SPECTRAL ASPECT SUBCONVEX BOUNDS FOR SOME L-FUNCTIONS

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Abstract. In this dissertation, we will prove subconvex bounds for GL(3) × GL(2) L-functions in the GL(2) spectral aspect.

1. Introduction

Let ζ be the Riemann zeta function defined by the following Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathfrak{Re}(s) > 1.$$

This function was extensively studied by B. Riemann in his famous memoir in 1860. It is well-known that this function has a meromorphic continuation with a simple pole at $s = 1$. In his memoir, Riemann proposed a conjucture, famously known as the Riemann hypothesis, about the 'non-trivial' zeros of ζ. It asserts that all the 'non-trivial' zeros lie on the 'critical' line $\mathfrak{Re}(s) = 1/2$. It has many remarkable applications in number theory. One of the main consequences of the Riemann hypothesis is the Lindelöf hypothesis which asserts that

$$\zeta(1/2 + it) \ll_\epsilon (1 + |t|)^\epsilon,$$

for any $\epsilon > 0$, where the symbol $A \ll_\epsilon B$ means $|A| \leq C(\epsilon)B$, for some constant $C(\epsilon)$ depending on ϵ only. We will keep using this convention throughtout the synopsis.

A similar phenomenon occurs for the Dirichlet L-functions $L(s, \chi)$, where χ is a Dirichlet character or more generally for any higher degree 'automorphic' L-functions $L(s, F)$, where F is some higher degree automorphic form. In fact, $\zeta(s)$ and $L(s, \chi)$ can be thought of as degree one automorphic L-function. It is one of the important and sought after problem to understand the size of $L(s, f)$ on the critical line $\mathfrak{Re}(s) = 1/2$. Using the 'functional equation' and the Phragmen Lindelöf convexity principle, it follows that

$$L(1/2 + it, F) \ll_\epsilon \mathcal{C}|^{1/4+\epsilon}, \quad \epsilon > 0, \tag{1}$$

where $\mathcal{C}$ is a function of t and other parameters (level, spectral parameters etc.) of the form f, known as the analytic conductor. The above bound is known as 'Convexity' bound. We expect (1) to be of the form (the generalized Lindelöf hypothesis)

$$L(1/2 + it, F) \ll_\epsilon |\mathcal{C}|^\epsilon.$$

However even getting an exponent of the form $1/4 - \delta$ for some $\delta > 0$, known as subconvexity exponent, is very challanging and out of reach (in most scenarios). Moreover, as the 'degree' of the L-function gets higher and higher, getting these bounds become more difficult. Mostly, in applications, one seeks subconvexity estimates with respect to some subfamily (i.e. only one of the parameters t, level or spectral parameters varies). If t varies (F is fixed), we call it t-aspect, if level varies, we call it level aspect and if spectral parameters vary, we call it spectral aspect.

For degree one L-functions subconvexity bounds were proved by H. Weyl, Hardy-Littlewood in the t-aspect and by Burgess in the level aspect. For degree two L-functions, these bounds are due to A. Good in the t-aspect, Duke-Friedlander-Iwaniec in level aspect and by Michel-Venkatesh in all aspects. For degree three L-functions, first t-aspect subconvex bounds were proved by Li [3] for self-dual forms. This result was generalised by Munshi to any GL(3) form using the delta symbol approach. In the spectral aspect, Blomer-Buttcane [1] resolved the subconvexity problem for those GL(3) forms whose spectral parameters are in 'generic' positions. For degree four onwards, sub-convexity problem is mostly open (except the Rankin-Selberg L-functions). We give a more detailed history in the introduction chapter, Chapter 0, of the thesis. For degree six L-functions (Rankin-Selberg L-functions associated to GL(3) and GL(2) forms), the first subconvex bound was proved by Li [3] in the GL(2) spectral aspect (for self-dual GL(3) forms) and by Munshi [5] in the t-aspect. In this thesis, we consider the subconvexity problem for GL(3) $\times$ GL(2) L-functions in the GL(2) spectral aspect.

2. Statement of results

Let π be a Hecke-Maass cusp form of type (ν_1, ν_2) for $SL(3, \mathbb{Z})$ with the Langlands parameters $(\alpha_1, \alpha_2, \alpha_3)$ and f be a holomorphic cusp form with weight k_f or a Hecke-Maass cusp form corresponding to the Laplacian eigenvalue $1/4 + \nu_f^2$, $\nu_f \geq 1$, for $SL(2, \mathbb{Z})$. The associated Rankin–Selberg L-series is given by

$$L(s, \pi \times f) = \sum\sum_{n,r \geq 1} \frac{\lambda_\pi(n,r)\, \lambda_f(n)}{(nr^2)^s}, \quad \Re(s) > 1. \tag{2}$$

2.1. **GL(2) spectral aspect.** As a first result, we prove the following theorem in this thesis, which gives the GL(2) spectral aspect subconvexity bound for the Rankin-Selberg L-functions associated to π and f.

Theorem 1. *Let π and f be as above. Let $\nu_f \asymp k_f$. Then we have*

$$L\left(1/2, \pi \times f\right) \ll_{\pi,\epsilon} k_f^{3/2-1/51+\epsilon}, \tag{3}$$

for any $\epsilon > 0$.

The above theorem generalises Li's GL(2) spectral aspect result [3] to all GL(3) forms.

3. Discussion on proof

The method of proofs of all the above results is motivatived by Munshi's t-aspect result [5], which we discuss briefly.

3.1. **Munshi's approach.** Munshi applied the delta method approach, which he developed in a series of articles, along with the conductor lowering trick which he introduced in [4]. More specifically, he used the delta method of Duke, Friedlander and Iwaniec (DFI) to separate the oscillatory factors. The key input in this paper was his observation that the character sum, emerging after the summation formulae, essentially boils down to an additive character, which is very specific to Rankin–Selberg convolutions of the type GL(n + 1) × GL(n).

We recall from (**??**) that Munshi proves the following result

$$L\left(1/2+it,\pi\times f\right)\ll_{\epsilon,f,\pi}(1+|t|)^{3/2-1/51+\epsilon},$$

for any $\epsilon>0$. Upon using the functional equation, the problem boils down to getting some cancellations in the following sum

$$\sum_{n\sim N}\lambda_\pi(n,1)\lambda_f(n)n^{-it},\quad N\asymp t^3.$$

He initiates the proof by applying the DFI delta method (to separate $\lambda_\pi(n,1)$ and $\lambda_f(n)n^{-it}$) along with the conductor lowering trick (to reduce the modulus in the DFI). Thus he ends up into

$$\int_K^{2K}\sum_{q\sim Q}\sum_{a\bmod q}^{\star}\sum_{n\sim N}\lambda_\pi(n,1)n^{iv}e\left(\frac{an}{q}\right)\sum_{m\sim N}\lambda_f(m)\,m^{-i(t+v)}e\left(\frac{-am}{q}\right)\mathrm{d}v,$$

where K is a parameter $K<t$ which he chooses optimally later and $Q=t^{3/2}/K$. Here the situation seems to be worse a priori, as we have lost N in the above sum. However he gains structurely and he manages to gain it back later. In the second step he applies summation formulae to the sum over n and m, and he saves $\sqrt{NK}/t$ in the m-sum and $N^{1/4}/K^{3/4}$ in the n-sum. Then he analyses the v-integral in which he gets square-root cancellations, in other words, he saves $\sqrt{K}$. The analysis of the a-sum also gives square-root cancellations and he saves $\sqrt{q}$ from it. Hence in total he saves N/t so far and he is left with the following sum

$$\sum_{q\sim Q}\sum_{n\sim K^{3/2}N^{1/2}}\lambda_\pi(1,n)\sum_{m\sim t^2/K}\lambda_f(m)\mathfrak{C}\,\mathfrak{I},$$

in which he needs to save t and a bit more, say, t^η. Here $\mathfrak{I}$ is an integral transform which oscillates like n^{iK} with respect to n, and the character sum $\mathfrak{C}$ is given by

$$\mathfrak{C}=\sum_{a\bmod q}^{*}S\left(\bar{a},n;q\right)\,e\left(\frac{\bar{a}m}{q}\right)\approx qe\left(-\frac{\bar{m}n}{q}\right).$$

Next he applies Cauchy to break the involution and arrives at

$$\sum_{n\sim K^{3/2}N^{1/2}}\left|\sum_{q\sim Q}\sum_{m\sim t^2/K}\lambda_f(m)e\left(-\frac{\bar{m}n}{q}\right)\mathfrak{I}\right|^2,$$

in which t^2 (plus extra) is needed to be saved. In the end game strategy, we applies the Poisson summation formula to the sum over n. In the zero frequency he saves t^2Q/K which is more then t^2 provided $K<t$. In the non-zero frequencies, he saves $K^{3/2}N^{1/2}/(\sqrt{Q^2K})\times Q$, which is good enough if $K>t^{1/2}$. Thus he succeeds by chosing K between $\sqrt{t}$ and t. Notice that there is an extra Q in the saving of the non-zero frequencies. It is a crucial factor which he obtains due to the additive (with respect to n) character $e\left(-\bar{m}n/q\right)$, which comes due to the $\mathrm{GL}(3)\times\mathrm{GL}(2)$ structure. This is the key input in this paper.

3.2. **Our approach.** On applying the functional equation, our problem boils down to getting cancellations in

$$S(N)=\sum_{n\sim N}\lambda_\pi(n,1)\lambda_f(n),$$

where $N\asymp k_f^3$ in Theorem 1 and $N\asymp T^3$ in Theorem **??** and Theorem **??**. To prove Theorem 1, following Munshi, we apply DFI delta method to separate $\lambda_\pi(n,1)$ and $\lambda_f(n)$ along with the conductor lower trick. Then applications of summation formulae followed by Cauchy and Poisson gives us the result. We also get the structural advantage of the $\mathrm{GL}(3)\times\mathrm{GL}(2)$ type and hence we are able to save more (then the usual) in the Poisson. The main technical input of this theorem is to get square-root cancellations in the integral transforms. Indeed, after summation formule, the integral transform (for f hololomorphic) looks like

$$\mathrm{I}=\int U(y)e(\mathfrak{a}y^{1/3})J_{k_f-1}(\mathfrak{b}\sqrt{y})\,\mathrm{d}y,$$

where $\mathfrak{a}\asymp\mathfrak{t}$, for some $\mathfrak{t}<k_f$ to be chosen later, $\mathfrak{b}\asymp k_f$ and U is a smooth bump function supported on $[1/2,5/2]$. Here the argument $\mathfrak{b}y^{1/2}$ of the Bessel function is in 'transitional range' ($\mathfrak{b}\asymp k_f$), in which case, a 'nice' asymptotic expansion (uniform in k_f) is not known. We get desired cancellations ($\mathrm{I}\ll 1/k_f$) using the integral representation

$$J_{k-1}(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e\left(\frac{(k-1)\tau-x\sin\tau}{2\pi}\right)\mathrm{d}\tau,$$

followed by a chain of stationary phase analysis. We discuss full details of the proof in Chapter 2.

References

[1] V. Blomer; J. Buttcane: *On the subconvexity problem for L-functions on GL(3)*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 6, 1441–1500.

[2] Sumit Kumar: *Subconvexity bounds for GL(3)×GL(2) L-functions in GL(2) spectral aspect)*, arXiv: 2007.05043.

[3] X. Li: *Bounds for GL(3) × GL(2) L-functions and GL(3) L-functions*, Annals of Math. **173** (2011), 301-336.

[4] R. Munshi: *The circle method and bounds for L-functions-III*, Journal of the American Mathematical Society **28** (2015), 913-938.

[5] R. Munshi: *Subconvexity for GL(3) × GL(2) L-functions in t-aspect*, J. Eur. Math. Soc. **23** (2021), no. 1. DOI: 10.4171/JEMS/1131.