INDIAN STATISTICAL INSTITUTE

B.STAT I YEAR

PERIODICAL EXAMINATION

Computing Techniques and Programming I

Time : 3 hours. Date : 6.9.04

Remarks : Answer as many parts as you like. Maximum you can score is 100.

Note : For all programming problems give the flowchart first and give data sets required for testing all branches of the flowchart. Then write the program in FORTRAN 90 based on the flowchart.

a) Convert the given numbers to the number systems required **(4X2+3+3=14)**

$(73)_{10}=(\ )_{2}$ $(73)_{8}=(\ )_{10}$ $(AF)_{16}=(\ )_{10}$ $(124)_{10}=(\ )_{8}$ $(110.01)_{2}=(\ )_{10}$ $(77.25)_{10}=(\ )_{2}$

b) Perform the following binary addition, multiplication, subtraction and division,
1001.101 + 100.1 11.01 - 10.11 101.1 * 11.01 100.1/1.1 **(2X2+3X3=10)**

c) Convert the following decimal numbers to binary numbers and represent the negative numbers i 2's complement form then perform the additions (assume that integers are stored in two bytes). Give the results in decimal after converting from the binary representation.
-137 + 21 137 - 21 **(5+5=10)**

a) Name three input devices to the computer that you have used.
b) Name three output devices for the computer that you have used.
c) Name two operating systems for a computer that you have used.
d) What is primary memory and what is secondary memory. **(3X2+2X2=10)**

.a) The following is a IEEE single-precision floating point number given in hexadecimal. What is th decimal number that it is representing ?
C 3 C 0 8 0 0 0 **(10)**

b) Why are the following memories not reasonable ?
Instruction word has a 7-bit address field and memory has 1024 words
Instruction word has a 16-bit address field and memory has 128 words **(2+2)**

c) Decode the following binary ASCII text 0110100 1100011 **(2)**

5. Write a function subprogram that calculates factorial(x) for an integer argument x non-negative and less than 30.

If x is a negative integer the function returns the value 0.

If x is an integer greater than or equal to 30 the function returns the value -1.

Write a main program which reads user given values from the keyboard and uses the ab defined function to print the factorial of the given integer if the integer is non-negative ar than 30.

For other integers suitable messages must be given to user. (6+6

6. The following 2 matrices are to be multiplied and the resulting matrix is to be printed. Write a program to declare necessary arrays and initialize them with the given values d declaration, then implement matrix multiplication using these arrays and print the resultin

5 7 9 8	2 7 5
9 6 5 4	8 0 3
2 3 4 7	4 3 8
	9 7 4

(10)

7.Write a program which writes the examination result of a student in a file "stud.res" after read overall average marks from another file "stud.ave".

The input file contains records whose format is as follows

record columns	information contained
1-6	Roll number of student (character string)
7-11	overall average marks with one digit after decimal point for example, 78.5 (real number)

Score greater than 100.0 means student did not appear. The output file format is as follows

record columns	information contained
1-6	Roll number of student (character string)
7[illegible]	(Integer) 4 if average marks < 40 3 if 40 <= average marks < 60 2 if 60 <= average marks < 80 1 if 80 <= average marks <= 100 5 if student did not appear

(10)

8.Write a program to check if a square matrix is symmetric or not. The dimension of the matrix is input through the keyboard and the array to hold the matrix should be dynamically allocated a runtime. All values of the matrix elements are to be input through the keyboard. **(10)**

9. Write a program to check if a string of maximum length 80 input through the keyboard has an number of "a"s followed by an equal number of "b"s. (1[illegible]

Indian Statistical Institute

B. Stat I (2004-05)

Mid-Semestral Examinations

Statistical Methods I

Time 3 hours. Maximum marks: 30

Date: 8 september, 2004.

Note: Answer as much as you can. The maximum you can score is 30.

1.The following table gives the frequency distribution of heights for 177 Indian adult males.

Height in cm. (class interval)	frequency
144.55-149.55	1
149.55-154.55	3
154.55-159.55	24
159.55-164.55	58
164.55-169.55	60
169.55-174.55	27
174.55-179.55	2
179.55-184.55	2
Total	177

(a)Compute the mean m, standard deviation s of the heights. Find what proportion of the heights lie in each of the intervals (i) $[m-s, m+s]$, (ii) $[m-3s, m+3s]$.
(b) Find the median. Compute the value of a measure of skewness which involves the mean, Median and the standard deviation.
© Compute the values of the β_1 and the β_2 coefficients.

[6+3+9]

2. Show that the mean deviation of a variable about a real number A may be obtained from the formula:

$$N\sum_1^N |x_i - A| = S_2 - S_1 + A(n_2 - n_1)$$

where S_2 is the sum of the x_i's which are greater than A and n_2 is the number of such x_i's while S_1 is the sum of the x_i's which are less than A and n_1 is the number of such x_i's. Using this give an alternative proof to show that the mean deviation is minimum when A is taken to be the median.

[7]

3. Prove the following inequalities:

(a) standard deviation $\geq$ |Mean-Median|

(b) $\beta_2 \geq \beta_1 + 1$

(c) $\delta^r_{r+1} \geq \delta_r^{r+1}$, where δ_r denotes the rth absolute moment about zero.

[3+3+3]

2. In a batch of 10 children, the height(in cm) of a very tall boy is 36 cm above the average height of the other 9 children. Show that the s.d. of the heights. of all the children can not be less than 10.8. If the standard deviation is actually 11.8 determine what the s.d. will be if this tall boy is left out.

[6]

B.Stat. I Year 2004-05
Mid-Semestral Examinations
Probability theory I

Maximum Marks 40 | Time : 3 hours
10 September, 2004

1. Consider the random experiment of placing n distinguishable objects of one kind and m indistinguishable objects of another kind – a total of $m+n$ objects – into r cells. Write down a sample space for the experiment. (You need not make the probability assignments.) [4]

2. From a closet containing n pairs of shoes, $2r$ shoes are chosen at random. Find the probability that there will be (a) exactly r complete pairs, (b) no complete pair, among the shoes chosen ($2r < n$). [6]

3. In a random arrangement of r_1 alphas and r_2 betas, find the probability that the arrangement contains exactly k runs of either kind. [6]

4. In the ballot box problem, if candidate I has m votes and II has n votes, find the probability that I leads (strictly) all through. [6]

5. In Polya's urn scheme with r red balls and b black balls to start with and $c = d$, show that the probability $p_k(n)$ of k black balls in n drawings, satisfies a recurrence relation

$$p_k(n+1) = \alpha\, p_k(n) + \beta\, p_{k-1}(n). \qquad [6]$$

Find α and β.

6. Seven balls are distributed randomly in seven cells. Let X_i be the number of cells containing exactly i balls. Using the probabilities tabulated on the page attached, write down the joint distribution of (X_2, X_3). [6]

7. Suppose that an airplane engine will fail, when in flight, with probability $1-p$ independently from engine to engine; suppose that the airplane will make a successful flight if at least 50 per cent of its engines remain operative. For what values of p is a four-engine plane preferable to a two-engine plane. [6]

P. T. O

ways. To each particular assignment of our occupancy numbers to the seven cells there correspond $7! \div (2! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 0! \cdot 0!) = 7! \div (2! \cdot 2!)$ different distributions of the $r = 7$ balls into the seven cells. Accordingly, *the total number of distributions such that the occupancy numbers coincide with* 2, 2, 1, 1, 1, 0, 0 *in some order is*

$$\frac{7!}{2!\,3!\,2!} \times \frac{7!}{2!\,2!}. \tag{5.4}$$

It will be noticed that this result has been derived by a *double application of* (4.7), namely to balls and to cells. The same result can be derived and rewritten in many ways,

Table 1

Random Distributions of 7 Balls in 7 Cells

Occupancy numbers	Number of arrangements equals 7! × 7! divided by	Probability (number of arrangements divided by 7^7)
1, 1, 1, 1, 1, 1, 1	7! × 1!	0.006 120
2, 1, 1, 1, 1, 1, 0	5! × 2!	0.128 518
2, 2, 1, 1, 1, 0, 0	2! 3! 2! × 2! 2!	0.321 295
2, 2, 2, 1, 0, 0, 0	3! 3! × 2! 2! 2!	0.107 098
3, 1, 1, 1, 1, 0, 0	4! 2! × 3!	0.107 098
3, 2, 1, 1, 0, 0, 0	2! 3! × 3! 2!	0.214 197
3, 2, 2, 0, 0, 0, 0	2! 4! × 3! 2! 2!	0.026 775
3, 3, 1, 0, 0, 0, 0	2! 4! × 3! 3!	0.017 850
4, 1, 1, 1, 0, 0, 0	3! 3! × 4!	0.035 699
4, 2, 1, 0, 0, 0, 0	4! × 4! 2!	0.026 775
4, 3, 0, 0, 0, 0, 0	5! × 4! 3!	0.001 785
5, 1, 1, 0, 0, 0, 0	2! 4! × 5!	0.005 355
5, 2, 0, 0, 0, 0, 0	5! × 5! 2!	0.001 071
6, 1, 0, 0, 0, 0, 0	5! × 6!	0.000 357
7, 0, 0, 0, 0, 0, 0	6! × 7!	0.000 008

but the present method provides the simplest routine technique for a great variety of problems. (Cf. problems 43–45 of section 10.) Table 1 contains the analogue to (5.4) and the probabilities for all possible configurations of occupancy numbers in the case $r = n = 7$. ►

(a) Bose-Einstein and Fermi-Dirac statistics

Consider a mechanical system of r indistinguishable particles. In statistical mechanics it is usual to subdivide the phase space into a large number, n, of small regions or cells so that each particle is assigned to one cell. In this way the state of the entire system is described in terms of a random distribution of the r particles in n cells. Offhand it would seem that (at least with an appropriate definition of the n cells) all n^r arrangements should have equal probabilities. If this is true, the physicist speaks

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: (2004-2005) I Semester

B.Stat. I year

Vectors and Matrices-I

Date: 14 Sep. 2004. Maximum Marks 30. Duration: 3 Hrs.

Note: Class-room notation is used. You can answer any part of any question.

1. Show that $\|Ax + By\| \geq \|Ax\|$ for all x and y implies every column of A is orthogonal to every column of B.

[5]

2. For any matrix A of order m×n, define Row space, Row rank, Column space, Column rank, Null space and Nullity of A, and show that the Row rank is equal to the Column rank.

[1 + 4 = 5]

3. Define generalized inverse of a matrix. Show the equivalence of the following:

a) G is a generalized inverse of A.
b) AGA = A.
c) AG is idempotent and r(AG) = r(A).

[2+6 = 8]

4. Let S be a subspace of R^n. Define ζ, the orthogonal projection of R^n onto S. Show that there exists a symmetric idempotent matrix A with $C(A) = S$ such that for all vectors x in R^n, $\zeta(x) = Ax$.

[1+6 = 7]

5. Prove or disprove the following:

a) $C(A) = R(A)$ implies A is symmetric.
b) A has unique left inverse implies A has unique right inverse.
c) Every symmetric matrix has a symmetric generalized inverse.
d) If A and B are two idempotent matrices such that $C(A) = C(B)$ and AB = BA then A = B.
e) $r(A) = r(A^2) \Leftrightarrow$ VU is nonsingular, where A = UV is any rank factorization of A.

[5×2 = 10]

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INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination
17 September 2004
B. Stat. First Year Class
Subject: Analysis I
Full marks: 100 Time: 2 hours

Answer Question No. 1, and any FOUR from the remaining five questions. Each question carries 20 marks.

1. Let $\{x_n\}$ be a sequence of real numbers such that

$$x_{n+1} = \sqrt{k + x_n},$$

where x_1 and k are positive. Show that the sequence is monotone decreasing or increasing according as x_1 is greater or less than the positive root of the equation

$$x^2 - x - k = 0,$$

and in either case, this positive root is the limit of the sequence.

2. Let $\{a_n\}$ be a sequence of positive terms. Show that

$$\liminf \left(\frac{a_{n+1}}{a_n}\right) \leq \liminf \sqrt[n]{a_n} \leq \limsup \sqrt[n]{a_n} \leq \limsup \left(\frac{a_{n+1}}{a_n}\right).$$

3. Let $\{K_n\}$ be a nested sequence of non-empty compact sets with

$$K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$$

Show that the intersection

$$\bigcap_{n=1}^{\infty} K_n$$

is a non-empty compact set.

4. Show that if S is a closed set of real numbers which contains all its limit points, then S is uncountable.

5. (a) Show that the seris

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

(b) Is the the series

$$\sum_{n=3}^{\infty} \frac{1}{n \log n \log(n \log n)}$$

converges or diverges? Justify your answer.

6. Find the radius of convergence of the power series

$$\frac{(x-1)}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{24} - \frac{(x-1)^4}{64} + \cdots + \frac{(-1)^{n+1}(x-1)^n}{2^n \cdot n} + \cdots,$$

and discuss its convergence at the end points of the interval of convergence.

INDIAN STATISTICAL INSTITUTE
First Semestral Examination (2004-05)
November 29, 2004
B. Stat. First Year
Subject: Analysis I
Full marks: 100 Time: 3 hours

Answer any FIVE questions. Each question carries 20 marks. The figures appearing in the margin at the end of a question indicate marks allotted for the question.

Answers should be complete as far as practicable. No credit will be given to incomplete answers.

1. (a) Let X and Y be sets of real numbers, and $f : X \longrightarrow Y$ a function. Show that the following conditions are equivalent:

(i) f is continuous.

(ii) G is an open set in Y implies $f^{-1}(G)$ is open in X.

(iii) F is a closed set in Y implies $f^{-1}(F)$ is closed in X.

[3 + 3 + 4 = 10]

(b) A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{q}, & \text{if } x \text{ is a rational number } \frac{p}{q} (p \text{ and } q \text{ are integers prime to each other}). \end{cases}$$

Show that f is continuous at every irrational point, and has a discontinuity at each rational point. [10]

2. (a) Show that if f is a monotone function on (a, b), then the set of points at which f is discontinuous is a countable set.

(b) Suppose that $f : K \longrightarrow \mathbb{R}$ is a function on a compact subset K of $\mathbb{R}$, and that f is injective so that the inverse function $g : f(K) \longrightarrow K$ is defined. Show that f if is continuous on K, then g is continuous on $f(K)$. [10 + 10 = 20]

3. (a) Define a uniformly continuous function.

Show that if f is continuous on a compact set of real numbers, then f is uniformly continuous. [4 + 8 = 12]

(b) Let $f : X \longrightarrow \mathbb{R}$ be a uniformly continuous function on a dense subset X of $\mathbb{R}$. Show that f can be extended to a uniformly continuous function $F : \mathbb{R} \longrightarrow \mathbb{R}$.

[8]

P.T.O

4. (a) Let f and g be differentiable functions on an open interval (a, b), and $g' \neq 0$ in (a, b). Let

$$\lim_{x \to a+} g(x) = \infty \text{ or } -\infty, \text{and}$$

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L.$$

Then show that

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L.$$

(b) Show that

$$\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{(1/x^2)} = \exp\left(-\frac{1}{6}\right).$$

[10 + 10 = 20]

5. (a) In the mean-value theorem applied to $f(x)$ in $[0, h]$, $h > 0$, that is, in

$$f(h) = f(0) + hf'(\theta h), \quad 0 < \theta < 1,$$

show that if if $f(x) = \cos x$, then

$$\lim_{h \to 0+} \theta = \frac{1}{2}.$$

OR

(a) If $f'(x)$ exists and is finite for every x in $(a - h, a + h)$, $h > 0$, and if f is continuous in $[a - h, a + h]$, then show that

$$f''(a) = \lim_{a \to 0} \frac{f(a - h) - 2f(a) + f(a + h)}{h^2},$$

provided f'' exists.

(b) If m is any real number and $|x| < 1$, show that the series

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!}x^n + \cdots$$

converges to $(1 + x)^m$.

[10 + 10 = 20]

6. (a) Sketch the graph of the function

$$f(x) = e^{(1/x)},$$

and the graph of its inverse (multiple-valued function) $y = g(x)$. Explain how to restrict the values of y to make it a single-valued function.

[6 + 2 = 8]

(b) If $y = \cosh(\sin^{-1} x)$, prove that

(i)

$$(1 - x^2)y_2 - xy_1 - y = 0.$$

(ii)

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + 1)y_n = 0.$$

Moreover, if

$$\cosh(\sin^{-1} x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,$$

prove that $a_{2n+1} = 0$, and

$$a_{2n} = \frac{4n^2 - 8n + 5}{2n(2n-1)} \cdot a_{2n-2}.$$

[6 + 6 = 12]

INDIAN STATISTICAL INSTITUTE
First Semester Examination: 2004-05
B. Stat. I Year
Probability Theory I

Date: 2.12.04 Maximum Marks: 60 Duration: 3 Hours

Answer any five questions.

(a) If r distinguishable balls are placed at random into n cells, find the probability that exactly m cells remain empty.

(b) Recall that in the ballot box problem where there are m votes for candidate A and n for candidate B, $m > n$, the probability that A leads (strictly) throughout the counting is $\frac{m-n}{m+n}$. Find the probability that at each stage before counting ends, among the votes yet to be counted, there are more for A than for B.

[6+6=12]

2.(a) Give an example of events A_1, A_2, A_3 which are pairwise independent, but not independent.

(b) Let $X_1, \ldots\ldots, X_n$ be independent having a common geometric distribution, $P(X_1 = k) = q^k p$, $k = 0, 1, \ldots\ldots$. Find the distribution of $U = \max(X_1, \ldots., X_n)$.

(c) In (b) above, show that the random variables $\max(X_1, X_2)$ and $X_1 - X_2$ are not independent.

[4+4+4=12]

3.(a) For a group of n people, find the expected number of multiple birthdays. (Assume there are 365 days in a year. A day is a multiple birthday if at least 2 of the n people have that as birthday.)

(b) Let the random variables X and Y take the values $a_1 < a_2 < \cdots < a_n$. $P(X = a_i) = p_i, P(Y = a_i) = q_i$, $i = 1, 2, \ldots\ldots, n$. Given that for some $k, 1 < k < n, p_i < q_i$ if $i \le k$ and $p_i > q_i$ if $i > k$, show that $E(X) > E(Y)$.

[6+6=12]

4.(a) Show that the Binomial distributions $B(n_i, p_i), i = 1, 2, 3, \ldots.$ converge to a Poisson distribution if $n_i \to \infty$ and $n_i p_i \to \lambda$ as $i \to \infty$.

(b) A book of n pages contains on the average λ misprints per page. Explaining how a Poisson distribution may reasonably apply here, estimate the probability that at least one page will contain more than k misprints.

[7+5=12]

[P.T.O.]

5.(a) For the random walk starting from zero which takes a step in the positive direction with probability p, let ϕ_i be the probability that the random walk reaches 1 for the first time at time i. Show that the generating function Φ for the sequence (φ_i) satisfies a second degree equation.

(b) Let for each fixed $n = 1,2,3,.....$ the sequence $(a_{0n}, a_{1n}, a_{2n}, ...)$ be a probability distributions on the non-negative integers, having the generating function A_n. Suppose $\lim_{n\to\infty} a_{k,n} = a_k$ exists for all $k = 0,1,2,....$. Show that for all s, $-1 < s < 1$, $\lim_{n\to\infty} A_n(s)$ exists.

[7+5=12]

6.(a) Let the random variable N have the Poisson distribution with parameter λ and let N balls be placed at random into 2 cells. Let X and Y stand for the number of balls in cell 1 and cell 2. Show that X and Y are independent.

(b) Let X ,Y be independent with the common negative binomial distribution with parameters r, p. Find the conditional distribution of X given $X + Y$.

[6+6=12]

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INDIAN STATISTICAL INSTITUTE

Semester Examination: (2004-2005) I Semester

B.Stat. I year

Vectors and Matrices-I

Date: 6 Dec. 2004. Maximum Marks 60. Duration: 3 Hrs.

Note: Class-room notation is used. You can answer any part of any question.

1. Show that the cardinality of any independent subset of a finite dimensional vector space V is less than or equal to the cardinality of any generating subset of V and hence show that every basis of V has the same cardinality.

[5 + 1 = 6]

2. a) Show that r(AB) = r(B) – d{ C(B) ∩ N(A) }.
b) Show that r(AB) + r(BC) ≤ r(B) + r(ABC)

[4 + 2 = 6]

3. Let A and B be matrices such that r(A + B) = r(A) + r(B) and C(A) + C(B) = R^n. Then show that every g-inverse of (A + B) is a g-inverse of A and hence or otherwise obtain the projection operator that projects vectors of R^n onto C(A) along C(B).

[4 +2 =6]

4. Obtain the rank, the determinant, a basis of row space, a basis of column space, a basis of null space, rank factorization and a g-inverse of the matrix

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 3 & 2 \end{bmatrix}$$

[12]

5. Prove the following:
a) y'Ax = 0 for all A implies either x or y is null.
b) y'Ay = 0 for all y implies A is skew symmetric.
c) r(A) = r(A^2) and AB = 0 = BA implies r(A + B) = r(A) + r(B).
d) C(A) ∩ C(B) = { ϕ } and C(A) ⊆ C(A + B) implies r(A + B) = r(A) + r(B).
e) ABA = 0 implies B can be expressed as C + D where AC = 0 and DA = 0.
f) | I + uv' | = 1 + v'u, where u and v are vectors.
g) r(A + B + C) = r(A) + r(B) + r(C) implies r(A + C) = r(A) + r(C).
h) A is an idempotent matrix of rank r implies A can be expressed as sum of r idempotent matrices of rank 1 each.
i) r(A) = r(A^2) = r implies A has a nonzero principal minor of order r.
j) Any independent set of vectors is an orthonormal set of vectors with respect to som inner product.

[10×3 = 3

--x--

NDIAN STATISTICAL INSTITUTE
B.STAT-I (2004-05)
Statistical Methods I (Semestral Examinations)
Time: 3 hours. Maximum marks: 60.

Date : **8 December, 2004**.

Note: Answer as many question as you like.The whole question paper carries 70 marks. The maximum you can score is 60.

1. (a) Let X be a variable taking values $1, 2, 3, \cdots, k$ with frequencies $f_1, f_2, f_3, \cdots, f_k$ respectively. Let the cumulative frequencies computed from below be given by:
$F_k = f_k,\ F_{k-1} = f_k + f_{k-1},\ \cdots,\ F_1 = f_k + f_{k-1} + \cdots + f_1$. Let the same procedure be applied to the F 's to obtain
$G_k = F_k,\ G_{k-1} = F_k + F_{k-1},\ \cdots, G_1 = F_k + F_{k-1} + \cdots + F_1$. Repeating the same, once more we get:
$H_k = G_k,\ H_{k-1} = G_k + G_{k-1},\ \cdots,\ H_1 = G_k + G_{k-1} + \cdots + G_1$
Show that the mean μ and the variance σ^2 of the variable X are given by:
$$\mu = \frac{G_1}{F_1},\ \sigma^2 = \frac{2H_1 - G_1}{F_1} - \left(\frac{G_1}{F_1}\right)^2.$$
(b) The Mean and the standard deviation of the statures of 1000 army recruits were reported as 69.5 in. and 1.3 in. respectively. Among the recruits there were 207 with stature more than 72in. and 51 with stature less than 66 in. Show that the data is inconsistent.

[10+5]

2. (a) In an examination, a candidate fails if he scores less than 40 (out of 100), passes with distinction if he scores 75 or more and gets an ordinary pass otherwise. In one year, the percntages of failures, ordinary passes, and passes with distinctions were respectively 23%, 62% and 15%. Assuming normality of the distribution of scores, find the mean and the standard deviation of the scores of all the candidates.
(b) Find the mode of the Poisson distribution with parameter λ.

[5+5]

3. (a)Let $X_1, X_2, \cdots, X_k$ be k variables such that
$$Var(X_i) = \sigma^2 \text{ for all } i \text{ and } Cov(X_i, X_j) = \rho \text{ for } i, j = 1, 2, \cdots, k.$$
Show that $Var\left(\frac{X_1+X_2+\cdots+X_k}{k}\right) = \frac{\sigma^2}{k}\left(1 + (k-1)\rho\right)$. Hence conclude that
$$\rho \geq -\frac{1}{k-1}.$$
(b) Define the correlation ratio $\eta_{Y,X}$ of a variable Y on a variable X. Show that
$$\eta^2_{Y,X} \geq \rho^2_{Y,X}$$
where $\rho^2_{Y,X}$ denote the square of the corelation coefficient between X and Y.
(c)Nine students were ranked with respect to intellectual ability by two teachers as given below:

	A	B	C	D	E	F	G	H	I
$Teacher\ I$	1	2	5	4	9	8	7	3	6
$Teacher\ II$	3	6	1	7	2	5	8	4	9

Compute the Spearman's rank correlation coefficient. Compute the Kendall's rank correlation coefficient. Interpret the results.

P.T.O

(*d*) The following is the variance of the total rainfall (1) for January to March, (2) for April to December, (3) for the whole year, at Greenwich in the eighty years 1841-1920, the unit being a millimetre :

Jan-Mar	$\sigma_1^2 = 1521$
April-Dec	$\sigma_2^2 = 8968$
Whole year	$\sigma_3^2 = 10754$

Find the correlation coefficient between the rainfall in January-March and April-December.

[5+(2+5)+(4+4)+

4. (a) Describe how you can use a fair coin to draw a random sample of size 2 without replacement from a population with 10 units.

1.(b) For a variable X, let m_r, $r = 1, 2, \cdots$, denote the rth central moments and let $\beta_1$and β_2 denote the usual moment cefficients defined in terms of m_2, m_3 and m_4.
Prove the following inequalities:
(i) $m_{2r+1}^2 \leq m_{2r} m_{2r+2}$.
(ii) $\beta_2 \geq \beta_1 + 1$.
(c) Define the p-th quantile of a distribution, where p is a number with $0 < p < 1$.
Give a formula for a suitable measure of skewness of a distribution using the quartiles.

[5+(4+6)+(2+

INDIAN STATISTICAL INSTITUTE
1st Semestral Examination
B. Stat. II year : 2004–2005
C & Data Structures

Date : 08. 12. 2004 Marks : 100 Time : 3 Hours

Answer all questions. Please try to write all the part answers of a question at the same place.

1. (a) Write a C program that takes the preorder traversal data of a binary search tree as input and outputs the tree itself.
   (b) Explain what happens when the following codes are executed.
      i. `char *p, *q; while (*p++ = *q++);`
      ii.
```
int i, k = 1, n = 5;
for (i = 0; k < n+1; i = k-i)
{ printf("%d\n", k); k = k+i; }
```
   (c) Write a function in C that finds a given substring in a circular string. As an example, the substring "abc" is absent in the string "cxyzsdfgab", but exists when the string is considered in a circular manner.

7+6+7 = 20

2. (a) Briefly explain three hashing strategies with clear description of collision resolution.
   (b) Select a specific hashing strategy among the above three and implement a function in C programming language that can manage search and insertion in a hash table.
   (c) Provide an analysis of average search/insertion time complexity for double hashing.

9+6+5 = 20

3. (a) Clearly explain the insertion algorithm in a balanced binary search tree.
   (b) Write down C routines for single and double rotations.
   (c) Provide specific examples to demonstrate single and double rotations while inserting a node in a balanced binary search tree.

10+5+5 = 20

P. T. O.

4. Given positive integers a, b, n, the integer b is called the inverse of a modulo n if $ab - 1$ is divisible by n.

   (a) Given a, n, write a function in C that finds the inverse of a modulo n.

   (b) Execute your function with (i) $a = 7, n = 51$ and (ii) $a = 9, n = 39$.

   (c) Describe the RSA public key cryptosystem and highlight where exactly the inverse finding algorithm is required.

$6+4+10 = 20$

5. (a) Briefly describe Linear Feedback Shift Register (LFSR).

   (b) What are its applications in cryptography?

   (c) Describe how you can efficiently implement an LFSR in C language.

   (d) Consider an LFSR having connection polynomial $x^{16} + x^{14} + x^{12} + x^{10} + x^7 + x^5 + x^4 + x^2 + 1$. Take the binary pattern of last two digits of your roll number and append zeros after that to get a 16-bit initial seed. Evolve the LFSR 32 times to produce 32 bits as output. Comment on the randomness of this 32-bit pattern.

$3+3+4+10 = 20$

INDIAN STATISTICAL INSTITUTE
B.STAT (HONS) I YEAR(2004-2005)
First Semester
SEMESTRAL EXAMINATION
Computing Techniques and Programming I

Time : 3 hours. Date : 10.12.04

Remarks : Answer as many parts as you like. Maximum you can score is 100.

Note : Write all the programs in FORTRAN 90. If you are required to give an algorithm or a flowchart for a programmming problem then the corresponding program must be based on the algorithm or flowchart that you have given,

1.i) Give the flowchart for floating point addition in computers. (6)

ii) What do you think will be printed in case the following program is compiled and run ? Why? (2+2=4)

```
PROGRAM TEST
  IMPLICIT NONE
  INTEGER:: I=5,J=2,K
  PRINT *,I+J
CALL ADD(I,J,K)
  PRINT *,K
END PROGRAM TEST

SUBROUTINE ADD(X,Y,Z)
  IMPLICIT NONE
  REAL,INTENT(IN)::X,Y
  REAL,INTENT(OUT)::Z
  Z=X+Y
END SUBROUTINE ADD
```

iii) Give the FORTRAN 90 rules for Implicit Type Declaration and Implicit Type Conversion. (2+2=4)

iv) Draw the block diagram of a computer and briefly describe each of its constituent part. (2+4=6)

2. You are required to read the roll numbers and marks (given in integer) for the subject English of 100 students one by one from the keyboard and print the frequencies of the marks in the following classes in a tabular form using some suitable FORMAT statements.

Range for the Classes : 0-24, 25-49, 50-74, 75-99, 100.

Draw a flowchart (for the programming problem) which uses only less than tests to cumulate the marks and detects and rejects out of range marks. (5)

Give test sets to follow all paths of the flowchart. (3)

Write the program using a structured IF statement. (5)

Suggest an arithmetic function of marks which would calculate the array index directly and eliminate the need for the structured IF statements that you have written. (2)

Write the program using the above function with the help of a CASE statement. (5)

(P.T.O.)

3. i) Write a subroutine to find out if an integer P is prime and then write a program to read two positive integers M and N from keyboard and find using the subroutine N consecutive prime numbers greater than M. (10)

ii) Give the flowchart and write a program to compute ln 2 correct to 4 places of decimal using the formula given below

$$\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+ \cdots\cdots \qquad (10)$$

4.i) Write a MODULE for a polymorphic function SORT to sort a given array of integers or a given array of characters in ascending order of values. (10)

ii) Draw the flowchart and write an integer valued function subprogram to implement binary search for a given integer in a sorted array of integers ; if not found, the function should return 0 else it must return the index value of the array position in which the integer is found. (10)

5. Write a program which reads integer values from the keyboard in an unsorted order and creates a linked list of nodes in descending order of values of the integers; each node containing one integer. The descending order of values of the integers must be maintained for insertion of each new node during creation.

Define and use derived types, pointer variables and subroutines as necessary. (20)

6. A file contains the names of continents, countries and capitals in that order in records of maximum length 50 using a dollar symbol to separate the names as shown in the following examples, (the records are not sorted in any order).

Examples:

EUROPE$FRANCE$PARIS
AFRICA$EGYPT$CAIRO
....
ASIA$INDIA$NEW_DELHI

Write a program which gives a menu to run either of the following two subroutines, (2)

i) Write a subroutine to read the file and then print on the monitor the list of names of all countries situated in a particular continent. (8)

ii) Write a subroutine which lists in a file the names and capitals of all countries whose names start with the English character P. (10)

INDIAN STATISTICAL INSTITUTE
First Semester Back Paper Examination 2004-05

B. STAT. I YEAR
Probability Theory I

Date: 17.01.2005 Maximum Marks : 100 Duration : 3 Hours

Answer all questions. All questions carry equal marks.

1. From an urn containing a red balls and b black balls $(b<a)$, balls are removed at random one by one. Find the probability that at any stage the set of withdrawn balls contain more red than black balls.

2. Let X, Y be independent random variables with the common distribution

0	1	2	. . .	n
$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$	. . .	$\frac{1}{n+1}$

Let U= max (X,Y), V = min (X,Y). Find the joint distribution of U and V.

3. From the numbers 1, 2,.....N a random sample $x_1,\ldots\ldots x_n$ is drawn without replacement. Find

E $(x_1+\ldots\ldots+x_n)$

and Var $(x_1+\ldots\ldots+x_n)$

4. Let $d_n = \log n! - \left(n+\frac{1}{2}\right)\log n + n \quad , \quad n=1,2,\ldots\ldots$

Show that $\lim_{n\to\infty} d_n$ exists.

5. a) If X is a nonnegative random variable, show that $P(X > \lambda) \le \frac{EX}{\lambda} \quad , \lambda > 0$

b) For a random variable X, prove that

$$(E|X|)^r \le E|X|^r \quad ,r > 0.$$

Contd...2/-

6. A coin with probability p for falling a head is tossed until 2 heads are obtained. Let X be the no. of tosses until the first head and Y the total number of tosses. Find the conditional distribution of X given Y.

7. Let the probability p_n that a family has exactly n children be αp^n when $n \geq 1$, and $p_0 = 1 - \alpha p \left(1 + p + p^2 + \ldots\right)$. Suppose that all sex distributions of n children have the same probability. Show that for $k \geq 1$ the probability that a family has exactly k boys is $2 \alpha p^k / (2-p)^{k-1}$.

8. Let the random variable N have the Poisson distribution with parameter λ and let N balls be placed at random in 3 cells. Find the expected number of cells left empty.

9. In a random walk starting from zero with probability p of taking a step in the positive direction, find the probabilities f_{2k} that the random walk returns to zero for the first time at $2k$, $k = 1, 2, \ldots$.

10. a) Give an example of random variables X, Y, Z such that X, Y, Z are not independent but any two of them are.

b) Give an example of a random variable X such that var (X) exists, but $E\left(X^3\right)$ does not exist.

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INDIAN STATISTICAL INSTITUTE
First Semester Back Paper Examination 2004-05

B. STAT. I YEAR
Vectors and Matrices I

Date: 18.01.05 Maximum Marks : 50 Duration : 3 Hours

Notation : Class room notation.

1. a) Define Inner Product and Norm of vectors in R^n.

b) State and prove Bessel's Inequality and deduce from it Cauchy-Shwarz inequality.

[3+5+2=10]

2. a) Let U and V be vector spaces over the same field F. Define a Linear Transformation from U into V. Show that it can be represented by a matrix of appropriate order.

b) Obtain a matrix representing the linear transformation from R^n to the row space of a given matrix of order $n \times m$.

[7+3=10]

Consider the system of Linear equations

$$A x = b$$

where A is a matrix of order $m \times n$.

a) Obtain a necessary and sufficient condition that the above system is consistent for all vectors b.

b) Obtain a necessary and sufficient condition for the above system to have unique solution given that it is consistent.

c) Obtain a necessary and sufficient condition for the above system to have unique solution for every vector b.

[4+4+2=10]

a) Show that $r(A + B) \leq r(A) + r(B)$ and equality occurs if and only if

$C(A) \cap C(B) = \{\varphi\}$ and $R(A) \cap R(B) = \{\varphi\}$.

Contd...2/-

b) Let A and B be matrices of order $n \times n$ such that $r(A+B) = r(A) + r(B) = r$. Obtain the matrix representing the projection of R^n onto $C(B)$ along $C(A)$.

[5+5=10]

5. Find the rank, the determinant, a basis of row space, a basis of column space, a basis of null space, a rank factorization and a g-inverse of the matrix.

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 3 & 1 & 3 & 1 \end{pmatrix}$$

[10]

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INDIAN STATISTICAL INSTITUTE
B.STAT(HONS) I YEAR, 2004-2005
First Semester
BACKPAPER EXAMINATION
Computing Techniques and Programming I

Time : 3 hours. Date : 19.1.05

Note : Write all the programs in FORTRAN 90. If you are required to give an algorithm or a flowchart for a programmming problem then the corresponding program must be based on the algorithm or flowchart that you have given,

1. I) The following are two IEEE single-precision floating point numbers given in hexadecimal. Add and express the normalized result in hexadecimal. (10)

3EE00000H
3D800000H

ii) Name and briefly explain the functions of the different hardware parts of the computer. (10)

2. i) Draw the flowchart and write a program to read two positive integers M and N from keyboard and find the sum of N consecutive numbers greater than M. (6)

ii) Give the flowchart and write a program to determine the roots of the quadratic equation given below, where a,b,c are read from the keyboard. The program should give the user the chance to repeat the calculation with different sets of values and the option to quit must be specified. Roots may be real or complex.

$ax^2+bx+c = 0$ (8)

iii) Write a recursive subprogram to find the nth fibonacci number given n. How many times the subprogram will be called for n=6 ? (4+2=6)

. Write a MODULE for datatype COMPLEX which should overload the arithmetic operators. (20)

. Write a subroutine which sorts a given linked list. (20)

. A file contains some records each having the roll number and name of a student together with his marks in three subjects. The records are sorted in order of roll numbers.

Write a program which displays a menu which allows the user to run either of the following two subroutines (4)

Write a subprogram to read the file and print the roll number, name and average marks of each student and also save this information in a file. (4+4=8)

Write another subprogram which prints roll number and names of those students who have passed in all subjects (pass marks are 40 for each subject) and also save this information in a file. (4+4=8)

INDIAN STATISTICAL INSTITUTE

First Semester Backpaper Examination (2004–05)

B. Stat. I Year

Analysis I

Date 20.01.2005 Maximum Marks : 100 Time : 3 hours

Answer any **five** questions. Each question carries 20 marks. The figures appearing in the margin at the end of a question indicate marks allotted for the question.

1. Let S be a countable subset of an open interval (a,b), and $\{x_1, x_2, \ldots, x_n, \ldots\}$ be an enumeration of S. Let $\sum c_n$ be a convergent series of positive terms. Define a function $f : (a,b) \to \mathbb{R}$ by

$$f(x) = \sum_{x_n < x} c_n, \quad a < x < b$$

where the sum is over those indices n for which $x_n < x$. Show that

   (a) f is monotonically increasing on (a,b),

   (b) f is discontinuous at every point of S. In fact,

$$\lim_{x \to x_n^+} f(x) - \lim_{x \to x_n^-} f(x) = c_n, \quad x_n \in S$$

   (c) f is continuous at every other point of (a,b). [$4 + 8 + 8 = 20$]

2. (a) A continuous function $f : \mathbb{R} \to \mathbb{R}$ has the property that

$$f(x+y) = f(x) + f(y)$$

   for all $x, y \in \mathbb{R}$. Show that f is of the form $f(x) = cx$, where is a constant.

   (b) Show that continuous image of a compact set is compact. [$10 + 10 = 20$]

3. (a) Let a function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

   Show that f is differentiable at all points x, but f' is not a continuous function.

Moreover, if $g(x) = \tan x$, then show that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$$

(b) Evaluate $\lim_{x \to 0} (\cos mx)^{n/x^2}$. [(6 + 6) + 8 = 20]

4. (a) State and prove Cauchy's mean value theorem.

If $f(x) = x^{1/3}$, $x = -a$ and $h = 2a$, then show that the value of θ which satisfies

$$f(x + h) = f(x) + hf'(x + \theta h)$$

is given by

$$\theta = \frac{1}{2} \pm \frac{1}{18}\sqrt{3}.$$

(b) If $c_0 + \frac{c_1}{2} + \cdots + \frac{c_{n-1}}{n} + \frac{c_n}{n+1} = 0$, where c_i's are real constants, show that the equation

$$c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n = 0$$

has at least one real root between 0 and 1. [10 + 10 = 20]

5. (a) If $P_n = \frac{d^n}{dx^n}(x^n \log x)$, show that

$$P_n = nP_{n-1} + (n-1)!,$$

and hence show that

$$P_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}\right).$$

(b) If $y = \sin(m \sin^{-1} x)$ show that

$$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0,$$

and hence obtain the expansion of $\sin(m \sin^{-1} x)$ in a power series of x.

[10 + 10 = 20]

6. (a) State and prove Taylor's theorem with a remainder in Lagrange's form.

How many terms of the Taylor's series do you need to compute $e^{1/10}$ to an accuracy of 10^{-3}?

(b) Show that

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad \text{provided } -1 < x \leq 1$$

[(2+6+4)+8=20]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2004-05

B. Stat. (Hons.) I Year

Probability Theory II

Date: 21.2.05 Maximum Marks: 30 Duration: 2 Hours

This paper carries a total of 34 marks. You may answer any number of questions; the maximum you can score is 30.

1.(a) Let F be a distribution function such that

$$0 < \sum_{x \in \mathbb{R}} F(x) - F(x-) < 1$$

Show that there exist distribution functions G and H, G continuous and H discrete (i.e. $\sum_{x \in \mathbb{R}} H(x) - H(x-) = 1$) such that

$$F(x) = \alpha G(x) + \beta H(x) \quad , \quad x \in \mathbb{R}$$

for real, positive α and β, $\alpha + \beta = 1$.

(b) Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned} F(x, y) &= 0 \quad \text{if} \quad x + 2y < 0 \\ &= 1 \quad \text{if} \quad x + 2y \geq 0 \end{aligned}$$

Show that F cannot be the bivariate distribution function of a pair of random variables (X, Y).

(6+4=10)

2. Let X, Y be i.i.d random variables with uniform distribution on $[-1,1]$. Find the density of the random variables

(a) $|X - Y|$ (b) $X \wedge Y^3$

(5+5=10)

3. Let $X, X_1, X_2, \ldots$ be random variables such that $P\left(|X_n - X| \leq \frac{1}{n}\right) = 1$

(a) Show that $F_{X_n}(t) \xrightarrow[n \to \infty]{} F_X(t)$ at all continuity points t of F_X.

(b) What can you say if the condition above is replaced by $P\left(|X_n - X| \leq \frac{1}{n}\right) \xrightarrow[n \to \infty]{} 1$?

(6+4=10)

4. Let X and Y be independent, each uniformly distributed on $[-1,1]$. Show that

$$P(X = Y) = 0$$

(4)

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INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: (2004-2005) II Semester

B.Stat. I year

Vectors and Matrices-II

Date: 25 Feb. 2005. Maximum Marks 30. Duration: 3 Hrs.

Note: Class-room notation is used. You can answer any part of any question.

1.

a) Given any square matrix A show that there exists a unitary matrix U and a triangular matrix T such that $A = UTU^*$.

b) Using the above result derive the spectral decomposition of a real symmetric matrix.

c) Define singular values of a matrix and derive singular value decomposition of a given real matrix.

[4+3+3=10]

2. Prove or disprove the following:

a) Let $\{x_1, x_2, \ldots, x_n\}$ be any orthonormal basis of $\mathbf{R}^n$. Consider the quadratic form $x'Ax$ where A is a real symmetric matrix with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then $\sum x_i'Ax_i = \sum \lambda_i$ where the summations are over $i = 1, 2, \ldots, n$.

b) For any pair of matrices A and B of order mxn, $|I+A'A|\,|I+B'B| \le |I+A'B|^2$.

c) Let A be an nnd matrix of rank r and B is any principal submatrix of order r of A. Then the product of nonzero eigenvalues of $A \ge |B|$.

d) λ is a nonzero eigenvalue of A and G is a reflexive g-inverse of A implies $1/\lambda$ is an eigenvalue of G.

e) μ is a singular value of A implies $1/\mu$ is a singular of A^+.

f) For any square matrix A the number of nonzero eigenvalues of $A \le r(A)$.

g) Let A be a square matrix of order n, rank r with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. If the index of A be greater than 1 then

$$(\sum \lambda_i)^2 / (\sum \lambda_i^2) < r.$$

h) For a square matrix A, the sum of squares of all the eigenvalues is equal to the sum of squares of all the singular values implies A is symmetric.

i) Let A = B + C and r(A) = r(B) + r(C). Then A is nnd and B is symmetric implies B and C are nnd.

j) Every square matrix is similar to a diagonal matrix.

[10×2 = 20]

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INDIAN STATISTICAL INSTITUTE

SECOND SEMESTER MIDTERM EXAMINATION (2004–05)

B. STAT. I YEAR

ANALYSIS II

: 28.02.2005 Maximum Marks : 100 Time : 3 hours

The question carries 105 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

(a) Define a compact set in $\mathbb{R}^k$. Define a connected set in $\mathbb{R}^k$. [$5 + 5 = 10$]

(b) Give an example of a set in $\mathbb{R}^k$ that is (*i*) both compact and connected; (*ii*) compact but not connected; (*iii*) connected but not compact; (*iv*) neither compact nor connected. [$2 + 2 + 2 + 2 = 8$]

(c) Give an example to show that the union of two connected sets need not be connected. Under what additional assumptions is the union of two connected sets connected? Justify your answer. [$2 + 5 = 7$]

(d) Show that if two connected subsets of $\mathbb{R}$ intersect, then the intersection is connected. Does the same conclusion hold in any $\mathbb{R}^k$ for $k > 1$? Justify your answer. [$4 + 4 = 8$]

(e) A nonempty set $A \subseteq \mathbb{R}^k$ has the following property : For any two points $x, y \in A$, there exists a connected subset $B \subseteq A$ such that $x, y \in B$. Show that A itself is connected. [10]

P. T. O.

2. (a) Without quoting the theorem on integrability of composite functions, show that if $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$. Is the converse true? [10 + 5]

(b) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Define

$$f_+(x) = \begin{cases} f(x) & \text{if} \quad f(x) > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } f_-(x) = \begin{cases} -f(x) & \text{if} \quad f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $f \in \mathcal{R}[a,b]$ if and only if both $f_+ \in \mathcal{R}[a,b]$ and $f_- \in \mathcal{R}[a,b]$ Moreover,

$$\int_a^b f(x)dx = \int_a^b f_+(x)dx - \int_a^b f_-(x)dx$$ [10

(c) Let $f : [a,b] \to \mathbb{R}$ be a function with a continuous derivative. Show that . is the sum of a continuous increasing function and a continuous decreasin function. [7

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function of period $p > 0$, *i.e.*,

$$f(x+p) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

(a) Show that the integral of f over any interval of length p is the same, *i.e.*,

$$\int_a^{a+p} f(x)dx = \int_b^{b+p} f(x)dx \quad \text{for all } a, b \in \mathbb{R}.$$ [10]

(b) Show that $\int_0^p [f(x+a) - f(x)]dx = 0$ for all $a \in \mathbb{R}$. And hence, show that given $a \in \mathbb{R}$, there exists $x \in [0,p]$ such that $f(x+a) = f(x)$. [4 + 6 = 10]

4. Let $f \in \mathcal{R}[a,b]$. Show that for every $\varepsilon > 0$, there is a continuous function g on $[a,b]$ such that

$$\int_a^b |f(x) - g(x)|dx < \varepsilon$$ [10]

INDIAN STATISTICAL INSTITUTE

Periodical Examination

Course: B.Stat.(Hons.) I Year Year: 2004-05
Subject: Computational Techniques and Programming II
Date of Examination: 2.3.2005

Maximum marks : 100 Duration : 3 hrs.

Answer all the questions.

Derive Lagrange's interpolation formula with error term. Find the sum of the lagrangian coefficient functions. [25]

Describe with justification , adaptive quadrature scheme based on Simpson's one-third rule. [20]

Write a program in FORTRAN for interpolation using Neville's algorithm. Interpolation points will be supplied in ascending order. [20]

Describe composite Simpson's one-third rule and obtain the error term. [15]

Determine the lagrangian coefficient functions, in explicit polynomial form, relative to the ordinates f(x) at four points x= -2, -1, 1 and 2. Use results to obtain approximate expressions for f(0) and f'(0) . [20]

INDIAN STATISTICAL INSTITUTE

Second Semestral Midterm Examination, 2004-05

B. Stat. (Hons.) I

Statistical Methods II

te: March 4, 2005 — Duration: 2 hours — Maximum Marks: 100

Answer all questions

Suppose a sample data was collected for 200 males and 200 females to learn whether re is any correlation between the height (H) and the (average) length of hair (L) of individual. It was found that the correlation between H and L is negative. Also, vas found that the the partial correlation between these two variables were almost o when the effect of a third variable was adjusted for. Explain the situation and ke a guess for the third variable with justifications. [15]

Let (ρ_{ij}), $i, j = 0, 1, 2$ be the correlation matrix of the variables Y, X_1, X_2 and (ρ^{ij}) its inverse. Prove the following identities.

(a) $$\rho_{02.1} = \frac{\rho_{02} - \rho_{01}\rho_{12}}{\sqrt{1-\rho_{01}^2}\sqrt{1-\rho_{12}^2}}.$$

(b) $$R^2_{0(12)} = 1 - \frac{1}{\rho^{00}},$$

ere $R^2_{0(12)}$ denotes the square of multiple correlation between Y and X_1, X_2.

[15+25=40]

). Let (σ_{ij}), $i, j = 0, 1, 2$ be the covariance matrix of the variables Y, X_1, X_2 d (σ^{ij}) be its inverse. Consider the linear regression of Y on X_1 and X_2 ($y = + \beta_1 x_1 + \beta_2 x_2$). Derive expressions of β_1 and β_2 as bi-linear functions of the entries (σ_{ij}) and (σ^{ij}).

). Prove the formula (standard notation)

$$(1 - R^2_{0(12)}) = (1 - \rho_{01}^2)(1 - \rho_{1.2}^2).$$

[25+20=45]

INDIAN STATISTICAL INSTITUTE

Second Semester Semestral Examination (2004–05)

B. Stat. I Year

Analysis II

Date : 04.05.2005 | Maximum Marks : 100 | Time : $3\frac{1}{2}$ hours

The question carries 125 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. Show that a connected set in $\mathbb{R}^k$ is either a singleton or uncountable. [10]

2. Let $f \in \mathcal{R}[a,b]$. Show that for every $\varepsilon > 0$, there is a continuous function g on $[a,b]$ such that
$$\int_a^b |f(x) - g(x)|dx < \varepsilon \qquad [10]$$

3. Let $f : [a,b] \to \mathbb{R}$ be bounded such that $f^2 \in \mathcal{R}[a,b]$. Does it follow that $f \in \mathcal{R}[a,b]$? What if $f^3 \in \mathcal{R}[a,b]$? [5 + 5 = 10]

4. Suppose $f : [a,b] \to \mathbb{R}$ is differentiable with monotonically decreasing derivative such that $f'(x) \geq m > 0$ for all $x \in [a,b]$. Prove that
$$\left|\int_a^b \cos f(x)dx\right| \leq \frac{2}{m} \qquad [10]$$
[Hint : Multiply and divide the integrand by $f'(x)$ and integrate by parts.]

5. Test the convergence of the integral
$$\int_0^\infty \frac{1}{x^2 + \sqrt{x}}dx \qquad [15]$$

6. Let $g : [0,1] \to \mathbb{R}$ be continuous. Let $f_n(x) = x^n g(x)$ for $x \in [0,1]$. Show that $\{f_n\}$ converges uniformly on $[0,1]$ if and only if $g(1) = 0$. [10]

[P.T.O.]

7. Starting from a suitable power series and precisely justifying all your steps prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$ [5]

8. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on the interval $[0,1]$. Prove or disprove

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)dx = \int_0^1 f(x)dx$$ [10]

9. Let f be a continuous function on $[a,b]$ such that for all $n = 0, 1, \ldots,$

$$\int_a^b f(x)x^n dx = 0.$$

Show that f is identically 0 on $[a,b]$. [10]

10. (a) Find all real x such that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

(b) Prove that the above series coincides with the Taylor series around 0 of the function $f(x) = \log \frac{1}{1-x}$. [5 + 10 = 15]

11. (a) Let $f(x) = (\pi - |x|)^2$, $x \in [-\pi, \pi]$. Compute the Fourier coefficients of f and show that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

(b) Use (a) to prove that

$$(i) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad (ii) \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

[10 + (4 + 6) = 20]

INDIAN STATISTICAL INSTITUTE
Second Semester Examination: 2004-05
B. Stat. I Year
Probability Theory II

Date: 6.5.05 | Maximum Marks: 70 | Duration: 3 Hours

Answer any five questions.

1 .(a) Let the random variable X have a bounded density function f. Show that $P(X \in C) = 0$ where C is the Cantor set.

(b) Let $X_1, X_2, \ldots$. be i.i.d random variables and N, a non-negative integer valued random variable independent of $X_1, X_2, \ldots$. If f is the density of X, find the density of $Y = X_1 + \cdots + X_N$.

(c) In (b), show that $EY = EX.EN$ if both EX and EN exist.

[6+4+4=14]

2 .(a) If (X,Y) is uniformly distributed on the unit circle $\{(x,y): x^2+y^2 \le 1\}$, show that X and Y are not independent.

(b) If (X,Y) is a pair of independent continuous random variables which are symmetric (i.e. $F_X(t) = 1 - F_X(-t)$ for all t, and similarly for Y), show that

$$P(|X+Y| > t) \ge \frac{1}{2} P(\max(|X|,|Y|) > t)$$

(c) If a random variable X is such that X is independent of itself, show that X is a constant with probability one.

[5+5+4=14]

3 .(a) On a probability space $(\Omega, \mathcal{A}, P)$, let X be a non-negative finite valued random variable such that $X(w) = \lim_{n\to\infty} X_n(w)$, where $X_1, X_2, \ldots$. is a non decreasing sequence of finite valued non-negative random variables. Show that

$$EX = \lim EX_n$$

using only properties of expectations for finite valued random variables.

(b) On $([0,1], \mathcal{B}_{[0,1]}, P)$, where $\mathcal{B}_{[0,1]}$ is the Borel σ-field and P corresponds to uniform distribution on $[0,1]$, let X be the random variable

$$X(t) = \varphi(t) \qquad t \in [0,1],$$

where φ is a non-negative Riemann integrable function on $[0,1]$. Show that $E(X) = \int_0^1 \varphi(t)dt$.

[7+7=14]

P.T.O.

4.(a) Let X_1, X_2 be i.i.d uniform $[0,1]$ random variables and let $U = \max(X_1, X_2)$ and $V = \min(X_1, X_2)$. Write down the conditional distribution of (i) X_1 given V (ii) $U - V$ given V. (No proof need be given.)

(b) Three persons A, B, C arrive at a post office and find the two counters in the post office free. The service of A and B commences immediately, but that of C commences at the time when the first among A or B is discharged. Given that the service times of A,B and C are independent uniform $(0,1)$, what is the probability that C would be the last to leave the post office.

[8+6=14]

5.(a) Let X be a Normal $(0,\sigma^2)$ random variable. Find the density function f_σ of the random variabl

$$U = \frac{1}{X^2}.$$

(b) (In this problem you may use the fact that the convolution of densities f_σ and $f_{\sigma'}$ as in (a) is density $f_{\sigma''}$ where σ'' can be computed from σ and σ'.) Let X and Y be independent Norma random variables with expectation zero and variances σ_1^2 and σ_2^2. Show that $Z = XY/\sqrt{X^2+Y^2}$ has Normal distribution.

[4+10=14

6.(a) Let $G_u(t) = P(Y \le t \mid X = u)$, $u \in \mathbb{R}$, $t \in \mathbb{R}$ be the conditional distribution function of Y give X. For $\alpha, \beta \in \mathbb{R}$, let $Z = 1_{(\alpha,\beta)}(X).Y$. Show that the conditional distribution of Z give $X = u$ is given by

$$\Gamma_u(t) = G_u(t) \text{ if } u \in (\alpha,\beta)$$
$$= 1_{[0,\infty)}(t) \text{ if } u \notin (\alpha,\beta).$$

(Here it is not given that (X,Y) has a joint density.)

(b) Let $a > 0$ and

$$f(x,y) = [(1+ax)(1+ay) - a]\, e^{-x-y-axy}, \quad x>0, y>0$$
$$= 0 \qquad \text{otherwise.}$$

Show that f is the joint density of a pair of random variables (X,Y). Find $E(Y|X)$.

[7+7=

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INDIAN STATISTICAL INSTITUTE
Second Semester Examination: 2004-2005
B. Stat. I Year
Vectors and Matrices II

Date: 9.5.05 | Maximum Marks: 60 | Duration: 3 Hours

This paper contains 67 marks. Maximum you can score is 60. You may answer any part of any question. Classroom notation is used.

1.(a) State and prove (Courant-Fischer's) min-max theorem on eigen values of a real symmetric matrix.

(b) State and prove Sturmian separation theorem of eigen values of real symmetric matrices.

[8+7=15]

2.(a) For a non-indefinite (real symmetric) matrix A, show that $a_{ii}=0$ implies i^{th} row of A is null.

(b) Show that a real matrix A is n.n.d implies $A = B'B$ for some real matrix B.

[5+5=10]

3. Show that for any pair of real matrices A and B of the same order, the characteristic polynomial of

$$H = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

is the product of the characteristic polynomials of the matrices $(A+iB)$ and $(A-iB)$.

[5]

Let A, B and C be real matrices such that $A = B + C$ and $r(A) = r(B) + r(C)$. Then show that A is n.n.d and B is symmetric implies B and C are n.n.d

[5]

Prove or disprove the following stating clearly the results used:

(a) A is a real symmetric matrix of rank r implies all nonzero r^{th} order principal minors have the same sign.

(b) If λ is a nonzero eigen value of A then $1/\lambda$ is an eigen value of every reflexive g-inverse of A.

(c) Given a pair of unit vectors x and y there exists an orthogonal matrix A such that $Ax = y$.

(d) Real part of every eigen value of a real skew symmetric matrix is zero.

(e) For any pair of matrices A and B of same order $A^*A = B^*B$ implies $A = BU$ for some unitary matrix U.

P.T.O.

(f) For any square matrix A, maximum singular value is greater than or equal to $|\lambda|$ for any eigen value λ of A.

(g) A is n.n.d implies A has an n.n.d g-inverse.

(h) Singular values of any idempotent matrix are greater than or equal to 1.

[8 × 4=32]

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INDIAN STATISTICAL INSTITUTE
Second Semestral Examination : (2004-2005)
B. Stat.(Hons.) I Year : Year 2004 - 05
Computing Techniques and Programming II
Date : 11.05.2005
Maximum Marks : 100 Duration : 3 Hours

Answer all the questions

1. Define divided difference of order n of a function $f(x)$. Derive Newton's divided difference formula for interpolation with error term. If $f(x) = 1/(a-x)$, show that

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{(a-x_0)(a-x_1)\ldots(a-x_n)}$$

[4+12+4]

2. (a) Derive Newton-Raphson iterative formula for solving $f(x) = 0$ and give its geometric interpretation. Show that Newton-Raphson iteration converges quadratically.
(b) Locate roots of $x^6 - 2x^2 + 3x - 4 = 0$ using Sturm's theorem. [12+8]

3. (a) Define degree of precision of a quadrature formula. Show that the degree of precision of the Lagrangian quadrature formula based on m points is at least $(m-1)$ and can not exceed $(2m-1)$.
(b) Let $P_0, P_1, \ldots, P_m$ be an orthogonal polynomial sequence over $[a, b]$ relative to weight $w(x)$ and $x_1, x_2, \ldots, x_m$ be the zeros of $P_m(x)$. Show that, for a function $f(x)$ defined over $[a, b]$, the quadrature formula given by

$$\int_a^b w(x) f(x) dx = \sum_{i=1}^{m} H_i f(x_i) + \text{Error}$$

where

$$H_i = \int_a^b w(x) l_i(x) dx,$$

$l_i(x)$ is the ith Lagrangian co-efficient for the points $x_1, x_2, \ldots, x_m$, has degree of precision $(2m-1)$.

(c) Calculate abscissas and weights for 3 point Legendre-Gauss quadrature formula and write down 3 point Legendre-Gauss Quadrature formula. [5+10+5]

4. Explain Jacobi's method for finding eigen values and eigen vectors of real symmetric matrix and write down the algorithm. [20]

5. Fit free cubic spline to the following data and find y for $x = 0.5$ and 2.5.

x	y
0	0
1	0
2	1
3	0
4	0

[20]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination, 2004-05

B. Stat. (Hons.) I

Statistical Methods II

Date: 13. 05. 05 Maximum Marks : 100 Time: 3 hours

The paper carries 110 points. Answer all questions.

1. Consider three random variables Y, X_1 and X_2 defined on the same (finite) sample space and define (a^*, b^*, c^*) to be the minimizer of the function $Q(a,b,c) = E(Y - a - b\,X_1 - c\,X_2)^2$ over $a, b, c \in I\!R$. Show that

(a) (a^*, b^*, c^*) satisfies

$$\begin{pmatrix} 1 & E(X_1) & E(X_2) \\ E(X_1) & E(X_1^2) & E(X_1X_2) \\ E(X_2) & E(X_1X_2) & E(X_2^2) \end{pmatrix} \begin{pmatrix} a^* \\ b^* \\ c^* \end{pmatrix} = \begin{pmatrix} E(Y) \\ E(Y\,X_1) \\ E(Y\,X_2) \end{pmatrix}.$$

(b) Hence or otherwise show that (b^*, c^*) satisfies

$$\begin{pmatrix} \mathrm{Var(X_1)} & \mathrm{Cov(X_1, X_2)} \\ \mathrm{Cov(X_1, X_2)} & \mathrm{Var(X_2)} \end{pmatrix} \begin{pmatrix} b^* \\ c^* \end{pmatrix} = \begin{pmatrix} \mathrm{Cov(Y, X_1)} \\ \mathrm{Cov(Y, X_2)} \end{pmatrix}.$$

[10+15=25]

2(a) Describe an algorithm for generating random samples from the density $f(x) = (0.5 + x)$, $0 \le x \le 1$ (accurate upto any given level of tolerance) starting from a sufficiently long sequence of fair Bernoulli trials.

2(b) Suppose $(X_1, X_2, \ldots, X_n)$ is a sequence of random samples from a Multinomial$(1; 0.25, 0.25, 0.25, 0.25)$ distribution. Using only uniform random numbers give an algorithm for choosing a subset $(Y_1, Y_2, \ldots, Y_S)$ of the original samples which may be thought of as a random sample from Multinomial$(1; 0.5, 0.15, 0.15, 0.2)$. (Here S is a random sample size which may become 0). [15 + 15=30]

3(a) Define the entropy of a probability vector $(p_1, p_2, \ldots, p_k)$.

3(b) For two probability vectors $(p_1, p_2, \ldots, p_k)$ and $(q_1, q_2, \ldots, q_k)$ (all p_i's and q_i's positive) show that $\sum_1^k p_i \log(p_i/q_i) \ge 0$. [5+10=15]

4(a) Let $(1, 2, 2, 0, 0, 0, 4, 3, 1, 2, 0, 0, 2)$ be a random sample from the p.m.f given by $p(0) = C\,\eta$, $p(x) = C\,\theta^x$, for $x = 1, 2, \ldots$. Solve the constant C so that p is a valid probability mass function.

P.T.O

4(b) Find out estimates of η and θ based on method of moments using the data given in (a).

4(c) Suppose the data is generated under the following scheme:

$$y_i = \sin(\frac{2\pi i}{2N+1}) + \epsilon_i,$$

where $i = 0, \pm1, \pm2, \ldots, N$ and $\{\epsilon_i\}$ are random samples from a $N(0,1)$ density. Find out an approximate mean and variance of $\sum_{-k}^{k} c_i y_i$ assuming that $N >> k$. Intuitively describe how you should choose the constants c_i's and k so that the approximate mean is close to zero while the variance is as small as possible.

[5+15+20 = 40]

INDIAN STATISTICAL INSTITUTE
Second Semester Back Paper Examination 2004-05

B. STAT. I YEAR
Vectors and Matrices II

Date: 21.7.05 | Maximum Marks : 100 | Duration : 3 Hours

Classroom notation is used.

1. a) Show that every real non negative definite matrix A can be written as $A = LL'$ for some lower triangular matrix L.

b) Let a real n.n.d. matrix A be partitioned as $A = \begin{pmatrix} A_1 & A_2 \\ A_2' & A_3 \end{pmatrix}$ where A_1 is square.

Then show that $\mathcal{C}(A_2) \subseteq \mathcal{C}(A_1)$.

[10+5=15]

2. a) Define algebraic multiplicity, $am(\lambda, A)$ and geometric multiplicity, $gm(\lambda, A)$ of an eigenvalue λ of a matrix A. Show that $am(\lambda, A) \geq gm(\lambda, A)$.

b) Show that $am(0, A) = gm(0, A)$ if and only if $r(A) = r(A^2)$.

c) Show that A is similar to a diagonal matrix if and only if $am(\lambda, A) = gm(\lambda, A)$ for each eigen value λ of A.

[2+8+5+5=20]

a) Show that every square matrix can be expressed as $A = UTU^*$ for some unitary matrix U and some upper triangular matrix T.

b) Define Normal matrix. Show that A is normal if and only if $A = UDU^*$ for some unitary matrix U and some diagonal matrix D.

[8+5=13]

a) Define Moore Penrose inverse of a matrix under eucledean inner products.

b) Show that every reflexive g-inverse is a Moore-Penrose inverse under suitable inner products.

[2+8=10]

P.T.O

5. Prove the following

a) Eigenvalues of a real symmetric matrix are real .

b) Eigenvalues of (real) skew symmetric have real parts zero.

c) Eigenvalues of an orthogonal matrix are of modules 1.

d) Eigenvalues of an idempotent matrix A are 1 and 0 with am(1,A) = r(A)

e) Eigenvalues of an n.n.d matrix are non negative.

f) For an n.n.d matrix the set of singularvalues is same as the set of nonzero eigenvalues.

g) If λ is a nonzero eigenvalue of the product AB then it is also an eigenvalue of BA (both AB and BA defined).

h) If A and B are of same order such that $\mathscr{C}$(A) = $\mathscr{C}$(B) Then show that A = BC for some non singular matrix C.

[8x4=32]

6\. a) Derive Spectral decomposition of a real symmetric matrix.

b) Define singular values of a real matrix A and derive singular value decomposition of A.

[5+5=10]

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INDIAN STATISTICAL INSTITUTE

Second Semestral Backpaper Examination, 2004-05

B. Stat. (Hons.) I

Statistical Methods II

Date: 22.7.05 Duration: $2\frac{1}{2}$ hours Maximum Marks: 90

Answer all questions

1\. Let (ρ_{ij}), $i, j = 0, 1, 2$ be the correlation matrix of the variables Y, X_1, X_2 and (ρ^{ij}) be its inverse. Prove the following identities.

(a) $$\rho_{02.1} = \frac{\rho_{02} - \rho_{01}\rho_{12}}{\sqrt{1-\rho_{01}^2}\sqrt{1-\rho_{12}^2}}.$$

(b) $$R^2_{0(12)} = 1 - \frac{1}{\rho^{00}},$$

where $R^2_{0(12)}$ denotes the square of multiple correlation between Y and X_1, X_2.

[15+15=30]

2(a) Let $X_1, X_2, \ldots, X_n$ $(n \geq 10)$be random samples from some exponential distribution. Find out the conditional distribution of $X_{(5)}$ given $X_{(1)}$ and $X_{(n)}$ where $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the order statistics. Describe a method of simulating from the above conditional distribution in terms of a uniform random number generator.

2(b) Find out the maximum likelihood estimate of the mean of the exponential distribution based on the first k order statistics only. Will the MLE have larger variance than the estimate based on the full sample, namely, $\bar{X}$ for every $k \leq n$? [15+(10+5)=30]

3(a) Describe an algorithm for generating random samples from the density $f(x) = (0.5 + x)$, $0 \leq x \leq 1$ (accurate upto any given level of tolerance) starting from a sufficiently long sequence of fair Bernoulli trials.

3(b) Suppose $(X_1, X_2, \ldots, X_n)$ is a sequence of random samples from a Multinomial$(1; 0.25, 0.25, 0.25, 0.25)$ distribution. Using only uniform random numbers give an algorithm for choosing a subset $(Y_1, Y_2, \ldots, Y_S)$ of the original samples which may be thought of as a random sample from Multinomial$(1; 0.1, 0.1, 0.7, 0.1)$. (Here S is a random sample size which may become 0). Obtain $E(S)$ for your scheme.

[15 + 15=30]

INDIAN STATISTICAL INSTITUTE
Second Semester Back Paper Examination 2004-05

B. STAT. I YEAR
Probability Theory II

Date: 25.7.05 | Maximum Marks : 100 | Duration : 3 Hours

Answer all questions.

1. Let (X, Y) have the joint density

$$f(x,y) = 2y \quad \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1$$
$$= 0 \quad \text{otherwise}$$

Find the distribution of the random variable $Z = XY$.

[10]

2. Let $X_1, X_2, \ldots$ be i.i.d uniform $(0,1)$ random variables and $S_k = X_1 + X_2 + \ldots + X_k$, $k = 1, 2, \ldots$. Denoting the distribution function of S_k by U_k and the corresponding density by u_k,

(a) show that $u_{n+1}(x) = U_n(x) - U_n(x-1)$.

(b) find u_3.

[12]

(a) Define the Cantor distribution function F.
(b) If X is a random variable which has the Cantor distribution function, find E X.

[6+8=14]

Let $X_1, X_2, \ldots$ be i.i.d random variables with the common distribution exponential (λ).

(a) For $t > 0$, define the random variable N_t to be the number of indices $k \geq 1$ such that $S_k \leq t$. Obtain the distribution of N_t.

(b) Let M be the random variable defined as the smallest $k \geq 2$ such that $X_k < X_1$. Find the distribution of M.

[8+8=16]

P.T.O.

5. Let $X_1, X_2, \ldots X_n$ be i.i.d Uniform (0,1) and let $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ be the corresponding order statistics, i.e., $X_{(1)} = \min(X_1, X_2, \ldots X_n)$, $X_{(2)}$ the next smallest among the $X_1, \ldots X_n$ etc. Find the density of $X_{(k)}$, $1 \le k \le n$.

[12]

6. Let X_1, X_2 be i.i.d exponential (λ) random variables and $X_{(1)} = \min(X_1, X_2)$, $X_{(2)} = \max(X_1, X_2)$. Find the distributions of $X_{(1)}$ and $X_{(2)}$ and the conditional distribution of $X_{(1)}$ given $X_{(2)}$.

[12]

7. Let $X_1, \ldots X_m$ be i.i.d. Normal (0, 1) and let $Y = X_1 + \ldots + X_n$, $Z = X_1 + \ldots + X_m$, $n < m$. Find the joint distribution of (Y, Z) and the conditional distribution of Y given Z.

[12]

8. Show that if $\{G_u(t), u \in \mathbb{R}, t \in \mathbb{R}\}$ and $\{\Gamma_u(t), u \in \mathbb{R}, t \in \mathbb{R}\}$ are both conditional distributions of Y given X, then

$$G_X(t) = \Gamma_X(t)$$

for all t, with probability one.

[12]