INDIAN STATISTICAL INSTITUTE

B. Stat. (Hons.) I Year (2002-03)

Vectors & Matrices I

Backpaper Exam.

Date : 10.02.03 Maximum Marks : 100 Duration : 3 Hours

... 5 ques... ly as possible.

1. Let $A = ((a_{ij}))$ be an $n \times n$ matrix with real entries. If Ax=0 has a unique solution, then show that Ax=b has a unique solution for every $b \in \mathbb{R}^n$ (Here, x and b are column vectors).

[10]

2. Let $u = (a_1, a_2, \ldots a_n)$ be a fixed vector in $\mathbb{R}^n$ and S be the subspace of $\mathbb{R}^n$ generated by the n! (not necessarily distinct) vectors obtained by permuting the components of u. Show that

$$\dim(S) = \begin{cases} 0 & \text{if} \quad \underset{\sim}{u} = 0 \\ 1 & \text{if} \quad a_1 = a_2 = \ldots = a_n \neq 0 \\ n-1 & \text{if} \quad \underset{\sim}{u} \neq 0 \text{ and } \sum_{i=1}^{n} a_i = 0 \\ n & \text{if} \quad otherwise \end{cases}$$

[20]

3.(a) Let $A = ((a_{ij})), B = ((b_{ij})), i, j = 1, 2, \ldots, n$ be real matrices of order n. Prove that Rank $Rank(AB) \leq Rank(B)$.

(b) Let A and B be as in (a) above. Suppose AB=I_n (the identify matrix of order). Show that both A and B are invertible and, hence deduce that BA=I_n.

(c) Let V be an n-dimensional vector space over $\mathbb{R}$ and $X = \{x_1, \ldots x_n\}$ and $Y = \{y_1, \ldots y_n\}$ be two bases in V. Suppose $T: V \to V$ is a linear transformation and [T ; X]=A, [T; X,Y]=B are the matrix representations of T with respect to the bases X and Y respectively. Find a real matrix C of order n such that A=CB.

[6+10+12=28]

(a) Let $x_1, x_2, \ldots\ldots, x_k$ be vectors in $\mathbb{R}^n$ and let y_i be the vector in $\mathbb{R}^{n-1}$ formed by the first (n-1) components of $x_i, i = 1, 2, \ldots, k$. Show that if $y_1, y_2, \ldots, y_k$ are linearly independent in $\mathbb{R}^{n-1}$, then $x_1, x_2, \ldots\ldots, x_k$ are linearly independent in $\mathbb{R}^n$. Is the converse true ? Justify your answer.

(b) Show that the subsets $\{1, \sqrt{2}\}, \{\sqrt{2}, \sqrt{3}\}$ *and* $\{\sqrt{2}, \sqrt{3}, \sqrt{6}\}$ of $\mathbb{R}$ are linearly independent over Q and that $\{\sqrt{2}, \sqrt{3}, \sqrt{12}\}$ is linearly dependent over Q.

(c) If $\{x_1, x_2, \ldots x_k\}$ and $\{y_1, \ldots, y_k\}$ are linearly independent sets of vectors in a finite-dimensional vector space V over a field $\mathfrak{F}$, show that there exists a linear isomorphism $T: V \to V$ such that $Tx_j = y_j, j = 1, 2, \ldots\ldots, k$.

Contd.(2)

(d) Let $A = ((\alpha_{ij})), i, j = 1,2,\ldots,n$ be an $n \times n$ real matrix such that

$$\alpha_{ij} = \alpha_i \delta_{ij}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are pairwise distinct real numbers. Let $B = ((\beta_{ij}))$ be a real matrix of order n such that BA=AB. Show that there exist scalars $\beta_1, \beta_2, \ldots, \beta_n$ such that $\beta_{ij} = \beta_i \delta_{ij}$.

[(4+4)+(2+2+3+3)+7+7=32]

Suppose that V is a vector space over $\mathbb{C}$, x_0 is a vector in V, and f_0 is a linear functional on V. Write $Tx = \langle x, f_0 \rangle x_0$ for every $x \in V$. Under what conditions on x_0 and f_0, T is a projection ?

[10]

Indian Statistical Institute
Semester 1 (2002-2003)
B. Stat 1st Year
Backpaper Exam.
Probability Theory 1

Date: 13.2.03 Time: 3 hours Total Points $6 \times 17 = 102$

Answers are to be justified with clear and precise arguments

1. If n dice are thrown at random what is the probability of having each of the numbers $1, 2, \cdots, 6$ appear at least once? Find the numerical value of this probability when $n = 10$.

2. If m things are distributed among a men and b women show that the chance that the number of things received by men is odd is

$$\frac{1}{2}\frac{(b+a)^m - (b-a)^m}{(b+a)^m}.$$

3. Four tickets numbered $00, 01, 10, 11$ respectively are placed in a bag. A ticket is drawn with replacement five times. Find the probability that the sum of the numbers on the tickets thus drawn is 23.

4. In a random walk show that

$$P(S_n > S_j, j = 0, 1, \cdots, n-1) = \frac{1}{2}u_{2\nu},$$

where $\nu = n/2$ or $(n-1)/2$ according as n is even or odd.

$$\left(\text{Recall: } u_{2\nu} = \frac{\binom{2\nu}{\nu}}{2^{2\nu}}\right)$$

5. The subsets $A_1, A_2, \cdots, A_r$ are chosen with replacement from the collection of all subsets of the set $S = \{1, 2, \cdots, N\}$. Find the probability that the sets $A_1, \cdots, A_r$ are pairwise disjoint.

6. An insect lays a random number N of eggs where $N \sim \text{Poisson}(\lambda)$. The eggs act independently and each gives rise to 1 offspring with probability p and no offspring with probability $1 - p, (0 < p < 1)$. Find the expected number of offsprings.

INDIAN STATISTICAL INSTITUTE
B. Stat. I Year (2002-2003), Analysis - II
Backpaper Examination

Time: 3 hrs: Max. Marks 100: 29.7.-2003.

1. (a) Let $f(t)$ be a positive decreasing function defined on $[0,\infty)$ such that $\lim_{t\to\infty} f(t) = 0$. Let, for $n = 1, 2, \cdots$

$$S_n = \sum_{k=1}^{n} f(k), I_n = \int_1^n f(t)\,dt \text{ and } D_n = S_n - I_n.$$

Show:

i. $0 < f(n+1) \leq D_{n+1} \leq D_n \leq f(1)$.

ii. $\lim_{n\to\infty} D_n$ exists.

iii. $\sum_{k=1}^{\infty} f(k)$ converges if, and only if the sequence $\{I_n\}$ converges.

iv. $0 \leq D_k - \lim_{n\to\infty} D_n \leq f(k)$ for every $k = 1, 2, \cdots$.

(b) Use (a) to show:

i. $\sum_{k=1}^{n} \frac{1}{k} = \log n + C + O(\frac{1}{n})$ where C is a constant.

ii. $\sum_{k=1}^{n} \frac{1}{k \log k} = \log(\log n) + B + O(\frac{1}{n \log n})$ where B is a constant.

[12+8=20]

2. (a) Let $\sum_{n=1}^{\infty} f_n(x)$ be a series of functions defined on a set X. Suppose there exists a sequence $\{M_n\}$ of positive numbers such that $\sum_{n=1}^{\infty} M_n < \infty$, and $|f_n(x)| \leq M_n$ for all $x \in X$. Show that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on X.

(b) Let $\sum_{n=1}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, show that the power series $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$.

P.T.O

(c) Let $\{f_n(x)\}$ be a sequence of functions which are continuous on an interval (a,b). Suppose that $\{f_n(x)\}$ converges to a function $f(x)$ uniformly on (a,b). Show that $f(x)$ is continuous on (a,b).

[7+6+7=20]

3. (a) Show that the improper integral $\int_0^\infty t^{x-1}e^{-t}\,dt$ converges for any $x>0$ and that the convergence is uniform for $x\in[a,b]$ for any $a>0$ and $b<\infty, a<b$.

   (b) Show that the improper integral $\int_0^\infty t^{x-1}e^{-t}\log t\,dt$ converges for any $x>0$ and that the convergence is uniform for $x\in[a,b]$ for any $a>0$ and $b<\infty, a<b$.

   (c) Show that the function $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt$ is differentiable on $(0,\infty)$ and $\Gamma'(x)=\int_0^\infty t^{x-1}e^{-t}\log t.dt$.

[7+7+6=20]

4. (a) Find the Fourier Series of the function

$$f(\theta)=(\pi-|\theta|)^2, -\pi\le\theta\le\pi \text{ and } f(\theta+2\pi)=f(\theta).$$

   (b) Use (a) to evaluate $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

   (c) Use (a) to obtain the Fourier Series of the function $f(\theta)=\pi-\theta, 0\le\theta\le 2\pi, f(\theta+2\pi)=f(\theta)$.

[8+6+6=20]

5. (a) Let $f(x)$ be Riemann integrable on $[a,b]$. Then, show that for each β

$$\lim_{\alpha\to\infty}\int_a^b f(t)\sin(\alpha t+\beta)\,dt=0.$$

(b) Suppose that $f(x)$ is Riemann integrable on $[a, b]$ for each $b > a$, and that the improper integral $\int_a^{\infty} |f(x)| dx$ is convergent. Show that, for each β

$$\lim_{\alpha \to \infty} \int_a^{\infty} f(t) \sin(\alpha t + \beta)\, dt = 0.$$

(c) Use (a) to show that

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

[8+4+8=20]

INDIAN STATISTICAL INSTITUTE
Back Paper Examination : (2002-2003)
B-Stat(Hons) I year, Second Semester
Computational Techniques and Programming - II

Date : 31. 7. 03 Maximum marks : 100 Duration : 3 hours

Note: *Answer all the questions.*

1. (a) Define the operators $\boldsymbol{E}, \boldsymbol{\nabla}, \delta$ and μ for a function f.
   (b) Derive the following formulae.
      i. $E^{\frac{1}{2}} - \frac{1}{2}\delta - \mu = 0$
      ii. $(E^{\frac{1}{2}} - \frac{1}{2}\delta)^2 - \frac{1}{4}\delta^2 = 1.$
   (c) If f is a polynomial, show that
   $E = \sum_{k=0}^{\infty} \nabla^k$ [2+(2+2)+4=10]

2. Find the formula for $\sum_{i=1}^{r} i^4$ with the help of operators Δ and E. [10]

3. Write an algorithm for finding determinant of a square matrix $A_{n\times n}$. [10]

4. Let $A = \begin{bmatrix} 4 & 1 & -2 \\ -1 & 6 & -1 \\ 1 & -3 & 8 \end{bmatrix}$, $\underline{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underline{x}_0^T = (0,0,0)$. Use Jacobi's method for finding solution to the equations $A\underline{x} = \underline{b}$ starting with the above $\underline{x}_0$. Find value of $\underline{x}$ correct upto 2 places after the decimal for each of the components. [15]

5. Use the power method to find the dominant eigen value and corresponding eigen vector for $A = \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}$ starting from $\underline{x}_0^T = (1,1)$. Note that eigen vector should be correct upto 2 places after the decimal for each of the components, and the eigen value should also be correct upto 2 places after the decimal. [15]

P.T.O.

6. Answer the following using C programming Language:

(a) Write a program to check whether the given positive integer is a prime number or not.

(b) Explain the use of bit-fields, with examples.

(c) Dynamically allocate a 2-dimensional array, so that the array can be accessed using 2 subscripts, as *arr[i][j]*, and the rows of the array should be stored in adjacent memory locations.

(d) Explain the use and advantage of *register* variables, with examples.

(e) Implement a stack of integers.

(f) Read two input matrices from a file and compute their product using a function. Write the results into an output file.

[6+6+6+6+8+8=4

——X——

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination - 2003-04

B. Stat (Hons.) I Year

Vector & Matrices I

Date : 15.09.03 Maximum marks : 60 Duration : 3 Hours

Note : (1) The paper carries 72 marks. The maximum that you can score is 60.
(2) $\mathbb{R}$ denotes the field of real numbers.

1. Determine all distinct solutions of the system of linear equations

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

in the field $F_5 (= \mathbb{Z}/5\mathbb{Z})$.

[6]

2. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, then compute A^{100}.

[5]

3. Compute the inverse of the following matrix :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

[5]

4. Prove or disprove the following statements for $(n \times n)$ matrices over $\mathbb{R}$:-

(i) If $A^2 = I$, then either $A = I$ or $A = -I$.
(ii) If A, B and A+B are invertible matrices, then A-B is an invertible matrix.
(iii) If $(A + I_n)^2 = 0$, then A is an invertible matrix.
(iv) If A and B are nilpotent matrices , then AB is a nilpotent matrix.

[5 × 4=20]

5. Let A and B be matrices over $\mathbb{R}$ of sizes $(m \times n)$ and $(n \times m)$ restively. Show that if $I_m - AB$ is an invertible $(m \times m)$-matrix, then $I_n - BA$ is an invertible $(n \times n)$-matrix.

[8]

[P.T.O.]

6. Let A be an $(n \times n)$-matrix with entries in $\mathbb{R}$ such that AB = BA for every $(n \times n)$-matrix B. Show that there exists $\lambda \in \mathbb{R}$ for which $A = \lambda I_n$.

[10]

7. An $(n \times n)$ matrix $A = ((a_{ij}))$ over $\mathbb{R}$ is called a Markov matrix if it satisfies the conditions :

$$(I) \quad 0 \le a_{ij} \le 1 \quad ; \quad 1 \le i, j \le n$$

$$(II) \quad \sum_{j=1}^{n} a_{ij} = 1 \quad ; \quad 1 \le i \le n$$

(i) Prove that the product of two $(n \times n)$- Markov matrices is a Markov matrix.

(ii) If M is a Markov matrix, show that the value of every element of column j of M^2 lies between the values of the minimal and maximal elements of column j of M.

(iii) Do the $(n \times n)$-Markov matrices over $\mathbb{R}$ form a group under matrix multiplication ?

[5+3+4=12]

8. Assignment

[6]

***

INDIAN STATISTICAL INSTITUTE

B.STAT I YEAR

PERIODICAL EXAMINATION

Computing Techniques and Programming I

ime : 3 hours. Date : 18.9.03

arks : Answer as many parts as you like. Maximum you can score is 100

Convert the given numbers to the number systems required (6X1=6)+(2X2

$(56)_{10} = (\ )_{2}$ $(56)_{10} = (\ )_{8}$ $(56)_{10} = (\ )_{16}$ $(56)_{16} = (\ )_{10}$

$(56)_{8} = (\ )_{10}$ $(1011011)_{2} = (\ )_{10}$ $(101.11)_{2} = (\ )_{10}$ $(56.625)_{10} = (\ )_{2}$

Perform the following binary addition and multiplication

1011.1011 + 110.1 1101.1 * 101.01 (2X2=4)

Convert the following decimal numbers to binary numbers and re
the negative numbers in 2's complement form then perform the a

-78 + 23 78 - 23 (3X3=6

Give the floating point addition and multiplication flowcharts.

e short notes on any three
What is a microprocessor ?
What are primary memory and secondary memory ?
What are input output devices ?
What is an operating system ?
What is machine language and what is a high level language ?

Write a program to read a five digit integer N and print the sum of the decimal digits in it. The program must check the number of digits in integer N and if it is not five then ask the user for a five digit integer again. The user should be given the option to exit the program if he enters the value 0 for N. (5+2+1=

Write a program that reads a non-negative integer x less than 30 and calculates factorial(x). If the number is not a non-negative integer less than 30 then the program gives a message and loops to read again. The user should be given the option to exit from the program by enterring 0. If the input is 0 then the program exits only after printing the answer as 1. (4+2+1=

Write a program to add the following 2 matrices and then print the resulting matrix. Declare arrays as neccessary and initialize them as required. The program must print the resulting matrix with rows in consequetive lines (not as a vector with all rows in same line).

$$\begin{pmatrix} 3 & 0 & 5 \\ 1 & 9 & 4 \\ 2 & 7 & 6 \end{pmatrix} \qquad \begin{pmatrix} 8 & 7 & 5 \\ 9 & 0 & 2 \\ 4 & 3 & 1 \end{pmatrix}$$

(2+5+1=8)

PTO

(b) Write a FORTRAN program which reads two numbers X and Y and determines if the point (X,Y) is within, outside, or, on the circle

```
     2     2
   X  +  Y    =  50
```

and gives appropriate messages accordingly. Use the if-then-elseif-else construct. (1+4+2

5.a) Write a program which calculates and prints the electric bill for a month after reading an input (which is an integer giving the units current consumed during the previous months).Invalid data values ar not accepted.

The bill is calculated according to the following rules:

```
  i) Upto < 50 units : charge only meter rent, no other charge;
 ii)    = 50 to < 100 : Rs.1.25/10 units for all units + meter rent;
iii)    =100 to < 150 : Rs.1.20/10 units for all units + meter rent;
 iv)    =150 to < 200 : Rs.1.10/10 units for all units + meter rent;
  v)    > 200 units   : Rs.0.90/10 units for all units + meter rent;
```

for all cases the meter rent is Rs.40.00 per month.

Use the case selection construct and a subroutine call for printing the bill.

The subroutine must print the bill in the following format :

Consumer_No.	Name	Meter_rent	Units_Consumed	Charge_per_units	Total_ch
4 digit integer	25 char maximum	2 digit integer	4 digit integer	4.2 digit floating_point	7.2 di floating_

6.a) Write a program which tests if a sentence read in a character array length at most 80 has any vowels, or not, with the help of a functi subprogram and answers the user "Yes" , or "No", as the case may be

The Function subprogram takes two arguments, a character and an ar of characters. The function has to return value .TRUE. if the chara is present in the array of integers, else it must return value .FAL Define the type of the function appropriately.

((1+2+4=7)+(2+2+4=8)

7. Write a program to multiply a (NxN) square matrix by a (Nx1) vec

The data is stored in a file matrix_vector.dat, as follows :

1) the first integer in the file is the value of N,
2) next (NXN)integers are square matrix data values rowwise,.
3) next (NX1) values are vector data values,

Use allocatable arrays after reading N from file then read other da values appropriately into the arrays from file.

4) Allocate and use another 1-dimensional (NX1) array to store th result of the multiplication,

Write the result file result.dat in which the first integer shoul be the value N and then the (NX1) values of the result. (

Indian Statistical Institute

B. Stat (Hons.) 1st year (2003–04)

Mid Semester Examination.

Analysis - 1

Time : 3 hrs Date: 22. 09. 03

Maximum you can score is 100

You are required to state clearly any result that you use

Question 1. *a)* Define limit points of a subset of $\mathbb{R}$.

Let

$$A = \{\frac{1}{2^p} + \frac{1}{3^q} : p, q \in \mathbb{N}\}$$

Determine all limit points of A.

b) Let A and B be non-empty subsets of $\mathbb{R}$ such that interiors of A and B are both empty. Assume in addition that A is closed. Prove that interior of $A \cup B$ is also empty.

5 + 5

Question 2. For each of the following statement decide whether it is true or false. Give reasons for your answer.

(a) Let f be a function defined for all $x \in \mathbb{R}$, $x \neq 1$ by $f(x) = \frac{2x}{x-1}$. Then the range of f is $\mathbb{R}$.

(b) For all $n \in \mathbb{N}, n^3 + 5n$ is divisible by 6.

(c) There exists a bijection from an open interval (a, b) onto $\mathbb{R}$.

(d) Let $0 < b < 1$. Then $\lim_{n\to\infty} nb^n = 0$.

(e) Let $(x_n)_{n\geq 1}$ be a Cauchy sequence with $x_n \in \mathbb{N}, \forall n$. Then there exists a positive integer K such that $x_n = x_K$, $\forall n \geq K$.

(f) $\sum(-1)^n \frac{n^2-1}{n^2+1}$ is a convergent series. 4 + 4 + 4 + 4 + 2 + 2.

Question 3. (a) Define a Cauchy sequence.

(b) Prove that every Cauchy sequence is bounded.

(c) Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be Cauchy sequences. Show directly from the definition that the sequence $(x_n + y_n)_{n\geq 1}$ is Cauchy.

(d) Let $x_1 = 1, x_2 = 2$ and $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$, $n > 2$. Show that the sequence $(x_n)_{n\geq 1}$ is convergent. 2 + 4 + 6 + 8.

Question 4. (a) Let $(x_n)_{n\geq 1}$ be a sequence such that the even and the odd subsequences of $(x_n)_{n\geq 1}$ converge to l. Show that $(x_n)_{n\geq 1}$ converges to l as well.

P. T. O

(b) Let $x_n = \sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}$, $n \geq 1$. Show that the sequence $(x_n)_{n\geq 1}$ converges. 8+12.

Question 5. (a) Let $(x_n)_{n\geq 1}$ be a sequence converging to l and $(y_n)_{n\geq 1}$ be any sequence with the property that given any $\varepsilon > 0$, there exists $M > 0$ such that $|y_n - x_n| < \varepsilon, \forall n \geq M$. Prove that $(y_n)_{n\geq 1}$ is convergent.

(b) Let $x_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$, $n \geq 1$. Prove that the sequence $(x_n)_{n\geq 1}$ converges. 10 + 10.

Question 6. (a) For each real numbers α, β test the convergence of

$$1 + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \cdots.$$

(b) Test the convergence of $\sum_{n\geq 1} a_n$ where,

$$a_n = \frac{3 \cdot 6 \cdot 9 \cdots 3n}{7 \cdot 10 \cdot 13 \cdots (3n+4)} x^n \text{ for } x > 0$$

10 + 10.

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination – Semester I : 2003-2004
B.Stat. (Hons.) I Year
Probability Theory I

Date : 24.9.03 | **Maximum Score : 70 pts** | Time : 3 Hours

Note : This paper carries questions worth a **total** of **90 points**. Answer as much as you can. The **maximum** you can score is **70 points**.

1. (a) If A and B are two events with $P(A) = 0.5, P(B) = 0.4$ and $P(A^c \cap B^c) = 0.2$, find the probability that *one and only one* of the two events occur.
(b) If A, B, C are events with $P(A) = 0.3, P(A^c \cap B) = 0.2$ and $P(A \cup B \cup C) = 0.7$, then what is the probability that *only* C occurs? (5+5)=[10]

2. In five tosses of a coin, let A, B and C denote respectively the events that there is no head, that no two successive tosses result in tails and that the last toss results in a tail. List the sample points in $A^c \cap B \cap C$ and hence find its probability. [7]

3. In six throws of a die, what is the probability that exactly two distinct faces show up? [10]

4. An urn contains eight balls, two of which are red and the rest are black. Balls are drawn one after another (without replacement) until both the red balls show up. What is the probability that (a) at least five draws will be needed, (b) exactly five draws will be needed? (10+6)=[16]

5. In a random arrangement of 10 α's and 6 β's, find the probability of having exactly three α-runs. [8]

6. In a football tournament, 32 players are representing India – 16 for the senior team and 16 for the junior team. If the 32 players are paired at random in groups of 2 for determining roommates, what is the probability that there will be exactly two senior-junior roommate pairs? [8]

7. From a box containing nine tickets numbered $1, \ldots, 9$, five tickets are picked at random. What is the probability that the second largest number among the tickets picked is at most 6? [10]

8. With the notation of random walk, find the following probabilities:
(a) $P(S_5 = 1, S_9 = -1)$;
(b) $P(S_{10} = 2, \max_{0\leq k\leq 10} S_k \geq 3)$;
(c) $P(S_{10} = 2, \max_{0\leq k\leq 10} S_k = 3)$. (5+8+8)=[21]

INDIAN STATISTICAL INSTITUTE

FIRST SEMESTRAL EXAMINATION: (2003-2004)

B.Stat – 1st Year

Remedial English

Date : 27.11.03 **Maximum Marks : 80** **Duration : 2½ hrs.**

Section A – 25 Marks
Literature

Q1. Attempt any one question : (10)

a) Give a study in contrast of the characters of mother and son in the story "The Boss Came to Dinner ".

b) How does G.B.Shaw distinguish between Spoken English and Broken English ?

c) How does advertising damage Nature, Art, Language and Youth ?

d) "I was too far gone to reason now" – In what connection does the author say this ? What makes him think that he has gone "too far" ?

Q2. Attempt any two questions : 2 x 5 = (10)

a) What problem did Shamnath face with his mother ? How was the problem solved ?

b) In what way does the author find advertising unproductive and wasteful ?

c) "Even in private intercourse with cultivated people you must not speak too well"- Why does G.B.Shaw say so ?

d) How was Shamnath able to dissuade his mother from going to Hardwar ?

Q3. Attempt any five questions : 5 x 1 = (5)

a) What is bric – a – brac ?

b) What was the Memsahib's preference ?

c) What is the Sixth Stage of a man's Life ?

d) What is Cockney Dialect ?

e) In what connection does Shaw use the term 'Cloxst" ?

f) What does Tagore mean by "dreary desert sand of dead habit" ?

g) Why does the author find the term 'Aurora Dawn' Ludicrous ?

h) Explain "the crescendo of Success".

Contd……2

Section – B – 20 Marks.
Vocabulary

Q4. One Word Substitution : 10 x 1/2 (5)

a) A style full of words.
b) Give and receive mutually.
c) Worship of images or idols.
d) A fictitious name used by an author.
e) An ordinary and commonplace remark.
f) Change of form or character.
g) To give one's authority to another.
h) One who always talks of oneself.
i) Hater of women.
j) A person who easily believes in others.

Q5. Give the meanings of <u>any five</u> of the following Idioms and Phrases and frame sentences with them :- 5 x1 (5)

a) Out of the frying pan into the fire.
b) To fish in troubled waters.
c) To burn one's boats.
d) To blow one's own trumpet.
e) A thankless task.
f) A wild goose chase.
g) To put heads together.
h) A square peg in a round hole.

Q6. Name the Figure of Speech in each of the following sentences :- 10 x 1/2 (5)

a) The ploughman homeward plods his weary way.
b) The path of glory leads to the grave.
c) So much to do ; so little is done.
d) She is a gem of a girl.
e) I appeal to the throne for justice.
f) No light, but darkness visible.
g) Out, out, brief candle !
h) To strive, to seek, to find and not to yield.
i) Much have I seen of men and manners.
j) He left the world with a heavy heart.

Q.7 Transform the following sentences (as indicated in the bracket) 5 x 1 = (5)

a) Vatican City is the smallest republic in the world (Begin with : "No……………..)
b) One should keep one's promise (Change the Voice)
c) I was glad to know of his success (Change into Complex Sentence)
d) The Patient may reach a <u>stable</u> condition after a week (Re-write using the Verb form of the word underlined)
e) As you sow, so you reap (Change into Past Perfect Continous Tense.)

Section – C - 35 Marks
Grammar

Q8. Attempt any One of the following :- **20**

a) Write a paragraph (within 200 –250 words) on any one of the following topics :-
   i) Desire is the cause of all sufferings.
   ii) Who knows most says least.

b) Write an original Short Story with any One of the following themes :-
   i) A friendship to be cherished.
   ii) The horrifying expedition.

Q9. Attempt any one of the following :- **15**

a) Make a substance of the given passage :-

We are free now, but we have got freedom without bloody struggle. So it has become very easy to many of us who think that freedom means the right to do or say whatever we like. There are young men, Women and Students who now think that they are quite at liberty to act, talk and behave as they like, to criticize their parents, teachers, elders, leaders and anyone who may not please them. They take pride in breaking the established rules and principles of behaviour and conduct, and delight in mobbing the referee in the playground, assaulting the invigilator in or outside the examination hall, traveling without ticket and rough handling the ticket checker. But a day will come when they will be young no longer ; and those who will be young then will pay them back in their own coin. They are sowing the seeds of indiscipline and lawlessness and therefore, they must reap the fruit.

b) Write a letter to your landlord, asking him to arrange for certain repairs in the house .

or

Write a letter to the Health Officer complaining about the insanitary condition of your locality.

****

INDIAN STATISTICAL INSTITUTE

Semestral Examination – Semester I : 2003-200
B.Stat. (Hons.) I Year
Probability Theory I

Date : 3.12.03 **Maximum Score : 130 pts** Time : $3\frac{1}{2}$ Hours

Note : This paper carries questions worth a **total of 155 points**. Answer as much as you can. The **maximum** you can score is **130 points.**

1. If n married couples are seated randomly in a round table, what is the probability that no wife sits next to her husband? [15]

2. A construction company gets its supply of cement from three godowns. While godown A supplies 40% of the cement required, godowns B and C supply 30% each. Also, of all the cement bags supplied by godown A, 10% are found to contain poor quality cement. For godowns B and C, these percentages are 15% and 5% respectively. If a randomly picked bag of cement was found to contain poor quality cement, what is the probability that the bag came from godown A? [15]

3. Players A and B take turns in rolling a pair of fair dice, with A starting first. The player who is the first to get a sum of 9 wins the game. Find the probability that B will win the game. [15]

4. Five distinct numbers are randomly distributed to players A, B, C, D and E. Whenever two players compare their numbers, the one with the higher number is declared the winner. Initially players A and B compare their numbers; the winner then compares with player C and so on. Let X denote the number of times player A is a winner. Find the probability mass function of X. [15]

5. With usual notations for random walk, prove the following:
(a) $u_0u_{2n} + u_2u_{2n-2} + \cdots + u_{2n}u_0 = 1$;
(b) $P[S_1 \geq 0, \ldots, S_{2n-1} \geq 0, S_{2n} = 0] = 2f_{2n+2}$. (10+10)=[20]

6. Let X be a Poisson(λ) random variable. Find $E(e^{tX})$ where t is any real number. Denoting $f(t) = E[e^{tX}]$, $t \in \mathbb{R}$, find $f'(1)$. [15]

7. For a random variable X taking non-negative integer values only, show that [15]
$$E[X(X+1)] = 2\sum_{n=0}^{\infty} nP[X \geq n].$$

8. An elevator starts going up from the ground floor with 12 passangers. Each passanger is equally likely to get off at any one of the 10 floors from the first to the tenth. The elevator stops at a floor only if at least one passanger gets off at that floor. Find the expected number of floors where the elevator will stop after starting from the ground floor. [15]

9. Ten items, of which three are defective, are made to pass through a scanner one after another in a random order. The scanner gives a warning beep every time it sees a defective item. Let X denote the time until the first beep and Y that between the second and third beeps. Find the joint distribution of (X, Y) and hence find $P[X + Y = 6]$. [15]

10. Let X and Y be independent random variables, distributed geometrically with parameters p_1 and p_2 respectively. Find $P[X < Y]$. [15]

INDIAN STATISTICAL INSTITUTE

FIRST SEMESTRAL EXAMINATION (2003-04)

B.STAT I YEAR

Computing Techniques and Programming I

Time : 3 hours. Date : 5. 12. 2003

Remarks : Answer as many parts as you like. Maximum you can score is 100.

. Draw the flowchart for the floating-point multiplication (z <-- X*Y). (10)

. Write a program to tabulate the function

$$f(X,Y) = (X^2 + Y^2) / (X^2 - Y^2)$$

for the following set of values of (X,Y) :

(0,-6), (2,-5), (4,-4), (6,-3), (8,-2), (10,-1) and (12,0).

Compute the function using a do loop with loop variable X.
Assign the initial value,final value and increment accordingly.
Compute Y values using corresponding X values.

Keep computed function values f(X,Y) in an array.

For each (X,Y) compute f(X,Y) in a Function subprogram.
Within the subprogram you must check whether the divisor is zero.
If it is then the value 0 is returned.

Print the values from the array in the following format:

```
 X        Y           f(x,y)
------------------------------
 00      -6           -1.00
  .       .             .
  .       .             .
 12       0            1.00
------------------------------
```

There is a header line followed by a dashed line, followed by lines for values of each pair X, Y and corresponding f(X,Y), and lastly another dashed line of equal length.

The function value is printed with two decimal points.

Print the string "inf" as value of the function if f(X,Y) value in array is 0.

(6+4+10=20)

P.T.O

A data-type set of positive integers can be implemented as a sorted linked list as follows:
Write (1) a main program and (2) a module where the module defines the data types and the functions given below:

.Data types:
i)Derived data type NODE for each integer in the set.
i)Derived data type SET is a linked list of derived data type NODE.
The NODES contain positive integers in ascending order of magnitude. (5

.Functions:
). READN is a function which always returns a positive integer only. (5

).A logical function SEARCH checks if a number is present in the SET.

). A subroutine CREATE creates a NODE by allocating space for it and initializes it by a given positive integer and nullifies the pointer. (1

iv). A subroutine REMOVE deallocates a NODE.

v). A subroutine INSERT adds a number to an existing SET.
It uses function READN to read a positive integer.
It uses function SEARCH to find if the number exists in the SET.
If the number exists then
a message "Number already present" is printed
else
it calls CREATE to create a NODE holding this number and inserts it maintaining the ascending order of the numbers in the linked list. (1

vi). A subroutine DELETE removes a number from the SET.
If the pointer is NULL it prints a message to that effect and returi
else
It uses function READN to read a positive integer.
It uses function SEARCH to find out if the number exists in the SE
If the number exists then
the NODE containing that number is delinked from the SET, and then REMOVE is called to deallocate the NODE
else
a message "Number not found" is printed. (1

. The main program
Declares a pointer to SET.
Displays a menu which gives options EXIT, INSERT, DELETE, SEARCH.
It calls the subroutine or function required using the option variabl as a case select variable in a case statement. After completion of task MENU is displayed.
On option EXIT program exits, on any other choice it loops to display MENU. (1

rite a recursive subprogram to compute the Ackermann function A(m,n) defined as follows :

A(0,n) = n+1
A(m,0) = A(m-1,1) for m>0
A(m,n) = A(m-1, A(m,n-1)) for m,n > 0 (10

INDIAN STATISTICAL INSTITUTE
First Semester Examination: 2003-2004

B. Stat. (Hons.) I Year
Analysis I

Date : 9.12.03 | Maximum Marks : 100 | Duration : 3 Hours

1.(a) Let $(x_n)_{n\geq 1}$ be a sequence of positive reals such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} < 1$. Show that $(x_n)_{n\geq 1}$ is Convergent and find its limit. [10]

(b) Show that the Series $\sum_{n\geq 1} a_n$ where $a_n = \left(\frac{2n+1}{2n} - \frac{2n+2}{2n+1}\right)$ is Convergent. [10]

2.(a) Define uniform continuity of a function $f : S \to \mathbb{R}$, $S \subseteq \mathbb{R}$. [2]

(b) Let $f(x) = x + \sqrt{x}$ for $x \geq 0$. Determine if f is uniformly continuous as $[0,\infty)$. [8]

(c) Let $f : [a,b] \to \mathbb{R}$ be a one-to-one continuous function. Such that $f(a) = 0$ and $f(b) = 1$. Determine the range of f. [8]

3.(a) Prove that $\wp(\mathbb{N})$ is uncountable. [10]

(b) Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$ are functions such that $hog : Y \to W$ and $gof : X \to Z$ are bijective. Show that f, g, h are all bijective. [10]

4.(a) Define differentiability of a function $f : (a,b) \to \mathbb{R}$ at a point $c \in (a,b)$. [2]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ when x is rational and $f(x) = 0$ otherwise. Show that f is differentiable at $x = 0$ and find its derivative at $x = 0$. [6]

(c) Suppose $f : (a,b) \to \mathbb{R}$ is a function such that $f''(x) \geq 0$ for all x. Show that for every $c \in (a,b)$ f is convex at c. [8]

[P.T.O.]

5.(a) Prove that $1-\frac{1}{2}x^2 \le Cos\ x$ for all $x \in \mathbb{R}$. [8]

(b) For the function $f(x) = Cos\ x - 1 + \frac{1}{2}x^2$, determine if $x = 0$ is a local extremum or not. [8]

6.(a) State Roll's Theorem. [2]

(b) Suppose that $f:[a,b] \to \mathbb{R}$ is a C^1-function such that $f''(x)$ exists for all $x \in (a,b)$. Suppose $f(a) = f(b) = 0$ and $f(c) = 0$ for some $c, a < c < b$. Prove that there exists a point $\xi \in (a,b)$ such that $f''(\xi) = 0$. [8]

***

X

INDIAN STATISTICAL INSTITUTE
B-Stat (First Year, First Semester)
Statistical Methods I
Back Paper Examination

TOTAL MARKS : 100 Date: 6.2.04 TIME ALLOWED : 3 hours

This is a closed book and closed notes examination. Answer all questions and you may use calculator for numerical computations.

(1). Given a bivariate data set $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$, describe the half space depth median and the simplicial area median. Describe how one can compute the second median indicating all the computational steps involved in a systematic way. Describe also how one can compute the half space depth of a given point (x, y) with respect to the above data in a systematic way.

[$(2+2)+6+5=15$ points]

(2). State clearly **the bivariate version** of Chebysev's inequality and prove it.

[15 points]

(3). Consider ordered univariate data $X_1 < X_2 < \ldots < X_n$. Determine θ (in terms of the X's) that will minimize $\sum_{i=1}^{n}\{|X_i - \theta| + (2\alpha - 1)(X_i - \theta)\}$ for $0 < \alpha < 1$. Justify your answer.

[20 points]

(4). Given $n \ge 3$ data points in the plane having distinct X and Y coordinates, show that the straight line $y = \alpha + \beta x$ fitted to the data by the method of least squares can be expressed as a weighted average of all the elemental straight lines each of which passes through two distinct data points.

[20 points]

(5). Consider the following data on the day-time (X-coordinate) and the night-time (Y-coordinate) temperatures (measured in centigrades) in a city over ten different days : (28.0, 24.0) , (29.5, 24.5) , (27.0, 25.5) , (29.0, 26.0), (30.0, 28.5) , (27.5, 28.0) , (29.0, 26.5) , (30.5, 27.5) , (28.5, 28.0) , (29.5, 27.5).

P. T. O

Fit a linear equation of the form $y = \beta x$ to this data using (i) the method of least squares and (ii) the method of least absolute deviations.

[8 + 12 = 20 points]

(6). Describe how one computes the mean and the measure of dispersion for directional data. Explain why the usual average and the standard deviation of the values of the angles are not used for directional data.

[10 points

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination 2003-2004

B. Stat. (Hons.) I Year
Analysis II

Date: 16.02.2004 **Maximum Marks : 50** **Duration : 3 Hours**

YOU ARE REQUIRED TO STATE CLEARLY ANY RESULT THAT YOU USE.

1 a) Define Riemann integrability of a bounded function $f : [a, b] \to \mathbb{R}$.

b) Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^3 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

Compute the upper and lower integrals of f on [0, 1] and hence decide whether f is Riemann integrable on [0, 1].

[5+5=10]

2. a) Suppose $f : [0, 1] \to \mathbb{R}$ is continuous.

Prove that for every integer $n \geq 1$ there exists a point $c_n \in [0, 1]$ such that

$$\int_0^1 x^n f(x)\,dx = \frac{1}{n+1} f(c_n).$$

b) Suppose $f : [a, b] \to \mathbb{R}$ is a non-negative continuous function with

$$\int_0^1 f(x)\,dx = 0$$

Prove that $f(x) = 0$ for all x in [a, b].

[5+5=10]

3. Suppose $f : [a, b] \to \mathbb{R}$ is such that $\lim f(x)$ exists for all

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination 2003-2004

B. Stat. (Hons.) I Year
Vectors and Matrices II

Date: 18.02.04 **Maximum Marks : 60** **Duration : 3 Hours**

1. Let W be a subspace of a vector space and $T : V \to V$ be a linear transformation such that $T(W) \subseteq W$.

   (i) Show that T induces a linear transformation

   $$\bar{T} : V/W \to V/W$$

   such that $\bar{T}(\bar{x}) = \overline{T(x)} \quad \forall x \in V$.

   (ii) Suppose that V is finite-dimensional. Show that T is an isomorphism if and only if both the induced linear transformations $T|_W$ and $\bar{T}$ are isomorphisms.

   (iii) Give an example of an infinite-dimensional vector space V, a subspace W of V and a linear isomorphism $T : V \to V$ satisfying $T(W) \subseteq W$ such that the induced map $T|_W : W \to W$ is not surjective.

[5+9+6=20]

2. Let W be a subspace of a vector space V and

   $$W^\circ = \{f \in V^* \mid f(w) = 0 \quad \forall w \in W\}.$$

   Show that V^*/W° is isomorphic to W^*.

[8]

3. Find an orthonormal basis for the vector space

   $$V = \{f \in \mathbb{R}[X] \mid \deg f \leq 2 \text{ and } f(1) = 0\}$$

   with the inner product

   $$\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt.$$

[12]

4. Let $\{u_1, \ldots, u_n\}$ be an orthonormal set in an inner product space V. Show that this set is a basis for V if and only if

   $$\sum_{i=1}^{n} |\langle x, u_i \rangle|^2 = \|x\|^2 \quad \forall x \in V.$$

[8]

5. Prove that any r-dimensional subspace of an n-dimensional vector space V is the intersection of (n-r) hyperspaces in V.

[8

6. Let V be a vector space of dimension n. Let T be a linear transformation from V into itself such that $T \circ T = 0$. Show that rank (T) $\leq \frac{n}{2}$.

[(

7. For what values of k (in $\mathbb{R}$) do the following planes in $\mathbb{R}^3$

$$x + y + 1 = 0$$
$$4x + y - z = k$$
$$5x - y - 2z = k^2$$

(i) intersect in a point ?
(ii) intersect in a line ?
(iii) form a triangular prism?

In case (ii) or (iii), describe the direction of the line or the direction of the axis the prism.

INDIAN STATISTICAL INSTITUTE

Periodical Examination – Semester II : 2003-2004
B.Stat. (Hons.) I Year
Probability Theory II

ate : 20.02.04 **Maximum Score : 70 pts** Time : 3 Hours

lote : This paper carries questions worth a **total** of **85 POINTS**. Answer as much as ɔu can. The **MAXIMUM** you can score is **70 POINTS**.

1. The number of accidents in a city over a certain period of time is a random variable X with $Poisson(\lambda)$-distribution. In each accident, the probability is p_k that exactly k persons are injured, $k = 0, 1, \ldots$, independently of the other accidents. Let Y_k, for each $k = 0, 1, \ldots$, denote the number of accidents in which k persons are injured.
(a) Find the conditional distribution of X given Y_2.
(b) Examine whether the random variables Y_2 and Y_5 are independent or not.

(8+8)=[16]

2. A random variable X has density function $f(x) = \begin{cases} c(1+x)^{-3} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$
(a) Find c and find the c.d.f. of X.
(b) Find the probability $P[X/(3+X^2) < 1/4]$.
(c) If lightbulbs produced by a manufacturer has life-time with the above density, what is the probability that out of four such bulbs, at least one would survive for at least 4 units of time?

(6+6+6)=[18]

3. (a) Prove that the function $F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 - (1/[x]) & \text{if } 1 \leq x < 10 \\ 1 - (1/3x) & \text{if } x \geq 10 \end{cases}$ is a c.d.f.
(b) If X has c.d.f. F, find (i) $P[X > 3]$; (ii) $P[X \leq 17/3]$; (iii) $P[9/2 \leq X < 32/3]$; (iv) $P[X$ is an integer].

(6+4×3)=[18]

4. Let X be a random variable with the $B(3,3)$-distribution.
(a) Find the density function of $Y = \log(1 - X^2)$.
(b) Find the c.d.f. of $Z = \min\{X, 1 - X\}$. Does it have a density? If so, find it.
(c) Find the c.d.f. of $W = \min\{X, 1/2\}$. Does it have a density? If so, find it.

(7+7+7)=[21]

5. A standard deck of cards is randomly distributed between two persons A and B, each getting 26 cards. It turns out that A gets 3 aces and B gets 1 ace. The cards of A are then randomly divided into two equal halves to be given to C and D. The 13 cards that C gets are then mixed with the 26 cards of B. If from these 39 cards, 5 are drawn at random, what is the expected number of aces among the drawn cards?

[12]

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination : Semester II (2003-2004)

B. Stat. Ist Year

Computational Techniques and Programming II

Date: 23. 2. 2004   Maximum marks: 50   Time: 2 hours.

Note: *Answer all questions*

1. (a) Define finite forward differences of order r, $r = 1, 2, \ldots$ of a function $h(x)$ of x, given values of $h(x)$ tabulated at uniformly spaced values of x, h being the spacing between successive x-values.

   (b) Show that if $f(x)$ and $g(x)$ are any two functions of x then, for $h = 1$,

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+1)},$$

   and hence evaluate

$$\Delta^2\left[\frac{x+2}{x^2+1}\right].$$

   (c) The following table gives the the values of a function $y = f(x)$ at certain values of x. For the problem of interpolating the value of y at $x = 99$, suggest which formula is the most appropriate, and use it to perform the interpolation.

x	y
93.0	11.38
96.2	12.80
100.0	14.70
104.2	17.07
108.7	19.91

[5+10+10=25]

2. (a) Derive the two-point trapezoidal rule for numerical evaluation of

$$\int_a^b f(x)dx,$$

   and extend it to the general rule involving values of $f(x)$ tabulated at $n+1$ points $x = a, a+h, a+2h, \ldots, a+nh = b$, where $h = (b-a)/n$.

   (b) Deduce the error involved in using the two-point trapezoidal rule for numerical integration.

   (c) Evaluate numerically

$$\int_0^{\pi/4} \cos x \, dx,$$

   by using values of $\cos x$ at 5 equidistant values of x in the interval $[0, \pi/4]$. [8+7+10=25]

****

INDIAN STATISTICAL INSTITUTE

B-Stat (First Year, Second Semester)

Statistical Methods II

Mid-Semester Examination (February 25, 2004)

TOTAL MARKS : 100 TIME ALLOWED : 3 hours

This is a closed book and closed notes examination. Answer all questions, and you may use calculator for numerical computations.

(1). Compute the Kullback-Leibler distance and the Hellinger distance between two Poisson distributions with paramters 5 and 10. Compute the Kolmogorov-Smirnov distance between two binomial distributions each with $n = 5$ but $p = 0.25$ and 0.5.

[15 + 15 = 30 points]

(2). State clearly **the multivariate version** of Chebysev's inequality and prove it.

[15 points]

(3). Suppose that a uniform random number generator produced the number 0.352144. If you are interested in generating a Poisson random variable with parmeter 6, what will be the value of that obtained using the uniform random number ?

[10 points]

(4). Suppose that we have four variables Y, X_1, X_2 and X_3. It was found from the data that the correlation coefficient between Y and any of the X-variables is 0.05, and the correlation coefficient between any two of the X-variables is 0.15. Compute the multiple correlation coefficient between Y and the X-variables.

[15 points]

(5). Given n data points in the 3-dimensional space, describe clearly all the computational steps involved in obtaining the spatial median and the simplicial volume median of these data points. Your description should be clear enough so that one should be able to code it into a computer program.

[10 + 10 = 20 points]

(6). Assignments. [10 points]

INDIAN STATISTICAL INSTITUTE
Second Semestral Examination: 2003-04
B. Stat. I Year
Vectors and Matrices II

Date: 26.04.04 Maximum Marks: 70 Duration: 3 Hours

1. Let $V = \{f \in \mathbb{R}[X] \mid \deg f \leq 3\}$ and T the linear transformation on V defined by $T(f) = f'$ (the derivative of f).

(i) Compute the rank of T and the eigenvalue(s) of T specifying the algebraic and geometric multiplicities. Describe the corresponding eigenspace(s).

(ii) Determine the characteristic polynomial and the minimum polynomial of T.

(iii) Examine whether there exists a basis of V for which the matrix of T is a diagonal matrix.

[8+4+3=15]

2. Let $V = \mathbb{R}[X]$ and W the ideal $X^3\mathbb{R}[X]$ $\left(= \{X^3 f \mid f \in \mathbb{R}[X]\}\right)$. Compute the dimension of V/W as a vector space of $\mathbb{R}$.

[5]

3. Find an orthonormal basis (over $\mathbb{R}$) for the plane

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

[10]

4. Prove that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix} = (a+b+c+d) \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

for all $a, b, c, d \in \mathbb{R}$.

[10]

5. Let A be an orthogonal (5×5)-matrix with entires in $\mathbb{R}$ such that $\det A = 1$. Show that 1 is an eigenvalue of A.

[10]

6. Let T be an injective linear transformation on $\mathbb{R}^3$. Show that there exists a line L in $\mathbb{R}^3$ such that $T(L) = L$.

[10]

[P.T.O.]

7. Let A be a real symmetric $(n \times n)$-matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation whose matrix is A. Let $V = \mathbb{R}^n$, $W = \ker T$ and $U = \operatorname{im} T$. Prove that

(i) $W \perp U$ and $V = W \oplus U$.

(ii) A is congruent to a matrix of the type

$$\begin{bmatrix} I_r & & \\ & -I_s & \\ & & 0 \end{bmatrix}$$

[7+7=14]

8. A field F is said to be algebraically closed if every non-constant polynomial in $F[x]$ of degree n has n roots in F.

(i) Let V be a finite dimensional vector space over an algebraically closed field F and T a linear transformation on V. Show that there exists an ordered basis $\mathcal{B}$ of V such that the matrix of T, relative to $\mathcal{B}$, is a triangular matrix.

(ii) Construct a linear transformation on a finite-dimensional vector space over $\mathbb{R}$ which cannot be represented by a triangular matrix with entries in $\mathbb{R}$.

[10+6=16]

9. Let V be a finite-dimensional inner product space. Let $f \in V^*$. Prove that there exists a unique $u \in V$ such that $f(x) = \langle x, u \rangle \; \forall x \in V$.

[10]

***

INDIAN STATISTICAL INSTITUTE
Second Semestral Examination: 2003-04
B. Stat. I Year
Analysis II

Date: 29.04.04 | Maximum Marks: 60 | Duration: 3 Hours

1.(a) Prove that the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ is uniformly convergent on any internal $[a,b], 0 < a < b$.

(b) Prove that the double series $\sum_{p,q} \frac{1}{p^4+q^4}$ is convergent.

[5+5=10]

2. Let $(g_n)_{n\ge1}$ be a sequence of twice differentiable functions on $[0,1]$ such that $g_n(0) = g_n'(0) = 0$ for all n. Suppose $|g_n''(x)| \le 1$ for all $x \in [0,1]$ and for all n. Prove that the sequence of functions $(g_n)_{n\ge1}$ and $(g_n')_{n\ge1}$ are uniformly bounded on $[0,1]$.
Also show that for any $x, y \in [0,1]$

$$|g_n(x) - g_n(y)| \le |x-y| .$$

[10]

3. Prove that the improper integral $\int_0^{\infty} \frac{\sin x}{x}\, dx$ is convergent and find it value.

[10]

4. Let $S = \{x_1, x_2, \ldots\ldots, x_n, \ldots.\}$ be any countable infinite subset of $[0,1]$. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = x_n \text{ for some } n \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is R-integrable on $[0,1]$.

[10]

5.(a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 x^n f(x)\,dx = 0$ for all $n = 0,1,2,\ldots\ldots.$
Prove that $f(x) = 0$ for all $x \in [0,1]$. (You are supposed to state any result that you use in proving this.)

[P.T.O.]

(b) Prove that for $m > 0, n > 0$

$$B(m+1,n) = \frac{m}{m+n} B(m,n).$$

[7+3=10]

6.(a) Let $f(x) = \sum_n a_n x^n$ for $|x| < r$.

Suppose $f(x) = f(-x)$ for all x.

Show that $a_n = 0$ for all odd n.

(b) Determine the interval of convergence of $\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \cdots\cdots\cdots$

[5+5=10]

***

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination (2003-2004)

B. Stat. Ist Year

Computational Techniques and Programming II

Date: 5.5.2004 | Maximum marks: 100 | Time: 3 hours

Note: Answer all questions. Maximum you an score is 100.

1. (a) Describe Euler's method for obtaining the solution of the first-order differential equation

$$\frac{dy}{dx} = f(x,y)$$

with $y = y_0$ at $x = x_0$, at the points $x_0 + ih$, $i = 1, 2, \ldots, n$, where h is a small positive quantity.

(b) Give an upper bound to the error in the resulting solution.

(c) There is an improved version of this method. Describe it.

(d) Use the Euler method as well as the improved version to solve

$$\frac{dy}{dx} = x - y$$

with $y = 1$ at $x = 0$, at the points $0, 0.2, 0.4, 0.6, 0.8$.

[5+4+5+6=20]

2. (a) Describe Jacobi's method for the iterative solution of a set of n linear equations

$$\boldsymbol{Ax} = \boldsymbol{b}$$

in n unknowns $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)'$.

(b) Discuss the convergence of this algorithm, and state clearly the condition under which convergence takes place.

(c) Use the Jacobi method to solve

$$\begin{aligned} 20x_1 + 2x_2 + 6x_3 &= 28 \\ x_1 + 20x_2 + 9x_3 &= -23 \\ 2x_1 - 7x_2 - 20x_3 &= -57 \end{aligned}$$

[6+6+8=20]

3. (a) Explain how it is possible to factorize a $p \times p$ matrix $\boldsymbol{A}$ as

$$\boldsymbol{A} = \boldsymbol{PLU},$$

where $\boldsymbol{P}$ is a $p \times p$ permutation matrix, $\boldsymbol{L}$ is a $p \times p$ lower triangular matrix, and $\boldsymbol{U}$ is a $p \times p$ upper triangular matrix.

P.T.O

(b) Discuss the utility of this factorization in the context of solution of a system of linea equations

$$Ax = b$$

(c) Use the above to compute the inverse of the matrix

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

(d) Describe the method based on this factorization that is used for the numerical computation of eigenvalues of A, utilizing matrices similar to A.

[6+4+6+4=20]

4. (a) Explain how a real solution to a nonlinear equation $g(x) = 0$ in x can be obtained using fixed point iteration.

(b) Deduce the conditions under which the sequence of iterates obtained by this approach converges to the actual solution.

(c) Use this approach to obtain a solution to the equation

$$2x^2 - 4x + 1 = 0,$$

starting with an initial approximation $x = 1$. What happens when you start with the inintial approximation $x = 2$? Explain with the help of the conditions for convergence deduced earlier.

[4+6+10=20]

5. (a) If the (n+1)-point Lagrange interpolation formula for a function $f(x)$ is integrated from x_0 to x_n, a formula for numerical integration of the form

$$\int_{x_0}^{x_n} f(x)dx = \sum_{i=0}^{n} c_i f(x_i)$$

is obtained. Determine the values of c_i, $i = 1, 2, \ldots, n$. When $n = 2$ and the x_i's are equispaced, show that the formula becomes the Simpson one-third rule.

(b) Discuss the general principles behind the Gaussian approach to numerical integration. Specifically, explain how

$$\int_a^b f(x)dx$$

is computed using Legendre-Gauss quadrature.

(c) Applying Legendre-Gauss quadrature, evaluate

$$\int_0^1 \frac{dx}{\sqrt{x^4+1}},$$

with $n = 5$. To find the roots of the Legendre polynomial of degree k, solve

$$\frac{d^k}{du^k}\left[u^2 - (\frac{1}{2})^2\right]^k = 0.$$

[6+6+8=2

6. (a) What are divided differences in the context of a table of values of a function $f(x)$ at $n+$ values of x? State two useful properties of divided differences.

(b) Derive the $(n+1)$-point Lagrange interpolation formula for $f(x)$, starting with its $(n+1)$-divided difference.

(c) Compute the divided differences of all possible orders for the data

x	y
0	1
1	1
2	2
4	5

and use them to find an interpolating polynomial for $y = f(x)$ by Newton's divided diffe ence formula.

[5+5+10=2

Second Semesteral Examination : 2003-04

B.Stat. I Year

Statistical Methods II

Date : 07.05.04 Maximum Marks : 100 Duration : 3 Hour

Answer all questions

1. Consider two exponential distributions with expected values 3 and 7. Compute the Hellinger distance, Kullback-Leiber distance and Kolmogorov-Smirnov distance between these distributions. [6+7+7=20]

2. Suppose that we have a dependent variable Y and three independent variables X_1, X_2, X_3 such that the correlation co-efficient between Y and X is 0.03 and the correlation co-efficient between X_i and X_j is 0.21 for all $1 \leq i \leq j \leq 3$. Compute the multiple correlation co-efficient between Y and (X_1, X_2, X_3) as well as the partial correlation co-efficient betw... Y and X_1 fixing the effects of X_2 and X_3.

3. A random number generator that generates independent Cauchy random variables with the standard Cauchy distribution produced numbers 3.45 and 12.86. Using these values, generate two independent standard normal random numbers. [20]

4. Consider a set $A = \{\pm\alpha_1, \pm\alpha_2, \ldots, \pm\alpha_n\}$ of $m = 2n$ elements where $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$. Now, assume that we have a bivariate data cloud consisting of m^2 distinct observations given by $\Omega = \{(x_1, x_2) : x_1, x_2 \in A\}$.

   (a) If $r(\eta) = (r_1, r_2)$ is the spatial rank of an observation $\eta = (\beta_1, \beta_2)$ with $|\beta_1| = |\beta_2|$, show that $r_1/r_2 = \beta_1/\beta_2$.

   (b) If δ_i is the spatial depth of $\eta_i = (\alpha_i, \alpha_i)$ for $i = 1, 2, \ldots, n$, show that $\delta_1 > \delta_2 > \ldots > \delta_n$.

   (c) Find out the spatial median of the data cloud and calculate its half space depth.

   (d) Sketch a spider web plot for this data set using four values of θ ($\theta = \frac{(2i-1)\pi}{4}$; $i = 1, 2, 3, 4$).

   (e) If $\sum_{i=1}^{n} \alpha_i^2 = n$, find out the volume of the smallest ellipsoid that contains at least 75% observations.

   (f) Taking $n = 2$ and $\alpha_i = i - 0.5$ for $i = 1, 2$, construct a univariate QQ plot for the distribution of $X_1 + X_2$ and hence comment on its normality.

[6+5+3+2+4+10=30]

5. Assignment [10]

P.T.O

Area under the Normal Curve from 0 to X

X	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00000	0.00399	0.00798	0.01197	0.01595	0.01994	0.02392	0.02790	0.03188	0.03586
0.1	0.03983	0.04380	0.04776	0.05172	0.05567	0.05962	0.06356	0.06749	0.07142	0.07535
0.2	0.07926	0.08317	0.08706	0.09095	0.09483	0.09871	0.10257	0.10642	0.11026	0.11409
0.3	0.11791	0.12172	0.12552	0.12930	0.13307	0.13683	0.14058	0.14431	0.14803	0.15173
0.4	0.15542	0.15910	0.16276	0.16640	0.17003	0.17364	0.17724	0.18082	0.18439	0.18793
0.5	0.19146	0.19497	0.19847	0.20194	0.20540	0.20884	0.21226	0.21566	0.21904	0.22240
0.6	0.22575	0.22907	0.23237	0.23565	0.23891	0.24215	0.24537	0.24857	0.25175	0.25490
0.7	0.25804	0.26115	0.26424	0.26730	0.27035	0.27337	0.27637	0.27935	0.28230	0.28524
0.8	0.28814	0.29103	0.29389	0.29673	0.29955	0.30234	0.30511	0.30785	0.31057	0.31327
0.9	0.31594	0.31859	0.32121	0.32381	0.32639	0.32894	0.33147	0.33398	0.33646	0.33891
1.0	0.34134	0.34375	0.34614	0.34849	0.35083	0.35314	0.35543	0.35769	0.35993	0.36214
1.1	0.36433	0.36650	0.36864	0.37076	0.37286	0.37493	0.37698	0.37900	0.38100	0.38298
1.2	0.38493	0.38686	0.38877	0.39065	0.39251	0.39435	0.39617	0.39796	0.39973	0.40147
1.3	0.40320	0.40490	0.40658	0.40824	0.40988	0.41149	0.41308	0.41466	0.41621	0.41774
1.4	0.41924	0.42073	0.42220	0.42364	0.42507	0.42647	0.42785	0.42922	0.43056	0.43189
1.5	0.43319	0.43448	0.43574	0.43699	0.43822	0.43943	0.44062	0.44179	0.44295	0.44408
1.6	0.44520	0.44630	0.44738	0.44845	0.44950	0.45053	0.45154	0.45254	0.45352	0.45449
1.7	0.45543	0.45637	0.45728	0.45818	0.45907	0.45994	0.46080	0.46164	0.46246	0.46327
1.8	0.46407	0.46485	0.46562	0.46638	0.46712	0.46784	0.46856	0.46926	0.46995	0.47062
1.9	0.47128	0.47193	0.47257	0.47320	0.47381	0.47441	0.47500	0.47558	0.47615	0.47670
2.0	0.47725	0.47778	0.47831	0.47882	0.47932	0.47982	0.48030	0.48077	0.48124	0.48169
2.1	0.48214	0.48257	0.48300	0.48341	0.48382	0.48422	0.48461	0.48500	0.48537	0.48574
2.2	0.48610	0.48645	0.48679	0.48713	0.48745	0.48778	0.48809	0.48840	0.48870	0.48899
2.3	0.48928	0.48956	0.48983	0.49010	0.49036	0.49061	0.49086	0.49111	0.49134	0.49158
2.4	0.49180	0.49202	0.49224	0.49245	0.49266	0.49286	0.49305	0.49324	0.49343	0.49361
2.5	0.49379	0.49396	0.49413	0.49430	0.49446	0.49461	0.49477	0.49492	0.49506	0.49520
2.6	0.49534	0.49547	0.49560	0.49573	0.49585	0.49598	0.49609	0.49621	0.49632	0.49643
2.7	0.49653	0.49664	0.49674	0.49683	0.49693	0.49702	0.49711	0.49720	0.49728	0.49736
2.8	0.49744	0.49752	0.49760	0.49767	0.49774	0.49781	0.49788	0.49795	0.49801	0.49807
2.9	0.49813	0.49819	0.49825	0.49831	0.49836	0.49841	0.49846	0.49851	0.49856	0.49861
3.0	0.49865	0.49869	0.49874	0.49878	0.49882	0.49886	0.49889	0.49893	0.49896	0.49900
3.1	0.49903	0.49906	0.49910	0.49913	0.49916	0.49918	0.49921	0.49924	0.49926	0.49929
3.2	0.49931	0.49934	0.49936	0.49938	0.49940	0.49942	0.49944	0.49946	0.49948	0.49950
3.3	0.49952	0.49953	0.49955	0.49957	0.49958	0.49960	0.49961	0.49962	0.49964	0.49965
3.4	0.49966	0.49968	0.49969	0.49970	0.49971	0.49972	0.49973	0.49974	0.49975	0.49976
3.5	0.49977	0.49978	0.49978	0.49979	0.49980	0.49981	0.49981	0.49982	0.49983	0.49983
3.6	0.49984	0.49985	0.49985	0.49986	0.49986	0.49987	0.49987	0.49988	0.49988	0.49989
3.7	0.49989	0.49990	0.49990	0.49990	0.49991	0.49991	0.49992	0.49992	0.49992	0.49992
3.8	0.49993	0.49993	0.49993	0.49994	0.49994	0.49994	0.49994	0.49995	0.49995	0.49995
3.9	0.49995	0.49995	0.49996	0.49996	0.49996	0.49996	0.49996	0.49996	0.49997	0.49997
4.0	0.49997	0.49997	0.49997	0.49997	0.49997	0.49997	0.49998	0.49998	0.49998	0.49998

INDIAN STATISTICAL INSTITUTE

Semestral Examination – Semester II : 2003-2004
B.Stat. (Hons.) I Year
Probability Theory II

Date : 12.05.04 **Maximum Score : 130 pts** **Time : $3\frac{1}{2}$ Hours**

Note : This paper carries questions worth a **total** of **140 marks**. Answer as much as you can. The **maximum marks** you can score is **130**.

1. If X has an exponential distribution with parameter λ, determine whether the random variables
(a) $Y = \sqrt{X}$,
(b) $Z = |2 - X|$
have densities and, if so, find the density functions. (12+13)=[25]

2. Suppose a random variable X has density function $f(x) = \dfrac{C}{(1+|x|)^4}$, $-\infty < x < \infty$.
(a) Find the constant C and the c.d.f. (cumulative distribution function) of X.
(b) Determine which of the moments of X are finite and find those moments.
(c) Find the expected value of $\min\{|X|, 1\}$.
(10+10+10)=[30]

3. If X has a $U(0,1)$ distribution, find a function $g : (0,1) \to \mathbb{R}$ such that $g(X)$ has a geometric distribution with parameter p where $0 < p < 1$. Justify your answer.
[15]

4. A fair die is rolled repeatedly until the face six shows up on two consecutive throws. Find the expected number of throws needed. [Hint: Conditioning is an option.]
[15]

5. (a) Define what is meant by the moment generating function of a random variable and show that it is always a convex function defined on an interval.
(b) If X has a Gamma distribution with parameters (λ, α), find the moment generating function of X.
(c) With X as in (b), show that, for any $a > 0$,
$$P[X \geq a] \leq e^{-ta}\left(\frac{\lambda}{\lambda - t}\right)^{\alpha}, \text{ for any } t \in [0, \lambda),$$
and hence deduce that
$$P\left[X \geq \frac{2\alpha}{\lambda}\right] \leq \left(\frac{2}{e}\right)^{\alpha}.$$
(10+10+15)=[35]

6. Let F be a c.d.f. (cumulative distribution function) on $\mathbb{R}$.
(a) State clearly what is meant by the notation $\int g(x)\, dF(x)$ for a borel function $g : \mathbb{R} \to \mathbb{R}$.
(b) Suppose G is another c.d.f. on $\mathbb{R}$. Show that the function
$$H(y) = \int G(y - x)\, dF(x), \; y \in \mathbb{R}$$
is well-defined and that H is a c.d.f. on $\mathbb{R}$. (5+15)=[20]