Stochastic Equations Driven by Lévy Processes

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Contents

Introduction

In this thesis we first study a stochastic heat equation driven by Lévy noise and understand the well-posedness of the associated martingale problem. We use the method of duality to establish the same. In the second part of the thesis we explore the method of Algebraic duality and establish weak-uniqueness for a class of infinite dimensional interacting diffusions. We conclude the thesis with some preliminary observations on how to construct path wise stochastic integrals under a Poisson random measure.

Since the publication of Walsh's monograph [Wal86] and the book of Da Prato and Zabczyk [DPZ14] the field of stochastic partial differential equations (SPDE) has seen rapid evolution. Studies of many types of SPDEs can be found in the vast literature accrued since then. However, most of these works consider equations whose forcing term is the Gaussian white noise. The primary reasons for this is that equations driven by Gaussian noise display many appealing characteristics such as finite second moment and continuity of solutions. On the other hand, many physical systems can be modeled with greater accuracy if one assumes that the noise involved is non-Gaussian.

In probability theory Lévy processes are a natural generalization of the most well-known of Gaussian processes, viz. the Brownian motion. Like the latter, the former have independent and stationary increments. But unlike the Brownian motion, Lévy processes are not assumed to be continuous. This attribute makes them particularly well-suited for various naturally occurring systems which display discontinuous behavior. In the area of SPDEs therefore, a two-parameter version of Lévy processes, which may be called the *Lévy noise*, are sometimes considered as a natural generalization of the (Gaussian) space-time white noise. In the first part of this thesis, we focus on a special type of Lévy noise which are derived from α-stable processes. Let us now briefly describe the precise objectives of this thesis.

In the first chapter we give a brief overview of the theory of *martingale measures* and the stochastic heat equation with (Gaussian) space-time white noise. Martingale measures give the mathematically rigorous formalism of two-parameter stochastic forcing and are therefore crucial to any theory of SPDE.

Chapter 2 is devoted to the main question addressed in this thesis, viz. does the stochastic heat equation driven by a stable noise have a unique solution? For us, this will involve showing that any two solutions of the aforementioned SPDE are equal in distribution. This is called *weak uniqueness* of the solution. See Theorem 2.2.3 for the precise statement. In Chapter 3 we complete the proof of this theorem. These two chapters are based on the preprint [Mai21].

The technique we will use to prove the weak uniqueness result mentioned above is known as *duality*, which is another point of focus for this thesis. Stochastic duality seeks to establish a relationship between a given process, say X, with another, say Y, so that investigating some attribute of Y helps to shed light on that of X. This is a well-known method and is regularly used in statistical physics, population genetics and other areas of modern probability theory. But due to the lack of any general theory of stochastic duality, it has been observed over the years that, even though applying duality can yield remarkable results in certain situation, finding the correct dual of a given process is not easy.

Using ideas from the theory of Lie algebras, a new and potentially unifying approach towards this problem has recently been proposed. In a joint work with Aritra Mandal [MM22], we use these ideas to obtain duality relations for some well-known Markov processes. This is the content of Chapter 4. The main results are recorded in Theorems 4.2.1 and 4.3.1.

Lastly we return to martingale measures (described in Chapter 1) and the theory of integration against them. It is important to observe here that, as will be discussed in Section 1.2, like Itô integrals, we can only construct integrals w.r.t. martingale measures only in a weak sense. Therefore it makes sense to ask the following question: Do the stochastic integrals with respect to martingale measures have a pathwise meaning? Undoubtedly, the class of all possible martingale measures is vast and we do not expect to answer this question in the most general form. But fortunately the Poisson random measures constitute a particularly well-understood subclass of martingale measures.

In Chapter 5 we thus ask the same question restricted only to PRMs. We present some preliminary observations from an ongoing joint work with Siva Athreya and Atul Shekhar [AMS22]. Using some ideas from rough path theory we show that, when integrands are simple enough, it is indeed possible to interpret stochastic integrals w.r.t. PRMs in a pathwise manner (see Theorem 5.4.4). This chapter also contains the basics of Young and rough integration theories which are required for our analysis.

Chapter 1

Overview of martingale measures and SPDEs driven by Gaussian white noise

In the first part of this thesis we will investigate uniqueness of solutions to a certain SPDE. But before this we give a brief overview of the subject of stochastic partial differential equations (SPDE) for the benefit of the uninitiated reader. Those already familiar with this area of probability theory can start reading this thesis directly from Chapter 2.

At present there are two main approaches to this subject, the random field approach initiated by Walsh [Wal86] and the semigroup approach of Da Prato and Zabczyk [DPZ14]. Our exposition follows the first one and we rely on the monographs of Walsh [Wal86] and Khoshnevisan [Kho09], [Kho14].

Many examples of interesting SPDEs arising out of a diverse range of disciplines can be found in [DPZ14, Chapter 0], [Hai09, Chapter 2], [Wal86, Chapter 3]. Since our focus will be on the stochastic heat equation on the real line, we start by considering the (ordinary) heat equation,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \quad u(0,\cdot) = u_0 \tag{1.0.1}$$

This describes diffusion of heat content over time in an infinite rod starting from some initial condition u_0. Now suppose that these systems are under the influence of some external force or noise and the noise is *not* dependent on the current state of the observed quantity. We can model such a phenomenon with the following equation.

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + F(t,x), \quad u(0,\cdot) = u_0 \tag{1.0.2}$$

Here $F(t,x)$ stands for the influence of the forcing at time $t \geq 0$ and space $x \in \mathbb{R}$. When F is nice enough,[1] we can apply the Duhamel's principle (cf. [Eva10, Section 2.3]) to write down an explicit solution for (1.0.2) as follows: for $t > 0$ and $x \in \mathbb{R}$,

$$u(t,x) = \int_{\mathbb{R}} p_t(x-y)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)F(s,y)\,ds\,dy. \tag{1.0.3}$$

Here $p_t(x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$ is the heat kernel.

[1] For instance, F can be taken to be smooth and compactly supported.

Unfortunately, the random forcing terms that we shall consider in the sequel are rather irregular and thus, to obtain an expression such as the one above, we must first define them precisely. This is done in the next section. In Section 1.2 we define integrals such as the second term in the R.H.S. of (1.0.3).

1.1 White noise and martingale measures

This section is devoted towards understanding and defining the noise term in (1.0.2). The most common type of noise that appear in the SPDE literature is known as *Gaussian white noise*, often denoted by $\dot{W}$. To visualize such an object, it helps to consider a discrete grid inside our domain $[0, \infty) \times \mathbb{R}$, say the integer lattice $\mathbb{N} \times \mathbb{Z}$. To each point (t, x) in the grid we assign a $N(0, 1)$-distributed random variable, denoted by $\dot{W}_{t,x}$ and impose the property that the collection $\{\dot{W}_{t,x} \mid (t, x) \in \mathbb{N} \times \mathbb{Z}\}$ be independent and identically distributed (i.i.d.). Formally we can write,

$$\mathbb{E}[\dot{W}_{t,x}\dot{W}_{s,y}] = \delta_{t-s}\delta_{x-y}$$

where $(s, y), (t, x) \in \mathbb{N} \times \mathbb{Z}$ and δ denotes the Dirac delta function. In practice this means that, at each point on the lattice, the observed quantity gets a random kick and the intensity of kicks are i.i.d.

Now we can ask how whether it is possible to implement the same idea in the continuous setting, i.e. on the whole $[0, \infty) \times \mathbb{R}$. This is indeed possible and we can think of this *space-time white noise* on the real line as the derivative of Brownian motion. But due to Brownian motion's highly irregular nature, we must interpret its derivative in the sense of Schwartz distributions. Although we shall work in standard Euclidean spaces, we define the white noise in the more general setting of an abstract *Polish space*, a separable and completely metrizable topological space.

Definition 1.1.1. Let $(E, \mathcal{E})$ be a measurable space with a σ-finite measure μ. Let $\Lambda = \{A \in \mathcal{E} \mid \mu(A) < \infty\}$. A *Gaussian white noise* (or simply *white noise*) W on E is a collection of random variables $\{W(A) \mid A \in \Lambda\}$ such that the following conditions hold.

- $W(A) \sim N(0, \mu(A))$ for each $A \in \Lambda$.
- When $A, B \in \Lambda$ are disjoint, $W(A)$ and $W(B)$ independent and

$$W(A \cup B) = W(A) + W(B).$$

From this definition it is clear that $\{W(A)\}_{A\in\Lambda}$ is a *Gaussian process*, i.e. for every finite collection of sets $A_1, \ldots, A_k \in \Lambda$, the vector $(W(A_1), \ldots, W(A_k))$ follows a multi-variate normal distribution. If we define C to be the covariance functional, then for $A, B \in \Lambda$

$$C(A, B) = \mathbb{E}[W(A)W(B)] = \mathbb{E}[W(A \cap B)^2] = \mu(A \cap B)$$

using the definition above. C is clearly symmetric. It is a known consequence of the Kolmogorov extension theorem that when C is non-negative definite, there is probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a Gaussian process W exists with covariance C. To see that C is indeed non-negative definite we take real numbers $a_1, \dots, a_k$ and sets $A_1, \dots, A_k \in \Lambda$ and observe that

$$\sum_{j=1}^{k}\sum_{i=1}^{k} a_i a_j C(A_i, A_j) = \sum_{j=1}^{k}\sum_{i=1}^{k} a_i a_j \mu(A_i \cap A_j) = \int_E \left(\sum_{i=1}^{k} a_i \mathbf{1}_{A_i}\right)^2 d\mu \geq 0.$$

From now on we shall assume that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which W and all other subsequent random objects are defined.

Some familiar processes can be recovered from the definition of the white noise. For example, if W is a white noise on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$[2] (with the Lebesgue measure), the function $t \mapsto W([0,t])$ defines a standard Brownian motion. And if W is a white noise on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ (again with the Lebesgue measure), the function $t \mapsto W([0,e^t] \times [0, e^{-t}])$ is an Ornstein-Uhlenbeck process.

Before going further we take note of an important feature of the white noise W. While W is, by definition, a finitely additive measure, in general it is not countably additive almost surely. Instead, if we take a countable collection of disjoint sets $A_1, A_2, \ldots \in \Lambda$ such that $A := \cup_{i=1}^{\infty} A_i \in \Lambda$, as $n \longrightarrow \infty$ by Definition 1.1.1 we have

$$\mathbb{E}\left|W\left(\sum_{i=1}^{n} A_i\right) - W(A)\right|^2 = \mathbb{E}\left|W\left(\sum_{i=n+1}^{\infty} A_i\right)\right|^2 = \mu\left(\sum_{i=n+1}^{\infty} A_i\right) = \sum_{i=n+1}^{\infty} \mu(A_i) \longrightarrow 0,$$

since $\mu(A) < \infty$. This can be written concisely as

$$\sum_{i=1}^{\infty} W(A_i) = W\left(\bigcup_{i=1}^{\infty} A_i\right) \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Thus W is a σ-finite $L^2(\Omega)$-valued measure. We now define this notion precisely. As usual, $(E, \mathcal{E})$ is a Polish space.

Let $U : \mathcal{E} \times \Omega \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ be a random set-function[3] and let $\mathcal{A} \subseteq \mathcal{E}$ be a *set algebra*, i.e. a collection of sets that contains the empty set, complements of all its members and is closed under finite unions and finite intersections. Assume that for all $A \in \mathcal{A}$, we have $\mathbb{E}[U(A)^2] < \infty$ and that U is almost surely finitely additive on $\mathcal{A}$.

Definition 1.1.2. (a) The function U is called *σ-finite* if there is an increasing sequence $\{E_n\}_n$ of $\mathcal{E}$-measurable such that,

- $E = \cup_{n=1}^{\infty} E_n$,
- for every $n \geq 1$, $\mathcal{E}_n := \{A \cap E_n \mid A \in \mathcal{E}\} \subseteq \mathcal{A}$, and
- for every $n \geq 1$, $\sup\{\mathbb{E}[U(A)^2] \mid A \in \mathcal{E}_n\} < \infty$.

(b) Let U be a σ-finite finitely additive random set-function. Suppose for every decreasing sequence $\{A_j\}_j$ in some fixed $\mathcal{E}_n$ with $A_j \downarrow \emptyset$, we have

$$\lim_{j \to \infty} \mathbb{E}[U(A_j)^2] = 0.$$

Then U will be called an *$L^2(\Omega)$-valued measure.*

[2] $\mathcal{B}(S)$ will denote the Borel σ-algebra on the non-empty set S.

[3] A function that takes sets as its input.

Note that the last condition ensures that U is countably additive on E_n for every n.

We are ready to define one of the central objects in the theory of stochastic PDEs, namely a martingale measure. This can easily be seen as a generalization of the space-time white noise on $[0, \infty) \times \mathbb{R}^d$. Unlike the white noise however, we demand that in one special "time" co-ordinate this new object behaves as a martingale. Recall that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ is our underlying filtered probability space.

Definition 1.1.3. The collection of processes $\{M_t(A)\}_{t\geq 0}$, $A \in \mathcal{A}$, is called a *martingale measure* if

(a) For each $A \in \mathcal{A}$, $M_0(A) = 0$ a.s.

(b) For each $t > 0$, M_t is a σ-finite $L^2(\Omega)$-valued measure.

(c) For each $A \in \mathcal{A}$, the process $\{M_t(A)\}_{t\geq 0}$ is an $\mathcal{F}_t$-martingale.

As mentioned before the space-time white noise on $[0, \infty)\times\mathbb{R}^d$ defined by $W_t(A) := W([0, t]\times A)$, $A \in \mathcal{B}(\mathbb{R}^d)$ is a martingale measure with respect to the natural filtration $(\mathcal{F}_t)_t$ generated by the processes $W_t(\cdot)$. To check this first note that for $0 \leq s < t$ and a fixed $A \in \mathcal{R}^d$ with finite Lebesgue measure, $W_t(A) - W_s(A) = W((s, t] \times A)$ is independent of $\mathcal{F}_s$. Thus

$$\mathbb{E}\left[W_t(A) \mid \mathcal{F}_s\right] = W_s(A) + \mathbb{E}\left[W_t(A) - W_s(A) \mid \mathcal{F}_s\right] = W_s(A) + \mathbb{E}\left[W_t(A) - W_s(A)\right] = W_s(A).$$

Another important class of martingale measures comes from *Poisson random measures*. Although they can be defined on any measure spaces, we shall define them on the Euclidean space $[0, \infty) \times \mathbb{R}^d$.

Definition 1.1.4. Suppose μ is a σ-finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. On an appropriate probability space let N be a random measure on $[0, \infty) \times \mathbb{R}^d$ having the following properties.

(i) Whenever $E \in \mathcal{B}(\mathbb{R}^d)$ with $\mu(E) < \infty$, $t \mapsto N([0, t) \times E) =: N_t(E)$ is a Poisson process with intensity $\mu(E) = \mathbb{E}N(1, E)$.

(ii) If $A_1, \ldots, A_k \in \mathcal{B}([0, \infty) \times \mathbb{R}^d)$ are disjoint, then the random variables $N(A_1), \ldots, N(A_k)$ are independent.

Then N will be called a Poisson random measure (PRM) on $[0, \infty) \times \mathbb{R}^d$ with intensity measure $dt \times \mu$ (dt denotes the Lebesgue measure on $\mathbb{R}_+$).

It follows from the above that, for $0 \leq a < b < \infty$ and $\mu(E) < \infty$, then we have $N([a, b)\times E) \sim \text{Poisson}((b - a)\mu(E))$. Now let us define the *compensated PRM* $\tilde{N}$ as follows. If a, b and E are as above,

$$\tilde{N}((a, b] \times E) := N((a, b] \times E) - (b - a)\mu(E).$$

We claim that this is a martingale measure in the sense of Definition 1.1.3. Clearly, a.s $\tilde{N}_0(E) = 0$. As the Lebesgue measure on $\mathbb{R}_+$ and μ on $\mathbb{R}^d$ are σ-finite, the second condition also holds. Lastly, as $\{N_t(E)\}_{t\geq 0}$ is a Poisson process, it has independent increments. Thus, for $s < t$ and $E \in \mathcal{B}(\mathbb{R}^d)$ with $\mu(E) < \infty$,

$$\mathbb{E}[\tilde{N}_t(E) \mid \mathcal{F}_s] = \mathbb{E}[N_t(E) - N_s(E)] - (t - s)\mu(E) + \mathbb{E}[N_s(E) - s\mu(E) \mid \mathcal{F}_s] = \tilde{N}_s(E),$$

showing that $\{N_t(E)\}_{t\geq 0}$ is an $\mathcal{F}_t$-martingale.

1.2 Stochastic integration with respect martingale measures

Since the equations that are of interest to us are *randomly* forced, standard Lebesgue integration theory is insufficient for giving meaning to their weak forms, such as the one in (1.0.3). The trick to address this issue is to adapt the Itô integration in the setting of martingale measures. As with Itô integrals, we start by first defining integration of elementary and simple functions.

Let $(E, \mathcal{E})$ be a Polish space as usual and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. An *elementary function* $f : [0,T] \times E \times \Omega \to \mathbb{R}$ is one having the form

$$f(s,x,\omega) = X(\omega) \cdot \mathbf{1}_{(a,b]}(s) \cdot \mathbf{1}_A(x)$$

where $0 \leq a \leq b \leq T$, $A \in \mathcal{E}$ and X is a bounded $\mathcal{F}_s$-measurable random variable. Finite sums of elementary functions are called *simple functions*. Let $\mathbf{S}$ be the collection of all simple functions and $\mathcal{P}$ denote the σ-algebra on $[0,T] \times E \times \Omega$ generated by $\mathbf{S}$. We refer to $\mathcal{P}$-measurable functions or stochastic processes as *predictable*.

Given a martingale measure M on E, we can now define the integral of an elementary function f against M as follows.

Definition 1.2.1. For $t \in [0,T]$ and $B \in \mathcal{E}$,

$$(f \cdot M)_t(B,\omega) = \left(\int_0^t \int_B f \, dM\right)(\omega) := X(\omega) \left[M_{t\wedge b}(A \cap B)(\omega) - M_{t\wedge a}(A \cap B)(\omega)\right]. \quad (1.2.1)$$

By linearity the same definition can be extended to simple functions.

We here observe that $f \cdot M$ is itself a martingale measure. This is because for elementary f, $B \in \mathcal{E}$ and $s < t$

$$\begin{aligned}\mathbb{E}\left[(f \cdot M)_t(B) \mid \mathcal{F}_s\right] =& X \cdot \mathbb{E}[M_{t\wedge b}(A \cap B) - M_{t\wedge a}(A \cap B) | \mathcal{F}_s] \\ =& X \cdot \left[M_{s\wedge b}(A \cap B) - M_{s\wedge a}(A \cap B)\right].\end{aligned}$$

$f \cdot M$ satisfies the other properties in Definition 1.1.3 trivially.

Before extending the integration defined above to a larger class of functions, we introduce some concepts.

Definition 1.2.2. If M is a martingale measure on E, then we define its *covariance functional.* For $A, B \in \mathcal{E}$,

$$\bar{Q}_t(A,B) := \langle M(A), M(B)\rangle_t, \quad t \in [0,T],$$

where $\langle \cdot, \cdot \rangle$ denotes the quadratic covariation. Also, for $A, B \in \mathcal{E}$ and $0 \leq s < t \leq T$, let

$$Q(A \times B \times (s,t]) := \bar{Q}_t(A,B) - \bar{Q}_s(A,B).$$

By [Pro05, Theorem II.22] we know that the process $t \mapsto \bar{Q}_t(A,A)$ is increasing and that $\bar{Q}_t(\cdot,\cdot)$ is bilinear[4]. Q is a measure-like functional on the collection of sets of the form $A \times B \times (s,t]$,

[4] Linear in both coordinates

called *rectangles.* If $A_i \times B_i \times (s_i, t_i]$, $i = 1, \dots, k$ are disjoint rectangles, we can define Q on their union by linearity,

$$Q\left(\bigcup_{i=1}^{k} A_i \times B_i \times (s_i, t_i]\right) := \sum_{i=1}^{k} Q(A_i \times B_i \times (s_i, t_i]).$$

One can also check that the above is well-defined.

As demonstrated in [Wal86, Chapter 2, p. 305], not all martingale measures can serve as an integrator for predictable functions. This lets us introduce the following special subclass. Recall that in Definition 1.1.2 we have stipulated the existence of a nested collection $\{E_n\}_{n\geq 1}$ in $(E, \mathcal{E})$ satisfying certain properties.

Definition 1.2.3. A martingale measure M on E is called *worthy* if there is a random σ-finite measure $K(ds, dy, dt, \omega)$ on $(E \times E \times [0, T], \mathcal{E} \times \mathcal{E} \times \mathcal{B}([0, T]))$ such that,

(i) K is symmetric in the first two "space" coordinates.

(ii) K is positive definite in the sense that, for any bounded measurable function $f : E \times [0, \infty) \longrightarrow \mathbb{R}$ we have,

$$\int_E \int_E \int_0^\infty f(x, t) f(y, t) K(dx, dy, dt) \geq 0,$$

whenever the above integral exists finitely.

(iii) For $A, B \in \mathcal{E}$ and $t \geq 0$, the process $\{K(A \times B \times (0, t])\}_{t\geq 0}$ is predictable.

(iv) For each $n \in \mathbb{N}$, $\mathbb{E}[K(E_n \times E_n \times [0, T)] < \infty$.

(v) For $A, B \in \mathcal{E}$ and $t \geq 0$, we have

$$|Q(A \times B \times [0, t)| \leq K(A \times B \times [0, t)) \quad \text{a.s.}$$

where Q is as in Definition 1.2.2

We call K the *dominating measure* of M.

Now we are in the position of extending Definition 1.2.1. We are going to assume that we have a worthy martingale measure M dominated by K. For a predictable f and g, let

$$\langle f, g \rangle_K := \int_E \int_E \int_0^\infty f(x, t) g(y, t) K(dx, dy, dt)$$

and

$$\|f\|_M := (\mathbb{E}\langle |f|, |f| \rangle_K)^{1/2}. \tag{1.2.2}$$

Let $\mathcal{P}_M$ denote the collection of all predictable functions f with the property that $\|f\|_M < \infty$. It is easy to check that that the above defines a norm on this space. We have the following results whose proof can be found in [Wal86, Proposition 2.3].

Proposition 1.2.4. *(i)* $(\mathcal{P}_M, \|\cdot\|_M)$ *is a Banach space.*

(ii) The set S *of all simple functions is dense in* $\mathcal{P}_M$ *in the norm* $\|\cdot\|_M$.

Let us now define integrals $f \cdot M$ for all $f \in \mathcal{P}_M$. Fix a $f \in \mathcal{P}_M$. Then the first part of the above proposition says that there exists a sequence of simple functions $\{f_n\}_{n\geq 1}$ such that $\|f_n - f\|_M \to 0$ as $n \to \infty$. Now, for each $n, m \geq 1$ and $E \in \mathcal{E}$,

$$\begin{aligned}\mathbb{E}[((f_n \cdot M)_t(E) - (f_m \cdot M)_t(E))^2] =&\mathbb{E}\left[\int_E \int_E \int_0^t f_{n,m}(x,s) f_{n,m}(y,s)\, Q(dx,dy,ds)\right] \\ \leq&\mathbb{E}\left[\int_E \int_E \int_0^t |f_{n,m}(x,s) f_{n,m}(y,s)|\, K(dx,dy,ds)\right] \\ =&\|f_n - f_m\|_M^2.\end{aligned}$$

where we have used the notation $f_{n,m} = f_n - f_m$. This shows that the sequence $\{(f_n \cdot M)_t(E)\}_{n\geq 1}$ is Cauchy in $L^2(\Omega)$. By completeness of the $L^2(\Omega)$, there exists a limit of this sequence. We denote this by

$$(f \cdot M)_t(E) = \int_0^t \int_F f(s,x)\, M(ds,dx),$$

and this is the required stochastic integral.

Finally, we state the a result which can be interpreted as a version of Itô isometry applied to integrals against martingale measures.

Proposition 1.2.5. *Suppose* M *is a worthy martingale measure with covariance* Q *and* $f \in \mathcal{P}_M$.

(i) Then $(f \cdot M)$ *is itself a worthy martingale measure. The covariance functional corresponding to* $(f \cdot M)$ *is given by*

$$Q_{f\cdot M}(dx,dy,dt) = f(x,t) f(y,t) Q(dx,dy,dt).$$

(ii) For all $E \in \mathcal{E}$ *and* $t \geq 0$,

$$\mathbb{E}[((f \cdot M)_t(E))^2] \leq \|f\|_M^2,$$

where $\|f\|_M$ *is defined in* (1.2.2).

1.3 The stochastic heat equation

Let us now consider the non-linear stochastic heat equation (SHE) with a multiplicative Gaussian space-time white noise W defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. In the following $\dot{W}$ will denote the distributional derivative of W. Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a non-random function. Our object of attention in this section is the equation,

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) =&\frac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{W}_{t,x}, \quad t \geq 0, x \in \mathbb{R} \\ u(0,\cdot) =&u_0,\end{aligned} \tag{1.3.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2}$. We shall assume that $\sigma : \mathbb{R} \to \mathbb{R}$ is a bounded Lipschitz function, i.e. there exists a finite constant $K > 0$ such that

$$|\sigma(x)| \leq K \text{ and } |\sigma(x) - \sigma(y)| \leq K|x - y|, \tag{1.3.2}$$

for all $x, y \in \mathbb{R}$.

The aim of this section is to prove the existence and uniqueness of the solution $u = u(t, x)$ to (1.3.1). Observe that (1.3.1) is purely formal. This is because, since we cannot *a priori* guarantee the smoothness of u in either time or in space, the derivatives appearing in (1.3.1) make little sense. Indeed, it is known that the u has a continuous modification which is only $(\frac{1}{4} - \epsilon)$-Hölder in time and $(\frac{1}{4} - \epsilon)$-Hölder in space, for any small $\epsilon > 0$ (cf. [Kho09, Theorem 6.7]). Therefore they must interpreted as (Schwartz) distributional derivatives. This gives rise to the so-called *mild formulation* of (1.3.1), which we presently write down.

$$u(t,x) = \int_{\mathbb{R}} p_t(x-y)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy), \quad t \geq 0, x \in \mathbb{R}, \tag{1.3.3}$$

where as before $p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$. Note that W is the (Gaussian) space-time white noise on $\mathbb{R}_+ + x\mathbb{R}$ and that the stochastic integral in the r.h.s. of the above with respect to W was defined in the previous section. We now state our main theorem precisely. Henceforth we shall treat (1.3.3) as the main equation of interest. The proof follows that of [Kho09, Theorem 6.4].

Theorem 1.3.1. *Assume that u_0 is bounded and that σ is a bounded Lipschitz with constant K (as in (1.3.2)). Then the following holds.*

(i) (1.3.3) *has a solution.*

(ii) If u and v are two solutions of (1.3.3) *defined on the same probability space, then a.s.*

$$u(t,x) = v(t,x)$$

for all $t \geq 0$ and $x \in \mathbb{R}$. This condition is sometimes called pathwise uniqueness.

(iii) The solution u has finite second moments. More precisely, for any $T > 0$ we have

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \mathbb{E}\left(|u(t,x)|^2\right) < \infty \tag{1.3.4}$$

The key tool for all these statements is the Grönwall lemma. This is the content of the next result.

Lemma 1.3.2. *Let $T > 0$ and f, g and h be non-negative integrable functions on $[0,T]$ satisfying the following inequality for all $t \in [0,T]$,*

$$f(t) \leq g(t) + \int_0^t h(s)f(s)\,ds. \tag{1.3.5}$$

Then for a.e. $t \in [0,T]$ we have,

$$f(t) \leq g(t) + \int_0^t g(s)h(s)\exp\left(\int_0^s h(r)\,dr\right)ds. \tag{1.3.6}$$

The proof is omitted as it is standard.

Proof of Theorem 1.3.1. Throughout the proof C will be used as a generic constant whose value may change from one line to the next. We start our proof with the last statement, assuming that a (1.3.3) has a solution u. By Itô isometry for stochastic integrals against martingale measures (see Proposition 1.2.5(ii) for a similar result) we have,

$$\mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)\right|^2\right) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)^2 \mathbb{E}\left[\sigma(u(s,y))^2\right] dsdy$$
$$\leq C \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)^2 \mathbb{E}\left[u(s,y)^2\right]^2 dsdy,$$

by (1.3.2). Jensen's inequality, on the other hand, gives,

$$\left(\int_{\mathbb{R}} p_t(x-y)u_0(y)\,dy\right)^2 \leq \int_{\mathbb{R}} p_t(x-y)u_0(y)^2\,dy \leq C,$$

as u_0 is assumed to be bounded. Therefore we get from (1.3.3),

$$\mathbb{E}(u(t,x)^2) \leq C + C\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)^2 \mathbb{E}\left[u(s,y)^2\right]^2 dsdy. \tag{1.3.7}$$

If we define

$$Z_t = \sup_{s\in[0,t]} \sup_{x\in\mathbb{R}} \mathbb{E}(u(s,x)^2),$$

from (1.3.7) we obtain,

$$Z_t \leq C + C\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)^2 Z_s\,ds\,dy = C + C\int_0^t \frac{Z_s}{\sqrt{t-s}}\,ds \tag{1.3.8}$$

Now let $p \in (1,2)$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's inequality,

$$\int_0^t \frac{Z_s}{\sqrt{t-s}}\,ds \leq \left(\int_0^t (t-s)^{-\frac{p}{2}}\,ds\right)^{\frac{1}{p}} \cdot \left(\int_0^t Z_s^q\,ds\right)^{\frac{1}{q}} \leq C\left(\int_0^t Z_s^q\,ds\right)^{\frac{1}{q}}. \tag{1.3.9}$$

From (1.3.8) we thus have,

$$Z_t^q \leq C + C\int_0^t Z_s^q\,ds \tag{1.3.10}$$

We are now in a position to apply Grönwall's inequality from Lemma 1.3.2. This shows that $Z_t \leq C$ for some constant $C < \infty$ and hence proves (1.3.4).

Now let us prove that (1.3.3) indeed has at least one solution. The key tool for proving existence is the *Picard iteration* scheme. We will define by induction a sequence $\{u_n\}_{n\geq 1}$ of two-parameter random fields as follows. Let $u_0(x,t) = u_0(x)$ for all $x \in \mathbb{R}$ and $t \geq 0$. Given u_n, let

$$u_{n+1}(t,x) := \int_{\mathbb{R}} p_t(x-y)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_n(s,y))W(ds,dy), \quad t \geq 0, x \in \mathbb{R}. \tag{1.3.11}$$

We will show that u_n converges to a limit in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. This limit will be the required solution.

For $n \geq 1$, let

$$H_n(t) = \sup_{s\in[0,t]} \sup_{x\in\mathbb{R}} \mathbb{E}\left(|u_{n+1}(s,x) - u_n(s,x)|^2\right).$$

From (1.3.11), Itô isometry and (1.3.2), we have

$$\begin{aligned}\mathbb{E}\left[|u_{n+1}(t,x) - u_n(t,x)|^2\right] \leq& \mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)[\sigma(u_n(s,y)) - \sigma(u_{n-1}(s,y))]W(ds,dy)\right|^2\right) \\ \leq& \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)^2 \mathbb{E}\left(|\sigma(u_n(s,y)) - \sigma(u_{n-1}(s,y))|^2\right) ds\, dy,\end{aligned}$$

and therefore,

$$H_n(t) \leq C \int_0^t \frac{H_{n-1}(s)}{\sqrt{t-s}}\, ds \tag{1.3.12}$$

for all $t \geq 0$ and $n \geq 1$. We can use the same argument from (1.3.9) to show,

$$H_n(t)^q \leq C \int_0^t H_{n-1}(s)^q\, ds \tag{1.3.13}$$

where $q > 1$.

An easy calculation shows that, for all $s \in [0,t]$ and $x \in \mathbb{R}$, we have $\mathbb{E}\left(|u_1(s,x) - u_0(x)|^2\right) \leq C\sqrt{s}$ and thus $H_0(t) \leq C\sqrt{t}$. Using this as an initial condition, (1.3.12) gives us by induction,

$$H_n(t)^q \leq C_1(T)\frac{Ct^{n-1}}{(n-1)!}$$

for all $t \in [0,T]$ and $n \geq 1$. This also means that $\sum_{n\geq 1} H_n(t) < \infty$. Therefore, the sequence in $\{u_n(t,x)\}_{n\geq 1}$ is Cauchy in $L^2(\Omega)$. Call its limit point $u(t,x)$ and this is our required solution.

We now turn to the question of uniqueness of solutions to (1.3.3). The argument is very similar to what we did before. Suppose u and v both satisfy (1.3.3). Denote

$$H(t) = \sup_{s\in[0,t]} \sup_{x\in\mathbb{R}} \mathbb{E}\left(|u(s,x) - v(s,x)|^2\right), \quad t \geq 0.$$

Similarly as in (1.3.8) we use the conditions on σ and Itô isometry to obtain,

$$H(t) \leq C \int_0^t \frac{H(t)}{\sqrt{t-s}}\, ds.$$

From this we can also derive that, for some $q > 1$

$$H(t)^q \leq C \int_0^t H(s)^q\, ds.$$

Now Grönwall's inequality shows that $H(t) = 0$. This shows that

$$\mathbb{P}(u(s,x) = v(s,x) \text{ for all } s \in [0,t] \text{ and } x \in \mathbb{R}) = 1$$

which completes the proof. □

As we have shown in the previous proof, when σ is Lipschitz, proving existence, uniqueness and various other properties of the solutions to (1.3.1) is not difficult. However, when σ is non-Lipschitz, there is no standard technique for answering these questions. Our main focus in this thesis will be the SHE where the noise is Lévy and σ is only Hölder continuous. We end this chapter by stating a weak existence result due to Mueller and Perkins [MP92] for the Gaussian white noise case with σ being Hölder.

Theorem 1.3.3. *Let $0 < \gamma < 1$ and u_0 be a compactly supported function on $\mathbb{R}$. Then on some probability space the equation*

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} =& \frac{1}{2}\Delta u(t,x) + u(t,x)^{\gamma}\dot{W}_{t,x}, \qquad t \geq 0, x \in \mathbb{R}\\ u(0,x) =& u_0(x),\end{aligned}$$

has a solution.

The authors show this result indirectly. They first construct a so-called *historical process* and then show that the martingale problem associated with it has a solution. Lastly, they prove that the density of the historical process satisfies the SHE displayed in the above theorem.

Chapter 2

The stochastic heat equation driven by stable noise

We now begin our study of the stochastic equation

$$\frac{\partial Y_t(x)}{\partial t} = \frac{1}{2}\Delta Y_t(x) + Y_{t-}(x)^\beta \dot{L}^\alpha_{t,x}, \qquad t \geq 0, x \in \mathbb{R} \tag{2.0.1}$$

where L^α is a stable noise of index α without negative jumps. Here, as in the last chapter, $\Delta = \frac{\partial^2}{\partial x^2}$. Our aim is to show that its solutions are unique in law when $1 < \alpha < 2$, $0 < \beta < \left(\frac{3}{\alpha} - 1\right) \wedge 1$ and $1 < \alpha\beta$ (see Theorem 2.2.3). Proving uniqueness in law, also called *weak uniqueness*, is an important step for establishing that a model of an underlying system of interacting particles converges to the limit. This chapter contains an overview of the literature related to this problem and various preliminary concepts.

2.1 Literature

When $\alpha = 2$ the above equation is similar to the following SPDE

$$\frac{\partial Y_t(x)}{\partial t} = \frac{1}{2}\Delta Y_t(x) + Y_t(x)^\beta \dot{W}_{t,x} \tag{2.1.1}$$

where $\dot{W}$ is the Gaussian space-time white noise. When $\beta = \frac{1}{2}$ this describes evolution of the density of the super-Brownian motion (SBM) in $\mathbb{R}$. This measure-valued diffusion is obtained as the scaling limit of interacting branching Brownian motions. The weak uniqueness of (2.1.1) with $\beta = \frac{1}{2}$ follows from the martingale problem formulation of SBM using exponential duality (see [Per02, Theorem II.5.1]). As mentioned before at the end of the previous chapter, in the $\beta \in (\frac{1}{2}, 1)$ case, the weak existence of (2.1.1) was proved by Mueller, Perkins [MP92]. Weak uniqueness was established by Mytnik [Myt98]. The question of path-wise uniqueness of (2.1.1) for $\beta > \frac{3}{4}$ was settled by Mytnik and Perkins in 2011 [MP11]. Some negative results are also known: [MMP14] proved path-wise non-uniqueness of solutions to (2.1.1) for $\beta \in (0, \frac{3}{4})$ and [BMP10] showed path-wise non-uniqueness for $\beta \in (0, \frac{1}{2})$ with an added non-trivial drift.

Let us now consider (2.0.1) where $x \in \mathbb{R}^d$. For $\alpha \neq 2$, Mueller [Mue98] proved a certain short time (strong) existence of solution to (2.0.1) under the relations $d < \frac{2(1-\alpha)}{\alpha\beta - (1-\alpha)}$, $\alpha \in (0,1)$. The weak existence was shown by [Myt02] under the relations $0 < \alpha\beta < \frac{2}{d}+1$, $1 < \alpha < \min(2, \frac{2}{d}+1)$. When $d = 1$ and $\alpha\beta = 1$ it is known that (2.0.1) describes the density of the super-Brownian motion with α-stable branching mechanism; see [MP03] for more details. The weak uniqueness for this case was proved in [Myt02] while the same for the general case was left open (see [Myt02, Remark 5.9]). See Figure 2.2 for a graphical representation of the (α, β)-parameter space.

As stated earlier in our main result we resolve the question for the case $d = 1$, $1 < \alpha < 2$, $0 < \beta < 1$ and $1 < \alpha\beta$. We note that the weak uniqueness for (2.0.1) when $d = 1$, $1 < \alpha < 2$, $\alpha\beta < 1$ still remains open.

It is known that path-wise uniqueness implies weak uniqueness for (2.0.1). But as the coefficient of the noise term in (2.0.1) is not Lipschitz, standard techniques such as Grönwall's inequality cannot be used to prove path-wise uniqueness of solutions. However, more recently Yang and Zhou [YZ17] have established path-wise uniqueness for (2.0.1) in the regime $\frac{2(\alpha-1)}{(2-\alpha)^2} < \beta < \frac{1}{\alpha}+\frac{\alpha-1}{2}$ and $d = 1$. This region partially overlaps with that stated in Theorem 2.2.3 when $\beta > \frac{1}{\alpha}$.

In the next subsection we precisely define our model and state the main theorem.

2.2 Model and main result

To define our model and state the main result we need to introduce the following notations. Let $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$ and $\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{1/p}$ for $p \geq 1$ be the norms of the spaces $\mathbf{L}^\infty(\mathbb{R})$ and $\mathbf{L}^p(\mathbb{R})$ respectively. The norms on $\mathbf{L}^\infty([0,T] \times R)$ and $\mathbf{L}^p([0,T] \times \mathbb{R})$ are defined similarly. By $\mathbf{L}^p_{loc}(\mathbb{R}_+ \times \mathbb{R})$ we will mean the collection of (equivalence classes of) measurable functions $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $\int_0^T \int_{\mathbb{R}} |f(s,x)|^p \, ds \, dx < \infty$ for all $T \in (0,\infty)$. We also define $\mathcal{S} \equiv \mathcal{S}(\mathbb{R})$ to be the space

$$\mathcal{S}(\mathbb{R}) = \left\{\varphi \in \mathcal{C}^\infty(\mathbb{R}) \mid \text{ for all } m, n \in \mathbb{N}, \sup_{x\in\mathbb{R}} |x^m \varphi^{(n)}(x)| < \infty\right\}$$

of all smooth rapidly-decreasing functions defined on $\mathbb{R}$ whose derivatives of all orders are also rapidly-decreasing. In the above, $\mathcal{C}^\infty(\mathbb{R})$ is the space of all infinitely differentiable real-valued functions on $\mathbb{R}$ and $\varphi^{(n)}$ stands for the n-th order derivative of φ. The subsets of $\mathbf{L}^p(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ containing all non-negative functions are denoted by $\mathbf{L}^p(\mathbb{R})_+$ and $\mathcal{S}_+ \equiv \mathcal{S}(\mathbb{R})_+$ respectively.

Let $M_F \equiv M_F(\mathbb{R})$ be the set of all non-negative finite measures on the real line, $\mathbb{R}$, equipped with the topology of weak convergence. We denote the space of all cádlág paths in M_F as $D \equiv D([0,\infty), M_F)$. This space is equipped with the topology of weak convergence and $\mathcal{B}(D)$ denotes the Borel σ-algebra on D. Similarly, $\mathcal{B}(\mathbb{R})$ denotes the σ-algebra of all Borel measurable subsets of $\mathbb{R}$ and for $E \in \mathcal{B}(\mathbb{R})$ we use $|E|$ for the Lebesgue measure of A. For $\mu \in M_F$ and $\varphi \in \mathcal{S}$ we denote $\langle \mu, \varphi \rangle = \int_{\mathbb{R}} \varphi \, d\mu$. We will often identify a measurable function $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to \mathbb{R}$ with $f(x)\,dx$, where dx denotes the Lebesgue measure. In this case, $\langle f, \varphi \rangle := \int_{\mathbb{R}} f(x)\varphi(x)\,dx$.

Definition 2.2.1. Let $\alpha \in (0,2)$. Suppose for each $E \in \mathcal{B}(\mathbb{R})$ with $|E| < \infty$, $\{L^\alpha_t(E)\}_{t\geq 0}$ is a martingale defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and

$$\mathbb{E} \exp\left(-\lambda L^\alpha_t(E)\right) = \exp\left(\lambda^\alpha t|E|\right), \tag{2.2.1}$$

for all $t \geq 0$, $\lambda \geq 0$. Then we call L^α an *α-stable martingale measure on* $[0,\infty)\times\mathbb{R}$ *without negative jumps.*

Observe that $L^\alpha(E \times [0,t]) := L^\alpha_t(E)$ is indeed a martingale measure in the sense of Walsh [Wal86, Chapter 2].

Definition 2.2.2. Let $Y_0 \in M_F$. Given an α-stable martingale measure L^α on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ without negative jumps, a two-parameter stochastic process $\{Y_t(x)\}_{t\geq 0, x\in\mathbb{R}}$ defined on the same probability space is said to solve (2.0.1) if the following hold.

(i) Y is adapted to $(\mathcal{F}_t)_{t\geq 0}$.

(ii) The map $t \mapsto Y_t(x)\,dx$ defines an M_F-valued cádlág process. In other words, $Y \in D([0,\infty), M_F)$ a.s..

(iii) For all $\psi \in \mathcal{S}(\mathbb{R})$ and $t \geq 0$,

$$\langle Y_t, \psi\rangle = \langle Y_0, \psi\rangle + \int_0^t \langle Y_s, \frac{1}{2}\Delta\psi\rangle\, ds + \int_{s\in[0,t]}\int_{x\in\mathbb{R}} (Y_{s-}(x))^\beta \psi(x) L^\alpha(dx,\, ds), \tag{2.2.2}$$

Recall that [Myt02, Theorem 1.5] guarantees the weak existence of such a solution $Y \equiv (Y_t)_{t\geq 0}$. Also, it was shown in [Myt02, Proposition 4.1] that

$$Y \in D([0,\infty), M_F) \cap \mathbf{L}^\rho_{loc}(\mathbb{R}_+ \times \mathbb{R}) \quad (1 < \rho < 3).$$

We need to recall some notions of existence and uniqueness for solutions to (2.0.1) that will be used in this article.

- (2.0.1) is said to admit a *weak solution* with initial condition Y_0 if there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and an $\{\mathcal{F}_t\}$-adapted pair (Y, L^α) such that L^α satisfies (2.2.1) and (2.2.2) holds.
- *Weak uniqueness* holds for (2.0.1) if whenever the pairs (Y, L^α) and $(\tilde{Y}, \tilde{L}^\alpha)$ satisfy (2.2.2) with the same initial condition, they have the same finite dimensional distributions.

Our main result is the following.

Theorem 2.2.3 (Theorem 1.3 of [Mai21])**.** *Assume that* $1 < \alpha < 2$ *and* $\frac{1}{\alpha} < \beta < \left(\frac{3}{\alpha} - 1\right) \wedge 1$ *and* $Y_0 \in M_F$. *Then weak uniqueness holds for solutions to* (2.2.2)*, i.e. if* (Y, L^α) *and* $(\tilde{Y}, L^\alpha)$ *are both weak solutions of* (2.2.2) *and* $Y_0 = \tilde{Y}_0$*, then* Y *and* $\tilde{Y}$ *have the same finite dimensional distributions.*

We will now describe our approach for proving this result.

2.3 Proof strategy

An approach for showing weak uniqueness of stochastic equations is the following. First, one shows that solutions to (2.2.2) are equivalently also solutions of an appropriate martingale problem. Then it is enough to show that any two solutions to the martingale problem have the same one-dimensional distributions (cf. [EK86, Theorem 4.4.2]). We may define the (local) martingale problem as follows. For $\psi \in \mathcal{S}_+$ and $t \geq 0$ let

$$M^Y_t(\psi) = e^{-\langle Y_t, \psi\rangle} - e^{-\langle Y_0, \psi\rangle} - \int_0^t e^{-\langle Y_{s-}, \psi\rangle}\left(-\langle Y_{s-}, \frac{1}{2}\Delta\psi\rangle + \langle Y^{\alpha\beta}_{s-}, \psi^\alpha\rangle\right) ds \tag{2.3.1}$$

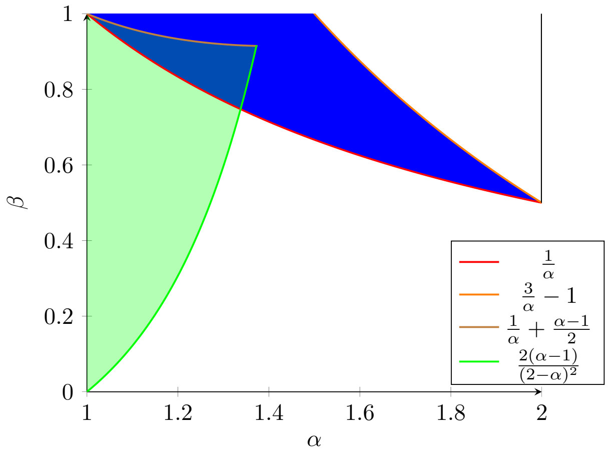

Figure 2.1: Mytnik proved weak uniqueness for of (2.0.1) when $\beta = \frac{1}{\alpha}$ (red line). Yang, Zhou showed pathwise uniqueness of this equation when (α, β) falls in the green region. Our main result, Theorem 2.2.3, proves weak uniqueness for (2.0.1) when (α, β) is in the blue region.

where $(Y_t)_{t\geq 0}$ are the coordinate maps on D, i.e. $Y_t(\omega) = \omega(t)$ for $\omega \in D$. A probability measure $\mathbb{P}$ on $(D, \mathcal{B}(D))$ is said to be a solution of the (local) martingale problem for (2.3.1) if for all $\psi \in \mathcal{S}_+$ we have that $\{M_t^Y(\psi)\}_t$ is a (local) martingale under $\mathbb{P}$. We know from [Myt02, Proposition 4.1] that a solution to the local martingale problem (2.3.1) exists with stopping times

$$\gamma^Y(k) := \inf\left\{s \geq 0 \middle| \int_0^s \|Y_r\|_{\alpha\beta}^{\alpha\beta}\, dr > k\right\}, \quad k \in \mathbb{N}. \tag{2.3.2}$$

where $\|\cdot\|_{\alpha\beta}$ denotes the norm of the space $\mathbf{L}^{\alpha\beta}(\mathbb{R})$. [Myt02, Proposition 4.1] also guarantees that, when $Y_0 \in M_F$, for all $t > 0$ we have $Y_t \in M_F$ as well.

Since $Y_0 \in M_F$ is chosen arbitrarily, in light of the above discussion we can rephrase Theorem 2.2.3 into the equivalent result concerning the martingale problem (2.3.1).

Theorem 2.3.1. *Under the assumptions of Theorem 2.2.3, any two solutions of (2.3.1) have same one-dimensional distributions.*

Remark 2.3.2. When $\alpha \in (1, 2)$, $\beta \in (\frac{1}{\alpha}, 1)$ and $Y_0 \in \mathcal{S}_+$, the statement of Theorem 2.3.1 remains valid. This can be shown by the same approximating duality idea discussed below and using the improved moment bound (see Remark 3.1.2). In other words, under these conditions the one-dimensional laws of $(Y_t)_{t\geq 0}$ are uniquely determined. However, we cannot prove the more general result of weak uniqueness (i.e. Theorem 2.2.3).

Motivated by [Myt98] we use an *approximating duality* argument and would like to show the following.

Theorem 2.3.3. *Let $\psi \in \mathcal{S}_+$ and Y be a solution to (2.3.1) where Y_0 is as in Theorem 2.2.3. Then there exists a sequence of processes $\{Z^{(n)}\}_{n\geq 1}$, independent of Y, with $Z_0^{(n)} = \psi$ for all $n \geq 1$ such that for each $t \geq 0$*

$$\mathbb{E}\exp(-\langle Y_t, \psi\rangle) = \lim_{n\to\infty} \mathbb{E}\exp(-\langle Y_0, Z_t^{(n)}\rangle). \tag{2.3.3}$$

By the virtue of [Myt96, Theorem 1.3] this will imply Theorem 2.3.1.

Now we shall discuss a formal strategy of how one would prove Theorem 2.3.3. Suppose Y solves (2.3.1) and Z solves SPDE given below

$$\frac{\partial Z_t(x)}{\partial t} = \frac{1}{2}\Delta Z_t(x) + Z_{t-}(x)^{\frac{1}{\beta}}\dot{L}^{\alpha\beta}, \qquad Z_0 = \psi \tag{2.3.4}$$

or equivalently the local martingale problem

$$M_t^Z(\varphi) = e^{-\langle\varphi, Z_t\rangle} - e^{-\langle\varphi, Z_0\rangle} - \int_0^t e^{-\langle\varphi, Z_{s-}\rangle}\left(-\langle\frac{1}{2}\Delta\varphi, Z_{s-}\rangle + \langle\varphi^{\alpha\beta}, Z_{s-}^{\alpha},\rangle\right) ds, \quad \varphi \in \mathcal{S}_+ \tag{2.3.5}$$

is an $\mathcal{F}_t^Z$-local martingale. Then one could try to establish the following exponential duality relation

$$\mathbb{E}\exp(-\langle Y_t, \psi\rangle) = \mathbb{E}\exp(-\langle\varphi, Z_t\rangle). \tag{2.3.6}$$

Although this duality relationship holds, the required integrability conditions will fail to hold (see [EK86, Theorem 4.4.11]). Thus one uses the approximate duality technique. In this approach we construct an approximating sequence $Z^{(n)}$ to Z using the framework of [Myt98] to prove the Theorem 2.3.3.

However, there are two key difficulties to overcome. First, we require a moment estimate of the solutions of (2.3.1) and the second difficulty is the fact that $M^Y(\psi)$, as defined above, are only local martingales. A moment estimate was shown in [YZ17, Lemma 2.4] for very general initial condition but we prove an improved estimate in Proposition 3.1.1 when $Y_0 \in \mathcal{S}(\mathbb{R})_+$ and for the range of α and β in Theorem 2.2.3. From this we can show that $M^Y(\psi)$ is indeed a martingale.

Remark 2.3.4. Note that the condition $\alpha\beta > 1$ is crucial for our argument and the technique of approximate duality. Consequently, the case when $\alpha\beta < 1$ is not covered by this method.

Throughout this chapter and the next we shall use the notations c, c_1, C, C_1 etc. to denote constants whose value may change from one line to the next. They will usually depend on the time horizon T and the initial condition Y_0. Wherever necessary we will denote their dependence on the relevant parameters.

2.4 Mild formulations of the SHE

This section contains notations that are used throughout the paper and some useful results regarding the mild forms of (2.2.2). Let $p_t(x) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)$ for all $t > 0, x \in \mathbb{R}$. For any function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and a measure $\mu \in M_F$, we will denote

$$P_t f(x) = \int_{\mathbb{R}} p_t(x-y) f(y)\, dy \text{ and } P_t\mu(x) = \int_{\mathbb{R}} p_t(x-y)\, \mu(dy).$$

As in [YZ17], define a measure on $\mathbb{R}$,

$$m_0(dz) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} z^{-1-\alpha}\mathbf{1}\{z > 0\}\, dz. \tag{2.4.1}$$

We first show that the solution Y_t in (2.2.2) of Definition 2.2.2 can be written in the following equivalent mild forms.

Proposition 2.4.1. *Let Y be a solution as in Definition 2.2.2 and $Y_0 \in M_F$. Then*

(a) *For $t \geq 0, x \in \mathbb{R}$,*

$$Y_t(x) = P_t Y_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) Y_s(y)^{\beta} L^{\alpha}(dy,\, ds). \tag{2.4.2}$$

(b) *There exists a Poisson random measure (PRM) N on $(0,\infty)^2 \times \mathbb{R}$ with intensity $ds\, m_0(dz) dx$ such that*

$$Y_t(x) = P_t Y_0(x) + \int_0^t \int_0^{\infty} \int_{\mathbb{R}} z\, p_{t-s}(x-y) Y_s(y)^{\beta}\, \tilde{N}(dy,\, dz,\, ds), \tag{2.4.3}$$

where $\tilde{N}(dy,\, dz,\, ds) = N(dy,\, dz,\, ds) - dy\, m_0(dz)\, ds$.

(c) *On an enlarged probability space there exists a PRM N_0 on $(0,\infty)^2 \times \mathbb{R} \times (0,\infty)$ with intensity $ds\, m_0(dz)\, dy\, dv$ such that, for all $t \geq 0$ and a.e. $x \in \mathbb{R}$,*

$$Y_t(x) = P_t Y_0(x) + \int_0^t \int_0^{\infty} \int_{\mathbb{R}} \int_0^{Y_s(y)^{\alpha\beta}} z\, p_{t-s}(x-y) \tilde{N}_0(\, dv,\, dy,\, dz,\, ds), \tag{2.4.4}$$

where $\tilde{N}_0(\, dv,\, dy,\, dz,\, ds) = N_0(\, dv,\, dy,\, dz,\, ds) -\, dv\, dy\, m_0(dz)\, ds$.

Proof. (a) This can be shown by an argument similar to the one in the proof of Theorem 1.1(a) in [MP03]. The only difference here is to show that for each $t > 0$,

$$\int_0^t \int_{\mathbb{R}} (t-s)^{-\alpha/2} Y_s(y)^{\alpha\beta}\, dy\, ds \leq \left(\sup_{s \leq t} \|Y_s\|_{\alpha\beta}^{\alpha\beta} \right) \int_0^t (t-s)^{-\alpha/2}\, ds < \infty \text{ a.s.} \tag{2.4.5}$$

This follows from the facts that $\int_0^t \|Y_s\|_{\alpha\beta}^{\alpha\beta}\, ds < \infty$ and $s \mapsto \|Y_s\|_{\alpha\beta} = \langle Y_s^{\alpha\beta}, 1\rangle^{\frac{1}{\alpha\beta}}$ is a cadlag map.

The claim in part (b) follows from the above and [MP03, Theorem 1.1(a)]. Using a change of variable type transformation as indicated in the proof of [YZ17, Proposition 2.1] we get part (c). □

Remark 2.4.2. To show that the stochastic integral in (2.4.4) is well-defined Yang and Zhou used the additional condition (see [YZ17, Assumption 1.4]) that there is a $q > \frac{3\alpha\beta}{3-\alpha}$ such that $\int_0^t \int_{\mathbb{R}} Y_s(x)^q\, dx\, ds < \infty$ for all $t > 0$ a.s.. We here observe that our proof of (2.4.4) above does not require this assumption.

2.5 Construction of the approximating sequence

As mentioned before we need to construct an approximating sequence $\{Z^{(n)}\}_{n\geq 1}$ to Z described in (2.3.4). We shall use the construction given in [Myt02, §3]. For completeness we only present the sketch below.

Define $Z_0^{(n)}(\, dx) = \psi(x)\, dx$ and let $b_n = \frac{\alpha\beta}{\Gamma(2-\alpha\beta)} n^{\alpha\beta-1}$. We know from [Fle88, Proposition A2] that given $\mu \in M_F$, there is a unique non-negative solution to the partial differential equation (PDE)

$$v_t = P_t\mu - \int_0^t P_{t-s}(b_n v_s^{\alpha})\, ds \tag{2.5.1}$$

where $(P_t\mu)(x) = \int_{\mathbb{R}} p_t(x-y)\mu(\,dy)$. Let us call this solution $V^n_\cdot(\mu)$. See Section 2.6.2 for some properties of the above PDE under nicer initial conditions.

The idea behind this $Z^{(n)}$ is as follows. $Z^{(n)}$ evolves according to the PDE (2.5.1), jumps after a random time given by Dirac measures at specified mass and location (denoted in the following by $\gamma^n(T^n_k)$, S^n_i and U^n_i respectively, see (2.5.3) for precise definition). More precisely, let

$$\tilde{T}^{Z,n}_i := \tilde{T}^n_i \sim \operatorname{Exp}\left(n^{\alpha\beta}\frac{(\alpha\beta-1)}{\Gamma(2-\alpha\beta)}\right), \quad i \in \mathbb{N},$$

be i.i.d. random variables and $T^n_i := \sum_{k=1}^{i} \tilde{T}^n_k$. The jump heights are given by i.i.d.random variables $\{S^n_i \mid i \in \mathbb{N}\}$ taking values in $[\frac{1}{n}, \infty)$. These are defined by

$$\mathbb{P}(S^n_i \geq b) = \frac{\int_{b\vee(1/n)}^{\infty} \lambda^{-\alpha\beta-1}\,d\lambda}{\int_{1/n}^{\infty} \lambda^{-\alpha\beta-1}\,d\lambda}, \qquad b \geq 0.$$

We observe that $\mathbb{E}[S^n_i] = \frac{\alpha\beta}{n(\alpha\beta-1)}$. Let

$$A^n_t := \sum_{k=1}^{\infty} S^n_k \mathbf{1}(T^n_k \leq t)$$

be the process that jumps by height S^n_i at time T^n_i for all $i \in \mathbb{N}$. By $(\mathcal{F}^{A^n}_t)_{t\geq 0}$ we will denote the filtration generated by A^n. For $0 \leq t \leq T^n_1$ define the time change

$$\gamma^n(t) = \inf\left\{s \geq 0 \middle| \int_{0+}^{s} \|V^n_r(\mu)\|^\alpha_\alpha\,dr > t\right\}.$$

We can define the approximating sequence $Z^{(n)}$ on the (random) interval $[0, \gamma^n(T^n_1))$ by

$$Z^{(n)}_t = V^n_t(Z^{(n)}_0), \quad 0 \leq t < \gamma^n(T^n_1), \tag{2.5.2}$$

where V^n is the solution of the PDE (2.5.1). For defining $Z^{(n)}$ at the time $t = \gamma^n(T^n_1)$, we proceed as follows. For each $f \in \mathbf{L}^\alpha(\mathbb{R})_+$ let $G(f,\cdot)$ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that for all $E \in \mathcal{B}(\mathbb{R})$,

$$G(f,E) := \frac{\int_E f(x)^\alpha\,dx}{\|f\|^\alpha_\alpha}.$$

Lastly, let U^n_1 be a $\mathbb{R}$-valued random variable defined by the relation

$$\mathbb{P}(U^n_1 \in E \mid \mathcal{F}^{A^n}_{T^n_1}) = G(Z^{(n)}_{\gamma^n(T^n_1)-}, E), \quad E \in \mathcal{B}(\mathbb{R}).$$

Then we can define

$$Z^{(n)}_{\gamma^n(T^n_1)} = Z^{(n)}_{\gamma^n(T^n_1)-} + S^n_1 \delta_{U^n_1}. \tag{2.5.3}$$

Thus we have constructed $Z^{(n)}$ on the interval $[0, \gamma^n(T^n_1)]$.

When $t > \gamma^n(T_1^n)$, $Z^{(n)}$ is defined inductively: for integers $k \geq 1$,

$$Z_t^{(n)} := \begin{cases} V^n_{t-\gamma^n(T_k^n)}(Z^{(n)}_{\gamma^n(T_k^n)}), & t \in [\gamma^n(T_k^n), \gamma^n(T_{k+1}^n)), \\ Z^{(n)}_{\gamma^n(T_k^n)-} + S^n_{k+1}\delta_{U^n_{k+1}}, & t = \gamma^n(T^n_{k+1}), \end{cases} \tag{2.5.4}$$

where

$$\gamma^n(t) = \inf\left\{ s \geq 0 \middle| T_k^n + \int_{0+}^{s-\gamma^n(T_k^n)} \|V_r^n(Z^{(n)}_{T_k^n})\|_\alpha^\alpha \, dr > t \right\}, \qquad T_k^n \leq t < T_{k+1}^n,$$

and

$$\mathbb{P}(U^n_{k+1} \in E \mid \mathcal{F}^{A^n}_{T^n_{k+1}}) := G(Z^{(n)}_{\gamma^n(T^n_{k+1})-}, E), \qquad E \in \mathcal{B}(\mathbb{R}).$$

This completes the construction of $Z^{(n)}$. It is known that $Z^{(n)}$ solves a local martingale problem as described by the following lemma. As usual, $\mathcal{F}^{Z^{(n)}}$ denotes the filtration generated by $Z^{(n)}$.

Lemma 2.5.1. *Let* $\eta := \frac{\alpha\beta(\alpha\beta-1)}{\Gamma(2-\alpha\beta)}$ *and*

$$g(r, y) := \int_{0+}^r (e^{-\lambda y} - 1 + \lambda y)\lambda^{-\alpha\beta-1}\, d\lambda. \tag{2.5.5}$$

For all $\varphi \in \mathcal{S}_+$ *and* $n \geq 1$

$$\begin{aligned} M_t^{Z,n}(\varphi) &= e^{-\langle \varphi, Z_t^{(n)}\rangle} - e^{-\langle \varphi, Z_0^{(n)}\rangle} \\ &- \int_0^t e^{-\langle \varphi, Z_{s-}^{(n)}\rangle} \left(-\langle \frac{1}{2}\Delta\varphi, Z_s^{(n)}\rangle + \langle \varphi(\cdot)^{\alpha\beta} - \eta g\left(1/n, \varphi(\cdot)\right), (Z_{s-}^{(n)}(\cdot))^\alpha \rangle \right) ds \end{aligned} \tag{2.5.6}$$

is an $\mathcal{F}^{Z^{(n)}}$*-local martingale with stopping times*

$$\gamma^{Z,n}(k) := \gamma^n(k) = \inf\left\{ s \geq 0 \middle| \int_0^s \|Z_r^{(n)}\|_\alpha^\alpha \, dr > k \right\}, \quad k \in \mathbb{N}. \tag{2.5.7}$$

Proof. See [Myt02, Lemma 3.7]. □

We conclude the section with a result describing the behavior of the compensator of $\mathcal{N}^{Z^{(n)}} = \mathcal{N}^n$ which is the counting measure tracking the jumps of $Z^{(n)}$.

Lemma 2.5.2. *The compensator of* $\mathcal{N}^n$ *is, for* $B_1 \in \mathcal{B}([0,\infty))$, $B_2 \in \mathcal{B}(\mathbb{R})$

$$\hat{\mathcal{N}}^n(t, B_1 \times B_2) = \eta \int_0^{t\wedge T_n^*} dr \int_{B_1} d\lambda \int_{B_2} dx \frac{\left(Z_{r-}^{(n)}(x)\right)^\alpha}{\|Z_{r-}^{(n)}\|_\alpha^\alpha} \mathbf{1}(\lambda > 1/n)\lambda^{-\alpha\beta-1} \tag{2.5.8}$$

where

$$T_n^* = \inf\{t \geq 0 \mid \gamma^n(t) = \infty\} = \inf\{t \geq 0 \mid A_t^n - b_n t = 0\}. \tag{2.5.9}$$

Proof. See [Myt02, Lemma 3.5]. □

2.6 Some tools from analysis

2.6.1 A Grönwall-type lemma

We use the general Grönwall lemma (see Lemma 1.3.2) to prove the required estimate.

Lemma 2.6.1. *Let $\gamma, \theta \in (0,1)$ and $f : (0,T] \longrightarrow [0,\infty)$ be an integrable function such that and for all $t \in [0,T]$*

$$f(t) \le ct^{-\theta} + c\int_0^t (t-r)^{-\gamma} f(r)\,dr \tag{2.6.1}$$

for some constant $c > 0$. Then there exists an integrable function $C_1 : (0,T] \longrightarrow [0,\infty)$ and a constant $C_2 > 0$ such that, for a.e. $t \in [0,T]$,

$$f(t) \le C_1(t) + \int_0^t C_1(s)\exp(C_2 s)\,ds. \tag{2.6.2}$$

Moreover C_1, C_2 are independent of the function f.

Proof. Let $k > 0$ be the smallest integer such that $\gamma < \frac{k}{k+1}$ and $t \in [0,T]$. We apply (2.6.1) and use the substitutions $w = \frac{r}{t}$ and $v = \frac{r-u}{t-u}$ for the first and second integrals in the RHS of the following computation.

$$\begin{aligned}
f(t) \le & ct^{-\theta} + c^2\int_0^t (t-r)^{-\gamma} t^{-\theta}\,dr + c^2\int_0^t\int_0^r (t-r)^{-\gamma}(r-u)^{-\gamma} f(u)\,du\,dr \\
= & ct^{-\theta} + c^2 t^{1-\gamma-\theta}\int_0^1 (1-w)^{-\gamma} w^{-\theta}\,dw + c^2\int_0^t du f(u)\int_u^t dr (t-r)^{-\gamma}(r-u)^{-\gamma} \\
= & c_1(t) + c^2\int_0^t f(u)(t-u)^{1-2\gamma}\,du\int_0^1 dv (1-v)^{-\gamma} v^{-\gamma} \\
= & c_1(t) + c_1'\int_0^t f(u)(t-u)^{1-2\gamma}\,du
\end{aligned} \tag{2.6.3}$$

where $c_1(t) = ct^{-\theta} + c^2 B(1-\gamma, 1-\theta)\, t^{1-\gamma-\theta}$ and $c_1' = c^2 B(1-\gamma, 1-\gamma)$ with B here denoting the Beta function. Again applying (2.6.1) to (2.6.3) we have

$$f(t) \le c_2(t) + c_2'\int_0^t f(u)(t-u)^{2-3\gamma}\,du,$$

where $c_2(t) = c_1(t) + c_1' c B(2-2\gamma, 1-\theta)\, t^{2-2\gamma-\theta}$ and $c_2' = c_1' c B(1-\gamma, 2-2\gamma)$. Continuing this process for k steps we get,

$$\begin{aligned}
f(t) \le & c_k(t) + c_k'\int_0^t f(u)(t-u)^{k-(k+1)\gamma}\,du \\
\le & c_k(t) + c_k' T^{k-(k+1)\gamma}\int_0^t f(u)\,du.
\end{aligned} \tag{2.6.4}$$

where the last step is obtained by our assumption on k. Also note that f is non-negative and integrable on $[0,T]$ by hypothesis. Therefore we can apply the standard Grönwall's inequality from Lemma 1.3.2 and have,

$$f(t) \leq c_k(t) + \int_0^t c_k(s)\exp(c'_k T^{k-(k+1)\gamma} s)\,ds$$

for a.e. $t \in [0,T]$. We can thus define $C_1(t) = c_k(t)$ and $C_2 = c'_k T^{k-(k+1)\gamma}$. Clearly these are independent of f. To see that C_1 is integrable on $(0,T]$ we only note that each $t^{m-m\gamma-\theta}$ $(m = 0,\dots,k)$ is integrable. □

2.6.2 Norm estimates for solutions of the evolution equation (2.5.1)

This section contains some useful properties of the solutions to the PDE

$$\begin{aligned} &\frac{\partial}{\partial t}v(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}v(t,x) - b_n v(t,x)^\alpha, \quad x \in \mathbb{R}, t \in [0,T], \\ &v(0,\cdot) = \varphi. \end{aligned} \tag{2.6.5}$$

where T is arbitrary but finite. When $\varphi \in \mathcal{S}(\mathbb{R})_+$, [Isc86, Theorem A] guarantees that this equation admits a unique solution.

Lemma 2.6.2. *If $v = v(t,x)$ solves the PDE (2.6.5) and $\varphi \in \mathcal{S}(\mathbb{R})_+$, then v satisfies all the hypotheses of Proposition 3.2.2.*

Proof. **(a)** It follows from [Fle88, Proposition A2] and proof of [Myt02, Lemma 2.1(c)] that $s \mapsto v(s) \in \mathbf{L}^\eta(\mathbb{R}) \cap \mathbf{L}^\rho(\mathbb{R})$ is continuous.

(b) We first prove that

$$\sup_{s\leq T}\left\|\frac{\partial^2}{\partial x^2}v(s)\right\|_\infty = \left\|\frac{\partial^2}{\partial x^2}v\right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R})} < \infty. \tag{2.6.6}$$

Note that as $\varphi \in \mathcal{S}_+$, by [Isc86, Theorem A], the solution $v : [0,T] \to C_0(\mathbb{R}_+)_+$ (continuous, non-negative functions vanishing at infinity) is a continuous map. Therefore by (2.6.5), to show (2.6.6) it is enough to prove that

$$\sup_{s\leq t}\|w(s)\|_\infty < \infty \tag{2.6.7}$$

where we have used the notation $w(s) = \dot{v}(s) = \frac{\partial}{\partial s}v(s)$.

From the proof of [Isc86, Theorem A] it follows that w must satisfy the PDE

$$w(t) = P_t(\tilde{\varphi}) - \alpha b_n \int_0^t P_{t-s}(v(s)^{\alpha-1}w(s))\,ds \tag{2.6.8}$$

where $\tilde{\varphi} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi - \frac{b_n}{2}\varphi^\alpha$. This gives us,

$$\|w(t)\|_\infty \leq \|\tilde{\varphi}\|_\infty + \alpha b_n \int_0^t \|(v(s)^{\alpha-1}\|_\infty \|w(s)\|_\infty\,ds, \tag{2.6.9}$$

from which using Grönwall's inequality (see [Eva10, Appendix B2]) we obtain

$$\|w(t)\|_\infty \leq \|\tilde{\varphi}\|_\infty \exp\left(\alpha b_n \int_0^t \|v(s)^{\alpha-1}\|_\infty \, ds\right) < \infty. \tag{2.6.10}$$

As v is continuously differentiable (see the proof of [Isc86, Theorem A]), $s \mapsto w(s, \cdot)$ is continuous. This fact along with the above gives us (2.6.7).

Next we show that the map $[0, T] \to \mathbf{L}^\infty(\mathbb{R})$, $t \mapsto \frac{\partial^2}{\partial x^2} v(t)$ is continuous, i.e.

$$\left\| \frac{\partial^2}{\partial x^2} v(s) - \frac{\partial^2}{\partial x^2} v(t) \right\|_\infty \to 0 \text{ as } s \to t \text{ in } [0, T]. \tag{2.6.11}$$

Similarly as above, since $v \in C_0(\mathbb{R})_+$, by (2.6.5) it is enough to show that

$$\|w(t) - w(s)\|_\infty \to 0 \text{ as } s \to t \tag{2.6.12}$$

and we use (2.6.8) for this purpose.

Let $0 \leq s < t \leq T$. Let $f_r = v(r)^{\alpha-1} w(r)$. Then from (2.6.8)

$$\begin{aligned} w(t) - w(s) =& P_t(\tilde{\varphi}) - P_s(\tilde{\varphi}) - \alpha b_n \left[\int_0^t P_{t-r}(f_r)\, dr - \int_0^s P_{s-r}(f_r)\, dr \right] \\ =& P_t(\tilde{\varphi}) - P_s(\tilde{\varphi}) - \alpha b_n \left[\int_s^t P_{t-r}(f_r)\, dr + \int_0^s \left(P_{t-r}(f_r) - P_{s-r}(f_r)\right)\, dr \right] \\ =& P_t(\tilde{\varphi}) - P_s(\tilde{\varphi}) - \alpha b_n \left[\int_s^t P_{t-r}(f_r)\, dr + \int_0^s \left(P_{t-s}(P_{s-r}(f_r)) - P_{s-r}(f_r)\right)\, dr \right] \end{aligned} \tag{2.6.13}$$

using that fact $P_{t-r} f_r = P_{t-s}(P_{s-r} f_r)$.

As $\varphi \in \mathcal{S}(\mathbb{R})_+$, by our definition $\tilde{\varphi} \in \mathcal{S}(\mathbb{R})$. Therefore when $s \to t$, $\|P_t \tilde{\varphi} - P_s \tilde{\varphi}\|_\infty \to 0$. For the second term in (2.6.13), note that if we can prove that

$$\sup_{r \leq t} \|P_{t-r}(f_r)\|_\infty < \infty,$$

it will follow that $\int_s^t P_{t-r}(f_r)\, dr \to 0$ in $\mathbf{L}^\infty(\mathbb{R})$ as $s \to t$. We have, for $x \in \mathbb{R}$

$$|P_{t-r}(f_r)(x)| \leq \left| \int_{\mathbb{R}} p_{t-r}(x-y) f_r(y)\, dr \right| \leq \|f_r\|_\infty = \|v(r)^{\alpha-1} w(r)\|_\infty < \infty.$$

since know $v(r) \in C_0(\mathbb{R})$ and we have already shown that $\sup_{r \leq t} \|w(r)\|_\infty < \infty$. Similarly, the third term in (2.6.13) can be shown to be converging to 0 in $\mathbf{L}^\infty(\mathbb{R})$ as $s \to t$. This proves (2.6.12) and hence (2.6.11).

(c) Let $\rho = \frac{\alpha\beta}{\alpha\beta - 1}$. To show $\sup_{s \leq t} \|\dot{v}(s)\|_\rho < \infty$ we again use (2.6.8). Note that

$$\int_{\mathbb{R}} |w(t,x)|^\rho \, dx \leq C \left[\int_{\mathbb{R}} |P_t(\tilde{\varphi})|^\rho \, dx + \alpha b_n \int_{\mathbb{R}} \left| \int_0^t P_{t-s}(v(s)^{\alpha-1} w(s))\, ds \right|^\rho dx \right]. \tag{2.6.14}$$

Using Jensen inequality,

$$\begin{aligned}\int_{\mathbb{R}} |P_t(\tilde{\varphi})|^\rho \, dx &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} p_t(x-y)\tilde{\varphi}(y)\, dy \right|^\rho \, dx \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} p_t(x-y)|\tilde{\varphi}(y)|^\rho \, dy\, dx \\ &= \|\tilde{\varphi}\|_\infty^\rho < \infty.\end{aligned}$$

By definition of P_{t-s} and using Jensen's inequality once more we have,

$$\begin{aligned}\int_{\mathbb{R}} \left| \int_0^t P_{t-s}(v(s)^{\alpha-1} w(s))\, ds \right|^\rho \, dx &= \int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) v(s,y)^{\alpha-1} w(s,y)\, dy\, ds \right|^\rho \, dx \\ &\leq t^{\rho-1} \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) |v(s,y)^{\alpha-1} w(s,y)|^\rho \, dy\, ds\, dx.\end{aligned}$$

Using Fubini's theorem, as all terms are non-negative, integrating out x in the above we have

$$\begin{aligned}\int_{\mathbb{R}} \left| \int_0^t P_{t-s}(v(s)^{\alpha-1} w(s))\, ds \right|^\rho \, dx &\leq t^{\rho-1} \int_0^t \int_{\mathbb{R}} |v(s,y)^{\alpha-1} w(s,y)|^\rho \, dy\, ds \\ &\leq C_1 \int_0^t \|v(s)\|_\infty^{\rho(\alpha-1)} \|w(s)\|_\rho^\rho \, ds. \qquad (2.6.15)\end{aligned}$$

Using (2.6.15) in (2.6.14) we have

$$\|w(t)\|_\rho^\rho \leq C\|\tilde{\varphi}\|_\infty^\rho + C \int_0^t \|v(s)\|_\infty^{\rho(\alpha-1)} \|w(s)\|_\rho^\rho \, ds. \qquad (2.6.16)$$

Again by Grönwall's inequality

$$\|w(t)\|_\rho^\rho \leq C\|\tilde{\varphi}\|_\infty^\rho \exp\left(\int_0^t \|v(s)\|_\infty^{\rho(\alpha-1)} \, ds \right) < \infty \qquad (2.6.17)$$

and the required result follows as in part (b). □

Chapter 3

Proof of Theorem 2.3.3

In this chapter we will prove several key technical tools and finish the proof of our main theorem. The first section establishes the key moment result for solutions Y of (2.2.2). Section 3.2 shows that $M^Y(\psi)$ defined in (2.3.1) is a martingale for all $\psi \in \mathcal{S}_+$. In Section 3.3 we give an overview of the proof of Theorem 2.3.3 and the last section finishes the argument. The last section of this chapter contains a proof of Proposition 3.2.2.

3.1 The moment estimate

The following is an alternative proof of the estimate presented in [YZ17, Lemma 2.4]. Recall from the statement of Theorem 2.2.3 that $Y_0 \in M_F$, the collection of all finite non-negative measures on $\mathbb{R}$.

Proposition 3.1.1. *Let $1 \le q < \alpha$. Then for a.e. $t \in [0, T]$ we have*

$$\sup_{x\in\mathbb{R}} \mathbb{E}(Y_t(x)^q) \le C\, t^{-\frac{q}{2}} + C \tag{3.1.1}$$

where $C = C(T, Y_0, \alpha, \beta) > 0$ is a constant.

Remark 3.1.2. We note in passing that when Y_0 is a bounded function on $\mathbb{R}$, the above estimate can be improved further. In this situation we will have,

$$\sup_{x\in\mathbb{R}} \mathbb{E}(Y_t(x)^q) \le C_1'(T, Y_0)e^{C_2'(T)t}, \quad t \in [0, T],$$

where C_1' and C_2' are positive constants.

Proof of Proposition 3.1.1. From (2.4.4) we have, for $t \in [0, T]$,

$$Y_t(x) = P_t Y_0(x) + \int_0^t \int_0^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z p_{t-r}(x - y)\tilde{N}_0(\,dv,\, dy,\, dz,\, dr). \tag{3.1.2}$$

Let us define

$$\tau_N = \inf\left\{t \ge 0 \mid \int_0^t \|Y_r\|_{\alpha\beta}^{\alpha\beta}\, dr \ge N\right\},$$

when $N \in \mathbb{N}$ and from [PZ07, Lemma 8.21] recall that the quadratic variation of

$$\int_0^s \int_0^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z p_{t-r}(x-y) \tilde{N}_0(\,dv,\, dy,\, dz,\, dr)$$

equals

$$\int_0^s \int_0^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z^2 p_{t-r}(x-y)^2 N_0(\,dv,\, dy,\, dz,\, dr),$$

for $s \in [0, T]$. By the Burkholder-Davis-Gundy inequality and the fact that $q < 2$ we have

$$\begin{aligned}
I :=& \mathbb{E}\left[\left|\int_0^t \int_0^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z p_{t-r}(x-y) \tilde{N}_0(\,dv,\, dy,\, dz,\, dr)\right|^q \mathbf{1}(t \le \tau_N)\right] \\
=& \mathbb{E}\left[\left|\int_0^t \int_0^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z p_{t-r}(x-y) \mathbf{1}(r \le \tau_N) \tilde{N}_0(\,dv,\, dy,\, dz,\, dr)\right|^q\right] \\
\le& c\mathbb{E}\left[\left|\int_0^t \int_0^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z^2 p_{t-r}(x-y)^2 \mathbf{1}(r \le \tau_N) N_0(\,dv,\, dy,\, dz,\, dr)\right|^{q/2}\right] \\
\le& c\mathbb{E}\left[\left|\int_0^t \int_0^1 \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z^2 p_{t-r}(x-y)^2 \mathbf{1}(r \le \tau_N) N_0(\,dv,\, dy,\, dz,\, dr)\right|^{q/2}\right] \\
&+ c\mathbb{E}\left[\left|\int_0^t \int_1^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z^2 p_{t-r}(x-y)^2 \mathbf{1}(r \le \tau_N) N_0(\,dv,\, dy,\, dz,\, dr)\right|^{q/2}\right].
\end{aligned} \tag{3.1.3}$$

Fix $p \in (\alpha, 2)$ such that $\frac{p-1}{2} + \frac{\alpha\beta}{2} \le 1$. This is possible because of our assumption that $\beta < \frac{3}{\alpha} - 1$. Applying Jensen's inequality to the above (noting that $p/q > 1$) we have

$$\begin{aligned}
I \le& c\mathbb{E}\left[\left|\int_0^t \int_0^1 \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z^2 p_{t-r}(x-y)^2 \mathbf{1}(r \le \tau_N) N_0(\,dv,\, dy,\, dz,\, dr)\right|^{p/2}\right]^{q/p} \\
&+ c\mathbb{E}\left[\left|\int_0^t \int_1^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z^2 p_{t-r}(x-y)^2 \mathbf{1}(r \le \tau_N) N_0(\,dv,\, dy,\, dz,\, dr)\right|^{q/2}\right] \\
\le& c\mathbb{E}\left[\int_0^t \int_0^1 \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z^p p_{t-r}(x-y)^p \mathbf{1}(r \le \tau_N) N_0(\,dv,\, dy,\, dz,\, dr)\right]^{q/p} \\
&+ c\mathbb{E}\left[\int_0^t \int_1^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z^q p_{t-r}(x-y)^q \mathbf{1}(r \le \tau_N) N_0(\,dv,\, dy,\, dz,\, dr)\right],
\end{aligned} \tag{3.1.4}$$

where the second inequality above is due a fact about random sums (see the proof of [PZ07, Lemma 8.22]). Now we use the definition of the PRM N_0, integrate out z and use the inequality

$u^{q/p} \leq u+1$ for $u \geq 0$.

$$\begin{aligned}
I \leq & c + c\mathbb{E}\left[\int_0^t \int_0^1 \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z^p p_{t-r}(x-y)^p \mathbf{1}(r \leq \tau_N) N_0(\,dv,\,dy,\,dz,\,dr)\right] \\
& + c\mathbb{E}\left[\int_0^t \int_1^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z^q p_{t-r}(x-y)^q \mathbf{1}(r \leq \tau_N) N_0(\,dv,\,dy,\,dz,\,dr)\right] \\
\leq & c + c\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} Y_r(y)^{\alpha\beta} p_{t-r}(x-y)^p \mathbf{1}(r \leq \tau_N)\,dy\,dr\right] \\
& + c\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} Y_r(y)^{\alpha\beta} p_{t-r}(x-y)^q \mathbf{1}(r \leq \tau_N)\,dy\,dr\right] \\
\leq & c + c\mathbb{E}\int_0^t \int_{\mathbb{R}} (p_{t-r}(x-y)^p + p_{t-r}(x-y)^q) Y_r(y)^{\alpha\beta} \mathbf{1}(r \leq \tau_N)\,dy\,dr,
\end{aligned} \tag{3.1.5}$$

as $\int_0^1 z^p m_0(dz) < \infty$ and $\int_0^\infty z^q m_0(dz) < \infty$. From (3.1.2) and (3.1.5) we have

$$\begin{aligned}
& \mathbb{E}\left[Y_t(x)^q \mathbf{1}(t < \tau_N)\right] \\
\leq & c(P_t Y_0(x))^q + c + c\mathbb{E}\int_0^t \int_{\mathbb{R}} (p_{t-r}(x-y)^p + p_{t-r}(x-y)^q) Y_r(y)^{\alpha\beta} \mathbf{1}(r < \tau_N)\,dr\,dy \\
= & c(P_t Y_0(x))^q + c + c\int_0^t \int_{\mathbb{R}} (p_{t-r}(x-y)^p + p_{t-r}(x-y)^q) \mathbb{E}\left[Y_r(y)^{\alpha\beta} \mathbf{1}(r < \tau_N)\right]\,dr\,dy.
\end{aligned} \tag{3.1.6}$$

by applying Fubini's theorem in the last line. Use the definition of $p_t(x)$ to get

$$\begin{aligned}
& \mathbb{E}\left[Y_t(x)^q \mathbf{1}(t < \tau_N)\right] \\
\leq & c(P_t Y_0(x))^q + c + c\int_0^t \,dr((t-r)^{-\frac{p-1}{2}} + (t-r)^{-\frac{q-1}{2}}) \int_{\mathbb{R}} p_{t-r}(x-y) \mathbb{E}\left[Y_r(y)^{\alpha\beta} \mathbf{1}(r < \tau_N)\right]\,dy \\
\leq & c(P_t Y_0(x))^q + c + c\int_0^t \,dr(t-r)^{-\frac{p-1}{2}} \int_{\mathbb{R}} p_{t-r}(x-y) \mathbb{E}\left[Y_r(y)^{\alpha\beta} \mathbf{1}(r < \tau_N)\right]\,dy
\end{aligned} \tag{3.1.7}$$

where in the last line we have used the fact that $(t-r)^{-\frac{q-1}{2}} \leq C_T(t-r)^{-\frac{p-1}{2}}$.

When $q = \alpha\beta$ this becomes

$$\begin{aligned}
& \mathbb{E}\left[Y_t(x)^{\alpha\beta} \mathbf{1}(t < \tau_N)\right] \\
\leq & c(P_t Y_0(x))^{\alpha\beta} + c + c\int_0^t \,dr(t-r)^{-\frac{p-1}{2}} \int_{\mathbb{R}} p_{t-r}(x-y) \mathbb{E}\left[Y_r(y)^{\alpha\beta} \mathbf{1}(r < \tau_N)\right]\,dy.
\end{aligned} \tag{3.1.8}$$

Let $s \in [0,T]$ be such that $s \geq t$. Apply P_{s-t} to both sides and use Fubini's theorem,

$$
\begin{aligned}
&\mathbb{E}\left[P_{s-t}\left(Y_t^{\alpha\beta}\right)(x)\mathbf{1}(t<\tau_N)\right]\\
\leq& c(P_sY_0(x))^{\alpha\beta}+c+c\int_0^t dr\,(t-r)^{-\frac{p-1}{2}}\int_{\mathbb{R}}p_{s-t}(x-y)\int_{\mathbb{R}}p_{t-r}(y-z)\mathbb{E}\left[Y_r(z)^{\alpha\beta}\mathbf{1}(r<\tau_N)\right]dz\,dy\\
\leq& c(P_sY_0(x))^{\alpha\beta}+c+c\int_0^t dr(t-r)^{-\frac{p-1}{2}}\int_{\mathbb{R}}p_{s-r}(x-z)\mathbb{E}\left[Y_r(z)^{\alpha\beta}\mathbf{1}(r<\tau_N)\right]dz\\
=& c(P_sY_0(x))^{\alpha\beta}+c+c\int_0^t dr\,(t-r)^{-\frac{p-1}{2}}\mathbb{E}\left[P_{s-r}\left(Y_r^{\alpha\beta}\right)(x)\mathbf{1}(r<\tau_N)\right]\\
\leq& c(Y_0)s^{-\frac{\alpha\beta}{2}}+c+c\int_0^t dr\,(t-r)^{-\frac{p-1}{2}}\mathbb{E}\left[P_{s-r}\left(Y_r^{\alpha\beta}\right)(x)\mathbf{1}(r<\tau_N)\right]
\end{aligned}
\tag{3.1.9}
$$

where we have used the assumption on Y_0 to obtain the bound on $(P_sY_0(x))^{\alpha\beta}$. The constants appearing hereafter all depend on Y_0. Since the above holds for every $t \in [0,s]$, by Lemma 2.6.1 there exists a function C_1 on $(0,T]$ and a constant $C_2(s) > 0$ such that for a.e. $t \leq s$,

$$
\mathbb{E}\left[P_{s-t}\left(Y_t^{\alpha\beta}\right)(x)\mathbf{1}(t<\tau_N)\right]\leq C_1(t)+\int_0^t C_1(r)e^{C_2(s)r}\,dr. \tag{3.1.10}
$$

Observe from the proof of Lemma 2.6.1 that

$$
C_1(t)=o(t^{-\frac{\alpha\beta}{2}}) \text{ as } t\downarrow 0
$$

and that the constant $C_2(s)$ is non-decreasing in s. So we have $C_2(s) \leq C_2(T)$ and (3.1.10) gives

$$
\mathbb{E}\left[P_{s-t}\left(Y_t^{\alpha\beta}\right)(x)\mathbf{1}(t<\tau_N)\right]\leq C_3t^{-\frac{\alpha\beta}{2}}+C_3\int_0^t r^{-\frac{\alpha\beta}{2}}e^{C_2(T)r}\,dr. \tag{3.1.11}
$$

for a.e. $t \leq s \leq T$. Here $C_3 = C_3(T) > 0$ is a constant. Now replace s by t in the above. We get,

$$
\mathbb{E}\left[(Y_t(x))^{\alpha\beta}\,\mathbf{1}(t<\tau_N)\right]\leq C_3t^{-\frac{\alpha\beta}{2}}+C_3\int_0^t r^{-\frac{\alpha\beta}{2}}e^{C_2(T)r}\,dr. \tag{3.1.12}
$$

for a.e. $t \in [0,T]$.

We now plug this into (3.1.7) to get,

$$
\begin{aligned}
&\mathbb{E}\left[Y_t(x)^q\mathbf{1}(t<\tau_N)\right]\\
\leq& c\,t^{-\frac{q}{2}}+c+c\int_0^t dr(t-r)^{-\frac{p-1}{2}}\int_{\mathbb{R}}p_{t-r}(x-y)\mathbb{E}\left[Y_r(y)^{\alpha\beta}\mathbf{1}(r<\tau_N)\right]dy\\
\leq& c\,t^{-\frac{q}{2}}+c+C_4\int_0^t dr(t-r)^{-\frac{p-1}{2}}r^{-\frac{\alpha\beta}{2}}+C_4\int_0^t dr(t-r)^{-\frac{p-1}{2}}r^{1-\frac{\alpha\beta}{2}}\\
=& c\,t^{-\frac{q}{2}}+c+C_5t^{1-\frac{p-1}{2}-\frac{\alpha\beta}{2}}+C_6t^{2-\frac{p-1}{2}-\frac{\alpha\beta}{2}}
\end{aligned}
\tag{3.1.13}
$$

where $C_4 = C_4(T) > 0$ is a constant and $C_5 = C_4\,B(1-\frac{p-1}{2}, 1-\frac{\alpha\beta}{2})$, $C_6 = C_4\,B(1-\frac{p-1}{2}, 2-\frac{\alpha\beta}{2})$ with B denoting the Beta function. By our assumption on p, the second term in RHS of (3.1.13) has non-negative exponent and clearly so does the third. Therefore, we can write,

$$
\mathbb{E}\left[Y_t(x)^q\mathbf{1}(t<\tau_N)\right]\leq C\,t^{-\frac{q}{2}}+C \tag{3.1.14}
$$

where C is constant depending on T, Y_0 and the parameters α, β.

Take $N \to \infty$ and we obtain the required result. □

We here observe that the previous moment estimate can be utilized to show that the stochastic integrals appearing in (2.4.2), (2.4.3) and (2.4.4) are martingales. For this we recall the notion of a *class DL* process (see [RY99, Definition IV.1.6]).

A real valued and adapted stochastic process X is said to be of *class DL* if for every $t > 0$, the set

$$\{X_\tau : \tau \leq t \text{ is a stopping time}\}$$

is uniformly integrable. And we know from [RY99, Proposition IV.1.7] that a local martingale X is a martingale if and only if it is of class DL. For practical purposes it is enough to show that there is an $\epsilon > 0$ such that

$$\sup_{\tau \leq t} \mathbb{E}(|X_\tau|^{1+\epsilon}) < \infty$$

where the supremum is taken over all stopping times $\tau \leq t$.

Lemma 3.1.3. *Let $t \in [0, T]$ be fixed. Then for $s \in [0, t]$ the stochastic integrals*

$$\int_0^s \int_{\mathbb{R}} p_{t-r}(x-y) Y_r(y)^\beta \, L^\alpha(dy, dr) = \int_0^s \int_0^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z \, p_{t-r}(x-y) \tilde{N}_0(\,dv, dy, dz, dr) \tag{3.1.15}$$

are martingales with respect to $\mathcal{F}^Y$, the filtration generated by Y.

Proof. Since the integrals in (3.1.15) are stochastic integrals with respect to martingale measure L^α and the compensated PRM $\tilde{N}_0$, they are clearly $\mathcal{F}^Y$-local martingales. Therefore, we have to show that

$$\int_0^s \int_0^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z \, p_{t-r}(x-y) \tilde{N}_0(\,dv, dy, dz, dr) \tag{3.1.16}$$

is of class DL.

Let $\tau \leq t$ be a stopping time. We omit some of the calculations since they are similar to the ones found in the previous proof. Choose q and p such that $1 < q < \alpha < p < 2$ and $\frac{p-1}{2} + \frac{\alpha\beta}{2} \geq 1$. Proceeding as in (3.1.3) and (3.1.4) we obtain,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^\tau \int_0^\infty \int_{\mathbb{R}} \int_0^{Y_r(y)^{\alpha\beta}} z \, p_{t-r}(x-y) \tilde{N}_0(\,dv, dy, dz, dr)\right|^q\right] \\
\leq& c + c\mathbb{E} \int_0^\tau \int_{\mathbb{R}} (p_{t-r}(x-y)^p + p_{t-r}(x-y)^q) Y_r(y)^{\alpha\beta} \, dy \, dr \\
\leq& c + c \int_0^t \int_{\mathbb{R}} (p_{t-r}(x-y)^p + p_{t-r}(x-y)^q) \mathbb{E}(Y_r(y)^{\alpha\beta}) \, dy \, dr \\
\leq& c + c_1 \int_0^t \int_{\mathbb{R}} (p_{t-r}(x-y)^p + p_{t-r}(x-y)^q) r^{-\frac{\alpha\beta}{2}} \, dy \, dr + c_1 \int_0^t \int_{\mathbb{R}} (p_{t-r}(x-y)^p + p_{t-r}(x-y)^q) \, dy \, dr \\
\leq& c + c_1 \int_0^t (t-r)^{-\frac{p-1}{2}} r^{-\frac{\alpha\beta}{2}} \, dr + c_1 \int_0^t (t-r)^{-\frac{p-1}{2}} \, dr \\
\leq& c + c_2 t^{1-\frac{p-1}{2}-\frac{\alpha\beta}{2}} + c_3 t^{1-\frac{p-1}{2}}
\end{aligned}$$

applying (3.1.1) at the end. The RHS of the above is finite on $[0, T]$ by our assumptions on p. Also, it is independent of the stopping times τ. This shows that (3.1.16) is in class DL and is therefore a martingale with respect to $\mathcal{F}^Y$. □

3.2 The martingale problem

Next we observe that $M_t^Y(\psi)$ defined in (2.3.1) is a martingale. This will be crucial for simplifying our approximate duality argument in the proof of Proposition 3.3.1.

Proposition 3.2.1. *For each $\psi \in \mathcal{S}_+$, the local martingale $M^Y(\psi)$ is in fact a martingale with respect to $\mathcal{F}^Y$.*

Proof. Recall that

$$M_t^Y(\psi) = e^{-\langle Y_t,\psi\rangle} - e^{-\langle Y_0,\psi\rangle} - \int_0^t I(Y_{s-},\psi)\,ds$$

is an $\mathcal{F}_t^Y$-local martingale, where

$$I(Y_{s-},\psi) = e^{-\langle Y_{s-},\psi\rangle}\left(-\langle Y_{s-},\frac{1}{2}\Delta\psi\rangle + \langle Y_{s-}^{\alpha\beta},\psi^\alpha\rangle\right)$$

To show that $M_t^Y(\psi)$ is a martingale we show that it is in class DL, i.e. for each $t > 0$,

$$\sup_{\tau\le t}\mathbb{E}\left(|M_\tau^Y(\psi)|^{1+\epsilon}\right) < \infty \tag{3.2.1}$$

for some $\epsilon > 0$. The supremum ranges over all $\mathcal{F}^Y$-stopping times τ that are bounded by t.

From the expression above it is enough to prove

$$\sup_{\tau\le t}\mathbb{E}\left(\left|\int_0^\tau I(Y_{s-},\psi)\,ds\right|^{1+\epsilon}\right) < \infty. \tag{3.2.2}$$

Fix a stopping time $\tau \le t$. By Jensen's inequality

$$\begin{aligned}\left|\int_0^\tau I(Y_{s-},\psi)\,ds\right|^{1+\epsilon} &= \left|\int_0^\tau e^{-\langle Y_{s-},\psi\rangle}\left(-\langle Y_{s-},\frac{1}{2}\Delta\psi\rangle + \langle Y_{s-}^{\alpha\beta},\psi^\alpha\rangle\right)ds\right|^{1+\epsilon}\\
&= \tau^{1+\epsilon}\left|\frac{1}{\tau}\int_0^\tau e^{-\langle Y_{s-},\psi\rangle}\left(-\langle Y_{s-},\frac{1}{2}\Delta\psi\rangle + \langle Y_{s-}^{\alpha\beta},\psi^\alpha\rangle\right)ds\right|^{1+\epsilon}\\
&\le \tau^{\epsilon}\int_0^\tau\left|-\langle Y_{s-},\frac{1}{2}\Delta\psi\rangle + \langle Y_{s-}^{\alpha\beta},\psi^\alpha\rangle\right|^{1+\epsilon}ds\\
&\le C_\epsilon t^\epsilon\int_0^t\left(|\langle Y_{s-},\frac{1}{2}\Delta\psi\rangle|^{1+\epsilon} + |\langle Y_{s-}^{\alpha\beta},\psi^\alpha\rangle|^{1+\epsilon}\right)ds.\end{aligned} \tag{3.2.3}$$

Let $0 < \epsilon < \frac{1}{\beta} - 1 < \alpha - 1 < 1$. Again apply Jensen's inequality, Fubini's Theorem and

Proposition 3.1.1.

$$\begin{aligned}\mathbb{E}\int_0^t |\langle Y_{s-}, \frac{1}{2}\Delta\psi\rangle|^{1+\epsilon}\, ds =& \frac{1}{2^{1+\epsilon}}\mathbb{E}\int_0^\tau \left|\int_{\mathbb{R}} Y_{s-}(x)\Delta\psi(x)\, dx\right|^{1+\epsilon} ds\\ \leq& \frac{\|\Delta\psi\|_1^{1+\epsilon}}{2^{1+\epsilon}}\mathbb{E}\int_0^t \left|\frac{1}{\|\Delta\psi\|_1}\int_{\mathbb{R}} Y_{s-}(x)|\Delta\psi(x)|\, dx\right|^{1+\epsilon} ds\\ \leq& \frac{\|\Delta\psi\|_1^{\epsilon}}{2^{1+\epsilon}}\int_0^t\int_{\mathbb{R}} \mathbb{E}Y_{s-}(x)^{1+\epsilon}|\Delta\psi(x)|\, dx\, ds\\ \leq& C_T\frac{\|\Delta\psi\|_1^{1+\epsilon}}{2^{1+\epsilon}}\int_0^t s^{-\frac{1+\epsilon}{2}}\, ds + C_T\frac{\|\Delta\psi\|_1^{1+\epsilon}}{2^{1+\epsilon}}\\ =& C_T\frac{\|\Delta\psi\|_1^{1+\epsilon}}{2^{1+\epsilon}}(1+t^{1-\frac{1+\epsilon}{2}}).\end{aligned} \tag{3.2.4}$$

Similarly,

$$\begin{aligned}\mathbb{E}\int_0^t |\langle Y_{s-}^{\alpha\beta}, \frac{1}{2}\Delta\psi\rangle|^{1+\epsilon}\, ds \leq& \|\psi^\alpha\|_1^\epsilon\int_0^t\int_{\mathbb{R}} \mathbb{E}Y_{s-}(x)^{\alpha\beta(1+\epsilon)}\psi(x)^\alpha\, dx\, ds\\ =& C_T\frac{\|\psi^\alpha\|_1^{1+\epsilon}}{2^{1+\epsilon}}\int_0^t s^{-\frac{\alpha\beta(1+\epsilon)}{2}}\, ds + C_T\frac{\|\psi^\alpha\|_1^{1+\epsilon}}{2^{1+\epsilon}}\\ =& C_{T,\psi,\alpha,\beta,\epsilon}(t^{1-\frac{\alpha\beta(1+\epsilon)}{2}}+1)\end{aligned} \tag{3.2.5}$$

Note that $1-\frac{1+\epsilon}{2}\geq 0$ and $1-\frac{\alpha\beta(1+\epsilon)}{2}\geq 0$ by our conditions on α, β and ϵ. Plugging (3.2.4) and (3.2.5) in (3.2.3) we get,

$$\mathbb{E}\left(\left|\int_0^\tau I(Y_{s-},\psi)\, ds\right|^{1+\epsilon}\right) \leq C_{T,\psi,\alpha,\beta,\epsilon}(1+t^{1-\frac{\alpha\beta(1+\epsilon)}{2}}+t^{1-\frac{1+\epsilon}{2}}) \tag{3.2.6}$$

Taking supremum over all $\tau\leq t$ gives (3.2.2). □

We now show the above result holds for $\psi:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ satisfying certain assumptions.

Proposition 3.2.2. *Let $T\in(0,\infty)$ If Y is a solution to the martingale problem (2.3.1) and $\psi:[0,T]\times\mathbb{R}\to\mathbb{R}$ is such that*

(i) The map $[0,T]\ni s\mapsto\psi_s\in\mathbf{L}^\eta(\mathbb{R})\cap\mathbf{L}^\rho(\mathbb{R})$ is continuous, for some fixed $\eta\in(\frac{1}{\beta},\alpha)$ and $\rho\in(\alpha,\frac{\alpha}{\beta}\wedge 2)$. (Note that, as $\frac{1}{\alpha}<\beta<1$ and $\alpha<2$, such η and ρ exist.)

(ii) $\sup_{s\leq T}\left\|\frac{\partial}{\partial s}\psi_s\right\|_{\mathbf{L}^{\frac{\alpha\beta}{\alpha\beta-1}}(\mathbb{R})}<\infty$.

(iii) The map $[0,T]\to\mathbf{L}^\infty(\mathbb{R})$, $s\mapsto\frac{\partial^2}{\partial x^2}\psi_s$ is continuous.

Then,

$$\tilde{M}_t^Y(\psi)=e^{-\langle Y_t,\psi_t\rangle}-e^{-\langle Y_0,\psi_0\rangle}-\int_0^t\tilde{I}(Y_{s-},\psi_s)\, ds \tag{3.2.7}$$

is an $\mathcal{F}_t^Y$ martingale, where

$$\tilde{I}(Y_{s-},\psi_s)=e^{-\langle Y_{s-},\psi_s\rangle}\left[-\langle Y_{s-},\frac{1}{2}\partial_{xx}^2\psi_s+\partial_s\psi_s\rangle+\langle Y_{s-}^{\alpha\beta},\psi_s^\alpha\rangle\right]. \tag{3.2.8}$$

We present it in the last section of this chapter (Section 3.5).

3.3 Overview of the proof

In this section we describe our plan for proving Theorem 2.3.3. Our proof follows the argument in [Myt98] and will be split into various propositions which we state in the following. At the end of this section we establish the theorem assuming these results.

The first proposition describes the behavior of Y when coupled with the solutions of the evolution equations used to construct $Z^{(n)}$. In what follows we denote by $\mathbb{E}_Y$ the expectation with respect to Y. In particular, under $\mathbb{E}_Y$ we treat all the random variables used to construct $Z^{(n)}$ in Section 2.5 as non-random owing to our assumption of independence.

Proposition 3.3.1. *Let Y be a solution to the martingale problem (2.3.1). Then for each $t \in [0,T]$, $n \geq 1$ and $\mu \in M_F$,*

$$\mathbb{E}_Y\left[e^{-\langle Y_{T-t}, V_t^n(\mu)\rangle}\right] = \mathbb{E}_Y\left[e^{-\langle Y_T, V_0^n(\mu)\rangle}\right] + \mathbb{E}_Y\left[\int_0^t \tilde{\mathfrak{I}}(Y_{(T-r)-}, V_r^n(\mu))\, dr\right] \tag{3.3.1}$$

where

$$\tilde{\mathfrak{I}}(Y_{(T-r)-}, V_r^n(\mu)) = e^{-\langle Y_{(T-r)-}, V_r^n(\mu)\rangle}\left\{-\langle Y^{\alpha\beta}_{(T-r)-}, \left(V_r^n(\mu)\right)^{\alpha}\rangle + \langle Y_{(T-r)-}, b_n \left(V_r^n(\mu)\right)^{\alpha}\rangle\right\},$$

and V^n is the solution of the PDE (2.5.1).

In the next proposition we describe the relationship between Y and the jumps of $Z^{(n)}$. Define

$$\tau^n(t) := \int_0^t \|Z^{(n)}_{r-}\|_\alpha^\alpha\, dr \tag{3.3.2}$$

and observe from (2.5.7) that τ^n is the inverse of γ^n: $\tau^n(\gamma^n(t)) = t$ and vice-versa.

Proposition 3.3.2. *If Y is a solution to the martingale problem (2.3.1), independent of $Z^{(n)}$'s, then for all $t \in [0,T]$,*

$$\begin{aligned}&\mathbb{E}_Y\left[e^{-\langle Y_{T-t}, Z_t^{(n)}\rangle}\right]\\ =&\mathbb{E}_Y\left[e^{-\langle Y_T, Z_0\rangle} + \int_0^t \tilde{\mathfrak{I}}(Y_{(T-r)-}, Z^{(n)}_{r-})\, dr + \int_0^{\tau^n(t)}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \theta_n(s,x,\lambda)\mathfrak{N}^n(d\lambda,\, dx,\, ds)\right]\end{aligned} \tag{3.3.3}$$

where

$$\theta_n(s,x,\lambda) = e^{-\langle Y_{T-\gamma^n(s)}, Z^{(n)}_{\gamma^n(s)-}\rangle}\left(e^{-\lambda Y_{T-\gamma^n(s)}(x)} - 1\right).$$

In the last proposition before we prove our main result we show that the previous result holds at the stopping time $\Upsilon_k^n(t) := \gamma^n(k) \wedge t$. Recall the definitions of η and g from Lemma 2.5.1.

Proposition 3.3.3. *If Y is a solution to the martingale problem (2.3.1), independent of $Z^{(n)}$'s, then for each $m \in \mathbb{N}$ and $t \in [0,T]$,*

$$\begin{aligned}\mathbb{E}[\exp(-\langle Y_{T-\Upsilon_m^n(t)}, Z_{\Upsilon_m^n(t)}\rangle)] =&\mathbb{E}[\exp\left(-\langle Y_T, Z_0\rangle\right)]\\ &-\frac{\eta}{2}\mathbb{E}\left[\int_0^{\Upsilon_m^n(t)} e^{-\langle Y_{(T-r)-}, Z^{(n)}_{r-}\rangle}\langle g(1/n, Y_{(T-r)-}(\cdot)), (Z^{(n)}_{r-})^{\alpha}\rangle\, dr\right].\end{aligned} \tag{3.3.4}$$

We note that Propositions 3.3.1 and 3.3.2 are used to prove Proposition 3.3.3. Now we present the proof of Theorem 2.3.3 assuming that the above propositions hold. We will prove them in the next section.

Proof of Theorem 2.3.3. Let Y and $Z^{(n)}$ be as in the statement of Theorem 2.3.3. Let

$$k_n = \ln n.$$

We will show that for a.e. $t \in [0, T]$,

$$\lim_{n\to\infty} |\mathbb{E}[e^{-\langle Y_0, Z^{(n)}_{\Upsilon^n_{k_n}(t)}\rangle}] - \mathbb{E}[e^{-\langle Y_t, Z_0\rangle}]| = 0. \tag{3.3.5}$$

This will prove the theorem with the approximate dual processes being $\tilde{Z}^{(n)}_t := Z^{(n)}_{\Upsilon^n_{k_n}(t)}$.

Towards this, we are first going to show that

$$\left|\mathbb{E}\exp\left(-\langle Y_{T-\Upsilon^n_{k_n}(t)}, Z^{(n)}_{\Upsilon^n_{k_n}(t)}\rangle\right) - \mathbb{E}e^{-\langle Y_T, Z_0\rangle}\right| \le C_{\alpha.\beta,T}((T-t)^{-\frac{\bar{p}}{2}} + 1)n^{-\frac{\alpha-\alpha\beta}{2}}k_n, \tag{3.3.6}$$

when $0 \le t < T$. Note that, as $k_n = \ln n$, the RHS converges to 0 as $n \to \infty$.

Note that for all $1 < p < 2$ and $\lambda \ge 0$,

$$e^{-\lambda} - 1 + \lambda \le \frac{\lambda^p}{p}.$$

Also by our assumptions on α and β we have $1 < \frac{\alpha(\beta+1)}{2} < \alpha < 2$. So,

$$\begin{aligned}
g\left(\frac{1}{n}, Y_{T-s}(x)\right) &= \int_{0+}^{1/n} \left(e^{-\lambda Y_{T-s}(x)} - 1 + \lambda Y_{T-s}(x)\right)\lambda^{-\alpha\beta-1}\,d\lambda \\
&\le \frac{2}{\alpha(\beta+1)}\int_{0+}^{1/n} (\lambda Y_{T-s}(x))^{\frac{\alpha(\beta+1)}{2}}\lambda^{-\alpha\beta-1}\,d\lambda \\
&= \frac{2}{\alpha(\beta+1)}Y_{T-s}(x)^{\frac{\alpha(\beta+1)}{2}}\int_{0+}^{1/n}\lambda^{\frac{\alpha(\beta+1)}{2}-\alpha\beta-1}\,d\lambda \\
&= \frac{2}{\alpha(\beta+1)}Y_{T-s}(x)^{\frac{\alpha(\beta+1)}{2}}\frac{2}{\alpha-\alpha\beta}n^{-\frac{\alpha-\alpha\beta}{2}}.
\end{aligned} \tag{3.3.7}$$

Eq. (3.3.4) and the above calculation gives us

$$\begin{aligned}
&\left|\mathbb{E}\exp\left(-\langle Y_{T-\Upsilon^n_{k_n}(t)}, Z^{(n)}_{\Upsilon^n_{k_n}(t)}\rangle\right) - \mathbb{E}e^{-\langle Y_T, Z_0\rangle}\right| \\
=&\left|\frac{\eta}{2}\mathbb{E}\left[\int_{0+}^{\Upsilon^n_{k_n}(t)} e^{-\langle Y_{(T-s)-}, Z^{(n)}_s\rangle}\langle g(1/n, Y_{(T-s)-}(\cdot)), \left(Z^{(n)}_{s-}\right)^\alpha\rangle\,ds\right]\right| \\
=&\frac{\eta}{2}\mathbb{E}\left[\int_0^{\Upsilon^n_{k_n}(t)}\int_{\mathbb{R}} Z^{(n)}_{s-}(x)^\alpha g(1/n, Y_{T-s}(x))\,dx\,ds\right] \\
\le&\frac{\eta}{2}\frac{2}{\alpha(\beta+1)}\frac{2}{\alpha-\alpha\beta}n^{-\frac{\alpha-\alpha\beta}{2}}\mathbb{E}\left[\int_0^{\Upsilon^n_{k_n}(t)}\int_{\mathbb{R}} Z^{(n)}_{s-}(x)^\alpha Y_{T-s}(x)^{\frac{\alpha(\beta+1)}{2}}\,dx\,ds\right]
\end{aligned} \tag{3.3.8}$$

using the fact that $Z_{s-}^{(n)}(\cdot) \geq 0$ and $Y_{(T-s)-}(\cdot) \geq 0$ for the second equality. Now use the estimate from Proposition 3.1.1 with $\bar{p} = \frac{\alpha(\beta+1)}{2}$. We have by Fubini's theorem

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\Upsilon_{k_n}^n(t)} \int_{\mathbb{R}} Z_{s-}^{(n)}(x)^\alpha Y_{T-s}(x)^{\frac{\alpha(\beta+1)}{2}}\, dx\, ds\right] =& \mathbb{E}_Z \mathbb{E}_Y \left[\int_0^{\Upsilon_{k_n}^n(t)} \int_{\mathbb{R}} Z_{s-}^{(n)}(x)^\alpha Y_{T-s}(x)^{\bar{p}}\, dx\, ds\right] \\
=& \mathbb{E}_Z \left[\int_0^{\Upsilon_{k_n}^n(t)} \int_{\mathbb{R}} Z_{s-}^{(n)}(x)^\alpha \mathbb{E}_Y \left(Y_{T-s}(x)^{\bar{p}}\right)\, dx\, ds\right] \\
\leq& C\mathbb{E}_Z \left[\int_0^{\Upsilon_{k_n}^n(t)} \int_{\mathbb{R}} Z_{s-}^{(n)}(x)^\alpha (T-s)^{-\frac{\bar{p}}{2}}\, dx\, ds\right] \\
&+ C\mathbb{E}_Z \left[\int_0^{\Upsilon_{k_n}^n(t)} \int_{\mathbb{R}} Z_{s-}^{(n)}(x)^\alpha\, dx\, ds\right] \\
\leq& C\mathbb{E}_Z \left[(T - \Upsilon_{k_n}^n(t))^{-\frac{\bar{p}}{2}} \int_0^{\Upsilon_{k_n}^n(t)} \|Z_{s-}^{(n)}\|_\alpha^\alpha\, ds\right] \\
&+ C\mathbb{E}_Z \left[\int_0^{\Upsilon_{k_n}^n(t)} \|Z_{s-}^{(n)}\|_\alpha^\alpha\, ds\right] \\
\leq& C((T-t)^{-\frac{\bar{p}}{2}} + 1)\mathbb{E}_Z \left[\int_0^{\Upsilon_{k_n}^n(t)} \|Z_{s-}^{(n)}\|_\alpha^\alpha\, ds\right] \\
\leq& C((T-t)^{-\frac{\bar{p}}{2}} + 1)k_n. \qquad (3.3.9)
\end{aligned}$$

The third inequality is due to the fact that $\Upsilon_k^n(t) = \gamma^n(k) \wedge t \leq t$ and the last inequality follows from the definition of γ^n (see (2.5.7)). Plugging this in (3.3.8) gives (3.3.6).

Next we turn our attention to (3.3.5). We can write,

$$\begin{aligned}
&|\mathbb{E}\exp(-\langle Y_0, Z_{\Upsilon_{k_n}^n(t)}^{(n)}\rangle) - \mathbb{E}\exp(-\langle Y_t, Z_0\rangle)| \\
\leq& |\mathbb{E}\exp(-\langle Y_0, Z_{\Upsilon_{k_n}^n(t)}^{(n)}\rangle) - \mathbb{E}\exp(-\langle Y_{t-\Upsilon_{k_n}^n(t-\frac{1}{k_n})}, Z_{\Upsilon_{k_n}^n(t-\frac{1}{k_n})}^{(n)}\rangle)| \\
&+ |\mathbb{E}\exp(-\langle Y_{t-\Upsilon_{k_n}^n(t-\frac{1}{k_n})}, Z_{\Upsilon_{k_n}^n(t-\frac{1}{k_n})}^{(n)}\rangle) - \mathbb{E}\exp(-\langle Y_t, Z_0\rangle)|. \qquad (3.3.10)
\end{aligned}$$

By (3.3.6) (with T and t replaced by t and $t - \frac{1}{k_n}$ respectively) we can bound the second term in the RHS of the above as follows,

$$\left|\mathbb{E}\exp\left(-\langle Y_{t-\Upsilon_{k_n}^n(t-\frac{1}{k_n})}, Z_{\Upsilon_{k_n}^n(t-\frac{1}{k_n})}^{(n)}\rangle\right) - \mathbb{E}e^{-\langle Y_t Z_0\rangle}\right| \leq C_{\alpha.\beta,t}(k_n^{\frac{\bar{p}}{2}} + 1)n^{-\frac{\alpha-\alpha\beta}{2}}k_n.$$

We note that the RHS of the above converges to 0 as $n \to \infty$.

Let us now consider the first term in the RHS of (3.3.10). By definition of Υ^n and T_n^* (see

(2.5.9)) ,

$$
\begin{aligned}
&|\mathbb{E}\exp(-\langle Y_{t-\Upsilon^n_{k_n}(t-\frac{1}{k_n})}, Z^{(n)}_{\Upsilon^n_{k_n}(t-\frac{1}{k_n})}\rangle) - \mathbb{E}\exp(-\langle Y_0, Z^{(n)}_{\Upsilon^n_{k_n}(t)}\rangle)| \\
=&|\mathbb{E}\left[\exp(-\langle Y_{t-\Upsilon^n_{k_n}(t-\frac{1}{k_n})}, Z^{(n)}_{\Upsilon^n_{k_n}(t-\frac{1}{k_n})}\rangle) - \exp(-\langle Y_0, Z^{(n)}_{\Upsilon^n_{k_n}(t)}\rangle); \Upsilon^n_{k_n}(t) < t-\frac{1}{k_n}\right]| \\
&+ |\mathbb{E}\left[\exp(-\langle Y_{t-\Upsilon^n_{k_n}(t-\frac{1}{k_n})}, Z^{(n)}_{\Upsilon^n_{k_n}(t-\frac{1}{k_n})}\rangle) - \exp(-\langle Y_0, Z^{(n)}_{\Upsilon^n_{k_n}(t)}\rangle); \Upsilon^n_{k_n}(t) \geq t-\frac{1}{k_n}\right]| \\
\leq&\mathbb{P}(\Upsilon^n_{k_n}(t) < t-\frac{1}{k_n}) + |\mathbb{E}\left[\exp(-\langle Y_{\frac{1}{k_n}}, Z^{(n)}_{t-\frac{1}{k_n}}\rangle) - \exp(-\langle Y_0, Z^{(n)}_{\Upsilon^n_{k_n}(t)}\rangle); \Upsilon^n_{k_n}(t) \geq t-\frac{1}{k_n}\right]|.
\end{aligned}
$$

The second term above converges to 0 since Y is right-continuous and $\Upsilon^n_{k_n}(t) = \gamma^n(k_n) \wedge t \longrightarrow t$ as $n \longrightarrow \infty$. Also as $\mathbb{P}(T^*_n < \infty) = 1$ (see [Myt02, eq. (3.14)]), we have

$$
\mathbb{P}(\Upsilon^n_{k_n}(t) < t-\frac{1}{k_n}) = \mathbb{P}(\gamma^n(k_n) < t-\frac{1}{k_n}) \leq \mathbb{P}(T^*_n > k_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
$$

This proves (3.3.5). □

3.4 Proof of key Propositions

We will prove the three propositions required for the proof of Theorem 2.3.3 in this section. For Proposition 3.3.1 we start by verifying (3.3.1) for measures having densities and then prove it for the case of general measures.

Proof of Proposition 3.3.1. Let $\varphi_l \in \mathcal{S}(\mathbb{R})_+$, $l \in \mathbb{N}$, be such that $\mu_l(dx) := \varphi_l(x)\,dx \implies \mu(dx)$ as $l \longrightarrow \infty$. Since n is fixed in this proof, let $v_l(\cdot) = V^n_{\cdot}(\mu_l)$ solve

$$
\begin{aligned}
\partial_t v_l(t) &= \frac{1}{2}\partial^2_{xx} v_l(t) - b_n v_l(t)^\alpha \\
v_l(0) &= \varphi_l.
\end{aligned}
\tag{3.4.1}
$$

Fix $l, k \in \mathbb{N}$. Let $\psi(s,x) := v_l(T-s,x) = V^n_{T-s}(\mu_l)(x)$. Lemma 2.6.2 says that ψ satisfies the conditions of Proposition 3.2.2.

From (3.2.8) recall that

$$
\tilde{I}(Y_{s-}, \psi_s) = e^{-\langle Y_{s-}, \psi_s\rangle}\left[-\langle Y_{s-}, \frac{1}{2}\Delta\psi_s\rangle + \langle Y^{\alpha\beta}_{s-}, \psi^\alpha_s\rangle - \langle Y_{s-}, \frac{\partial}{\partial s}\psi_s\rangle\right].
$$

Then by Proposition 3.2.2 for each $k \in \mathbb{N}$ and $t \in [0,T)$,

$$
\mathbb{E}_Y\left[\tilde{M}^Y_{T-t}(\psi)\right] = \mathbb{E}_Y\left[\tilde{M}^Y_T(\psi)\right]
$$

which implies,

$$
\mathbb{E}_Y \exp\left(-\langle Y_{T-t}, \psi_{T-t}\rangle\right) = \mathbb{E}_Y\left[e^{-\langle Y_T \psi_T\rangle} - \int_{T-t}^T \tilde{I}(Y_{s-}, \psi_s)\right].
\tag{3.4.2}
$$

From the above definition of $\tilde{I}(Y,\psi)$ and (3.4.1) we get,

$$
\begin{aligned}
&\int_{T-t}^{T} \tilde{I}(Y_{s-},\psi_s)\,ds\\
&= \int_{T-t}^{T} e^{-\langle Y_{s-},\psi_s\rangle}\left\{-\langle Y_{s-},\frac{1}{2}\Delta\psi_s\rangle + \langle Y_{s-}^{\alpha\beta},\psi_s^{\alpha}\rangle - \langle Y_{s-},\frac{\partial}{\partial s}\psi_s\rangle\right\} ds\\
&= \int_{T-t}^{T} e^{-\langle Y_{s-},v_l(T-s)\rangle}\left\{-\langle Y_{s-},\frac{1}{2}\Delta v_l(T-s)\rangle + \langle Y_{s-}^{\alpha\beta},v_l(T-s)^{\alpha}\rangle - \langle Y_{s-},\frac{\partial}{\partial s}v_l(T-s)\rangle\right\} ds\\
&= -\int_{t}^{0} e^{-\langle Y_{(T-r)-},v_l(r)\rangle}\left\{-\langle Y_{(T-r)-},\frac{1}{2}\Delta v_l(r)\rangle + \langle Y_{(T-r)-}^{\alpha\beta},v_l(r)^{\alpha}\rangle + \langle Y_{(T-r)-},\frac{\partial}{\partial r}v_l(r)\rangle\right\} dr,\\
&= -\int_{t}^{0} e^{-\langle Y_{(T-r)-},v_l(r)\rangle}\left\{\langle Y_{(T-r)-}^{\alpha\beta},v_l(r)^{\alpha}\rangle + \langle Y_{(T-r)-},\frac{\partial}{\partial r}v_l(r) - \frac{1}{2}\Delta v_l(r)\rangle\right\} dr\\
&= -\int_{t}^{0} e^{-\langle Y_{(T-r)-},v_l(r)\rangle}\left\{\langle Y_{(T-r)-}^{\alpha\beta},v_l(r)^{\alpha}\rangle - \langle Y_{(T-r)-},b_n v_l(r)^{\alpha}\rangle\right\} dr, \qquad (3.4.3)
\end{aligned}
$$

using the substitution $r = T - s$ for the third equality.

By (3.4.2) and (3.4.3),

$$\mathbb{E}_Y \exp\left(-\langle Y_{T-t}, v_l(t)\rangle\right) = \mathbb{E}_Y \exp\left(-\langle Y_T, v_l(0)\rangle\right) + \mathbb{E}_Y \int_0^t \tilde{\mathfrak{I}}(Y_{(T-r)-}, v_l(r))\,dr \qquad (3.4.4)$$

where

$$\tilde{\mathfrak{I}}(Y_{(T-r)-}, v_l(r)) = e^{-\langle Y_{(T-r)-},v_l(r)\rangle}\frac{1}{2}\left(\langle Y_{(T-r)-}, b_n\,(v_l(r))^{\alpha}\rangle - \langle Y_{(T-r)-}^{\alpha\beta}, (v_l(r))^{\alpha}\rangle\right)$$

We now have to check whether this holds when $\mu := w - \lim_{l\to\infty}\mu_l$.

Let $v(r) = V_r^n(\mu)$,

$$R_l := \mathbb{E}_Y \int_0^t \tilde{\mathfrak{I}}(Y_{(T-r)-}, V_r^n(\mu_l))\,dr = \mathbb{E}_Y \int_0^t \tilde{\mathfrak{I}}(Y_{(T-r)-}, v_l(r)))\,dr,$$

and

$$R := \mathbb{E}_Y \int_0^t \tilde{\mathfrak{I}}(Y_{(T-r)-}, V_r^n(\mu))\,dr = \mathbb{E}_Y \int_0^t \tilde{\mathfrak{I}}(Y_{(T-r)-}, v(r))\,dr.$$

We only have to prove $R_l \to R$ as $l \to \infty$. In $R_l - R$ adding and subtracting the term

$$e^{-\langle Y_{(T-r)-},v(r)\rangle}\left(b_n\langle Y_{(T-r)-}, v_l(r)^{\alpha}\rangle - \langle Y_{(T-r)-}^{\alpha\beta}, v_l(r)^{\alpha}\rangle\right)$$

we have

$$|R_l - R| \le \frac{1}{2}(I_1^l + I_2^l),$$

where

$$
\begin{aligned}
I_1^l =&\mathbb{E}_Y\left|\int_0^t \left(e^{-\langle Y_{(T-r)-},v_l(r)\rangle} - e^{-\langle Y_{(T-r)-},v(r)\rangle}\right)\left(b_n\langle Y_{(T-r)-}, v_l(r)^{\alpha}\rangle - \langle Y_{(T-r)-}^{\alpha\beta}, v_l(r)^{\alpha}\rangle\right) dr\right|\\
I_2^l =&\mathbb{E}_Y\left|\int_0^t e^{-\langle Y_{(T-r)-},v(r)\rangle}\left(b_n\langle Y_{(T-r)-}, v_l(r)^{\alpha} - v(r)^{\alpha}\rangle - \langle Y_{(T-r)-}^{\alpha\beta}, v_l(r)^{\alpha} - v(r)^{\alpha}\rangle\right) dr\right|.
\end{aligned}
$$

To prove $|R_l - R| \longrightarrow 0$ as $l \longrightarrow \infty$ we need to show
(i): $I_1^l \longrightarrow 0$; and
(ii): $I_2^l \longrightarrow 0$,
as $l \longrightarrow \infty$.

Proof of (i): Let $1 < q < \frac{1}{\beta}$ and $p > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Note that

$$\begin{aligned}
I_1^l &\leq \int_0^t \mathbb{E}_Y \left| \left(e^{-\langle Y_{(T-r)-}, v_l(r) \rangle} - e^{-\langle Y_{(T-r)-}, v(r) \rangle} \right) \left(b_n \langle Y_{(T-r)-}, v_l(r)^\alpha \rangle - \langle Y^{\alpha\beta}_{(T-r)-}, v_l(r)^\alpha \rangle \right) \right| dr \\
&\leq \int_0^t \mathbb{E}_Y \left(\left| e^{-\langle Y_{(T-r)-}, v_l(r) \rangle} - e^{-\langle Y_{(T-r)-}, v(r) \rangle} \right|^p \right)^{1/p} \mathbb{E}_Y \left(\left| b_n \langle Y_{(T-r)-}, v_l(r)^\alpha \rangle - \langle Y^{\alpha\beta}_{(T-r)-}, v_l(r)^\alpha \rangle \right|^q \right)^{1/q} dr \\
&=: \int_0^t I_{11}^l(r) I_{12}^l(r)\, dr \qquad (3.4.5)
\end{aligned}$$

using Hölder's inequality in the second line. Here $I_{11}^l(r)$ and $I_{12}^l(r)$ denote the first and second terms of the integrand in the above.

Now let us use a notation from Fleischmann [Fle88]:

$$\|v\|_{\mathbf{L}^{\alpha,\mathrm{T}}} := \sup_{0 \leq t \leq T} \|v(t)\|_{\mathbf{L}^\alpha(\mathbb{R})}.$$

By [Fle88, Proposition A2], we have $v_l \longrightarrow v$ in $\mathbf{L}^{\alpha,\mathrm{T}}$ as $l \longrightarrow \infty$. Thus there is a subsequence of v_l, which we also denote as v_l by a slight abuse of notation, such that $v_l(t,x) \longrightarrow v(t,x)$ as $l \longrightarrow \infty$ for a.e. $t \in [0,T]$ and $x \in \mathbb{R}$. As the term inside the expectation of $I_{11}^l(r)$ is dominated by 2, dominated convergence theorem this gives us

$$\lim_{l \to \infty} I_{11}^l(r) = 0$$

for each $r \in [0,t]$. Since $|I_{11}^l(r)| \leq 2$ for all l and r, again by the dominated convergence theorem, to prove (i) as above we only have to show that $I_{12}^l(r) \leq C < \infty$ for some constant $C = C_t$ independent of l.

$$\begin{aligned}
I_{12}^l(r) &= \mathbb{E}_Y \left(\left| b_n \langle Y_{(T-r)-}, v_l(r)^\alpha \rangle - \langle Y^{\alpha\beta}_{(T-r)-}, v_l(r)^\alpha \rangle \right|^q \right)^{1/q} \\
&\leq b_n \mathbb{E}_Y \left(\left| \langle Y_{(T-r)-}, v_l(r)^\alpha \rangle \right|^q \right)^{1/q} + \mathbb{E}_Y \left(\langle Y^{\alpha\beta}_{(T-r)-}, v_l(r)^\alpha \rangle^q \right)^{1/q} \\
&:= b_n I_{121}^l(r) + I_{122}^l(r) \qquad (3.4.6)
\end{aligned}$$

using Minkowski's inequality.

For all $r < t$,

$$\begin{aligned}
I^l_{121}(r) =& \mathbb{E}_Y \left(\left| \langle Y_{(T-r)-}, v_l(r)^\alpha \rangle \right|^q \right)^{1/q} \\
=& \|v_l(r)\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})} \mathbb{E}_Y \left[\left(\int_{\mathbb{R}} \frac{1}{\|v_l(r)\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})}} v_l(r,x)^\alpha Y_{T-r}(x)\, dx \right)^q \right]^{1/q} \\
\leq& \|v_l(r)\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})} \left[\mathbb{E}_Y \left(\frac{1}{\|v_l(r)\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})}} \int_{\mathbb{R}} v_l(r,x)^\alpha Y_{T-r}(x)^q\, dx \right) \right]^{1/q} \\
=& \|v_l(r)\|^{\alpha-\alpha/q}_{\mathrm{L}^\alpha(\mathbb{R})} \left[\int_{\mathbb{R}} v_l(r,x)^\alpha \mathbb{E}_Y (Y_{T-r}(x)^q)\, dx \right]^{1/q} \\
\leq& C_T \|v_l(r)\|^{\alpha-\alpha/q}_{\mathrm{L}^\alpha(\mathbb{R})} \left[\int_{\mathbb{R}} v_l(r,x)^\alpha (T-r)^{-\frac{q}{2}}\, dx + \int_{\mathbb{R}} v_l(r,x)^\alpha\, dx \right]^{1/q} \\
\leq& C_T \|v_l(r)\|^{\alpha-\alpha/q}_{\mathrm{L}^\alpha(\mathbb{R})} ((T-t)^{-\frac{q}{2}} + 1)^{1/q} \left[\int_{\mathbb{R}} v_l(r,x)^\alpha\, dx \right]^{1/q} \\
=& C_T \|v_l(r)\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})} ((T-t)^{-\frac{q}{2}} + 1)^{1/q}.
\end{aligned} \tag{3.4.7}$$

Here we have used Jensen's inequality and Proposition 3.1.1 (applicable by our assumption that $q < \alpha$) in the first and second inequalities respectively. [Fle88, Proposition A2] implies that for large enough $l \in \mathbb{N}$, $\|v_l(r)\|_{\mathrm{L}^\alpha(\mathbb{R})} \leq \|v\|_{\mathrm{L}^{\alpha,\mathrm{T}}} + 1$ for all $r \in [0,t]$. Therefore (3.4.7) gives us

$$I^l_{121}(r) \leq C_T (\|v\|_{\mathrm{L}^{\alpha,\mathrm{T}}} + 1)^\alpha ((T-t)^{-\frac{q}{2}} + 1)^{1/q}, \tag{3.4.8}$$

when l is large.

For the term I^l_{122} we again proceed as in the calculation (3.4.7). Note that, as $\alpha\beta q < \alpha$ by our assumption, we can again apply Proposition 3.1.1 in the following. Let $r < t$.

$$\begin{aligned}
I^l_{122}(r) =& \mathbb{E}_Y \left(\langle Y^{\alpha\beta}_{(T-r)-}, v_l(r)^\alpha \rangle^q \right)^{1/q} \\
\leq& \|v_l(r)\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})} \left[\mathbb{E}^A_Y \left(\frac{1}{\|v_l(r)\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})}} \int_{\mathbb{R}} v_l(r,x)^\alpha Y_{T-r}(x)^{\alpha\beta q}\, dx \right) \right]^{1/q} \\
=& \|v_l(r)\|^{\alpha-\alpha/q}_{\mathrm{L}^\alpha(\mathbb{R})} \left[\int_{\mathbb{R}} v_l(r,x)^\alpha \mathbb{E}_Y (Y_{T-r}(x)^{\alpha\beta q})\, dx \right]^{1/q} \\
\leq& C_T \|v_l(r)\|^{\alpha-\alpha/q}_{\mathrm{L}^\alpha(\mathbb{R})} \left[\int_{\mathbb{R}} v_l(r,x)^\alpha (T-r)^{-\frac{\alpha\beta q}{2}}\, dx + \int_{\mathbb{R}} v_l(r,x)^\alpha\, dx \right]^{1/q} \\
\leq& C_T \|v_l(r)\|^\alpha_{\mathrm{L}^\alpha(\mathbb{R})} ((T-t)^{-\frac{\alpha\beta q}{2}} + 1)^{1/q} \leq C_T (\|v\|_{\mathrm{L}^{\alpha,\mathrm{T}}} + 1)^\alpha ((T-t)^{-\frac{\alpha\beta q}{2}} + 1)^{1/q}
\end{aligned} \tag{3.4.9}$$

for large l. We can observe that (3.4.8) and (3.4.9) together show that $I^l_{12} \leq C_{t,T}$ where $C_{t,T}$ is independent of l. Thus **(i)** is proved.

Proof of (ii): First note that $v_l \to v$ in $\mathbf{L}^{\alpha,\mathrm{T}}$ implies the following almost everywhere convergence along a sub-sequence: there exists a sequence $(l_i)_i$ of natural numbers such that

$$v_{l_i}(r,x) \to v(r,x) \text{ as } i \to \infty$$

for a.e. $(r,x) \in [0,T]\times\mathbb{R}$. We will abuse our notation again and use l to denote this subsequence.

Also observe that since the stochastic integration part in (2.4.2) is a martingale (see Lemma 3.1.3) we have $\mathbb{E}Y_t(x) = P_tY_0(x)$. By Proposition 3.1.1,

$$\begin{aligned}
I_2^l =&\mathbb{E}_Y\left|\int_0^t e^{-\langle Y_{(T-r)-},v(r)\rangle}\left(b_n\langle Y_{(T-r)-},v_l(r)^\alpha - v(r)^\alpha\rangle - \langle Y^{\alpha\beta}_{(T-r)-},v_l(r)^\alpha - v(r)^\alpha\rangle\right)dr\right| \\
\leq &\int_0^t\int_{\mathbb{R}}|v_l(r,x)^\alpha - v(r,x)^\alpha|\cdot\mathbb{E}_Y\left(b_nY_{(T-r)-}(x)+Y^{\alpha\beta}_{(T-r)-}(x)\right)dr\,dx \\
\leq &C_T\int_0^t\int_{\mathbb{R}}\left[b_nP_{T-r}Y_0(x)+(T-r)^{-\frac{\alpha\beta}{2}}+1\right]|v_l(r,x)^\alpha - v(r,x)^\alpha|\,dr\,dx \\
\leq &C_T(b_n(T-t)^{-\frac{1}{2}}+(T-t)^{-\frac{\alpha\beta}{2}}+1)\int_0^t\int_{\mathbb{R}}|v_l(r,x)^\alpha - v(r,x)^\alpha|\,dr\,dx
\end{aligned} \tag{3.4.10}$$

using the fact that $P_{T-r}Y_0(x)\leq C(T-r)^{-\frac{1}{2}}$ with C being independent of x and l. The right hand side converges to 0 as $l\to\infty$ as $v_l\to v$ in $\mathbf{L}^{\alpha,\mathrm{T}}$. This proves **(ii)** .

□

Next we prove Proposition 3.3.2. For the proof we will need to understand how Y behaves when $Z^{(n)}$ jumps. Since n is fixed in this proof, we drop it to simplify the notations introduced in Section 2.5. We shall write $Z=Z^{(n)}$, $V=V^n$, $S=S^n$, $U=U^n$, $T_l=T_l^n$, $\tau=\tau^n$, $\mathcal{N}=\mathcal{N}^n$, $\hat{\mathcal{N}}=\hat{\mathcal{N}}^n$, $\gamma(s):=\gamma^n(s)$ and $\gamma_l:=\gamma^n(T_l^n)$ for $l\in\mathbb{N}$. Also recall the notation

$$\theta(s,x,\lambda):=\theta_n(s,x,\lambda)=e^{-\langle Y_{T-\gamma^n(s)},Z^{(n)}_{\gamma^n(s)-}\rangle}\left(e^{-\lambda Y_{T-\gamma^n(s)}(x)}-1\right).$$

Proof of Proposition 3.3.2. Fix $t\in[0,T)$ and let

$$\theta_j:=\theta(T_j,U_j,S_j)=e^{-\langle Y_{T-\gamma_j},Z_{\gamma_j}\rangle}-e^{-\langle Y_{T-\gamma_j},Z_{\gamma_j-}\rangle}=e^{-\langle Y_{T-\gamma_j},Z_{\gamma_j-}\rangle}\left(e^{-S_jY_{T-\gamma_j}(U_j)}-1\right).$$

Suppose we show that on the event $\{\gamma_l\leq t<\gamma_{l+1}\}$ we have,

$$\mathbb{E}_Y\left[e^{-\langle Y_{T-t},Z_t\rangle}\right]=\mathbb{E}_Y\left[e^{-\langle Y_T,Z_0\rangle}+\int_0^t\tilde{\mathfrak{I}}(Y_{(T-r)-},Z_{r-})\,dr+\sum_{i=1}^l\theta_i\right], \tag{3.4.11}$$

then we can write

$$\sum_{i=1}^l\theta_i=\int_0^{\tau(t)}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\theta(s,x,\lambda)\mathcal{N}(\,d\lambda,\,dx,\,ds),$$

since for $\gamma_l\leq t<\gamma_{l+1}$ by definition (see (2.5.7) and (3.3.2)) $\tau(t)\in[T_l,T_{l+1})$. Replace the above in (3.4.11) and we obtain (3.3.3). So, to complete the proof of (3.3.3) we need to establish (3.4.11).

We will prove this by induction on $l=0,1,2,\ldots$ and use (3.3.1) repeatedly in the following. Note that (3.4.11) for $t=0$ is trivial. When $0=\gamma_0<t<\gamma_1$, by our convention $l=0$. In this case (3.4.11) is

$$\mathbb{E}_Y\left[e^{-\langle Y_{T-t},Z_t\rangle}\right]=\mathbb{E}_Y\left[e^{-\langle Y_T,Z_0\rangle}+\int_0^t\tilde{\mathfrak{I}}(Y_{(T-r)-},Z_{r-})\,dr\right] \tag{3.4.12}$$

and this follows directly from (3.3.1).

Now assume that (3.4.11) holds on the event $\{\gamma_l \leq s < \gamma_{l+1}\}$. We first show that

$$
\begin{aligned}
&\mathbb{E}_Y\left[e^{-\langle Y_{T-\gamma_{l+1}}, Z_{\gamma_{l+1}}\rangle}\right] \\
=&\mathbb{E}_Y\left[e^{-\langle Y_T, Z_0\rangle} + \int_0^{\gamma_{l+1}} \tilde{\mathfrak{I}}(Y_{(T-r)-}, Z_{r-})\, dr + \sum_{i=1}^{l+1} \theta_i\right].
\end{aligned}
\tag{3.4.13}
$$

By definition of θ_{l+1} and induction hypothesis,

$$
\begin{aligned}
&\mathbb{E}_Y\left[e^{-\langle Y_{T-\gamma_{l+1}}, Z_{\gamma_{l+1}}\rangle}\right] = \mathbb{E}_Y[\theta_{l+1}] + \mathbb{E}_Y\left[e^{-\langle Y_{T-\gamma_{l+1}}, Z_{\gamma_{l+1}-}\rangle}\right] \\
=&\mathbb{E}_Y[\theta_{l+1}] + \lim_{\substack{s\uparrow\gamma_{l+1}\\ \gamma_l\leq s<\gamma_{l+1}}} \mathbb{E}_Y e^{-\langle Y_{T-s}, Z_s\rangle} \\
=&\mathbb{E}_Y[\theta_{l+1}] + \lim_{\substack{s\uparrow\gamma_{l+1}\\ \gamma_l\leq s<\gamma_{l+1}}} \mathbb{E}_Y\left[e^{-\langle Y_T, Z_0\rangle} + \int_0^{s} \tilde{\mathfrak{I}}(Y_{(T-r)-}, Z_{r-})\, dr + \sum_{i=1}^{l} \theta_i\right] \\
=&\mathbb{E}_Y[\theta_{l+1}] + \mathbb{E}_Y\left[e^{-\langle Y_T, Z_0\rangle} + \int_0^{\gamma_{l+1}} \tilde{\mathfrak{I}}(Y_{(T-r)-}, Z_{r-})\, dr + \sum_{i=1}^{l} \theta_i\right] \\
=&\mathbb{E}_Y\left[e^{-\langle Y_T, Z_0\rangle} + \int_0^{\gamma_{l+1}} \tilde{\mathfrak{I}}(Y_{(T-r)-}, Z_{r-})\, dr + \sum_{i=1}^{l+1} \theta_i\right].
\end{aligned}
$$

This proves (3.4.13).

The last step of the induction is to prove (3.4.11) when l is replaced with $l+1$ and $\gamma_{l+1} < t < \gamma_{l+2}$. We use (3.3.1) with $T-\gamma_{l+1}$, $t-\gamma_{l+1}$ instead of T, t and then apply (3.4.13) to get,

$$
\begin{aligned}
&\mathbb{E}_Y\left[e^{-\langle Y_{T-t}, Z_t\rangle}\right] = \mathbb{E}_Y\left[\exp\left(-\langle Y_{T-t}, V_{t-\gamma_{l+1}}(Z_{\gamma_{l+1}})\rangle\right)\right] \\
=&\mathbb{E}_Y\left[\exp\left(-\langle Y_{T-\gamma_{l+1}}, V_0(Z_{\gamma_{l+1}})\rangle\right)\right] + \mathbb{E}_Y\left[\int_0^{t-\gamma_{l+1}} \tilde{\mathfrak{I}}(Y_{(T-\gamma_{l+1}-r)-}, V_r(Z_{\gamma_{l+1}}))\, dr\right] \\
=&\mathbb{E}_Y\left[e^{-\langle Y_T, Z_0\rangle} + \int_0^{\gamma_{l+1}} \tilde{\mathfrak{I}}(Y_{(T-r)-}, Z_{r-})\, dr + \sum_{i=1}^{l+1} \theta_i\right] \\
&+ \mathbb{E}_Y\left[\int_{\gamma_{l+1}}^{t} \tilde{\mathfrak{I}}(Y_{(T-r)-}, Z_{r-})\, dr\right],
\end{aligned}
\tag{3.4.14}
$$

which is the required expression. This completes the induction argument and proves (3.4.11).

□

For the proof of our final proposition, we continue to suppress n and use the notations introduced before the previous proof. Define

$$
M_s = \int_0^s \int_{\mathbb{R}} \int_0^\infty \theta(r, x, \lambda)[\mathcal{N}(d\lambda,\, dx,\, dr) - \hat{\mathcal{N}}(d\lambda,\, dx,\, dr)] \tag{3.4.15}
$$

and note that M is an $\mathcal{F}^{Z^{(n)}}$-martingale.

Proof of Proposition 3.3.3. We recall that

$$\eta = \frac{\alpha\beta(\alpha\beta - 1)}{\Gamma(2-\alpha\beta)} \text{ and } g(r,y) = \int_{0+}^{r} (e^{-\lambda y} - 1 + \lambda y)\lambda^{-\alpha\beta-1}\, d\lambda, \quad r, y \geq 0.$$

Since for all $y \geq 0$,

$$y^{\alpha\beta} = \eta \int_{0+}^{\infty} (e^{-\lambda y} - 1 + \lambda y)\lambda^{-\alpha\beta-1}\, d\lambda = \eta g(1/n, y) + \eta \int_{1/n}^{\infty} (e^{-\lambda y} - 1)\lambda^{-\alpha\beta-1}\, d\lambda + b_n y,$$

we can write

$$\begin{aligned}
&\mathbb{E}_Y\left[\int_0^t \tilde{\mathfrak{I}}(Y_{(T-r)-}, Z_{r-})\, dr\right] = \mathbb{E}_Y\left[\int_0^t e^{-\langle Y_{(T-r)-}, Z_{r-}\rangle}\frac{1}{2}\langle b_n Y_{(T-r)-} - Y^{\alpha\beta}_{(T-r)-}, Z^{\alpha}_{r-}\rangle\, dr\right]\\
=& -\frac{\eta}{2}\mathbb{E}_Y\left[\int_0^t e^{-\langle Y_{(T-r)-}, Z_{r-}\rangle}\langle g(1/n, Y_{(T-r)-}(\cdot)), Z^{\alpha}_{r-}\rangle\, dr\right]\\
&-\frac{\eta}{2}\mathbb{E}_Y\left[\int_0^t e^{-\langle Y_{(T-r)-}, Z_{r-}\rangle}\langle \int_{1/n}^{\infty} (e^{-\lambda Y_{(T-r)-}(\cdot)} - 1)\lambda^{-\alpha\beta-1}\, d\lambda, Z^{\alpha}_{r-}\rangle\, dr\right].
\end{aligned} \tag{3.4.16}$$

Let

$$\begin{aligned}
h(r) =& e^{-\langle Y_{T-r}, Z_{r-}\rangle}\int_{\mathbb{R}} \frac{(Z_{r-}(x))^{\alpha}}{\|Z_{r-}\|^{\alpha}_{\alpha}} \int_{1/n}^{\infty} \left(e^{-\lambda Y_{T-r}(x)} - 1\right)\lambda^{-\alpha\beta-1}\, d\lambda\, dx\\
\beta(r) =& \frac{1}{2}\|Z_{r-}\|^{\alpha}_{\alpha}.
\end{aligned}$$

Then by definition $\gamma(t) = \inf\{s \geq 0 \mid \int_0^s \beta(s)\, ds > t\}$ and also recall from (3.3.2) that $\gamma(\tau(s)) = s$. Given Y, applying [EK86, Exercise 6.12], for any $s \geq 0$, we have

$$\begin{aligned}
&\frac{\eta}{2}\int_0^s e^{-\langle Y_{(T-r)-}, Z_{r-}\rangle}\langle \int_{1/n}^{\infty} (e^{-\lambda Y_{(T-r)-}(\cdot)} - 1)\lambda^{-\alpha\beta-1}\, d\lambda, Z^{\alpha}_{r-}\rangle\, dr\\
=& \eta\int_0^s h(r)\beta(r)\, dr = \eta\int_0^{\gamma(\tau(s))} h(r)\beta(r)\, dr = \eta\int_0^{\tau(s)} h(\gamma(r))\, dr\\
=& \eta\int_0^{\tau(s)} e^{-\langle Y_{T-\gamma(r)}, Z_{\gamma(r)-}\rangle}\int_{\mathbb{R}} \frac{(Z_{\gamma(r)-}(x))^{\alpha}}{\|Z_{\gamma(r)-}\|^{\alpha}_{\alpha}}\int_{1/n}^{\infty}\left(e^{-\lambda Y_{T-\gamma(r)}(x)} - 1\right)\lambda^{-\alpha\beta-1}\, d\lambda\, dx\, dr\\
=& \eta\int_0^{\tau(s)}\int_{\mathbb{R}}\int_0^{\infty}\theta(r,x,\lambda)\frac{(Z_{\gamma(r)-}(x))^{\alpha}}{\|Z_{\gamma(r)-}\|^{\alpha}_{\alpha}}\mathbf{1}(\lambda > 1/n)\lambda^{-\alpha\beta-1} d\lambda\, dx\, dr\\
=& \int_0^{\tau(s)}\int_{\mathbb{R}}\int_0^{\infty}\theta(r,x,\lambda)\hat{\mathcal{N}}(d\lambda,\, dx,\, dr)
\end{aligned} \tag{3.4.17}$$

using Lemma 2.5.2 in the last line.

Combining (3.4.15) and the calculations in (3.4.16), (3.4.17) we get,

$$\begin{aligned}
&\mathbb{E}_Y\left[\int_0^t \tilde{\mathfrak{I}}(Y_{(T-r)-}, Z_{r-})\, dr + \int_0^{\tau(t)}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\theta(s,x,\lambda)\mathcal{N}(d\lambda,\, dx,\, ds)\right]\\
=& \mathbb{E}_Y\left[M_{\tau(t)} - \frac{\eta}{2}\int_0^t e^{-\langle Y_{(T-r)-}, Z_{r-}\rangle}\langle g(1/n, Y_{(T-r)-}(\cdot)), Z^{\alpha}_{r-}\rangle\, dr\right]
\end{aligned} \tag{3.4.18}$$

We can now use (3.4.18) to rewrite (3.3.3).

$$\begin{aligned}&\mathbb{E}_Y\left[e^{-\langle Y_{T-t}, Z_t\rangle}\right]\\&=\mathbb{E}_Y\left[M_{\tau(t)}\right]-\mathbb{E}_Y\left[\frac{\eta}{2}\int_0^t e^{-\langle Y_{(T-r)-}, Z_{r-}\rangle}\langle g(1/n, Y_{(T-r)-}(\cdot)), Z^\alpha_{r-}\rangle\, dr\right]\end{aligned}\tag{3.4.19}$$

Recall the notation $\Upsilon_m(t)=\gamma(m)\wedge t$ and observe that for any $m\in\mathbb{N}$, $\tau(\Upsilon_m(t))=\tau(t)\wedge m$. We localize the above as follows.

$$\begin{aligned}&\mathbb{E}_Y\left[e^{-\langle Y_{T-\Upsilon_m(t)}, Z_{\Upsilon_m(t)}\rangle}\right]\\=&\mathbb{E}_Y\left[e^{-\langle Y_T, Z_0\rangle}\right]+\mathbb{E}_Y\left[M_{\tau(t)\wedge m}\right]\\&-\frac{\eta}{2}\mathbb{E}_Y\left[\int_0^{\Upsilon_m(t)} e^{-\langle Y_{(T-r)-}, Z_{r-}\rangle}\langle g(1/n, Y_{(T-r)-}(\cdot)), Z^\alpha_{r-}\rangle\, dr\right]\end{aligned}\tag{3.4.20}$$

Apply $\mathbb{E}_Z$ to the above. As

$$\mathbb{E}_Z\mathbb{E}_Y(M_{\tau^Z(t)\wedge m})=\mathbb{E}_Y\mathbb{E}_Z(M_{\tau^Z(t)\wedge m})=0$$

we have,

$$\begin{aligned}\mathbb{E}\left[e^{-\langle Y_{T-\Upsilon_m(t)}, Z_{\Upsilon_m(t)}\rangle}\right]=&\mathbb{E}\left[e^{-\langle Y_T, Z_0\rangle}\right]\\&-\frac{\eta}{2}\mathbb{E}\left[\int_0^{\Upsilon_m(t)} e^{-\langle Y_{(T-r)-}, Z_{r-}\rangle}\langle g(1/n, Y_{(T-r)-}(\cdot)), Z^\alpha_{r-}\rangle\, dr\right].\end{aligned}\tag{3.4.21}$$

This is the required expression. □

3.5 Proof of Proposition 3.2.2

Since the proof is a little long, we carry it out in two steps. The first shows that a solution of the weak form (2.2.2) also satisfies a time-dependent version as described in (3.5.1). The proof follows the argument of [Shi94, Theorem 2.1].

Lemma 3.5.1. *Let $T>0$ be fixed and assume that Y satisfies (2.2.2) and the following conditions hold for $\psi:[0,T]\times\mathbb{R}\to[0,\infty)$.*

(i) The map $[0,T]\ni s\mapsto\psi_s\in\mathbf{L}^\eta(\mathbb{R})\cap\mathbf{L}^\rho(\mathbb{R})$ is continuous, for some fixed $\eta\in(\frac{1}{\beta},\alpha)$ and $\rho\in(\alpha,\frac{\alpha}{\beta}\wedge 2)$.

(ii) $\sup_{s\le T}\|\frac{\partial}{\partial s}\psi_s\|_{\frac{\alpha\beta}{\alpha\beta-1}}<\infty$, and

(iii) $s\mapsto\frac{\partial^2}{\partial x^2}\psi_s$ is continuous in $\mathbf{L}^\infty(\mathbb{R})$, i.e. $\|\frac{\partial^2}{\partial x^2}\psi_s-\frac{\partial^2}{\partial x^2}\psi_t\|_\infty\to 0$ as $|s-t|\to 0$.

Then for each $t\in[0,T]$, we have

$$\langle Y_t,\psi_t\rangle=\langle Y_0,\psi_0\rangle+\int_0^t\langle Y_s,\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial s}\right)\psi_s\rangle\, ds+\int_0^t\int_{\mathbb{R}}(Y_{s-}(x))^\beta\psi_s(x)L^\alpha(dx,ds).\tag{3.5.1}$$

Proof. Fix $0 \le t \le T$ and let $\Delta = \{0 = t_0 < t_1 < \cdots < t_N = t\}$ be a partition of $[0, t]$. For all $s \in [t_{i-1}, t_i]$, denote $\pi_\Delta(s) = t_{i-1}$ and $\bar{\pi}_\Delta(s) = t_i$. Then we have

$$
\begin{aligned}
&\langle Y_t, \psi_t\rangle - \langle Y_0, \psi_0\rangle \\
&= \sum_{i=1}^{N} \left(\langle Y_{t_i}, \psi_{t_i} - \psi_{t_{i-1}}\rangle - \langle Y_{t_i} - Y_{t_{i-1}}, \psi_{t_{i-1}}\rangle\right) \\
&= \sum_{i=1}^{N} \left[\int_{t_{i-1}}^{t_i} \langle Y_{\bar{\pi}_\Delta(s)}, \dot{\psi}_s\rangle\, ds + \int_{t_{i-1}}^{t_i} \langle Y_s, \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi_{\pi_\Delta(s)}\rangle\, ds + \int_{t_{i-1}}^{t_i}\int_{\mathbb{R}} Y_{s-}(x)^\beta \psi_{\pi_\Delta(s)}(x) L^\alpha(dx,\, ds)\right] \\
&= \int_0^t \left(\langle Y_{\bar{\pi}_\Delta(s)}, \dot{\psi}_s\rangle + \langle Y_s, \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi_{\pi_\Delta(s)}\rangle\right) ds + \int_0^t\int_{\mathbb{R}} Y_{s-}(x)^\beta \psi_{\pi_\Delta(s)}(x) L^\alpha(dx,\, ds) \qquad (3.5.2)
\end{aligned}
$$

To prove the lemma we have to show that, as $|\Delta| \to 0$

(a) $\int_0^t \langle Y_{\bar{\pi}_\Delta(s)}, \dot{\psi}_s\rangle\, ds \longrightarrow \int_0^t \langle Y_s, \dot{\psi}_s\rangle\, ds$ a.s. ,

(b) $\int_0^t \langle Y_s, \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi_{\pi_\Delta(s)}\rangle\, ds \longrightarrow \int_0^t \langle Y_s, \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi_s\rangle\, ds$ a.s. , and

(c) $\int_0^t \int_{\mathbb{R}} Y_{s-}(x)^\beta \psi_{\pi_\Delta(s)}(x) L^\alpha(\, ds,\, dx) \longrightarrow \int_0^t \int_{\mathbb{R}^d} (Y_{s-}(x))^\beta \psi_s(x) L^\alpha(dx,\, ds)$ in probability.

For (a) and (b), we need to show that the integrand converges pointwise (i.e. for each s) and that the dominated convergence theorem (DCT) can be applied.

(a) Recall that $s \mapsto Y_s$ is right continuous measure-valued a.s. and by the definition of weak convergence we have, for each $s \in [0, t]$

$$|\langle Y_{\bar{\pi}_\Delta(s)} - Y_s, \dot{\psi}_s\rangle| \longrightarrow 0, \text{ a.s.}$$

as $\dot{\psi}_s$ is bounded and continuous (in the space variable).

By Hölder's inequality, as $\alpha\beta > 1$, we have a.s.

$$
\begin{aligned}
|\langle Y_{\bar{\pi}_\Delta(s)} - Y_s, \dot{\psi}_s\rangle| &\le \int_{\mathbb{R}} |Y_s(x)||\dot{\psi}_s(x)|\, dx + \int_{\mathbb{R}} |Y_{\bar{\pi}_\Delta(s)}(x)||\dot{\psi}_s(x)|\, dx \\
&\le \|Y_s\|_{\alpha\beta}\|\dot{\psi}_s\|_{\frac{\alpha\beta}{\alpha\beta-1}} + \|Y_{\bar{\pi}_\Delta(s)}\|_{\alpha\beta}\|\dot{\psi}_s\|_{\frac{\alpha\beta}{\alpha\beta-1}}.
\end{aligned}
$$

Therefore, a.s.

$$
\begin{aligned}
\int_0^t |\langle Y_{\bar{\pi}_\Delta(s)} - Y_s, \dot{\psi}_s\rangle|\, ds &\le (\sup_{s\le t}\|\dot{\psi}_s\|_{\frac{\alpha\beta}{\alpha\beta-1}}) \left[\int_0^t \|Y_s\|_{\alpha\beta}\, ds + \int_0^t \|Y_{\bar{\pi}_\Delta(s)}\|_{\alpha\beta}\, ds\right] \\
&= (\sup_{s\le t}\|\dot{\psi}_s\|_{\frac{\alpha\beta}{\alpha\beta-1}}) \left[\left(\int_0^t \|Y_s\|_{\alpha\beta}\, ds\right)^{\frac{\alpha\beta}{\alpha\beta}} + \left(\int_0^t \|Y_{\bar{\pi}_\Delta(s)}\|_{\alpha\beta}\, ds\right)^{\frac{\alpha\beta}{\alpha\beta}}\right] \\
&\le (\sup_{s\le t}\|\dot{\psi}_s\|_{\frac{\alpha\beta}{\alpha\beta-1}}) \left[\left(t^{\alpha\beta}\frac{1}{t}\int_0^t \|Y_s\|_{\alpha\beta}^{\alpha\beta}\, ds\right)^{\frac{1}{\alpha\beta}} + \left(t^{\alpha\beta}\frac{1}{t}\int_0^t \|Y_{\bar{\pi}_\Delta(s)}\|_{\alpha\beta}^{\alpha\beta}\, ds\right)^{\frac{1}{\alpha\beta}}\right],
\end{aligned}
$$

using Jensen in the last line. The quantity above is finite by assumption (ii) and the fact that $Y \in L_{loc}^{\alpha\beta}(\mathbb{R}_+ \times \mathbb{R})$. This implies that $s \mapsto |\langle Y_{\bar{\pi}_\Delta(s)} - Y_s, \dot{\psi}_s\rangle|$ is a.s. integrable on $[0, t]$.

(b) Fix $s \in [0,t]$. Then

$$|\langle Y_s, \frac{\partial^2}{\partial x^2}(\psi_s - \psi_{\pi_\Delta(s)})\rangle| \le \left(\sup_{x\in\mathbb{R}} |\frac{\partial^2}{\partial x^2}(\psi_s - \psi_{\pi_\Delta(s)})(x)|\right)\langle Y_s, 1\rangle, \text{ a.s.}$$

We know that Y_s is a finite measure, i.e. $\langle Y_s, 1\rangle < \infty$. Thus the RHS above converges to 0 by our assumption (iii).

Let us introduce a new stopping time: for $k \in \mathbb{N}$,

$$\sigma_k = \inf\{s \ge 0 \mid \langle Y_s, 1\rangle > k\}.$$

For $s \le \sigma_k \wedge t$ we have, a.s.

$$\begin{aligned}|\langle Y_s, \frac{\partial^2}{\partial x^2}(\psi_s - \psi_{\pi_\Delta(s)})\rangle| &\le \left(\sup_{x\in\mathbb{R}} |\frac{\partial^2}{\partial x^2}(\psi_s - \psi_{\pi_\Delta(s)})(x)|\right)\langle Y_s, 1\rangle \\ &\le 2k\left(\sup_{\substack{s\le t\\ x\in\mathbb{R}}} |\frac{\partial^2}{\partial x^2}\psi_s(x)|\right) < \infty\end{aligned}$$

by hypothesis. As $\sigma_k \to \infty$ as $k \to \infty$ the above is true for all $s \le t$. Thus we can apply DCT to obtain (b).

(c) Recall the notations introduced in the beginning of Section 2.5. We have

$$L^\alpha(dx, ds) = \int_0^\infty z\tilde{N}(dx, dz, ds) \tag{3.5.3}$$

where $N(dx, dz, ds)$ is a PRM on $\mathbb{R}\times(0,\infty)^2$ with intensity $dx\, m_0(dz)\, ds$.

Note that

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}} Y_{s-}(x)^\beta(\psi_{\pi_\Delta(s)}(x) - \psi_s(x))L^\alpha(dx, ds)\\ &= \int_0^t\int_0^1\int_{\mathbb{R}} Y_{s-}(x)^\beta(\psi_{\pi_\Delta(s)}(x) - \psi_s(x))z\tilde{N}(dx, dz, ds)\\ &\quad+ \int_0^t\int_1^\infty\int_{\mathbb{R}} Y_{s-}(x)^\beta(\psi_{\pi_\Delta(s)}(x) - \psi_s(x))z\tilde{N}(dx, dz, ds).\end{aligned} \tag{3.5.4}$$

Here we note that $\int_0^1 z^\rho m_0(dz) < \infty$ as $\rho \ge \alpha$ and $\int_1^\infty z^\eta m_0(dz) < \infty$ as $\eta < \alpha$. Using the Burkholder-Davis-Gundy inequality for the first term, Fubini's theorem and Proposition 3.1.1

we have

$$
\begin{aligned}
&\mathbb{E}\left(\left|\int_0^t\int_0^1\int_{\mathbb{R}} Y_{s-}(x)^\beta(\psi_{\pi_\Delta(s)}(x)-\psi_s(x))z\tilde{N}(dx,\,dz,\,ds)\right|^\rho\right)\\
\le& C_\rho\mathbb{E}\left(\left|\int_0^t\int_0^1\int_{\mathbb{R}} Y_{s-}(x)^{2\beta}|\psi_{\pi_\Delta(s)}(x)-\psi_s(x)|^2z^2N(dx,\,dz,\,ds)\right|^{\rho/2}\right)\\
\le& C_\rho\mathbb{E}\left(\int_0^t\int_0^1\int_{\mathbb{R}} Y_{s-}(x)^{\rho\beta}|\psi_{\pi_\Delta(s)}(x)-\psi_s(x)|^\rho z^\rho N(dx,\,dz,\,ds)\right)\\
\le& C_\rho\int_0^t\int_{\mathbb{R}}\left|\psi_{\pi_\Delta(s)}(x)-\psi_s(x)\right|^\rho\mathbb{E}(Y_{s-}(x)^{\rho\beta})dx\,ds\\
\le& C_\rho\left(\sup_s\|\psi_{\pi_\Delta(s)}-\psi_s\|_\rho\right)^\rho\left(\int_0^t s^{-\rho\beta/2}\,ds\right)\\
=& C_\rho\left(\sup_s\|\psi_{\pi_\Delta(s)}-\psi_s\|_\rho\right)^\rho t^{1-\rho\beta/2}\longrightarrow 0
\end{aligned}
\tag{3.5.5}
$$

as $|\Delta|\longrightarrow 0$. The second inequality again uses the fact about random sums described in [PZ07, Lemma 8.22] as $\rho/2<1$. The last line follows from our assumption (i) of continuity of the map $s\mapsto\psi_s\in\mathbf{L}^\rho(\mathbb{R})$, which implies uniform continuity of the same on $[0,t]$. For the second term in (3.5.4) we proceed as in the previous calculation. Observe that $1<\eta\beta<\alpha$ by assumption and thus Proposition 3.1.1 is applicable in the following.

$$
\begin{aligned}
&\mathbb{E}\left(\left|\int_0^t\int_1^\infty\int_{\mathbb{R}} Y_{s-}(x)^\beta(\psi_{\pi_\Delta(s)}(x)-\psi_s(x))z\tilde{N}(dx,\,dz,\,ds)\right|^\eta\right)\\
\le&\mathbb{E}\left(\left|\int_0^t\int_1^\infty\int_{\mathbb{R}} Y_{s-}(x)^{2\beta}(\psi_{\pi_\Delta(s)}(x)-\psi_s(x))^2z^2N(dx,\,dz,\,ds)\right|^{\eta/2}\right)\\
\le&\mathbb{E}\int_0^t\int_1^\infty\int_{\mathbb{R}}\left|Y_{s-}(x)^{\eta\beta}(\psi_{\pi_\Delta(s)}(x)-\psi_s(x))^\eta z^\eta\right|N(\,dx,\,dz,\,ds)\\
=&\mathbb{E}\int_0^t\int_1^\infty\int_{\mathbb{R}}|Y_{s-}(x)|^{\eta\beta}\left|\psi_{\pi_\Delta(s)}(x)-\psi_s(x)\right|^\eta z^\eta\,dx\,m_0(dz)\,ds\\
\le& C_\eta\int_0^t\int_{\mathbb{R}}\mathbb{E}\left(|Y_{s-}(x)|^{\eta\beta}\right)\left|\psi_{\pi_\Delta(s)}(x)-\psi_s(x)\right|^\eta\,dx\,ds\\
\le& C_\eta\int_0^t\int_{\mathbb{R}} s^{\eta\beta/2}\left|\psi_{\pi_\Delta(s)}(x)-\psi_s(x)\right|^\eta\,dx\,ds\\
\le& C_\eta\left(\sup_s\|\psi_{\pi_\Delta(s)}-\psi_s\|_\eta\right)^\eta t^{1-\eta\beta/2}.
\end{aligned}
\tag{3.5.6}
$$

By assumption (i) the RHS above converges to 0 as $|\Delta|\longrightarrow 0$. The calculations (3.5.4), (3.5.5) and (3.5.6) together prove (c). □

In the last lemma we show how to turn the time dependent weak form of our SPDE (3.5.1), which was proved in the previous result, into a martingale. This will complete the proof of Proposition 3.2.2.

Lemma 3.5.2. *If* (3.5.1) *holds for some smooth* $\psi : [0,T]\times R \to \mathbb{R}$ *and* $0 \le t \le T$*, then*

$$M_t^Y(\psi) = e^{-\langle Y_t,\psi_t\rangle} - e^{-\langle Y_0,\psi_0\rangle} - \int_0^t e^{-\langle Y_{s-},\psi_s\rangle}\left(-\langle Y_{s-}, \frac{1}{2}\Delta\psi_s + \frac{\partial}{\partial s}\psi_s\rangle + \langle Y_{s-}^{\alpha\beta}, \psi_s^{\alpha}\rangle\right) ds \tag{3.5.7}$$

is an $\mathcal{F}^Y$*-martingale.*

Proof. The proof is an application of Ito's formula. Using the representation (3.5.3) and some algebraic manipulations we have,

$$\begin{aligned}
\langle Y_t, \psi\rangle =& \langle Y_0,\psi_0\rangle + \int_0^t \langle Y_s, \frac{1}{2}\Delta\psi_s + \frac{\partial}{\partial s}\psi_s\rangle\, ds + \int_0^t\int_0^\infty\int_{\mathbb{R}} Y_{s-}(x)^\beta\psi_s(x)z\tilde{N}(\,dx,\,dz,\,ds)\\
=& \langle Y_0,\psi_0\rangle + \int_0^t \langle Y_s, \frac{1}{2}\Delta\psi_s + \frac{\partial}{\partial s}\psi_s\rangle\, ds + \int_0^t\int_0^1\int_{\mathbb{R}} Y_{s-}(x)^\beta\psi_s(x)z\tilde{N}(\,dx,\,dz,\,ds)\\
&+ \int_0^t\int_1^\infty\int_{\mathbb{R}} Y_{s-}(x)^\beta\psi_s(x)zN(\,dx,\,dz,\,ds) - \int_0^t\int_1^\infty\int_{\mathbb{R}} Y_{s-}(x)^\beta\psi_s(x)z\,dx\,m_0(dz)\,ds\\
=& \langle Y_0,\psi_0\rangle + \int_0^t \left[\langle Y_s, \frac{1}{2}\Delta\psi_s + \frac{\partial}{\partial s}\psi_s\rangle - \int_1^\infty z\langle Y_{s-}(\cdot)^\beta, \psi_s\rangle\, m_0(dz)\right] ds\\
&+ \int_0^t\int_0^1\int_{\mathbb{R}} Y_{s-}(x)^\beta\psi_s(x)z\tilde{N}(\,dx,\,dz,\,dt) + \int_0^t\int_1^\infty\int_{\mathbb{R}} Y_{s-}(x)^\beta\psi_s(x)zN(\,dx,\,dz,\,dt)
\end{aligned} \tag{3.5.8}$$

Since $\int_1^\infty z\, m_0(dz) = \frac{\alpha}{\Gamma(2-\alpha)}$ the above can be written formally as

$$\begin{aligned}
d\langle Y_t,\psi\rangle =& \left(\langle Y_t, \frac{\Delta}{2}\psi_t + \frac{\partial}{\partial t}\psi_t\rangle - \frac{\alpha}{\Gamma(2-\alpha)}\langle Y_{t-}^\beta, \psi_t\rangle\right) dt\\
&+ \int_0^1\int_{\mathbb{R}} Y_{t-}(x)^\beta\psi_t(x)z\tilde{N}(\,dt,\,dz,\,dx) + \int_1^\infty\int_{\mathbb{R}} Y_{t-}(x)^\beta\psi_t(x)zN(\,dt,\,dz,\,dx).
\end{aligned} \tag{3.5.9}$$

Ito's formula as given in [App09, Theorem 4.4.7] can now be applied with $f(x) = e^{-x}$ and

$G(t) = \left(\langle Y_t, \frac{\Delta}{2}\psi_t + \frac{\partial}{\partial t}\psi_t\rangle - \frac{\alpha}{\Gamma(2-\alpha)}\langle Y_{t-}^{\beta}, \psi_t\rangle\right)$. We have from (3.5.9),

$$
\begin{aligned}
& e^{-\langle Y_t,\psi_t\rangle} - e^{-\langle Y_0,\psi_0\rangle} = f(\langle Y_t, \psi_t\rangle) - f(\langle Y_0, \psi_0\rangle) \\
&= \int_0^t f'(\langle Y_{s-}, \psi_s\rangle)G(s)\, ds + \int_0^t \int_1^\infty \int_{\mathbb{R}} \left[f(\langle Y_{s-}, \psi_s\rangle + Y_{s-}(x)^\beta \psi_s(x) z) - f(\langle Y_{s-}, \psi_s\rangle)\right] N(\, dx,\, dz,\, ds) \\
&\quad + \int_0^t \int_0^1 \int_{\mathbb{R}} \left[f(\langle Y_{s-}, \psi_s\rangle + Y_{s-}(x)^\beta \psi_s(x) z) - f(\langle Y_{s-}, \psi_s\rangle)\right] \tilde{N}(\, dx,\, dz,\, ds) \\
&\quad + \int_0^t \int_0^1 \int_{\mathbb{R}} \left[f(\langle Y_{s-}, \psi_s\rangle + Y_{s-}(x)^\beta \psi_s(x) z) - f(\langle Y_{s-}, \psi_s\rangle) - Y_{s-}(x)^\beta \psi_s(x) z f'(\langle Y_{s-}, \psi_s\rangle)\right]\, ds\, m_0(dz)\, dx \\
&= -\int_0^t e^{-\langle Y_{s-},\psi_s\rangle} \left(\langle Y_{s-}, \frac{\Delta}{2}\psi_s + \frac{\partial}{\partial s}\psi_s\rangle - \frac{\alpha}{\Gamma(2-\alpha)}\langle Y_{s-}^\beta, \psi_s\rangle\right)\, ds \\
&\quad + \int_0^t \int_1^\infty \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1\right) N(\, dx,\, dz,\, ds) \\
&\quad + \int_0^t \int_0^1 \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1\right) \tilde{N}(\, dx,\, dz,\, ds) \\
&\quad + \int_0^t \int_0^1 \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1 + Y_{s-}(x)^\beta \psi_s(x) z\right)\, dx\, m_0(dz)\, ds \\
&= -\int_0^t e^{-\langle Y_{s-},\psi_s\rangle} \left(\langle Y_{s-}, \frac{\Delta}{2}\psi_s + \frac{\partial}{\partial s}\psi_s\rangle - \frac{\alpha}{\Gamma(2-\alpha)}\langle Y_{s-}^\beta, \psi_s\rangle\right)\, ds \\
&\quad + \Bigg[\int_0^t \int_1^\infty \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1\right) \tilde{N}(\, dx,\, dz,\, ds) \\
&\quad + \int_0^t \int_0^1 \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1\right) \tilde{N}(\, dx,\, dz,\, ds)\Bigg] \\
&\quad + \int_0^t \int_1^\infty \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1\right)\, dx\, m_0(dz)\, ds \\
&\quad + \int_0^t \int_0^1 \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1 + Y_{s-}(x)^\beta \psi_s(x) z\right)\, dx\, m_0(dz)\, ds
\end{aligned}
\tag{3.5.10}
$$

adding and subtracting the term $\int_0^t \int_1^\infty \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1\right)\, ds\, m_0(dz)\, dx$ in the last line. Note that the term in the square bracket above is

$$
M_t(\psi) = \int_0^t \int_0^\infty \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1\right) \tilde{N}(\, dx,\, dz,\, ds)
$$

and it is an $\mathcal{F}^Y$-martingale. We consider the last term in the RHS of (3.5.10). Recall the definition of m_0 from (2.4.1) and the fact that for $y \geq 0$,

$$
\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_{0+}^\infty \left(e^{-\lambda y} - 1 + \lambda y\right) \lambda^{-\alpha-1}\, d\lambda = y^\alpha.
$$

From these we can get,

$$
\begin{aligned}
&\int_0^t \int_0^1 \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1 + Y_{s-}(x)^\beta \psi_s(x) z\right) dx\, m_0(dz)\, ds \\
&= \int_0^t \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left[\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1 + Y_{s-}(x)^\beta \psi_s(x) z\right) z^{-1-\alpha}\, dz\right] dx\, ds \\
&\quad - \int_0^t \int_1^\infty \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1 + Y_{s-}(x)^\beta \psi_s(x) z\right) ds\, m_0(dz)\, ds \\
&= \int_0^t \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} Y_{s-}(x)^{\alpha\beta} \psi_s(x)^\alpha\, dx\, ds \\
&\quad - \int_0^t \int_1^\infty \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1 + Y_{s-}(x)^\beta \psi_s(x) z\right) dx\, m_0(dz)\, ds. \qquad (3.5.11)
\end{aligned}
$$

To finish the proof use the result of (3.5.11) in (3.5.10). By algebraic manipulations we have,

$$
\begin{aligned}
&e^{-\langle Y_t,\psi_t\rangle} - e^{-\langle Y_0,\psi_0\rangle} \\
&= -\int_0^t e^{-\langle Y_{s-},\psi_s\rangle} \left(\langle Y_{s-}, \frac{\Delta}{2}\psi_s + \frac{\partial}{\partial s}\psi_s\rangle - \frac{\alpha}{\Gamma(2-\alpha)}\langle Y_{s-}^\beta, \psi_s\rangle\right) ds + M_t(\psi) \\
&\quad + \int_0^t \int_1^\infty \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1\right) ds\, m_0(dz)\, dx + \int_0^t e^{-\langle Y_{s-},\psi_s\rangle} \langle Y_{s-}^{\alpha\beta}, \psi_s^\alpha\rangle\, ds \\
&\quad - \int_0^t \int_1^\infty \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} \left(e^{-Y_{s-}(x)^\beta \psi_s(x) z} - 1 + Y_{s-}(x)^\beta \psi_s(x) z\right) dx\, m_0(dz)\, ds, \\
&= -\int_0^t e^{-\langle Y_{s-},\psi_s\rangle} \left(\langle Y_{s-}, \frac{\Delta}{2}\psi_s + \frac{\partial}{\partial s}\psi_s\rangle - \frac{\alpha}{\Gamma(2-\alpha)}\langle Y_{s-}^\beta, \psi_s\rangle + \langle Y_{s-}^{\alpha\beta}, \psi_s^\alpha\rangle\right) ds + M_t(\psi) \\
&\quad - \int_0^t \int_1^\infty \int_{\mathbb{R}} e^{-\langle Y_{s-},\psi_s\rangle} Y_{s-}(x)^\beta \psi(x) z\, dx\, m_0(dz)\, ds, \\
&= -\int_0^t e^{-\langle Y_{s-},\psi_s\rangle} \left(\langle Y_{s-}, \frac{\Delta}{2}\psi_s + \frac{\partial}{\partial s}\psi_s\rangle + \langle Y_{s-}^{\alpha\beta}, \psi_s^\alpha\rangle\right) ds + M_t(\psi) \qquad (3.5.12)
\end{aligned}
$$

again using the fact that $\int_1^\infty z\, m_0(dz) = \frac{\alpha}{\Gamma(2-\alpha)}$. □

Chapter 4

Lie algebraic duality for some Markov processes

Two Markov processes X and Y, taking values in state spaces E and F, are said to be *dual* to each other if there exists a function $D : E \times F \longrightarrow \mathbb{R}$ such that for all $x \in E$ and $y \in F$,

$$\mathbb{E}_x D(X_t, y) = \mathbb{E}^y D(x, Y_t) \tag{4.0.1}$$

for each $t \geq 0$. D is called the *duality function*. Here $\mathbb{E}_x$ and $\mathbb{E}^y$ denote the expectations taken with respect to the law $\mathbb{P}_x$ of the processes X starting at $x \in E$ and the law $\mathbb{P}^y$ of Y starting at $y \in F$ respectively.

This notion of stochastic duality between two Markov processes is a powerful tool used to understand many of their properties. These processes arise from different branches of probability theory including statistical physics, population genetics, stochastic partial differential equations etc. However, given an arbitrary Markov process, there does not exist a general technique to obtain its dual and an associated duality function. See the survey article of Jansen and Kurt [JK14] for a more detailed overview of the various types of duality found in the literature.

Our aim is to use the techniques from the theory of Lie algebra to establish duality relations for some infinite dimensional Markov processes. The papers [GKRV09], [CGGR15] of Giardina, Redig and others have successfully used these ideas to obtain dual processes of the finite dimensional Wright-Fisher diffusion, the Brownian momentum process, the symmetric exclusion process etc. Their approach is based on viewing Markov generators as sums and product of other, simpler operators, such that the latter form a basis for a representation of a Lie algebra.

Following their initiative, in the present article we consider two models whose dual processes have previously been obtained by classical methods, viz. the Feller diffusion process in dimension one and the interacting Wright-Fisher diffusion in infinite dimensions.

This chapter is organized as follows. In Section 4.1 we have provided basic details related to duality of Markov processes and the Lie algebraic method mentioned above. Section 4.2 contains our result regarding the Feller diffusion and in Section 4.3 we discuss the interacting Wright-Fisher diffusion. We end the chapter with some open questions in Section 4.4.

4.1 Preliminaries

We start by giving a brief introduction to the classical duality theory of Markov processes and the new algebraic method.

4.1.1 Duality of Markov processes

Let $(\Omega, \mathcal{F})$ be a measurable space and E, F be Polish spaces endowed with their Borel σ-algebra $\mathcal{B}(E)$ and $\mathcal{B}(F)$ respectively. Let $X = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, \{\mathbb{P}_x\}_{x\in E})$ and $Y = (\Omega, \mathcal{F}, (Y_t)_{t\geq 0}, \{\mathbb{P}^y\}_{y\in F})$ be two Markov processes taking values in E and F. See [JK14, p. 61] for this notation and the definition of a Markov process. Recall that, in (4.0.1) we have already defined the meaning of duality between X and Y with respect to a function D.

Let $(P_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ denote the semi-groups of X and Y, i.e. for measurable functions $f : E \longrightarrow \mathbb{R}$, $g : F \longrightarrow \mathbb{R}$ and $t \geq 0$,

$$(P_t f)(x) = \mathbb{E}_x[f(X_t)] \text{ and } (Q_t g)(y) = \mathbb{E}^y[g(Y_t)],$$

whenever the above exist. Assume now that X and Y have infinitesimal generators $\mathcal{L}^X$ and $\mathcal{L}^Y$ with domains $\mathcal{D}(\mathcal{L}^X)$ and $\mathcal{D}(\mathcal{L}^Y)$ respectively.

The following result connects (4.0.1) with a definition of duality expressed through the generators $\mathcal{L}^X$ and $\mathcal{L}^Y$. Its proof can be found in [JK14, Proposition 1.2]. See also [SSV18, Lemma 1].

Proposition 4.1.1. *Let X and Y be as defined above and $D : E \times F \longrightarrow \mathbb{R}$ be a bounded and continuous function. Moreover, assume that for each $x \in E, y \in F$ and $t \geq 0$ we have $D(x,\cdot), P_t D(x,\cdot) \in \mathcal{D}(\mathcal{L}^Y)$ and $D(\cdot,y), P_t D(\cdot,y) \in \mathcal{D}(\mathcal{L}^X)$. Then X and Y are dual to each other with respect to D if and only if,*

$$\mathcal{L}^X D(\cdot,y)(x) = \mathcal{L}^Y D(x,\cdot)(y), \tag{4.1.1}$$

for all $x \in E$ and $y \in F$.

In the following we give three well-known examples of stochastic processes and their duals.

Example 4.1.2. Let $B = (B_t)_{t\geq 0}$ be the Brownian motion in $\mathbb{R}$ starting from some $x > 0$. Let $\tau = \inf\{t \geq 0 \mid B_t = 0\}$ be the first hitting time of the Brownian motion at 0. Then

$$X_t = B_{\tau\wedge t}, \quad t \geq 0$$

is called the *absorbing Brownian motion* with absorption at the origin. Also let us define the process,

$$Y_t = |B_t|, \quad t \geq 0.$$

This is known as the *reflected Brownian motion.* See [Bre92, Section 16.3] for detailed discussions regarding these processes. It is well-known that (cf. [Lig05, Section II.3]) X and Y are dual to each other with the following duality relation:

$$\mathbb{P}_x(X_t \leq y) = \mathbb{P}^y(Y_t \leq x) \text{ for all } x > 0, y > 0, t \geq 0.$$

Example 4.1.3. The one-dimensional Wright-Fisher diffusion is defined to be the solutions of the following SDE:

$$dX_t = \sqrt{X_t(1 - X_t)}\, dB_t$$

where B_t is a standard Brownian motion on $\mathbb{R}$. Clearly, the infinitesimal generator of X is given by

$$\mathcal{L}^X f(x) = \frac{1}{2} x(1 - x) \frac{\partial^2 f}{\partial x^2}, \quad x \in [0, 1] \tag{4.1.2}$$

where $f \in \mathcal{C}_b^2(\mathbb{R})$, the space of all real-valued twice continuously differentiable function on $\mathbb{R}$ with bounded derivatives. Then X is dual to the process defined by the following generator,

$$\mathcal{L}^Y g(n) = \binom{n}{2} (g(n - 1) - g(n)), \quad n \in \mathbb{N}, \tag{4.1.3}$$

defined for any function $g : \mathbb{N} \to \mathbb{R}$. The corresponding duality function is $D(x, n) = x^n$. This generator represents the process known as *Kingman's coalescent* (see [Eth11, Section 2.1]).

Example 4.1.4. Our final example comes from the theory of superprocesses. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space and M_F be the collection of non-negative finite measures on $\mathbb{R}$. For a function φ on $\mathbb{R}$ and $\mu \in M_F$, recall that $\langle \varphi, \mu \rangle = \int \varphi(x)\, \mu(dx)$.

We say that an a.s. continuous M_F-valued process $X \equiv (X_t)_{t \geq 0}$, starting from some non-random $X_0 \in M_F$, is a *super-Brownian motion* if

$$M_t(\varphi) = \langle \varphi, X_t \rangle - \langle \varphi, X_0 \rangle - \int_0^t \langle \frac{1}{2} \Delta \varphi, X_t \rangle\, ds$$

is a $\mathcal{F}_t$–local martingale with $\langle M(\varphi) \rangle_t = \int_0^t X_s(\varphi^2)\, ds$, for all $\varphi \in \mathcal{C}_b^2(\mathbb{R})_+$. Here Δ is the one-dimensional Laplacian and $\langle M(\varphi) \rangle_.$ denotes the quadratic variation of $M(\varphi)$.

Now suppose $V \equiv V(\varphi)_s(x)$ is the unique non-negative solution of the PDE:

$$\frac{\partial V_t(x)}{\partial t} = \frac{1}{2} \Delta V_t(x) - \frac{1}{2} V_t^2, \quad V_0 = \varphi,$$

for $t \geq 0, x \in \mathbb{R}$ and some $\varphi \in \mathcal{C}_b^2(\mathbb{R})_+$. Then we know from [Per02, Eq. (II.5.7), p. 168] that X and V are dual to each other with respect to the exponential function. More precisely,

$$\mathbb{E}_{\delta_{X_0}} \exp\left(-\langle \varphi, X_t \rangle\right) = \exp\left(-\langle X_0, V(\varphi)_t \rangle\right),$$

for all $\varphi \in \mathcal{C}_b^2(\mathbb{R})_+$.

4.1.2 Some Lie algebra preliminaries

Lie algebras and their representations are well studied and have vast literature. Here we give a very brief exposition of the basics and direct the interested reader to the survey article of Sturm, Swart and Völlering [SSV18] for a more detailed treatment of these topics. Let us first define a Lie algebra and their homomorphism.

Definition 4.1.5. A (complex) Lie algebra $\mathfrak{g}$ is a finite dimensional vector space over $\mathbb{C}$ endowed with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the following relations. For all $x, y, z \in \mathfrak{g}$,

(i) $[x, y] = -[y, x]$ (skew-symmetry) and,

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity).

The map $[\cdot,\cdot]$ is called a *Lie bracket*.

From every Lie algebra $\mathfrak{g}$ we can obtain its *conjugate Lie algebra* $\bar{\mathfrak{g}} := \{\bar{x} \mid x \in \mathfrak{g}\}$ defined via a conjugate linear bijection $x \mapsto \bar{x}$ from $\mathfrak{g}$ onto $\bar{\mathfrak{g}}$ such that $[\bar{x}, \bar{y}] = [y, x]$ holds for all $x, y \in \mathfrak{g}$.

Definition 4.1.6. Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, a *Lie algebra homomorphism* is a linear map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ that preserves the Lie algebra structure, i.e. for all $x, y \in \mathfrak{g}$,

$$[\varphi(x), \varphi(y)] = \varphi([x, y])$$

where the Lie brackets in the l.h.s and the r.h.s. are associated with $\mathfrak{h}$ and $\mathfrak{g}$ respectively.

A conjugate linear map $\mathfrak{g} \longrightarrow \mathfrak{g}$, $x \mapsto x^*$ will be called an *adjoint* if $(x^*)^* = x$ and $[x^*, y^*] = [y, x]^*$ for every $x, y \in \mathfrak{g}$. When $\mathfrak{g}$ has a basis $\{x_1, x_2, \ldots, x_n\}$, the Lie bracket on $\mathfrak{g}$ is completely determined by the so-called *commutation relations*,

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ijk} x_k, \quad i < j. \tag{4.1.4}$$

Note that when V is a finite dimensional complex vector space, the space of all linear maps $L(V, V)$ from V to itself is a Lie algebra with the Lie bracket given by

$$[A, B] = AB - BA, \quad A, B \in L(V, V),$$

where AB denotes the composition of linear maps. We are now ready to define representations of a Lie algebra $\mathfrak{g}$.

Definition 4.1.7. A representation of a complex Lie algebra $\mathfrak{g}$ is (Lie algebra) homomorphism $\pi : \mathfrak{g} \longrightarrow L(V, V)$ where V is a complex linear space.

Let V be a complex vector space with $\dim(V) \geq 1$ and suppose $X_1, \ldots, X_n \in L(V, V)$ satisfy commutation relations

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k, \quad i < j.$$

Then the map defined by $x_i \mapsto X_i$, $i \in \mathbb{N}$, defines a representation of $\mathfrak{g}$ by virtue of (4.1.4).

As an example of a Lie algebra, we define the *Heisenberg algebra*, denoted by $\mathfrak{h}$. This is a three dimensional complex Lie algebra with basis $\{a^0, a^-, a^+\}$ and the following commutation relations (see (4.1.4)),

$$[a^-, a^+] = a^0, \quad [a^-, a^0] = 0, \quad [a^+, a^0] = 0. \tag{4.1.5}$$

One can check easily that the operators on $L^2(\mathbb{R})$,

$$A^+ f(x) = x f(x), \quad A^- f(x) = \frac{\partial}{\partial x} f(X) \text{ and } A^0 f(x) = f(x) \tag{4.1.6}$$

satisfy the commutation relations $[A^-, A^+] = A^0$, $[A^\pm, A^0] = 0$. This is known as the Schrödinger representation of $\mathfrak{h}$.

4.1.3 Algebraic duality

For the purpose of studying duality between two processes, the notion of an intertwiner or homomorphism between two representations of a Lie algebra is crucial.

Definition 4.1.8. Let V and W be two vector spaces and $\pi_V : \mathfrak{g} \to L(V,V)$, $\pi_W : \mathfrak{g} \to L(W,W)$ be two representations of $\mathfrak{g}$. A linear map $\Phi : W \longrightarrow V$ is called an *intertwiner* between these two representations if

$$\pi_V(x) \circ \Phi = \Phi \circ \pi_W(x) \tag{4.1.7}$$

for all $x \in \mathfrak{g}$. When Φ is an isomorphism, we say that the two representations are *equivalent*.

Suppose (Ω, μ), $(\hat{\Omega}, \hat{\mu})$ are measure spaces and V, W are linear subspaces of $L^2(\Omega, \mu)$ and $L^2(\hat{\Omega}, \hat{\mu})$ respectively. V and W inherit the natural (real) L^2-inner products $\langle \cdot, \cdot \rangle_\mu$ and $\langle \cdot, \cdot \rangle_{\hat{\mu}}$ from their parent spaces. Let $\mathfrak{g}$ be a Lie algebra having basis $\{b_k \mid k \in I\}$ (I being a countable – finite or infinite – index set). Also let $T_k, k \in I$ and $S_k, k \in I$ be operators on V and W such that the correspondences $b_k \mapsto T_k$ and $\bar{b}_k \mapsto S_k$ define representations of $\mathfrak{g}$ and its conjugate Lie algebra $\bar{\mathfrak{g}}$. Note that the operators S_k^* (adjoints of S_k with respect to the inner product $\langle \cdot, \cdot \rangle_{\hat{\mu}}$ on W) define a representation of $\mathfrak{g}$.

The idea behind algebraic duality is this: if we can express the generator L of a Markov process using the basis $\{T_k\}_{k \in I}$ of a representation of a Lie algebra $\mathfrak{g}$, roughly speaking, a dual of the Markov process can be obtained by replacing $\{T_k\}_{k \in I}$ by those coming from the representation $\{S_k^*\}_{k \in I}$ of $\mathfrak{g}$, provided that these two representations are related in a nice manner. The next result makes this idea precise. This is the infinite dimensional version of [SSV18, Proposition 10] and can be proved similarly. This will be used for calculating the duality function in the next section.

Proposition 4.1.9 (Intertwiners and duality functions)**.** *Let $\Phi : W \longrightarrow V$ be a linear map having the form*

$$\Phi g(x) = \int_{\hat{\Omega}} g(y) D(x,y) \hat{\mu}(\,dy) \tag{4.1.8}$$

for some measurable function $D : \Omega \times \hat{\Omega} \longrightarrow \mathbb{R}$. Then the following are equivalent:

(a) Φ is an intertwiner between the representations $\{T_k \mid k \in I\}$ and $\{S_k^* \mid k \in I\}$ of $\mathfrak{g}$; i.e. for all $k \in I$,

$$T_k \Phi = \Phi S_k^*. \tag{4.1.9}$$

(b) For all $x \in \Omega$, $y \in \hat{\Omega}$ and $k \in I$,

$$T_k D(\cdot, y)(x) = S_k D(x, \cdot)(y). \tag{4.1.10}$$

Proof. Assume statement (a) and fix $k \in I$. Note that, to prove (4.1.10), it is enough to show that for all $f \in V$ and $g \in W$, we have

$$\int_\Omega \mu(dx)\, f(x) \int_{\hat{\Omega}} \hat{\mu}(dy)\, g(y)\, S_k D(x, \cdot)(y) = \int_{\hat{\Omega}} \hat{\mu}(dy)\, g(y) \int_\Omega \mu(dx)\, f(x)\, T_k D(\cdot, y)(x). \tag{4.1.11}$$

Fix $f \in V, g \in W$. Then, by definition of Φ from (4.1.8) and Fubini's theorem,

$$\begin{aligned}
\langle f, T_k \Phi g\rangle_\mu =& \langle T_k^* f, \Phi g\rangle_\mu \\
=& \int_\Omega \mu(dx)\, T_k^* f(x)(\Phi g)(x) \\
=& \int_{\hat\Omega} \hat\mu(dy)\, g(y) \int_\Omega \mu(dx)\, T_k^* f(x) D(x,y) \\
=& \int_{\hat\Omega} \hat\mu(dy)\, g(y)\, \langle T_k^* f, D(\cdot,y)\rangle_\mu \\
=& \int_{\hat\Omega} \hat\mu(dy)\, g(y)\, \langle f, T_k D(\cdot,y)\rangle_\mu \\
=& \int_{\hat\Omega} \hat\mu(dy)\, g(y) \int_\Omega \mu(dx)\, f(x)\, T_k D(\cdot,y)(x).
\end{aligned}$$

This is the r.h.s. of (4.1.11). Similarly one can show that

$$\langle f, \Phi S_k^* g\rangle_\mu = \int_\Omega \mu(dx)\, f(x) \int_{\hat\Omega} \hat\mu(dy)\, g(y)\, S_k D(x,\cdot)(y),$$

which is the l.h.s. of (4.1.11). Since $\langle f, T_k\Phi g\rangle_\mu = \langle f, \Phi S_k^* g\rangle_\mu$ by our assumption, (4.1.11) holds and we are done. The above computations also show that the converse is trivial. □

We show how to treat duality with this method in the simple instance of the Wright-Fisher diffusion which was already introduced in Example 4.1.3.

Example 4.1.10. Recall the Schrödinger representations (4.1.6) of the three dimensional Heisenberg algebra $\mathfrak{h}$. With these we can write down the generator $\mathcal{L}^X$ of the Wright-Fisher diffusion defined in (4.1.2) as

$$\mathcal{L}^X = \frac{1}{2}[A^+ - (A^+)^2](A^-)^2. \tag{4.1.12}$$

Now consider the operators

$$B^- g(n) = n g(n-1), \quad B^+ g(n) = g(n+1) \text{ and } B^0 g(n) = g(n), \quad x \in \mathbb{N},$$

where $g : \mathbb{N} \to \mathbb{R}$ is any function. One can check easily that $[B^-, B^+] = -B^0$. This therefore gives a representation of the conjugate Heisenberg algebra $\bar{\mathfrak{h}}$. Thus their adjoints, $B^* := \{(B^-)^*, (B^+)^*, (B^0)^*\}$ is a representation of $\mathfrak{h}$. Assume that the intertwiner Φ between the representations A and B^* has the integral form given in (4.1.8). Then we can apply Proposition 4.1.9.

We have the intertwiner relations $A^-\Phi = \Phi(B^-)^*$ and $A^+\Phi = \Phi(B^+)^*$. Now composing $\mathcal{L}^X$ with Φ from the right (4.1.12) gives us,

$$\begin{aligned}
\mathcal{L}^X \Phi =& \frac{1}{2}\left([A^+ - (A^+)^2](A^-)^2\right)\Phi \\
=& \frac{1}{2}\Phi\left([(B^+)^* - ((B^+)^*)^2]((B^-)^*)^2\right) \\
=& \frac{1}{2}\Phi\left((B^-)^2[B^+ - (B^+)^2]\right)^*,
\end{aligned}$$

using the fact that $T^*S^* = (ST)^*$ for two operators S and T. The above definitions of $B^{\pm}, B^0$ and some computations will show that

$$\frac{1}{2}\left((B^-)^2[B^+ - (B^+)^2]\right) g(n) = \binom{n}{2}(g(n-1) - g(n)),$$

which equals the generator of the Kingman's coalescent $\mathcal{L}^Y$ defined in (4.1.3).

In Section 4.3 we are going to consider an infinite dimensional generalization of this model.

4.2 The one-dimensional Feller diffusion

The Feller diffusion is a real-valued process defined by the stochastic differential equation (SDE)

$$X_t = x + \int_0^t \sqrt{X_s}\, dB_s, \qquad t \geq 0, \tag{4.2.1}$$

where $x > 0$ and B is the Brownian Motion in $\mathbb{R}$. It arises naturally as the weak limit of the rescaled critical Galton-Watson branching process (cf. [EK86, Theorem 9.1.3]). The existence and uniqueness of solution to this SDE follows from Theorem 2.3 and Theorem 3.2 of [IW89, Chapter IV]

Theorem 4.2.1 (Theorem 3.1 of [MM22]). *Suppose $y > 0$ and let Y be the solution $Y : [0, \infty] \to \mathbb{R}_+$ of the ordinary differential equation,*

$$Y_t = y - 2\int_0^t Y_s^2\, ds, \qquad t \geq 0. \tag{4.2.2}$$

Then X, defined according to (4.2.1), and Y are dual to each other with respect to the duality function $D(x, y) := \exp(-2xy)$ defined on $\mathbb{R}_+^2$.

By an application of Proposition 4.1.1, the above claim can be checked easily once we note that

$$\mathcal{L}^X f(x) = \frac{1}{2}x\frac{\partial^2 f}{\partial x^2} \text{ and } \mathcal{L}^Y g(y) = -2y^2\frac{\partial g}{\partial y^2} \tag{4.2.3}$$

(defined for all $f, g \in C_c^2(\mathbb{R})$) are the generators of X and Y respectively. We give a proof of this fact using only the algebraic method discussed in the last section.

Proof of Theorem 4.2.1. The three dimensional Heisenberg algebra $\mathfrak{h}$ has the following (rescaled) Schrödinger representation given by the operators

$$A^- f(x) := -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x}f(x), \quad A^+ f(x) := \sqrt{2}xf(x), \quad A^0 f(x) := -f(x),$$

defined on a suitable subspace of $L^2(\mathbb{R})$. In terms of this representation of $\mathfrak{h}$ we can write $\mathcal{L}^X$ as

$$L := \mathcal{L}^X = \frac{1}{\sqrt{2}}A^+(A^-)^2. \tag{4.2.4}$$

Now define

$$B^- := A^+, \quad B^+ := A^-, \quad B^0 := A^0. \tag{4.2.5}$$

Clearly $[B^-, B^+] = -B^0$ and $[B^\pm, B^0] = 0$. Therefore $\{B^+, B^-, B^0\}$ defines a representation of the conjugate Heisenberg algebra $\bar{\mathfrak{h}}$. Thus $\{(B^+)^*(B^-)^*, (B^0)^*\}$ defines a representation of the Heisenberg algebra $\mathfrak{h}$. It can be seen by the Stone-von Neumann's theorem (see [SSV18, Section 2.5]) that A and B^* define equivalent representations of $\mathfrak{h}$. In other words, there exists a bijective intertwiner Φ such that

$$A^+\Phi = \Phi(B^+)^*, A^-\Phi = \Phi(B^-)^*, A^0\Phi = \Phi(B^0)^*. \tag{4.2.6}$$

First, let us derive the generator $\hat{L}$ of the dual Markov process. Applying Φ to L (as defined in (4.2.4)) from the right we get,

$$L\Phi = \frac{1}{\sqrt{2}}[A^+(A^-)^2]\Phi = \Phi(\frac{1}{\sqrt{2}}(B^-)^2B^+)^*. \tag{4.2.7}$$

Using the relations (4.1.6) and (4.2.5) we can explicitly write down the (differential) form of $\hat{L} := \frac{1}{\sqrt{2}}(B^-)^2B^+$. We note that this matches with the generator $\mathcal{L}^Y$ given in (4.2.3).

To obtain the duality function, let us assume that Φ has an integral kernel denoted by $D : \mathbb{R}_+^2 \to \mathbb{R}_+$, i.e. it has the form given in (4.1.8), with $\hat{\mu}$ being the ordinary Lebesgue measure. By Proposition 4.1.9 we can thus write

$$A^\pm D(., y)(x) = B^\pm D(x, .)(y). \tag{4.2.8}$$

By (4.1.6) and (4.2.5), the above is a system of ODEs from which we can compute D. Solving these we find that, up to a multiplicative constant, the duality function is given by

$$D(x, y) = e^{-2xy}, \quad x, y > 0.$$

This completes our proof. □

4.3 Interacting Wright-Fisher diffusion on infinite sites

Let Λ be a countable set. Let $\mathcal{X} = (\mathcal{X}_\cdot(i))_{i\in\Lambda}$ be defined by the following systems of SDEs:

$$d\mathcal{X}_t(i) = \sum_{j\in\Lambda} q_{ij}(\mathcal{X}_t(j) - \mathcal{X}_t(i))\,dt + \sqrt{2\mathcal{X}_t(i)(1 - \mathcal{X}_t(i))}\,dB_t(i), \tag{4.3.1}$$

for $t \geq 0, i \in \Lambda$, where $\{B(i) \mid i \in \Lambda\}$ are independent standard Brownian motions and $q_{ij} \geq 0$ $(i, j \in \Lambda, i \neq j)$ denote the transition rates of a continuous time Markov process on Λ. q_{ij}'s are assumed to satisfy the following properties.

(i) $q_{ii} = 0$ for all $i \in \Lambda$.

(ii) $\sup_{i\in\Lambda} \sum_{j\in\Lambda} q_{ij} < \infty$. (Summability)

(iii) $Q = (q_{ij})_{i,j\in\Lambda}$ is irreducible in the following sense. For every non-empty $\Delta \subset \Lambda$ there exist $i \in \Delta$ and $j \in \Lambda \setminus \Delta$ such that $q_{ij} > 0$ or $q_{ji} > 0$.

(iv) $\sum_{j\in\Lambda} q_{ji} = \sum_{j\in\Lambda} q_{ij}$ for all $i \in \Lambda$. (Weak-symmetry)

$\mathcal{X}$ describes a system of linearly interacting Wright-Fisher diffusions on Λ with resampling. In the papers [AS05], [AS12] of Athreya and Swart, it was shown that the process $\mathcal{X}$ is a dual to the system X of jumping and coalescing particle given by the generator

$$G\phi(n) = \sum_{i,j\in\Lambda} q_{ij} n_i(\phi(n+\delta_j-\delta_i)-\phi(n)) + \sum_{i\in\Lambda} n_i(n_i-1)(\phi(n-\delta_i)-\phi(n)), \tag{4.3.2}$$

where $n = (n_i)_i \in \mathbb{N}^\Lambda$. The above is defined for every $\phi : \mathbb{N}^\Lambda \to \mathbb{R}$ [1] for which the sums exist finitely. In this section we prove this only with the help of Proposition 4.1.9.

First, we set up some notations. Let Λ_k ($k \geq 1$) be finite subsets of Λ such that $\Lambda_k \subseteq \Lambda_{k+1}$ and $\cup_{k\in\mathbb{N}}\Lambda_k = \Lambda$. We call a function $f : [0,1]^\Lambda \to \mathbb{R}$ *cylindrical* when it is determined by finitely many co-ordinates. In other words, there exists a $k \geq 1$ and a function $\tilde{f} : [0,1]^{\Lambda_k} \to \mathbb{R}$ such that $f((x_i)_{i\in\Lambda}) = \tilde{f}((x_i)_{i\in\Lambda_k})$ for all $x = (x_i)_{i\in\Lambda} \in [0,1]^\Lambda$. Define the space

$$C^2_{cyl} = C^2_{cyl}([0,1]^\Lambda) := \left\{ f : [0,1]^\Lambda \to \mathbb{R} | f \text{ is cylindrical and } \tilde{f} \text{ is twice differentiable} \right\}. \tag{4.3.3}$$

On $C^2_{cyl}([0,1]^\Lambda)$ we define the following inner product: if $f, g \in C^2_{cyl}$ and $k \geq 1$ is the smallest integer such that $\tilde{f}$ and $\tilde{g}$ can both be defined on Λ_k, then

$$\langle f, g\rangle_{C^2_{cyl}} := \int_{[0,1]^\Lambda} \tilde{f} \cdot \tilde{g}\, d\lambda^{\Lambda_k}$$

where $\lambda^{\Lambda_k} = \otimes_{i\in\Lambda_k}\lambda^i$ is the product of one-dimensional Lebesgue measures on $[0,1]$. The integral above is always finite by the regularity conditions of $\tilde{f}$ and $\tilde{g}$ and the fact that $[0,1]^{\Lambda_k}$ is compact. The bilinearity is obvious.

Also, let us define

$$N(\Lambda) := \{n = (n_i)_{i\in\Lambda} \in \mathbb{N}^\Lambda | n_i = 0 \text{ for all but finitely many } i \in \Lambda\}.$$

For every $\phi : N(\Lambda) \to \mathbb{R}$, define the support $\operatorname{supp}\phi$ to be the collection of points $n \in N(\Lambda)$ such that $\phi(n) \neq 0$. Let $S_{fin}(N(\Lambda))$ be the set of all finitely supported functions on $N(\Lambda)$.

We can define an inner product on $S_{fin}(N(\Lambda))$ as follows: for $\phi, \psi \in S_{fin}(N(\Lambda))$,

$$\langle \phi, \psi\rangle_{S_{fin}(N(\Lambda))} := \sum_{n\in N(\Lambda)} \phi(n)\psi(n). \tag{4.3.4}$$

By definition of $S_{fin}(N(\Lambda))$, the above sum is always finite.[2] We also note that it is enough to define the generator G in (4.3.2) for functions in $S_{fin}(N(\Lambda))$.

Theorem 4.3.1 (Proposition 1.1 of [AS12], Theorem 4.1 of [MM22]). *The process $\mathcal{X}$ defined above is a dual to X with respect to the duality function*

$$D(x,n) = x^n := \Pi_{i\in\Lambda} x(i)^{n(i)},$$

where $x = (x(i))_{i\in\Lambda} \in [0,1]^\Lambda$ and $n = (n(i))_{i\in\Lambda} \in \mathbb{N}^\Lambda$.

[1] For us $\mathbb{N} = \{0,1,2,\ldots\}$

[2] The sum will be finite whenever the sequences $(g_1(n))_{n\in N(\Lambda)}$ and $(g_2(n))_{n\in N(\Lambda)}$ are in $l^2(N(\Lambda))$.

Remark 4.3.2. The dual process X of $\mathcal{X}$ matches with the more general dual stated in *Proposition 1.1* of [1]. But the duality function obtained by here is slightly different from the one found in [AS12], which is $(x, n) \to (1-x)^n$

Proof of Theorem 3.1. For our purpose, we will use representations of the Heisenberg algebra having basis indexed by Λ,

$$\mathfrak{h}(\Lambda) = \operatorname{span}\{a^0, a_i^\pm \mid i \in \Lambda\}.$$

where the following commutation relations hold:

$$[a_i^-, a_j^+] = \delta_{ij} a^0, \quad [a_i^\pm, a^0] = 0.$$

Also throughout the proof we will let $V = C^2_{cyl}([0,1]^\Lambda)$ and $W = S_{fin}(N(\Lambda))$.

$\mathfrak{h}(\Lambda)$ has the Schrödinger representation given by

$$A_i^+ f(x) = x_i f(x), \quad A_i^- f(x) = \frac{\partial f}{\partial x_i}(x), \quad A^0 f(x) = f(x),$$

where $i \in \Lambda$, $x \in (0,1)^\Lambda$ and $f \in C^2_{cyl}([0,1]^\Lambda)$.

Also, for $\phi \in S_{fin}(N(\Lambda))$, we define the following operators

$$B_i^+ \phi(n) = \phi(n + \delta_i), B_i^- \phi(n) = \mathbf{1}_{(n_i \geq 1)} n_i \phi(n - \delta_i), B^0 \phi(n) = \phi(n),$$

where $n \in N(\Lambda)$. Note that these are the basis of a representation of $\overline{\mathfrak{h}(\Lambda)}$, i.e. $[B_i^-, B_j^+] = -\delta_{ij} B^0$ for $i, j \in \Lambda$.

Denote by $(B_i^\pm)^*, (B^0)^*$ the adjoints of $B_i^\pm, B^0$ with respect to the inner product $\langle \cdot, \cdot \rangle_{S_{fin}}$ defined above. It can be seen easily that

$$(B_i^+)^* \phi(n) = \mathbf{1}_{(n_i \geq 1)} \phi(n - \delta_i), (B_i^-)^* \phi(n) = (n_i + 1) \phi(n + \delta_i), (B^0)^* = B^0$$

and that these give a representation of $\mathfrak{h}(\mathbf{\Lambda})$. We first explicitly give a bijective intertwiner showing that this representation is actually equivalent to the one given by $\{A_i^\pm, A_i^0 \mid i \in \Lambda\}$.

Define $\Phi : S_{fin} \to C^2_{cyl}$ by

$$(\Phi\phi)(x) = \sum_{n \in N(\Lambda)} \phi(n) x^n$$

where $\phi \in S_{fin}, x = (x_i)_{i \in \Lambda} \in [0,1]^\Lambda$ and $x^n = \prod_{i \in \Lambda} x_i^{n_i}$ if $n = (n_i)_\Lambda$. Note that the function $\Phi\phi$ is cylindrical: only finitely many Λ-coordinates appear in each term as $n_i = 0$ for all but finitely many i and there are finite such terms in the sum as ϕ has finite support.

It is also clear that Φ is injective and thus to prove that it is the required equivalence between the two representations one only has to show that Φ is linear and Φ is homomorphism between the representations, i.e.

$$\Phi (B_i^\pm)^* = (A_i^\pm) \Phi \tag{4.3.5}$$

on S_{fin}. These are easy to check.

To obtain a dual of $\mathcal{G}$ we only need to write it in terms of A_i's and apply Φ from right. We have

$$\mathcal{G} = \sum_{i,j\in\Lambda} q_{ij}(A_j^+ - A_i^+)A_i^- + \sum_{i\in\Lambda} \left[A_i^+ - (A_i^+)^2\right](A_i^-)^2 \tag{4.3.6}$$

and (4.3.5) gives

$$\mathcal{G}\Phi = \Phi\left(\sum_{i,j\in\Lambda} q_{ij}B_i^-(B_j^+ - B_i^+) + \sum_{i\in\Lambda}(B_i^-)^2 B_i^+(1-B_i^+)\right)^* . \tag{4.3.7}$$

If we call G the expression inside the round bracket and write down the definitions of the operators $B_i^\pm$, we end up with the (4.3.2).

The only remaining part is to give the duality function D such that $\mathcal{G}D = GD$ happens. For this we use proposition 4.1.9. By this proposition, because of the relations (4.3.5), we have the following

$$A_i^\pm D(x,n) = B_i^\mp D(x,n) \tag{4.3.8}$$

for all $x \in [0,1]^\Lambda, n \in N(\Lambda)$, where $D(x,n) := x^n$ is the integral kernel of Φ. This directly shows that $\mathcal{G}D = GD$ and thus $D(x,n) = x^n$ is the required duality function. □

4.4 Conclusion and some open questions

As we have seen in the previous sections, using algebraic method one can usually obtain the dual process when the duality function is already known. Also, given a stochastic process with a generator $\mathcal{L}$, choosing the correct Lie algebra $\mathfrak{g}$ to express it in terms of its representation, remains an *ad hoc* procedure. Moreover, one does not know *a priori* that the dual $\hat{\mathcal{L}}$, obtained using Proposition 4.1.9 or [SSV18, Proposition 9], will be the generator of a Markov process. Owing to these issues, we have faced some difficulties when trying to apply algebraic techniques to stochastic processes whose duals have been well-studied. We list two such problems that are currently outstanding.

(i) [AS12] considers a more general system of diffusion processes $\mathcal{X}$ than the one we studied in Section 4.3. This system includes selection and mutation with rates s and m respectively. However, when trying to apply the Lie algebraic framework of using the Schrödinger representation of the Heisenberg algebra, we end up with an error term in the required duality relation $\mathcal{G}D = GD$.

(ii) The Dawson-Watanabe superprocess is the high density weak limit of the critical branching Brownian motions. They can be seen as a spatial generalization of the Feller diffusion that we considered in Section 4.2. See Example 4.1.4 for the dual relation of the super-Brownian motion, a particular instance of duality involving superprocess. The ideas of algebraic duality remain to be applied here.

Chapter 5

Rough paths and integration with respect to Poisson random measures

In the final chapter we shall aim to discuss the pathwise interpretation to the theory of stochastic integration with respect to Poisson random measures (PRMs). We will present some initial results and observations from an ongoing work. We begin with a description of the problem.

Let $T > 0$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space. Let $Y : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a predictable function (cf. Section 1.2) and N be a Poisson random measure on $[0, \infty) \times \mathbb{R}^d$ with intensity $dt \times \mu(dx)$ as in Definition 1.1.4. μ is assumed to be a Levy measure on $\mathbb{R}^d$, i.e. it satisfies the condition

$$\int_{\mathbb{R}^d} (1 \wedge |x|^2)\mu(dx) < \infty. \tag{5.0.1}$$

Let $\tilde{N}(dt, dx) := N(dt, dx) - dt\, \mu(dx)$. We know that when

$$\int_0^T \int_{\mathbb{R}^d} |Y(t, x)|^2\, dt\, \mu(dx) < \infty \qquad \text{a.s.}, \tag{5.0.2}$$

the stochastic integral

$$\int_0^t \int_{\mathbb{R}^d} Y(s, x)\tilde{N}(ds, dx), t \geq 0, \tag{5.0.3}$$

exists as a limit in probability (cf. [App09, p. 227]). We want to give pathwise meaning to the integrals such as (5.0.3).

The pathwise interpretations of stochastic integrals with respect to one-parameter objects (i.e. stochastic processes) are well understood by now and have found applications in the numerical simulations of various SDEs. Here, on the other hand, we have a situation where our integrands are functions of two variables – time and space. Further, it is worth observing that a pathwise meaning of integrals such as (5.0.3) will help us solve the SDE (cf. [App09, Eq. (6.12)])

$$dY_t = b(Y_{t-})\, dt + \sigma(Y_{t-})\, dB_t + \int_{|x|\geq 1} G(Y_{t-}, x)N(dt, dx) + \int_{|x|<1} F(Y_{t-}, x)\tilde{N}(dt, dx) \tag{5.0.4}$$

in a pathwise sense. Here B_t is the Brownian motion and b, σ , G and F are appropriately defined functions.

We will need some basic techniques from the theory of rough paths. The theory of rough paths was introduced by Lyons [Lyo98]. There are a number of excellent references for studying this area, e.g. [LCL07] and [FV10]. We borrow our exposition mainly from the book of Friz and Hairer [FH14], but instead of treating only Hölder continuous rough paths as these authors have done, we deal with the more general case of possibly discontinuous paths having finite p-variation. This is the approach adopted in the article by Friz and Shekhar [FS17]. We also note that our discussion avoids the general algebraic theory, such as the one found in [FV10].

This chapter is structured as follows. Section 5.1 introduces the central notions of p-variations of a path and that of the control function. In Section 5.2 we take a close look at the theory of Young integration. Section 5.3 introduces rough paths and their integration. We present our observations on pathwise integrals against PRMs in the final section.

5.1 p-variation of paths and control functions

We begin by precisely defining the notion of p-variation of a path in $\mathbb{R}^d$ which is a quantitative way of capturing the regularity (smoothness vs. roughness) of the path in question. By a *partition* $\mathcal{P}$ of an interval $[a, b]$ we mean a finite collection of points $a = u_0 < u_1 < \cdots < u_{k-1} < u_k = b$. Throughout the rest of this chapter and the next one we will use this term interchangeably to mean a collection of sub-intervals arising out of the consecutive points in this list, i.e.

$$\mathcal{P} = \{[a, u_1], [u_1, u_2], \ldots, [u_{k-2}, u_{k-1}], [u_{k-1}, b]\}.$$

$PP[a, b]$ denotes the collection of all partitions of $[a, b]$.

Definition 5.1.1. Let $a < b$ be real numbers and $p > 0$. The *p-variation* of a path $X : [a, b] \to \mathbb{R}^d$ is defined to be

$$\|X\|_{p-var;[a,b]} := \left[\sup_{\mathcal{P} \in PP[a,b]} \sum_{[u,v] \in \mathcal{P}} |X_v - X_u|^p \right]^{\frac{1}{p}}.$$

The above defines a semi-norm on the space of all $\mathbb{R}^d$-valued paths defined on $[0, T]$. We will sometimes denote $X(t) = X_t$ and shorten the notations for $X_t - X_s$ by using $X_{s,t}$. Also the p-variation norm of X will be denoted by $\|X\|_{p-var}$ when the interval in question is clear from the context. We next define oscillation of the path X as follows,

$$\mathrm{Osc}(X; [0, T]) = \sup_{u,v \in [0,T]} |X_v - X_u|.$$

It is a well-known fact that when X is cádlág (right-continuous with left limits) or cáglád (left-continuous with right limits), it has finite oscillation over $[0, T]$ (cf. [Bil99, Lemma 3.1]). This can be used to show that when $0 < p < p'$, $\|X\|_{p-var} < \infty$ implies $\|X\|_{p'-var} < \infty$.

Now let us denote by $\Delta[0, T]$ the two-dimensional simplex $\{(s, t) \in \mathbb{R}^2 \mid 0 \leq s \leq t \leq T\}$.

Definition 5.1.2. A function $w : \Delta[0, T] \to [0, \infty)$ is called a *control function* if it satisfies the following super-additive property: whenever $0 \leq s \leq u \leq t \leq T$, we have

$$w(s, u) + w(u, t) \leq w(s, t).$$

It is easy to see that if $X : [0,T] \to \mathbb{R}^d$ has finite p-variation then for $0 \le s \le u \le t \le T$ and any two partitions $\mathcal{P}_1 \in PP[s,u]$ and $\mathcal{P}_2 \in PP[u,t]$ we have

$$\sum_{[a,b]\in\mathcal{P}_1} |X_{a,b}|^p + \sum_{[a,b]\in\mathcal{P}_2} |X_{a,b}|^p \le \|X\|^p_{p-var;[s,t]},$$

as $\mathcal{P}_1 \cup \mathcal{P}_2$ is a partition of $[s,t]$. Taking the supremum over all partitions of $[s,t]$ the above relation gives,

$$\|X\|^p_{p-var;[s,u]} + \|X\|^p_{p-var;[u,t]} \le \|X\|^p_{p-var;[s,t]}.$$

Thus $w(s,t) = \|X\|^p_{p-var;[s,t]}$ defines a control. These will serve as the building blocks for all the control functions to be defined in the sequel. Although we do not require our control functions to be continuous for the most part, it is useful to observe that if the path X is continuous, so is the control w defined from it.

Lemma 5.1.3. *Let $X : [0,T] \to \mathbb{R}^d$ be a continuous path of finite p-variation. Then the map $t \mapsto \|X\|_{p-var;[0,t]}$ is continuous.*

Proof. See [FV10, Propositon 5.8]. □

The following two results are important properties of control function.

Lemma 5.1.4. *Suppose $w_1, w_2 : \Delta[0,T] \to [0,\infty)$ are control functions and $\alpha, \beta \ge 0$ with $\alpha+\beta \ge 1$. Then $w := w_1^\alpha w_2^\beta$ is again a control function.*

Proof. To prove the super-additivity of w it is enough to prove the following claim: if $x, y, a, b \ge 0$ and $\alpha + \beta \ge 1$ then $x^\alpha y^\beta + a^\alpha b^\beta \le (x+a)^\alpha (y+b)^\beta$. As this is equivalent to

$$\left(\frac{x}{x+a}\right)^\alpha \left(\frac{y}{y+b}\right)^\beta + \left(\frac{a}{x+a}\right)^\alpha \left(\frac{b}{y+b}\right)^\beta \le 1 \tag{5.1.1}$$

we can assume that $x + a = y + b = 1$. Using the fact $\frac{1}{(\alpha+\beta)/\alpha} + \frac{1}{(\alpha+\beta)/\beta} = 1$, by Young's inequality for products we get

$$x^\alpha y^\beta \le \frac{x^{\alpha\frac{\alpha+\beta}{\alpha}}}{\frac{\alpha+\beta}{\alpha}} + \frac{y^{\beta\frac{\alpha+\beta}{\beta}}}{\frac{\alpha+\beta}{\beta}} = \frac{\alpha x^{\alpha+\beta} + \beta y^{\alpha+\beta}}{\alpha+\beta} \text{ and } a^\alpha b^\beta \le \frac{\alpha a^{\alpha+\beta} + \beta b^{\alpha+\beta}}{\alpha+\beta}.$$

Hence

$$x^\alpha y^\beta + a^\alpha b^\beta \le \frac{\alpha(x+a)^{\alpha+\beta} + \beta(y+b)^{\alpha+\beta}}{\alpha+\beta} = \frac{\alpha+\beta}{\alpha+\beta} = 1$$

using our assumptions $\alpha + \beta \ge 1$ and $x + a = y + b = 1$. This proves (5.1.1). □

Lemma 5.1.5. *Suppose $w : \Delta[0,T] \to [0,\infty)$ is control function and $\mathcal{P}$ is a partition of some interval $[s,t] \subseteq [0,T]$ containing at least three points. Then there exist three consecutive points $u- < u < u+$ in $\mathcal{P}$ such that*

$$w(u-, u+) \le \frac{2}{r-1} w(s,t) \tag{5.1.2}$$

where $r + 1 = \#(\mathcal{P})$, the cardinality of $\mathcal{P}$.

Proof. If not, then there is a partition $\mathcal{P} = \{s = u_0 < u_1 < \cdots < u_r = t\}$ such that for all $i = 1, \ldots, r-1$,

$$w(u_{i-1}, u_{i+1}) > \frac{2}{r-1} w(s,t).$$

Suppose r is even. Then

$$\begin{aligned}
2w(s,t) &< \sum_{i=1}^{r-1} w(u_{i-1}, u_{i+1}) \\
&= \sum_{\substack{i=1 \\ i \text{ odd}}}^{r-1} w(u_{i-1}, u_{i+1}) + \sum_{\substack{i=1 \\ i \text{ even}}}^{r-1} w(u_{i-1}, u_{i+1}) \\
&= [w(u_0, u_2) + \cdots + w(u_{r-2}, u_r)] + [w(u_1, u_3) + \cdots + w(u_{r-3}, u_{r-1})] \\
&\leq w(u_0, u_r) + w(u_1, u_{r-1}) \leq 2w(u_0, u_r) = 2w(s,t),
\end{aligned}$$

which is a contradiction. Similar conclusion can be reached when r is odd. □

5.2 The Young integral

In this section we present the theory of integration developed by Young [You36]. This provides a blueprint for the theory of rough paths which will come in the next section. For a path $X : [0,T] \to \mathbb{R}^d$ to be *regulated* we mean that the left and right hand limits of X at each point $t \in [0,T]$ exists.

Definition 5.2.1. Let $X : [0,T] \to \mathbb{R}^d$ and $Y : [0,T] \to \mathbb{R}^d$ be regulated paths. We define the Young integral of Y with respect to X to be

$$\int_0^T Y_r \, dX_r = \lim_{\substack{|\mathcal{P}| \to 0 \\ \mathcal{P} \in PP[0,T]}} \sum_{[s,t] \in \mathcal{P}} Y_s (X_t - X_s) \tag{5.2.1}$$

where $|\mathcal{P}|$ denotes the mesh size of the partition $\mathcal{P}$.

The limit above is said to exist and equal to L if for each $\epsilon > 0$ there is a delta $\delta > 0$ such that whenever $|\mathcal{P}| < \delta$, we have $|\sum_{[s,t] \in \mathcal{P}} Y_s(X_t - X_s) - L| < \epsilon$. This is sometimes called the *Mesh Riemann-Stieltjes convergence* of Riemann sums (cf. [FS17, Definition 1]). The main result concerning Young integrals is the following.

Theorem 5.2.2. *Suppose X and Y are both regulated paths with finite p-variations for some $p < 2$. Moreover assume that X is càdlàg. Then the Young integral $\int_0^T Y_r \, dX_r$, as defined in (5.2.1), exists.*

Before the proof of this theorem we need a lemma that allows us to control the jumps of X and Y together on small intervals.

Lemma 5.2.3. *If $X, Y : [0,T] \to \mathbb{R}^d$ are regulated paths and X is càdlàg, then for every $\epsilon > 0$ there is some $\delta > 0$ such that*

$$\text{either } |Y_a - Y_c| < \epsilon \text{ or } |X_c - X_b| < \epsilon$$

whenever $0 \leq a \leq c \leq b \leq T$ with $b - a < \delta$.

Proof. Suppose not. Then we can find an $\epsilon > 0$ and sequences $\{s_n\}_{n\geq 1}, \{u_n\}_{n\geq 1}, \{t_n\}_{n\geq 1}$ in $[0, T]$ having the following properties: for all $n \in \mathbb{N}$, (i) $s_n < u_n < t_n$, (ii) $|t_n - s_n| < \frac{1}{n}$ and (iii) $|Y(s_n) - Y(u_n)| \geq \epsilon$ and $|X(t_n) - X(u_n)| \geq \epsilon$.

As $\{u_n\}_n$ is a sequence in a compact set $[0, T]$, we can a find sub-sequence $\{u_{n_k}\}_{k\geq 1}$ of $\{u_n\}_{n\geq 1}$ so that $u_{n_k} \to u$ as $k \to \infty$. The conditions on the other two sequences then imply that $s_{n_k} \to u$ and $t_{n_k} \to u$ as $k \to \infty$.

Now several situations arise. We split them in the following cases.

Case (a): (When $u_{n_k} = u$ for all but finitely many k's) By right continuity of X, we know $X(u+) = X(u)$. But this contradicts the conclusion that

$$\epsilon \leq \lim_{k\to\infty} |X(t_{n_k}) - X(u)| = |X(u+) - X(u)|.$$

Case (b): (When $u_{n_k} < u$ for infinitely many k) Without loss of generality assume that $u_{n_k} < u$ for all k. As $s_{n_k} < u_{n_k}$, $\lim_{k\to\infty} s_{n_k} = \lim_{k\to\infty} u_{n_k} = u$ and $Y(u-)$ exists, we get

$$\epsilon \leq \lim_{k\to\infty} |Y(u_{n_k}) - Y(s_{n_k})| = |Y(u-) - Y(u-)| = 0,$$

which is a contradiction.

Case (c): (When $u_{n_k} > u$ for infinitely many k) Using the same argument as in the last case and the fact that $X(u+)$ exists we have,

$$\epsilon \leq \lim_{k\to\infty} |X(u_{n_k}) - X(t_{n_k})| = |X(u+) - X(u+)| = 0,$$

we again arrive at a contradiction.

The above three cases together imply the statement of the lemma. $\square$

Proof of Theorem 5.2.2. Fix a càdlàg path X and a regulated path Y defined on $[0, T]$ and taking values in $\mathbb{R}^d$. For each partition $\mathcal{P}$ of $[0, T]$, let us use the notation

$$S(\mathcal{P}) = \sum_{[s,t]\in\mathcal{P}} Y_s(X_t - X_s)$$

for the Riemann sum corresponding to $\mathcal{P}$.

As per our definition of Young integrals we only have to show that the limit in RHS of (5.2.1) converges. For this purpose it is enough to prove the following Cauchy criterion: For a given $\epsilon > 0$, there is a $\delta > 0$ such that whenever $\mathcal{P}_1, \mathcal{P}_2 \in PP[0, T]$ with $|\mathcal{P}_1|, |\mathcal{P}_2| < \delta$ we have $|S(\mathcal{P}_1) - S(\mathcal{P}_2)| < \epsilon$.

So, fix an $\epsilon > 0$. Let $\delta > 0$ be such that Lemma 5.2.3 holds with ϵ. Let $\mathcal{P}_1, \mathcal{P}_2 \in PP[0, T]$ be such that $|\mathcal{P}_1|, |\mathcal{P}_2| < \delta$. We can safely assume that $\mathcal{P}_2$ is a *refinement* of $\mathcal{P}_1$, i.e. $\mathcal{P}_1 \subseteq \mathcal{P}_2$. When $0 \leq s < t \leq T$, we denote by $\mathcal{P}_2[s, t]$ all the points in $\mathcal{P}_2$ (and hence their corresponding sub-intervals) between s and t. We have,

$$S(\mathcal{P}_2) - S(\mathcal{P}_1) = \sum_{[s,t]\in\mathcal{P}_1} \left(\sum_{[u,v]\in\mathcal{P}_2[s,t]} Y_u(X_v - X_u) - Y_s(X_t - X_s) \right) \tag{5.2.2}$$

Denote $\theta_{s,t} := \sum_{[u,v]\in\mathcal{P}_2[s,t]} Y_u(X_v - X_u) - Y_s(X_t - X_s)$.

Let $\eta \in [\frac{1}{2}, \frac{1}{p})$. As $\|X\|_{p-var;[0,T]} < \infty$ and $\|Y\|_{p-var;[0,T]} < \infty$ by our assumption, the function

$$w(s,t) := \|X\|^{p\eta}_{p-var;[s,t]} \cdot \|Y\|^{p\eta}_{p-var;[s,t]} \tag{5.2.3}$$

for $0 \leq s \leq t \leq T$ is a control by Lemma 5.1.4.

Now fix $[s,t] \in \mathcal{P}_1$. For some $\alpha \in (1, \frac{1}{p\eta})$ we have,

$$\begin{aligned}
&|S(\mathcal{P}_2[s,t]) - S(\mathcal{P}_2[s,t] \setminus \{u\})| \\
=&|Y_{u-}(X_u - X_{u-}) + Y_u(X_{u+} - X_u) - Y_{u-}(X_{u+} - X_{u-})| \\
=&|Y_u - Y_{u-}| \cdot |X_{u+} - X_u| \\
=&|Y_u - Y_{u-}|^{1-p\eta\alpha} \cdot |X_{u+} - X_u|^{1-p\eta\alpha} \cdot |Y_u - Y_{u-}|^{p\eta\alpha} \cdot |X_{u+} - X_u|^{p\eta\alpha}.
\end{aligned} \tag{5.2.4}$$

Note that, as X and Y are both regulated, there is a finite constant $C > 0$ such that

$$\mathrm{Osc}(X; [0,T]) \leq C, \mathrm{Osc}(Y; [0,T]) \leq C.$$

From (5.2.4) therefore

$$\begin{aligned}
&|S(\mathcal{P}_2[s,t]) - S(\mathcal{P}_2[s,t] \setminus \{u\})| \\
\leq&\epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} \cdot \left(|Y_u - Y_{u-}|^p\right)^{\eta\alpha} \cdot \left(|X_{u+} - X_u|^p\right)^{\eta\alpha} \\
\leq&\epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} \cdot \left(\|Y\|^{p\eta}_{p-var;[u-,u]} \cdot \|X\|^{p\eta}_{p-var;[u,u+]}\right)^{\alpha} \\
\leq&\epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} \cdot w(u-, u+)^{\alpha},
\end{aligned} \tag{5.2.5}$$

where we have used Lemma 5.2.3 and our assumptions on the partitions $\mathcal{P}_1$, $\mathcal{P}_2$ for the first inequality.

As $\mathcal{P}_1 \subseteq \mathcal{P}_2$, we must have $\#(\mathcal{P}_2[s,t]) \geq 2$. If $\#(\mathcal{P}_2[s,t]) = 2$, clearly $\theta_{s,t} = 0$. So we can assume that $r := \#(\mathcal{P}_2[s,t]) \geq 3$. Therefore by Lemma 5.1.5, for the control function w, there are three consecutive points $u- < u < u+$ in $\mathcal{P}_2[s,t]$ such that

$$w(u-, u+) \leq \frac{2}{(r-1)-1} w(s,t). \tag{5.2.6}$$

We can use this in the calculations (5.2.5) above to get,

$$\begin{aligned}
|S(\mathcal{P}_2[s,t]) - S(\mathcal{P}_2[s,t] \setminus \{u\})| \leq&\epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} \cdot w(u-, u+)^{\alpha}, \\
\leq&\epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} \cdot \left(\frac{2}{r-2}\right)^{\alpha} w(s,t)^{\alpha}.
\end{aligned} \tag{5.2.7}$$

Recall that $r = \#(\mathcal{P}_2[s,t])$. Since Lemma 5.1.5 holds as long as the partition has at least three points in it, we can estimate $|S(\mathcal{P}_2[s,t]) - Y_s(X_t - X_s)|$ by inductively defining a decreasing sequence of sub-partitions $\mathcal{Q}_k$ $(1 \leq k \leq r-3)$ of $\mathcal{P}_2[s,t]$ as follows: Let $\mathcal{Q}_0 := \mathcal{P}_2[s,t], \mathcal{Q}_1 := \mathcal{P}_2[s,t] \setminus \{u\}$ where u satisfies (5.2.6). In $\mathcal{Q}_k$, again there are $u- < u < u+$ (according to Lemma 5.1.5) such that (5.2.6) holds for $r - k = \#(\mathcal{Q}_k) \geq 3$. Let $\mathcal{Q}_{k+1} := \mathcal{Q}_k \setminus \{u\}$. For each k, by the same argument used to obtain (5.2.7), we can get

$$|S(\mathcal{Q}_k) - S(\mathcal{Q}_{k+1})| \leq \epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} \cdot \left(\frac{2}{r-k-2}\right)^{\alpha} w(s,t)^{\alpha}, \tag{5.2.8}$$

with the same $\alpha \in (1, \frac{1}{p\eta})$ and the control function defined by (5.2.3).

From the above computations we now have,

$$\begin{aligned}
|\theta_{s,t}| \leq& \sum_{k=0}^{r-3} |S(\mathcal{Q}_{k+1}) - S(\mathcal{Q}_k)| \\
\leq& \epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} \cdot \sum_{k=0}^{r-3} \left(\frac{2}{r-k-2}\right)^{\alpha} w(s,t)^{\alpha} \\
\leq& \epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} 2^{\alpha} \left(\sum_{k=0}^{\infty} \frac{1}{k^{\alpha}}\right) w(s,t)^{\alpha} \\
=& \epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} 2^{\alpha} \zeta(\alpha) w(s,t)^{\alpha}
\end{aligned} \tag{5.2.9}$$

using (5.2.8) in the second inequality. Here ζ denotes the Riemann zeta function. From this and (5.2.2), it easily follows that

$$\begin{aligned}
|S(\mathcal{P}_2) - S(\mathcal{P}_1)| \leq& \sum_{[s,t] \in \mathcal{P}_1} |\theta_{s,t}| \\
\leq& \epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} 2^{\alpha} \zeta(\alpha) \left(\sum_{[s,t] \in \mathcal{P}_1} w(s,t)^{\alpha}\right) \\
\leq& \epsilon^{1-p\eta\alpha} C^{1-p\eta\alpha} 2^{\alpha} \zeta(\alpha) w(0,T)^{\alpha},
\end{aligned} \tag{5.2.10}$$

where we have used the super-additivity of the control function w and the fact $\alpha > 1$ to get the final inequality. Since (5.2.10) holds for any $\epsilon > 0$, we have proved the required Cauchy criterion. □

The technique of controlling the difference of Riemann sums for the purpose of proving the existence of a certain integral will recur throughout this chapter. A more refined statement of this, called the *sewing lemma*, can be found in the next section (cf. Proposition 5.3.6). Analogous to the bound (5.2.9) on $\theta_{s,t}$ obtained in the above proof, we have the following estimate, called the Loeve-Young inequality. Let us recall from (5.2.1) that

$$\int_0^T Y_r \, dX_r = \lim_{\substack{|\mathcal{P}| \to 0 \\ \mathcal{P} \in PP[0,T]}} \sum_{[s,t] \in \mathcal{P}} Y_s (X_t - X_s),$$

and that the previous theorem guarantees the existence of the limit above.

Lemma 5.2.4 (Loeve-Young inequality). *Under the hypotheses of the above theorem, for any* $0 \leq s \leq t \leq T$, *we can find a* $\beta = \beta(p) \in (0,1)$ *such that*

$$\left| \int_s^t Y_r \, dX_r - Y_s (X_t - X_s) \right| \leq C_1 \sigma_{s,t}^{1-\beta} \|X\|_{p-var;[s,t]}^{\beta} \|Y\|_{p-var;[s,t]}^{\beta} \tag{5.2.11}$$

where $\sigma_{s,t} = \sigma_{s,t}(X,Y)$ *satisfies the condition that,*

$$\lim_{\delta \downarrow 0} \left(\sup_{\substack{s,t \in [0,T] \\ |t-s| \leq \delta}} \sigma_{s,t}(X,Y) \right) = 0. \tag{5.2.12}$$

Proof. We use the following notations, already used in the proof of Theorem 5.2.2. We have $\frac{1}{2} \leq \eta < \frac{1}{p}$, $1 < \alpha < \frac{1}{p\alpha}$ and w is the control function defined in (5.2.3). Let $\mathcal{P}$ be an arbitrary partition of $[0, T]$. Suppose $r = \#(\mathcal{P}[s, t]) \geq 3$. Then by Lemma 5.1.5 there are points $u- < u < u+$ in $\mathcal{P}[s, t]$ such that (5.2.6) holds. We have,

$$\begin{aligned}
&|S(\mathcal{P}[s,t]) - S(\mathcal{P}[s,t] \setminus \{u\})| \\
\leq &|Y_u - Y_{u-}|^{1-p\eta\alpha} \cdot |X_{u+} - X_u|^{1-p\eta\alpha} \cdot |Y_u - Y_{u-}|^{p\eta\alpha} \cdot |X_{u+} - X_u|^{p\eta\alpha} \\
\leq &|Y_u - Y_{u-}|^{1-p\eta\alpha} \cdot |X_{u+} - X_u|^{1-p\eta\alpha} w(u-, u+)^{\alpha} \\
\leq &|Y_u - Y_{u-}|^{1-p\eta\alpha} \cdot |X_{u+} - X_u|^{1-p\eta\alpha} \left(\frac{2}{r-2}\right)^{\alpha} w(s,t)^{\alpha},
\end{aligned} \tag{5.2.13}$$

applying the definition of w in the second inequality and (5.2.6) for the third.

As before, we let $C > 0$ be a finite constant bounding the oscillations of X and Y on $[0, T]$. For $0 \leq s \leq t \leq T$ define,

$$\sigma_{s,t} = \sigma_{s,t}(X, Y) := C \cdot \sup_{s \leq a \leq c \leq b \leq t} |Y_c - Y_a| \wedge |X_b - X_c|. \tag{5.2.14}$$

Clearly

$$|Y_u - Y_{u-}| \cdot |X_{u+} - X_u| \leq \sigma_{s,t}(X, Y)$$

for any $s \leq u- < u < u+ \leq t$. Also observe that σ satisfies (5.2.12) by Lemma 5.2.3. Hence (5.2.13) gives

$$|S(\mathcal{P}[s,t]) - S(\mathcal{P}[s,t] \setminus \{u\})| \leq \sigma_{s,t}(X, Y)^{1-p\eta\alpha} \left(\frac{2}{r-1}\right)^{\alpha} w(s,t)^{\alpha}.$$

Recall that $\theta_{s,t}(\mathcal{P}) = S(\mathcal{P}[s,t]) - Y_s(X_t - X_s)$. Similarly as in (5.2.9) we have,

$$|\theta_{s,t}(\mathcal{P})| \leq \sigma_{s,t}(X, Y)^{1-p\eta\alpha} 2^{\alpha} \zeta(\alpha) w(s,t)^{\alpha} = C_1 \sigma_{s,t}(X, Y)^{1-p\eta\alpha} \|X\|_{p-var;[s,t]}^{p\eta\alpha} \|Y\|_{p-var;[s,t]}^{p\eta\alpha}.$$

As the above is true for any $\mathcal{P}$, we obtain

$$\begin{aligned}
\left| \int_s^t Y_r \, dX_r - Y_s(X_t - X_s) \right| &= \lim_{|\mathcal{P}| \to 0} |\theta_{s,t}(\mathcal{P})| \\
&\leq C_1 \sigma_{s,t}(X, Y)^{1-p\eta\alpha} \|X\|_{p-var;[s,t]}^{p\eta\alpha} \|Y\|_{p-var;[s,t]}^{p\eta\alpha}.
\end{aligned}$$

This proves the lemma with $\beta = p\eta\alpha$. □

5.3 Rough path and rough integral

We are now ready to introduce the notion of rough paths. For a two-parameter function, say $G : \Delta[0, T] \to W$ (where $(W, |\cdot|)$ is normed vector space), its p-variation is defined similarly as in Definition 5.1.1.

$$\|G\|_{p-var;[0,T]} := \left[\sup_{\mathcal{P} \in PP[0,T]} \sum_{[u,v] \in \mathcal{P}} |G(s,t)|^p \right].$$

As is customary in the literature, we use tensor product notations in this section. By $\mathbb{R}^m \otimes \mathbb{R}^n$ we will denote the space $\mathbb{R}^{m\times n}$ of $m \times n$ matrices with real entries. Given two (column) vectors $X, Y \in \mathbb{R}^m$, the notation $X \otimes Y$ will denote the $m \times m$ matrix XY^T obtained by the method of matrix multiplication.

Definition 5.3.1. Let $p \in [2, 3)$ and $X : [0, T] \to \mathbb{R}^d$, $\mathbb{X} : \Delta[0, T] \to \mathbb{R}^d \otimes \mathbb{R}^d$ be functions such that

(i) $\|X\|_{p-var;[0,T]} < \infty$, $\|\mathbb{X}\|_{p/2-var;[0,T]} < \infty$.

(ii) The map $t \mapsto (X_{0,t}, \mathbb{X}_{0,t})$ is cádlág.

(iii) For all $0 \le s < u < t \le T$,

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t} \qquad \text{(Chen's relation)}.$$

Then the pair $\mathbf{X} = (X, \mathbb{X})$ is called a *cádlág rough path of p-variation.*

Since we are interested in doing integration against rough paths, we next introduce our integrands. Given two vector spaces V and W, by $\mathcal{L}(V, W)$ we will denote the collection of all the linear maps from V to W.

Definition 5.3.2. Suppose (X, X') is a p-variation cádlág rough path as defined above. Let $Y : [0, T] \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)$ and $Y' : [0, T] \to \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)) \cong \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^m)$ be two paths. Then the pair $\mathbf{Y} = (Y, Y')$ is called an X- *controlled rough path* if

(a) $\|Y\|_{p-var;[0,T]} + \|Y'\|_{p-var;[0,T]} < \infty$

(b) If $R^Y_{s,t} := Y_{s,t} - Y'_s(X_{s,t})$ then $\|R^Y\|_{p/2-var;[0,T]} < \infty$. $R^Y : \Delta[0, T] \to \mathbb{R}^d$ is called the *remainder term.*

When the maps $t \mapsto (X_t, \mathbb{X}_{0,t})$ and $t \mapsto (Y_t, Y'_t)$ are continuous instead of cádlág, their p-variation (or $p/2$-variation) norms are equivalent to their $1/p$-Hölder ($2/p$-Hölder) norms. In the next section we will work with continuous rough paths.

Definition 5.3.3. Let $\Xi(s, t) := Y_s(X_{s,t}) + Y'_s(\mathbb{X}_{s,t})$ for $0 \le s \le t \le T$. We define the rough integral of $\mathbf{Y} = (Y, Y')$ with respect to $\mathbf{X} = (X, \mathbb{X})$ as

$$\int_0^T \mathrm{Y}_r \, d\mathbf{X}_r := \lim_{\substack{|\mathcal{P}|\to 0 \\ \mathcal{P}\in PP[0,T]}} \sum_{[u,v]\in\mathcal{P}} \Xi(u, v). \tag{5.3.1}$$

The first step towards proving the existence of the limit in (5.3.1) is to prove a result similar to Lemma 5.2.3, but this time for $\mathbf{X}$ and $\mathbf{Y}$, instead of just X and Y.

Lemma 5.3.4. *Let* $\mathbf{X}$ *be a cádlág rough path and* $\mathbf{Y}$ *be an* $\mathbf{X}$*-controlled rough path. Then for every* $\epsilon > 0$ *there is a* $\delta > 0$ *such that whenever* $|t - s| < \delta$ *and* $0 \le s < u < t \le T$, *we have*

(a) $|R^Y_{s,u}| < \epsilon$ *or* $|X_{u,t}| < \epsilon$, *and*

(b) $|Y'_{s,u}| < \epsilon$ *or* $|\mathbb{X}_{u,t}| < \epsilon$.

Proof. The statements can be proved, *mutatis mutandis*, with the same argument used in Lemma 5.2.3. □

Note that the conclusion of the above lemma is true if we replace $|R^Y_{s,u}|$, $|X_{u,t}|$, $|Y'_{s,u}|$ and $|\mathbb{X}_{u,t}|$ by $|R^Y_{s,u}|^\eta, \ldots, |\mathbb{X}_{u,t}|^\eta$ respectively, where $\eta > 0$ is any number. And therefore, we can write the statement of the lemma in the following compact form:

$$\lim_{\delta \downarrow 0} \left[\sup_{t-s\leq\delta} C\sigma_{s,t}(\mathbf{X}, \mathbf{Y}) \right] = 0, \tag{5.3.2}$$

where

$$\sigma_{s,t}(\mathbf{X}, \mathbf{Y}) = \sup_{u\in[s,t]} \left(|R^Y_{s,u}|^\eta \wedge |X_{u,t}|^\eta + |Y'_{s,u}|^\eta \wedge |\mathbb{X}_{u,t}|^\eta \right) \tag{5.3.3}$$

and C is a constant that bounds $\|R^Y\|_\infty := \sup_{0\leq u\leq v\leq T} |R^Y_{u,v}|, \mathrm{Osc}(X; [0,T]), \mathrm{Osc}(Y'; [0,T])$ and $\|\mathbb{X}\|_\infty := \sup_{0\leq u\leq v\leq T} |\mathbb{X}_{u,v}|$.

Proposition 5.3.5. *Let* $\mathbf{X} = (X, \mathbb{X})$ *be a cádlág* p*-variation rough path with* $2 \leq p < 3$ *and* $\mathbf{Y} = (Y, Y')$ *is an* X*-controlled rough path as in the previous definitions. Suppose for* $0 \leq s < u < t \leq T$ *we have* $\Xi(s,t) := Y_s(X_{s,t}) + Y'_s(\mathbb{X}_{s,t})$ *and*

$$\delta\Xi(s,u,t) = \Xi(s,t) - \Xi(s,u) - \Xi(u,t).$$

Then there is a control function w *on* $[0,T]$ *and* $\beta > 1$ *such that*

$$\lim_{\delta\downarrow 0} \left[\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}} \left(\sup_{u\in[s,t]} \frac{|\delta\Xi(s,u,t)|}{w(s,t)^\beta} \right) \right] = 0. \tag{5.3.4}$$

Proof. Our approach here is similar to the first part of the proof of Theorem 5.2.2. For $u \in [s,t]$, by the definition of Ξ

$$\begin{aligned}
\delta\Xi(s,u,t) =& \Xi(s,t) - \Xi(s,u) - \Xi(u,t)\\
=& Y_s(X_{s,t}) + Y'_s(\mathbb{X}_{s,t}) - Y_s(X_{s,u}) - Y'_s(\mathbb{X}_{s,u}) - Y_u(X_{u,t}) - Y'_u(\mathbb{X}_{u,t})\\
=& Y_s(X_{u,t}) + Y'_s(\mathbb{X}_{s,t} - \mathbb{X}_{s,u}) - Y_u(X_{u,t}) - Y'_u(\mathbb{X}_{u,t})\\
=& (Y_s - Y_u)(X_{u,t}) + Y'_s(\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t}) + Y'_s(\mathbb{X}_{u,t}) - Y'_u(\mathbb{X}_{u,t})\\
=& (Y_s - Y_u)(X_{u,t}) + Y'_s(X_{s,u} \otimes X_{u,t}) + (Y'_s - Y'_u)(\mathbb{X}_{u,t}),
\end{aligned}$$

using algebraic manipulations and applying the Chen's relation in the last line. As $Y'_s(X_{s,u} \otimes X_{u,t}) = Y'_s(X_{s,u})(X_{u,t})$, we have

$$\delta\Xi(s,u,t) = Y_{s,u}(X_{u,t}) + Y'_s(X_{s,u})(X_{u,t}) + Y'_{s,u}(\mathbb{X}_{u,t}) = R^Y_{s,u}(X_{u,t}) + Y'_{s,u}(\mathbb{X}_{u,t}). \tag{5.3.5}$$

Now choose parameters β and η such that $1 < \beta < \frac{3}{p}$ and $0 < \eta < 1 - \frac{p\beta}{3}$. Then

$$w(s,t) := \|R^Y\|_{p/2-var;[s,t]}^{\frac{1-\eta}{\beta}} \|X\|_{p-var;[s,t]}^{\frac{1-\eta}{\beta}} + \|Y'\|_{p-var;[s,t]}^{\frac{1-\eta}{\beta}} \|\mathbb{X}\|_{p/2-var;[s,t]}^{\frac{1-\eta}{\beta}} \tag{5.3.6}$$

defines a control function by Lemma 5.1.4, as $\frac{1-\eta}{\beta}\frac{1}{p} + \frac{1-\eta}{\beta}\frac{2}{p} = \frac{1-\eta}{\beta}\frac{3}{p} \geq 1$.

Recall the definition of $\sigma_{s,t}(\mathbf{X},\mathbf{Y})$ from (5.3.3). Equation (5.3.5) gives,

$$\begin{aligned}|\delta\Xi(s,u,t)| \leq & |R^Y_{s,u}|^{\eta}|X_{u,t}|^{\eta}|R^Y_{s,u}|^{1-\eta}|X_{u,t}|^{1-\eta} + |Y'_{s,u}|^{\eta}|\mathbb{X}_{u,t}|^{\eta}|Y'_{s,u}|^{1-\eta}|\mathbb{X}_{u,t}|^{1-\eta} \\ \leq & \sigma_{s,t}(\mathbf{X},\mathbf{Y})\left(|R^Y_{s,u}|^{1-\eta}|X_{u,t}|^{1-\eta} + |Y'_{s,u}|^{1-\eta}|\mathbb{X}_{u,t}|^{1-\eta}\right).\end{aligned}$$

As $\beta > 1$ we can use the inequality $(x+y)^{\beta} \geq x^{\beta} + y^{\beta}$ for $x, y \geq 0$ to get

$$\begin{aligned}& |R^Y_{s,u}|^{1-\eta}|X_{u,t}|^{1-\eta} + |Y'_{s,u}|^{1-\eta}|\mathbb{X}_{u,t}|^{1-\eta} \\ \leq & \left(\|R^Y\|^{\frac{1-\eta}{\beta}}_{p/2-var;[s,u]}\|X\|^{\frac{1-\eta}{\beta}}_{p-var;[u,t]}\right)^{\beta} + \left(\|Y'\|^{\frac{1-\eta}{\beta}}_{p/2-var;[s,u]}\|\mathbb{X}\|^{\frac{1-\eta}{\beta}}_{p/2-var;[u,t]}\right)^{\beta} \\ \leq & \left[\left(\|R^Y\|^{\frac{1-\eta}{\beta}}_{p/2-var;[s,u]}\|X\|^{\frac{1-\eta}{\beta}}_{p-var;[u,t]}\right) + \left(\|Y'\|^{\frac{1-\eta}{\beta}}_{p/2-var;[s,u]}\|\mathbb{X}\|^{\frac{1-\eta}{\beta}}_{p/2-var;[u,t]}\right)\right]^{\beta} \\ \leq & \left[\left(\|R^Y\|^{\frac{1-\eta}{\beta}}_{p/2-var;[s,t]}\|X\|^{\frac{1-\eta}{\beta}}_{p-var;[s,t]}\right) + \left(\|Y'\|^{\frac{1-\eta}{\beta}}_{p/2-var;[s,t]}\|\mathbb{X}\|^{\frac{1-\eta}{\beta}}_{p/2-var;[s,t]}\right)\right]^{\beta} \\ = & w(s,t)^{\beta},\end{aligned}$$

using the facts that $\|R^Y\|_{p/2-var;[s,u]} \leq \|R^Y\|_{p/2-var;[s,t]}$, $\|X\|_{p-var;[u,t]} \leq \|X\|_{p-var;[s,t]}$ etc. and the definition of control function from (5.3.6). We can combine the above calculations to write,

$$|\delta\Xi(s,u,t)| \leq \sigma_{s,t}(\mathbf{X},\mathbf{Y})w(s,t)^{\beta}.$$

Therefore,

$$\frac{|\delta\Xi(s,u,t)|}{w(s,t)^{\beta}} \leq \sigma_{s,t}(\mathbf{X},\mathbf{Y}).$$

The conclusion follows from (5.3.2). □

In the next theorem we prove that the integral defined in Definition 5.3.3 exists when $\mathbf{X}$ and $\mathbf{Y}$ are as in the previous proposition. These types of results are often called *sewing lemmas* and its proof is reminiscent of the last part of the proof of Theorem 5.2.2. Notice that the statement below is quite abstract as it does not use the specific properties of $\mathbf{X}$ and $\mathbf{Y}$. This will allow us to re-use the statement in other contexts as well.

Proposition 5.3.6. *Suppose $\beta > 1$ and $\Xi : \Delta[0,T] \to \mathbb{R}^d$ satisfies (5.3.4) for some control function w. Then for each $s \leq t$ in $[0,T]$,*

$$\lim_{\substack{\mathcal{P}\in PP[s,t] \\ |\mathcal{P}|\to 0}} \sum_{[u,v]\in\mathcal{P}} \Xi(u,v) \tag{5.3.7}$$

exists. Moreover, if we denote the limit to be $\varphi(s,t)$, the function $\varphi : \Delta[0,T] \to \mathbb{R}^d$ is additive in the sense that $\varphi(0,s) + \varphi(s,t) = \varphi(0,t)$ for all $s \leq t$.

Proof. We will use the notations

$$\sigma_{s,t}(\Xi) = \sup_{u\in[s,t]} \frac{|\delta\Xi(s,u,t)|}{w(s,t)^{\beta}} \text{ and } \|\sigma(\Xi)\|_{\infty,\delta} = \sup_{0\leq t-s\leq\delta} \sigma_{s,t}(\Xi),$$

where $0 \le s \le t \le T$. Using these let us first rewrite (5.3.4) as follows: For each $\epsilon > 0$ there is a $\delta > 0$ such that whenever $s \le u \le t$ with $|t-s| \le \delta$, $\|\sigma(\Xi)\|_{\infty,\delta} < \epsilon$.

We will use the Cauchy criterion to prove the existence of the desired limit. Let $\mathcal{P}$ be a partition of $[s,t] \subseteq [0,T]$ having $r \ge 3$ points and $|\mathcal{P}| < \delta$. Then there are three consecutive points $u- < u < u+$ in $\mathcal{P}$ having the property that $w(u-,u+) \le \frac{2}{r-1} w(s,t)$ by Lemma 5.1.5. If $S(\Xi,\mathcal{P}) = \sum_{[u,v]\in\mathcal{P}} \Xi(u,v)$ then

$$\begin{aligned}|S(\Xi,\mathcal{P}) - S(\Xi,\mathcal{P}\setminus\{u\})| =&|\delta\Xi(u_-,u,u_+)| \le \|\sigma(\Xi)\|_{\infty,\delta} w(u-,u+)^\beta \\ \le&\frac{2^\beta \|\sigma(\Xi)\|_{\infty,\delta}}{(r-1)^\beta} w(s,t)^\beta.\end{aligned}$$

Hence by iterating this procedure as we did for Theorem 5.2.2 we arrive at the following:

$$\begin{aligned}|S(\Xi,\mathcal{P}) - \Xi(s,t)| \le&\|\sigma(\Xi)\|_{\infty,\delta} 2^\beta \left(\sum_{k=1}^{r-1} \frac{1}{k^\beta}\right) w(s,t)^\beta \\ \le&\|\sigma(\Xi)\|_{\infty,\delta} 2^\beta \zeta(\beta) w(s,t)^\beta. \end{aligned} \tag{5.3.8}$$

Now for any two partitions in $\mathcal{P}$ and $\mathcal{P}'$ of $[0,r] \subseteq [0,T]$, with $\mathcal{P} \subset \mathcal{P}'$ and $|\mathcal{P}'| < \delta$, we can write

$$\begin{aligned}|S(\Xi,\mathcal{P}') - S(\Xi,\mathcal{P})| =&\left|\sum_{[s,t]\in\mathcal{P}} \left(S(\Xi,\mathcal{P}'[s,t]) - \Xi(s,t)\right)\right| \\ \le&\|\sigma(\Xi)\|_{\infty,\delta} 2^\beta \zeta(\beta) \sum_{[s,t]\in\mathcal{P}} w(s,t)^\beta \\ \le&\|\sigma(\Xi)\|_{\infty,\delta} 2^\alpha \zeta(\beta) w(0,T)^\beta \\ =&O(\|\sigma(\Xi)\|_{\infty,\delta}) = O(\epsilon),\end{aligned}$$

where we used the super-additivity of w and the fact that $\beta > 1$ to get the second inequality. This proves that

$$\varphi(0,r) := \lim_{\substack{\mathcal{P}\in PP[0,r] \\ |\mathcal{P}|\to 0}} \sum_{[u,v]\in\mathcal{P}} \Xi(u,v)$$

exists for each $r \in [0,T]$. Similarly one can show the existence of $\varphi(s,t)$ when $0 \le s \le t \le T$. It is also easy to see that,

$$\varphi(0,t) - \varphi(0,s) = \lim_{\substack{\mathcal{P}\in PP[s,t] \\ |\mathcal{P}|\to 0}} S(\Xi,\mathcal{P}) = \varphi(s,t),$$

proving the additivity of φ required by the statement of the proposition. □

Theorem 5.3.7. *Suppose* $\mathbf{X}$ *is a p-variation cádlág rough path and* $\mathbf{Y}$ *is a p-variation X-controlled rough path for some $p \in [2,3)$. Then rough integral* $\int_0^t \mathbf{Y}_r \, d\mathbf{X}_r$*, as defined in* (5.3.1)*, exists.*

Proof. That the limit in (5.3.1) exists is a consequence of combining the propositions 5.3.5 and 5.3.6. By definition this is $\int_0^t \mathbf{Y}_r \, d\mathbf{X}_r$. □

To obtain an estimate of the type given in Lemma 5.2.4 in our present situation of integration driven by a cádlág rough path, we do the following familiar analysis. Let $\Xi(u,v) = Y_u(X_{u,v}) + Y'_u(\mathbb{X}_{u,v})$ and w be the control defined in (5.3.6). Assume that in a partition $\mathcal{P}$ of $[s,t]$, $u- < u < u+$ are three consecutive points satisfying $w(u-.u+) \leq \frac{2}{r-2}w(s,t)$ where $r = \#(\mathcal{P}) \geq 3$. Then,

$$
\begin{aligned}
|S(\Xi,\mathcal{P}) - S(\Xi,\mathcal{P}\setminus\{u\})| = |\delta\Xi(u-,u,u+)| &= \frac{|\delta\Xi(u-,u,u+)|}{w(u-,u+)^{\beta}} w(u-,u+)^{\beta} \\
&\leq \frac{2^{\beta}}{(r-2)^{\beta}} \sigma_{s,t}(\mathbf{X},\mathbf{Y}) w(s,t)^{\beta}.
\end{aligned}
$$

Iterating, we get

$$
|S(\Xi,\mathcal{P}) - \Xi(s,t)| \leq 2^{\beta}\zeta(\beta)\sigma_{s,t}(\mathbf{X},\mathbf{Y})w(s,t)^{\beta}.
$$

Now take the limit of the LHS as $|\mathcal{P}| \to 0$. We record this in the following lemma, for the sake of completeness.

Lemma 5.3.8 (Loeve-Young inequality)**.** *For* $\mathbf{X}$, $\mathbf{Y}$ *as in Theorem 5.3.7 and* $0 \leq s \leq t \leq T$, *we have*

$$
\left| \int_s^t \mathbf{Y}_r \, d\mathbf{X}_r - Y_s(X_{s,t}) - Y'_s(\mathbb{X}_{s,t}) \right| \leq C\sigma_{s,t}(\mathbf{X},\mathbf{Y})w(s,t)^{\beta}
$$

where w *is the control function defined in* (5.3.6).

The last two lemmas of this section give recipes for constructing new controlled rough paths from a given one. To simplify notations let us use $V = \mathbb{R}^d$ and $W = \mathcal{L}(R^d,\mathbb{R}^m) \cong \mathbb{R}^{m\times d}$ where m is a fixed natural number. Also, let $\mathcal{C}_b^2(W,\mathcal{L}(V,W))$ be the collection of bounded and twice continuously differentiable functions $W \to \mathcal{L}(V,W)$ with bounded derivatives of all orders.

Lemma 5.3.9. *Suppose* $\mathbf{X}$ *is a rough path and* $\mathbf{Y}$ *is an* X*-controlled rough path. If, for* $f \in \mathcal{C}_b^2(W,\mathcal{L}(V,W))$,

$$
f(Y)_t := f(Y_t) \in \mathcal{L}(V,W) \text{ and } f(Y)'_t := Df(Y_t)\circ Y'_t \in \mathcal{L}(V,\mathcal{L}(V,W)),
$$

then the pair $\mathbf{f}(\mathbf{Y}) := (f(Y), f(Y)')$ *defines an* X*-controlled rough path.*

Proof. We have to show that the $\mathbf{f}(\mathbf{Y})$ satisfies the conditions given in Definition 5.3.2. Let $\mathcal{P}$ be a partition of $[0,T]$. By the mean value theorem,

$$
\begin{aligned}
\sum_{[u,v]\in\mathcal{P}} |f(Y_v) - f(Y_u)|^p &\leq \sum_{[u,v]\in\mathcal{P}} \|Df(Y_\cdot)\|^p_{\infty;[u,v]} |Y_v - Y_u|^p \\
&\leq \|Df\|^p_{\infty;[0,T]} \sum_{[u,v]\in\mathcal{P}} |Y_v - Y_u|^p,
\end{aligned}
$$

where $\|Df(Y_\cdot)\|_{\infty;[u,v]}$ denotes the supremum of $\{|Df(Y_s)| \mid s \in [u,v]\}$. Hence $\|f(Y)\|_{p-var;[0,T]} \leq \|Df\|_{\infty;[0,T]}\|Y\|_{p-var;[0,T]} < \infty$. By Minkowski's inequality,

$$
\begin{aligned}
&\left[\sum_{[u,v]\in\mathcal{P}} |f(Y)'_v - f(Y)'_u|^p\right]^{\frac{1}{p}} = \left[\sum_{[u,v]\in\mathcal{P}} |Df(Y_v)\circ Y'_v - Df(Y_u)\circ Y'_u|^p\right]^{\frac{1}{p}} \\
\leq &\left[\sum_{[u,v]\in\mathcal{P}} |Df(Y_v)\circ Y'_v - Df(Y_u)\circ Y'_v|^p\right]^{\frac{1}{p}} + \left[\sum_{[u,v]\in\mathcal{P}} |Df(Y_u)\circ Y'_v - Df(Y_u)\circ Y'_u|^p\right]^{\frac{1}{p}}. \quad (5.3.9)
\end{aligned}
$$

Since

$$\begin{aligned}\sum_{[u,v]\in\mathcal{P}} |Df(Y_v)\circ Y'_v - Df(Y_u)\circ Y'_v|^p \leq & \sum_{[u,v]\in\mathcal{P}} |Df(Y_v)-Df(Y_u)|^p|Y'_v|^p \\ \leq & \|Y'\|^p_{\infty;[0,T]} \sum_{[u,v]\in\mathcal{P}} |Df(Y_v)-Df(Y_u)|^p \\ \leq & \|Y'\|^p_{\infty;[0,T]}\|D^2f\|^p_{\infty;[0,T]} \sum_{[u,v]\in\mathcal{P}} |Y_v-Y_u|^p\end{aligned}$$

and

$$\sum_{[u,v]\in\mathcal{P}} |Df(Y_u)\circ Y'_v - Df(Y_u)\circ Y'_u|^p \leq \|Df\|^p_{\infty;[0,T]} \sum_{[u,v]\in\mathcal{P}} |Y'_v - Y'_u|^p,$$

from (5.3.9) we have,

$$\|f(Y)'\|_{p-var;[0,T]} \leq \|Y'\|_{\infty;[0,T]}\|D^2f\|_{\infty;[0,T]}\|Y\|_{p-var;[0,T]} + \|Df\|_{\infty;[0,T]}\|Y'\|_{p-var;[0,T]},$$

which is finite by assumption. This proves the first part of Definition 5.3.2.

Now by definition, $R^{f(Y)}_{s,t} := f(Y_t)-f(Y_s)-f(Y)'_s(X_{s,t})$ and we have to show that $\|R^{f(Y)}\|_{p/2-var;[0,T]} < \infty$. Let $\mathcal{P}$ be a partition of $[0,T]$ and recall that $R^Y_{s,t} = Y_t - Y_s - Y'_s(X_{s,t})$. We have,

$$\begin{aligned}\sum_{[u,v]\in\mathcal{P}} |R^{f(Y)}_{u,v}|^{\frac{p}{2}} = & \sum_{[u,v]\in\mathcal{P}} |f(Y_v)-f(Y_u)-Df(Y_u)(Y'_u(X_{u,v}))|^{\frac{p}{2}} \\ = & \sum_{[u,v]\in\mathcal{P}} |f(Y_v)-f(Y_u)-Df(Y_u)(Y_{u,v})+Df(Y_u)(R^Y_{u,v})|^{\frac{p}{2}} \\ \leq & C_p\left[\sum_{[u,v]\in\mathcal{P}} |f(Y_v)-f(Y_u)-Df(Y_u)(Y_{u,v})|^{\frac{p}{2}} + \sum_{[u,v]\in\mathcal{P}} |Df(Y_u)(R^Y_{u,v})|^{\frac{p}{2}}\right] \\ \leq & C_p\left[\sum_{[u,v]\in\mathcal{P}} \left(\frac{1}{2}|D^2f(Y_u)||Y_{u,v}|^2\right)^{\frac{p}{2}} + \|Df\|^{\frac{p}{2}}_{\infty;[0,T]} \sum_{[u,v]\in\mathcal{P}} |R^Y_{u,v}|^{\frac{p}{2}}\right]\end{aligned}$$

by using Taylor series expansion of f. Hence,

$$\begin{aligned}&\|R^{f(Y)}\|^{\frac{p}{2}}_{p/2-var;[0,T]} \\ \leq & C_p\left[\|D^2f\|^{\frac{p}{2}}_{\infty;[0,T]}\left(\sup_{\mathcal{P}}\sum_{[u,v]\in\mathcal{P}} |Y_{u,v}|^p\right) + \|Df\|^{\frac{p}{2}}_{\infty;[0,T]}\left(\sup_{\mathcal{P}}\sum_{[u,v]\in\mathcal{P}} |R^Y_{u,v}|^{\frac{p}{2}}\right)\right] \\ = & C_p\left[\|D^2f\|^{\frac{p}{2}}_{\infty;[0,T]}(\|Y\|_{p-var;[0,T]}) + \|Df\|^{\frac{p}{2}}_{\infty;[0,T]}(\|R^Y\|_{p/2-var;[0,T]})\right] < \infty,\end{aligned}$$

and this proves the second part of Definition 5.3.2. □

Lemma 5.3.10. *If* Y *is an* X*-controlled rough path, then*

$$t \mapsto \left(\int_0^t \mathrm{Y}_r\, d\mathbf{X}_r, Y_t\right)$$

is also an X*-controlled rough path. Here the integral appearing in the definition above is a rough integral in the sense of Definition 5.3.3.*

Proof. Denote by $I_t = \int_0^t \mathbf{Y}_r \, d\mathbf{X}_r$. Then for a partition $\mathcal{P}$ of $[0,T]$,

$$\sum_{[u,v]\in\mathcal{P}} \left|I_{u,v}\right|^p = \sum_{[u,v]\in\mathcal{P}} \left|\int_u^v \mathbf{Y}_r \, d\mathbf{X}_r - \Xi(u,v) + \Xi(u,v)\right|^p$$

$$\leq C_p \left[\sum_{[u,v]\in\mathcal{P}} \left|\int_u^v \mathbf{Y} \, d\mathbf{X} - \Xi(u,v)\right|^p + \sum_{[u,v]\in\mathcal{P}} |\Xi(u,v)|^p \right] \tag{5.3.10}$$

By Lemma 5.3.8 we have

$$\begin{aligned}\sum_{[u,v]\in\mathcal{P}} \left|\int_u^v \mathbf{Y} \, d\mathbf{X} - \Xi(u,v)\right|^p &\leq C \sum_{[u,v]\in\mathcal{P}} \sigma_{u,v}(\mathbf{X},\mathbf{Y})^p w(u,v)^{p\beta} \\ &\leq C \sum_{[u,v]\in\mathcal{P}} w(u,v)^{p\beta}\end{aligned}$$

because $\sigma_{s,t}(\mathbf{X},\mathbf{Y})$ is bounded on $[0,T]$ by definition. As w is super-additive and $p\beta > 1$, it now follows that

$$\sum_{[u,v]\in\mathcal{P}} \left|\int_u^v \mathbf{Y} \, d\mathbf{X} - \Xi(u,v)\right|^p \leq Cw(0,T)^{p\beta}.$$

The second term in (5.3.10) is also bounded because $\Xi(u,v) = Y_u(X_{u,v}) + Y_u'(\mathbb{X}_{s,t})$ and the paths involved have finite p-variation (by hypothesis). These show that $\|I\|_{p-var;[0,T]} < \infty$. As we already know that $\|Y\|_{p-var;[0,T]} < \infty$, we have shown part (a) of Definition 5.3.2.

Now let $R^I_{s,t} := I_{s,t} - Y_s(X_{s,t})$. Again by Lemma 5.3.8,

$$\begin{aligned}\sum_{[u,v]\in\mathcal{P}} |R^I_{u,v}|^{\frac{p}{2}} &= \sum_{[u,v]\in\mathcal{P}} \left|\int_u^v \mathbf{Y} \, d\mathbf{X}_r - \Xi(u,v) + Y_u'(\mathbb{X}_{u,v})\right|^{\frac{p}{2}} \\ &\leq C \left[\sum_{[u,v]\in\mathcal{P}} \left|\int_u^v \mathbf{Y} \, d\mathbf{X}_r - \Xi(u,v)\right|^{\frac{p}{2}} + \sum_{[u,v]\in\mathcal{P}} |Y_u'(\mathbb{X}_{u,v})|^{\frac{p}{2}} \right] \\ &\leq C \left[\sum_{[u,v]\in\mathcal{P}} \sigma_{u,v}(\mathbf{X},\mathbf{Y})^{\frac{p}{2}} w(u,v)^{\frac{p\beta}{2}} + \sum_{[u,v]\in\mathcal{P}} |Y_u'(\mathbb{X}_{u,v})|^{\frac{p}{2}} \right]\end{aligned} \tag{5.3.11}$$

As $\sigma_{s,t}(\mathbf{X},\mathbf{Y})$ is bounded and $p\beta \geq 2$ (recall that $p \geq 2$ and $\beta > 1$), by super-additivity of w,

$$\sum_{[u,v]\in\mathcal{P}} \sigma_{u,v}(\mathbf{X},\mathbf{Y})^{\frac{p}{2}} w(u,v)^{\frac{p\beta}{2}} \leq C \sum_{[u,v]\in\mathcal{P}} w(u,v)^{\frac{p\beta}{2}} \leq Cw(0,T)^{\frac{p\beta}{2}}.$$

For the second term in (5.3.11), note that

$$\sum_{[u,v]\in\mathcal{P}} |Y_u'(\mathbb{X}_{u,v})|^{\frac{p}{2}} \leq \sum_{[u,v]\in\mathcal{P}} |Y_u'|^{\frac{p}{2}} |\mathbb{X}_{u,v}|^{\frac{p}{2}} \leq \|Y'\|^{\frac{p}{2}}_{\infty;[0,T]} \sum_{[u,v]\in\mathcal{P}} |\mathbb{X}_{u,v}|^{\frac{p}{2}}.$$

We therefore have, from (5.3.11),

$$\|R^I\|_{p/2-var;[0,T]} \leq C \left[w(0,T)^{\frac{p\beta}{2}} + \|\mathbb{X}\|_{p/2-var;[0,T]} \right] < \infty.$$

This proves that $t \mapsto \left(\int_0^t \mathbf{Y}_r \, d\mathbf{X}_r, Y_t\right)$ is also an X-controlled rough path. $\square$

5.4 Integration with respect to PRM

Let us now consider the stochastic integrals

$$\int_0^t \int_{\mathbb{R}^d} Y(s,x)\tilde{N}(ds,dx), t \geq 0, \tag{5.4.1}$$

where the integrand Y is a predictable real-valued function defined on $[0,T] \times \mathbb{R}^d \times \Omega$ and N is a Poisson random measure on $[0,\infty) \times \mathbb{R}^d$ with intensity $dt\, \mu(dx)$.

Let us recall an important connection between PRM and Lévy process. Let

$$X_t = \int_0^t \int_{\mathbb{R}^d} x\tilde{N}(ds,dx). \tag{5.4.2}$$

It is well-known that X is a Levy process. As X is a cádlág, it has a.s. at most finitely many jumps of size greater than 1 and therefore the integral

$$\int_0^t \int_{|x|\geq 1} Y(s,x)N(ds,dx) = \sum_{s\in[0,t]} Y(s,\Delta X_s)\mathbf{1}_{|x|\geq 1}(\Delta X_s) \tag{5.4.3}$$

is only a random finite sum. Here $\Delta X_s = X_s - X_{s-}$ denotes the size of the jump of X at time s. Thus it is enough to pay our attention to the integral over "small jumps", i.e.

$$Z_t := \int_0^t \int_{|x|<1} Y(s,x)\tilde{N}(ds,dx). \tag{5.4.4}$$

It is worth noting that, unlike (5.4.3), the above integral must be defined as a limit in probability.

At present we can only give a pathwise meaning to Z_t when the time and the space variables in the integrand Y can be separated. In the following, we shall always assume that

$$Y(t,x) = h(x)g(W_t), \quad t \geq 0, x \in \mathbb{R}^d \tag{5.4.5}$$

where $g \in \mathcal{C}_b^2(\mathbb{R}^d,\mathbb{R})$ (twice differentiable with bounded derivatives of all orders), W is an $\mathbb{R}^d$-valued cádlág adapted process and the function $h : \mathbb{R}^d \to \mathbb{R}$ is measurable satisfying the condition that

$$c_q = \int_{|x|<1} |h(x)|^q \mu(dx) < \infty. \tag{5.4.6}$$

for some $q \in (1,2]$. In this situation we have the following intuition.

$$Z_t := \int_0^t \int_{|x|<1} Y(s,x)\tilde{N}(ds,dx) = \int_0^t g(W_s)\, d\left(\int_{|x<1|} h(x)\tilde{N}(s,dx)\right). \tag{5.4.7}$$

Let us denote by X^h the process inside the round bracket above, i.e.

$$X_s^h := \int_0^s \int_{|x|<1} h(x)\tilde{N}(du,dx).$$

In the next section we briefly discuss the path properties of X^h.

Properties of X^h

From [App09, p. 122] we recall that X^h is defined as the following $L^2(\Omega)$ limit,

$$X_t^h = L^2(\Omega) - \lim_{n\to\infty} \int_{\frac{1}{n}<|x|<1} h(x)\tilde{N}((0,t],dx). \tag{5.4.8}$$

Since for each $n \in \mathbb{N}$, $t \mapsto \int_{1/n<|x|<1} h(x)\tilde{N}((0,t],dx)$ is a cádlág Levy process, [App09, Theorem 1.3.7] and the proof of [App09, Theorem 2.4.11] show that X^h has the same properties. Moreover, as X^h is the Itô integral with respect to a martingale measure, X^h is itself a martingale.

To understand the regularity properties of X^h, we state a result due to Manstavičius [Man04, Theorem 1.3] that connects the p-variation (as in Definition 5.1.1) of a strong Markov process (SMP) with its transition probabilities. Given a real-valued SMP $\{X_t\}_{t\in[0,T]}$ let us define the quantity,

$$\alpha(\eta,a) = \sup\{\mathbb{P}(|X_t - z| > a \mid X_s = z) \mid 0 \le s < t \le (s+\eta)\wedge T, z \in \mathbb{R}^d\} \tag{5.4.9}$$

where $\eta \in [0,T]$ and $a > 0$. For $\beta \ge 1$ and $\gamma > 0$, let $\mathcal{M}(\beta,\gamma)$ denote the class of all SMP X for which there exist constants $a_0 > 0$ and $K > 0$ such that

$$\alpha(\eta,a) \le K\frac{\eta^\beta}{(a\wedge a_0)^\gamma}$$

for all $\eta \in [0,T]$ and $a > 0$. Then we have the following result.

Theorem 5.4.1 ([Man04]). *Suppose X is in $\mathcal{M}(\beta,\gamma)$ for some $\beta \ge 1$, $\gamma > 0$. Then,*

$$\|X\|_{p-var;[0,T]} < \infty \quad a.s.,$$

for every $p > \frac{\gamma}{\beta}$.

Although this result is quite technical, we can summarize the broad strategy of its proof as follows. Given an SMP X satisfying the above hypothesis one first notices that it is possible to work with its cádlág modification, which we also denote by X. This allows us to give a (random) finite bound on the oscillations of its sample paths, say $M = M(\omega)$. Now, the p-variation norm of X can be bounded in the following manner. Given a partition $\mathcal{P} \in PP[0,T]$ and $r \in \mathbb{N}$ we define the set $K_r(\omega,\mathcal{P})$ containing those intervals of $\mathcal{P}$ on which the jump size of X is in $[2^{-r-1},2^{-r})$, i.e.

$$K_r(\omega,\mathcal{P}) = \{[u,v] \in \mathcal{P} \mid |X_v(\omega) - X_u(\omega)| \in [2^{-r-1},2^{-r})\}.$$

Let $r_1 \in \mathbb{N}$ be such that $2^{-r+3} \ge a_0$ (a_0 as above) for all $r \le r_1$. Then,

$$\begin{aligned}\|X(\omega)\|_{p-var}^p \le \sum_{r>r_1} \sup_{\mathcal{P}\in PP[0,T]} \sum_{[u,v]\in K_r(\omega,\mathcal{P})} |X_v(\omega) - X_u(\omega)|^p \\ + \sup_{\mathcal{P}\in PP[0,T]} \sum_{r\le r_1} \sum_{[u,v]\in K_r(\omega,\mathcal{P})} |X_v(\omega) - X_u(\omega)|^p,\end{aligned} \tag{5.4.10}$$

where the second term can be obtained using the fact that the outer sum is finite. Also note that the second term contains all the large jumps of the path $X(\omega)$. Since there can only be a finite number of such jumps, say $\nu_0(\omega)$, this term can be bounded by $(2M(\omega)\nu_0(\omega))^p$. On the other hand, showing finiteness for the first term of (5.4.10) requires more careful considerations. In [Man04], this is done by computing precise bounds to $\mathbb{E}\sup_{\mathcal{P}\in PP[0,T]}|K_r(\omega,\mathcal{P})|$ and concluding that the series

$$\sum_{r>r_1} 2^{-rp}\left(\sup_{\mathcal{P}\in PP[0,T]}|K_r(\omega,\mathcal{P})|\right)$$

is summable. Here $|K_r|$ denotes the cardinality of K_r. See [Man04, §3] for more details.

We can now use Theorem 5.4.1 to compute the p-variation norms for X^h. Recall the relationship between the function h and the exponent $q\in(1,2]$ from (5.4.6). By a slight abuse of notations, we will denote X^h by X when the corresponding h (and q) is clear from the context.

Proposition 5.4.2. *X^h has finite p-variation on $[0,T]$ for any $p>q$, almost surely.*

Proof. Since $X_t = X_t^h$ is a martingale, we can apply Burkholder's inequality. For all $t\in[0,T]$

$$\begin{aligned}
\mathbb{E}(|X_t|^q) \leq& \mathbb{E}\left[\left|\int_0^t\int_{|x|<1} h(x)^2 N(dr,dx)\right|^{q/2}\right]\\
\leq& \mathbb{E}\left[\left|\int_0^t\int_{|x|<1} h(x)^q N(dr,dx)\right|\right] \qquad (q/2\leq 1)\\
=& \mathbb{E}\int_0^t\int_{|x|<1} |h(x)|^q N(dr,dx)\\
=& tc_q,
\end{aligned}$$

which is finite by our assumption on h. Therefore for $z\in\mathbb{R}^d$, $\eta, a>0$ and $0<s<t\leq(s+\eta)\wedge T$,

$$\begin{aligned}
\mathbb{P}(|X_t - z|>a \mid X_s = z) =& \mathbb{P}_z(|X_{t-s}-z|>a) = \mathbb{P}(|X_{t-s}|>a)\\
\leq& \frac{\mathbb{E}(|X_{t-s}|^q)}{a^q} \leq c_q\frac{(t-s)}{a^q} \leq c_q\frac{\eta}{a^q}
\end{aligned} \tag{5.4.11}$$

using the fact that X is a Levy process and Markov's inequality. Since X is clearly strong Markov, Theorem 5.4.1 implies that $\|X\|_{p-var;[0,T]}<\infty$ a.s. for all $p>q$. □

Pathwise interpretation of Z in the case of separable variables

Let us recall our definition of $Y(t,x)$ from (5.4.5). Depending on the $q\in(1,2]$ for which h satisfies (5.4.6) we will consider Z_t (defined in (5.4.4)) either as a Young integral or as a rough integral.

When $1<q<2$ and $\|W\|_{p-var}<\infty$ a.s. for some $1\leq p<2$, by Lemma 5.4.2 we have $\|X^h\|_{q'-var}<\infty$ a.s. for all $q'>q$. Therefore, the integral

$$Z_t = \int_0^t g(W_s)\,dX_s^h$$

can be given a pathwise meaning in the Young sense by the virtue of Theorem 5.2.2.

Now suppose $q=2$. As the theory of Young integration is not available to us in this case, we must look to rough integrals for a pathwise understanding of Z. Indeed [CF19, Section 4] shows that, since $X \equiv X^h$ is a martingale, X has a rough-path lift $\mathbf{X}$. Therefore, it is possible to define Z in a pathwise manner when W is controlled rough path with respect to X.

For the sake of completeness let us explicitly define $\mathbf{X}$ and the controlled rough paths with respect to X. To keep our analysis simple we make the following assumption which is stronger than (5.0.1).

Assumption 1. There is a $\gamma \in (0, 1/2)$ such that,

$$c_{1+\gamma} := \int_{|x|<1} |h(x)|^{1+\gamma} \mu(dx) < \infty. \tag{5.4.12}$$

This condition holds for example, when μ is the Levy measure of an γ-stable process with $\gamma < 1/2$ and $h(x) = x$.

Now we define $\mathbf{X}$ and discuss some of its properties. For $s \leq t$ in $[0, T]$ let,

$$\mathbb{X}_{s,t} := \int_s^t (X_{u-} - X_s)\, dX_u, \tag{5.4.13}$$

where the above is a stochastic integral with respect to the martingale X.

Lemma 5.4.3. *We have the following properties,*

(a) Chen's relation holds for $(X, \mathbb{X})$, i.e. for $s < u < t$,

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = (X_u - X_s)(X_t - X_u). \tag{5.4.14}$$

(b) The maps $t \mapsto X_t$ and $t \mapsto \mathbb{X}_{0,t}$ are cádlág.

(c) For any $p \in (2 + 2\gamma, 3)$ we have $\|X\|_{p-var;[0,T]} < \infty$ and $\|\mathbb{X}\|_{p/2-var;[0,T]} < \infty$.

The first and second statements are trivial. The last statement follows from a calculation similar to Proposition 5.4.2. We therefore have a cádlág rough path $\mathbf{X} = (X, \mathbb{X})$ in the sense of Definition 5.3.1.

Suppose $\mathbf{W} = (W, W')$ is a controlled rough path associated with $\mathbf{X}$ as in Definition 5.3.2. Then we record our observation from (5.4.7) in the form of the following result.

Theorem 5.4.4. *Let $Y(t, x) = h(x)g(W_t)$ where $h : \mathbb{R}^d \to \mathbb{R}$ is a measurable function satisfying (5.4.12), $g : \mathbb{R}^d \to \mathbb{R}$ is in $\mathcal{C}_b^2$ and (W, W') is as above. Then*

$$Z_t = \int_0^t \int_{|x|<1} Y(u, x)\tilde{N}(du, dx) = \int_0^t g(W_u)\, dX_u$$

exists a.s. as a rough integral in the sense of Definition 5.3.3.

The proof of this result follows directly from Lemma 5.3.9 and Theorem 5.3.7.

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