V_0}|\,d\mu, \tag{5.2.5}$$

for all $0<\gamma<1$, *we have then*

$$\begin{aligned}&\mu\big(\{x\in V_0^*: M_{\mathcal{F}}(f-C_{V_0^*})(x)>K\lambda, M^\sharp_{\mathcal{F}}f(x)\le\gamma\lambda\}\big)\\ &\lesssim\ \gamma\cdot\mu\big(\{x\in V_0^*: M_{\mathcal{F}}(f-C_{V_0^*})(x)>\lambda\}\big).\end{aligned}$$

We postpone the proof of this proposition and continue with the proof of the inequality (5.2.4).

For any λ satisfying the condition (5.2.5), it follows from Proposition 5.2.2 that

$$\begin{aligned}\mu(\Omega_{K\lambda}) &= \mu\big(\{x\in V_0^*: M_{\mathcal{F}}(f-C_{V_0^*})(x)>K\lambda, M^\sharp_{\mathcal{F}}f(x)\le\gamma\lambda\}\big)\\ &\quad+\mu\big(\{x\in V_0^*: M_{\mathcal{F}}(f-C_{V_0^*})(x)>K\lambda, M^\sharp_{\mathcal{F}}f(x)>\gamma\lambda\}\big)\\ &\lesssim \gamma\cdot\mu\big(\{x\in V_0^*: M_{\mathcal{F}}(f-C_{V_0^*})(x)>\lambda\}\big)\\ &\quad+\mu\big(\{x\in V_0^*: M^\sharp_{\mathcal{F}}f(x)>\gamma\lambda\}\big).\end{aligned}$$

Therefore, for all $\lambda>0$, we obtain

$$\begin{aligned}\mu(\Omega_{K\lambda}) &\lesssim \gamma\cdot\mu(\Omega_\lambda)+\mu\big(\{x\in V_0^*: M^\sharp_{\mathcal{F}}f(x)>\gamma\lambda\}\big)\\ &\quad+\chi_{\{0<\lambda<D^{11}\fint_{V_0^*}|f-C^*_{V_0}|\,d\mu\}}(\lambda)\cdot\mu(V_0^*).\end{aligned} \tag{5.2.6}$$

We now consider the following quantity

$$\mathcal{J}_N := \sup_{0<\lambda\le N} \lambda^p \mu(\Omega_\lambda), \quad N>0.$$

From (5.2.6), it follows that

$$\begin{aligned}
\mathcal{J}_N &= K^p \sup_{0<\lambda\le \frac{N}{K}} \lambda^p \mu(\Omega_{K\lambda})\\
&\lesssim K^p\,\gamma \sup_{0<\lambda\le \frac{N}{K}} \lambda^p \mu(\Omega_\lambda)\\
&\quad + K^p \sup_{0<\lambda\le \frac{N}{K}} \lambda^p\,\mu\big(\{x\in V_0^* : M_{\mathcal{F}}^\sharp f(x) > \gamma\lambda\}\big)\\
&\quad + K^p \left(D^{11} \fint_{V_0^*} |f - C_{V_0^*}|\,d\mu\right)^p \mu(V_0^*)\\
&\lesssim K^p\,\gamma\,\mathcal{J}_N + \frac{K^p}{\gamma^p} \sup_{0<\lambda\le \frac{N\gamma}{K}} \lambda^p \mu\big(\{x\in V_0^* : M_{\mathcal{F}}^\sharp f(x) > \lambda\}\big)\\
&\quad + K^p \left(D^{11} \fint_{V_0^*} |f - C_{V_0^*}|\,d\mu\right)^p \mu(V_0^*)\\
&= K^p\,\gamma\,\mathcal{J}_N + \frac{K^p}{\gamma^p} \|M_{\mathcal{F}}^\sharp f\|_{L^{p,\infty}(V_0^*)}^p\\
&\quad + K^p \left(D^{11} \fint_{V_0^*} |f - C_{V_0^*}|\,d\mu\right)^p \mu(V_0^*).
\end{aligned}$$

Now, choosing γ small enough in the last estimate, we see that

$$\mathcal{J}_N \lesssim \|M_{\mathcal{F}}^\sharp f\|_{L^{p,\infty}(V_0^*)}^p + \left(D^{11} \fint_{V_0^*} |f - C_{V_0^*}|\,d\mu\right)^p \mu(V_0^*).$$

Taking $N\to\infty$, we then obtain

$$\begin{aligned}
&\|M_{\mathcal{F}}(f - C_{V_0^*})\|_{L^{p,\infty}(V_0^*)}^p\\
&\quad\lesssim \|M_{\mathcal{F}}^\sharp f\|_{L^{p,\infty}(V_0^*)}^p + \left(D^{11} \fint_{V_0^*} |f - C_{V_0^*}|\,d\mu\right)^p \mu(V_0^*)\\
&\quad\lesssim \left[\|M_{\mathcal{F}}^\sharp f\|_{L^{p,\infty}(V_0^*)} + \left(D^{11} \fint_{V_0^*} |f - C_{V_0^*}|\,d\mu\right) \mu(V_0^*)^{\frac{1}{p}}\right]^p.
\end{aligned}$$

That is,

$$\begin{aligned}
&\|M_{\mathcal{F}}(f - C_{V_0^*})\|_{L^{p,\infty}(V_0^*)}\\
&\quad\lesssim \|M_{\mathcal{F}}^\sharp f\|_{L^{p,\infty}(V_0^*)} + \left(D^{11} \fint_{V_0^*} |f - C_{V_0^*}|\,d\mu\right) \mu(V_0^*)^{\frac{1}{p}}\\
&\quad=: I + II. \qquad (5.2.7)
\end{aligned}$$

We now estimate I and II separately. The quantity II is relatively easier to handle. We have

$$\begin{aligned}\|f\|_{JN_p(V_0^*)} &\geq \Big[\big(\inf_{c\in\mathbb{R}} \fint_{V_0^*} |f-c|\,d\mu\big)^p \mu(V_0^*)\Big]^{\frac{1}{p}}\\ &= \Big(\inf_{c\in\mathbb{R}} \fint_{V_0^*} |f-c|\,d\mu\Big)\mu(V_0^*)^{\frac{1}{p}}\\ &= \mu(V_0^*)^{\frac{1}{p}} \fint_{V_0^*} |f-C_{V_0^*}|\,d\mu,\end{aligned}$$

where we have used the definition of $JN_p(V_0^*)$ and equation (5.2.2). Hence,

$$II \lesssim \|f\|_{JN_p(V_0^*)}.$$

Now we turn our attention to estimate I. Take any $x \in V_0^*$ so that $M^\sharp_{\mathcal{F}} f(x) > \lambda$. By the definition of $M^\sharp_{\mathcal{F}} f$, there exists a base set $V_x \in \mathcal{F}$ containing x such that $\fint_{V_x} |f - C_{V_x}|\,d\mu > \lambda$. By an argument similar to that used in Lemma 2.2.1 in [32], we deduce the following conclusion. There exists a countable family of pairwise disjoint base sets $\{V_{x_j}\}_j$ from the collection $\{V_x : \fint_{V_x} |f - C_{V_x}|\,d\mu > \lambda\}$ such that

$$\{x \in V_0^* : M^\sharp_{\mathcal{F}} f(x) > \lambda\} \subseteq \bigcup_j V_{x_j}^{**}.$$

It then follows, from the last fact, that

$$\begin{aligned}\mu\big(\{x \in V_0^* : M^\sharp_{\mathcal{F}} f > \lambda\}\big) &\leq D^2 \sum_j \mu(V_{x_j})\\ &\leq \frac{D^2}{\lambda^p} \sum_j \mu(V_{x_j})\Big(\fint_{V_{x_j}} |f - C_{V_{x_j}}|\,d\mu\Big)^p\\ &\leq \frac{D^2}{\lambda^p} \|f\|^p_{JN_p(V_0^*)}.\end{aligned}$$

In other words,

$$I = \|M^\sharp_{\mathcal{F}} f\|_{L^{p,\infty}(V_0^*)} \lesssim \|f\|_{JN_p(V_0^*)}.$$

Putting these estimates together in (5.2.7), we conclude that

$$\|M_{\mathcal{F}}(f - C_{V_0^*})\|_{L^{p,\infty}(V_0^*)} \lesssim \|f\|_{JN_p(V_0^*)}.$$

Thus, inequality (5.2.4), to which we reduced the proof of Theorem 5.2.1, holds. Therefore, in order to complete the proof of the Theorem 5.2.1, we are left with the task of proving Proposition 5.2.2. To do so, we need the following lemmas.

Lemma 5.2.3. *Let $V = x + U_i \in \mathcal{F}$ such that*

$$\fint_V |f - C_{V_0^*}|\, d\mu \geq D^{11} \fint_{V_0^*} |f - C_{V_0^*}|\, d\mu. \tag{5.2.8}$$

Then $V^{6*} \in \mathcal{F}$.

Proof. It suffices to show that $\theta^6(i) \leq i_0$. Assume that $\theta^6(i) > i_0$. Then, $\theta^8(i) \geq \theta^2(i_0)$. Since $x \in V_0 \cap V$, we have $x \in V_0^{2*} \cap V^{8*} (\neq \phi)$. By Lemma 4.1.3, it follows that $V_0^{2*} \subseteq V^{10*}$. Altogether,

$$D^{11} \leq \frac{\fint_V |f - C_{V_0^*}|\, d\mu}{\fint_{V_0^*} |f - C_{V_0^*}|\, d\mu} \leq \frac{\mu(V_0^*)}{\mu(V)} \leq \frac{\mu(V_0^{2*})}{\mu(V)} \leq \frac{\mu(V^{10*})}{\mu(V)} \leq D^9,$$

which is a contradiction. Hence, the desired result follows. □

The proof of the following lemma is omitted as it is quite similar to that of Lemma 4.3.6.

Lemma 5.2.4. *Let Ω_λ be the set defined in (5.2.3) and assume that λ satisfies (5.2.5). If Ω_λ is nonempty, then there exists a sequence of pairwise disjoint base sets $\{V_i\}_i = \{y_i + U_{\alpha_i}\}_i$ from $\mathcal{F}$ such that*

(a) $\bigcup_i V_i \subseteq \Omega_\lambda \subseteq \bigcup_i V_i^{4*}$;

(b) $V_i^{6*} \in \mathcal{F}$ *for all i;*

(c) $\fint_{V_i} |f - C_{V_0^*}|\, d\mu > \lambda$ *for all i;*

(d) if $\alpha_i < r \leq i_0$ for some i, then $\fint_{y_i + U_r} |f - C_{V_0^*}|\, d\mu \leq \lambda$.

We are now ready to present the proof of Proposition 5.2.2.

Proof of Proposition 5.2.2. Recall that the goal is to prove that for all γ with $0 < \gamma < 1$ and $K = \max\{3, D^6\}$, the inequality

$$\mu\big(\{x \in V_0^* : M_{\mathcal{F}}(f - C_{V_0^*})(x) > K\lambda, M_{\mathcal{F}}^{\sharp} f(x) \le \gamma\lambda\}\big) \lesssim \gamma \cdot \mu\big(\{x \in V_0^* : M_{\mathcal{F}}(f - C_{V_0^*})(x) > \lambda\}\big) \tag{5.2.9}$$

holds provided that

$$\lambda \ge D^{11} \fint_{V_0^*} |f - C_{V_0}^*| \, d\mu.$$

First we apply the decomposition Lemma 5.2.4 to λ satisfying the above condition to get a family of base sets $\{V_i\}_i$ satisfying (a) through (d). As a next step to prove (5.2.9), we claim the following:

$$\{x \in V_i^{4*} : M_{\mathcal{F}}(f - C_{V_0^*})(x) > \lambda K\} = \{x \in V_i^{4*} : M_{\mathcal{F}}((f - C_{V_0}^*)\chi_{V_i^{6*}})(x) > \lambda K\}. \tag{5.2.10}$$

The direction $\supseteq$ is obvious. So, we only prove the other direction. Suppose x lies in the set on the left. Then there exists a base set $V = y + U_j$ containing x with $j \le i_0$, $y \in V_0$ such that

$$\fint_V |f - C_{V_o^*}| \, d\mu > \lambda K \ge \lambda D^6. \tag{5.2.11}$$

Therefore, $x \in V_i^{4*} \cap V$, and hence, $x = y_i + u = y + v$, with $u \in U_{\theta^4(\alpha_i)}, v \in U_j$. We claim that $j \le \theta^4(\alpha_i)$. Suppose this is not the case. Then $j > \theta^4(\alpha_i)$, which in turn implies that $U_{\theta^4(\alpha_i)} \subseteq U_j$. Let $z = y + w \in V$ for some $w \in U_j$. We can write

$$z = y + w = y + v - v + w = y_i + u - v + w.$$

This means $z \in y_i + U_j - U_j + U_j \subseteq y_i + U_{\theta^2(j)}$. So, we find that

$$V \subseteq y_i + U_{\theta^2(j)}. \tag{5.2.12}$$

Also, $x \in V \cap V_i^{4*} \subset V \cap (y_i + U_j) = (y + U_j) \cap (y_i + U_j)$. So, by Lemma 4.1.3, we obtain $y_i + U_j \subseteq y + U_{\theta^2(j)} = V^{**}$. From this fact, it follows that

$$\frac{\mu(y_i + U_{\theta^2(j)})}{\mu(y + U_j)} \leq \frac{D^2\mu(y_i + U_j)}{\mu(y + U_j)} \leq \frac{D^2\mu(y + U_{\theta^2(j)})}{\mu(y + U_j)} \leq D^4. \tag{5.2.13}$$

On the other hand,

$$\begin{aligned} ⨍_V |f - C_{V_0^*}|\, d\mu > \lambda K \geq \lambda D^6 \quad &> \quad D^{17} ⨍_{V_0^*} |f - C_{V_0^*}|\, d\mu \\ &> \quad D^{11} ⨍_{V_0^*} |f - C_{V_0^*}|\, d\mu. \end{aligned}$$

So by lemma 5.2.3, we have $\theta^6(j) \leq i_0$ and hence $i_0 > \theta^2(j) > \theta^4(\alpha_i) > \alpha_i$. This fact, together with Lemma 5.2.4 (d), implies that

$$⨍_{y_i + U_{\theta^2(j)}} |f - C_{V_0^*}|\, d\mu \leq \lambda.$$

Combining this with (5.2.12), (5.2.11) and (5.2.13), we conclude that

$$\lambda D^6 \leq \lambda K \leq ⨍_V |f - C_{V_0^*}|\, d\mu \leq D^4 \lambda,$$

which is a contradiction. Hence, the claim $j \leq \theta^4(\alpha_i)$ holds true. So by Lemma 4.1.3, we have $V \subseteq V_i^{6*}$, and therefore,

$$\begin{aligned} \lambda D^6 < ⨍_V |f - C_{V_0^*}|\, d\mu \quad &= \quad ⨍_V |f - C_{V_0^*}| \chi_{V_i^{6*}}\, d\mu \\ &\leq \quad M_{\mathcal{F}}((f - C_{V_0^*})\chi_{V_i^{6*}})(x). \end{aligned}$$

This means that x also lies in the set on the right of (5.2.10). Thus, equation (5.2.10) holds.

In the next step, we claim that

$$M_{\mathcal{F}}((f - C_{V_i^{6*}})\chi_{V_i^{6*}})(x) \geq (K-2)\lambda \quad \text{if } x \in \Sigma_\lambda \cap V_i^{4*}. \tag{5.2.14}$$

To see this, we first observe that from triangle inequality, we have

$$⨍_{V_i^{6*}} |C_{V_i^{6*}} - C_{V_0^*}|\, \chi_{V_i^{6*}}(x) d\mu$$

$$\begin{aligned} &\le \fint_{V_i^{6*}} |f - C_{V_i^{6*}}|\, d\mu + \fint_{V_i^{6*}} |f - C_{V_0^*}|\, d\mu \\ &\le 2 \fint_{V_i^{6*}} |f - C_{V_0^*}|\, d\mu\,. \end{aligned}$$

This, via equation (5.2.10), Lemma 5.2.4 (b) and (d), if $x \in \Sigma_\lambda \cap V_i^{4*}$, then

$$\begin{aligned} K\lambda &< M_{\mathcal{F}}((f - C_{V_i^{6*}})\chi_{V_i^{6*}})(x) \\ &\le M_{\mathcal{F}}((f - C_{V_i^{6*}})\chi_{V_i^{6*}})(x) + M_{\mathcal{F}}\big((C_{V_i^{6*}} - C_{V_0^*})\chi_{V_i^{6*}}\big)(x) \\ &\le M_{\mathcal{F}}((f - C_{V_i^{6*}})\chi_{V_i^{6*}})(x) + 2\fint_{V_i^{6*}} |f - C_{V_0^*}|\, d\mu \\ &\le M_{\mathcal{F}}((f - C_{V_i^{6*}})\chi_{V_i^{6*}})(x) + 2\lambda. \end{aligned}$$

That is, $M_{\mathcal{F}}((f - C_{V_i^{6*}})\chi_{V_i^{6*}})(x) \ge (K-2)\lambda$, and hence, (5.2.14) holds true.

Now, for each i, pick a point $\bar{x}_i \in \Sigma_\lambda \cap V_i^{4*}$. Since, $\Sigma_\lambda \subseteq \Omega_\lambda$, therefore, by Lemma 5.2.4, equation (5.2.14) and the weak $(1,1)$ property of $M_{\mathcal{F}}$, we obtain

$$\begin{aligned} \mu(\Sigma_\lambda) &= \mu(\Sigma_\lambda \cap \Omega_\lambda) \\ &\le \sum_i \mu(\Sigma_\lambda \cap V_i^{4*}) \\ &\le \sum_i \mu\big(\{x \in V_i^{4*} : M_{\mathcal{F}}((f - C_{V_i^{6*}})\chi_{V_i^{6*}})(x) > (K-2)\lambda, M_{\mathcal{F}}^{\sharp} f \le \gamma\lambda\}\big) \\ &\le \sum_i \frac{C}{(K-2)\lambda} \int_{V_i^{6*}} |f - C_{V_i^{6*}}|\, d\mu \\ &\le \frac{CD^6}{(K-2)\lambda} \sum_i \mu(V_i) \fint_{V_i^{6*}} |f - C_{V_i^{6*}}|\, d\mu \\ &\le \frac{CD^6}{(K-2)\lambda} \sum_i \mu(V_i) M_{\mathcal{F}}^{\sharp} f(\bar{x}_i) \\ &\le \frac{CD^6}{(K-2)\lambda} \sum_i \mu(V_i)\, \gamma\, \lambda \le \frac{CD^6}{(K-2)} \gamma\, \mu(\Omega_\lambda). \end{aligned}$$

Hence,

$$\mu\Big(\{x \in V_0^* : M_{\mathcal{F}}(f - C_{V_0^*}) > K\lambda, f_{\mathcal{F}^*} \le \gamma\lambda\}\Big) \le \frac{CD^6}{K-2} \cdot \gamma \cdot \mu(\Omega_\lambda),$$

the desire inequality holds. □

Theorem 5.2.1 is now completely proven. □

Remark 5.2.5. *We conclude this chapter by briefly comparing several variants of John–Nirenberg inequality in a variety of contexts. In the Euclidean spaces [65], the John–Nirenberg inequality tells us that the space* $JN_p(Q_0)$ *can be embedded into weak-*$L^p(Q_0)$*. The corresponding inequality in the setting of doubling metric measure spaces also shows that for any ball* B*,* $JN_p(B)$ *can be embedded into weak-*$L^p(B)$ *(see [1]). Note that in [1],* f *is said to be in* $JN_p(B)$ *if* $f \in L^1_{\mathrm{loc}}(11B)$ *and satisfies a condition similar to the one in* (1.4.1)*. In [7], the authors improved this result. They showed that in a space of homogeneous type, if* $B = B(x, r)$ *is a ball of radius* r *centred at* x *and* $\widehat{B} = B(x, (1+\delta)r)$*, where* $\delta > 0$*, then* $JN_p(\widehat{B})$ *can be embedded into weak-*$L^p(B)$*. In the present context, we have shown that for any base set* V*,* $JN_p(V^*)$ *can be embedded into weak-*$L^p(V)$*.*

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List of Publications

1. Molla, Md N., H^1, BMO, and John–Nirenberg inequality on LCA groups, *Mediterr. J. Math.* **20**, 55 (2023).

2. Molla, Md N., John–Nirenberg space on LCA groups, *Anal. Math. Phys.* **12**, 119(2022).

3. Molla, Md N. and Behera, B., Weighted norm inequalities for maximal operator of Fourier series, *Adv. Oper. Theory*, **7**, Paper No. 13 (2022).

4. Behera, B. and Molla, Md N., Characterization of Schauder basis property of Gabor systems in local fields, *Acta Sci. Math.*, **87** (2021), 517–539.

5. Molla, Md N. and Behera, B., Inequalities for maximal operators associated with a family of general sets, *preprint.*