Equivariant Homology Decompositions for Projective Spaces and Associated Results

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Dedicated to my mother and my sisters

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Contents

Notations and Abbreviations

Throughout this document, G denotes the cyclic group of order n. The notation p is used for a prime. The unit sphere of an orthogonal G-representation V is denoted by $S(V)$, the unit disk by $D(V)$, and S^V the one-point compactification $\cong D(V)/S(V)$. We write $1_{\mathbb{C}}$ for the trivial complex representation and 1 for the real trivial representation, and ρ or $\rho_{\mathbb{C}}$, in the complex case or $\rho_{\mathbb{H}}$, in the quaternionic case, to denote the regular representation. The irreducible complex representations of G are 1-dimensional, and up to isomorphism are listed as $1_{\mathbb{C}}, \lambda, \lambda^2, \ldots, \lambda^{n-1}$ where λ sends the generator of G to $e^{2\pi i/n}$, the n^{th} root of unity. We sometimes use the notation λ^0 instead of $1_{\mathbb{C}}$. The non-trivial real irreducible representations are the realizations of these. The realization of λ^i is also denoted by the same notation. Note that λ^i and λ^{n-i} are conjugate and hence their realizations are isomorphic by the natural $\mathbb{R}$-linear map $z \mapsto \bar{z}$ which reverses orientation. Unless specified, the cohomology groups are taken with $\underline{\mathbb{Z}}$-coefficients and suppressed from the notation. The notation $\mathcal{U}$ is used for the complete G-universe.

Introduction

The purpose of this thesis is to discuss new calculations for the equivariant cohomology of complex projective spaces. Given a complex representation V of a group G, one obtains a "linear" G-action on $P(V)$ = the space of lines in V. The underlying space here is $\mathbb{C}P^{\dim(V)-1}$ whose homology computation is well-known. The Borel-equivariant cohomology, which is the cohomology of the Borel construction, is easy to calculate as the space $P(V)$ has non-empty fixed points.

The equivariant cohomology used in this paper refers to an "ordinary" cohomology theory represented in the equivariant stable homotopy category [34, Ch. XIII §4]. In this context, the word "ordinary" means that the equivariant homotopy groups of the representing spectrum HM are concentrated in degree 0. At this point, one notes that the equivariant stable homotopy category possesses additional structure which makes the equivariant homotopy groups part of a Mackey functor [34, Ch. IX §4]. Conversely, given a Mackey functor $\underline{M}$, one has an associated "ordinary" equivariant cohomology theory represented by an Eilenberg-MacLane spectrum $H\underline{M}$ [16, Theorem 5.3], which is unique up to homotopy.

A Mackey functor $\underline{M}$ [11] comprises a pair of functors $(\underline{M}_*, \underline{M}^*)$ from the category $\mathcal{O}_G$ (of finite G-sets) to Abelian groups, such that $\underline{M}_*$ is covariant and $\underline{M}^*$ is contravariant, taking the same value on a given G-set S, denoted $\underline{M}(S)$. These are required to be compatible in the sense of a double coset formula [37]. The covariant structure gives restriction maps

$$\mathrm{res}^H_K : \underline{M}(H) \to \underline{M}(K), \text{ for } K \subset H,$$

and the contravariant structure gives transfer maps

$$\mathrm{tr}^H_K : \underline{M}(K) \to \underline{M}(H), \text{ for } K \subset H.$$

The two important examples in the context of equivariant cohomology are the Burnside ring Mackey functor $\underline{A}$, and the constant Mackey functor $\underline{\mathbb{Z}}$. The Mackey functor $\underline{\mathbb{Z}}$ sends each $G/H \mapsto \mathbb{Z}$, with $\mathrm{res}^H_K = \mathrm{Id}$ and tr^H_K = multiplication by the index $[H : K]$. The Burnside ring Mackey functor $\underline{A}$ sends $G/H \mapsto A(H)$, the ring generated by isomorphism classes of finite H-sets. The restriction maps res^H_K for $\underline{A}$ are described as the restriction of the action of H to K, and the transfer maps tr^H_K are described by induction $S \mapsto H \times_K S$.

The category of Mackey functors is an Abelian category. It also has a symmetric monoidal structure given by $\square$, whose unit object is the Burnside ring Mackey functor $\underline{A}$. The constant Mackey functor $\underline{\mathbb{Z}}$ is also a commutative monoid. The cohomology groups associated to the commutative monoids possess a graded commutative ring structure. This means that $H\underline{A}$ is a homotopy commutative ring spectrum, and the commutative monoids give homotopy commutative $H\underline{A}$-algebras. However, if one tries to rigidify the construction of Eilenberg-MacLane spectra into a functor taking values in equivariant orthogonal spectra, there are obstructions coming from norm maps [38, 20].

Our focus is on computations of equivariant cohomology for G-spaces. A simple-minded approach would be to break up the spaces into equivariant cells, and compute via cellular homology. An equivariant G-CW complex has cells of the form $G/H \times D^n$, which are attached along maps from $G/H \times S^{n-1}$ onto lower skeleta. Via this argument, one shows $H^n_G(X; \underline{\mathbb{Z}}) \cong H^n(X/G; \mathbb{Z})$. While working through concrete examples like the projective spaces $P(V)$, or for G-manifolds, we see that there is no systematic way of breaking these up into cells of the form $G/H \times D^n$ or identifying the space X/G. In these cases, the spaces may be naturally built out of cells of the form $G \times_H D(V)$[40], where V is a unitary H-representation, and $D(V)$ stands for the unit disk

$$D(V) = \{v \in V \mid \langle v, v\rangle \leq 1\}.$$

In the equivariant stable homotopy category, the representation spheres S^V (defined as the one-point compactification of V) are invertible in the sense that there is S^{-V}, such that $S^{-V} \wedge S^V \simeq S^0$. Consequently, the equivariant cohomology becomes $RO(G)$-graded [34, Ch. XIII], and so, computations for the G-CW complexes above require the knowledge of $H^\alpha_G(G/H; \underline{M})$ for $\alpha \in RO(G)$ and $H \subset G$. Such computations exist in the literature only for a handful of finite groups, namely, for $G = C_p$ for p prime [30], $G = C_{p_1 \cdots p_k}$ for distinct primes p_i [6, 7], and $G = C_{p^2}$ [43]. For restricted α belonging to certain sectors of $RO(G)$ more computations are known [20, 22, 27, 3].

0.1 Homology of projective spaces

Equivariant homology and cohomology of G-spaces have been carried out for many of the groups above [19, 26]. Most of these computations are done in the case where the complete calculations for $H^\alpha_G(G/H; \underline{\mathbb{Z}})$ are known. Many of these occur in cases where the cohomology is a free module over $\pi_\bigstar H\underline{\mathbb{Z}}$. There are various structural results which imply the conclusion that the cohomology is a free module [33, 8]. For the equivariant projective spaces, this is particularly relevant, and although, the entire knowledge of $\pi_{\bigstar}^{C_n} H\underline{\mathbb{Z}}$ is still unknown, we are able to prove that the cohomology $P(V)$ is free when V is a direct sum of copies of the regular representation. (see Theorems 3.2.1 and 3.4.1)

Theorem 0.1.1. Let $G = C_n$. We have the following decompositions.
a) Write $\phi_0 = 0$ and $\phi_i = \lambda^{-i}(1_{\mathbb{C}} + \lambda + \lambda^2 + \cdots + \lambda^{i-1})$ for $i > 0$. Then,

$$H\underline{\mathbb{Z}} \wedge P(m\rho_{\mathbb{C}})_+ \simeq \bigvee_{i=0}^{nm-1} H\underline{\mathbb{Z}} \wedge S^{\phi_i},$$

$$H\underline{\mathbb{Z}} \wedge B_G S^1_+ \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{\phi_i}.$$

b) For the quaternionic case, we have for $W_k = \lambda^{-k}\Big(\sum_{i=0}^{k-1}(\lambda^i + \lambda^{-i})\Big)$,

$$H\underline{\mathbb{Z}} \wedge P_{\mathbb{H}}(m\rho_{\mathbb{H}})_+ \simeq \bigvee_{i=0}^{mn-1} H\underline{\mathbb{Z}} \wedge S^{W_i},$$

$$H\underline{\mathbb{Z}} \wedge B_G S^3_+ \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{W_i}.$$

In the above expression, λ refers to the one-dimensional complex C_n-representation which sends a fixed generator g to $e^{\frac{2\pi i}{n}}$, and it's powers are taken with respect to the complex tensor product. The second implications above come from the identification $B_G S^1 \simeq P(\mathcal{U}) \simeq \varinjlim_m P(m\rho)$, and $B_G S^3 \simeq P_{\mathbb{H}}(\mathbb{H} \otimes_{\mathbb{C}} \mathcal{U}) \simeq \varinjlim_m P_{\mathbb{H}}(m\rho_{\mathbb{H}})$, where $\mathcal{U}$ is a complete G-universe. We also carry out the computation for general V when the group G equals C_p (Theorem 3.1.2). One may view this as a simplification of the results in [30] in the case of the Mackey functor $\underline{\mathbb{Z}}$. For the group C_2, we consider $\mathbb{C}P^n_\tau$ where C_2 acts on $\mathbb{C}P^n$ by complex conjugation, and compute it's equivariant homology (Theorem 3.3.1). In this context, one should note that results such as the above theorem are not expected for $\underline{A}$-coefficients once the group contains either C_{p^2} or $C_p \times C_p$ [31, Remark 2.2],[12].

As an application for the homology decomposition in Theorem 0.1.1 , we reprove a theorem of Caruso [9] stating that the cohomology operations expressable as a product of the even degree Steenrod squares over $\mathbb{Z}/2$ do not occur as restriction of integer degree C_2-equivariant cohmology operations. If we allow the more general $RO(C_2)$-graded operations, there are those that restrict to Sq^i for every i [39]. For the group C_p, the same result holds for the products of the Steenrod powers P^i.

A careful analysis of the cellular filtration of the projective spaces $P(V)$ for $V = m\rho$ shows that the induced filtration on $\Sigma^2 H\underline{\mathbb{Z}} \wedge P(V)$ matches the slice filtration. The slice tower was defined as the equivariant analogue of the Postnikov tower using the localizing subcategory generated by $\{G/H \wedge S^{k\rho_H} \mid k|H| \geq n\}$ instead of the spheres of the form $\{G/H_+ \wedge S^n\}$ (Here ρ_H is the regular representation of H). The slice filtration played a critical role in the proof of the Kervaire invariant one problem [20] and has been widely studied since. Usually, the slice tower is an involved computation even for the spectra $\Sigma^n H\underline{\mathbb{Z}}$. However, for the complex projective spaces and the quaternionic projective spaces, we discover that the slice tower for the $\underline{\mathbb{Z}}$-homology becomes amazingly simple. More precisely, we prove the following theorem in this regard. (see Theorems 3.6.4 and 3.6.5)

Theorem 0.1.2. Let $G = C_n$.
a) The slice towers of $\Sigma^2 P(\mathcal{U})_+ \wedge H\underline{\mathbb{Z}}$ and $\Sigma^2 P(m\rho)_+ \wedge H\underline{\mathbb{Z}}$ are degenerate and these spectra are a wedge of slices of the form $S^V \wedge H\underline{\mathbb{Z}}$.
b) The slice towers of $\Sigma^4 P(\mathcal{U}_{\mathbb{H}})_+ \wedge H\underline{\mathbb{Z}}$ and $\Sigma^4 P(m\rho_{\mathbb{H}})_+ \wedge H\underline{\mathbb{Z}}$ are degenerate and these spectra are a wedge of slices of the form $S^V \wedge H\underline{\mathbb{Z}}$.

0.2 The cohomology ring structure

We proceed to describe the cohomology ring structure for the projective spaces. The basic approach towards the computation comes from [30], which deals with the case $G = C_p$. In this case, we observe that the generators are easy to describe, but the relations become complicated once the order of the group increases. For $G = C_n$, we show that $H^\bigstar_G(P(V);\underline{\mathbb{Z}})$ are are multiplicatively generated by classes α_{ϕ_d} for $d \mid n$, in degree $\sum_{i=-d}^{-1} \lambda^i$ (Proposition 5.1.2). However, the relations turn out to be difficult to write down in general, so we restrict our attention to prime powers n. In the process of figuring out the generators, we realize that there are exactly m relations ρ_j for $1 \leq j \leq m$ ($n = p^m$), of the form

$$u_{\lambda^{p^{j-1}} - \lambda^{p^j}} \alpha_{\phi_{p^j}} = \alpha^p_{\phi_{p^{j-1}}} + \text{ lower order terms.}$$

The explicit form turns out to be quite involved with $\underline{\mathbb{Z}}$-coefficients, so we determine them modulo p, and prove the following results. (see Theorems 5.3.6, 5.4.1, and 5.5.1)

Theorem 0.2.1. a) $H^\bigstar_{C_2}(\mathbb{C}P^\infty_\tau;\underline{\mathbb{Z}}) \cong H^\bigstar_{C_2}(\mathrm{pt})[\epsilon_{1+\sigma}]$.
b) The cohomology ring

$$H^\bigstar_G(B_G S^1;\underline{\mathbb{Z}/p}) \cong H^\bigstar_G(\mathrm{pt};\underline{\mathbb{Z}/p})[\alpha_{\phi_0},\cdots,\alpha_{\phi_m}]/(\rho_1,\cdots,\rho_m).$$

The relations ρ_r are described by

$$\rho_r = u_{\lambda^{p^{r-1}} - \lambda^{p^r}} \alpha_{\phi_{p^r}} - \mathcal{T}^p_{r-1} + a^{p-1}_{\lambda^{p^{r-1}}} \mathcal{T}_{r-1} (\prod_{i=0}^{r-2} \mathcal{A}_i)^{p-1},$$

where $\mathcal{T}_j$ and $\mathcal{A}_j$ are defined in (5.3.3).

c) The cohomology ring

$$H^{\bigstar}_G(B_G S^3; \underline{\mathbb{Z}/p}) \cong H^{\bigstar}_G(\mathrm{pt}; \underline{\mathbb{Z}/p})[\beta_{2\phi_0}, \cdots, \beta_{2\phi_m}]/(\mu_1, \cdots, \mu_m).$$

The relations μ_r are described by

$$\mu_r = (u_{\lambda^{p^{r-1}} - \lambda^{p^r}})^2 \beta_{2\phi_{p^r}} - \mathcal{L}^p_{r-1} + a^{2(p-1)}_{\lambda^{p^{r-1}}} \mathcal{L}_{r-1} \big(\prod_{i=0}^{r-2} \mathcal{C}_i \big)^{p-1},$$

where $\mathcal{L}_j$ and $\mathcal{C}_i$ are defined in Theorem 5.4.1.

0.3 Homology of connected sums

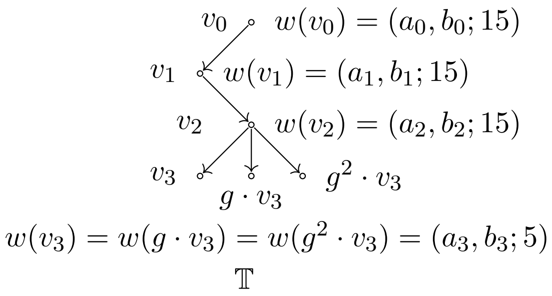
Simply connected 4-manifolds form an important category of spaces from the point of view of both topologists and geometers. Their homotopy type is determined by the intersection form. The ones with positive definite intersection form are homotopy equivalent to a connected sum of copies of $\mathbb{C}P^2$. We study the equivariant homotopy type of certain cyclic group actions on these 4-manifolds defined in [18], and splitting results for the equivariant homology with constant coefficients.

Recall that a simply connected 4-manifold M possesses a CW-complex structure whose 2-skeleton is a wedge of spheres, and outside the 2-skeleton, there is a single 4-cell. It follows that the homology is torsion-free, and non-zero in only three degrees 0, 2 and 4, with $H_0(M) = \mathbb{Z}$ and $H_4(M) = \mathbb{Z}$. If k is the second Betti number of M, in the stable homotopy category, we obtain the decomposition $H\mathbb{Z} \wedge M_+ \simeq H\mathbb{Z} \vee (\bigvee_{i=1}^k \Sigma^2 H\mathbb{Z}) \vee \Sigma^4 H\mathbb{Z}$.

Equivariant homology decompositions have been studied extensively for spaces built up using even cells of the type $G \times_H D(V)$, specially in the case $G = C_p$ or C_n for n square free. We prove decomposition results for cyclic group actions on a connected sum of copies of $\mathbb{C}P^2$ with constant $\mathbb{Z}$-coefficients. The homology decompositions for X are usually proved by building up a cellular filtration of X, and then showing that after smashing with the spectrum $H\underline{\mathbb{Z}}$, the connecting maps are all trivial. For this purpose, the cells are taken of the form $D(V)$, a disk in a unitary G-representation V, so that the filtration quotients are wedges of S^V, the one-point compactification of V.

The G-manifolds $X(\mathbb{T})$ are defined using *admissible weighted trees* $\mathbb{T}$ [18], which are directed rooted trees with G-action, with each vertex carrying a weight comprising 3 integers a, b, m of gcd 1 (see figure below). The underlying manifolds $\mathbb{C}P^2(a, b; m)$ are copies of $\mathbb{C}P^2$, which have an action of the group C_m, by identifying $\mathbb{C}P^2$ as the space of complex lines in a three-dimensional complex representation of C_m. The numbers a, b are used to describe the irreducible representations therein. We fix λ as a complex

1-dimensional representation where C_m acts via m^{th}-roots of unity, and in these terms $\mathbb{C}P^2(a,b;m) = P(\lambda^a + \lambda^b + 1_{\mathbb{C}})$. The admissible part of the definition of the tree allows us to construct the equivariant connected sum in the figure.



$$X(\mathbb{T}) = \Big[\mathbb{C}P^2(a_0,b_0;15) \#\mathbb{C}P^2(a_1,b_1;15)\#\mathbb{C}P^2(a_2,b_2;15)\Big] \#C_{15}\times_{C_5}\mathbb{C}P^2(a_3,b_3;5)$$

$$C_{15} = \langle g \mid g^{15} = 1\rangle$$

FIGURE 0.3.0: Example of an equivariant connected sum defined using a tree.

For a cyclic group C_m of odd order, we prove two decomposition results (see Theorem 4.2.5 and Theorem 4.2.9), where $\Sigma^V H\underline{\mathbb{Z}}$ denotes $H\underline{\mathbb{Z}} \wedge S^V$. In the theorem below, the notation $\mathbb{T}_0$ stands for the C_m-fixed points of $\mathbb{T}$, and $\mathbb{T}_d$ refers to the vertices whose stabilizer is C_d for $d \mid m$.

Theorem 0.3.1. a) If $\mathbb{T}$ is an admissible weighted tree such that all fixed vertices v with weight $w(v) = (a_v, b_v; m_v)$ satisfy $\gcd(a_v - b_v, m_v) = 1$, then

$$\begin{aligned} H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0}+\lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\mathbb{T}_0} \Sigma^{\lambda} H\underline{\mathbb{Z}}\Big) \\ \vee \Big(\bigvee_{[\mu]\in\mathbb{T}_d/C_m, d\neq m} C_m/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}}\Big) \end{aligned}$$

where $w(v_0) = (a_0, b_0; m)$ is the weight of the root vertex.
b) Let $\mathbb{T}$ be an admissible weighted tree such that for the root vertex v_0 with weight $w(v_0) = (a_0, b_0; m)$, one of a_0 or b_0 is zero. Then,

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{[\mu]\in\mathbb{T}_d/C_m} C_m/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}}\Big).$$

For example in Figure 0.3.0, if $\gcd(a_i - b_i, 15) = 1$ for $0 \le i \le 2$, we are in the case a) of Theorem 0.3.1 which implies the decomposition

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0}+\lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big((\Sigma^{\lambda} H\underline{\mathbb{Z}})^{\vee 3}\Big) \vee \Big(C_{15}/C_{5+} \wedge \Sigma^{\lambda^{a_3 - b_3}} H\underline{\mathbb{Z}}\Big).$$

On the other hand if $a_0 = 0$, we are in case b) of Theorem 0.3.1, and the second summand here includes $d = m$. Also observe that the condition implies that $a_0 - b_0$ is relatively prime to 15. Therefore, we have

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_1-b_1}} H\underline{\mathbb{Z}} \\ \vee \Sigma^{\lambda^{a_2-b_2}} H\underline{\mathbb{Z}} \vee \Big(C_{15}/C_{5+} \wedge \Sigma^{\lambda^{a_3-b_3}} H\underline{\mathbb{Z}} \Big).$$

The results in Theorem 0.3.1 depend on a hypothesis on the weights at vertices which are fixed under the C_m-action. We further prove that these hypotheses may be removed when the group is of prime power order. (See Theorem 4.3.7 and Theorem 4.3.10)

Theorem 0.3.2. a) Let $\mathbb{T}$ be an admissible weighted C_p-equivariant tree such that $p \nmid a_0, b_0$ but $p \mid a_v - b_v$ for some fixed vertex v. Then,

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\phi(\mathbb{T})+1} \Sigma^{\lambda} H\underline{\mathbb{Z}} \Big) \\ \vee \Big(\bigvee_{\psi(\mathbb{T})-1} \Sigma^{2} H\underline{\mathbb{Z}} \Big) \vee \Big(\bigvee_{\mathbb{T}_e/C_p} C_p/e_+ \wedge \Sigma^{2} H\underline{\mathbb{Z}} \Big).$$

b) Let $\mathbb{T}$ be an admissible weighted C_{p^n}-equivariant tree. Suppose $\tau > 0$ is the maximum power of p that divides $a_v - b_v$ among the fixed vertices v and $p \nmid a_0, b_0$. Then

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+\lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}} H\underline{\mathbb{Z}})^{\vee W_{\mathbb{T}}(i)} \Big) \\ \vee \Big(\bigvee_{[\mu] \in \mathbb{T}_d/C_{p^n},\ d \neq p^n} C_{p^n}/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \Big).$$

In the statements of Theorem 0.3.2, we observe that the complementary cases are proved in Theorem 0.3.1. The notations $\phi(\mathbb{T})$, $\psi(\mathbb{T})$ and $W_{\mathbb{T}}(i)$ are clarified later in the document. The techniques used in the proof are the cellular filtration of the manifolds $X(\mathbb{T})$, and the following result about the $RO(C_m)$-graded homotopy groups of $H\underline{\mathbb{Z}}$. (See Theorem 2.2.10)

0.4 Organization

We start with a discussion of results in equivariant stable homotopy theory useful from the point of view of ordinary cohomology in Chapter 1. In Chapter 2, we recall computational methods for $\underline{\mathbb{Z}}$-coefficients, and perform the necessary computations in the context of freeness theorems. We apply these calculations in Chapter 3 to write down the additive structure of the cohomology of projective spaces, and deduce applications to cohomology operations, and the slice filtration. In Chapter 4, we discuss homology decompositions of an equivariant connected sum of projective planes. In Chapter

5, we write down the ring structure for the cohomology of projective spaces, and the equivariant classifying spaces of S^1 and S^3.

Chapter 1

Preliminaries on Equivariant Homotopy

In this chapter, we discuss structural results about equivariant stable homotopy theory, which are used later in the thesis. Most of this may be found in [34] and [32]. We restrict our attention to finite groups G. A G-representation V may then be equipped with a G-invariant inner product $\langle -, - \rangle$. We denote the unit disk in V as

$$D(V) = \{v \in V \mid \langle v, v \rangle \leq 1\},$$

the unit sphere as

$$S(V) = \{v \in V \mid \langle v, v \rangle = 1\},$$

and

$$S^V = \text{ one point compactification of } V \cong D(V)/S(V).$$

1.1 Equivariant CW-Complex

Definition 1.1.1. A G-CW complex is a G-space X with filtration $\{X^{(n)}\}_{n\geq 0}$, where
(1) $X^{(0)}$ is a disjoint union of G-orbits,
(2) $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching cells of the form $G/H \times D^n$ along equivariant maps $G/H \times S^{n-1} \to X^{(n-1)}$ where $H \leq G$ and the action of G on $D^n$and S^{n-1} are trivial.
(3) A subspace A of X is closed if $A \cap X^{(n)}$ is closed $\forall n$.

The space $X^{(n)}$ is said to be the n^{th} skeleton of X. The attaching map $G/H \times S^{(n-1)} \to X^{(n-1)}$ is equivalent to the map $S^{n-1} \to (X^{(n-1)})^H = (X^H)^{(n-1)}$.

Example 1.1.2. Consider the one-dimensional real C_2-representation σ where the action of C_2 on the real line is defined by $x \to -x$. Then S^σ has a C_2-CW-structure as follows. Its 0-skeleton $X^{(0)}$ contains two fixed points $C_2/C_2 \times D^0$ and $C_2/C_2 \times D^0$ and $X^{(1)}$ contains the equivariant 1-cell $C_2/e \times D^1$.

Example 1.1.3. Look at $S(\lambda)$ where λ is a complex C_n-representation defined by the C_n-action which takes $z \to e^{\frac{2\pi i}{n}} z$. It has the G-CW complex structure consisting of a zero disk $C_n/e \times D^0$ and a 1-disk $C_n/e \times D^1$.

Theorem 1.1.4 (Equivariant Whitehead Theorem)**.** Let $e : Y \to Z$ be a weak equivalence of G-CW complexes. Then e is a G-homotopy equivaence.

One may deduce that the category of G-CW complexes is equivalent to the functor category from $\mathcal{O}_G^{op}$ to the category of CW-complexes, where $\mathcal{O}_G$ is the orbit category of G with objects, the finite G-sets and morphisms, the G-equivariant maps between G-sets. Elmendorf's theorem provides another model for $GTop$ as Top-valued presheves on the orbit category.

1.2 Equivariant Stable Homotopy Category

The category of based G-spaces and G-equivariant maps is denoted by $GTop_*$. We also have an enriched category T_G with based G-spaces and all maps with G acting on the mapping spaces by conjugation. The equivariant stable homotopy category is constructed from the category of equivariant orthogonal spectra using the positive stable model category structure.

Definition 1.2.1. A G-universe $\mathcal{U}$ is a countably infinite dimensional representation of G with an inner product such that $\mathcal{U}$ contains countably many copies of each finite dimensional G-representation.

Definition 1.2.2. An equivariant orthogonal spectrum X has based G-spaces $X(V)$ with $O(V)$-action together with G-equivariant structure maps

$$\sigma : X(V) \wedge S^W \to X(V \bigoplus W)$$

that are $O(V) \times O(W)$-equivariant.

We use the notation $\mathcal{J}_G\mathcal{S}$ for the category of G-equivariant orthogonal spectra, and the notation $\{\_,\_\}^G$ for the homotopy classes of maps. From now on we denote the sphere spectrum S_G by S^0. Using smash products with G-spaces, we define the G-spectra S^V for representations V of G, and using the shift desuspension functors, we define S^α for $\alpha \in RO(G)$.

For a subgroup $H \leq G$, with the inclusion functor denoted by i, one has an induced forgetful functor i^* on G-spaces whose left adjoint sends a based H-space Y to $G_+ \wedge_H Y$. For G-spectra this construction is carried out in [32, §V.2], and one has

$$\mathcal{J}_G\mathcal{S}(G_+ \wedge_H Y, X) \simeq \mathcal{J}_H\mathcal{S}(Y, i^*X)$$

where the H-universe is i^* applied to the G-universe. An Ω-G-spectrum X is positive, stable fibrant [32, §V] and an H-CW-complex Y is stable cofibrant and we have an isomorphism

$$\{G_+ \wedge_H Y, X\}^G \simeq \{Y, i^*X\}^H$$

.

For a normal subgroup K of G, we write $\epsilon : G \to G/K$ for the natural map. Note that, $\mathcal{U}^K$ is a G/K-universe. We use the construction of the K-fixed point spectrum [32, §V.3] which takes an object $X \in \mathcal{J}_G\mathcal{S}$ to the object X^K. It is $X(W)^K$ when evaluated on a subspace $W \subseteq \mathcal{U}^K$. For G/K-spaces Y, we have

$$\mathcal{J}_G\mathcal{S}(\epsilon^*Y, X) \simeq \mathcal{J}_{G/K}\mathcal{S}(Y, X^K)$$

. Also, assuming that X is an Ω-G-spectrum, we can use the fact that it is fibrant in the model structure on $\mathcal{J}_G\mathcal{S}$. Then the equivalence passes to homotopy classes to give

$$\{\epsilon^*Y, X\}^G \cong \{Y, X^K\}^{G/K}.$$

1.3 Mackey Functors

Equivariant Eilenberg-MacLane spectra are spectra whose integer-graded homotopy groups are non-zero only in degree 0. Some of them may be constructed from $\mathbb{Z}[G]$-modules, But the more general Mackey functors also yield these spectra.

Definition 1.3.1. A Mackey functor consists of a pair $\underline{M} = (\underline{M}_*, \underline{M}^*)$ of functors from the category of finite G-sets to $\mathcal{A}b$ with $\underline{M}_*$ covariant and $\underline{M}^*$ contravariant. On every object S, $\underline{M}_*$ and $\underline{M}^*$ have the same value, denoted by $\underline{M}(S)$ and $\underline{M}$ carries disjoint unions to direct sum. The functors are required to satisfy that for every pull-back diagram of finite G-sets as below

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "\gamma"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & Z \end{tikzcd}$$

we get the commutative diagram

$$\begin{tikzcd} \underline{M}(P) \arrow[r, "\underline{M}_*(\delta)"] & \underline{M}(X) \\ \underline{M}(Y) \arrow[u, "\underline{M}^*(\gamma)"] \arrow[r, "\underline{M}_*(\beta)"'] & \underline{M}(Z) \arrow[u, "\underline{M}^*(\alpha)"'] \end{tikzcd}$$

Mackey functors are naturally contravariant functors from the Burnside category $Burn_G$ of G to abelian groups. The objects of $Burn_G$ are finite G-sets and the morphisms are formed by group completing the monoid of correspondences. The representable functor associated to the G-set G/G is called Burnside ring Mackey functor $\underline{A}$. For a finite G-set S, $\underline{A}_S$ is the representable functor associated to S.

Example: For an abelian group C, an immediate example for a Mackey functor is the constant Mackey functor $\underline{C}$ defined by the assignment $\underline{C}(S) = Map^G(S, C)$, the set of G-maps from the G-orbit S to C with trivial G-action.

we recall the equivariant cohomology with coefficients in a Mackey functor, with a particular emphasis on $\underline{\mathbb{Z}}$-coefficients. We restrict our attention to cyclic groups G. For such G, a G-Mackey functor $\underline{M}$ consists an Abelian group, $\underline{M}(G/H)$, which has a G/H-action for every subgroup $H \leq G$, and they are related via the following maps.

1. The *transfer* map $\mathrm{tr}_K^H \colon \underline{M}(G/K) \to \underline{M}(G/H)$

2. The *restriction* map $\mathrm{res}_K^H \colon \underline{M}(G/H) \to \underline{M}(G/K)$

for $K \leq H \leq G$. The composite $\mathrm{res}_L^H \mathrm{tr}_K^H$ satisfies a double coset formula (see [20, §3]). Let us see some examples.

Example 1.3.2. The Burnside ring Mackey functor, $\underline{A}$ is defined by $\underline{A}(G/H) = A(H)$. Here $A(H)$ is the Burnside ring of H, i.e., the group completion of the monoid of finite H-sets up to isomorphism. The transfer maps are defined by inducing up the action : $S \mapsto H \times_K S$ for $K \leq H$, and the restriction maps are given by restricting the action. For the K-set K/K, the double coset formula takes the form $\mathrm{res}_L^H \mathrm{tr}_K^H(K/K) = \mathrm{res}_L^H(H/K) =$ union of double cosets $L \backslash H/K$.

Example 1.3.3. The constant G-Mackey functor $\underline{\mathbb{Z}}$ and $\underline{\mathbb{Z}/p}$ are defined as follows. For an Abelian group C, the constant G-Mackey functor $\underline{C}$ is defined as

$$\underline{C}(G/H) = C, \mathrm{res}_K^H = \mathrm{Id}, \mathrm{tr}_K^H = [H : K].$$

for $K \leq H \leq G$. The double coset formula is given by $\mathrm{res}_L^H \mathrm{tr}_K^H(x) = [H : K]x$, for an element $x \in C$. We may also define its dual Mackey functor $\underline{C}^*$ by

$$\underline{C}^*(G/H) = C, \mathrm{res}_K^H = [H : K], \mathrm{tr}_K^H = \mathrm{Id}$$

Example 1.3.4. For an Abelian group C, the Mackey functor $\langle C \rangle$ is described by

$$\langle C \rangle(G/H) = \begin{cases} C & \text{if } H = G \\ 0 & \text{otherwise.} \end{cases}$$

Given a G-module $\underline{M}$, we may define the Mackey functor $\underline{M}$ given by $\underline{M}(G/H) = M^H$. Restriction maps are given by inclusion. The transfer maps are defined by $\mathrm{tr}_K^H(x) = \sum_{h \in H/K} hx$. If the action of G is trivial, then we get the constant Mackey functor.

Example 1.3.5. For the group C_p, the Mackey functor $\langle \mathbb{Z}/p \rangle$ is defined by

$$\langle \mathbb{Z}/p \rangle(C_p/C_p) = \mathbb{Z}/p, \quad \langle \mathbb{Z}/p \rangle(C_p/e) = 0, \quad \mathrm{res}_e^{C_p} = 0, \quad \mathrm{tr}_e^{C_p} = 0.$$

The following Mackey functor will appear in §3.5.

Example 1.3.6. For the group C_2, we have the Mackey functor $\langle C_2\rangle = \langle \Lambda\rangle$ described by

$$\langle \Lambda\rangle(C_2/e) = \mathbb{Z}/2, \quad \langle \Lambda\rangle(C_2/C_2) = 0, \quad \mathrm{res}_e^{C_2} = 0, \quad \mathrm{tr}_e^{C_2} = 0.$$

The equivariant stable homotopy category is the homotopy category of equivariant orthogonal spectra [32]. The Eilenberg-MacLane spectra are those whose integer graded homotopy groups vanishes except in degree 0. The following describes its relation with the Mackey functors.

Theorem 1.3.7. [16, Theorem 5.3] Corresponding to a Mackey functor $\underline{M}$, there is an Eilenberg-MacLane G-spectrum $H\underline{M}$ which is unique up to isomorphism in the equivariant stable homotopy category.

For a cyclic group C_n of odd order, the irreducible complex G-representations are one-dimensional, and up to isomorphism these are $1_{\mathbb{C}}, \lambda, \dots, \lambda^{n-1}$ where λ sends g to $e^{2\pi i/n}$, the n^{th} root of unity. The non-trivial real irreducible representations are obtained as realizations of these, and these are two-dimensional. Let r denote the "realization" functor which takes a complex representation to the underlying real representation. We often use the same notation, λ^i, for the realization $r(\lambda^i)$. Since λ^i and λ^{n-i} are conjugate, as a real representation $r(\lambda^i) \cong r(\lambda^{n-i})$. The isomorphism is given by the $\mathbb{R}$-linear map $z \mapsto \bar{z}$, which has degree -1.

For an even order cyclic group C_{2n}, the irreducible complex representations are powers of λ as described above. In addition, there is a one-dimensional irreducible real representation, namely the sign representation σ. This satisfies $2\sigma = r(\lambda^n)$.

In the equivariant stable homotopy category the objects S^V are invertible for a representation V. For a G-spectrum X, the equivariant homotopy groups have the structure of a Mackey functor $\underline{\pi}_n^G(X)$, which on objects assigns the value

$$\underline{\pi}_n^G(X)(G/H) := \pi_n(X^H).$$

The grading may be extended to $\alpha \in RO(G)$, the real representation ring of G, as

$$\underline{\pi}_\alpha^G(X)(G/K) \cong \text{ Ho-}G\text{-spectra } (S^\alpha \wedge G/K_+, X),$$

which is isomorphic to $\pi_\alpha^K(X)$. Similarly, the Mackey functor valued cohomology theory and homology theory are $RO(G)$-graded. Moreover, they have the structure of a Mackey functor which on objects is defined by

$$\underline{H}^{\alpha}_{G}(X;\underline{M})(G/K) \cong \text{ Ho-}G\text{-spectra } (X \wedge G/K_+, \Sigma^{\alpha} H\underline{M}),$$

$$\underline{H}^{G}_{\alpha}(X;\underline{M})(G/K) \cong \text{ Ho-}G\text{-spectra } (S^{\alpha} \wedge G/K_+, X \wedge H\underline{M}).$$

For a constant Mackey functor $\underline{C}$, the integer graded groups at orbit G/G compute the cohomology of the orbit space of X under the G-action, i.e., $H^n_G(X;\underline{C}) = H^n(X/G;C)$.

The Mackey functor $\underline{\mathbb{Z}}$ has a multiplicative structure which makes it a *commutative Green functor* [34, Chapter XIII.5]. The consequence of this multiplication is that the cohomology $H^{\bigstar}_G(X;\underline{\mathbb{Z}})$ has a graded commutative ring structure. The multiplicative structure also allows us to consider the Mackey functors which are $\underline{\mathbb{Z}}$-modules, and examples of these are the homology and the cohomology Mackey functors $\underline{H}^G_{\alpha}(X;\underline{\mathbb{Z}})$ and $\underline{H}^{\alpha}_G(X;\underline{\mathbb{Z}})$.

Remark 1.3.8. For any $\underline{M} \in \underline{\mathbb{Z}}\text{-Mod}_G$, $\operatorname{tr}^H_K \operatorname{res}^H_K$ equals the multiplication by index $[H:K]$ for $K \leq H \leq G$ [42, Theorem 4.3].

The spectrum $H\underline{\mathbb{Z}}$ also has the following well-known relation after smashing with representation spheres.

Proposition 1.3.9. If $\gcd(d,m) = 1$, then

$$H\underline{\mathbb{Z}} \wedge S^{\lambda^k} \simeq H\underline{\mathbb{Z}} \wedge S^{\lambda^{dk}}.$$

where λ^i is an irreducible complex representation of C_m for $i \in \mathbb{Z}$.

Proof. We check that $S^{-\lambda^{dk}} \wedge H\underline{\mathbb{Z}}$ is an Eilenberg MacLane spectrum up to suspension, whose underlying Mackey functor depends on k but not on d. For this note that

$$\underline{\pi}^G_i(S^{-\lambda^{dk}} \wedge H\underline{\mathbb{Z}})(G/L) \cong \underline{H}^{-i}_G(S^{\lambda^{dk}};\underline{\mathbb{Z}})(G/L) \cong \tilde{H}^{-i}_L(S^{\lambda^{dk}};\underline{\mathbb{Z}}) \cong \tilde{H}^{-i}(S^{\lambda^{dk}}/L;\mathbb{Z}).$$

Now observe that the orbit space $S^{\lambda^{dk}}/L \simeq S^2$, so that the groups above are $\mathbb{Z}$ for $i = -2$, and 0 otherwise. Therefore, we obtain

$$S^{-\lambda^{dk}} \wedge H\underline{\mathbb{Z}} \simeq \Sigma^{-2} H\underline{M},$$

where $\underline{M}(G/L) = \mathbb{Z}$ for $L \leq G$. For a subgroup $K = C_{r_1}$ of another subgroup $L = C_{r_2}$ of G, the restriction $\underline{M}(G/L) \to \underline{M}(G/K)$ is induced by the quotient $S^{\lambda^{dk}}/K \to S^{\lambda^{dk}}/L$, which is a map of degree $\frac{r_2 \cdot \gcd(r_1, dk)}{r_1 \cdot \gcd(r_2, dk)}$ which equals $\frac{r_2 \cdot \gcd(r_1, k)}{r_1 \cdot \gcd(r_2, k)}$ as $\gcd(d, m) = 1$. The transfer maps are then determined by Remark 1.3.8. It follows that $S^{-\lambda^{dk}} \wedge H\underline{\mathbb{Z}} \simeq S^{-\lambda^k} \wedge H\underline{\mathbb{Z}}$ if $\gcd(d, m) = 1$, so that $H\underline{\mathbb{Z}} \wedge S^{\lambda^k} \simeq H\underline{\mathbb{Z}} \wedge S^{\lambda^{dk}}$. $\square$

Remark 1.3.10. The above proposition shows that up to equivalence there is only one nontrivial suspension by a one dimensional representation for a group of prime order. We will use this fact repeatedly elsewhere in this thesis.

1.4 $RO(G)$-graded Cohomology

The category with finite G-set as objects, and homotopy classes of spectrum maps as morphisms is naturally isomorphic to Burnside category. Therefore, the homotopy groups of G-spectra are naturally Mackey functors. Lewis, May and McClure [29] construct $RO(G)$-graded cohomology theoroies associated to Mackey functors, which on the objects G/H gave the underlying Mackey functor.

Theorem 1.4.1. For a Mackey functor $\underline{M}$, there is an Eilenberg-MacLane G-spectrum $H\underline{M}$ which is unique up to isomorphism in the equivariant stable homotopy category. For Mackey functors $\underline{M}$ and $\underline{M}'$, $\{H\underline{M}, H\underline{M}'\}^G \cong Hom_{\mathcal{M}_G}(\underline{M}, \underline{M}')$.

For a G-Mackey functor $\underline{M}$, we fix a model of the Eilenberg-MacLane spectrum $H\underline{M}$ which is an Ω-G-spectrum and fibrant in the model structure on $\mathcal{J}_G\mathcal{S}$. This allows us to compute homotopy classes of maps to it's fixed point spectra via adjunctions.

We used the notation $\tilde{H}^{\alpha}_G(X; \underline{M}), \alpha \in RO(G)$, for the reduced cohomology of a based G-space X for the cohomology theory represented by $H\underline{M}$. That is,

$$\tilde{H}^{\alpha}_G(X; \underline{M}) \cong \{X, S^{\alpha} \wedge H\underline{M}\}^G$$

We recall that there are change of groups functors on equivariant spectra. The restriction functor from G-spectra to H-spectra has a left adjoint given by smashing with G/H_+. This also induces an isomorphism on cohomology with Mackey functor coefficients

$$\tilde{H}^{\alpha}_G(G/H_+ \wedge X; \underline{M}) \cong \tilde{H}^{\alpha}_H(X; i^*H(\underline{M}))$$

For the Mackey functors $\underline{A}$ and $\underline{\mathbb{Z}}$, we have $i^*(\underline{A}) = \underline{A}$ and $i^*(\underline{\mathbb{Z}}) = \underline{\mathbb{Z}}$. Therefore we have

$$H^\alpha_G(G/H_+ \wedge X; \underline{A}) \cong \tilde{H}^\alpha_H(X; \underline{A}).$$

The $RO(G)$-graded theories may also be assumed to be Mackey functor-valued as in the definition below.

Definition 1.4.2. Let X be a pointed G-space, $\underline{M}$ be any Mackey functor, $\alpha \in RO(G)$. Then the Mackey functor valued cohomology $\underline{H}^\alpha_G(X; \underline{M})$ is defined as

$$\underline{H}^\alpha_G(X; \underline{M})(G/K) = \tilde{H}^\alpha_G(G/K_+ \wedge X; \underline{M})$$

In particular, for $H = \{e\}$, we have

$$\tilde{H}^\alpha(G_+ \wedge X; \underline{A}) \cong \tilde{H}^{|\alpha|}(X; Z).$$

The restriction and transfer maps are induced by the appropiate maps of G-spectra.

Chapter 2

Equivariant cohomology with integer coefficients

In this chapter, we recall the salient features of equivariant cohomology with $\underline{\mathbb{Z}}$-coefficients. We restrict our attention to cyclic groups G. Most of the results here form part of [5].

As $H\underline{\mathbb{Z}}$ is a ring spectrum, the Mackey functor $\underline{\mathbb{Z}}$ has a multiplicative structure, i.e, it is a commutative Green functor [34, chapter XIII.5]. As a consequence, $\underline{H}^{\alpha}_{G}(X;\underline{\mathbb{Z}})$ are $\underline{\mathbb{Z}}$-modules. These modules satisfy the following property.

Proposition 2.0.1. [42, Theorem 4.3] For any $\underline{\mathbb{Z}}$-module G-Mackey functor $\underline{M}$, $\mathrm{tr}^H_K \mathrm{res}^H_K$ is multiplication by the index $[H : K]$ for $K \leq H \leq G$.

2.1 Reduction to divisor-like gradings

For an element $\alpha \in RO(G)$ such that $|\alpha^H| = 0$ for all $H \leq G$, the representation sphere S^α belongs to the Picard group of G-spectra. Theorem 5.1 of [1] provides a description of the Mackey functor $\underline{\pi}^G_0(S^\alpha \wedge H\underline{A})$. The equivalence

$$S^\alpha \wedge H\underline{\mathbb{Z}} \simeq S^\alpha \wedge H\underline{A} \wedge_{H\underline{A}} H\underline{\mathbb{Z}},$$

helps us determine $\underline{\pi}^G_0(S^\alpha \wedge H\underline{\mathbb{Z}})$. In particular, let $G = C_{p^m}$ and $\alpha = \lambda^{si} - \lambda^i$ where $\gcd(s, p) = 1$. Let $i = p^r t$ be such that $p \nmid t$. Then we get by applying Proposition 1.3.9

$$\underline{\pi}^G_0(H\underline{\mathbb{Z}} \wedge S^{\lambda^{si}-\lambda^i}) \cong \underline{A}[\tau]\Box\underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}, \tag{2.1.1}$$

where $\tau = (\tau_d) \in \prod_{d|p^m} \mathbb{Z}$ be such that $\tau_{p^r} = s$ and 1 if $d \neq p^r$.

We recall a few important classes which generate a portion of the ring $\pi^G_\bigstar(H\underline{\mathbb{Z}})$.

Definition 2.1.2. [20, §3] For a G-representation V, consider the inclusion $S^0 \hookrightarrow S^V$. Composing with the unit map $S^0 \to H\underline{\mathbb{Z}}$, we obtain $S^0 \hookrightarrow S^V \to S^V \wedge H\underline{\mathbb{Z}}$ which represents an element in $\pi^G_{-V}(H\underline{\mathbb{Z}}) \cong \tilde{H}^V_G(S^0;\underline{\mathbb{Z}})$. This class is denoted by a_V.

Definition 2.1.3. [20, §3] For an oriented G-representation of dimension n, $\underline{\pi}^G_{n-V}(H\underline{\mathbb{Z}}) = \underline{\mathbb{Z}}$ [20, Example 3.10]. Define $u_V \in \pi^G_{n-V}(H\underline{\mathbb{Z}}) \cong \tilde{H}^G_n(S^V;\underline{\mathbb{Z}}) \cong \mathbb{Z}$ to be the generator that restricts to the choice of orientation in $\underline{H}^G_n(S^V;\underline{\mathbb{Z}})(G/e) \cong \tilde{H}_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$.

Proposition 1.3.9 simplifies calculations for $\pi^G_\bigstar(H\underline{\mathbb{Z}})$, and implies $\Sigma^{\lambda^i - \lambda^{si}} H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}$ if $\gcd(s,n) = 1$. That is, in the graded commutative ring $\pi^G_\bigstar H\underline{\mathbb{Z}}$, there are invertible elements $u_{\lambda^{si}-\lambda^i}$ in degrees $\lambda^{si} - \lambda^i$ whenever $\gcd(s,n) = 1$. Therefore to determine the ring $\pi^G_\bigstar H\underline{\mathbb{Z}}$ it is enough to consider the gradings which are linear combinations of λ^d for $d \mid n$. We recall the following computation of $\underline{\pi}^{C_p}_\bigstar(H\underline{\mathbb{Z}})$ [13, Appendix B].

$$\underline{\pi}^{C_p}_\alpha(H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}} & \text{if } |\alpha| = 0,\ |\alpha^{C_p}| \geq 0 \\ \underline{\mathbb{Z}}^* & \text{if } |\alpha| = 0,\ |\alpha^{C_p}| < 0 \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| < 0,\ |\alpha^{C_p}| \geq 0, \text{ and } |\alpha| \text{ even} \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| > 0,\ |\alpha^{C_p}| < -1, \text{ and } |\alpha| \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.4}$$

The linear combinations $\ell - (\Sigma_{d_i|n} \ b_i \lambda^{d_i}) + \epsilon\sigma \in RO(G)$ with $\ell, b_i \in \mathbb{Z}$ and $\epsilon \in \{0,1\}$ are denoted by $\bigstar_{\text{div}}$. The last term $\epsilon\sigma$ occurs only when $|G|$ is even. In the case of $H\underline{\mathbb{Z}}$, the ring $\pi^G_{\bigstar_{\text{div}}}(H\underline{\mathbb{Z}})$ is also obtained from $\pi^G_\bigstar(H\underline{\mathbb{Z}})$ by identifying all the $u_{\lambda^{si}-\lambda^i}$ with 1. More precisely,

$$\pi^G_{\bigstar_{\text{div}}}(H\underline{\mathbb{Z}}) \cong \pi^G_\bigstar(H\underline{\mathbb{Z}})/(u_{\lambda^{si}-\lambda^i} - 1), \quad \text{and} \quad \pi^G_\bigstar(H\underline{\mathbb{Z}}) \cong \pi^G_{\bigstar_{\text{div}}}(H\underline{\mathbb{Z}})[u_{\lambda^{si}-\lambda^i} \mid \gcd(s,n) = 1].$$

In fact, we make a choice of $u_{\lambda^{si}-\lambda^i}$ such that $\mathrm{res}^G_e(u_{\lambda^{si}-\lambda^i}) = 1 \in H^0(\mathrm{pt})$. This implies that $u_{\lambda^{si}-\lambda^i} \cdot u_{\lambda^i} = u_{\lambda^{si}}$. The following proposition describes the relation between $a_{\lambda^{si}}$ and a_{λ^i} in $\pi^G_{\bigstar_{\text{div}}}(H\underline{\mathbb{Z}})$ in case G is of prime power order.

Proposition 2.1.5. Let $G = C_{p^m}$. If $\gcd(s, p) = 1$, then

$$u_{\lambda^i - \lambda^{si}} a_{\lambda^{si}} = s a_{\lambda^i}.$$

That is, in the ring $\pi^G_{\bigstar_{\mathrm{div}}}(H\underline{\mathbb{Z}})$,

$$a_{\lambda^{si}} = s a_{\lambda^i}.$$

Proof. By Proposition 1.3.9 we have

$$H\underline{\mathbb{Z}} \wedge S^{\lambda^i} \simeq H\underline{\mathbb{Z}} \wedge S^{\lambda^{si}}. \tag{2.1.6}$$

Hence there exists $u_{\lambda^{si}-\lambda^i} \in \pi^G_0(H\underline{\mathbb{Z}} \wedge S^{\lambda^{si}-\lambda^i})$ such that the composition

$$\psi : H\underline{\mathbb{Z}} \wedge S^{\lambda^i} \xrightarrow{id \wedge u_{\lambda^{si}-\lambda^i}} H\underline{\mathbb{Z}} \wedge S^{\lambda^i} \wedge H\underline{\mathbb{Z}} \wedge S^{\lambda^{si}-\lambda^i} \to H\underline{\mathbb{Z}} \wedge S^{\lambda^{si}}$$

induces the equivalence (2.1.6). Since all the fixed point dimensions of $\lambda^{si} - \lambda^i$ are zero,

$$\pi^G_0(H\underline{\mathbb{Z}} \wedge S^{\lambda^{si}-\lambda^i}) \cong \underline{\mathbb{Z}}$$

by (2.1.1). Let us choose $u_{\lambda^{si}-\lambda^i} \in \pi^G_0(H\underline{\mathbb{Z}} \wedge S^{\lambda^{si}-\lambda^i})$ to be the element such that $\mathrm{res}^G_e(u_{\lambda^{si}-\lambda^i}) = 1$. The induced map

$$\psi_* : \pi^G_\alpha(H\underline{\mathbb{Z}} \wedge S^{\lambda^i}) \to \pi^G_\alpha(H\underline{\mathbb{Z}} \wedge S^{\lambda^{si}})$$

sends a_{λ^i} to $u_{\lambda^{si}-\lambda^i} \cdot a_{\lambda^i}$. Consider the map

$$\phi_{\lambda^i - \lambda^{si}} : S^{\lambda^i} \to S^{\lambda^{si}}.$$

under which $z \mapsto z^s$, that is, the underlying degree of $\phi_{\lambda^i-\lambda^{si}}$ is s. Since $\gcd(s, p) = 1$, we may choose s' such that $s \cdot s' = 1 + tp^m$. Then

$$u_{\lambda^{si}-\lambda^i} = s' \cdot \phi^{H\underline{\mathbb{Z}}}_{\lambda^i-\lambda^{si}} - t \cdot \mathrm{tr}^G_e(1)$$

as $\mathrm{res}^G_e(u_{\lambda^{si}-\lambda^i}) = s \cdot s' - tp^m = 1$, where $\phi^{H\underline{\mathbb{Z}}}_{\lambda^i-\lambda^{si}}$ is the Hurewicz image of $\phi_{\lambda^i-\lambda^{si}}$. This implies

$$u_{\lambda^{si}-\lambda^i} \cdot a_{\lambda^i} = s' \cdot \phi_{\lambda^i-\lambda^{si}} \cdot a_{\lambda^i} - t \cdot \mathrm{tr}^G_e(1) \cdot a_{\lambda^i} = s' \cdot a_{\lambda^{si}}.$$

Consequently $a_{\lambda^{si}} = s \cdot a_{\lambda^i}$. □

In particular, we obtain the following

Proposition 2.1.7. Let $d = p^k$ be a divisor of p^m and $1 \leq i < d$. Note that $i-d$ and i have the same p-adic valuation. Then

$$a_{\lambda^{i-d}} = \Theta_{i,d} \cdot a_{\lambda^i},$$

where $\Theta_{i,d} = \frac{i-d}{i}$ which is well defined in $\mathbb{Z}/p^m$.

The following computations of $\underline{\pi}_{\bigstar}^{C_p}(H\underline{\mathbb{Z}/p})$ [13, Appendix B] will help us in the following sections. For an odd prime p we have

$$\underline{\pi}_{\alpha}^{C_p}(H\underline{\mathbb{Z}/p}) = \begin{cases} \underline{\mathbb{Z}/p} & \text{if } |\alpha| = 0,\ |\alpha^{C_p}| \geq 0 \\ \underline{\mathbb{Z}/p}^* & \text{if } |\alpha| = 0,\ |\alpha^{C_p}| < 0 \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| < 0,\ |\alpha^{C_p}| \geq 0, \text{ and } |\alpha| \text{ even} \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| > 0,\ |\alpha^{C_p}| < -1, \text{ and } |\alpha| \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.8}$$

For the group C_2, we recall the following computation of $\underline{\pi}_{\bigstar}^{C_2}(H\underline{\mathbb{Z}/2})$ [13, Appendix B].

$$\underline{\pi}_{\alpha}^{C_2}(H\underline{\mathbb{Z}/2}) = \begin{cases} \underline{\mathbb{Z}/2} & \text{if } |\alpha| = 0,\ |\alpha^{C_2}| \geq 0 \\ \underline{\mathbb{Z}/2}^* & \text{if } |\alpha| = 0,\ |\alpha^{C_2}| < 0 \\ \langle \Lambda \rangle & \text{if } |\alpha| = 0,\ |\alpha^{C_2}| = -1, \\ \langle \mathbb{Z}/2 \rangle & \text{if } |\alpha| < 0,\ |\alpha^{C_2}| \geq 0, \\ \langle \mathbb{Z}/2 \rangle & \text{if } |\alpha| > 0,\ |\alpha^{C_2}| < -1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.9}$$

2.1.10. Anderson duality. Let $I_{\mathbb{Q}}$ and $I_{\mathbb{Q}/\mathbb{Z}}$ be the spectra representing the cohomology theories given by $X \mapsto Hom(\pi^G_{-*}(X), \mathbb{Q})$ and $X \mapsto Hom(\pi^G_{-*}(X), \mathbb{Q}/\mathbb{Z})$ respectively. The natural map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces the spectrum map $I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}}$, and the homotopy fibre is denoted by $I_{\mathbb{Z}}$. For a G-spectrum X, the *Anderson dual* $I_{\mathbb{Z}}X$ of X, is the function spectrum $F(X, I_{\mathbb{Z}})$. For $X = H\underline{\mathbb{Z}}$, one easily computes $I_{\mathbb{Z}}H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}^* \simeq \Sigma^{2-\lambda}H\underline{\mathbb{Z}}$ [7, 43] in the case G is a cyclic group.

In general, for G-spectra E, X, and $\alpha \in RO(G)$, there is short exact sequence

$$0 \to Ext_L(\underline{E}_{\alpha-1}(X), \mathbb{Z}) \to I_{\mathbb{Z}}(E)^{\alpha}(X) \to Hom_L(\underline{E}_{\alpha}(X), \mathbb{Z}) \to 0. \tag{2.1.11}$$

In (2.1.11), Ext_L and Hom_L refers to level-wise Ext and Hom, which turn out to be Mackey functors. In particular, for $E = H\mathbb{Z}$ and $X = S^0$, we have the equivalence $\underline{E}_\alpha(X) \cong \underline{\pi}^G_\alpha(S^0; \underline{\mathbb{Z}})$. Therefore, one may rewrite (2.1.11) as

$$0 \to Ext_L(\underline{\pi}^G_{\alpha+\lambda-3}(H\underline{\mathbb{Z}}), \mathbb{Z}) \to \underline{\pi}^G_{-\alpha}(H\underline{\mathbb{Z}}) \to Hom_L(\underline{\pi}^G_{\alpha+\lambda-2}(H\underline{\mathbb{Z}}), \mathbb{Z}) \to 0 \tag{2.1.12}$$

for each $\alpha \in RO(G)$.

2.2 $\pi^G_\bigstar(H\mathbb{Z})$ for cyclic groups

This section describes various structural results of $\pi^G_\bigstar(H\mathbb{Z})$ which helps us to construct the homology decompositions in the later sections. With Burnside ring coefficients, Lewis [30] first described $\pi^{C_p}_\bigstar(H\underline{A})$. The portion of the $RO(C_{p^n})$-graded homotopy of $H\underline{\mathbb{Z}}$ in dimensions of the form $k - V$ was described in [23, 22]. Using the Tate square, $\pi^{C_2}_\bigstar(H\underline{\mathbb{Z}})$ was determined in [15] and $\pi^{C_p}_\bigstar(H\underline{\mathbb{Z}/p})$ in [6]. For groups of square free order $\pi^G_\bigstar(H\underline{\mathbb{Z}})$ was explored in [7], and for the group C_{p^2}, $\pi^G_\bigstar(H\underline{\mathbb{Z}})$ appeared in [43].

Theorem 2.2.1 ([3])**.** The subalgebra $\pi^G_{\bigstar_{div}}(H\underline{\mathbb{Z}})$ of $\pi^G_\bigstar(H\underline{\mathbb{Z}})$ is generated over $\mathbb{Z}$ by the classes $a_{\lambda^d}, u_{\lambda^d}$ where d is a divisor of n, $d \neq n$ with only possible relations are:

$$\frac{n}{d} a_{\lambda^d} = 0, \tag{2.2.2}$$

$$\frac{d}{\gcd(d,s)} a_{\lambda^s} u_{\lambda^d} = \frac{s}{\gcd(d,s)} a_{\lambda^d} u_{\lambda^s}. \tag{2.2.3}$$

For a general cyclic group G and $\alpha \in \bigstar^e$, where $\bigstar^e$ consists of the representations which do not have a copy of sign representation in it, we observe that the expression above implies that $\pi^G_\alpha(H\underline{\mathbb{Z}})$ is cyclic. This is generated by a product of the corresponding u-classes and a-classes, and the relation (2.2.3) implies that they assemble together into a cyclic group. The order of this is the least common multiple of the order of a product of a-classes and u-classes occuring in $\pi^G_\alpha(H\underline{\mathbb{Z}})$.

In $\underline{\mathbb{Z}/p}$-coefficients the relation (2.2.3) simplified to the following

Proposition 2.2.4. In $\underline{\mathbb{Z}/p}$-coefficients, we have the following relation

$$a_{\lambda^{pd}}u_{\lambda^d} = pa_{\lambda^d}u_{\lambda^{pd}} = 0. \tag{2.2.5}$$

Note that the space $S(\lambda^{p^r})$ fits into a cofibre sequence

$$G/C_{p^r}{}_+ \stackrel{1-g}{\to} G/C_{p^r}{}_+ \to S(\lambda^{p^r})_+.$$

It follows that

$$\underline{\pi}_\alpha^{C_{p^r}}(H\underline{\mathbb{Z}}) = 0 \text{ and } \underline{\pi}_{\alpha-1}^{C_{p^r}}(H\underline{\mathbb{Z}}) = 0 \implies \underline{\pi}_\alpha^G(S(\lambda^{p^r})) = 0. \tag{2.2.6}$$

In the case $r = 0$, we may make a complete computation to obtain [7, Chapter 2, Example 2.13]

$$\underline{\pi}_\alpha(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}}^* & \text{if } |\alpha| = 0, \\ \underline{\mathbb{Z}} & \text{if } |\alpha| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.7}$$

Suppose α satisfies $|\alpha^H| > 0$ for all subgroups H. This means that S^α has a cell structure with cells of the type $G/H \times D^n$ for $n \geq |\alpha^H|$. Therefore,

$$|\alpha^H| > 0 \text{ for all subgroups } H \implies \underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) = 0. \tag{2.2.8}$$

Now if α satisfies $|\alpha^H| < 0$ for all subgroups H, $\beta = -\alpha$ satisfies the above condition. As S^α is the Spanier-Whitehead dual of S^β, we may construct it using cells of the type $G/H \times D^n$ for $n < 0$. Again, $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) = 0$. Thus,

$$|\alpha^H| < 0 \text{ for all subgroups } H \implies \underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) = 0. \tag{2.2.9}$$

The following theorem may be viewed as an extension of (2.2.8). It also provides the necessary input in proving homology decompositions.

Theorem 2.2.10. Let $\alpha \in RO(G)$ be such that $|\alpha^H|$ is odd for all subgroups H, and $|\alpha^H| > -1$ implies $|\alpha^K| \geq -1$ for all subgroups $K \supseteq H$. Then $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) = 0$.

Note that the second condition stated here may be equivalently expressed as $|\alpha^H| < -1$ implies $|\alpha^K| \leq -1$ for all subgroups K of H which is just the contrapositive of the condition. The first condition implies that α does not contain any multiples of the sign representation in the case $|G|$ is even. This happens because of the following. Observe that, if $|G| = 2n$ and H is a subgroup of G of order n, then the sign representation σ can be defined by the action of $C_2 \cong G/H$ on $\mathbb{R}$ which takes x to $-x$. Thus the action of H on σ is trivial which implies $|\sigma^H| = 1$. Also note that the G-fixed point subspace of σ, that is $\sigma^G = \{0\}$, consequently $|\sigma^G| = 0$. Now, let $\alpha \in RO(G)$ where $|G| = 2n$. Then α is a sum of some multiples of $\lambda^i, 0 \leq i \leq 2n-1$, some multiples of the trivial representation, say x many, and a copy of σ (or $-\sigma$), since 2σ is isomorphic to λ^n. Thus $|\alpha^H| = 2k + x + 1$ (or $2k + x - 1$) for some k and $|\alpha^G| = 2j + x$ for some j. In any case, either $2k + x + 1$ is even or $2j + x$ is . Thus α fails to satisfy the hypothesis of the Theorem.

Proof. It suffices to prove this for G of prime power order, via Proposition 2.0.1. For groups of odd prime power order, this is proved in [4].

Let $\mathcal{F}_G = \{\alpha \in RO(G) \mid \forall\ H \subseteq G,\ |\alpha^H| > -1 \implies |\alpha^K| \geq -1 \text{ for all } H \subset K\}$. We would like to show that $\alpha \in \mathcal{F}_G \implies \underline{\pi}_\alpha(H\underline{\mathbb{Z}}) = 0$. If $\alpha \in \mathcal{F}_G$ with $|\alpha^G| < -1$, the hypothesis implies $|\alpha^H| \leq -1$ for all subgroups H of G. For these α, $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) = 0$ by (2.2.9) as all the fixed points are negative.

Now let $\alpha \in \mathcal{F}_G$, and $H = C_{p^r}$ is a subgroup such that $|\alpha^H| < -1$. This implies that for all $K \subset H$, $|\alpha^K| \leq -1$. For such an α, $\alpha - \lambda^{p^s} \in \mathcal{F}_G$ for $s \leq r$. Also the cofibre sequence

$$S(\lambda^{p^s})_+ \to S^0 \to S^{\lambda^{p^s}},$$

implies the long exact sequence of Mackey functors

$$\underline{\pi}_\alpha^G(S(\lambda^{p^s})_+ \wedge H\underline{\mathbb{Z}}) \to \underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha-\lambda^{p^s}}^G(H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha-1}(S(\lambda^{p^s})_+ \wedge H\underline{\mathbb{Z}}). \tag{2.2.11}$$

The given condition on α implies that $\alpha - \lambda^{p^s}$ and $\alpha - \lambda^{p^s} \pm 1$ have negative dimensional fixed points for subgroups of C_{p^s}. It follows from (2.2.6) and (2.2.9) that $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \cong \underline{\pi}_{\alpha-\lambda^{p^s}}(H\underline{\mathbb{Z}})$. In this way by adding and subtracting copies of λ^{p^s} for $s \leq r$ while adhering to the condition $|\alpha^K| \leq -1$ for all subgroups K of H, we may find a new $\beta \in \mathcal{F}_G$, satisfying $|\beta^K| = -1$ for all $K = C_{p^s}$ for $s \leq r$, and $\underline{\pi}_\beta^G(H\underline{\mathbb{Z}}) \cong \underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})$. A consequence of this manoeuvre is that it suffices to prove the result for those $\alpha \in \mathcal{F}_G$ such that $|\alpha^H| \geq -1$ for all subgroups H. Call this collection $\mathcal{F}_G^{\geq -1} \subset \mathcal{F}_G$.

A small observation will now allow us to assume $|\alpha| \geq 1$ in $\mathcal{F}_G^{\geq -1}$. For, we have the long exact sequence

$$\underline{\pi}_\alpha^G(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}) \to \underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha-\lambda}^G(H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha-1}(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}),$$

by putting $s = 0$ in (2.2.11). Applying the computation of (2.2.7), we deduce $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \cong \underline{\pi}_{\alpha-\lambda}^G(H\underline{\mathbb{Z}})$ if $|\alpha| \geq 3$, and if $|\alpha| = 1$, $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) = 0 \implies \underline{\pi}_{\alpha-\lambda}^G(H\underline{\mathbb{Z}}) = 0$. The last conclusion is true because for $\nu = \alpha - 1$, $|\nu| = 0$, and the map

$$\underline{\mathbb{Z}} \cong \underline{\pi}_\nu^G(S(\lambda)_+ \wedge H\underline{\mathbb{Z}})) \to \underline{\pi}_\nu^G(H\underline{\mathbb{Z}}),$$

is an isomorphism at G/e, and hence injective at all levels.

Suppose that $\alpha \in \mathcal{F}_G^{\geq -1}$ and $|\alpha| \geq 1$. By Proposition 2.0.1, we have $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})$ only features torsion elements as $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})(G/e) = 0$. Applying Anderson duality (2.1.12), we obtain

$$\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \cong Ext_L(\underline{\pi}_{\lambda-\alpha-3}^G(H\underline{\mathbb{Z}}), \mathbb{Z}).$$

Now note that all the fixed points of $\lambda - \alpha - 3$ are negative if $\alpha \in \mathcal{F}_G^{\geq -1}$ and $|\alpha| \geq 1$. Therefore, $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) = 0$ and the proof is complete. □

We also have a calculation of $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})$ if all the fixed points are ≥ 0.

Proposition 2.2.12. Let $\alpha \in RO(G)$ be such that $|\alpha| = 0$, and $|\alpha^K| \geq 0$ even for all subgroups $K \neq e$. Then $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})$ is isomorphic to the Mackey functor $\underline{\mathbb{Z}}$.

Proof. The cofibre sequence

$$S(\lambda)_+ \to S^0 \to S^\lambda,$$

implies the long exact sequence (by taking Mackey functor valued homotopy groups after smashing with $H\underline{\mathbb{Z}}$)

$$\cdots \underline{\pi}_{\alpha+\lambda}^G(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha+\lambda}^G(H\underline{\mathbb{Z}}) \to \underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha+\lambda-1}^G(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha+\lambda-1}^G(H\underline{\mathbb{Z}}) \to \cdots$$

putting $s = 0$ in (2.2.11). Note that $\alpha + \lambda - 1$ satisfies the hypothesis of Theorem 2.2.10. The element $\alpha + \lambda$ has dimension 2, so by Proposition 2.0.1, the Mackey functor

$\underline{\pi}_{\alpha+\lambda}^{G}(H\underline{\mathbb{Z}})$ features only torsion elements. By Anderson duality (2.1.12), we have

$$\underline{\pi}_{\alpha+\lambda}^{G}(H\underline{\mathbb{Z}}) \cong Ext_L(\underline{\pi}_{-3-\alpha}^{G}(H\underline{\mathbb{Z}}), \mathbb{Z}).$$

Clearly from the given hypothesis, all the fixed points of $-3-\alpha$ are negative, therefore by (2.2.9), $\underline{\pi}_{\alpha+\lambda}^{G}(H\underline{\mathbb{Z}}) = 0$. We obtain

$$\underline{\pi}_{\alpha}^{G}(H\underline{\mathbb{Z}}) \cong \underline{\pi}_{\alpha+\lambda-1}^{G}(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}},$$

by (2.2.7). This completes the proof. □

This helps us define the following classes.

Definition 2.2.13. Let j be a multiple of i. Then by Proposition 2.2.12, the Mackey functor $\underline{\pi}_{\lambda^i-\lambda^j}^{G}(H\underline{\mathbb{Z}})$ is isomorphic to $\underline{\mathbb{Z}}$. Define the class $u_{\lambda^i-\lambda^j} \in \pi_{\lambda^i-\lambda^j}^{G}(H\underline{\mathbb{Z}})$ to be the element which under restriction to the orbit G/e corresponds to $1 \in \mathbb{Z}$.

The multiplication of the class $u_{\lambda^k-\lambda^{dk}}$ with $a_{\lambda^{dk}}$ is a multiple of a_{λ^k}. Similar description also appeared in [23, p. 395].

Proposition 2.2.14. We have the following relation

$$u_{\lambda^k-\lambda^{dk}}\, a_{\lambda^{dk}} = d a_{\lambda^k}$$

$$u_{\lambda^k-\lambda^{dk}}\, u_{\lambda^{dk}} = u_{\lambda^k}.$$

Proof. Let us denote $a_{\lambda^{dk}/\lambda^k}$ to be the map

$$a_{\lambda^{dk}/\lambda^k} : S^{\lambda^k} \to S^{\lambda^{dk}},$$

under which $z \mapsto z^d$. Therefore, the underlying degree of this map is d. Moreover,

$$a_{\lambda^{dk}/\lambda^k}\, a_{\lambda^k} = a_{\lambda^{dk}}.$$

Hence $u_{\lambda^k-\lambda^{dk}}\, a_{\lambda^{dk}} = u_{\lambda^k-\lambda^{dk}}\, a_{\lambda^{dk}/\lambda^k}\, a_{\lambda^k}$, where the element

$$u_{\lambda^k-\lambda^{dk}}\, a_{\lambda^{dk}/\lambda^k} \in \underline{H}_G^0(S^0, \underline{\mathbb{Z}})(G/G) \cong \mathbb{Z}.$$

Since $\mathrm{res}_e^G(u_{\lambda^k-\lambda^{dk}}) = 1$ and $\mathrm{res}_e^G(a_{\lambda^{dk}/\lambda^k}) = d$, we obtain $u_{\lambda^k-\lambda^{dk}}\, a_{\lambda^{dk}} = d a_{\lambda^k}$.

Similarly, since $\mathrm{res}_e^G(u_{\lambda^k-\lambda^{dk}}\, u_{\lambda^{dk}}) = \mathrm{res}_e^G(u_{\lambda^k}) = 1$, we have $u_{\lambda^k-\lambda^{dk}}\, u_{\lambda^{dk}} = u_{\lambda^k}$. □

The following will be used in subsequent sections.

Proposition 2.2.15. Let $\alpha = \lambda^{i_1} + \cdots + \lambda^{i_k}$. Then the group

$$H_G^{-\alpha}(S^0) = 0.$$

Proof. For a representation λ^{i_s}, we have the cofibre sequence $S(\lambda^{i_s})_+ \xrightarrow{r} S^0 \to S^{\lambda^{i_s}}$. If H_s is the kernel of the representation λ^{i_s}, then we have the cofibre sequence $G/H_{s+} \to G/H_{s+} \to S(\lambda^{i_s})_+$. To see $H_G^{-\lambda^{i_1}}(S^0) = 0$, consider the long exact sequence

$$0 \to H_G^{-1}(S(\lambda^{i_1})_+) \to H_G^{-\lambda^{i_1}}(S^0) \to H_G^0(S^0) \xrightarrow{r^*} H_G^0(S(\lambda^{i_1})_+) \to \cdots$$

The first term is zero as $H_G^j(G/H_{1+}) = 0$ for $j \leq -1$. Also, $H_G^0(S(\lambda^{i_1})_+) \cong H_G^0(G/H_{1+}) \cong \mathbb{Z}$. Moreover, the map r^* is the restriction map $res_{H_1}^G(\underline{\mathbb{Z}})$, hence r^* is an isomorphism. Thus $H_G^{-\lambda^{i_1}}(S^0) = 0$. Using similar arguments, the result follows by induction. □

Example 2.2.16. The Mackey functor $\underline{H}_{C_2}^0(S^\sigma; \underline{\mathbb{Z}})$ is zero. To see this consider the cofibre sequence

$$C_2/e_+ \to S^0 \to S^\sigma$$

and the associated long exact sequence in cohomology

$$0 \to \underline{H}_{C_2}^0(S^\sigma; \underline{\mathbb{Z}}) \to \underline{H}_{C_2}^0(S^0; \underline{\mathbb{Z}}) \to \underline{H}_{C_2}^0(C_2/e_+; \underline{\mathbb{Z}}) \to \underline{H}_{C_2}^1(S^\sigma; \underline{\mathbb{Z}}) \to \dots$$

The restriction map $\underline{H}_{C_2}^0(S^0; \underline{\mathbb{Z}}) \to \underline{H}_{C_2}^0(C_2/e_+; \underline{\mathbb{Z}})$ is injective. Thus we have $\underline{H}_{C_2}^0(S^\sigma; \underline{\mathbb{Z}}) = 0$.

Next we prove a homology decomposition theorem for a cyclic group by generalizing Lewis's approach.

Theorem 2.2.17. Let X be a generalised G-cell complex with only even dimensional cells of the form $D(W)$. If for cells $D(W), D(V)$ we have the condition $\dim W < \dim V \implies |W^H| \leq |V^H|$ for every subgroup H of G, then

$$H\underline{\mathbb{Z}} \wedge X_+ \simeq \bigvee_{i=0}^{k} \sum\nolimits^{W_i} H\underline{\mathbb{Z}}$$

where $D(W_i)$ is a cell of X and $S^{W_0} = S^0$.

Proof. The main step of the proof involves a pushout diagram of the form

$$\begin{tikzcd} S(V) \arrow[r] \arrow[d] & D(V) \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{2.2.18}$$

where we know that $H\underline{\mathbb{Z}} \wedge X \simeq \bigvee_{i=1}^k H\underline{\mathbb{Z}} \wedge S^{W_i}$ for G-representations W_i with $|W_i| \leq |V|$. Look at the cofibre sequence

$$H\underline{\mathbb{Z}} \wedge X \to H\underline{\mathbb{Z}} \wedge Y \to H\underline{\mathbb{Z}} \wedge S^V.$$

The connecting map goes from $H\underline{\mathbb{Z}} \wedge S^V$ to $\bigvee_{i=1}^k H\underline{\mathbb{Z}} \wedge S^{W_i+1}$. For each i, $H\underline{\mathbb{Z}} \wedge S^V \to H\underline{\mathbb{Z}} \wedge S^{W_i+1}$ is in $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})$ where $\alpha = V - W_i - 1$. Using Theorem 2.2.10 for this α we get the connecting map to be 0. Hence the result follows. □

Theorem 2.2.17 is useful to prove that cohomology of G-spaces which are constructed by attaching even cells of the type $D(V)$, is a free module over the cohomology of a point $\underline{H}_G^\bigstar(S^0;\underline{\mathbb{Z}})$. Such results have been proved in [30], [6], [7], in the context of equivariant projective spaces and Grassmannians. A more careful argument has also been used in [13] and [12], where the free module property has been proved for all finite complexes obtained by attaching cells of the type $D(V)$ in even dimensions.

Chapter 3

Homology Decompositions for Projective Spaces

One of the first homology computations encountered in algebraic topology after the calculations for spheres is that of complex projective spaces. The main argument there is that the complex projective spaces have a CW-complex structure with only even cells, and so, the cellular chain complex has zero differential. In the equivariant case, one may try for an analogous result, but the naive construction of a G-CW complex is not particularly useful for this purpose.

3.0.1. Cellular filtration of complex projective spaces. For a complex representation V of G, the equivariant complex projective space $P(V)$ is the set of complex lines in V. It is constructed by attaching even dimensional cells of the type $D(\phi)$ for representations ϕ. We note that $P(V)$ and $P(V \otimes \eta)$ are homeomorphic as G-spaces for a one dimensional G-representation η. As G is abelian, the complex representation V is a direct sum of η_i where $\dim(\eta_i) = 1$. If we write $V = V' \oplus \eta$ for a one-dimensional representation η, we have a cofibre sequence

$$P(V') \to P(V) \to S^{\eta^{-1} \otimes V'}.$$

As a consequence, we obtain a cellular filtration of $P(V)$, which we proceed to describe now. Write $V = \eta_1 + \eta_2 + \cdots + \eta_n$ and $V_i = \eta_1 + \eta_2 + \cdots + \eta_i$. The cellular filtration is given by

$$P(V_1) \subseteq P(V_2) \subseteq \cdots \subseteq P(V_n) = P(V)$$

with cofibre sequences

$$P(V_i) \to P(V_{i+1}) \to S^{\eta_{i+1}^{-1}\otimes V_i}.$$

This shows that $P(V_{i+1})$ is obtained from $P(V_i)$ by attaching a cell of the type $D(\phi_i)$ for $\phi_i = \eta_{i+1}^{-1} \otimes V_i$. Note that this filtration depends on the choice of the ordering of the η_i. Via Theorem 2.2.17, we try for decompositions of $H\underline{\mathbb{Z}} \wedge P(V)$ as a wedge of suspensions of $H\underline{\mathbb{Z}}$.

The results of this chapter form a part of the paper [5]

3.1 Additive homology decompositions for projective spaces

In this section, we discuss computations of the homology of equivariant projective spaces. More precisely, we show that $H\underline{\mathbb{Z}} \wedge P(V)$ is a wedge of suspensions of $H\underline{\mathbb{Z}}$ in many examples. Along the way, we also construct suitable bases for the homology which are used in later sections. We start with an example.

Example 3.1.1. Consider $P(V)$ for $V = \lambda^0+2\lambda+\lambda^2$. We may write $V = \lambda^0+\lambda+\lambda+\lambda^2$ and obtain the corresponding cellular filtration on $P(V)$ following the construction in 3.0.1. The corresponding cells are $D(\phi_i)$ for $\phi_i = \eta_i^{-1} \otimes V_{i-1}$ for $i \leq 4$. Observe that $|\phi_3| < |\phi_4|$ but $|\phi_3^{C_p}| = 2 > 0 = |\phi_4^G|$ which means that the hypothesis of Theorem 2.2.17 is not satisfied. However a simple rearrangement allows us to write down a homology decomposition. Write $V = \lambda^0 + \lambda + \lambda^2 + \lambda$, and we now see that the resulting ϕ_i satisfy the hypothesis of Theorem 2.2.17. With the help of Proposition 1.3.9 this implies

$$H\underline{\mathbb{Z}} \wedge P(V)_+ \simeq H\underline{\mathbb{Z}} \bigvee H\underline{\mathbb{Z}} \wedge S^{\lambda} \bigvee H\underline{\mathbb{Z}} \wedge S^{2\lambda} \bigvee H\underline{\mathbb{Z}} \wedge S^{2\lambda+2}.$$

For the following theorem we fix some notation. Let $V = n_0\lambda^0+n_1\lambda^1+\cdots+n_{p-1}\lambda^{p-1}$ be any C_p-representation. Except for the fact that the n_i's are non-negative, no other condition is imposed on n_i. We may assume $n_0 \geq n_i$ by replacing V with $V \otimes \lambda^{-j}$ if necessary.

Theorem 3.1.2. Let $V = n_0\lambda^0 + n_1\lambda^1 + \cdots + n_{p-1}\lambda^{p-1}$ be a complex C_p-representation and $n_0 \geq n_i \geq 0$ for all i. Then

$$H\underline{\mathbb{Z}} \wedge P(V)_+ \simeq H\underline{\mathbb{Z}} \bigvee_{i=1}^{a_1-1} \Sigma^{i\lambda} H\underline{\mathbb{Z}} \bigvee_{i=a_1-1}^{a_1+a_2-2} \Sigma^{i\lambda+2} H\underline{\mathbb{Z}} \bigvee \cdots \bigvee_{i=(\sum_{j=1}^{n_0-1} a_j)-(n_0-1)}^{(\sum_{j=1}^{n_0} a_j)-n_0} \Sigma^{i\lambda+2(n_0-1)} H\underline{\mathbb{Z}}$$

where a_i is the cardinality of the set $\{n_j : n_j \geq i\}$.

Proof. We arrange the irreducible representations in V in such a way that

$$V = A_1 + A_2 + \cdots + A_{n_0}$$

where $A_1 = \sum_{n_i \geq 1} \lambda^i, A_2 = \sum_{n_i \geq 2} \lambda^i, \ldots, A_{n_0} = \sum_{n_i \geq n_0} \lambda^i$. Then, a_i is the number of summands appearing in A_i.

In this arrangement for V, we rename the irreducible representations to get the cell complex structure on $P(V)$ as in 3.0.1 associated to $V = \sum_{i=1}^{\dim_{\mathbb{C}} V} \eta_i$ where η_i's are defined by

$$\eta_{a_{j-1}+1} + \eta_{a_{j-1}+2} + \cdots + \eta_{a_{j-1}+a_j} = A_j = \sum_{n_i \geq j} \lambda^i$$

for $j \geq 1$, assuming $a_0 = 0$, and the powers of λ above are arranged in increasing order from 0 to $p-1$. To prove the statement, we use induction on the sum $n_0+n_1+\cdots+n_{p-1} = \dim_{\mathbb{C}} V$.

When $n_0 + \cdots + n_{p-1} = 1$, that is, $n_0 = 1$ and $n_i = 0 \ \forall\ 1 \leq i \leq p-1$, then $V = \lambda^0$. Thus $P(V)_+ = S^0$ and

$$H\underline{\mathbb{Z}} \wedge P(V)_+ \simeq H\underline{\mathbb{Z}} \wedge S^0 \simeq H\underline{\mathbb{Z}}.$$

Now suppose that the statement is true for integers less than $n_0 + n_1 + \cdots + n_{p-1}$. Using the notation $V_k = \sum_{i=1}^{k} \eta_i$ as above, the inductive hypothesis implies the result for $X = P(V_k)$ whenever $k < \dim_{\mathbb{C}} V$. In particular, letting $m = \sum n_i - 1$,

$$V = V_m + \eta_{\dim_{\mathbb{C}} V} = V_m + \lambda^s$$

for some integer s. Let a_i', n_i' and A_i' denote the values for V_m that correspond to a_i, n_i and A_i for V. Observe that $a_i' = a_i$ if $i < n_0$, $a_{n_0}' = a_{n_0} - 1$, and our choice of the η_i

implies that $n_s = n_0$. The induction hypothesis implies

$$H\underline{\mathbb{Z}} \wedge P(V_m)_+ \simeq H\underline{\mathbb{Z}} \bigvee_{i=1}^{a_1'-1} \Sigma^{i\lambda} H\underline{\mathbb{Z}} \bigvee_{i=a_1'-1}^{a_1'+a_2'-2} \Sigma^{i\lambda+2} H\underline{\mathbb{Z}} \bigvee \cdots \bigvee_{i=(\sum_{j=1}^{n_0'-1} a_j')-(n_0'-1)}^{(\sum_{j=1}^{n_0'} a_j')-n_0'} \Sigma^{i\lambda+2(n_0'-1)} H\underline{\mathbb{Z}} \tag{3.1.3}$$

We see that, either $s = 0$ or if $s \neq 0$, $n_s = n_0$. We first consider the latter case. Then $n_0' = n_0$, $a_i' = a_i$ whenever $i < n_0$ and $a_{n_0}' = a_{n_0} - 1$. Thus (3.1.3) reduces to

$$H\underline{\mathbb{Z}} \wedge P(V_m)_+ \simeq H\underline{\mathbb{Z}} \bigvee_{i=1}^{a_1-1} \Sigma^{i\lambda} H\underline{\mathbb{Z}} \bigvee_{i=a_1-1}^{a_1+a_2-2} \Sigma^{i\lambda+2} H\underline{\mathbb{Z}} \bigvee \cdots \bigvee_{i=(\sum_{j=1}^{n_0-1} a_j)-(n_0-1)}^{(\sum_{j=1}^{n_0} a_j)-(n_0+1)} \Sigma^{i\lambda+2(n_0-1)} H\underline{\mathbb{Z}}$$

As $n_0 = n_s$, the coefficient of λ^s in $V_m = V - \lambda^s$ is $n_s - 1 = n_0 - 1$, which in turn implies the coefficient of λ^0 in $\lambda^{-s} \otimes V_m$ is $n_0 - 1$. Thus $|(\lambda^{-s} \otimes V_m)^{C_p}| = 2(n_0 - 1)$. We look at representations $i\lambda + j\lambda^0$ where $i \in \{0, 1, \ldots, a_0 + a_1 + \cdots + a_{n_0-1} - 1 - n_0\}$ and $j \in \{0, 1, \ldots, n_0 - 1\}$. Then,

$$|(i\lambda + j\lambda^0)^{C_p}| = 2j \leq 2(n_0 - 1) = |(\lambda^{-s} \otimes V_m)^{C_p}|$$

When $s = 0$ we have $n_0' = n_0 - 1$, $a_i' = a_i$ for all $i < n_0'$. In this case, (3.1.3) reduces to,

$$H\underline{\mathbb{Z}} \wedge P(V_m)_+ \simeq H\underline{\mathbb{Z}} \bigvee_{i=1}^{a_1-1} \Sigma^{i\lambda} H\underline{\mathbb{Z}} \bigvee_{i=a_1-1}^{a_1+a_2-2} \Sigma^{i\lambda+2} H\underline{\mathbb{Z}} \bigvee \cdots \bigvee_{i=(\sum_{j=1}^{n_0-2} a_j)-(n_0-2)}^{(\sum_{j=1}^{n_0-1} a_j)-(n_0-1)} \Sigma^{i\lambda+2(n_0-2)} H\underline{\mathbb{Z}}$$

Note that $a_{n_0} = 1$ that is $a_{n_0} - 1 = 0$. So we can rewrite the equation as

$$H\underline{\mathbb{Z}} \wedge P(V_m)_+ \simeq H\underline{\mathbb{Z}} \bigvee_{i=1}^{a_1-1} \Sigma^{i\lambda} H\underline{\mathbb{Z}} \bigvee_{i=a_1-1}^{a_1+a_2-2} \Sigma^{i\lambda+2} H\underline{\mathbb{Z}} \bigvee \cdots \bigvee_{i=(\sum_{j=1}^{n_0-2} a_j)-(n_0-2)}^{(\sum_{j=1}^{n_0} a_j)-n_0} \Sigma^{i\lambda+2(n_0-2)} H\underline{\mathbb{Z}}$$

Since $s = 0, \lambda^{-s} \otimes V_m = V_m$. Now for all representations of the form $i\lambda + j\lambda^0$ where $i \in \{0, 1, \ldots, (a_0 + a_1 + \cdots + a_{n_0-1} - n_0)\}$ and $j \in \{0, 1, \ldots, (n_0 - 2)\}$ we have

$$|(i\lambda + j\lambda^0)^{C_p}| = 2j \leq 2(n_0 - 2) < 2(n_0 - 1) = |V_m^{C_p}| = |(\lambda^{-s} \otimes V_m)^{C_p}|.$$

These calculations imply that both cases satisfy the hypothesis of Theorem 2.2.17. Thus, we obtain

$$H\underline{\mathbb{Z}} \wedge P(V)_+ \simeq H\underline{\mathbb{Z}} \wedge P(V_m)_+ \bigvee H\underline{\mathbb{Z}} \wedge S^{\lambda^{-s}\otimes V_m}.$$

Using the facts $H\underline{\mathbb{Z}} \wedge S^{\lambda^i} \simeq H\underline{\mathbb{Z}} \wedge S^{\lambda}$ and $H\underline{\mathbb{Z}} \wedge S^{\lambda^0} \simeq H\underline{\mathbb{Z}} \wedge S^2$,

$$H\underline{\mathbb{Z}} \wedge P(V)_+ \simeq H\underline{\mathbb{Z}} \wedge P(V_m)_+ \bigvee H\underline{\mathbb{Z}} \wedge S^{(\sum_{j=1}^{n_0} a_j - n_0)\lambda + 2(n_0-1)}$$

for both $s = 0$ and $s \neq 0$.

□

We elaborate this theorem with an example below.

Example 3.1.4. Let $V = 3\lambda^0 + 2\lambda + 4\lambda^2$ be a complex C_p-representation. Since among the three coefficients appearing in the expression for V here, 4 is the largest, we consider $V \otimes \lambda^{-2} = 4\lambda^0 + 3\lambda^{p-2} + 2\lambda^{p-1}$. For simplicity, we call this also as V. We now write V as a sum of A_i, that is, $V = A_1 + A_2 + A_3 + A_4$, where $A_1 = \lambda^0 + \lambda^{p-2} + \lambda^{p-1}$, $A_2 = \lambda^0 + \lambda^{p-2} + \lambda^{p-1}$, $A_3 = \lambda^0 + \lambda^{p-2}$, $A_4 = \lambda^0$.

Note that $a_1 = 3, a_2 = 3, a_3 = 2$ and $a_4 = 1$. From Theorem 3.1.2, we conclude that

$$H\underline{\mathbb{Z}} \wedge P(V)_+ \simeq H\underline{\mathbb{Z}} \wedge P(V_m)_+ \bigvee H\underline{\mathbb{Z}} \wedge S^{V_m}$$

which is same as

$$H\underline{\mathbb{Z}} \bigvee_{i=1}^{2} \Sigma^{i\lambda} H\underline{\mathbb{Z}} \bigvee_{i=2}^{4} \Sigma^{i\lambda+2} H\underline{\mathbb{Z}} \bigvee_{i=4}^{5} \Sigma^{i\lambda+4} H\underline{\mathbb{Z}} \bigvee \Sigma^{5\lambda+6} H\underline{\mathbb{Z}}.$$

3.2 Decompositions over general cyclic groups

We now proceed towards the discussion of equivariant homology decomposition of complex projective spaces where the group G is any cyclic group C_n. For the next result, note that a complete C_n-universe $\mathcal{U}$ may be constructed as

$$\mathcal{U} = \varinjlim_{m} m\rho$$

where ρ is the regular representation. In the remaining part of the section, we stick to these representations to avoid the involved expressions as in Theorem 3.1.2 for the general cyclic groups.

Theorem 3.2.1. We have the decomposition

$$H\underline{\mathbb{Z}} \wedge P(m\rho)_+ \simeq \bigvee_{i=0}^{nm-1} H\underline{\mathbb{Z}} \wedge S^{\phi_i}$$

where $\phi_0 = 0$ and $\phi_i = \lambda^{-i}(1 + \lambda + \lambda^2 + \cdots + \lambda^{i-1})$ for $i > 0$. Passing to the homotopy colimit, for a C_n-universe $\mathcal{U}$, we obtain

$$H\underline{\mathbb{Z}} \wedge P(\mathcal{U})_+ \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{\phi_i}$$

Proof. We use induction on k to show that

$$H\underline{\mathbb{Z}} \wedge P(V_k)_+ \simeq \bigvee_{i=0}^{k} H\underline{\mathbb{Z}} \wedge S^{\phi_i}$$

for $V_k = \sum_{i=0}^{k} \lambda^i$. The statement holds for $k = 0$ since $V_0 = \lambda^0 = 1_{\mathbb{C}}$ and $H\underline{\mathbb{Z}} \wedge P(V_0)_+ \simeq H\underline{\mathbb{Z}} \wedge S^0$.

Now let the statement be true for V_k i.e $H\underline{\mathbb{Z}} \wedge P(V_k)_+ \simeq \bigvee_{i=0}^{k} H\underline{\mathbb{Z}} \wedge S^{\phi_i}$. We have the cofiber sequence

$$P(V_k)_+ \to P(V_{k+1})_+ \to S^{\phi_{k+1}}.$$

Smashing with $H\underline{\mathbb{Z}}$ we have

$$H\underline{\mathbb{Z}} \wedge P(V_k)_+ \to H\underline{\mathbb{Z}} \wedge P(V_{k+1})_+ \to H\underline{\mathbb{Z}} \wedge S^{\phi_{k+1}}$$

Thus the connecting map goes from $H\underline{\mathbb{Z}} \wedge S^{\phi_{k+1}}$ to $\bigvee_{i=0}^{k} H\underline{\mathbb{Z}} \wedge S^{\phi_i+1}$. For each $i \leq k$ the map $H\underline{\mathbb{Z}} \wedge S^{\phi_{k+1}} \to H\underline{\mathbb{Z}} \wedge S^{\phi_i+1}$ belongs to $\pi_0^{C_n}(H\underline{\mathbb{Z}} \wedge S^{\phi_i+1-\phi_{k+1}}) \simeq \pi_{\phi_{k+1}-\phi_i-1}^{C_n}(H\underline{\mathbb{Z}})$. Taking $\alpha = \phi_{k+1} - \phi_i - 1$, we have $|\alpha|$ is odd. Note that

$$|\alpha^H| = |\phi_{k+1}^H| - |\phi_i^H| - 1 = |(\sum_{j=i+1}^{k+1} \lambda^j)^H| - 1 \geq -1.$$

Applying Theorem 2.2.10 we have that the connecting map is 0. Thus as the above equation is a cofiber sequence of $H\underline{\mathbb{Z}}$-modules and the maps are all $H\underline{\mathbb{Z}}$-module maps, we have

$$H\underline{\mathbb{Z}} \wedge P(V_{k+1})_+ \simeq H\underline{\mathbb{Z}} \wedge P(V_k)_+ \bigvee H\underline{\mathbb{Z}} \wedge S^{\phi_{k+1}}$$

Therefore, by hypothesis we have,

$$H\underline{\mathbb{Z}} \wedge P(V_{k+1})_+ \simeq \bigvee_{i=0}^{k+1} H\underline{\mathbb{Z}} \wedge S^{\phi_i}.$$

Hence the theorem follows. □

3.3 Action by complex conjugation

We may also consider a variant in the case $G = C_2$ which acts on $\mathbb{C}P^n$ by complex conjugation. The resulting C_2-space is denoted $\mathbb{C}P^n{}_\tau$ for $1 \leq n \leq \infty$. Then, the fixed points $(\mathbb{C}P^n{}_\tau)^{C_2} \simeq \mathbb{R}P^n$ which shows that this example is homotopically different from the example above.

Theorem 3.3.1. We have the decomposition

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^n_\tau{}_+ \simeq \bigvee_{i=0}^{n} H\underline{\mathbb{Z}} \wedge S^{i+i\sigma} \simeq \bigvee_{i=0}^{n} H\underline{\mathbb{Z}} \wedge S^{i\rho}.$$

Passing to the homotopy colimit we have

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^\infty_\tau{}_+ \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{i+i\sigma} \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{i\rho}.$$

Proof. The method of the proof is exactly same as 3.2.1. The main step involves the cofibre sequence

$$\mathbb{C}P^{n-1}_\tau{}_+ \hookrightarrow \mathbb{C}P^n_\tau{}_+ \to S^{n+n\sigma}$$

Smashing with $H\underline{\mathbb{Z}}$ gives

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^{n-1}_\tau{}_+ \to H\underline{\mathbb{Z}} \wedge \mathbb{C}P^n_\tau{}_+ \to H\underline{\mathbb{Z}} \wedge S^{n+n\sigma}.$$

The connecting homomorphism goes from

$$H\underline{\mathbb{Z}} \wedge S^{n+n\sigma} \to \bigvee_{i=0}^{n-1} H\underline{\mathbb{Z}} \wedge S^{i+i\sigma+1}.$$

as $H\underline{\mathbb{Z}} \wedge \mathbb{C}P^{n-1}_{\tau}{}_{+} \simeq \bigvee_{i=0}^{n-1} H\underline{\mathbb{Z}} \wedge S^{n+n\sigma}$, by the inductive hypothesis. For each $0 \leq i \leq n-1$, upto homotopy, the map $H\underline{\mathbb{Z}} \wedge S^{n+n\sigma} \to H\underline{\mathbb{Z}} \wedge S^{i+i\sigma+1}$ belongs to $\pi_0(S^{i+1-n+(i-n)\sigma} \wedge H\underline{\mathbb{Z}}) \cong H_G^{i+1-n+(i-n)\sigma}(S^0)$, which is 0 when $i < n-1$ as all the fixed-point dimensions are negative. At $i = n-1$, we get the Mackey functor $\underline{\pi}_0(S^{-\sigma} \wedge H\underline{\mathbb{Z}}) = \underline{H}_{C_2}^{-\sigma}(S^0; \underline{\mathbb{Z}})$, which is zero by Example 2.2.16. Hence the Theorem follows. □

3.4 Quaternionic projective spaces

As in the complex case, the quaternionic projective spaces may be equipped with a cell structure that turn out to be useful from the perspective of homology decompositions [10]. The quaternionic C_n-representation ψ^r is given as multiplication by $e^{\frac{2\pi i r}{n}}$ on $\mathbb{H}$. As a complex C_n-representation, $\psi^r \cong \lambda^r + \lambda^{-r}$. The equivariant projective space $P_{\mathbb{H}}(V)$ for a quaternionic representation V, is the set of lines in V, that is, $P_{\mathbb{H}}(V) = V \setminus \{0\}/\sim$ where $v \sim hv \ \forall \ v \in V \setminus \{0\}$, and $\forall \ h \in \mathbb{H} \setminus \{0\}$. Define $\rho_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{C}} \rho_G$, and note that as $\mathbb{H}$-representations,

$$\rho_{\mathbb{H}} = \sum_{i=0}^{n-1} \psi^i.$$

We write $V_k = \sum_{i=0}^{k-1} \psi^i$ and $\xi_k = \lambda^{-k} \otimes_{\mathbb{C}} V_k$. We recall from [10] that $P_{\mathbb{H}}(m\rho_{\mathbb{H}})$ is a G-cell complex with cells of the form $D(\xi_k)$ for $k \leq mn-1$.

Theorem 3.4.1. Let $G = C_n$. We have the splitting

$$H\underline{\mathbb{Z}} \wedge P_{\mathbb{H}}(m\rho_{\mathbb{H}})_+ \simeq \bigvee_{i=0}^{mn-1} H\underline{\mathbb{Z}} \wedge S^{\xi_i}.$$

and passing to the homotopy colimit, we obtain

$$H\underline{\mathbb{Z}} \wedge B_G S^3_+ \simeq H\underline{\mathbb{Z}} \wedge P_{\mathbb{H}}(\mathcal{U}_{\mathbb{H}})_+ \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{\xi_i}$$

where $\xi_0 = 0$ and $\xi_k = \lambda^{-k} \otimes_{\mathbb{C}} V_k$

Proof. We proceed as in Theorem 3.2.1 by showing via induction on k that

$$H\underline{\mathbb{Z}} \wedge P(V_k)_+ \simeq \bigvee_{i=0}^{k-1} H\underline{\mathbb{Z}} \wedge S^{\xi_i}.$$

Along the way we are required to check $|\xi_k^H - \xi_i^H - 1| \geq -1$ for $i < k$ which implies Theorem 2.2.10 applies, to prove the result. Indeed, it is true as

$$|\xi_i^{C_d}| = \begin{cases} \lfloor \frac{2i-1}{d} \rfloor + 1 & \text{if } d \mid i \\ \lfloor \frac{2i-1}{d} \rfloor & \text{if } d \nmid i, \end{cases}$$

is a non-decreasing function of i. Hence our result follows. □

Let $G = C_n$.

Next we define the classes α_{ϕ_ℓ} which serve as additive generators of $H_G^\bigstar(P(\mathcal{U}))$.

3.4.2. Construction of a homology basis. Let V_ℓ and ϕ_ℓ be the representations

$$V_\ell := 1_\mathbb{C} + \lambda + \cdots + \lambda^\ell \quad \text{and} \quad \phi_\ell := \lambda^{-\ell}(1_\mathbb{C} + \lambda + \cdots + \lambda^{\ell-1}). \tag{3.4.3}$$

Consider the cofibre sequence

$$P(V_{\ell-1})_+ \hookrightarrow P(V_\ell)_+ \xrightarrow{\chi} S^{\phi_\ell}.$$

At $\deg \phi_\ell$, the associated long exact sequence is

$$\cdots \to \tilde{H}_G^{\phi_\ell - 1}(P(V_{\ell-1})_+) \to \tilde{H}_G^{\phi_\ell}(S^{\phi_\ell}) \xrightarrow{\chi^*} \tilde{H}_G^{\phi_\ell}(P(V_\ell)_+) \to \tilde{H}_G^{\phi_\ell}(P(V_{\ell-1})_+) \to \cdots$$

Note that, $\underline{H}_G^{\phi_\ell - 1}(P(V_{\ell-1})_+)(G/e) \cong H^{2\ell-1}(P(\mathbb{C}^\ell)) \cong 0$. So restriction of the map χ^* at the orbit G/e is an isomorphism. Hence, the Mackey functor diagram says that the image of $1 \in \tilde{H}_G^{\phi_\ell}(S^{\phi_\ell}) \cong H_G^0(\mathrm{pt}) \cong \mathbb{Z}$ under the map χ^* is nonzero. Define $\alpha_{\phi_\ell}^{V_\ell}$ to be the element $\chi^*(1)$. We often omit the superscript and simply write α_{ϕ_ℓ}.

Now we lift α_{ϕ_ℓ} by induction to get the generator $\alpha_{\phi_\ell}^{\mathcal{U}}$ (or simply α_{ϕ_ℓ}) which belongs to $H_G^{\phi_\ell}(P(\mathcal{U}))$. For this we successively add representations (one at a time) to V_ℓ in a proper order to reach $\mathcal{U}$. Let $U' \subseteq \mathcal{U}$ be a representation containing V_ℓ. Assume that for U' the class α_{ϕ_ℓ} has been defined for $H_G^{\phi_\ell}(P(U'))$. Suppose $U = U' + \lambda^j$. Consider the cofibre sequence

$$P(U')_+ \xrightarrow{\theta} P(U)_+ \to S^{\lambda^{-j}U'}$$

and thus the long exact sequence

$$\cdots \to \tilde{H}_G^{\phi_\ell}(S^{\lambda^{-j}U'}) \to \tilde{H}_G^{\phi_\ell}(P(U)_+) \xrightarrow{\theta^*} \tilde{H}_G^{\phi_\ell}(P(U')_+) \to \tilde{H}_G^{\phi_\ell+1}(S^{\lambda^{-j}U'}) \to \cdots \quad (3.4.4)$$

By Proposition 2.2.15 and 2.2.10, the first and the fourth term in 3.4.4 is zero. So the map θ^* is an isomorphism. Hence a unique lift of α_{ϕ_ℓ} exists in $H_G^{\phi_\ell}(P(U))$ along the map

$$\theta^* : \tilde{H}_G^{\phi_\ell}(P(U)_+) \to \tilde{H}_G^{\phi_\ell}(P(U')_+).$$

Since the restriction of the map χ^* to the orbit G/e is an isomorphism, we get

$$\mathrm{res}_e^G(\alpha_{\phi_\ell}) = x^\ell. \quad (3.4.5)$$

3.4.6. Additive generators of $H_G^{\bigstar}(P(\mathcal{U}))$. By Theorem 3.2.1, we may express the additive structure of the cohomology of $P(\mathcal{U})$ as

$$H_G^{\bigstar}(P(\mathcal{U})) = \bigoplus_{k=0}^{\infty}\bigoplus_{i=0}^{n-1} H_G^{\bigstar-k\phi_n-\phi_i}(\mathrm{pt}), \quad (3.4.7)$$

where $\phi_i = \lambda^{-i}(1_{\mathbb{C}} + \lambda + \cdots + \lambda^{i-1})$ and $\phi_0 = 0$. The above construction defines the generators $\alpha_{k\phi_n+\phi_i} \in H_G^{k\phi_n+\phi_i}(P(\mathcal{U}))$ which corresponds to the factor $H_G^{\bigstar-k\phi_n-\phi_i}(\mathrm{pt})$ in (3.4.7). Summarizing the above, we get

Proposition 3.4.8. Let $G = C_n$. Then additively, the classes $\alpha_{k\phi_n+\phi_i}$ generates $H_G^{\bigstar}(P(\mathcal{U}))$ as a module over $H_G^{\bigstar}(\mathrm{pt})$ where $0 \le k < \infty$ and $0 \le i \le n-1$.

3.5 Application for cohomology operations

Let $G = C_p$ (p not necessarily odd). Recall that $\mathcal{A}_p$ is the mod p Steenrod algebra. We consider

$$\mathcal{A}_G^n = \{H\underline{\mathbb{Z}/p}, \Sigma^n H\underline{\mathbb{Z}/p}\}^G,$$

for $n \in \mathbb{Z}$, and the map

$$\Omega : \mathcal{A}_G^* \to \mathcal{A}_p,$$

as its restriction to the identity subgroup. We demonstrate that the additive decomposition of §3.1 recovers the following result of Caruso [9].

Theorem 3.5.1. Let θ be a degree r (r is even) cohomology operation not involving the Böckstein β. For such θ, there does not exist an equivariant cohomology operation

$$\tilde{\theta} : H\underline{\mathbb{Z}/p} \to \Sigma^r H\underline{\mathbb{Z}/p}$$

such that $\Omega(\tilde{\theta}) = \theta$.

Before going into the proof, let us look at some examples.

Example 3.5.2. Let p be odd. We claim that there does not exist an equivariant cohomology operation

$$\tilde{P^1} : H\underline{\mathbb{Z}/p} \to \Sigma^{2p-2} H\underline{\mathbb{Z}/p}$$

such that $\Omega(\tilde{P^1}) = P^1$, where P^1 is the power operation. The existence of such a $\tilde{P^1}$ will lead to a map of Mackey functors

$$\underline{H}^{\alpha}_G(X;\underline{\mathbb{Z}/p}) \to \underline{H}^{\alpha+2p-2}_G(X;\underline{\mathbb{Z}/p}) \qquad \text{for } \alpha \in RO(G),$$

which is natural in X. In particular, let us take $X = P(\mathcal{U})$ and $\alpha = \lambda$. We observe that

$$\begin{tikzcd} \underline{H}^{\lambda}_G(P(\mathcal{U});\underline{\mathbb{Z}/p}) \arrow[r, "\tilde{P^1}"] \arrow[d, "\cong"] & \underline{H}^{\lambda+2p-2}_G(P(\mathcal{U});\underline{\mathbb{Z}/p}) \arrow[d, "\cong"] \\ \underline{\mathbb{Z}/p} & \underline{\mathbb{Z}/p}^*. \end{tikzcd} \tag{3.5.3}$$

An explanation for this is as follows. For the group C_p, the additive decomposition in Theorem 3.2.1 tells us

$$P(\mathcal{U}) \wedge H\underline{\mathbb{Z}} \simeq \bigvee_{k=0}^{\infty} \bigvee_{i=0}^{p-1} S^{k\phi_p + i\lambda} \qquad \text{such that } (k,i) \neq (0,0). \tag{3.5.4}$$

As a result,

$$\tilde{H}^{\bigstar}_G(P(\mathcal{U});\underline{\mathbb{Z}/p}) \cong \bigoplus_{k=0}^{\infty} \bigoplus_{i=0}^{p-1} \tilde{H}^{\bigstar - k\phi_p - i\lambda}_G(S^0;\underline{\mathbb{Z}/p}), \qquad (k,i) \neq (0,0).$$

Let $\alpha = \lambda + 2p - 2 - k\phi_p - i\lambda$. So $|\alpha| = 2p - 2kp - 2i$ and $|\alpha^{C_p}| = 2p - 2 - 2k$. Applying (2.1.8), we conclude that $\underline{H}^{\lambda+2p-2}_G(P(\mathcal{U});\underline{\mathbb{Z}/p}) \cong \underline{\mathbb{Z}/p}^*$.

As $P^1(x) = x^p$ for a generator $x \in H^2(\mathbb{C}P^\infty)$, the diagram 3.5.3 yields a commutative diagram

$$\begin{tikzcd} \mathbb{Z}/p \arrow[r, "\tilde{P^1}"] \arrow[d, bend right, "1"'] & \mathbb{Z}/p \arrow[d, bend right, "0"'] \\ \mathbb{Z}/p \arrow[r, "\cong"] \arrow[u, bend right, "0"'] & \mathbb{Z}/p, \arrow[u, bend right, "1"'] \end{tikzcd}$$

which is a contradiction.

The technique used in example 3.5.2 does not work when $p = 2$. This is because the Mackey functor $\underline{\mathbb{Z}/2}$ may now appear in the right hand side of the diagram 3.5.3, and so the contradiction drawn out by comparing the Mackey functor diagram fails. Below we argue differently to show that Sq^2 is not in the image of Ω.

Example 3.5.5. Let $p = 2$. There does not exist an equivariant cohomology operation

$$\tilde{Sq^2} : H\underline{\mathbb{Z}/2} \to \Sigma^2 H\underline{\mathbb{Z}/2}$$

such that $\Omega(\tilde{Sq^2}) = Sq^2$. The existence of such will lead to a map of Mackey functors

$$\underline{H}^\alpha_G(X; \underline{\mathbb{Z}/2}) \to \underline{H}^{\alpha+2}_G(X; \underline{\mathbb{Z}/2}) \quad \text{for } \alpha \in RO(G),$$

Let $X = \mathbb{C}P^2_\tau$, the complex projective space with the conjugation action. Taking $\alpha = \rho = 1 + \sigma$

$$\begin{tikzcd} \underline{H}^\rho_G(\mathbb{C}P^2_\tau; \underline{\mathbb{Z}/2}) \arrow[r, "\tilde{Sq^2}"] \arrow[d, "\cong"] & \underline{H}^{\rho+2}_G(\mathbb{C}P^2_\tau; \underline{\mathbb{Z}/2}) \arrow[d, "\cong"] \\ \underline{\mathbb{Z}/2} & \langle \Lambda \rangle. \end{tikzcd} \tag{3.5.6}$$

To see this recall from Proposition 3.3.1 that

$$\mathbb{C}P^2_\tau \wedge H\mathbb{Z} \simeq \bigvee_{i=1}^{2} S^{i+i\sigma} \wedge H\underline{\mathbb{Z}}.$$

Hence

$$\tilde{H}^\bigstar_{C_2}(\mathbb{C}P^2_\tau; \underline{\mathbb{Z}/2}) \cong \bigoplus_{i=1}^{2} \tilde{H}^{\bigstar - i - i\sigma}_{C_2}(S^0; \underline{\mathbb{Z}/2}).$$

Applying (2.1.9), we obtain the required Mackey functors in (3.5.6). Since $Sq^2(x) = x^2$ for a generator $x \in H^2(\mathbb{C}P^\infty)$, the diagram (3.5.6) yields the following commutative diagram

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r, "\tilde{Sq}^2"] \arrow[d, bend right, "1"'] & 0 \arrow[d, bend right, "0"'] \\ \mathbb{Z}/2 \arrow[r, "\cong"] \arrow[u, bend right, "0"'] & \mathbb{Z}/2, \arrow[u, bend right, "0"'] \end{tikzcd}$$

which is a contradiction.

Now we demonstrate the result in general. Let $P(\mathcal{U})^{\wedge r}$ be the smash product of r-copies of $P(\mathcal{U})$. Equation (3.5.4) gives us

$$\begin{aligned} F(P(\mathcal{U})^{\wedge r}, H\underline{\mathbb{Z}/p}) &\simeq F_{H\underline{\mathbb{Z}}-\mathrm{mod}}(P(\mathcal{U})^{\wedge r} \wedge H\underline{\mathbb{Z}}, H\underline{\mathbb{Z}/p}) \\ &\simeq \bigvee_{k_j=0}^{\infty} \bigvee_{i_j=0}^{p-1} S^{-(k_1\phi_p+i_1\lambda+\cdots+k_r\phi_p+i_r\lambda)} \wedge H\underline{\mathbb{Z}/p} \end{aligned}$$

where $j \in \{1, \cdots, r\}$ and $(k_j, i_j) \neq (0,0)$. The last equivalence comes from the fact that $P(\mathcal{U})^{\wedge r} \wedge H\underline{\mathbb{Z}}$ is a wedge of suspensions of $H\underline{\mathbb{Z}}$ with finitely many $\Sigma^V H\underline{\mathbb{Z}}$ of a given dimension V. Hence

$$\tilde{H}^{\bigstar}_{C_p}(P(\mathcal{U})^{\wedge r}; \underline{\mathbb{Z}/p}) \cong \bigoplus_{k_j=0}^{\infty} \bigoplus_{i_j=0}^{p-1} \tilde{H}_G^{\bigstar-(k_1\phi_p+i_1\lambda+\cdots+k_r\phi_p+i_r\lambda)}(S^0; \underline{\mathbb{Z}/p})$$

where $j \in \{1, \cdots, r\}$ and $(k_j, i_j) \neq (0,0)$. We are now ready to prove Theorem 3.5.1.

Proof of the Theorem 3.5.1 when p is odd. Let θ be a cohomology operation of degree r for r even, such that θ does not involve the Böckstein. Note that this condition implies $(p-1) \mid r$. Let $s = \frac{r}{p-1}$. Consider the element $x_1 \otimes \cdots \otimes x_s \in \tilde{H}^{2s}(\mathbb{C}P^{\infty \wedge s}; \mathbb{Z}/p)$, where each x_i is a generator of $\tilde{H}^2(\mathbb{C}P^\infty; \mathbb{Z}/p)$. By an argument analogous to [35, Ch. 3, Theorem 2] for the prime 2, we obtain $\theta(x_1 \otimes \cdots \otimes x_s) \neq 0$. Now suppose we have a map

$$\tilde{\theta} : H\underline{\mathbb{Z}/p} \to \Sigma^r H\underline{\mathbb{Z}/p}$$

such that $\Omega(\tilde{\theta}) = \theta$. This will lead to a map of Mackey functors

$$\tilde{\theta} : \underline{H}^{\alpha}_G(X; \underline{\mathbb{Z}/p}) \to \underline{H}^{\alpha+r}_G(X; \underline{\mathbb{Z}/p}).$$

Let us take $X = P(\mathcal{U})^{\wedge s}$ and $\alpha = s\lambda$. We observe that

$$\begin{tikzcd} \underline{H}_G^{s\lambda}(P(\mathcal{U})^{\wedge s};\underline{\mathbb{Z}/p}) \arrow[r, "\tilde{\theta}"] \arrow[d, "\cong"] & \underline{H}_G^{s\lambda+r}(P(\mathcal{U})^{\wedge s};\underline{\mathbb{Z}/p}) \arrow[r, "proj"] \arrow[d, "\cong"] & \underline{\mathbb{Z}/p}^* \\ \underline{\mathbb{Z}/p} & \underline{\mathbb{Z}/p}^{*\oplus t} \oplus \langle \mathbb{Z}/p \rangle^{\oplus \ell} & \end{tikzcd} \tag{3.5.7}$$

for some integer $t \geq 1$ and $\ell \geq 0$. To see this, let $\alpha = s\lambda - k_1\phi_p - i_1\lambda - \cdots - k_s\phi_p - i_s\lambda$ and $\tilde{\alpha} = s\lambda + r - \tilde{k}_1\phi_p - \tilde{i}_1\lambda - \cdots - \tilde{k}_s\phi_p - \tilde{i}_s\lambda$. The condition $(k_j, i_j) \neq (0,0)$ implies $|\alpha| = 2s - 2p(k_1 + \cdots + k_s) - 2(i_1 + \cdots + i_s) \leq 0$. If $|\alpha| < 0$, then the Mackey functor is zero by 2.1.8. So the left hand side of diagram 3.5.7 turns out to be $\underline{\mathbb{Z}/p}$. However the Mackey functor $\underline{\mathbb{Z}/p}$ can not appear in the right hand side as the condition $|\tilde{\alpha}| = 0$ forces $|\tilde{\alpha}^{C_p}|$ to be > 0. This is because

$$|\tilde{\alpha}| = 2s + r - 2p(\tilde{k}_1 + \cdots + \tilde{k}_s) - 2(\tilde{i}_1 + \cdots + \tilde{i}_s) = 0$$

implies some $i_j \neq 0$. Now if $|\tilde{\alpha}^{C_p}| = r - 2(\tilde{k}_1 + \cdots + \tilde{k}_r) \leq 0$ then $|\tilde{\alpha}| \leq 2s + (2 - 2p)(\tilde{k}_1 + \cdots + \tilde{k}_r) - 2(\tilde{i}_1 + \cdots + \tilde{i}_r) < 0$. So $|\tilde{\alpha}^{C_p}| > 0$. Since $\theta(x_1 \otimes \cdots \otimes x_s) \neq 0$, the map $\underline{\mathbb{Z}/p}(G/e) \to \underline{\mathbb{Z}/p}^*(G/e)$ is an isomorphism. As in the example 3.5.2, this gives a contradiction. $\square$

As before $p = 2$ case require a different argument which we detail below.

Proof of the Theorem 3.5.1 when $p = 2$. Let θ be a cohomology operation of degree $2r$, such that θ does not involve the Böckstein. By an argument analogous to [35, Ch. 3, Theorem 2], we obtain $\theta(x_1 \otimes \cdots \otimes x_r) \neq 0$ where each x_i is a generator of $\tilde{H}^2(\mathbb{C}P^\infty; \mathbb{Z}/p)$ and $x_1 \otimes \cdots \otimes x_r \in \tilde{H}^{2r}(\mathbb{C}P^{\infty\wedge r}; \mathbb{Z}/p)$. Now suppose we have a map

$$\tilde{\theta} : H\underline{\mathbb{Z}/2} \to \Sigma^{2r} H\underline{\mathbb{Z}/2}$$

such that $\Omega(\tilde{\theta}) = \theta$. This will lead to a map of Mackey functors

$$\tilde{\theta} : \underline{H}_G^{\alpha}(X; \underline{\mathbb{Z}/2}) \to \underline{H}_G^{\alpha+2r}(X; \underline{\mathbb{Z}/2}).$$

Let us take $X = \mathbb{C}P_\tau^{\infty\wedge r}$ and $\alpha = r\rho = r + r\sigma$. With the help of Proposition 3.3.1 and (2.1.9) we derive that

$$\begin{tikzcd} \underline{H}_G^{r\rho}(\mathbb{C}P_\tau^{\infty\wedge r};\underline{\mathbb{Z}/2}) \arrow[r, "\tilde{\theta}"] \arrow[d, "\cong"] & \underline{H}_G^{r\rho+2r}(\mathbb{C}P_\tau^{\infty\wedge r};\underline{\mathbb{Z}/2}) \arrow[r, "proj"] \arrow[d, "\cong"] & \underline{\mathbb{Z}/2}^* \\ \underline{\mathbb{Z}/2} & \underline{\mathbb{Z}/2}^* \oplus \langle\mathbb{Z}/2\rangle^{\oplus\ell} & \end{tikzcd}$$

for some integer $\ell \geq 0$. Since $\theta(x_1 \otimes \cdots \otimes x_r) \neq 0$, the map $\underline{\mathbb{Z}/2}(G/e) \to \underline{\mathbb{Z}/2}^*(G/e)$ is an isomorphism. As in the examples, this gives a contradiction. $\square$

Note that here we are doing this application specifically for integer grading. But in general this computation is not true in $RO(G)$-grading as visible from computation in [28].

3.6 Slice tower of $P(V)_+ \wedge H\mathbb{Z}$

The slice filtration in the equivariant stable homotopy category was introduced by D. Dugger and used by Hill, Hopkins and Ravenel in their proof of the Kervaire invariant one problem [20] (see also [21]). The associated slice tower is an equivariant analogue of the Postnikov tower. We use the regular slice filtration on equivariant spectra (see [38]), which differs from the original formulation by a shift of one [38, Proposition 3.1].

Definition 3.6.1. Let $\tau_{\geq n}$ denote the localizing subcategory of genuine G-spectra which is generated by G-spectra of the form $G_+ \wedge_H S^{k\rho_H}$, where ρ_H is the (real) regular representation of H and $k|H| \geq n$.

Let E be a G-spectrum. Then E is said to be *slice n-connective* (written as $E \geq n$) if $E \in \tau_{\geq n}$, and E is said to be *slice n-coconnective* (written as $E < n$) if

$$[S^{k\rho_H + r}, E]^H = 0$$

for all subgroup $H \leq G$ such that $k|H| \geq n$ and for all $r \geq 0$. We say E is an *n-slice* if $n \leq E \leq n$.

In [25], the authors provide an alternative criterion for something being slice connective using the geometric fixed point functor.

Theorem 3.6.2. [25, Theorem 3.2] The representation sphere S^V is in $\tau_{\geq n}$ if and only if for all $H \subset G$,

$$\dim V^H \geq n/|H|.$$

We note the following result from [21, Proposition 2.23].

Proposition 3.6.3. If X is in $\tau_{\geq 0}$ and Y is in $\tau_{\geq n}$, then $X \wedge Y$ is in $\tau_{\geq n}$.

There has been a large number of computations of slices for equivariant spectra. They are either carried out in the case of MU or its variants ([20, 23, 24]) or for spectra of the form $\Sigma^n H\underline{\mathbb{Z}}$ ([22, 41, 17, 14, 36]). We show that our cellular filtration 3.0.1 actually yields the slice filtration for $P(\mathcal{U})_+ \wedge H\underline{\mathbb{Z}}$ up to a suspension and in addition, the additive decomposition proves that the slice tower is degenerate in the sense that the maps possess sections.

Theorem 3.6.4. The slice towers of $\Sigma^2 P(\mathcal{U})_+ \wedge H\underline{\mathbb{Z}}$ and $\Sigma^2 P(n\rho)_+ \wedge H\underline{\mathbb{Z}}$ are degenerate and these spectra are a wedge of slices of the form $S^\phi \wedge H\underline{\mathbb{Z}}$.

Proof. Theorem 3.2.1 allows us to write

$$P(\mathcal{U})_+ \wedge H\underline{\mathbb{Z}} \simeq \bigvee_{\ell=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{\phi_\ell},$$

where $\phi_\ell = \lambda^{-l}(\lambda^0 + \cdots + \lambda^{\ell-1})$ and $\phi_0 = 0$. We claim that each of $\Sigma^2 S^{\phi_\ell} \wedge H\underline{\mathbb{Z}}$ is a $2\ell+2$-slice. Let $H = C_m$, a subgroup of G. We verify

$$[S^{k\rho_H + r}, S^{res_H(\phi_\ell)+2} \wedge H\underline{\mathbb{Z}}]^H \cong H^\alpha_H(S^0; \underline{\mathbb{Z}}) = 0$$

for $k|H| > 2\ell + 2$ and $\alpha = \mathrm{res}_H(\phi_\ell) + 2 - r - k\rho_H$. We may write $\ell = qm + s$, where $s < m$, and thus $\mathrm{res}_H(\phi_\ell) = 2q\rho_H + \lambda + \cdots + \lambda^s$. Since $km > 2\ell + 2 = 2qm + 2s + 2$, we get $k > 2q$. Let $k = 2q + j$, so that

$$\alpha = -r - (\lambda^{s+1} + \cdots + \lambda^{m-1}) - (j-1)\rho_H.$$

Therefore either by Proposition 2.2.15, or using the fact that all the fixed point dimensions of α are negative, we have $H^\alpha_H(S^0; \underline{\mathbb{Z}}) = 0$.

To show $S^{\phi_\ell+2} \wedge H\underline{\mathbb{Z}} \geq 2\ell+2$, using Proposition 3.6.3, it is enough to prove $S^{\phi_\ell+2} \in \tau_{\geq 2\ell+2}$ as $H\underline{\mathbb{Z}}$ is a 0-slice [20, Proposition 4.50]. Appealing to Theorem 3.6.2, $S^{\phi_\ell+2} \in$

$\tau_{\geq 2\ell+2}$ as

$$\frac{2\ell+2}{m} = 2q + \frac{2s+2}{m} \leq 2q+2 = \dim(S^{\phi_\ell+2})^H \quad \text{for all } H \subset G.$$

□

Theorem 3.6.5. The slice towers of $\Sigma^4 P(\mathcal{U}_{\mathbb{H}})_+ \wedge H\underline{\mathbb{Z}}$ and $\Sigma^4 P(n\rho_{\mathbb{H}})_+ \wedge H\underline{\mathbb{Z}}$ are degenerate and these spectra are a wedge of slices of the form $S^\xi \wedge H\underline{\mathbb{Z}}$.

Proof. Theorem 3.4.1 allows us to write

$$P(\mathcal{U}_{\mathbb{H}})_+ \wedge H\underline{\mathbb{Z}} \simeq \bigvee_{\ell=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{\xi_\ell},$$

where $\xi_\ell = \lambda^{-\ell} \otimes (\psi^1 + \cdots + \psi^{\ell-1})$ and $\xi_0 = 0$. Let $H = C_m$, a subgroup of G. We may write $\ell = qm + s$, where $s < m$. Consequently, $\mathrm{res}_H(\xi_\ell) = 4q\rho_H + \lambda + \cdots + \lambda^s$. Proceeding as in the case of Theorem 3.6.4, we deduce that each of $\Sigma^4 S^{\xi_\ell} \wedge H\underline{\mathbb{Z}}$ is a $4\ell + 4$-slice. □

Chapter 4

Homology decompositions for connected sums

In this chapter, we discuss the equivariant homology of a connected sum of copies of $\mathbb{C}P^2$. These connected sums possess an action by a cyclic group of odd order, obtained by coherently amalgamating individual actions on each copy. The data of the amalgamation is expressed via trees where weights are systematically assigned to each vertex. We define these below, and prove that in many cases, the homology over $\underline{\mathbb{Z}}$ splits as a wedge of suspensions of $\pi_\bigstar(H\underline{\mathbb{Z}})$. The results in this section appear in the paper [4].

4.1 Tree manifolds

In this section we discuss the construction of connected sum of G-manifolds focussing on the special case of a connected sum of complex projective planes in the case $\mathcal{G} = C_m$, m odd. In the latter case, the construction is governed through a system of explicit combinatorial data expressed as admissible weighted trees (see [18] for details). We refer to these as *tree manifolds*.

4.1.1. Equivariant connected sums. Let X and Y be two smooth G-manifolds of the same dimension n. The equivariant connected sum $X\#Y$ depends on the following data

1) Points $x \in X^G$, $y \in Y^G$.

2) An orientation reversing isomorphism of real G-representations $\varphi : T_xX \to T_yY$.

Given the data above, one may conjugate φ with the exponential map to obtain a diffeomorphism of punctured disks near x and y. This identification is then performed on $X\setminus\{x\}\sqcup Y\setminus\{y\}$ to obtain the equivariant connected sum $X\#Y$. One readily observes the following homotopy cofibration sequences

$$X\setminus\{x\}\to X\#Y\to Y, \text{ and } \quad Y\setminus\{y\}\to X\#Y\to X. \tag{4.1.2}$$

An additional feature in the G-equivariant situation is the orbit-wise connected sum. Let X be a G-manifold and Y an N-manifold for a subgroup N. The data underlying an orbit-wise connected sum is

1) A point $y\in Y^N$, and a point $x\in X$ such that the stabilizer of x is N.

2) An orientation reversing isomorphism of real N-representations $\varphi: T_xX\to T_yY$.

The condition 1) implies that x induces the inclusion of an orbit $i_x: G/N\to X$. Now we may again use the exponential map to conjugate φ and identify punctured disks at points of $G/N\hookrightarrow X$ with those at points of $G/N\times\{y\}\hookrightarrow G\times_N Y$. The resulting connected sum is denoted by $X\#G\times_N Y$. The direct analogues of (4.1.2) are

$$X\setminus\{x\}\to X\#G\times_N Y\to \frac{G\times_N Y}{G/N\times\{y\}},$$

and,

$$G\times_N(Y\setminus\{y\})\to X\#G\times_N Y\to X/i_x(G/N).$$

The second sequence has a refinement in the form of a homotopy pushout

$$\begin{tikzcd} G\times_N(Y-\{y\}) \arrow[r] \arrow[d] & G\times_N C(Y-\{y\}) \arrow[d, "i_x\circ\pi_1"] \\ X\#G\times_N Y \arrow[r] & X. \end{tikzcd} \tag{4.1.3}$$

4.1.4. Linear actions on projective spaces. The principal construction of interest in this chapter is the equivariant connected sum of projective spaces. A method to construct a G-action on a complex projective space $\mathbb{C}P^n$ is to write it as $P(V)$, the projectivization of a unitary representation V. We call these *linear actions*. If ν is a 1-dimensional complex representation of G, there is an equivariant homeomorphism $P(V)\cong P(V\otimes\nu)$. We mention here that the notation G is used for cyclic group of any order whereas $\mathcal{G}$ is the notation for cyclic groups of odd order in particular.

Our principal objects of interest are linear $\mathcal{G}$-actions on $\mathbb{C}P^2$, that is, we write $\mathbb{C}P^2$ as

$P(V)$ where V is a 3-dimensional complex representation of $\mathcal{G}$. In terms of the notation above, V is a sum $\lambda^a + \lambda^b + \lambda^c$ for some integers a, b and c viewed $\pmod m$. As $P(V) \cong P(V \otimes \nu)$ for 1-dimensional ν, we may assume $c = 0$ in the expression for V. We denote this by $\mathbb{C}P^2(a, b; m)$. Often we use the notation $\mathbb{C}P^2(a, b; m')$ for a divisor m' of m. This denotes the $C_{m'}$-space $P(1_{\mathbb{C}} + \lambda^a + \lambda^b)$. In this expression, note that the restriction of $\mathbb{C}P^2(a, b; m)$ to the subgroup $C_{m'}$ is $\mathbb{C}P^2(a, b; m')$.

Proposition 4.1.5. The manifolds $\mathbb{C}P^2(a, b; m)$ satisfy the following properties.
1) If $\gcd(a, b, m) = d$, then $\mathbb{C}P^2(a, b; m) \cong \pi^*\mathbb{C}P^2(\frac{a}{d}, \frac{b}{d}; \frac{m}{d})$ the pullback via $\pi : C_m \to C_m/C_d \cong C_{m/d}$.
2) There are $\mathcal{G}$-homeomorphisms

$$\mathbb{C}P^2(a, b; m) \cong \mathbb{C}P^2(a - b, -b; m) \cong \mathbb{C}P^2(-a, b - a; m),$$

and $\mathbb{C}P^2(a, b; m) \cong \mathbb{C}P^2(b, a; m)$.
3) The points $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$, and $p_3 = [0, 0, 1]$ are fixed by $\mathcal{G}$. Their tangential representations are given by

$$T_{p_1}\mathbb{C}P^2(a, b; m) \cong \lambda^{b-a} + \lambda^{-a}, T_{p_2}\mathbb{C}P^2(a, b; m) \cong \lambda^{a-b} + \lambda^{-b},$$
$$T_{p_3}\mathbb{C}P^2(a, b; m) \cong \lambda^a + \lambda^b.$$

The proof of the above easily follows from the homeomorphism $P(V) \cong P(V \otimes \nu)$, and the identification of the tangent bundle of $\mathbb{C}P^2$ as $Hom(\gamma, \gamma^\perp)$, where γ is the canonical line bundle. We call the numbers (a, b) associated to the representation $\lambda^a + \lambda^b$ *rotation numbers*. As in [18], we assume that for the manifold denoted by $\mathbb{C}P^2(a, b; m)$, $\gcd(a, b, m) = 1$.

We also denote $S^4(a, b; m)$ for the $\mathcal{G}$-action on S^4 by identifying it with $S^{\lambda^a + \lambda^b}$. This may also be described as $S(1 + \lambda^a + \lambda^b)$, where 1 is the trivial real representation of dimension 1. This action has fixed points 0 and ∞, and the tangential representations are $\lambda^a + \lambda^b$ and $\lambda^{-a} + \lambda^{-b}$ respectively.

We now list the conditions required to form equivariant connected sums of copies of $\mathbb{C}P^2(a, b; m)$ and $S^4(a, b; m)$.

Proposition 4.1.6. 1) The connected sum $\mathbb{C}P^2(a, b; m)\#\mathbb{C}P^2(a', b'; m)$ may be formed if and only if for one of the equivalent choices of (a', b') as in 2) of Proposition 4.1.5,

$\pm(a',b') \in \{(a,-b),(a-b,b),(a,b-a)\}$. Once this condition is satisfied, there is a natural choice of data for the connected sum unless $a=b$ or one of a, b is 0.

2) The connected sum $\mathbb{C}P^2(a,b;m)\#S^4(a',b';m)$ may be formed if and only if $\pm(a',b') \in \{(a,-b),(a-b,b),(a,b-a)\}$. Here, $\mathbb{C}P^2(a,b;m)\#S^4(a',b';m)$ is $\mathcal{G}$-homeomorphic to $\mathbb{C}P^2(a,b;m)$.

3) For $m' \mid m$ but $m' \neq m$, the connected sum $\mathbb{C}P^2(a,b;m)\#\mathcal{G}\times_{C_{m'}}\mathbb{C}P^2(a',b';m')$ may be defined if and only if the following are satisfied

a) One of a, b, or $a-b$ satisfies the equation $\gcd(x,m)=m'$.

b) One of a', b', or $a'-b'$ is 0, and the others are up to sign two of the numbers in $\{-b,-a,b-a\}$ not divisible by m'.

The statements 1) and 2) follow from the examination of tangential representations at fixed points of $\mathbb{C}P^2(a,b;m)$, and 3) follows from the results in [18, §1.C].

4.1.7. Admissible weighted trees. The observations above inform us that the $\mathcal{G}$-connected sums of different $\mathbb{C}P^2(a,b;m)$s and $S^4(a,b;m)$s may be formed only when certain relations are satisfied between the weights involved. We now lay down the sequence of combinatorial criteria which allow us to form such a connected sum. These are written in the form of weights attached to trees with a $\mathcal{G}$-action satisfying required conditions, called *admissible weighted trees*.

Recall that a group action on a tree is given by an action on the vertices which preserves the adjacency relation. We define two types of trees called *Type I* and *Type II*.

Definition 4.1.8. An *admissible weighted tree* is a tree with $\mathcal{G}$-action having the following properties

1. There is a $\mathcal{G}$-fixed vertex v_0 called the root vertex of the tree. In case of a type II tree, v_0 is the unique $\mathcal{G}$-fixed vertex.

2. The vertices of $\mathbb{T}$ are arranged in levels starting from zero according to the distance from the root vertex with edge length considered to be 1. Observe that $\mathcal{G}$ preserves the levels and every edge goes from level L to $L+1$ for some L.

3. Each vertex v is equipped with a weight $w(v)=(a_v,b_v;m_v)$ (defined up to equivalence $(a_v,b_v;m_v)\sim(b_v,a_v;m_v)$) such that $m_v \mid m$ and $m_{v_0}=m=|\mathcal{G}|$, and $\gcd(a_v,b_v,m_v)=1$ for all v.

4. For every vertex v, $Stab(v) = C_{m_v} \subset \mathcal{G}$. Also $w(g.v) = w(v)$, so that weights of vertices in the same orbit are equal.

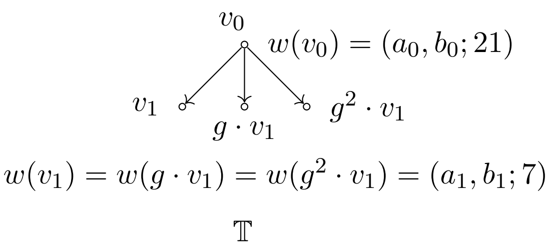
5. In the case of type I trees, there are at most three vertices v of level 1 such that $m_v = m$. Each of these vertices have distinct weights (up to equivalence) among $\{\pm(a_{v_0}, -b_{v_0}), \pm(a_{v_0}, b_{v_0} - a_{v_0}), \pm(b_{v_0}, a_{v_0} - b_{v_0})\}$.

6. Vertices with the same weight (up to equivalence and sign) do not have a common neighbour unless they are related by the $\mathcal{G}$-action.

7. Suppose there is an edge e from v in level L to u in level $L+1$. Then $m_u \mid m_v$ and

   (a) If $m_u = m_v$ and v is not the root vertex, then $\pm(a_u, b_u) \in \{(a_v, b_v - a_v), (a_v - b_v, b_v)\}$.

   (b) If $m_u \neq m_v$ then one of $a_v, b_v, a_v - b_v$ satisfies the equation $\gcd(x, m) = m_u$, accordingly $b_u = 0$, and a_u equals a value among $\pm\{-a_v, -b_v, b_v - a_v\}$ not divisible by m_u.



$X(\mathbb{T}) = S^4(a_0, b_0; 21)$
$\# C_{21} \times_{C_7} \mathbb{C}P^2(a_1, b_1; 7)$

FIGURE 4.1.8: Example of a type II tree

The properties (1) and (5) distinguish between type I and type II trees. As far as Definition 4.1.8 is concerned, type II trees are only a special subset of type I trees, but the construction of the tree manifolds associated to them will be different. In the case of type I trees (for example in Figure 0.3.0) the tree manifold associated to it is a connected sum of copies of $\mathbb{C}P^2$ with appropriate weights. For type II trees (see Figure 4.1.8), the root vertex gives a copy of S^4 in the connected sum and the rest of the vertices contribute $\mathbb{C}P^2$. One should observe here that if the root S^4 is connected to a $\mathbb{C}P^2$ associated to a fixed vertex, one may express the resultant tree manifold as one arising from a type I tree.

Remark 4.1.9. The definition of admissible, weighted tree above is the same definition as [18, §1.D]. To see this, one may observe the following

- The tree as defined inherits a direction, where an edge e moving from level L to $L+1$ is directed so that $\partial_0 e$ lies in level L, and $\partial_1 e$ lies in level $L+1$. One also observes that a vertex in level $L > 0$ is connected to a unique vertex in level $L-1$.

- The partial order may be generated from the condition that $\partial_0 e < \partial_1 e$. This implies that two vertices are comparable if they are connected by a sequence of edges, and in this case the order relation is determined by the level.

- The weights $w(v) = (a_v, b_v; m_v)$ are so defined that we obtain an equivalent weight under the operations $(a_v, b_v) \mapsto (b_v, a_v)$. This is the equivalence of weights referred to in the definition above.

- The conditions (5) and (7) above reflect the condition "pair of matching fixed components" of [18]. As we shall see, this is a slightly stronger condition that also includes the data required for us to form the corresponding equivariant connected sum.

4.1.10 Notation. The number $n(\mathbb{T})$ associated to an admissible, weighted tree $\mathbb{T}$ with vertex set $V(\mathbb{T})$ is defined as

$$n(\mathbb{T}) = \begin{cases} \#(V(\mathbb{T})) & \text{if } \mathbb{T} \text{ is of type I} \\ \#(V(\mathbb{T})) - 1 & \text{if } \mathbb{T} \text{ if of type II.} \end{cases}$$

The significance of the notation $n(\mathbb{T})$ is that we associate to an admissible weighted tree $\mathbb{T}$, a $\mathcal{G}$-manifold $X(\mathbb{T})$ whose underlying space is $\#^{n(\mathbb{T})}\mathbb{C}P^2$. We will use the notation $\mathbb{T}_0$ for $\mathbb{T}^{\mathcal{G}}$, and $\mathbb{T}_d = \{v \in \mathbb{T} \mid Stab(v) = C_d\}$. Observe that $\mathcal{G}/C_d$ acts freely on $\mathbb{T}_d$. Also note that $\mathbb{T}_0$ is always a sub-tree of $\mathbb{T}$, while $\mathbb{T}_d$ is not.

4.1.11. Construction of connected sums along trees. The construction of $X(\mathbb{T})$, the $\mathcal{G}$-manifold obtained by the connected sum of linear actions according to the data described in the tree $\mathbb{T}$ is carried out in [18, Theorem 1.7]. We describe it's main features below. For a vertex $v \in V(\mathbb{T})$, we use the notation

$$\mathbb{C}P^2_v := \mathbb{C}P^2(a_v, b_v; m_v).$$

Proposition 4.1.12. Given an admissible, weighted tree $\mathbb{T}$, there is a $\mathcal{G}$-manifold $X(\mathbb{T})$ such that

1. The underlying space of $X(\mathbb{T})$ is $\#^{n(\mathbb{T})}\mathbb{C}P^2$.

2. If $\mathbb{T}$ is of type I, then $X(\mathbb{T}) \cong$ a connected sum of copies of $\mathbb{C}P^2_v$ for every vertex v of $\mathbb{T}$. In case the vertex v is stabilized by a proper subgroup C_d of $\mathcal{G}$, the $\mathbb{C}P^2_{g\cdot v}$ assemble together equivariantly as $\mathcal{G} \times_{C_d} \mathbb{C}P^2_v$.

3. If $\mathbb{T}$ is of type II, then $X(\mathbb{T}) \cong$ a connected sum of copies of $\mathbb{C}P^2_v$ for every non-root vertex v of $\mathbb{T}$, and a copy of $S^4(a_{v_0}, b_{v_0}; m)$. As in (2), if the vertex v is stabilized by a proper subgroup C_d of $\mathcal{G}$, the $\mathbb{C}P^2_{g\cdot v}$ assemble together equivariantly as $\mathcal{G}\times_{C_d}\mathbb{C}P^2_v$.

4. For a non-root vertex v in level L, which is connected to w in level $L-1$ with $m_v = m_w$, the points where the connected sum is performed are $[0,0,1] \in \mathbb{C}P^2_v$, and the one in $\mathbb{C}P^2_w$ determined by the condition (7)(a) of Definition 4.1.8 if w is not the root vertex, or by (5) of Definition 4.1.8 if $w = v_0$.

5. For a non-root vertex v in level L connected to w in level $L-1$ with $m_v < m_w$, the points where the connected sum is performed are $[0,0,1] \in \mathbb{C}P^2_v$, and some equivalent choice of point in $\mathbb{C}P^2_w$ determined by the condition (7) (b) of Definition 4.1.8. Equivalent choices of the latter give equivalent manifolds [18, Lemma 1.2].

We now elaborate further on (4) and (5) of Proposition 4.1.12 above. We start with an example.

Example 4.1.13. In order to see if $\mathbb{C}P^2(a,b;m)\#\mathbb{C}P^2(a',b';m)$ is definable we may apply 1) of Proposition 4.1.6. Another method of saying this is that there is an expression of the second summand as $\mathbb{C}P^2(a',b';m)$ such that $\pm(a',b') \in \{(a,-b),(a-b,b),(a,b-a)\}$. Once this choice is made, say $(a',b') = (a,-b)$, we get a natural data for the equivariant connected sum as

1. The point $p \in \mathbb{C}P^2(a,b;m)$ used in the connected sum is $[0,0,1]$, and the corresponding tangential representation is $\lambda^a + \lambda^b$.

2. The point $q \in \mathbb{C}P^2(a',b';m)$ used in the connected sum is $[0,0,1]$, and the corresponding tangential representation is $\lambda^{a'} + \lambda^{b'} = \lambda^a + \lambda^{-b}$.

3. The natural orientation reversing isomorphism $T_p\mathbb{C}P^2(a,b;m) \to T_q\mathbb{C}P^2(a',b';m)$ is given by identity on the factor λ^a and complex conjugation on the factor λ^b.

In (4) of Proposition 4.1.12, the choice of (a_v, b_v) implies that

$$T_{[0,0,1]}\mathbb{C}P^2_v = \lambda^{a_v} + \lambda^{b_v}$$

equals one of $\lambda^{a_w} + \lambda^{b_w - a_w}$, $\lambda^{-a_w} + \lambda^{a_w - b_w}$, $\lambda^{b_w} + \lambda^{a_w - b_w}$, $\lambda^{-b_w} + \lambda^{b_w - a_w}$, in the case w is not the root vertex. We also note the tangential representations

$$T_{[1,0,0]}\mathbb{C}P^2_w = \lambda^{-a_w} + \lambda^{b_w - a_w},\ T_{[0,1,0]}\mathbb{C}P^2_w = \lambda^{-b_w} + \lambda^{a_w - b_w}.$$

Among the possibilities for $T_{[0,0,1]}\mathbb{C}P^2_v$, the first two are compatible with $T_{[1,0,0]}\mathbb{C}P^2_w$, and the second two are compatible with $T_{[0,1,0]}\mathbb{C}P^2_w$. This demonstrates how the weights imply the choice of connected sum point in $\mathbb{C}P^2_w$. The argument in Example 4.1.13 applies here to construct a canonical orientation reversing isomorphism among the tangential representations. Finally the condition (6) of Definition 4.1.8 implies that the choice of connected sum point is not the same as that of any other vertex.

We now look at (5) of Proposition 4.1.12. The condition $m_v < m_w$ implies that m_v is a proper divisor of m_w. Consider $\mathcal{O} \cong C_{m_w}/C_{m_v}$, the orbit of v under the C_{m_w}-action. From (4) of Definition 4.1.8, we observe that all the vertices in $\mathcal{O}$ have weight $w(v)$. The connected sum formed here is $\mathbb{C}P^2_w \# C_{m_w} \times_{C_{m_v}} \mathbb{C}P^2_v$, which connects the manifolds at all the vertices in $\mathcal{O}$ to $\mathbb{C}P^2_w$ at one go by writing

$$C_{m_w} \times_{C_{m_v}} \mathbb{C}P^2 \cong \coprod_{h \in C_{m_w}/C_{m_v}} h \cdot \mathbb{C}P^2_v = \coprod_{h \in C_{m_w}/C_{m_v}} \mathbb{C}P^2_{h \cdot v}.$$

In this case, we have $b_v = 0$ and m_v divides one of the numbers a_w, b_w, $b_w - a_w$ but not more than one (unless $m_v = 1$) as $\gcd(a_w, b_w, m_w) = 1$. We may assume $\gcd(b_w, m_w) = m_v$ without loss of generality, and it implies $a_v = \pm a_w$. The first part of the equivariant data for the connected sum is the point $[0,0,1]$ in $\mathbb{C}P^2_v$ with tangential C_{m_v}-representation $\lambda^{a_v} + 1_{\mathbb{C}}$ (as a complex representation). The next part is a choice of connected sum point which is required to have stabilizer C_{m_v}, and hence belongs to

$$P(\lambda^{b_w} + 1_{\mathbb{C}}) - \{[0,1,0],[0,0,1]\} \subset P(\lambda^{a_w} + \lambda^{b_w} + 1_{\mathbb{C}}) = \mathbb{C}P^2_w.$$

For any point $q \in P(\lambda^{b_w} + 1_{\mathbb{C}}) - \{[0,1,0],[0,0,1]\}$ the tangential C_{m_v}-representation is

$1_{\mathbb{C}}+\lambda^{a_w}$. We now have a canonical orientation reversing isomorphism between $T_{[0,0,1]}\mathbb{C}P^2_v$ and $T_q\mathbb{C}P^2_w$ which is conjugation on λ^{a_v} if $a_v = -a_w$, or conjugation on the other factor if $a_v = a_w$. As $P(\lambda^{b_w} + 1_{\mathbb{C}}) - \{[0,1,0],[0,0,1]\}$ is connected, this defines the equivariant connected sum up to diffeomorphism ([18, Lemma 1.2]). Note also that there is a completely analogous version of the above if w was the root vertex of a type II tree, and $\mathbb{C}P^2_w$ was replaced by $S^{\lambda^{a_w}+\lambda^{b_w}}$.

4.2 Equivariant homology decompositions for tree manifolds

In this section, we obtain homology decompositions for the tree manifolds defined in Section 4.1. Recall that, $\mathbb{C}P^2(a,b;m)$ serves as a building block for these manifolds. We describe a cellular decomposition of complex projective spaces, which has been studied along with cohomology of such spaces in [30, §3] and [6, §8.1].

4.2.1. Cellular filtration of projective spaces. The equivariant complex projective space $P(V)$ is built up by attaching even dimensional cells of the type $D(W)$ for the realization of complex representations W. To see this, let V_n be a complex C_m-representation that decomposes in terms of irreducible factors as $V_n = \sum_{i=0}^{n} \phi_i$, and let W_n denote the C_m-representation $\phi_n^{-1} \otimes \sum_{i=0}^{n-1} \phi_i$. Consider the C_m-equivariant map $D(W_n) \to P(V_n) \cong P(\phi_n^{-1} \otimes V_n)$ defined by

$$(z_0, z_1, \ldots, z_{n-1}) \mapsto [z_0, z_1, \ldots, z_{n-1}, 1 - \sum_{i=0}^{n-1} |z_i^2|],$$

where $z_i \in \phi_n^{-1} \otimes \phi_i$. Restricting this map to $S(W_n)$, we see that its image lies in $P(V_{n-1})$ (which may be regarded as a subspace of $P(V_n)$ in the obvious way), and it is a homeomorphism from $D(W_n) \setminus S(W_n)$ to $P(V_n) \setminus P(V_{n-1})$. Thus, $P(V_n)$ is obtained from $P(V_{n-1})$ by attaching the cell $D(W_n)$ along this boundary map. Observe that this filtration depends on the choice of the ordering of the ϕ_i's.

Returning to our example $\mathbb{C}P^2(a,b;m) = P(\lambda^a \oplus \lambda^b \oplus 1_{\mathbb{C}})$, we see that there are six possible ways to build it. This choices will play a crucial role in proving the homology decomposition theorems, as we will see below.

Example 4.2.2. Writing $\mathbb{C}P^2(a,b;m) = P(\lambda^a \oplus \lambda^b \oplus 1_{\mathbb{C}})$ in this order, the cellular filtration above gives us the following cofibre sequence (using the fact that $P(\lambda^a \oplus \lambda^b) \cong$

$S^{\lambda^{a-b}}$)

$$S^{\lambda^{a-b}} \to \mathbb{C}P^2(a,b;m) \to S^{\lambda^a+\lambda^b}.$$

Using the other orderings, we also obtain the following cofibre sequences

$$S^{\lambda^a} \to \mathbb{C}P^2(a,b;m) \to S^{\lambda^{a-b}+\lambda^{-b}}, \quad S^{\lambda^b} \to \mathbb{C}P^2(a,b;m) \to S^{\lambda^{b-a}+\lambda^{-a}}.$$

The homology decomposition is obtained by smashing these cofibre sequences with $H\underline{\mathbb{Z}}$ and trying to prove a splitting. For example, in the cofibre sequence

$$\Sigma^{\lambda^{a-b}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a,b;m) \to \Sigma^{\lambda^a+\lambda^b} H\underline{\mathbb{Z}}, \tag{4.2.3}$$

the connecting map $\Sigma^{\lambda^a+\lambda^b} H\underline{\mathbb{Z}} \to \Sigma^{\lambda^{a-b}+1} H\underline{\mathbb{Z}}$ is a $H\underline{\mathbb{Z}}$-module map which is classified up to homotopy by $\pi_0^{C_m}(\Sigma^{\lambda^{a-b}+1-\lambda^a-\lambda^b} H\underline{\mathbb{Z}})$. This group is now analyzed using Theorem 2.2.10 at $\alpha = -\lambda^{a-b} - 1 + \lambda^a + \lambda^b$. Note that

$$|(-\lambda^{a-b} - 1 + \lambda^a + \lambda^b)^{C_d}| = \begin{cases} -1 & \text{if } d \text{ does not divide any of } a,\ b \text{ or } a-b \\ 1 & \text{if } d \text{ divides exactly one of } a,\ b \text{ but not } a-b \\ -3 & \text{if } d \text{ divides } a-b \text{ but not } a \text{ or } b. \end{cases}$$

Observe that $|\alpha| = 1 > 0$, so in order to show $\pi_\alpha^{C_m} H\underline{\mathbb{Z}} = 0$, we need $|\alpha^{C_d}| \geq -1$ for all $d \mid m$. Under the condition $\gcd(a,b,m) = 1$, this is true if and only if $a-b$ is relatively prime to m, and in this case,

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a,b;m) \simeq \Sigma^{\lambda^{a-b}} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^a+\lambda^b} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^a+\lambda^b} H\underline{\mathbb{Z}}.$$

The last equivalence follows from Proposition 1.3.9. Using the other two cofibre sequences for $\mathbb{C}P^2(a,b;m)$, we see that a homology decomposition is obtained if one of a, b, or $a-b$ is relatively prime to m.

In the case of connected sums, we carry forward the homology decomposition argument of Example 4.2.2. We illustrate this in the following example.

Example 4.2.4. Let $X = \mathbb{C}P^2(a,b;m)\#\mathbb{C}P^2(a',b';m)$ where $\gcd(a'-b',m) = 1$ and $\gcd(a-b,m) = 1$. We assume that the connected sum point p in $\mathbb{C}P^2(a',b';m)$ has tangential representation $\lambda^{a'} \oplus \lambda^{b'}$ as in 1) of Proposition 4.1.6. Example 4.2.2 shows that

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a,b;m) \simeq \Sigma^{\lambda^a+\lambda^b} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}.$$

To compute $H\underline{\mathbb{Z}} \wedge X$, we use the cofibre sequence (4.1.2). We note that

$$\mathbb{C}P^2(a',b';m) \setminus \{p\} \simeq P(\lambda^{a'} \oplus \lambda^{b'}) \simeq S^{\lambda^{a'-b'}}.$$

Therefore we obtain a cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\Sigma^{\lambda^{a'-b'}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X \to \Sigma^{\lambda^a+\lambda^b} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}.$$

From Proposition 1.3.9, we have $\Sigma^{\lambda^{a'-b'}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda} H\underline{\mathbb{Z}}$, and now Theorem 2.2.10 implies that the above sequence splits. Consequently, we obtain

$$H\underline{\mathbb{Z}} \wedge X_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^a+\lambda^b} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}.$$

We now prove the main theorems of this section. Example 4.2.2 points out the necessity of the hypothesis in the theorem.

Theorem 4.2.5. If $\mathbb{T}$ is an admissible weighted tree of type I with C_m-action such that for all vertices $v \in \mathbb{T}_0$ with $w(v) = (a_v, b_v; m_v)$, $\gcd(a_v - b_v, m_v) = 1$, then, the $H\underline{\mathbb{Z}}$-module $H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+$ admits the decomposition

$$\begin{aligned} H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0}+\lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\mathbb{T}_0} \Sigma^{\lambda} H\underline{\mathbb{Z}}\Big) \\ \vee \Big(\bigvee_{[\mu]\in\mathbb{T}_d/C_m, d\neq m} C_m/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}}\Big) \end{aligned}$$

where $w(v_0) = (a_0, b_0; m)$. If $\mathbb{T}$ is of type II,

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0}+\lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{[\mu]\in\mathbb{T}_d/C_m,\ d\neq m} C_m/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}}\Big).$$

Proof. We proceed by induction on $L(\mathbb{T})$, the maximum level reached by vertices of the tree. The induction starts from a tree with only the root vertex. In the type I case, this is computed in Example 4.2.2. In case of type II, the manifold is $S^4(a_0, b_0; m)$, for which we have the following decomposition

$$H\underline{\mathbb{Z}} \wedge S^4(a_0, b_0; m)_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0}+\lambda^{b_0}} H\underline{\mathbb{Z}}.$$

Assume that the statement holds whenever $L(\mathbb{T}) \leq L$. We prove it for trees with $L(\mathbb{T}) = L+1$. Given a tree $\mathbb{T}$ we denote by $\mathbb{T}(L)$ the part of it up to level L, so that the result holds for $X(\mathbb{T}(L))$. We attach orbits of the level $L+1$ vertices one at a time. We write down the argument for a type I tree, as the other case is entirely analogous. Let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ denote the orbits of the level $L+1$ vertices. It suffices to prove the case when an orbit $\mathcal{O}_i$ is added to $\mathbb{T}(L)$ together with the attaching edges, which we denote by $\mathbb{T}(L) + \mathcal{O}_i$. The stabilizer for the vertices in $\mathcal{O}_i$ can be either the whole group C_m or a smaller subgroup C_d. We deal these cases separately.

Case 1: The stabilizer for the vertices in $\mathcal{O}_i$ is C_m, that is, $\mathcal{O}_i = \{v_i\}$. Suppose $w(v_i) = (a_i, b_i; m)$. By Proposition 4.1.12, this implies that the tangential representation at the connected sum point of $\mathbb{C}P^2_{v_i}$ is $\lambda^{a_i} \oplus \lambda^{b_i}$. We have the following cofibre sequence from (4.1.2)

$$\mathbb{C}P^2(a_i, b_i; m) \setminus D(\lambda^{a_i} \oplus \lambda^{b_i}) \to X\big(\mathbb{T}(L) + \mathcal{O}_i\big) \to X\big(\mathbb{T}(L)\big).$$

The left hand term can be simplified further as

$$\mathbb{C}P^2(a_i, b_i; m) \setminus D(\lambda^{a_i} \oplus \lambda^{b_i}) \simeq P(\lambda^{a_i} \oplus \lambda^{b_i}) \simeq S^{\lambda^{a_i - b_i}}.$$

We now apply Proposition 1.3.9 to note that $\Sigma^{\lambda^{a_i-b_i}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda} H\underline{\mathbb{Z}}$. Applying the induction hypothesis on $X\big(\mathbb{T}(L)\big)$, we get a cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\Sigma^{\lambda} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X\big(\mathbb{T}(L) + \mathcal{O}_i\big) \to \Sigma^{\lambda^{a_0}+\lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\mathbb{T}(L)_d/C_m} C_m/C_{d+} \wedge \Sigma^{\lambda} H\underline{\mathbb{Z}} \Big). \quad (4.2.6)$$

In the second summand of the right side of (4.2.6), d is also allowed to equal m. Next we observe that the cofibre sequence splits by showing that up to homotopy, the connecting map from each summand of the right hand side of equation (4.2.6) to $\Sigma^{\lambda+1} H\underline{\mathbb{Z}}$ is zero. For the first summand, this follows from Example 4.2.2. In the second summand, if $d = m$, the connecting map is classified up to homotopy by an element of $\underline{\pi}_{-1}(H\underline{\mathbb{Z}}) = 0$. If $d \neq m$, this is classified by an element of $\underline{\pi}_{-1}(H\underline{\mathbb{Z}})(C_m/C_d)$, which is also 0. Using

Theorem 2.2.10 in this manner, we obtain the required homology decomposition for $X(\mathbb{T}(L)+\mathcal{O}_i)$.

Case 2: The vertices in $\mathcal{O}_i$ have stabilizer $C_{m_i} < C_m$ and $w(v_i) = (a_i, b_i; m_i)$. Here, we are considering the connected sum of the form $X\big(\mathbb{T}(L)\big)\#C_m \times_{C_{m_i}} \mathbb{C}P^2(a_i, b_i; m_i)$. Consider the following homotopy pushout square of $\mathcal{G}$-spaces (4.1.3)

$$\begin{tikzcd}
\mathcal{G} \times_{C_{m_i}} S^{\lambda^{a_i - b_i}} \arrow[r] \arrow[d] & \mathcal{G} \times_{C_{m_i}} C(S^{\lambda^{a_i - b_i}}) \arrow[d] \\
X\big(\mathbb{T}(L)+\mathcal{O}_i\big) \arrow[r] & X\big(\mathbb{T}(L)\big),
\end{tikzcd}$$

where $C(S^{\lambda^{a_i-b_i}})$ denotes the cone of $S^{\lambda^{a_i-b_i}}$. Observe that the C_{m_i}-representation λ is also the restriction of a C_m-representation that we have denoted also using λ. Further using the shearing homeomorphism, we may write $\mathcal{G} \times_{C_{m_i}} S^{\lambda^{a_i-b_i}} \cong G/C_{m_i} \times S^{\lambda^{a_i-b_i}}$. In $\mathcal{G}$-spectra, this gives rise to the cofibre sequence

$$\mathcal{G}/C_{m_i+} \wedge S_+^{\lambda^{a_i-b_i}} \to X\big(\mathbb{T}(L)+\mathcal{O}_i\big)_+ \vee \mathcal{G}/C_{m_i+} \to X\big(\mathbb{T}(L)\big)_+.$$

We use

$$\mathcal{G}/C_{m_i+} \wedge S_+^{\lambda^{a_i-b_i}} \simeq (\mathcal{G}/C_{m_i+} \wedge S^{\lambda^{a_i-b_i}}) \vee \mathcal{G}/C_{m_i+},$$

to deduce the cofibre sequence

$$\mathcal{G}/C_{m_i+} \wedge S^{\lambda^{a_i-b_i}} \to X\big(\mathbb{T}(L)+\mathcal{O}_i\big)_+ \to X\big(\mathbb{T}(L)\big)_+.$$

Now we take the smash product with $H\underline{\mathbb{Z}}$ to get the following cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\mathcal{G}/C_{m_i+} \wedge \Sigma^{\lambda^{a_i-b_i}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X\big(\mathbb{T}(L)+\mathcal{O}_i\big)_+ \to H\underline{\mathbb{Z}} \wedge X(\mathbb{T}(L))_+. \tag{4.2.7}$$

Note that the summands of the right hand side of the form $\Sigma^{\lambda} H\underline{\mathbb{Z}}$ and $\mathcal{G}/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}}$ support a trivial connecting map using Theorem 2.2.10, and the facts

a) $|(\lambda^r - \lambda^s - 1)^K| > -1$ if $|K| \mid r$ but not s.

b) $|(\lambda^r - \lambda^s - 1)^K| < -1$ if $|K| \mid s$ but not r.

We now note that if the stabilizer C_{m_v} of a vertex v satisfies $m_v < m$, then m_v must divide a_0 or b_0 under the given hypothesis. In the case of type II trees, this is clear from

(7)(b) of Definition 4.1.8 as the maximum value of m_v is reached among the vertices at level 1. For a type I tree, the analogous role is played by vertices v with $m_v < m$ that are joined to a vertex w of $\mathbb{T}_0$. The same condition now implies that one of a_w, b_w, $a_w - b_w$ is divisible by m_v. The hypothesis rules out the third case. Now we repeatedly apply condition (7)(a) of Definition 4.1.8 along the path from w to the root vertex v_0 with the hypothesis ruling out the fact that m_v divides $a_u - b_u$ for any vertex u along the path. It follows that m_v divides either a_u or b_u for every vertex along this path. Therefore, m_v divides either a_0 or b_0. Now by Theorem 2.2.10 using the fact that m_i divides either a_0 or b_0, the connecting map on the summand $\Sigma^{\lambda^{a_0}+\lambda^{b_0}}H\underline{\mathbb{Z}}$ is 0. This completes the proof. $\square$

Remark 4.2.8. Observe that Theorem 4.2.5 has no hypothesis if the tree $\mathbb{T}$ is of type II. Henceforth, we prove further results for trees of type I. The hypothesis in Theorem 4.2.5 is required crucially in the proof. For example, observe that if $a_0 - b_0 \equiv 0 \pmod{m}$, then the the cofibre sequence (4.2.3) gives rise to the connecting map

$$\Sigma^{\lambda^{a_0}+\lambda^{b_0}}H\underline{\mathbb{Z}} \to \Sigma^3 H\underline{\mathbb{Z}},$$

which is determined by

$$\pi^{\mathcal{G}}_{\lambda^{a_0}+\lambda^{b_0}-3}(H\underline{\mathbb{Z}}).$$

This group may be non-zero.

In the following theorem, we observe that if one of the rotation numbers at the root vertex is 0, then we obtain a decomposition result with no further hypothesis on the weights.

Theorem 4.2.9. Let $\mathbb{T}$ be an admissible weighted tree with $\mathcal{G}$-action of type I such that for the root vertex v_0 with $w(v_0) = (a_0, b_0; m)$, one of a_0 or b_0 is zero. Then, $H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+$ admits the following decomposition

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2}H\underline{\mathbb{Z}} \vee \Big(\bigvee_{[\mu]\in\mathbb{T}_d/\mathcal{G}} \mathcal{G}/C_{d_+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \Big).$$

Note that in the rightmost summand d may be equal to m.

Proof. We proceed by induction on $L(\mathbb{T})$, the maximum level reached by the tree as in Theorem 4.2.5. We may assume b_0 is zero, so we have, $\gcd(a_0, m) = 1$. At the initial case $L(\mathbb{T}) = 0$, $X(\mathbb{T}) = \mathbb{C}P^2(a_0, 0; m)$, and for this manifold, the cellular decomposition gives us the following cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\Sigma^{\lambda^{a_0}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a_0, 0; m) \to \Sigma^{\lambda^{a_0}+2} H\underline{\mathbb{Z}}. \tag{4.2.10}$$

Note that $\Sigma^{\lambda^{a_0}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda} H\underline{\mathbb{Z}}$. The connecting map in (4.2.10),

$$\Sigma^{\lambda+2} H\underline{\mathbb{Z}} \to \Sigma^{\lambda+1} H\underline{\mathbb{Z}}$$

is trivial up to homotopy, hence the cofibre sequence splits and we obtain

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a_0, 0; m)_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}.$$

For the inductive step, assume the statement is true for the tree up to level L, $\mathbb{T}(L)$ and we attach one orbit $\mathcal{O}_i$ of the level $L+1$ vertices to $\mathbb{T}(L)$. We prove the case when the stabilizer for the vertices in $\mathcal{O}_i$ is $\mathcal{G}$. An analogous reasoning applies to the other cases.

Suppose $\mathcal{O}_i = \{v_i\}$ and $w(v_i) = (a_i, b_i; m)$. As in the proof of Theorem 4.2.5, we get the following cofibre sequence

$$S^{\lambda^{a_i - b_i}} \to X\big(\mathbb{T}(L) + \mathcal{O}_i\big) \to X\big(\mathbb{T}(L)\big).$$

Applying the induction hypothesis on $X\big(\mathbb{T}(L)\big)$, we get a cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\Sigma^{\lambda^{a_i - b_i}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X\big(\mathbb{T}(L) + \mathcal{O}_i\big) \to \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{[\mu] \in \mathbb{T}(L)_d/\mathcal{G}} \mathcal{G}/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \Big). \tag{4.2.11}$$

We claim that the connecting map is zero from each summand of the right hand side of the equation to $\Sigma^{\lambda^{a_i - b_i}+1} H\underline{\mathbb{Z}}$. For the first summand note that the group

$$\pi^{\mathcal{G}}_{\lambda+1-\lambda^{a_i - b_i}}(H\underline{\mathbb{Z}}) = 0$$

as it satisfies the condition given in Theorem 2.2.10. To show the map

$$\mathcal{G}/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \to \Sigma^{\lambda^{a_i - b_i}+1} H\underline{\mathbb{Z}}$$

is trivial consider the group

$$\pi_{\alpha}^{C_d}(H\underline{\mathbb{Z}}), \quad \alpha = \lambda^{a_\mu - b_\mu} - \lambda^{a_i - b_i} - 1$$

Then $|\alpha| = -1$, and for all subgroups C_j, $|\alpha^{C_j}| \leq 1$. Equality holds if and only if $j \mid (a_\mu - b_\mu)$ and $j \nmid (a_i - b_i)$. Then for any subgroup $C_k \supset C_j$, k does not divide $(a_i - b_i)$. So $|\alpha^{C_k}| \geq -1$. Thus α satisfies the condition given in Theorem 2.2.10. Hence the cofibre sequence in (4.2.11) splits and we obtain the required decomposition. This completes the proof. □

4.3 Homology decompositions for $\mathcal{G} = C_{p^n}$

In this section we derive homology decompositions for tree manifolds in the case $\mathcal{G} = C_{p^n}$ without any restriction on weight. We start with an example pointing out the need for a judicious choice of cellular filtration and later, we discuss how a reorientation may help to solve this. As a result, we observe some dimension shifting phenomena among the summands in the homology decomposition.

Example 4.3.1. Let $p \mid a - b$. We know from the cellular filtration of projective spaces (Example 4.2.2) that $\mathbb{C}P^2(a, b; p) = P(\lambda^a \oplus 1_{\mathbb{C}} \oplus \lambda^b)$ gives us the cofibre sequence

$$\Sigma^{\lambda^a} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \to \Sigma^{\lambda^{a-b} + \lambda^{-b}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda+2} H\underline{\mathbb{Z}}.$$

Since the connecting map is zero by Theorem 2.2.10, we obtain

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \simeq \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}. \tag{4.3.2}$$

One may also write $\mathbb{C}P^2(a, b; p) = P(\lambda^a \oplus \lambda^b \oplus 1_{\mathbb{C}})$ which yields the cofibre sequence

$$\Sigma^{\lambda^{a-b}} H\underline{\mathbb{Z}} \simeq \Sigma^2 H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \to \Sigma^{2\lambda} H\underline{\mathbb{Z}}. \tag{4.3.3}$$

Note that the group of possible connecting maps is $\mathbb{Z}/p$. We claim that the connecting map of (4.3.3) is non-zero. Suppose on the contrary that the connecting map is trivial. Then we have the splitting

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \simeq \Sigma^{2\lambda} H\underline{\mathbb{Z}} \vee \Sigma^2 H\underline{\mathbb{Z}}. \tag{4.3.4}$$

The Mackey functor

$$\underline{\pi}_{\lambda}^{C_p}(H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a,b;m))$$

is isomorphic (2.1.4) to $\underline{\mathbb{Z}}$ from (4.3.2), while isomorphic to $\underline{\mathbb{Z}}^* \oplus \langle \mathbb{Z}/p \rangle$ if (4.3.4) were true, a contradiction. Hence, the connecting map of (4.3.3) should be non-trivial.

Recall from Proposition 4.1.12 (4) that for a non-root vertex $v \in V(\mathbb{T})$, the connected sum is performed at the point $[0,0,1] \in \mathbb{C}P^2_v$. At the root vertex v_0 with weight $(a_0, b_0; m)$ we may change the weights to $(-a_0, b_0 - a_0; m)$ or $(a_0 - b_0, -b_0; m)$ to obtain a $\mathcal{G}$-homeomorphic manifold. This fact will be used in the proof of the theorems below.

The following example summarizes how a reorientation is performed and the resulting modifications of weights. We also demonstrate how this allows us to make a judicious choice of cellular filtration of $X(\mathbb{T})$.

Example 4.3.5. Suppose we have an admissible weighted $\mathcal{G}$-equivariant tree $\mathbb{T}$ depicted as in the left of the figure below with root vertex v_0, $w(v_0) = (a_0, b_0; m)$, and all other vertices have weight as mentioned therein. We reorient $\mathbb{T}$ to obtain a new tree $\mathbb{T}'$ whose root vertex is $u_0 = v_3$ with $w(u_0) = (a_3 - b_3, -b_3; m)$, and let for $i = 1, 2, 3$, the vertices $u_i \in \mathbb{T}'$ represent the vertices v_{3-i}. Proposition 4.1.12 (4) tells us that the connected sum is performed at $[0,0,1] \in \mathbb{C}P^2_{v_3}$ for which

$$T_{[0,0,1]}\mathbb{C}P^2_{v_3} = \lambda^{a_3} + \lambda^{b_3}.$$

This means there exists a $\mathcal{G}$-fixed point p in $\mathbb{C}P^2_{v_2}$ so that

$$T_p\mathbb{C}P^2_{v_2} = \lambda^{a_3} + \lambda^{-b_3} \text{ or } \lambda^{-a_3} + \lambda^{b_3}.$$

Then, if necessary, we apply a suitable $\mathcal{G}$-homeomorphism to map the point p to $[0,0,1]$, which allow us to perform the connected sum at the point $[0,0,1]$ of $\mathbb{C}P^2_{u_1}$ with $\mathbb{C}P^2_{u_0}$. This explains the weights in the new tree $\mathbb{T}'$. Note that $X(\mathbb{T})$ is $\mathcal{G}$ homeomorphic to $X(\mathbb{T}')$.

Suppose in the tree $\mathbb{T}$, $m \nmid a_0, b_0, a_1 - b_1, a_2 - b_2$ but $m \mid a_3 - b_3$. For the $\mathcal{G}$-manifold $X(\mathbb{T}) = \#_{i=0}^3 \mathbb{C}P^2_{v_i}$ if we proceed as in the proof of Theorem 4.2.5, we obtain the cofibre

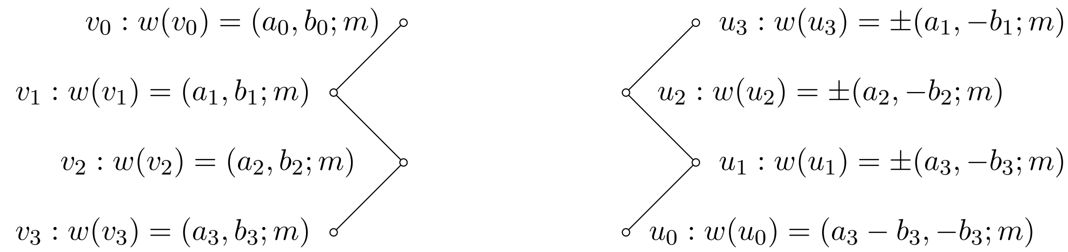


FIGURE 4.3.5: An example of reorientation: The left tree has root vertex v_0 and the right one has root vertex $u_0 = v_3$.

sequence

$$\begin{tikzcd}
\Sigma^{\lambda^{a_3-b_3}} H\underline{\mathbb{Z}} \arrow[r] \arrow[d, "\simeq"] & H\underline{\mathbb{Z}} \wedge X(\mathbb{T}) \arrow[r] & H\underline{\mathbb{Z}} \wedge \#_{i=0}^{2} \mathbb{C}P^2_{v_i} \arrow[d, "\simeq"] \\
\Sigma^2 H\underline{\mathbb{Z}} & & \Sigma^{2\lambda} H\underline{\mathbb{Z}} \vee \left(\bigvee_{i=0}^{2} \Sigma^{\lambda} H\underline{\mathbb{Z}} \right)
\end{tikzcd}$$

Observe that the connecting map may be non-zero here. On the other hand, for $\mathbb{T}'$, we obtain

$$\begin{tikzcd}
\Sigma^{\lambda^{\pm(a_1+b_1)}} H\underline{\mathbb{Z}} \arrow[r] \arrow[d, "\simeq"] & H\underline{\mathbb{Z}} \wedge X(\mathbb{T}') \arrow[r] & H\underline{\mathbb{Z}} \wedge \#_{i=0}^{2} \mathbb{C}P^2_{u_i} \arrow[d, "\simeq"] \\
\Sigma^{\lambda} H\underline{\mathbb{Z}} & & \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \left(\bigvee_{i=0}^{2} \Sigma^{\lambda} H\underline{\mathbb{Z}} \right)
\end{tikzcd} \tag{4.3.6}$$

The right vertical equivalence comes from Theorem 4.2.9 and the equivalence $\Sigma^{\lambda^{a_i+b_i}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda} H\underline{\mathbb{Z}}$ for $i = 1, 2, 3$. To see this, note from 7(a) of 4.1.8 that

$$\pm(a_1, b_1) \in \{(a_0, -b_0), (a_0 - b_0, b_0), (a_0, b_0 - a_0)\}.$$

Our condition $m \nmid a_0, b_0, a_0 - b_0$ implies $m \nmid a_1, b_1, a_1 + b_1$. Iterating this process the desired equivalence follows. Observe that, the connecting map in cofibre sequence (4.3.6) is trivial.

We now proceed towards the decomposition result in the case $\mathcal{G} = C_p$. For the tree $\mathbb{T}$, if $p \nmid a_v - b_v$ for all vertices $v \in \mathbb{T}_0$, the result is obtained from Theorem 4.2.5 as

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0}+\lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\mathbb{T}_0} \Sigma^{\lambda} H\underline{\mathbb{Z}} \Big) \vee \Big(\bigvee_{\mathbb{T}_e/\mathcal{G}} \mathcal{G}/e_+ \wedge \Sigma^2 H\underline{\mathbb{Z}} \Big).$$

where $\mathbb{T}_e = \{v \in \mathbb{T} \mid Stab(v) = e\}$. In the complementary situation p must divide $a_v - b_v$ for some $v \in \mathbb{T}_0$. If further $p \mid a_0$ or b_0, we are in the situation dealt in Theorem 4.2.9, so that we have

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\phi(\mathbb{T})} \Sigma^{\lambda} H\underline{\mathbb{Z}} \Big) \vee (\bigvee_{\psi(\mathbb{T})} \Sigma^2 H\underline{\mathbb{Z}}) \vee \Big(\bigvee_{\mathbb{T}_e/\mathcal{G}} \mathcal{G}/e_+ \wedge \Sigma^2 H\underline{\mathbb{Z}} \Big).$$

where $\phi(\mathbb{T}) = \#\{v \in \mathbb{T}_0 \mid p \nmid a_v - b_v\}$, and $\psi(\mathbb{T}) = \#\{v \in \mathbb{T}_0 \mid p \text{ divides } a_v - b_v\}$. For the remaining case, we prove

Theorem 4.3.7. Let $\mathcal{G} = C_p$ and $\mathbb{T}$ be an admissible weighted $\mathcal{G}$-equivariant tree of type I such that $p \nmid a_0, b_0$ but $p \mid a_v - b_v$ for some $v \in \mathbb{T}_0$. Then,

$$\begin{aligned} H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\phi(\mathbb{T})+1} \Sigma^{\lambda} H\underline{\mathbb{Z}} \Big) \\ \vee \Big(\bigvee_{\psi(\mathbb{T})-1} \Sigma^2 H\underline{\mathbb{Z}} \Big) \vee \Big(\bigvee_{\mathbb{T}_e/\mathcal{G}} \mathcal{G}/e_+ \wedge \Sigma^2 H\underline{\mathbb{Z}} \Big). \end{aligned}$$

Proof. By given hypothesis, there exist a vertex $v_\ell \in \mathbb{T}_0$ with $w(v_\ell) = (a_\ell, b_\ell; p)$ such that $a_\ell - b_\ell \equiv 0 \pmod p$. We further assume that v_ℓ is closest to the root vertex v_0 in terms of number of edges from v_0 to v_ℓ. We prove the statement for $\mathbb{T}_0 \subseteq \mathbb{T}$. The result follows from this because when we attach a free orbit to any level, the resulting connecting map becomes a map of underlying non-equivariant spectra which is trivial as it is a map from $H\underline{\mathbb{Z}}$ smashed with connected sum of $\mathbb{C}P^2$ to $H\underline{\mathbb{Z}}$ smashed with copies of wedge of S^3.

Let Γ denote the path from v_0 to v_ℓ passing through vertices $v_0, v_1, \ldots, v_\ell$, and let for $v_i \in \Gamma$, $w(v_i) = (a_i, b_i; p)$. We reorient Γ, as in Example 4.3.5, so that now v_ℓ becomes the root vertex u_0; $v_i \in \Gamma$ becomes the vertex $u_{\ell-i}$. Observe that the weight at $u_i = v_{\ell-i}$ becomes $\pm(a_{\ell-i+1}, -b_{\ell-i+1}; p)$ and the weight at u_0 is $(a_\ell - b_\ell, -b_\ell; p)$. Since $a_\ell - b_\ell \equiv 0 \pmod p$, the choice of cellular filtration of $\mathbb{C}P^2(a_\ell, b_\ell; p)$, as in Example 4.3.1, leads to

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a_\ell, b_\ell; p) \simeq \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}.$$

Theorem 4.2.9 implies

$$\begin{aligned} H\underline{\mathbb{Z}} \wedge X(\Gamma) &\simeq \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=1}^{\ell} \Sigma^{\lambda^{a_{\ell-i+1}+b_{\ell-i+1}}} H\underline{\mathbb{Z}} \Big) \\ &\simeq \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\ell} \Sigma^{\lambda} H\underline{\mathbb{Z}} \Big) \end{aligned}$$

To deduce the second equivalence we show that for $i = 1$ to ℓ, $p \nmid a_i + b_i$. The condition $p \nmid a_0, b_0, a_0 - b_0$ in turn implies $p \nmid a_1, b_1, a_1 + b_1$ as by $7(a)$ of Definition 4.1.8 $\pm(a_1, b_1) \in \{(a_0, -b_0), (a_0 - b_0, b_0), (a_0, b_0 - a_0)\}$. The fact that for $1 \leq i \leq \ell$, $p \nmid a_{i-1} - b_{i-1}$, let us continue this process up to $\ell - 1$.

Next we attach vertices of $\mathbb{T} \setminus \Gamma$ to Γ proceeding by induction as in Theorem 4.2.9, and observe that in each step the connecting map for the cofibre sequence is null. This completes the proof. $\square$

The rest of the section is devoted to proving homology decompositions in the case $\mathcal{G} = C_{p^n}$. We start with the following observation

Lemma 4.3.8. Let $\mathcal{G} = C_{p^n}$ and $\mathbb{T}$ be an admissible weighted $\mathcal{G}$-equivariant tree of type I. Let τ be the maximum power of p that divides $a_v - b_v$ among the vertices of $\mathbb{T}_0$, and p^τ does not divide a_0 and b_0. Then for $v \in \mathbb{T} \setminus \mathbb{T}_0$, $Stab(v) \leq C_{p^\tau}$.

Proof. Choose a vertex $u \in \mathbb{T} \setminus \mathbb{T}_0$ such that $Stab(u)$ is maximum among vertices of $\mathbb{T} \setminus \mathbb{T}_0$, and u is closest to the root vertex. This implies if u is in level L, then the vertex v in level $L - 1$ connected to u belongs to $\mathbb{T}_0$. So let $w(v) = (a_v, b_v, p^n)$. We claim that $C_{m_u} = Stab(u) \leq C_{p^\tau}$. On the contrary suppose $C_{m_u} > C_{p^\tau}$. From $7(b)$ of Definition 4.1.8, we get m_u divides one of $a_v, b_v, a_v - b_v$. Since m_u can not divide $a_v - b_v$, m_u divides a_v or b_v. Further, if v is connected to a level $L - 2$ vertex v' with $w(v') = (a_{v'}, b_{v'}; p^n)$, then

$$\pm(a_v, b_v) \in \{(a_{v'}, -b_{v'}), (a_{v'}, b_{v'} - a_{v'}), (a_{v'} - b_{v'}, b_{v'})\}.$$

This means m_u divides $a_{v'}$ or $b_{v'}$. Continuing this process we end up with p^τ dividing a_0 or b_0. Hence a contradiction. $\square$

Proceeding as in the C_p-case, we have that if $p \nmid a_v - b_v$ for all $v \in \mathbb{T}_0$, by Theorem 4.2.5

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0} + \lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\mathbb{T}_0} \Sigma^{\lambda} H\underline{\mathbb{Z}}\Big) \vee \Big(\bigvee_{\mathbb{T}_d/\mathcal{G},\ d \neq p^n} \mathcal{G}/C_{d+} \wedge \Sigma^{\lambda} H\underline{\mathbb{Z}}\Big).$$

In the complementary case p divides $a_v - b_v$ for some $v \in \mathbb{T}$. Let $\tau > 0$ be the maximum power of p that divides $a_v - b_v$ among the vertices of $\mathbb{T}_0$. If further $p^\tau \mid a_0$ or b_0, then proceeding exactly as in Theorem 4.2.9 we obtain

$$\begin{aligned} H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0}+\lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}} H\underline{\mathbb{Z}})^{\vee Z_{\mathbb{T}}(i)} \Big) \\ \vee \Big(\bigvee_{[\mu]\in \mathbb{T}_d/\mathcal{G},\ d\neq p^n} \mathcal{G}/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \Big) \end{aligned} \tag{4.3.9}$$

where for fixed $0 \le i \le n$, $Z_{\mathbb{T}}(i) := \#\{v \in \mathbb{T}_0, w(v) = (a_v, b_v; p^n) \mid \gcd(a_v - b_v, p^n) = p^i\}$.

We also define

$$W_{\mathbb{T}}(i) = \begin{cases} Z_{\mathbb{T}}(i) + 1 & \text{if } i = 0 \\ Z_{\mathbb{T}}(i) - 1 & \text{if } i = \tau \\ Z_{\mathbb{T}}(i) & \text{otherwise.} \end{cases}$$

Observe that the conditions on weights in Definition 4.1.8 does not change if we replace the weight $(a_0, b_0; p^n)$ at the root vertex by one of $(a_0 - b_0, -b_0; p^n)$ or $(b_0 - a_0, -a_0; p^n)$. This allows us to further assume $p \nmid a_0, b_0$ in the theorem below.

Theorem 4.3.10. Let $\mathcal{G} = C_{p^n}$ and $\mathbb{T}$ be an admissible weighted $\mathcal{G}$-equivariant tree of type I. Suppose $\tau > 0$ is the maximum power of p that divides $a_v - b_v$ among the vertices of $\mathbb{T}_0$ and $p \nmid a_0, b_0$. Then

$$\begin{aligned} H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+\lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}})^{\vee W_{\mathbb{T}}(i)} H\underline{\mathbb{Z}} \Big) \\ \vee \Big(\bigvee_{[\mu]\in \mathbb{T}_d/\mathcal{G},\ d\neq p^n} \mathcal{G}/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \Big). \end{aligned}$$

Proof. Let $v_\ell \in \mathbb{T}_0$ with $w(v_\ell) = (a_\ell, b_\ell; p^n)$ be a vertex for which $p^\tau \mid a_\ell - b_\ell$, and v_ℓ is closest to the root in terms of number of edges. Let Γ denote the path from v_0 to v_ℓ passing through the vertices $v_0, v_1, \dots, v_\ell$ with $w(v_i) = (a_i, b_i; p^n)$. We first compute $H\underline{\mathbb{Z}} \wedge X(\Gamma)$, then $H\underline{\mathbb{Z}} \wedge X(\mathbb{T}_0)$ and finally $H\underline{\mathbb{Z}} \wedge X(\mathbb{T})$.

We reorient Γ so that now v_ℓ becomes the root vertex u_0, and $v_{\ell-i}$ becomes the vertex u_i. Observe that the weight at $u_i = v_{\ell-i}$ becomes $\pm(a_{\ell-i+1}, -b_{\ell-i+1}; p^n)$ and the weight at u_0 is $(a_\ell - b_\ell, -b_\ell; p^n)$. Since $\gcd(a_\ell - b_\ell, p^n) = p^\tau$,

$$\Sigma^{\lambda^{a_\ell - b_\ell}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda^{p^\tau}} H\underline{\mathbb{Z}}. \tag{4.3.11}$$

We use the following cofibre sequence for $\mathbb{C}P^2(a_\ell, b_\ell; p^n)$

$$S^{\lambda^{a_\ell}} \to \mathbb{C}P^2(a_\ell, b_\ell; p^n) \to S^{\lambda^{a_\ell - b_\ell} + \lambda^{-b_\ell}}.$$

This together with the identification (4.3.11) leads to the following decomposition

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a_\ell, b_\ell; p^n)_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}.$$

Proceeding as in the proof of Theorem 4.3.7 we see that for $i = 0$ to $\ell - 1$, $p^\tau \nmid a_{i+1} + b_{i+1}$. Now (4.3.9) implies

$$\begin{aligned} H\underline{\mathbb{Z}} \wedge X(\Gamma) &\simeq \Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=1}^{\ell} \Sigma^{\lambda^{a_{\ell-i+1} + b_{\ell-i+1}}} H\underline{\mathbb{Z}} \Big) \\ &\simeq \Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}} H\underline{\mathbb{Z}})^{\vee W_\Gamma(i)} \Big), \end{aligned}$$

The last equivalence is obtained from Proposition 1.3.9 and the following claim

Claim: Given τ and Γ as above, we have for $s < \tau$

$$\#\{1 \leq i \leq \ell \mid \nu_p(a_i + b_i) = s\} = \#\{0 \leq j \leq \ell - 1 \mid \nu_p(a_j - b_j) = s\},$$

where $\nu_p(r) = \max\{k \mid p^k \text{ divides } r\}$ is the p-adic valuation.

Proof of the claim. Suppose p^s divides $a_i + b_i$ for some $1 \leq i \leq \ell$ and v_{i-1} is not the root vertex. Applying $7(a)$ of Definition 4.1.8, we see p^s divides a_{i-1} or b_{i-1} (this implies $p \nmid a_{i-1} - b_{i-1}$). Going one step further we see p^s divides one of $a_{i-2} - b_{i-2}, a_{i-2}$ or b_{i-2}, and the other two are relatively prime to p. Since $p \nmid a_0, b_0$, continuing this process we end up with p^s dividing $a_j - b_j$ for some $0 \leq j < i - 1$ and $p \nmid a_q - b_q$ or $a_q + b_q$ for $j < q \leq i - 1$. If $p^s \mid a_1 + b_1$, we must have $p^s \mid a_0 - b_0$ by (5) of 4.1.8 and the fact that $p \nmid a_0, b_0$. Therefore, the left-hand side is less than or equal to right-hand side.

In the reverse direction, suppose $p^s \mid a_j - b_j$ for some $0 \leq j \leq \ell - 1$ and v_j is not the root vertex. Then by $7(a)$ of Definition 4.1.8, we see p^s divides one of a_{j+1}, b_{j+1}, and hence, p divides neither of $a_{j+1} + b_{j+1}$ or $a_{j+1} - b_{j+1}$. Continuing further we see that p^s divides one of $a_{j+2} + b_{j+2}, a_{j+2}$ or b_{j+2}. Since $p \mid a_\ell - b_\ell$, $p \nmid a_\ell, b_\ell$. Thus iterating this process we see p^s divides $a_i + b_i$ for some $j + 1 < i < \ell$ and $p \nmid a_q + b_q$ or $a_q - b_q$ for $j + 1 \leq q < i$. If $p^s \mid a_0 - b_0$, then by (5) of Definition 4.1.8, p^s divides one $a_1 + b_1, a_1$ or

b_1. If $p^s \nmid a_1 + b_1$, then continuing one step further we see p^s divides one of $a_2 + b_2, a_2$ or b_2. Iterating this way we obtain the required. This completes the proof of the claim.

Next we attach vertices of $\mathbb{T}_0 \setminus \Gamma$ to Γ proceeding by induction as in the proof of Theorem 4.2.9, except the fact that now levels are defined according to the distance from Γ instead of the root vertex. For the inductive step suppose the statement holds for the tree up to level L, $\mathbb{T}_0(L)$ and we adjoin an orbit $\mathcal{O}_x$ containing a level $L+1$-vertex v_x to $\mathbb{T}_0(L)$. Suppose $w(v_x) = (a_x, b_x; p^n)$. By Proposition 1.3.9, we may write $\Sigma^{\lambda^{a_x - b_x}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda^{p^t}} H\underline{\mathbb{Z}}$ for some $0 \leq t < \tau$. Then we obtain the following cofibre sequence of $H\underline{\mathbb{Z}}$-modules after applying the induction hypothesis

$$\Sigma^{\lambda^{p^t}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X(\mathbb{T}_0(L) + \mathcal{O}_x) \to \Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}} H\underline{\mathbb{Z}})^{\vee W_{\mathbb{T}_0(L)}(i)}\Big).$$

The fact that p^τ is the highest power ensures the connecting maps are zero by Theorem 2.2.10. Hence the above cofibre sequence splits and the required homology decomposition is obtained.

Finally, to complete the proof, we adjoin vertices of $\mathbb{T} \setminus \mathbb{T}_0$ to $\mathbb{T}_0$, i.e., vertices on which $\mathcal{G}$ acts non-trivially. Again we proceed by induction on levels where levels are defined according to the distance from $\mathbb{T}_0$. Assume the statement holds for the tree up to level L', $\mathbb{T}(L')$ and we attach an orbit $\mathcal{O}_y$ of the level $L'+1$-vertices to $\mathbb{T}(L')$. Suppose for $v_y \in \mathcal{O}_y$, $w(v_y) = (a_y, b_y; C_{m_y})$, and $\Sigma^{\lambda^{a_y - b_y}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda^{p^{t'}}} H\underline{\mathbb{Z}}$ for some $0 \leq t' \leq n$. Proceeding along the lines of Theorem 4.2.5, we obtain the following cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\mathcal{G}/C_{m_y +} \wedge \Sigma^{\lambda^{p^{t'}}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X\big(\mathbb{T}(L') + \mathcal{O}_y\big)_+ \to H\underline{\mathbb{Z}} \wedge X(\mathbb{T}(L'))_+ \tag{4.3.12}$$

where by the induction hypothesis

$$\begin{aligned} H\underline{\mathbb{Z}} \wedge X(\mathbb{T}(L'))_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}} H\underline{\mathbb{Z}})^{\vee W_{\mathbb{T}(L')}(i)}\Big) \\ \vee \Big(\bigvee_{[\mu] \in \mathbb{T}(L')_d/\mathcal{G}} \mathcal{G}/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}}\Big). \end{aligned}$$

The connecting map

$$\Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \to \mathcal{G}/C_{m_y +} \wedge \Sigma^{\lambda^{p^{t'}} + 1} H\underline{\mathbb{Z}}$$

is classified up to homotopy by $\pi_0^{C_{m_y}}(\Sigma^{\lambda^{p^{t'}} + 1 - \lambda - \lambda^{p^\tau}} H\underline{\mathbb{Z}})$. Since Lemma 4.3.8 asserts that $C_{m_y} \leq C_{p^\tau}$, the above group reduces to $\pi_0^{C_{m_y}}(\Sigma^{\lambda^{p^{t'}}-\lambda-1}H\underline{\mathbb{Z}})$, which is trivial by Theorem 2.2.10. Analogously all other connecting maps can be seen to be zero. Hence the cofibre sequence (4.3.12) splits and we obtain the required decomposition. $\square$

Chapter 5

Ring Structure for Projective Spaces

The objective of this chapter is to compute the equivariant cohomology ring $H^{\bigstar}_G(B_G S^1)$ for $G = C_{p^m}$, p prime and $m \geq 1$. We use the identification $B_G S^1 \simeq P(\mathcal{U})$ where $\mathcal{U}$ is a complete G-universe. The additive decomposition of §3.1 already provides a basis for the cohomology as a module over $H^{\bigstar}_G(\mathrm{pt}) = \pi^G_{-\bigstar}(H\underline{\mathbb{Z}})$. These computaions form part of the paper [5]

5.1 Multiplicative generators of $H^{\bigstar}_G(P(\mathcal{U}))$

The fixed point space $P(\mathcal{U})^G$ is a disjoint union of n-copies of $\mathbb{C}P^\infty$ which are included in $P(\mathcal{U})$ as $P(\infty\lambda^i) = \mathrm{colim}_k P(k\lambda^i)$ for $0 \leq i \leq n-1$. Let $q_i : P(\infty\lambda^i) \to P(\mathcal{U})$ denote the inclusion. In particular, we have

$$q_0 : P(\mathbb{C}^\infty) \to P(\mathcal{U}). \tag{5.1.1}$$

For the trivial G-action on $\mathbb{C}P^\infty$, we get

$$H^{\bigstar}_G(\mathbb{C}P^\infty) = H^*(\mathbb{C}P^\infty) \otimes H^{\bigstar}_G(\mathrm{pt}) = H^{\bigstar}_G(\mathrm{pt})[x],$$

where x is the multiplicative generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

Recall from §3.4.2 that the classes $\alpha_{k\phi_n+\phi_i}$ form an additive generating set for $H_G^{\bigstar}(P(\mathcal{U}))$. The following notes the multiplicative generating set.

Proposition 5.1.2. The collection $\{\alpha_{\phi_d} \mid d \text{ divides } n\}$ generate $H_G^{\bigstar}(P(\mathcal{U}))$ as an algebra over $H_G^{\bigstar}(\mathrm{pt})$.

Proof. We show by induction that each $\alpha_{k\phi_n+\phi_i}$, for $0 \le k < \infty$ and $0 \le i \le n-1$, may be expressed as a sum of monomials on the α_{ϕ_d}. The induction process is carried on both k and i. Suppose we know that all generators with degree lower than $\alpha_{k\phi_n+\phi_i}$ can be written in terms of α_{ϕ_d}'s. Let $i = \sum_{j=0}^{\ell} r_j p^j$ where $0 \le r_j \le p-1$. The class $\alpha_{\phi_n}^k \alpha_{\phi_{p^\ell}}^{r_\ell} \cdots \alpha_{\phi_1}^{r_0}$ also belongs to $H_G^{k\phi_n+\phi_i}(P(\mathcal{U}))$. So we may express the class as

$$\alpha_{\phi_n}^k \alpha_{\phi_{p^\ell}}^{r_\ell} \cdots \alpha_{\phi_1}^{r_0} = c_{\phi_1}\alpha_{\phi_1} + \cdots + c_{k\phi_n+\phi_i}\alpha_{k\phi_n+\phi_i}, \qquad \text{where } c_{\phi_j} \in H_G^{\bigstar}(\mathrm{pt}). \tag{5.1.3}$$

Note that in (5.1.3), a generator $\alpha_{t\phi_n+\phi_s}$ with degree greater than the degree of $\alpha_{k\phi_n+\phi_i}$ cannot appear; this is because for $\zeta = k\phi_n + \phi_i - (t\phi_n + \phi_s)$, the group $H_G^{\zeta}(\mathrm{pt}) = 0$ by (2.2.8) as all the fixed point dimensions of ζ are negative. Now, except the class $c_{k\phi_n+\phi_i}$, the degree of the other c_ϕ must be greater than zero, hence their restriction to G/e is zero. Thus, $\mathrm{res}_e^G(\alpha_{\phi_n}^k \alpha_{\phi_{p^\ell}}^{r_\ell} \cdots \alpha_{\phi_1}^{r_0}) = x^{n+i} = \mathrm{res}_e^G(\alpha_{k\phi_n+\phi_i})$ by (3.4.5). Therefore, $c_{k\phi_n+\phi_i}$ must be 1. $\square$

Example 5.1.4. Let $G = C_p$. The two classes α_{ϕ_1} and α_{ϕ_p} generate $H_{C_p}^{\bigstar}(P(\mathcal{U}))$. The computations of Lewis [30, §5] may be adapted to prove

$$H_G^{\bigstar}(P(\mathcal{U})) \cong H_G^{\bigstar}(\mathrm{pt})[\alpha_{\phi_1}, \alpha_{\phi_p}]/(u_\lambda \alpha_{\phi_p} - \alpha_{\phi_1} \prod_{i=1}^{p-1}(ia_\lambda + \alpha_{\phi_1}).). \tag{5.1.5}$$

The relation is obtained by restriction to various fixed points.

Lewis in his paper [30] described the ring structure of $H_G^{\bigstar}(P(\mathcal{U})_+; \underline{A})$. In the case of $\underline{\mathbb{Z}}$-coefficients, the classes $\alpha_\lambda \in H_G^{\lambda}(P(\mathcal{U})_+)$ and $\alpha_{\phi_p} \in H_G^{\phi_p}(P(\mathcal{U})_+)$ generate $H_G^{\bigstar}(P(\mathcal{U}_+))$ as a module over $H_G^{\bigstar}(S^0)$ and satisfy

$$q_0^*(\alpha_\lambda) = u_\lambda x, \quad q_0^*(\alpha_{\phi_p}) = x(a_\lambda + u_\lambda x)^{p-1}.$$

To determine the relation in $H_G^{\bigstar}(P(\mathcal{U}_+))$, suppose that

$$u_\lambda \alpha_{\phi_p} = \sum_{i=0}^{p} m_i a_\lambda^{p-i} \alpha_\lambda^i. \tag{5.1.6}$$

Then restricting to the orbit G/e, we get that $m_p = 1$. Further applying q_0^* to equation (5.1.6), we deduce that $m_i = \binom{p-1}{i-1}$. Summarizing this, we have the above relations.

The proof of Proposition 5.1.2 also demonstrates that one may change the basis of $H_G^{\bigstar}(P(\mathcal{U}))$ over $H_G^{\bigstar}(\mathrm{pt})$ from $\{\alpha_{k\phi_n+\phi_i}\}$ to $\{\alpha_{\phi_n}^k \alpha_{\phi_{p^\ell}}^{r_\ell} \cdots \alpha_{\phi_1}^{r_0}\}$ where $0 \leq r_j \leq p-1$. Therefore, there exists a relation of the form

$$\alpha_{\phi_{p^j}}^p = c\alpha_{\phi_{p^{j+1}}} + \text{ lower order terms.}$$

By restricting to G/e, we see that c must be $u_{\lambda^{p^j} - \lambda^{p^{j+1}}}$. The lower order terms will be calculated by restriction to fixed points.

The next result describes $q_0^*(\alpha_\lambda)$.

Proposition 5.1.7. $q_0^*(\alpha_{\phi_1}) = u_\lambda x$.

Proof. At deg λ, $H_G^{\bigstar}(\mathbb{C}P^\infty)$ has a basis given by a_λ and $u_\lambda x$. So $q_0^*(\alpha_{\phi_1}) = c_1 a_\lambda + c_2 u_\lambda x$. In the notation of (3.4.3), $W_1 = 1_{\mathbb{C}} + \lambda$. Consider the map $i : P(W_1) = S^{\lambda^{-1}} \hookrightarrow P(\mathcal{U})$ and $f : \mathrm{pt} \hookrightarrow \mathbb{C}P^\infty$. Consider the commutative diagram

$$\begin{array}{ccccc}
H^2(\mathbb{C}P^\infty) & \xleftarrow{\mathrm{res}_e^G} & H_G^\lambda(P(\mathcal{U})) & \xrightarrow{i^*} & H_G^\lambda(S^{\lambda^{-1}}) \\
\downarrow{\cong} & & \downarrow{q_0^*} & & \downarrow{q_0^*} \\
H^2(\mathbb{C}P^\infty) & \xleftarrow{\mathrm{res}_e^G} & H_G^\lambda(\mathbb{C}P^\infty) & \xrightarrow{f^*} & H_G^\lambda(\mathrm{pt}).
\end{array}$$

By (3.4.5), $\mathrm{res}_e^G(\alpha_{\phi_1}) = x$. So the left commutative square implies c_2 must be 1. Next, observe that the map i^* sends α_{ϕ_1} to the generator corresponding to $1 \in H_G^0(S^0) \simeq \tilde{H}_G^\lambda(S^\lambda) \subseteq H_G^\lambda(S^\lambda)$.

The cofibre sequence $P(1_{\mathbb{C}})_+ \simeq S^0 \xrightarrow{q_0} P(W_1)_+ \to S^\lambda$ implies $q_0^* i^*(\alpha_\lambda) = 0$. So $c_1 = 0$, and thus $q_0^*(\alpha_{\phi_1}) = u_\lambda x$. $\square$

5.2 Restrictions to fixed points

We adapt the approach of Lewis [30] to our case for calculating $q_0^*(\alpha_{\phi_d})$. For a subset I of $\underline{d-1} := \{1, 2, \cdots, d-1\}$, denote

$$\omega_I = \lambda^{-d}(1_{\mathbb{C}} + \Sigma_{i \in I} \lambda^i),$$

and

$$V_{I,k} = 1_{\mathbb{C}} + \Sigma_{i \in I} \lambda^i + \lambda^d + k \cdot 1_{\mathbb{C}},$$

for $k \geq 0$. Consider the cofibre sequence

$$P(V_{I,0} - \lambda^d)_+ \to P(V_{I,0})_+ \xrightarrow{\chi} S^{\omega_I},$$

which implies the long exact sequence

$$\cdots \to \tilde{H}_G^{\omega_I - 1}(P(V_{I,0} - \lambda^d)_+) \to \tilde{H}_G^{\omega_I}(S^{\omega_I}) \xrightarrow{\chi^*} \tilde{H}_G^{\omega_I}(P(V_{I,0})_+) \to \tilde{H}_G^{\omega_I}(P(V_{I,0} - \lambda^d)_+) \to \cdots$$

Define $\Delta_{\omega_I}^{V_{I,0}} \in H_G^{\omega_I}(P(V_{I,0}))$ to be $\chi^*(1)$. Next we lift the class $\Delta_{\omega_I}^{V_{I,0}}$ uniquely to the class $\Delta_{\omega_I}^{V_{I,k}} \in H_G^{\omega_I}(P(V_{I,k}))$ with the help of the cofibre sequence $P(V_{I,\ell})_+ \xrightarrow{\theta_\ell} P(V_{I,\ell+1})_+ \to S^{V_{I,\ell}}$. At degree ω_I we get

$$\cdots \to \tilde{H}_G^{\omega_I}(S^{V_{I,\ell}}) \to \tilde{H}_G^{\omega_I}(P(V_{I,\ell+1})_+) \xrightarrow{\theta_\ell^*} \tilde{H}_G^{\omega_I}(P(V_{I,\ell})_+) \to \tilde{H}_G^{\omega_I + 1}(S^{V_{I,\ell}}) \to \cdots$$

We claim θ_ℓ^* is an isomorphism. To see this we observe that as $i < d$ and d is a power of p, all the fixed-point dimensions of λ^{i-d} and λ^i are same. Hence all the fixed-point dimensions of $\omega_I - V_{I,\ell}$ are ≤ -2, so $H_G^{\omega_I - V_{I,\ell}}(\mathrm{pt}) = 0$ and $H_G^{\omega_I + 1 - V_{I,\ell}}(\mathrm{pt}) = 0$ for $\ell \geq 0$.

As the restriction of χ^* to the orbit G/e is an isomorphism, we have

$$\mathrm{res}_e^G(\Delta_{\omega_I}^{V_{I,k}}) = x^{|I|+1}. \tag{5.2.1}$$

In the same spirit, using the cofibre sequence

$$P(V_{I,\ell})_+ \hookrightarrow P(V_{I,\ell+1})_+ \xrightarrow{\chi} S^{V_{I,\ell}},$$

we may define the class $\Omega_{V_{I,\ell}}^{V_{I,\ell+1}} \in H_G^{V_{I,\ell}}(P(V_{I,\ell+1}))$ to be the image of $\chi^*(1)$ where

$1 \in \tilde{H}_G^{V_{I,\ell}}(S^{V_{I,\ell}}) \cong \mathbb{Z}$. As for $\Delta_{\omega_I}^{V_{I,k}}$, this lifts uniquely to define the class $\Omega_{V_{I,\ell}}^{V_{I,k}} \in H_G^{V_{I,\ell}}(P(V_{I,k}))$. We define $\Omega_{V_{I,k}}^{V_{I,k}} = 0$. Its restriction to the orbit G/e is

$$\mathrm{res}_e^G(\Omega_{V_{I,\ell}}^{V_{I,k}}) = x^{|I|+\ell+2}.$$

For $i \in I$, let $\tau_{i,k}$ (or simply τ_i) be the inclusion map $P(V_{I\setminus\{i\},k}) \hookrightarrow P(V_{I,k})$.

Proposition 5.2.2. For the map $\tau_{i,k}$ we get the following

1. $\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}}) = \Theta_{i,d} \cdot a_{\lambda^i}\Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},k}} + u_{\lambda^i}\Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},k}}$.
2. $\tau_{i,k}^*(\Omega_{V_{I,\ell}}^{V_{I,k}}) = a_{\lambda^i}\Omega_{V_{I\setminus\{i\},\ell}}^{V_{I\setminus\{i\},k}} + u_{\lambda^i}\Omega_{V_{I\setminus\{i\},\ell+1}}^{V_{I\setminus\{i\},k}}$ for $0 \leq \ell < k$.

Proof. We prove (1). The proof of (2) is analogous. We start with the representation $V_{I,0}$, and successively add $1_{\mathbb{C}}$ to reach $V_{I,k}$. First consider the following diagram.

$$\begin{tikzcd}
P(V_{I\setminus\{i\},0} - \lambda^d)_+ \arrow[r, hook] \arrow[d, hook] & P(V_{I,0} - \lambda^d)_+ \arrow[d, hook] \\
P(V_{I\setminus\{i\},0})_+ \arrow[r, hook, "\tau_{i,0}"] \arrow[d] & P(V_{I,0})_+ \arrow[d] \\
S^{\omega_{I\setminus\{i\}}} \arrow[r, "a_{\lambda^{i-d}}"'] & S^{\omega_I}
\end{tikzcd} \tag{5.2.3}$$

Since $i < d$, the p-adic valuation of $i - d$ is same as i. We have

$$a_{\lambda^{i-d}} = \Theta_{i,d} \cdot a_{\lambda^i} \tag{5.2.4}$$

by Proposition 2.1.7. At $\deg \omega_I$, the bottom commutative square gives us

$$\tau_{i,0}^*(\Delta_{\omega_I}^{V_{I,0}}) = \Theta_{i,d} \cdot a_{\lambda^i}\Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},0}}.$$

In the next step, we add $1_{\mathbb{C}}$ to the representations in the middle row of the diagram (5.2.3). This fits into the following

$$\begin{tikzcd}
P(V_{I\setminus\{i\},0})_+ \arrow[r, hook, "\tau_{i,0}"] \arrow[d, hook, "+1_{\mathbb{C}}"'] & P(V_{I,0})_+ \arrow[d, hook, "+1_{\mathbb{C}}"] \\
P(V_{I\setminus\{i\},1})_+ \arrow[r, hook, "\tau_{i,1}"] \arrow[d] & P(V_{I,1})_+ \arrow[d] \\
S^{V_{I\setminus\{i\},0}} \arrow[r] & S^{V_{I,0}}
\end{tikzcd} \tag{5.2.5}$$

Since we have built $P(V_{I\setminus\{i\},0})$ by attaching cells in a proper order, the boundary maps in the cohomology long exact sequence induced by the left-hand cofibration sequence are trivial. Moreover, $H_G^{\bigstar}(P(V_{I\setminus\{i\},0}))$ is free as $H_G^{\bigstar}(\mathrm{pt})$-module. So

$$H_G^{\omega_I}(P(V_{I\setminus\{i\},1}) \cong H_G^{\omega_I}(P(V_{I\setminus\{i\},0})) \oplus H_G^{\omega_I - V_{I\setminus\{i\},0}}(\mathrm{pt}).$$

Further, $H_G^{\omega_I - V_{I\setminus\{i\},0}}(\mathrm{pt}) \cong \mathbb{Z}$ generated by the class u_{λ^i}. So the diagram (5.2.5) implies

$$\tau_{i,1}^*(\Delta_{\omega_I}^{V_{I,1}}) = \Theta_{i,d} \cdot a_{\lambda^i} \Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},1}} + c \cdot u_{\lambda^i} \Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},1}}$$

for some $c \in \mathbb{Z}$. We claim that $c = 1$. This is because from the next step onwards (where in each step we add a copy of $1_{\mathbb{C}}$) we have $H_G^{\omega_I}(P(V_{I\setminus\{i\},k})) \cong H_G^{\omega_I}(P(V_{I\setminus\{i\},1}))$ as $H_G^{\omega_I - V_{I,\ell}}(\mathrm{pt}) = 0$ for $\ell \geq 1$. As a consequence,

$$\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}}) = \Theta_{i,d} \cdot a_{\lambda^i} \Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},k}} + c \cdot u_{\lambda^i} \Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},k}}.$$

The map $\tau_{i,k}^*$ at the orbit G/e is an isomorphism. Moreover, by (5.2.1) $\mathrm{res}_e^G(\Delta_{\omega_I}^{V_{I,k}}) = x^{|I|+1} = \mathrm{res}_e^G(\Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},k}})$. So c must be 1, otherwise, we get a contradiction by restricting to the orbit G/e. This completes the proof of (1). □

In case $d = p^m$, then the following simplification occurs as $\frac{p^m}{i} a_{\lambda^i} = 0$ (2.2.2).

Proposition 5.2.6. For the map $\tau_{i,k}$ we get the following

$$\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}}) = a_{\lambda^i} \Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},k}} + u_{\lambda^i} \Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},k}}.$$

If we work with $\underline{\mathbb{Z}/p}$-coefficients, then $a_{\lambda^{i-d}} = a_{\lambda^i}$ as $\Theta_{i,d} \equiv 1 \pmod p$ by Proposition (2.1.7). This simplification leads us to

Proposition 5.2.7. In $\underline{\mathbb{Z}/p}$-coefficient, for the map $\tau_{i,k}$ we get the following

$$\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}}) = a_{\lambda^i} \Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},k}} + u_{\lambda^i} \Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},k}}.$$

The value of $\tau_{i,k}^*(\Omega_{V_{I,\ell}}^{V_{I,k}})$ remains same as in Proposition 5.2.2.

Remark 5.2.8. Note that, for the group C_p, we have $P(V_{\varnothing,k}) = P(\mathbb{C}^{k+2})$. Hence the class $\Delta_{\omega_\varnothing}^{V_{\varnothing,k}}$ is the class $x \in H_G^2(P(\mathbb{C}^{k+2}))$, and the class $\Omega_{V_{\varnothing,\ell}}^{V_{\varnothing,k}}$ is the class $x^{\ell+2} \in H_G^{2\ell+4}(P(\mathbb{C}^{k+2}))$.

Proposition 5.2.9. In the case when $I = \emptyset$, we obtain

1. $\tau_{d,k}^*(\Delta_{\omega_\emptyset}^{V_{\emptyset,k}}) = u_{\lambda^d} \cdot x$.
2. $\tau_{d,k}^*(\Omega_{V_{\emptyset,\ell}}^{V_{\emptyset,k}}) = a_{\lambda^d} \cdot x^{\ell+1} + u_{\lambda^d} \cdot x^{\ell+2}$.

Proof. Recall that $V_{\emptyset,k} = 1_{\mathbb{C}} + \lambda^d + k \cdot 1_{\mathbb{C}}$. The cofibre sequence

$$P(1_{\mathbb{C}})_+ \xrightarrow{\tau_{d,0}} P(1_{\mathbb{C}} + \lambda^d)_+ \to S^{\omega_\emptyset}$$

implies $\tau_{d,0}^*(\Delta_{\omega_\emptyset}^{V_{\emptyset,0}}) = 0$. The rest of the proof is quite similar to Proposition 5.2.2. So we describe it briefly. In the next step, we have

$$\begin{tikzcd}
P(1_{\mathbb{C}})_+ \arrow[r, hook, "\tau_{d,0}"] \arrow[d, hook] & P(V_{\emptyset,0})_+ \arrow[d, hook] \\
P(2 \cdot 1_{\mathbb{C}})_+ \arrow[r, hook, "\tau_{d,1}"] \arrow[d] & P(V_{\emptyset,1})_+ \arrow[d] \\
S^2 \arrow[r] & S^{V_{\emptyset,0}}.
\end{tikzcd}$$

At degree $\omega_\emptyset$, $H_G^{\omega_\emptyset - 2}(\mathrm{pt}) \cong \mathbb{Z}\{u_{\lambda^d}\}$, and $H_G^{\omega_\emptyset - V_{\emptyset,0}}(\mathrm{pt}) = 0$. Using restriction to the orbit G/e, we may conclude that $\tau_{d,1}^*(\Delta_{\omega_\emptyset}^{V_{\emptyset,k}}) = \tau_{d,k}^*(\Delta_{\omega_\emptyset}^{V_{\emptyset,k}}) = u_{\lambda^d} \cdot x$. □

The following is a direct calculation

Proposition 5.2.10. Let $\mathcal{I} = \{i_1, \cdots, i_r\}$. Then

$$\tau_d \tau_{i_1} \cdots \tau_{i_r}(\Omega_{V_{I,t}}^{V_{I,k}}) = x^{t+1}(a_{\lambda^d} + u_{\lambda^d} \cdot x) \prod_{s=1}^{r} (a_{\lambda^{i_s}} + x u_{\lambda^{i_s}})$$

Proof. Let $\mathcal{I} = \{i_1, \cdots, i_r\}$. Proposition 5.2.2 and Proposition 5.2.9 gives us

$$\tau_d \tau_{i_1} \cdots \tau_{i_r}(\Omega_{V_{I,t}}^{V_{I,k}}) = \tau_d \Big[\sum_{\ell=t}^{t+r} \big(\Omega_{V_{\emptyset,\ell}}^{V_{\emptyset,k}} \sum_{\{j_1,\cdots,j_{\ell-t}\} \subseteq \mathcal{I}} u_{\lambda^{j_1}} \cdots u_{\lambda^{j_{\ell-t}}} a_{\lambda^{j_{\ell-t+1}}} \cdots a_{\lambda^{j_r}} \big) \Big].$$

Further applying τ_d to $\Omega_{V_{\emptyset,\ell}}^{V_{\emptyset,k}}$, and taking x^{t+1} common, we simplify the right hand side as

$$x^{t+1}(a_{\lambda^d} + u_{\lambda^d} \cdot x) \sum_{\ell=0}^{r} x^\ell \big(\sum_{\{j_1,\cdots,j_\ell\} \subseteq \mathcal{I}} u_{\lambda^{j_1}} \cdots u_{\lambda^{j_\ell}} a_{\lambda^{j_{\ell+1}}} \cdots a_{\lambda^{j_r}} \big).$$

This easily factorizes to imply the result. □

Now we are in a position to determine $q_0^*(\alpha_{\phi_d})$.

Proposition 5.2.11. $q_0^*(\alpha_{\phi_d}) = \sum_{i=0}^{d-1}\Big[(\prod_{j=1}^{i}\Theta_{j,d}a_{\lambda^j})u_{\lambda^{i+1}}x\prod_{s=i+2}^{d}(a_{\lambda^s}+u_{\lambda^s}x)\Big]$.

Proof. Consider the map $i : P(V_{\underline{d-1},d}) \to P(\mathcal{U})$ given by inclusion. We claim $i^*(\alpha_{\phi_d}) = \Delta_{\omega_{\underline{d-1}}}^{V_{d-1,d}}$. The reason is as follows: we started with the classes $\alpha_{\phi_d}^{W_d}$ and $\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},0}}$ which were essentially same. Then we extended these classes through a chain of isomorphisms by successively adding representation λ^i (resp. $1_{\mathbb{C}}$) to define the class α_{ϕ_d} (resp. $\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}}$). Since in the end, we have $P(V_{\underline{d-1},d}) \hookrightarrow P(\mathcal{U})$, so $i^*(\alpha_{\phi_d}) = \Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}}$.

To determine $q_0^*(\alpha_{\phi_d})$, it is enough to work out $q_0^*(\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}})$. For this, we successively remove all the nontrivial representations from $V_{\underline{d-1},d}$. Now $q_0^*(\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}}) = \tau_d\cdots\tau_2\tau_1(\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}})$. Applying Proposition 5.2.7, this becomes

$$\tau_d\cdots\tau_2(\Theta_{1,d}\cdot a_\lambda\Delta_{\omega_{\underline{d-1}\backslash\{1\}}}^{V_{\underline{d-1}\backslash\{1\},d}}) + \tau_d\cdots\tau_2(u_\lambda\Omega_{V_{\underline{d-1}\backslash\{1\},0}}^{V_{\underline{d-1}\backslash\{1\},d}}). \tag{5.2.12}$$

Let $z_s = a_{\lambda^s} + u_{\lambda^s}x$. The second term can be simplified by Proposition 5.2.10 to $u_\lambda x\prod_{s=2}^{d}(a_{\lambda^s}+u_{\lambda^s}x) = u_\lambda x\prod_{s=2}^{d}z_s$. Now applying τ_2 in (5.2.12) and repeating the above procedure, we get

$$\tau_d\cdots\tau_3(\Theta_{1,d}\cdot a_\lambda\Theta_{2,d}\cdot a_{\lambda^2}\Delta_{\omega_{\underline{d-1}\backslash\{1,2\}}}^{V_{\underline{d-1}\backslash\{1,2\},d}}) + u_\lambda x\prod_{s=2}^{d}z_s + \Theta_{1,d}\cdot a_\lambda u_\lambda^2 x\prod_{s=3}^{d}z_s.$$

Repeating this process up to τ_d we obtain the required. □

When $d = p^m$, in the expression of $\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}})$, the numbers $\Theta_{i,d}$ becomes 1 (cf. Proposition 5.2.6 and 5.2.2). Using Proposition 5.2.6 we obtain the following simplification.

Proposition 5.2.13. $q_0^*(\alpha_{\phi_{p^m}}) = \prod_{i=1}^{p^m}(a_{\lambda^i}+xu_{\lambda^i}) - \prod_{i=1}^{p^m}a_{\lambda^i}$.

Proof. We use $\frac{p^m}{i}a_{\lambda^i} = 0$ (2.2.2) so that the terms Θ_{j,p^m} does not appear in this case as in the expression proved in Proposition 5.2.11. Write $z_s = a_{\lambda^s} + u_{\lambda^s}x$. We have

$$\begin{aligned}
q_0^*(\alpha_{\phi_{p^m}}) &= \sum_{i=0}^{p^m-1}\Big[(\prod_{j=1}^{i} a_{\lambda^j})u_{\lambda^{i+1}}x \prod_{s=i+2}^{p^m}(a_{\lambda^s}+u_{\lambda^s}x)\Big] \\
&= \sum_{i=0}^{p^m-1}\Big[(\prod_{j=1}^{i} a_{\lambda^j})u_{\lambda^{i+1}}x \prod_{s=i+2}^{p^m} z_s\Big] \\
&= \sum_{i=0}^{p^m-1}\Big[(\prod_{j=1}^{i} a_{\lambda^j}) \prod_{s=i+1}^{p^m} z_s - (\prod_{j=1}^{i+1} a_{\lambda^j}) \prod_{s=i+2}^{p^m} z_s\Big] \\
&= \prod_{i=1}^{p^m} z_s - \prod_{i=1}^{p^m} a_{\lambda^i}.
\end{aligned}$$

□

Either taken in $\underline{\mathbb{Z}/p}$-coefficient or in case of $d = p^m$, the expression of $\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}})$ is same (cf. Proposition 5.2.6 and 5.2.7). The coefficient in 5.2.13 is $\underline{\mathbb{Z}}$. But this expression does not work for $d = p^i, i < m$. Therefore, we go to $\underline{\mathbb{Z}/p}$ coefficients and as a result we may proceed as in Proposition 5.2.13 to obtain the following

Proposition 5.2.14. With $\underline{\mathbb{Z}/p}$-coefficients, we have

$$q_0^*(\alpha_{\phi_d}) = \prod_{i=1}^{d}(a_{\lambda^i} + xu_{\lambda^i}) - \prod_{i=1}^{d} a_{\lambda^i}.$$

Once we have the expressions for q_0 on the multiplicative generators, we relate them to obtain the relations in the cohomology ring. The following proposition states that q_0^* is injective, which means that the image of q_0^* may be used to detect relations.

Proposition 5.2.15. The map q_0^* is injective at each degree $\zeta_\ell^w = \lambda + \cdots + \lambda^\ell + \lambda^w$ where $w \leq \ell$.

Proof. Recall from the additive decomposition of Theorem 3.2.1 that

$$H_G^{\bigstar}(P(\mathcal{U})) \cong \bigoplus_{i\geq 0} H_G^{\bigstar-\phi_i}(\mathrm{pt})\{\alpha_{\phi_i}\}.$$

The notation here means that α_{ϕ_i} generates the factor $H_G^{\bigstar-\phi_i}(\mathrm{pt})$ of the free $H_G^{\bigstar}(\mathrm{pt})$-module $H_G^{\bigstar}(P(\mathcal{U}))$. Thus, $H_G^{\zeta_\ell^w}(P(\mathcal{U})) \cong \bigoplus_{i=0}^{\ell} H_G^{\zeta_\ell^w-\phi_i}(\mathrm{pt})\{\alpha_{\phi_i}\}$. By Theorem 2.2.1, the element $a_{\zeta_\ell^w-\phi_i} := a_{\lambda^{i+1}}\cdots a_{\lambda^\ell}a_{\lambda^w}$ generates the group $H_G^{\zeta_\ell^w-\phi_i}(\mathrm{pt}) \cong \mathbb{Z}/p^{r_{\ell,i}^w}$ (look at the discussion following Theorem 2.2.1), where

$$r^w_{\ell,i} = m - \text{ highest power of } p \text{ dividing at least one of } i+1, \cdots, \ell, w.$$

Therefore the element $a_{\zeta^w_\ell - \phi_i}\alpha_{\phi_i} \in H^{\zeta^w_\ell}_G(P(\mathcal{U}))$ also has order $p^{r^w_{\ell,i}}$. Hence,

$$H^{\zeta^w_\ell}_G(P(\mathcal{U}) \cong \mathbb{Z} \bigoplus \mathbb{Z}/p^{r^w_{\ell,\ell-1}} \bigoplus \cdots \bigoplus \mathbb{Z}/p^{r^w_{\ell,0}}.$$

Also, we have

$$H^{\bigstar}_G(P(\infty 1_{\mathbb{C}})) \cong \bigoplus_{i\geq 0} H^{\bigstar - 2i}_G(\mathrm{pt})\{x^i\}.$$

So $H^{\zeta^w_\ell}_G(P(\infty 1_{\mathbb{C}})) \cong \bigoplus_{i=0}^{\ell} H^{\zeta^w_\ell - 2i}_G(\mathrm{pt})\{x^i\}$. By the discussion following Theorem 2.2.1, the group $H^{\zeta^w_\ell - 2i}_G(\mathrm{pt}) \cong \mathbb{Z}/p^{t^w_{\ell,i}}$, where

$$t^w_{\ell,i} = \max\big[m - \max\{v_p(s_1), \cdots, v_p(s_{\ell-i})\} \;\mid\; s_1, \cdots, s_{\ell-i} \in \{1, \cdots, \ell, w\}\big].$$

This tells us

$$H^{\zeta^w_\ell}_G(P(\infty 1_{\mathbb{C}})) \cong \mathbb{Z} \bigoplus \mathbb{Z}/p^{t^w_{\ell,\ell-1}} \bigoplus \cdots \bigoplus \mathbb{Z}/p^{t^w_{\ell,0}}.$$

We observe that the element

$$u_{\phi_i} a_{\zeta^w_\ell - \phi_i} := u_\lambda u_{\lambda^2} \cdots u_{\lambda^i} a_{\lambda^{i+1}} \cdots a_{\lambda^\ell} a_{\lambda^w} \in H^{\zeta^w_\ell - 2i}_G(\mathrm{pt})$$

has order $p^{r^w_{\ell,i}}$. So the element $a_{\zeta^w_\ell - \phi_i} u_{\phi_i} x^i \in H^{\zeta^w_\ell}_G(P(\infty 1_{\mathbb{C}}))$ is also of order $p^{r^w_{\ell,i}}$ in $\mathbb{Z}/t_{\ell,i}$. Since $\mathrm{res}^G_e(\alpha_{\phi_i}) = x^i$, we have $\mathrm{res}^G_e(q_0^*(\alpha_{\phi_i})) = x^i$. This implies $q_0^*(\alpha_{\phi_i}) = u_{\phi_i} x^i + \Sigma_{j=0}^{i-1} c_j x^j$, for some coefficients c_j. Thus

$$q_0^*(a_{\zeta^w_\ell - \phi_i}\alpha_{\phi_i}) = a_{\zeta^w_\ell - \phi_i} q_0^*(\alpha_{\phi_i}) = a_{\zeta^w_\ell - \phi_i} u_{\phi_i} x^i + a_{\zeta^w_\ell - \phi_i} \Sigma_{j=0}^{i-1} c_j x^j.$$

Therefore q_0^* as a map

$$\mathbb{Z} \bigoplus \mathbb{Z}/p^{r^w_{\ell,\ell-1}} \bigoplus \cdots \bigoplus \mathbb{Z}/p^{r_{\ell,0}} \to \mathbb{Z} \bigoplus \mathbb{Z}/p^{t^w_{\ell,\ell-1}} \bigoplus \cdots \bigoplus \mathbb{Z}/p^{t^w_{\ell,0}}$$

is a lower triangular matrix of the form

$$\begin{pmatrix} 1 & & & \\ * & k_{\ell,\ell-1} & & \text{\huge 0} \\ \vdots & \vdots & \ddots & \\ * & * & \dots & k_{\ell,0} \end{pmatrix}$$

where $k_{\ell,i} = p^{t^w_{\ell,i} - r^w_{\ell,i}}$. Hence q_0^* is injective at the degree ζ_ℓ^w. □

5.3 Relations for complex projective spaces

Note that if $d = p^m$, then $a_{\lambda^d} = 0$ and $u_{\lambda^d} = 1$, so Proposition 5.2.13 simplified to $q_0^*(\alpha_{\phi_n}) = \prod_{i=1}^{n-1} x(a_{\lambda^i} + xu_{\lambda^i})$. For the group C_p, using Proposition 2.1.7, this further reduces to $q_0^*(\alpha_{\phi_p}) = x\prod_{i=1}^{p-1}(ia_\lambda + u_\lambda x)$. Using the fact that $q_0^*(\alpha_{\phi_1}) = u_\lambda x$ from (5.1.7), we see that

$$q_0^*(u_\lambda \alpha_{\phi_p} - \alpha_{\phi_1}\prod_{i=1}^{p-1}(ia_\lambda + \alpha_{\phi_1})) = 0.$$

Moreover, Proposition 5.2.15 tells us that q_0^* is injective. So the relation we obtain for C_p is

$$u_\lambda \alpha_{\phi_p} - \alpha_{\phi_1}\prod_{i=1}^{p-1}(ia_\lambda + \alpha_{\phi_1}).$$

For general C_{p^m} of order $n = p^m$, there are m relations of the form

$$u_{\lambda^{p^{i-1}} - \lambda^{p^i}} \alpha_{\phi_{p^i}} = \alpha^p_{\phi_{p^{i-1}}} + \text{ lower order terms},$$

for $1 \le i \le m$. In fact, the proof of Proposition 5.1.2 implies that the coefficients of the lower order terms are expressible as a sum of monomials with coefficients that are linear combinations of products of a_{λ^j}. The naive idea is to apply q_0^* to such an equation to determine all the coefficients. However, the expression in Proposition 5.2.11 does not directly yield a simple closed relation. We are able to obtain a simple expression after mapping to $\mathbb{Z}/p$-coefficients.

The first observation when we look at $\mathbb{Z}/p$-coefficients is that q_0^* is no longer injective. For, in the proof of Proposition 5.2.15, the diagonal entries in the lower triangular matrix other than at the top corner, turns out to be $0 \pmod p$. We use the formula for q_0^* and that it is injective with $\mathbb{Z}$-coefficients. Let $\mathcal{R}_d$ denote the algebra

$$\mathbb{Z}[u_{\lambda^i}, a_{\lambda^j}, u_{\lambda^{p^{d-1}}-\lambda^{p^d}}]/I,$$

where I is the ideal generated by the relations (2.2.2), those in Proposition 2.1.5 and

$$u_{\lambda^{p^{d-1}}-\lambda^{p^d}} u_{\lambda^{p^d}} = u_{\lambda^{p^{d-1}}},\ u_{\lambda^{p^{d-1}}-\lambda^{p^d}} a_{\lambda^{p^d}} = p a_{\lambda^{p^{d-1}}},$$

which maps to $\pi_{-\bigstar} H\underline{\mathbb{Z}} = H^{\bigstar}_G(\mathrm{pt})$. The algebra $\mathcal{R}_d$ contains the classes u_{λ^i} and a_{λ^j} but they are not required to satisfy the relation (2.2.3). Form the algebraic q_0^* map

$$Q_0 : \mathcal{R}_d[\alpha_{\phi_{p^j}} \mid 0 \le j \le m] \to \mathcal{R}_d[x]$$

given by the formula in Proposition 5.2.11. In the absence of the relation (2.2.3) in $\mathcal{R}_d$, the lower triangular matrix in the proof of Proposition 5.2.15 gets replaced by one where the diagonal entries are inclusions of the corresponding summand. This becomes injective even after tensoring with $\mathbb{Z}/p$. The algebra $\mathcal{R}_d[\alpha_{\phi_{p^j}} \mid 0 \le j \le m]$ is denoted $\mathcal{R}_d(P(\mathcal{U}))$.

We thus work with the diagram

$$\begin{tikzcd}
\mathcal{R}_d(P(\mathcal{U})) \arrow[r, "\nu_p"] \arrow[d, "Q_0"] & \mathbb{Z}/p \otimes \mathcal{R}_d(P(\mathcal{U})) \arrow[d, "Q_0"] \\
\mathcal{R}_d[x] \arrow[r, "\nu_p"] & \mathbb{Z}/p \otimes \mathcal{R}_d[x],
\end{tikzcd}$$

and seek relations χ which maps to 0 in $\mathbb{Z}/p \otimes \mathcal{R}_d[x]$. It follows that $\chi \equiv 0 \pmod p$ in $\mathcal{R}_d(P(\mathcal{U}))$ and thus gives a relation in $H^{\bigstar}_G(P(\mathcal{U}); \underline{\mathbb{Z}/p})$. We note from Proposition 2.1.7 that

$$a_{\lambda^{kp^{r-1}+i}} = (1 + kp^{r-1} \cdot i^{-1}) a_{\lambda^i}, \implies a_{\lambda^{kp^{r-1}+i}} \equiv a_{\lambda^i} \pmod p, \text{ for } i < p^{r-1}. \quad (5.3.1)$$

The following is a consequence of the identity $\prod_{i=1}^{p-1}(x+i) \equiv x^{p-1} - 1 \pmod p$.

Lemma 5.3.2.

$$\prod_{i=1}^{p-1} (i a_{\lambda^{p^{r-1}}} + x u_{\lambda^{p^{r-1}}}) = (x u_{\lambda^{p^{r-1}}})^{p-1} - (a_{\lambda^{p^{r-1}}})^{p-1}$$

We write $P(z,w) = (z-w)^{p-1} - w^{p-1}$. Define the following notations

$$\begin{aligned}
&\mathcal{B}_r = \prod_{i=1}^{p^r-1} (a_{\lambda^i} + u_{\lambda^i} x) \in \mathcal{R}_d[x],\\
&\mathcal{T}_r = \alpha_{\phi_{p^j}} + \prod_{i=1}^{p^j} a_{\lambda^i} \in \mathcal{R}_d(P(\mathcal{U})),\\
&\mathbb{T}_r = Q_0(\mathcal{T}_r) = \mathcal{B}_r \cdot (x u_{\lambda^{p^r}} + a_{\lambda^{p^r}}) \pmod p, \text{ by Proposition 5.2.14},\\
&\mathbb{A}_0 = P(\mathbb{T}_0, a_\lambda), \text{ and inductively, } \mathbb{A}_j = P(\mathbb{T}_j, a_{\lambda^{p^j}} \prod_{i=0}^{j-1} \mathbb{A}_i)\\
&\mathcal{A}_0 = P(\mathcal{T}_0, a_\lambda), \text{ and, } \mathcal{A}_j = P(\mathcal{T}_j, a_{\lambda^{p^j}} \prod_{i=0}^{j-1} \mathcal{A}_i), \text{ so that } Q_0(\mathcal{A}_j) = \mathbb{A}_j.
\end{aligned} \tag{5.3.3}$$

We now have the following relation with $\mathbb{Z}/p$ coefficients.

$$\begin{aligned}
\mathcal{B}_r &= \prod_{i=1}^{p^r-1} (a_{\lambda^i} + u_{\lambda^i} x)\\
&= \prod_{i=1, p^{r-1}\nmid i}^{p^r-1} (a_{\lambda^i} + u_{\lambda^i} x) \prod_{j=1}^{p-1} (a_{\lambda^{jp^{r-1}}} + u_{\lambda^{jp^{r-1}}} x)\\
&= \mathcal{B}_{r-1}^p \prod_{j=1}^{p-1} (j a_{\lambda^{p^{r-1}}} + u_{\lambda^{p^{r-1}}} x) \text{ by (5.3.1), and Proposition 2.1.7}\\
&= \mathcal{B}_{r-1}^p \big((x u_{\lambda^{p^{r-1}}})^{p-1} - a_{\lambda^{p^{r-1}}}^{p-1} \big)\\
&= \mathcal{B}_{r-1} \big((\mathbb{T}_{r-1} - \mathcal{B}_{r-1} a_{\lambda^{p^{r-1}}})^{p-1} - (\mathcal{B}_{r-1} a_{\lambda^{p^{r-1}}})^{p-1} \big)\\
&= \mathcal{B}_{r-1} P(\mathbb{T}_{r-1}, a_{\lambda^{p^{r-1}}} \mathcal{B}_{r-1}).
\end{aligned}$$

From the expression, it inductively follows that

$$\mathcal{B}_r = \prod_{i=0}^{r-1} \mathbb{A}_r. \tag{5.3.4}$$

We finally obtain

Proposition 5.3.5. With $\mathbb{Z}/p$-coefficients, the class $\alpha_{\phi_{p^r}}$ satisfies the following relation

$$u_{\lambda^{p^{r-1}} - \lambda^{p^r}} \alpha_{\phi_{p^r}} = \mathcal{T}_{r-1}^p - a_{\lambda^{p^{r-1}}}^{p-1} \mathcal{T}_{r-1} \big(\prod_{i=0}^{r-2} \mathcal{A}_i \big)^{p-1}.$$

Proof. As observed above, it suffices to prove with $\mathbb{Z}/p$-coefficients

$$Q_0(u_{\lambda^{p^{r-1}} - \lambda^{p^r}} \alpha_{\phi_{p^r}}) = \mathbb{T}_{r-1}^p - a_{\lambda^{p^{r-1}}}^{p-1} \mathbb{T}_{r-1} \big(\prod_{i=0}^{r-2} \mathbb{A}_i \big)^{p-1} = \mathbb{T}_{r-1}^p - a_{\lambda^{p^{r-1}}}^{p-1} \mathbb{T}_{r-1} \mathcal{B}_{r-1}^{p-1}.$$

We verify

$$\begin{aligned}
Q_0(u_{\lambda^{p^{r-1}}-\lambda^{p^r}}\alpha_{\phi_{p^r}}) &= u_{\lambda^{p^{r-1}}-\lambda^{p^r}}\big(\prod_{j=1}^{p^r}(xu_{\lambda^j}+a_{\lambda^j})-\prod_{j=1}^{p^r}a_{\lambda^j}\big)\\
&= \mathcal{B}_r xu_{\lambda^{p^{r-1}}}\\
&= \mathcal{B}_{r-1}^p\big((xu_{\lambda^{p^{r-1}}})^p - a_{\lambda^{p^{r-1}}}^{p-1}xu_{\lambda^{p^{r-1}}}\big)\\
&= \mathbb{T}_{r-1}^p - \mathcal{B}_{r-1}^p(a_{\lambda^{p^{r-1}}}^p + a_{\lambda^{p^{r-1}}}^{p-1}xu_{\lambda^{p^{r-1}}})\\
&= \mathbb{T}_{r-1}^p - a_{\lambda^{p^{r-1}}}^{p-1}\mathbb{T}_{r-1}\mathcal{B}_{r-1}^{p-1}.
\end{aligned}$$

This completes the proof. □

We now summarize the computation in the following theorem.

Theorem 5.3.6. The cohomology ring

$$H_G^{\bigstar}(B_GS^1;\underline{\mathbb{Z}/p}) \cong H_G^{\bigstar}(\mathrm{pt};\underline{\mathbb{Z}/p})[\alpha_{\phi_0},\cdots,\alpha_{\phi_m}]/(\rho_1,\cdots,\rho_m).$$

The relations ρ_r are described by

$$\rho_r = u_{\lambda^{p^{r-1}}-\lambda^{p^r}}\alpha_{\phi_{p^r}} - \mathcal{T}_{r-1}^p + a_{\lambda^{p^{r-1}}}^{p-1}\mathcal{T}_{r-1}\big(\prod_{i=0}^{r-2}\mathcal{A}_i\big)^{p-1},$$

where $\mathcal{T}_j$ and $\mathcal{A}_j$ are defined in (5.3.3).

5.4 Ring Structure of $B_GSU(2)$

As in the complex case, we get the multiplicative generators $\beta_{2\phi_d}$ of $H_G^{\bigstar}(P(\mathcal{U}_{\mathbb{H}}))$ at the degrees $2\phi_d$ for divisors d of n. The construction is same as the previous construction of the class α_{ϕ_d}. As in Section 3.4 consider the representation

$$V_{d+1} := 1_{\mathbb{H}} + \psi^1 + \psi^2 + \cdots + \psi^d.$$

We have the cofibre sequence

$$P(V_d)_+ \to P(V_{d+1})_+ \to S^{\xi_d}$$

where $\xi_d = \lambda^{-d}\otimes_{\mathbb{C}} V_d$.

At degree $2\phi_d$ the associated long exact sequence is

$$\cdots \to \tilde{H}^{2\phi_d-1}(P(V_d)_+) \to \tilde{H}^{2\phi_d}(S^{2\phi_d}) \to \tilde{H}^{2\phi_d}(P(V_{d+1})_+) \to \tilde{H}^{2\phi_d}(P(V_d)_+) \to \cdots$$

since $\lambda^{-d} \otimes_{\mathbb{C}} \sum_{i=0}^{d-1}(\lambda^i + \lambda^{-i}) = \sum_{i=0}^{d-1} 2\lambda^{i-d} = 2\phi_d$. We define the classes $\beta_{2\phi_d}$ to be the image of 1 in $\tilde{H}^{2\phi_d}(S^{2\phi_d}) \cong \mathbb{Z}$. By induction, we extend $\beta_{2\phi_d}$ to get the generator of $B_G SU(2)$ at degree $2\phi_d$. We use the notation $\mathcal{L}_j$ for $(\beta_{2\phi_{p^j}} + \prod_{i=1}^{p^j}(a_{\lambda^i})^2)$. As in the complex case with $\underline{\mathbb{Z}/p}$-coefficients we have

Theorem 5.4.1. The cohomology ring

$$H_G^\bigstar(B_G S^3; \underline{\mathbb{Z}/p}) \cong H_G^\bigstar(\mathrm{pt}; \underline{\mathbb{Z}/p})[\beta_{2\phi_0}, \cdots, \beta_{2\phi_m}]/(\mu_1, \cdots, \mu_m).$$

The relations μ_r are described by

$$\mu_r = (u_{\lambda^{p^{r-1}} - \lambda^{p^r}})^2 \beta_{2\phi_{p^r}} - \mathcal{L}_{r-1}^p + a^{2(p-1)}_{\lambda^{p^{r-1}}} \mathcal{L}_{r-1} (\prod_{i=0}^{r-2} \mathcal{C}_i)^{p-1},$$

where $\mathcal{C}_i$ is inductively defined as $\mathcal{C}_0 = P(\mathcal{L}_0, a_\lambda^2)$, and, $\mathcal{C}_j = P(\mathcal{L}_j, a^2_{\lambda^{p^j}} \prod_{i=0}^{j-1} \mathcal{C}_i)$.

The relations μ_r can be proved by restricting to fixed points as in the case of $BG(S^1)$.

We conclude this section with the ring structure computation of $\mathbb{C}P^\infty_\tau$.

5.5 Ring Structure for $\mathbb{C}P^\infty_\tau$

Recall the cofibre sequence from Proposition 3.3.1

$$\mathbb{C}P^{n-1}_{\tau}{}_+ \hookrightarrow \mathbb{C}P^n_{\tau}{}_+ \xrightarrow{\chi} S^{n+n\sigma}.$$

This implies the long exact sequence

$$\cdots \to \tilde{H}^{n+n\sigma}_{C_2}(S^{n+n\sigma}) \xrightarrow{\chi^*} \tilde{H}^{n+n\sigma}_{C_2}(\mathbb{C}P^n_{\tau}{}_+) \to \tilde{H}^{n+n\sigma}_{C_2}(\mathbb{C}P^{n-1}_{\tau}{}_+) \to \cdots$$

Observe that $\chi^*(1)$ is nonzero where $1 \in \tilde{H}^{n+n\sigma}_{C_2}(S^{n+n\sigma}) \cong \mathbb{Z}$, as the connecting homomorphism is zero. Let $\epsilon_{n+n\sigma} \in \tilde{H}^{n+n\sigma}_{C_2}(\mathbb{C}P^n_{\tau}{}_+)$ be the element $\chi^*(1)$. As the restriction of χ^* to the orbit C_2/e is an isomorphism, we have $\mathrm{res}^{C_2}_e(\epsilon_{n+n\sigma}) = x^n \in \tilde{H}^{2n}(\mathbb{C}P^n)$. We claim $H^{\bigstar}_{C_2}(\mathbb{C}P^{\infty}_{\tau}{}_+)$ is the polynomial ring $H^{\bigstar}_{C_2}(\mathrm{pt})[\epsilon_{1+\sigma}]$. This follows from the fact that $\underline{H}^{n+n\sigma}_{C_2}(\mathbb{C}P^{\infty}_{\tau}{}_+) \cong \underline{\mathbb{Z}}$ and $\mathrm{res}^{C_2}_e(\epsilon_{n+n\sigma}) = \mathrm{res}^{C_2}_e(\epsilon^n_{1+\sigma})$. Hence $\epsilon_{n+n\sigma} = \epsilon^n_{1+\sigma}$. Therefore,

Theorem 5.5.1. We have an isomorphism of cohomology rings

$$H^{\bigstar}_{C_2}(\mathbb{C}P^{\infty}_{\tau}) \cong H^{\bigstar}_{C_2}(\mathrm{pt})[\epsilon_{1+\sigma}].$$

The ring structure in this case was known to Araki [2] and is discussed by Hu-Kriz [28].

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