INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination
First semester 2009–2010
B. Stat (First year)
Vectors and Matrices I

Date: September 1, 2009

Maximum Marks: 60
Duration: 2 hours 30 minutes

Answer all questions.

State clearly any result that you use in your answer. Unless stated otherwise V is a finite dimensional vector space over a field F. $e_1, e_2, \ldots, e_n$ is the canonical (standard) basis of F^n.

(1) Let ℓ_0 denote the set of all sequences $\{x_n\}_{n\geq 1}$, where $x_n \in \mathbb{R}$ and $\lim_{n\to\infty} x_n = 0$. Define addition and scalar multiplication as follows:

$$\{x_n\} + \{y_n\} = \{x_n + z_n\} \text{ and } \lambda\{x_n\} = \{\lambda x_n\}$$

for $\{x_n\}, \{y_n\} \in \ell_0$ and $\lambda \in \mathbb{R}$.

(a) Prove that ℓ_0 is a vector space over $\mathbb{R}$.

(b) For each positive integer α, let χ_α denote the sequence whose α-th entry is 1 and all other entries are zero. Let $S = \{\chi_\alpha | \alpha = 1, 2, 3, \ldots\}$.
Prove that S is a linearly independent subset of ℓ_0 but it is not a basis.

(c) Determine the span of S in ℓ_0.

4+8+4

(2) Let $S = \{(x, y) \in \mathbb{R}^2 : x + 2y = 0\}$.

(a) Prove that S is a 1-dimensional subspace of $\mathbb{R}^2$.

(b) Define cosets of a subspace in an arbitrary vector space V. Describe geometrically the cosets of S in $\mathbb{R}^2$ and justify your answer.

4+(4+7)

(3) Let $T : V \to V$ be a linear transformation.

(a) Prove that $\dim V = \dim \ker T + \dim \operatorname{Im} T$.

(b) Prove that T is injective if and only if it is surjective.

(c) Consider the following system of linear equations :

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

where $a_i, b_i, c_i, d_i, i = 1, 2, 3$ are all real numbers. Suppose that this system has a unique solution when $d_1 = 0, d_2 = 0$ and $d_3 = 0$. Prove that the system has a unique solution for any set of values of d_1, d_2, d_3.

6+4+6

P.T.O

(4) Let $M_n(\mathbb{R})$ denote the vector space of all $n \times n$ matrices with real entries.

(a) Show that $M_n(\mathbb{R})$ is isomorphic with $\mathbb{R}^{n^2}$.

(b) Prove that for any A in $M_n(\mathbb{R})$ there is a positive integer k such that

$$a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_k A^k = 0$$

for some real numbers a_i, $i = 0, \ldots, k$. Here A^m denotes the m-fold product of A with itself.

4+6

(5) Let P_n denote the vector space of all polynomials of degree less than or equal to n with real coefficients (under usual addition and scalar multiplication). Define two linear transformations $S : P_n \to P_{n+1}$ and $T : P_{n+1} \to P_n$ as follows: For any polynomial $p(x)$ in P_n,

$$S(p(x)) = xp(x) \quad \text{and} \quad T(p(x)) = p'(x),$$

where $p'(x)$ denotes the derivative of the polynomial $p(x)$.

(a) Write down an ordered basis for P_n for any n.

(b) If A and B denote the matrices of S and T respectively relative to these ordered bases, then find the product matrix $A.B$.

2+7

INDIAN STATISTICAL INSTITUTE

MID-TERM EXAMINATION (2009–10)

B. STAT. I YEAR

ANALYSIS I

Date : 04.09.2009 Maximum Marks : 80 Time : $2\frac{1}{2}$ hours

The paper carries 85 marks. Maximum you can score is 80. Precisely justify all your steps. Carefully state all the results you are using. You may use properties of $\sin x$, $\log x$ etc. learnt in 10+2 without proof.

1. In each of the following cases, decide whether the given set $A \subseteq \mathbb{R}$ is bounded above (and/or below) and if it is, find its supremum (and/or infimum).

(a) $A = \{x \in \mathbb{R} : 3x^2 - 10x + 3 < 0\}$.

(b) $A = \left\{ \left| x + \frac{1}{x} \right| : x \neq 0 \right\}$. [3 + 4 = 7]

2. Let $A \subseteq \mathbb{R}$ be bounded above. Show that there is a sequence $\{a_n\}$ taking values in A such that $\lim_{n\to\infty} a_n = \sup A$. [5]

3. Given two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, define a sequence $\{c_n\}$ as follows : $c_{2n} = a_n$ and $c_{2n-1} = b_n$.

(a) If $\{a_n\}$ converges and $\lim_{n\to\infty}(a_n - b_n) = 0$, prove that $\{c_n\}$ converges. [8]

(b) Can you drop the condition "$\{a_n\}$ converges" in (a)? [5]

4. In each of the following cases, if the given sequence $\{a_n\}$ is convergent, find its limit. If it is not, find its lim sup and lim inf. [5 + 5 = 10]

(a) $a_n = \dfrac{3^n + (-2)^n}{3^{n+1} + (-2)^{n+1}}$ (b) $a_n = \dfrac{n}{7} - \left[\dfrac{n}{7}\right]$,

where $[x]$ is the largest integer $\leq x$.

P. T. O

5. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers such that $\{a_n\}$ is bounded and $\lim_{n\to\infty} a_n/b_n = \infty$. What is the strongest conclusion you can draw about $\{b_n\}$? [10]

6. For any sequence $\{a_n\}$ (not necessarily bounded), let $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} a_k$. Show that

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} \sigma_n \le \limsup_{n\to\infty} \sigma_n \le \limsup_{n\to\infty} a_n.$$

In this context, explain the significance of the two examples : (i) $a_n = (-1)^{n+1}$ and (ii) $a_n = (-1)^{n+1}n$. [15 + 5 = 20]

7. Test the convergence of the following series : [5 + 5 = 10]

$$(a)\ \sum_{n=1}^{\infty} \frac{n^3[\sqrt{2}+(-1)^n]^n}{3^n}, \qquad (b)\ \sum_{n=1}^{\infty} \sin\left(\frac{n^2}{2^n}\right)$$

8. Find all real x and s such that the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^s}$ converges. For what values of x and s does the series converge absolutely? [10]

Probability Theory I
B. Stat. Ist Year Semester 1
Indian Statistical Institute

Mid-semestral Examination

Time: 2 Hours 30 minutes   Full Marks: 30

Date: September 7, 2009

1. During IPL Season II, Boomer was running a promotion campaign where each packet of Boomer contained a logo of one of the eight competing teams. I bought 10 packets of Boomer. What is the probability that I got the logos of both the finalists of IPL I, as well as, that of Kolkata Knight Riders (who was not one of the finalists of IPL I)? [5]

2. An urn contains r red and b black balls. Balls are selected at random without replacement. What is the probability that the $(k+1)$-th ball drawn will be the first red ball? [4]

3. Conclude, with justification, whether the following statements are true or false:
   (a) If $P(A|B) = 1$ and $P(A|B^c) = 1$, then $P(A) = 1$. [4]
   (b) The event A is said to *attract* the event B, if $P(A|B) > P(A)$. If A attracts B and B attracts C, then A attracts C. [5]

4. Prove, using induction, that for any $n (\ge 2)$ events $A_1, \ldots, A_n$,

$$P\left(\cup_{r=1}^n A_r\right) \le \sum_{r=1}^{n} P(A_r) - \sum_{r=2}^{n} P(A_r \cap A_1).$$

Hence, or otherwise, prove Kounias' inequality:

$$P\left(\cup_{r=1}^n A_r\right) \le \min_{1\le k\le n} \left\{ \sum_{r=1}^{n} P(A_r) - \sum_{\substack{r=1\\ r\ne k}}^{n} P(A_r \cap A_k) \right\}.$$

[4+2=6]

5. In the random walk problem, let R_n be the number of distinct values taken by $S_0, S_1, \ldots, S_n$. Show that

$$P[R_n = R_{n-1} + 1] = P[S_1 S_2 \ldots S_n \ne 0].$$ [6]

(Hint: You may use approapriate dual events obtained by reversing a path.)

INDIAN STATISTICAL INSTITUTE

Midsemester Examination : (2009-2010)

B. Stat 1st Year

Statistical Methods -I

Date: 09. 09. 2009 Maximum marks: 60 Duration: 2 hours.

Note: This paper carries 66 points. Maximum you can score is 60.

1. (a) In a (hypothetical) study, it was found that the percentage of incidence of hypertension was less for people who regularly exercised. The study concluded that regular exercise prevents hypertension. Give reasons for your agreement or disageement.

   (b) Dividing the two groups (people who regularly exercise and who don't) into young and old, illustrate with hypothetical numbers how the percentage of incidence of hypertension can be higher for people who regularly exercise whether young or old, even when the overall percentages agree with the study.

   (c) Explain briefly how will you design a statistical study to check their conclusion? [5+10+5]

2. Suppose there are two sets of values of a varible x with n_1 and n_2 values respectively.

   (a) Let the geometric means of the two sets be G_1, and G_2 respectively. Show that the geometric mean of all $n_1 + n_2$ values taken together will lie between G_1 and G_2.

   (b) Suppose the harmonic means of the two sets are H_1 and H_2 respectively. Will the harmonic mean of all $n_1 + n_2$ values taken together will lie between H_1 and H_2? Justify your answer. [8+8]

3. Using the Cauchy-Scwartz inequality or otherwise, prove the following results:

   (a) Gini's mean difference cannot be greater than $\sqrt(2)$ times the standard deviation.

   (b) The measure of Kurtosis β_2 cannot be smaller than the measure of skewness β_1 for any variable. [10+10]

4. The first four moments of a distribution about the value 4 (i.e. $\frac{1}{n}\sum_{i=1}^{n}(X_i - 4)^r, r = 1, \ldots, 4$) are -1.5, 17, -30 and 108, respectively. Find the first four central moments. [10]

Indian Statistical Institute
B.Stat Mid-Semester Examination-2009
Computational Techniques and Programming-I
Full marks: 60
Time: ~~Three~~ Two hrs.
Date: 11/09/2009

Answer all the questions.

1. A) Convert the following and justify with mathematical reasons why your conversion procedure will lead to correct results.

i) $(11011)_2 = (\; ? \;)_{10}$

ii) $(95269)_{10} = (\; ? \;)_8$ [4]

B) I) Write a C-program with full documentation for the above (ii) conversion procedure. You should provide the flowchart/algorithm for the procedure.

II) Discuss the following three cases of designing compilers (notations have usual meanings).

(A) $C_A{}^{LA}$ is available, to design $C_B{}^{LB}$

(B) $C_A{}^{LA}$ is available, to design $C_A{}^{L'A}$

(C) Nothing is available, to design $C_A{}^{LA}$

You should explain sufficiently the necessary steps of your design formulations.

[8+8]

2. On using at least two '**for**' loops write the 'C' program for each of these patterns with complete documentation.

A)
```
     6
    5 6
   4 5 6
  3 4 5 6
 2 3 4 5 6
1 2 3 4 5 6
```

B)
```
        1
      1 0 1
    1 0 1 0 1
  1 0 1 0 1 0 1
1 0 1 0 1 0 1 0 1
```

```
C) * * * * * *        D) 1 0 1 0 1 0 1 0 1 0 1
   * * * * *                1 0 1 0 1 0 1 0 1
   * * * *                    1 0 1 0 1 0 1
   * * *                        1 0 1 0 1
   * *                            1 0 1
   *                                1
```

[

3. (a) Explain elaborately why NOR and NAND logics are considered as universal?

(b) Why any Boolean function can be expressed as sum of product form (SOP) as well product of sum form (POS)? Explain with examples.

(c) Explain why 2's complement representation is more important and useful than oth of representations?

(d) (I) In connection with computer memories, what is meant by "random access"?
(II) Whether both RAM and ROM are 'random accessed'? [4

INDIAN STATISTICAL INSTITUTE

Semester Examination
First semester
B. Stat (First year) 2009–2010
Vectors and Matrices I

Date: November 23, 2009 — Maximum Marks: 70
Duration: 3 hours

Answer all questions.
State clearly any result that you use in your answer.
$M_n(\mathbb{R})$ denotes the vector space of all $n \times n$ matrices over reals.
For any matrix A, $\rho(A)$ denotes the rank of A.

(1) (a) Let $T, S : V \to W$ be two linear transformations from a vector space V to another vector space W. Show that if T and S take the same value on a basis of V, then $T = S$.
(b) Let $f : M_n(\mathbb{R}) \to \mathbb{R}$ be a linear transformation that satisfies the relation $f(AB) = f(BA)$ for all $A, B \in M_n(\mathbb{R})$ and $f(I_n) = n$. Show that $f(A) = \text{trace}(A)$ for all $A \in M_n(\mathbb{R})$.

3+12

(2) Find a non-singular matrix P such that PAP^T is diagonal, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 9 \end{pmatrix}.$$

Determine the rank of A. 8

(3) Suppose that an $n \times n$ matrix H is in the Hermite canonical form.
(a) Prove that $H^2 = H$. Hence, show that the sum of the column space of H and the column space of $I - H$ is a direct sum.
(b) Suppose that $Hx = b$ is a consistent system. Prove that, a general solution of $Hx = b$ is given by $b + (I - H)z$, where z is an n-vector.

(7+3)+5

(4) Let $a \neq b$ and

$$A = \begin{pmatrix} a & 0 & b \\ 2a & 0 & 2b \\ 3a & 0 & 3b \end{pmatrix}.$$

Find a rank factorisation of A. 5

P. T. O

(5) If A is a non-singular $n \times n$ matrix, then prove that the rank of a $k \times l$ submatrix of A is at least $k + l - n$. 6

(6) (a) Let A be a matrix such that $\rho(A) = r$. Prove that there is a submatrix of order $r \times r$ which is non-singular. If an $r \times r$ submatrix of A is non-singular, then what can you say about $\rho(A)$? Justify your answer.

(b) Let A and B be two real matrices of the same order. Suppose that the rank of $\rho(A) = 2$. Prove that there exists a real number $\delta > 0$ such that the rank of $\rho(A + \lambda B) \geq 2$ for $|\lambda| < \delta$. (Hint: A 2×2 matrix is non-singular iff its determinant is non-zero.) (5+3)+8

(7) Let $a_i x + b_i y = c_i$, $i = 1, 2, 3$ represent three distinct straight lines in $\mathbb{R}^2$. Suppose that the lines are not concurrent but any two of them intersect in a point. What can you say about the consistency of the system? Determine the rank of the matrix A whose rows are (a_i, b_i, c_i), $i = 1, 2, 3$. 6

INDIAN STATISTICAL INSTITUTE

FIRST SEMESTER EXAMINATION (2009-2010)

B. STAT (First Year)

Computing Techniques and Programming I

Full Marks-100, Duration-Three hours, Date-25/11/09

Note: Answer all the Questions.

1. (a) Give the algorithm/flowchart of producing a 3×3 Magic Square where the entire row sums, column sums, and diagonal sums are equal. You have to give the dry run of your algorithm.
(b) (I) Give the corresponding C program of the above algorithm/flowchart with full documentation.
(II) Give also the C program to test in all possible ways whether a given square is really a magic square. [8+ (6+6)]

2. (a) With the help of C program you have to prepare two files (input file and output file) which have to be program controlled. In the input file go on adding arbitrary real numbers in the form of two-dimensional array, say $m\times n$. In the output file print all the elements of the said two-dimensional array ($m\times n$) and also the average of all the numbers.
(b) (I) Write a C program to count the number of characters including the blank spaces in a given text.
(II) Write a C program to count the number of words in the above text file
[10+ (4+6)]

3. (a) Given any real number x, you have to calculate:
$p_0(x) = a_0 + a_1x + a_2x^2 + a_3x^3$
$= a_0 + x.p_1(x)$, where $p_1(x)$ is recursively defined as $p_1(x) = a_1 + x.p_2(x)$
and in general $p_n(x) = a_n + x.p_{n+1}(x)$.
For the purpose of the calculation of $p_0(x)$, you have to use one '**for**' loop and one given value of x.

(b) (i) What do you understand by big O in the context of the complexity of algorithms?
Most of the classical sort algorithms take the time ranging from $O(n.\log n)$ to $O(n^2)$ where n denotes the number of elements to be sorted.

Consider two different values for n. The first value is n_0 whereas the second one is $100.n_0$. What would be the effect of the two values on $O(n.\log n)$ to $O(n^2)$?

(ii) For a cyclic redundancy code, an n-bit encoded message (where k is the number of message bits) has to be formed. What should be the length of check bits?

(iii) For four message bits 1100 and the generator polynomial x^3+x+1, find the encoded message and hence explain how you would proceed to locate all the single bit errors.

[6 + (6 +2 +6)]

4. (a) Distinguish between

(i) Recursive algorithm and Iterative algorithm

(ii) Subroutine call and Interrupt scheme

(b) What is a concurrent program? Distinguish between a procedure call and a process creation. [10 + 10]

5. (a) In the theory of 2's complement arithmetic describe all the cases of addition of two numbers (which may be positive or negative).

(b) We find that in two occasions we encounter overflow or underflow. In the event of any such occasion how would it be notified within the computer for the purpose of creating an interrupt?

[10+10]

INDIAN STATISTICAL INSTITUTE

First Semestral Examination : (2009-2010)

B. Stat 1st Year

Statistical Methods I

Date: 27. 11. 2009 Maximum marks: 100 Duration: 3 hours.

Note: Answer all questions. This paper carries 106 points. Maximum you can score is 100.

1. Suppose that male and female students are applying to a particular college for admission in three different streams. Illustrate Simpson's paradox with hypothetical data for this scenario. [12 points]

2. Consider a 2 x 2 contingency table with two attributes A and B each having two forms A and α, and B and β, respectively. Suppose the total frquency is n.

   Writing the cell frequencies $f_{AB}, f_{A\beta}, f_{\alpha B}$ and $f_{\alpha\beta}$ as a, b, c and d respectively, express the coefficients of contingency C and T in terms of the cell frequencies. [15 points]

3. Using the Cauchy-Scwartz inequality or otherwise, prove the following results:

   (a) The difference between the mean and median of a variable never exceeds the standard deviation for any variable. [10 points]

   (b) The measure of Kurtosis β_2 cannot be smaller than $\beta_1 + 1$ for any variable where β_1 is the measure of skewness. [10 points]

4. Let there be k groups of data on x and y, with means $\bar{x}_i$ and $\bar{y}_i$, variances s^2_{xi} and s^2_{yi}, correlations r_i, and number of observations n_i for $i = 1, 2, \ldots, k$.

   (a) Express the correlation of the combined data in terms of the above quantities. [12 points]

   (b) using part(a) or otherwise, give a concrete example with specific numbers where even though each r_i is positive, the correlation of the combined data is negative. [12 points]

[P.T.O.]

Consider two different values for n. The first value is n_0 whereas the second one is $100.n_0$. What would be the effect of the two values on $O(n.\log n)$ to $O(n^2)$?

(ii) For a cyclic redundancy code, an n-bit encoded message (where k is the number of message bits) has to be formed. What should be the length of check bits?

(iii) For four message bits 1100 and the generator polynomial x^3+x+1, find the encoded message and hence explain how you would proceed to locate all the single bit errors.

[6 + (6 +2 +6)]

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[10+10]

INDIAN STATISTICAL INSTITUTE

First Semestral Examination : (2009-2010)

B. Stat 1st Year

Statistical Methods I

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Note: Answer all questions. This paper carries 106 points. Maximum you can score is 100.

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2. Consider a 2 x 2 contingency table with two attributes A and B each having two forms A and α, and B and β, respectively. Suppose the total frquency is n.

   Writing the cell frequencies $f_{AB}, f_{A\beta}, f_{\alpha B}$ and $f_{\alpha\beta}$ as a, b, c and d respectively, express the coefficients of contingency C and T in terms of the cell frequencies. [15 points]

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   (a) The difference between the mean and median of a variable never exceeds the standard deviation for any variable. [10 points]

   (b) The measure of Kurtosis β_2 cannot be smaller than $\beta_1 + 1$ for any variable where β_1 is the measure of skewness. [10 points]

4. Let there be k groups of data on x and y, with means $\bar{x}_i$ and $\bar{y}_i$, variances s_{xi}^2 and s_{yi}^2, correlations r_i, and number of observations n_i for $i = 1, 2, \ldots, k$.

   (a) Express the correlation of the combined data in terms of the above quantities. [12 points]

   (b) using part(a) or otherwise, give a concrete example with specific numbers where even though each r_i is positive, the correlation of the combined data is negative. [12 points]

[P.T.O.]

5. (a) Consider r_i and s_i, $i = 1, \ldots, n$, data on ranks assigned (without ties) by two judges to n participants in a competition. Suppose $r_i = s_i + 1$ for $1 = 1, 2, \ldots, n-1$, and $r_n = 1$ and $s_n = n$. If the Spearman's rank correlation coefficient is 0.5, find the value of n. [10 points]

   (b) Derive the intra-class correlation coefficient when there are equal number of members in each class. [10 points]

6. The average height and weight of a group of students turned out to be 5 ft 6 inches and 65 kilograms respectively. The correlation between heights and weights was found to be 0.6. Using the regression equation for predicting weight from height, the estimated weight of a 6 ft tall student was calculated to be 80 kilograms. Predict the height of a student whose weight is 60 kilograms. [15 points]

PROBABILITY THEORY I

B. STAT. IST YEAR SEMESTER 1

INDIAN STATISTICAL INSTITUTE

Semestral Examination

Time: 3 Hours Full Marks: 50

Date: December 1, 2009

1. Let $X \sim \mathrm{Bin}(n; p)$ and $Y \sim \mathrm{NB}(k; p)$. Show that $\mathrm{P}[X \geqslant k] = \mathrm{P}[Y \leqslant n-k]$. [6]

2. An urn has n tickets numbered $1, 2, \ldots, n$, which are drawn at random, one at a time, without replacement. What is the probability that for at least one ticket, its number will coincide with its number of draw? [8]

3. Let $X \sim \mathrm{Poi}(\lambda)$. Show that $\mathrm{P}[X \leq \lambda/2] \leq (2/e)^{\lambda/2}$. [10]

4. The conditional distribution of X given Y is Poisson with parameter Y and Y has Poisson distribution with parameter μ. Show that $X+Y$ has probability generating function given by $\exp\left(\mu(se^{s-1} - 1)\right)$. [8]

5. A gardener in ISI has planted an exotic plant in December 2008. A plant from one seed can have only 1 or 2 flowers in the first winter with probability $\frac{1}{2}$ and $\frac{1}{4}$ respectively, otherwise it dies without blooming. Each flower produces 2 or 3 seeds with equal probability. During a summer the plant dies, but each seed survives independently with probability $\frac{1}{5}$.
   What is the expected number of flowers in December this year?
   Write down the generating function of the number of seeds available for planting in December this year. [3+5=8]

6. Show that $\max\{u, v\} = \frac{u+v}{2} + \frac{|u-v|}{2}$.
   Hence, using Cauchy-Schwarz inequality, show that, if X and Y are random variables with common mean 0, common variance 1 and correlation ρ, then

$$\mathrm{E}\left[\max\{X^2, Y^2\}\right] \leq 1 + \sqrt{1-\rho^2}.$$

[2+8=10]

INDIAN STATISTICAL INSTITUTE

FIRST SEMESTER SEMESTRAL EXAMINATION (2009–10)

B. STAT. I YEAR

ANALYSIS I

ate : 04.12.2009 Maximum Marks : 100 Time : 3 hours

The question carries 105 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. (a) For every $m \in \mathbb{N}$, show that $\frac{1}{\sqrt{m+1}} < 2(\sqrt{m+1} - \sqrt{m}) < \frac{1}{\sqrt{m}}$.

   (b) Deduce that for every $n \in \mathbb{N}$, $2(\sqrt{n} - 1) < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1$.

   (c) If $a_n = 2\sqrt{n} - \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$, show that $\{a_n\}$ converges. [5 + 5 + 10 = 20]

2. (a) Find all $x \in \mathbb{R}$ for which the power series $\sum_{n=1}^{\infty} nx^n$ converges.

   (b) Assuming that you are allowed to differentiate a geometric series "term by term", find a closed form expression for the sum of the above series.

   (c) Show that wherever the series converges, it converges to the above expression.

   [5 + 5 + 10 = 20]

   [Hint : Show that the difference between the n-th partial sum and the "sum" equals $\frac{(n+1)x^{n+1} - nx^{n+2}}{(1-x)^2}$.]

3. Let $f, g : I \to \mathbb{R}$. Define

$$h(x) = \max\{f(x), g(x)\}, \quad x \in I.$$

   (a) If f and g are continuous at $x_0 \in I$, show that h is continuous at x_0.

   (b) If f and g are differentiable at x_0, is h differentiable at x_0? [10 + 5 = 15]

4. Let f be a continuous real function on $\mathbb{R}$ such that for all $x \neq 0$, $f'(x)$ exists and $\lim_{x \to 0} f'(x) = 3$. Is f differentiable at 0? [5]

P. T. O

5. Let $f(x) = 1 - x^{2/3}$. Show that $f(1) = f(-1) = 0$, but $f'(x)$ is never zero o[n] $[-1, 1]$. This seems to contradict a theorem you know. Which theorem? Does i[t] really contradict the theorem? Justify! [10

6. Let $f : (0, 1] \to \mathbb{R}$ be a differentiable function with f' bounded on $(0, 1]$. Define

$$a_n = f\left(\frac{1}{n}\right), \quad n \geq 1.$$

Show that $\{a_n\}$ is a convergent sequence. [10

7. Let f be a thrice differentiable function on $(0, 1)$ such that $f(x) \geq 0$ for all $x \in (0, 1)$ If $f(x) = 0$ for *at least two* values of $x \in (0, 1)$, prove that $f'''(c) = 0$ for som[e] $c \in (0, 1)$. [10

8. Let $f : I \to \mathbb{R}$. We say that f is *locally bounded* if for every $x \in I$, there is an interva[l] $(x - \delta, x + \delta)$ such that f is bounded on $(x - \delta, x + \delta) \cap I$.

   (a) Give an example to show that a locally bounded function need not be bounded

   (b) Show that any locally bounded function on $I = [a, b]$ is bounded. [5 + 10 = 15

First Semester Examinations (2009-2010))
B Stat – 1 year
Remedial English
100 marks
1 ½ hours

Date: 04.12.2009

1. Write an essay on any one of the following topics. Five paragraphs are expected.
a) Autobiography of a Pencil.
b) Child Labour.
c) A Winter Morning. (60 marks)

2. Fill in the blanks with appropriate prepositions:
a) Sir Leoline is weak _____ health.
b) I went _____ the school _____ the other side of the road.
c) We argued _____ the matter _____ dawn.
d) Can we meet _____ Mocambo _____ lunch?
e) If she were _____ roast you alive _____ fire....
f) I was angry _____ him _____ lying.
g) I wish I could leave _____ you _____ Japan.
h) Don't throw stones _____ the poor animal because she is _____ pain.
i) I don't wish _____ eat samosas.
j) There is every reason _____ us _____ move out.
k) This building could be developed _____ a theatre.
l) Let us try and protect the species _____ extinction. (20 marks)

3. Fill in the blanks with appropriate words:
Plotinus (A.D. 204-70), the founder _____ Neoplatonism, is _____ last of the great philosophers of antiquity. _____ life is almost coextensive with _____ of the most disastrous periods _____ Roman history. Shortly before _____ birth, the army had become conscious of _____ power, and _____ adopted the practice _____ choosing emperors in return _____ monetary rewards and assassinating _____ afterwards to give occasion for a renewed sale of _____ empire. _____ preoccupations unfitted the soldiers for the defence _____ the frontier, and permitted vigorous incursions _____ Germans _____ the north and Persians _____ the east. War and pestilence diminished the population of _____ empire by about a third, while increased taxation _____ diminished resources caused financial ruin _____ even those provinces to which no hostile forces penetrated. (20 marks)

INDIAN STATISTICAL INSTITUTE

Semester Examination (Backpaper)
First semester

B. Stat (First year)2009–2010

Vectors and Matrices I

Date: 8.1.2010 | Maximum Marks: 100
Duration: 4 hours

Answer all questions.
State clearly any result that you use in your answer.
Unless stated otherwise, a matrix is defined over an arbitrary field F.
For any matrix A, $\rho(A)$ denotes the rank of A.

(1) Let $T : \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation with kernel S. What are the possible dimensions of S? Justify your answer. 5

(2) Let $P : V \to V$ be a projection operator. Define a projection operator Q such that $P + Q = \text{Id}$ and $P \circ Q = 0$. 6

(3) Let $T : \mathbb{R}^2 \to \mathbb{R}$ be a non-zero linear functional with kernel K.
   (a) Prove that the sets $T^{-1}(x)$, where $x \in \mathbb{R}^2$, are precisely the cosets of K.
   (b) 'A complementary subspace of K intersects $T^{-1}(x)$ in exactly one point' - Explain this geometrically. 5+5

(4) Let V be a vector space with a basis $B = \{v_1, v_2, \dots, v_n\}$ and let $T : V \to V$ be a linear transformation such that $T(B) = B$. If P denotes the matrix of T relative to the ordered basis $v_1, v_2, \dots, v_n$ then prove that $PP^T = P^T P = I_n$. 10

(5) Let S be an $n \times n$ real matrix such that $ST = TS$ for any $n \times n$ matrix T. Prove that S is a scalar matrix. 10

(6) (a) Let A_1 and A_2 be two matrices of the same order such that $\rho(A_1 + A_2) = \rho(A_1) + \rho(A_2)$. If (P_1, Q_1) is a rank factorisation of A_1 and (P_2, Q_2) is a rank factorisation of A_2, then show that (P, Q) is a rank factorisation of $A_1 + A_2$, where $P = [P_1\ P_2]$ and $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$.

P.T.O

(b) Let A be an $n \times n$ matrix of rank r. Find a matrix B of rank $n-r$ such that $A+B$ is non-singular.

7+7

(7) Let A be a real $m \times n$ matrix.

(a) Show that $\rho(A^T A) = \rho(A)$.

(b) If $D = \text{diag}(d_1, d_2, \ldots, d_n)$ is a real diagonal matrix, where d_i's have the same sign, then prove that $\rho(A^T DA) = \rho(A)$.

5+7

(8) Let A be a non-singular matrix of order n.

(a) If C is any matrix of order $m \times n$, then show that there exists a matrix X satisfying $XA = C$.

(b) Let

$$M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}.$$

Prove that $\rho(M) = \rho(A) + \rho(CA^{-1}B)$.

5+

(9) Find when the following system is consistent:

$$\begin{aligned} x + y + z &= 0 \\ \alpha x + \beta y + \alpha z &= 0 \\ \beta x + \alpha y + \alpha z &= \gamma \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

8

(10) Let A be a real symmetric matrix of order n. Prove that there exists a non-singular matrix P such that PAP^T is of the form $\begin{pmatrix} d_1 & 0 \\ 0 & B \end{pmatrix}$, where $d_1 \in \mathbb{R}$ and B is a matrix of order $n-1$. Hence prove that there exists a non-singular matrix Q such that QAQ^T is a diagonal matrix.

8+5

INDIAN STATISTICAL INSTITUTE

First Semester Backpaper Examination (2009–10)

B. Stat. I Year

Analysis I

te : 11. 1. 2010 | Maximum Marks : 100 | Time : 3 hours

Precisely justify all your steps. Carefully state all the results you are using.

1. (a) Given nonempty subsets S and T of $\mathbb{R}$ such that

$$s \leq t, \qquad \text{for every } s \in S \text{ and } t \in T, \tag{1}$$

show that S has a supremum, T has an infimum and

$$\sup S \leq \inf T. \tag{2}$$

(b) "If the inequality in (1) is strict for every $s \in S$ and $t \in T$, then the inequality in (2) is also strict." — True or false? Justify. [10 + 5 = 15]

2. Given that $\sum_{n=1}^{\infty} a_n$ converges absolutely, show that each of the following series also converges absolutely

$$(a)\ \sum_{n=1}^{\infty} a_n^2 \qquad (b)\ \sum_{n:a_n \neq -1} \frac{a_n}{a_n + 1} \qquad (c)\ \sum_{n=1}^{\infty} \frac{a_n^2}{a_n^2 + 1}$$

[6 + 7 + 2 = 15]

3. If x is not an integer multiple of 2π, show that $\sum_{k=1}^{n} \cos kx = \dfrac{\sin(n+1/2)x - \sin(x/2)}{2\sin(x/2)}$.

Using this show that $\sum_{n=1}^{\infty} \dfrac{\cos nx}{n^s}$ converges for all $s > 0$. [5 + 5 = 10]

4. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and define $g : [a, b] \longrightarrow \mathbb{R}$ by putting $g(a) = f(a)$ and

$$g(x) = \max\{f(t) : t \in [a, x]\} \quad \text{for } x \in (a, b].$$

Show that g is continuous on $[a, b]$. [15]

P.T.O

5. Define uniform continuity. Let I be an interval. Suppose $f : I \to \mathbb{R}$ is differentiabl and the derivative is bounded on I. Show that f is uniformly continuous on I [5+5=1C

6. Let f be a continuous real function on $\mathbb{R}$ such that for all $x \neq 0$, $f'(x)$ exists an $\lim_{x \to 0} f'(x) = 3$. Is f differentiable at 0? [5

7. Let $f(x) = 1 - x^{2/3}$. Show that $f(1) = f(-1) = 0$, but $f'(x)$ is never zero o $[-1, 1]$. This seems to contradict a theorem you know. Which theorem? Does i really contradict the theorem? Justify! [1(

8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. [1(

9. Let f be a thrice differentiable function on $(0,1)$ such that $f(x) \geq 0$ for all $x \in (0,1$ If $f(x) = 0$ for *at least two* values of $x \in (0,1)$, prove that $f'''(c) = 0$ for som $c \in (0,1)$. [1(

PROBABILITY THEORY I

B. STAT. IST YEAR SEMESTER 1

INDIAN STATISTICAL INSTITUTE

Backpaper Examination

Time: 3 Hours Full Marks: 100

Date: January 14, 2010

1. A hat contains n coins, k of which are unbiased and the rest produce heads with probability $\frac{2}{3}$. A coin is drawn at random from the hat and tossed twice giving a head first and then a tail. Given this information, find the conditional probability that an unbiased coin was chosen. [15]

2. If X and Y are independent random variables, show that $\mathrm{Var}[XY] = \mathrm{Var}[X]\,\mathrm{Var}[Y] + \mathrm{Var}[X](\mathrm{E}[Y])^2 + \mathrm{Var}[Y](\mathrm{E}[X])^2$. [15]

3. Suppose $\{p_k\}_{k \geq 1}$ be a sequence of numbers in $(0, 1)$ so that $\lim_{k \to \infty} k(1 - p_k) = \lambda$. Then show that $\mathrm{NB}(k; p_k)$ probability mass functions converge to $\mathrm{Poi}(\lambda)$ probability mass function. [15]

4. Let $X \sim \mathrm{Poi}(\lambda)$. Show that $\mathrm{P}[X \geq 2\lambda] \leq (e/4)^{\lambda}$. [20]

5. Assume $0 < p < 1/2$, $p + q = 1$. Show that $\varphi(s) = \sqrt{q - p}\sqrt{\frac{q+ps}{q-ps}}$ is an infinitely divisible probability generating function. [15]

6. In a simple symmetric random walk, let T be the time of first return to zero.
   Write down the probability mass function of T. (No proof required.)
   Using Stirling's formula or otherwise, show that $\mathrm{E}[T^{\alpha}] < \infty$ if and only if $\alpha < \frac{1}{2}$. [5+15=20]

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: 2009-2010

B.Stat. (Hons.) 1st Year. 2nd Semester

Vectors and Matrices II

Date: February 22, 2010 **Maximum Marks: 70** **Duration: 3 and 1/2 hours**

• This question paper carries 75 points. Answer as much as you can. However, the maximum you can score is 70.

• You should provide as much details as possible while answering a question. You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

---

1. Let f be a map from $\mathcal{F}^n \times \cdots \times \mathcal{F}^n$ (n copies) to $\mathcal{F}$. Suppose f satisfies the following conditions: (i) $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = 0$ whenever two of the $\mathbf{x}_i$'s are equal, (ii) $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is linear in each $\mathbf{x}_i$ when all the other $\mathbf{x}_j$'s are kept fixed. Show that $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = c|\mathbf{A}|$ for some constant $c \in \mathcal{F}$, where $\mathbf{A} = [\mathbf{x}_1 : \cdots : \mathbf{x}_n]^{\mathrm{T}}$. [12]

2. Let

$$\mathbf{A}_1 := [a],\ \mathbf{A}_2 := \begin{bmatrix} a & b \\ c & a \end{bmatrix},\ \mathbf{A}_n := \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{bmatrix}_{n\times n} \quad \text{for } n \geq 3.$$

Let $a = 1 + bc$. Show that $|\mathbf{A}_n| = 1 + bc + (bc)^2 + \cdots + (bc)^n$. [11]

3. Show that the intersection of the two distinct planes $a_1x + b_1y + c_1z = 0$ and $a_2x + b_2y + c_2z = 0$ in $\mathbb{R}^3$ is

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

provided the denominators are non-zero. [7]

4. Let $N(\cdot)$ be a norm on $\mathbb{R}^n$. For an $n{\times}n$ matrix $\mathbf{A}$, define $\|\mathbf{A}\|_N = \sup_{\mathbf{x}\neq 0}[N(\mathbf{A}\mathbf{x})]/[N(\mathbf{x})]$. Write $\mathbf{A} = ((a_{ij}))$, $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$.

(a) Show that $\|\cdot\|_N$ is a norm on $\mathbb{R}^{n\times n}$.

(b) Let $N(\mathbf{x}) := \sum_{i=1}^n |x_i|$. Show that $\|\mathbf{A}\|_N = \max_j \sum_{i=1}^n |a_{ij}|$. [5+7=12]

[P.T.O.]

5. Consider the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{A} \mathbf{x}$ on $\mathbb{R}^3$ where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}.$$

Find an orthonormal basis B of $S := \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$ and then extend it to an orthonormal basis C of $\mathbb{R}^3$. [6+4=10]

6. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ form a basis of $\mathbb{C}^n$.

(a) Show that $\mathbf{A} = \left(\sum \mathbf{u}_i \mathbf{u}_i^*\right)^{-1}$ exists.

(b) Show that $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* \mathbf{A} \mathbf{x}$ is an inner product on $\mathbb{C}^n$ and that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ form an orthonormal basis with respect to it. [5+(5+5)=15]

7. Given $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ with $\|\mathbf{u}\| = \|\mathbf{v}\|$, explain how an orthogonal matrix $\mathbf{C}$ can be obtained so that $\mathbf{C}\mathbf{u} = \mathbf{v}$. [8]

****** *Best of Luck!* ******

INDIAN STATISTICAL INSTITUTE

MID-TERM EXAMINATION (2009–10)

B. STAT. I YEAR

ANALYSIS II

Date : 26.02.2010 | Maximum Marks : 80 | Time : $2\frac{1}{2}$ hours

Precisely justify all your steps. Carefully state all the results you are using.

1. A function f is defined on [0,1] by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ x^3 & \text{if } x \text{ is irrational} \end{cases}$$

Compute $\underline{\int_0^1} f(x)\,dx$, $\overline{\int_0^1} f(x)\,dx$ and decide whether $f \in \mathcal{R}[0,1]$. [10]

2. Let $f \in \mathcal{R}[a,b]$. Show that for every $\varepsilon > 0$, there is a continuous function g on $[a,b]$ such that [10]

$$\int_a^b |f(x) - g(x)|dx < \varepsilon.$$

3. (a) Let $f : [a,b] \to \mathbb{R}$ be continuous, and P a partition of $[a,b]$. Show that there is a marking of P such that the Riemann sum $S(P,f) = \int_a^b f(x)\,dx$. [10]

(b) Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq M$ for all $x \in (0,1)$. Show that [10]

$$\left| \int_0^1 f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leq \frac{M}{n}.$$

4. Test the convergence of the integral $\int_0^1 |\log x|\,dx$. [10]

5. Show that the improper integral

$$\int_1^\infty \frac{\sin x}{x^p}\,dx$$

converges absolutely for $p > 1$ and conditionally for $0 < p \leq 1$. [15]

6. Show that the improper integral

$$\int_0^\infty \frac{1}{x^p(1+x)^q}\,dx$$

converges if and only if $0 < 1 - p < q$. [15]

INDIAN STATISTICAL INSTITUTE

Midsemester Examination : (2009-2010)

B. Stat 1st Year

Statistical Methods -II

Date: 1. 3. 10 Maximum marks: 40 Duration: 2 hours.

Note: This paper carries 44 points. Maximum you can score is 40. You may use any results proved in the class by stating the results clearly. If you use other results not discussed in class, they need to be proved. You may use calculators.

1. Suppose Y is regressed on X_1, X_2 and X_3 with an intercept term, and the following matrices are computed.

$$\text{Y'Y}= 5000 \quad \text{Y'X}= (20, 30, 50, -40), \quad \text{X'X}= \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 19 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

   (a) Compute the regression equation. [6 points]

   (b) Compute the analysis of variance of Y by deriving model, regression and residual sum of squares. Produce the ANOVA table. [8 points]

   (c) Compute the estimate of error variance. Find the variances of each regression coefficient. Also find the covariances between them. [6 points]

2. Consider data on three variables X_1, X_2 and X_3.

   (a) Derive the partial correlation between X_1 and X_3 adjusting for X_2, and express it in terms of the pairwise correlations between X_1, X_2 and X_3. [6 points]

   (b) Suppose a simple regression of X_1 on X_2 is performed to get the equations $\hat{X}_1 = \hat{\beta}_0 + \hat{\beta}_2 X_2$. If a multiple regression of X_1 on X_2 and X_3 is performed now by how much will the regression coefficient of X_2 change? [8 points]

   (c) Suppose the simple regression equations $\hat{X}_1 = 3 + 3X_2$, $\hat{X}_1 = 2 + 4X_3$, $\hat{X}_2 = -1 + 2X_3$ and the pairwise correlations ($r_{12} = r_{23} = r_{13} = 0.5$) are available but the original data is lost. Using part (b) or otherwise, can you find the multiple regression equation of X_1 on X_2 and X_3 based on those information? [10 points]

INDIAN STATISTICAL INSTITUTE
B.STAT-I (2009-10) Mid-Semestral examinations
Theory of Probability and its Applications - II
Maximum marks: 40. Time: 3 hours. Date : 3 March, 2010.

Note: Answer as many questions as you wish. The whole question paper carries 48 marks.The maximum you can score is 40.

1. The Distribution Function (D.F.) of a random variable X is given by

$$F(x) = \begin{cases} \frac{1}{8}e^x & \text{if } x < 0 \\ \frac{1}{8} & \text{if } 0 \le x < 1 \\ \frac{1}{8}x + \frac{1}{16} & \text{if } 1 \le x < 2 \\ \frac{1}{64}x^2 + \frac{1}{4} & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$$

(a) Find the discontinuity points of this D.F.
(b) Find $P(-0.5 \le X \le 2.5)$ and $P(X^2 \ge 3X)$.
(c) Find two Distribution Functions F_1, F_2 and a real number α, $0 \le \alpha \le 1$, such that F_1 is purely discrete, F_2 is continuous and $F(x)$ can be expressed as :

$$F(x) = \alpha F_1(x) + (1-\alpha)F_2(x)\,, \quad \text{for all } x.$$

[2+2+6]

2. Let Θ be a random variable which is uniformly distributed in $(-\pi, \pi)$. Let

$$Y = \begin{cases} \cos\Theta & \text{if } \Theta < 0 \\ \sin\Theta & \text{if } \Theta \ge 0. \end{cases}$$

Find the D.F and the density function of the random variable Y.

[6]

3. The speed of a molecule in a uniform gas at equilibrium is a random variable X whose probability density function is given by:

$$f(x) = \begin{cases} ax^2e^{-bx^2} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

where $b = \frac{m}{2kT}$ and k, T, m denote, respectively, the Boltzmann's constant, the absolute temperature, and the mass of the molecule.
(a) Evaluate a in terms of b.
(b) Let $Y = \frac{1}{2}mX^2$. Find the density of the random variable Y.

[2+5]

(Please Turn overleaf)

4. Let X and Y be independent random variables each having the exponential distribution with parameter λ. Find the $P(X - Y \leq a)$, for $a \in \mathbb{R}$ (consider the two cases $a \geq 0$ and $a < 0$ separately). Hence find the density of the random variable $Z = X - Y$.

[8+2]

5. Let X be a random variable such that $X \sim \Gamma(\alpha, n)$, i.e. the density of X is given by $g_n(x) = \frac{\alpha^n}{(n-1)!} e^{-\alpha x} x^{n-1} I_{(0,\infty)}(x)$, where $\alpha > 0$ and $n \geq 1$, is an integer. Show that the D.F. $G_n(x)$ of X is given by:

$$G_n(x) = 1 - \sum_{i=0}^{n-1} \frac{(\alpha x)^i}{i!} e^{-\alpha x}, \quad x>0, \text{ for } n \geq 1, x>0.$$

[5]

6. Let the random variable $X \sim N(0,1)$.
(a) Find the density of the random variable $Y = |X|$.
(b) Evaluate

$$\lim_{x \to \infty} \frac{P(X > x + \frac{a}{x})}{P(X > x)}.$$

[5+5]

INDIAN STATISTICAL INSTITUTE
B. Stat I
Computational Techniques and Programming II
(Mid Semester Exam)

Date: 5. 3. 10 — Time: 2 hours 30 mins.

Answer as many questions as you like. Maximum you can score is 40.

1. (a) Define Absolute error and relative error. An approximate number a=25,253 has a relative error of 1%. How many correct digits it has?
(1+1)+2

(b) The base of a cylinder has radius R$\approx$2m, the altitude of the cylinder is H$\approx$3m. With what absolute errors must be determined R and H so that the volume V may be computed to within 0.1m^3.
4

(c) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots\ldots$

How many terms of the above series are required to compute $\cos x$ upto 5 decimal place accuracy for $|x| < \frac{\pi}{2}$.
4

[4+4+4=12]

2. (a) Obtain Newton's formula for polynomial interpolation using divided difference in case of n equally spaced data points when the unknown argument is situated at the beginning of the data set and at the end of the data set.

(b) Let f(-2)= -50, f(-1)= 6, f(0)= 10, f(1)=10, f(2)= 30, f(3)= 190. Find approximate values of $f'(-2)$ and $f'(3)$ by stating clearly the formula you are going to use.

[6+6=12]

3. (a) Derive Newton-Raphson iterative formula for solving f(x) = 0 and give its geometrical interpretation. Also show that Newton Raphson method converges quadratically.
(3+2+3)

(b) Find the number and the position of the real roots of the equation $x^4 - 6x^3 + 10x^2 - 6x + 1 = 0$ using Sturm's method.
4

[8+4=12]

P.T.O

4. (a) Write down the Mechanical Quadrature Formula when the interpolating Polynomial is based on the Lagrange's formula. Hence deduce the quadrature formula when the interpolating points are equally spaced. Find out the error committed in this method also. (2+4+2

(b) Define degree of precision of a quadrature formula. Write down 3- point Legendre-Gauss quadrature formula and its degree of precision. (1+2+1

[6+6=12

----x----

INDIAN STATISTICAL INSTITUTE

Second Semester Examination: 2009-2010

B.Stat. (Hons.) 1st Year. 2nd Semester

Vectors and Matrices II

Date: April 30, 2010 **Maximum Marks: 100** **Duration: 4 hours**

- This question paper carries 110 points. Answer as much as you can. However, the maximum you can score is 100.

- You should provide as much details as possible while answering a question. You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

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1. Suppose $\mathbf{A} = ((a_{ij}))$ is an $(n+1) \times (n+1)$ real symmetric matrix such that all its principal minors are non-negative, and that $a_{n+1,n+1} > 0$. Let $\mathbf{B} := \mathbf{A}(1,\ldots,n|1,\ldots,n), \mathbf{c} := \mathbf{A}(1,\ldots,n|n+1)$. Show using facts about determinants that all the principal minors of the $n \times n$ matrix $\mathbf{D} := \mathbf{B} - \mathbf{c}\mathbf{c}^{\mathrm{T}}/a_{n+1,n+1}$ are non-negative. [12]

2. Let

$$\mathbf{A} := \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}.$$

Write $\mathbf{A}$ as a polynomial in a suitable permutation matrix $\mathbf{P}$. Hence, or otherwise, show that the characteristic roots of $\mathbf{A}$ are $a_1 + a_2\omega_i + a_3\omega_i^2 + \cdots + a_n\omega_i^{n-1}$, $i = 1,\ldots,n$, where $\omega_1, \omega_2, \ldots, \omega_n$ are the n-th roots of unity. [7+6 = 13]

3. Suppose $\mathbf{A}_1$ and $\mathbf{A}_2$ are $n \times r_1$ and $n \times r_2$ matrices, respectively, such that the columns of both $\mathbf{A}_1$ and $\mathbf{A}_2$ form orthonormal sets and $\mathcal{C}(\mathbf{A}_1) \perp \mathcal{C}(\mathbf{A}_2)$. Suppose $\mathbf{A}_3$ and $\mathbf{A}_4$ are $r_1 \times p$ and $r_2 \times p$ matrices, respectively, such that the columns of both $\mathbf{A}_3^{\mathrm{T}}$ and $\mathbf{A}_4^{\mathrm{T}}$ form orthonormal sets and $\mathcal{C}(\mathbf{A}_3^{\mathrm{T}}) \subsetneq \mathcal{C}(\mathbf{A}_4^{\mathrm{T}})^{\perp}$. Assume that $n \geq p$. Let $\mathbf{A} := \mathbf{A}_1\mathbf{A}_3\mathbf{A}_4^{\mathrm{T}}\mathbf{A}_2^{\mathrm{T}}$. Decide, with reasons, if $\mathbf{A}$ is semi-simple. [12]

4. Let

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{D} \end{bmatrix}$$

be symmetric, where $\mathbf{A}$ is square. Prove that $\mathbf{M}$ is positive definite if and only if $\mathbf{A}$ and $\mathbf{D} - \mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B}$ are positive definite. [13]

P.T.O.

5. Show that

$$\mathbf{A} := \frac{1}{15}\begin{bmatrix} 11 & -2 & -2 & -6 \\ -2 & 14 & -1 & -3 \\ -2 & -1 & 14 & -3 \\ -6 & -3 & -3 & 6 \end{bmatrix}$$

is an orthogonal projector. Obtain a spectral decomposition of $\mathbf{A}$. [4+9 = 13]

6. For $\rho \in \mathbb{R}$, consider the quadratic form $Q_\rho(\mathbf{x}) := \sum_{i=1}^n x_i^2 + 2\rho \sum_{i=1}^n \sum_{j=i+1}^n x_i x_j$, where $\mathbf{x} \equiv (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ $(n > 2)$. Let $A := \{\rho \in \mathbb{R} : Q_\rho \text{ is indefinite}\}$. Find A. [10]

7. Let $\mathbf{A} = ((a_{ij}))_{n\times n}$ $(n > 2)$ be defined by $a_{ij} = \min(i, j)$. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, i.e., $S^{n-1} := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$. Consider the function $f : S^{n-1} \longrightarrow \mathbb{R}$ defined by

$$f(\mathbf{x}) = \frac{(x_1 - x_2)^2}{\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j}, \quad \mathbf{x} \equiv (x_1, \ldots, x_n)^{\mathrm{T}},$$

assuming the denominator to be non-zero for $\mathbf{x} \in S^{n-1}$. Use tools of linear algebra to find $M := \sup\{f(\mathbf{x}) : \mathbf{x} \in S^{n-1}\}$ and the set $A := \{\mathbf{x} \in S^{n-1} : f(\mathbf{x}) = M\}$. [11+3 = 14]

8. Suppose $\boldsymbol{\Sigma}$ is a known $n \times n$ positive definite matrix. Also, suppose $\mathbf{x}_0 \in \mathbb{R}^n$ is known. Let $Q(\mathbf{x}) := \det(\boldsymbol{\Sigma} + (\mathbf{x} - \mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^{\mathrm{T}})$, $\mathbf{x} \in \mathbb{R}^n$. Consider the problem of minimizing $Q(\mathbf{x})$ subject to the condition $x_1 + \cdots + x_n = 0$, where $\mathbf{x} \equiv (x_1, \ldots, x_n)^{\mathrm{T}}$. Use tools of linear algebra to show that Q attains its minimum at a unique point in $\mathbb{R}^n$, denoted $\mathbf{x}^*$, to be obtained by you and find the minimum. [9+4 = 13]

9. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ positive definite matrices, prove that

$$|\mathbf{A} + \mathbf{B}|^{1/n} \geq |\mathbf{A}|^{1/n} + |\mathbf{B}|^{1/n}.$$

[**Note.** You may use any relevant inequality about real numbers for proving this result. You need not prove it.] [10]

****** *Best of Luck!* ******

INDIAN STATISTICAL INSTITUTE

SECOND SEMESTER SEMESTRAL EXAMINATION (2009–10)

B. STAT. I YEAR

ANALYSIS II

ıte : 04.05.2010 | Maximum Marks : 100 | Time : $3\frac{1}{2}$ hours

The question carries 120 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. (a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } f_-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $f \in \mathcal{R}[a, b]$ if and only if both $f_+ \in \mathcal{R}[a, b]$ and $f_- \in \mathcal{R}[a, b]$. Moreover,

$$\int_a^b f(x)dx = \int_a^b f_+(x)dx - \int_a^b f_-(x)dx. \qquad [10]$$

(b) Let $f : [a, b] \to \mathbb{R}$ be such that f' is continuous. Show that f is the sum of a continuous increasing function and a continuous decreasing function. [7]

2. A function f is continuous for all $x \geq 0$ and $f(x) \neq 0$ for all $x > 0$. If $f(x)^2 = 2\int_0^x f(t)\, dt$ prove that $f(x) = x$ for all $x \geq 0$. [10]

3. Test the convergence of the integral

$$\int_0^\infty \frac{1}{x^2 + \sqrt{x}} dx \qquad [15]$$

4. Let $g : [0, 1] \to \mathbb{R}$ be continuous. Let $f_n(x) = x^n g(x)$ for $x \in [0, 1]$. Show that $\{f_n\}$ converges uniformly on $[0, 1]$ if and only if $g(1) = 0$. [10]

5. Starting from a geometric series and precisely justifying all your steps prove that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$. [10]

6. Decide whether the power series $\sum_{n=1}^{\infty} \frac{n^3[\sqrt{2} + (-1)^n]^n}{3^n} x^n$ converges or diverges at the points $x = 1$ and $x = 2$. [10]

7. Let $\{f_n\}$ be a sequence of continuous functions on $[0,1]$ decreasing pointwise to the constant function 0. Is it true that

$$\int_0^1 f_n(x)dx \longrightarrow 0 \text{ as } n \to \infty?$$

[8]

Briefly justify your answer.

8. Prove that for all $p > 0$, the series $\sum_1^\infty \frac{(-1)^{n-1}x^n}{n^p(1+x^n)}$ converges uniformly on $[0,1]$. [20]

[Hint : If $p > 1$, use Weierstrass' M-test. If $0 < p \leq 1$, use Abel's test.]

9. (a) Compute the Fourier series of the function $f(x) = |x|$. [10]

(b) Show that [10]

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}, \quad \text{and hence,} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination : (2009-2010)

B. Stat 1st Year

Statistical Methods -II

Date: 7. 5. 10 Maximum marks: 100 Duration: 3 hours.

Note: This paper carries 105 points. Maximum you can score is 100.

1. Consider the analysis of variance for the multiple linear regression model. The total sum of squares can be decomposed in terms of the correction factor, the sum of squares due to the model and the residual sum of squares.

(a) Express each of these sum of squares as quadratic forms in Y, the vector of observed values of the dependent variable. [12]

(b) Show that the corresponding matrices in the quadratic forms in part (a) are all symmetric, idempotent and orthogonal to each other. Find the ranks of these matrices. [10 + 3]

2. The following table gives the correlation coefficients between 4 x-variables. Using them, find (a) $r_{14.23}$ and (b) $R^2_{1(234)}$. [12 + 8]

1	.8	.6	.6
.8	1	.8	.6
.6	.8	1	.8
.6	.6	.8	1

3. Present a method to generate a value of the random variable X which has the following distribution:

$$P\{X = j\} = \left(\frac{1}{2}\right)^{j+1} + \frac{\left(\frac{1}{2}\right) 3^{j-1}}{4^j}, \quad j = 1, 2, \ldots$$

[15]

4. Suppose we want to generate a random number from the standard normal distribution. Give an algorithm based on the rejection method for this problem. Choose an appropriate value of the parameter for the distribution that you choose for this purpose so that the efficiency of this algorithm is maximized. [12 + 8]

[P.T.O.]

5. In an ecological study of the feeding behaviour of birds, the number of hops between flights was counted for several birds. For the data given below, see whether an appropriate geometric distribution can give a good fit.

Number of hops	Frequency
1	48
2	31
3	20
4	9
5	6
6	5
7	4
8	2
9	1
10	1
11	2
12	1

(a) Check whether the method of moments and method of ML give same estimates of the parameter. [10]

(b) Calculate the chi-square statistic for goodness of fit. Comment on the results. [15]

INDIAN STATISTICAL INSTITUTE
B.STAT-I (2009-10)
Probability Theory - II
Semestral Examinations
Maximum marks: 60. Time: 3 hours.

Date: //.5.2010

Note: Answer as many questions as you wish.
The whole question paper carries 70 marks.
The maximum you can score is 60.

1. (a) Let X be a random variable with a continuous Distribtion Function $F(x)$. Show that the random variable $Y = F(X) \sim U(0,1)$.
(b) Let X and Y be independent and identically distributed random variables with common density

$$f(x) = \begin{cases} 2^{-\alpha}\alpha x^{\alpha-1} & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 1$. Find the density of the random variable $Z = |X^\alpha - Y^\alpha|$.
[7+7]

2. (a)Let X be a random variable which has $N(0,1)$ distribution. Let Z be independent of X taking values +1 and -1 with probability $\frac{1}{2}$ each. Let $Y = ZX$. Show that Y is again standard Normal. Show that X and Y are uncorrelated but not independent. Is there any contradiction here because X and Y are both normal ? Give reasons.
(b) Let X be a random variable with a density function $f(x)$ satisfying $f(x) = f(-x)$ for all $x \in \mathbb{R}$, such that the random variable $Y = \frac{1}{X^2}$ has the density

$$g(y) = \begin{cases} \frac{\alpha}{\sqrt{2\pi}} y^{-\frac{3}{2}} e^{-\frac{\alpha^2}{2y}} & \text{for } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $X \sim N(0, \frac{1}{\alpha^2})$.
[(4+2+2)+6]

3. Let (X,Y) have a bivariate normal density with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 1$ and $E(XY) = \rho$.
(a) Find the conditional density of X given $Y = y$.
(b) Find the correlation coefficient between X^2 and Y^2.
(c) Consider the usual polar trasformation (R, Θ) where $X = R\cos\Theta, Y = R\sin\Theta$, $R > 0$, $0 < \Theta < 2\pi$. Show that Θ has a density given by

$$f(\theta) = \begin{cases} \frac{\sqrt{(1-\rho^2)}}{2\pi(1-2\rho\sin\theta\cos\theta)} & \text{for } 0 < \theta < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

[4+3+7]

4. (a) Prove the following duplication formula for Gamma functions:

$$\Gamma(2r) = \frac{1}{\sqrt{\pi}} 2^{2r-1}\Gamma(r)\Gamma(r+\tfrac{1}{2})$$

(b)Let X and Y be independent random variables such that $X \sim Be(r,s)$ and $Y \sim Be(r+\frac{1}{2}, s)$. Let $Z = \sqrt{XY}$. Compute the moments $E(Z^k)$, $k = 1, 2, \cdots$ of the random variable Z and check that they are exactly equal to the respective moments of a

(Please Turn Overleaf)

random variable U such that $U \sim Be(2r, 2s)$. What conclusion can you draw from this? State any result that you are using to draw this conclusion.

[4+(6+4)]

5. Let (X, Y) be a pair of random varibles with joint density

$$f(x,y) = \begin{cases} C\frac{x\sqrt{y}}{\sqrt{(1-x-y}} & \text{for } x, y > 0 \text{ and } 0 < x+y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the value of C.

(b) Find the marginal density of Y. Find the density of $X+Y$. Find the conditional density of $Y \big| X+Y=u$, and $E(Y|X+Y=u)$, where $0 < u < 1$.

[2+(3+3+6)]

INDIAN STATISTICAL INSTITUTE
B. Stat I, Year (2009-10)
Computational Techniques and Programming II
(End Semester Exam)
Maximum Marks: 100

Date: 14.5.10 Time: 3 hours 30 mins

This paper carries 110 marks. Answer as many questions as you like. Maximum you can score is 100.

1. (a) Define absolute error and relative error. How are they related? How are absolute errors combined when two numbers are added together? How are relative errors combined when two numbers are multiplied together?

[2+1+4+4=11]

(b)(i) Explain the term loss of significance in terms of absolute error and relative error.

(ii)Let θ_s, θ_c be the relative errors in the values of $\sin\theta$, $\cos\theta$ respectively. Find the worst-case relative errors in evaluating each of the formulae

$\sin 2\theta = 2\sin\theta \cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

For what values of $\cos\theta$ does the second formula display loss of significance?

OR

What are the two sources of error in the formula

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

and how does each type of error behave as h increases?

[4+7=11]

2. (a) Derive a closed type general quadrature formula in case of numerical integration for equispaced points. Hence find the formula when a quadratic interpolating polynomial is used to approximate the integrating function. Interpret this formula geometrically also.

[5+5+4=14]

(b) Compare the error terms of two-point and three-point quadrature formulae. [8]

P. T. O.

3. (a) With reference to solution of differential equation $y' = f(x,y)$, explain the conventional notations x_n, $y(x_n)$, y_n, f_n. Explain the terms local error, global error, and order of a method.

[4+3=7]

(b) Without deriving any formulae, describe the general method for obtaining multistep formulae.
Milne's method uses the multistep formulae

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_n - f_{n-1} + 2f_{n-2})$$
$$y_{n+1} = y_{n-1} + \frac{h}{3}(\widetilde{f}_{n+1} + 4f_n + f_{n-1})$$

Each of which has local error O(h^5). What is the meaning of the term $\widetilde{f}_{n+1}$? Suggest a suitable starting procedure and explain how the Milne formulae are derived.

[3+2+5+5=15]

4. (a) How are the eigen-values and eigen-vectors of a matrix defined? What is the power method for finding the eigen-value of largest magnitude? What advantage does the scaled power method have over the power method?

[4+4+2=10]

(b) Apply five iterations of the normal and scaled power methods to the following 3 x 3 matrix to find the eigen-value of largest magnitude.

$$\begin{bmatrix} 2 & 6 & 4 \\ 6 & 19 & 12 \\ 2 & 8 & 14 \end{bmatrix}$$

[6+6=12]

5. A matrix A and a vector b are given below. Reduce A to upper triangular form using
(a) Householder Transformation
(b) Given's Transformation

Hence solve Ax=b.

$$A = \begin{bmatrix} 5 & 7 & 6 \\ 7 & 10 & 8 \\ 6 & 8 & 10 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

[9+9+4]=22

INDIAN STATISTICAL INSTITUTE

Backpaper Examination: 2009-2010
B.Stat. (Hons.) 1st Year. 2nd Semester
Vectors and Matrices II

Date: July 28, 2010 **Maximum Marks: 100** **Duration: 3 and 1/2 hours**

- Answer all the questions.
- You should provide as much details as possible while answering a question. You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

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1. Prove that

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$$

[10]

2. Show that

$$\begin{vmatrix} 1+x_1 & 1 & \cdots & 1 \\ 1 & 1+x_2 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1+x_n \end{vmatrix} = \left(1+\sum_{i=1}^{n}\frac{1}{x_i}\right)\left(\prod_{i=1}^{n} x_i\right)$$

if $x_1, x_2, \ldots, x_n$ are non-zero.

[10]

3. Find the orthogonal projector into the column space of

$$A := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 3 & -2 \\ -1 & -3 & 2 \end{bmatrix}.$$

[11]

4. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ form a basis of $\mathbb{C}^n$.

(a) Show that $\mathbf{A} = \left(\sum \mathbf{u}_i\mathbf{u}_i^*\right)^{-1}$ exists.

(b) Show that $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^*\mathbf{A}\mathbf{x}$ is an inner product on $\mathbb{C}^n$ and that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ form an orthonormal basis with respect to it.

[5+(5+5)=15]

P.T.O.

5. Let $\mathbf{A}$ be an $n \times n$ matrix such that $\mathbf{A}^2 = \mathbf{I}$. Suppose, moreover, that $\mathbf{A} \neq \alpha\mathbf{I}$ for any α. Show that $|\text{tr}(\mathbf{A})| < n$. [10]

6. Show that

$$\mathbf{A} := \frac{1}{15}\begin{bmatrix} 11 & -2 & -2 & -6 \\ -2 & 14 & -1 & -3 \\ -2 & -1 & 14 & -3 \\ -6 & -3 & -3 & 6 \end{bmatrix}$$

is an orthogonal projector. Obtain a spectral decomposition of $\mathbf{A}$. [4+9 = 13]

7. Determine the rank and signature of the quadratic form $Q(\mathbf{x}) := \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2$, where $\mathbf{x} \equiv (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ $(n > 2)$. [4+7 = 11]

8. Let $\mathbf{A} = ((a_{ij}))$ be an $n \times n$ positive definite matrix. Prove that $(\mathbf{A}^{-1})_{ii} \geq 1/a_{ii}$ for all $i = 1, \ldots, n$. [10]

9. If $\mathbf{A}$ and $\mathbf{B}$ are positive definite matrices of the same order and $0 < \alpha < 1$, prove that

$$|\alpha\mathbf{A} + (1-\alpha)\mathbf{B}| \geq |\mathbf{A}|^{\alpha}|\mathbf{B}|^{1-\alpha}.$$ [10]

****** *Best of Luck!* ******

INDIAN STATISTICAL INSTITUTE
B.STAT-I (2009-10)
Probability Theory - II
Backpaper Examinations
Maximum marks: 100. Time: 3 hours.

Date : 9/8/2010.

Note: Answer all questions.

1. (a) Prove that the set of discontinuity points of the Distribution Function(D.F.) of a random variable X is a countable set.
(b) Is it possible that the function $F(x, y) = I_{(x+3y\geq 0)}(x, y)$ on $\mathbb{R}^2$ is a bivariate Distribution Function? Give reasons for your answer.
(c)Let X, Y, Z be independent random variables. Let the D.F. of X and Y be F and G respectively. Let Z be $0-1$ valued random variable with
$$P(Z = 1) = \alpha = 1 - P(Z = 0), 0 < \alpha < 1.$$
Let $W = ZX + (1-Z)U$, where $U = Min(X, Y)$. Write down the D.F. of the random variable W in terms of α, F and G.
[7+8 + 5]

2. Let Θ be a random variable having uniform distribution $U(0, 2\pi)$. Find the density of the random variable $X = Sin\Theta$. Show that that the random variable $Y = Sin 2\Theta$ has the same density as that of X. Use this to show that the random variable $W = \frac{UV}{\sqrt{U^2+V^2}} \sim N(0, \frac{1}{4})$, where U, V are independent $N(0, 1)$ random variables.
[5+3+7]

3. Let (X, Y) have joint density
$$f(x, y) = \begin{cases} Cx^{\alpha-1}(y-x)^{\beta-1}e^{-\lambda y} & , 0 < x < y < \infty \\ 0 & , \text{otherwise} \end{cases}$$
(a) Find the constant C and the marginal densities of $X$and Y.
(b)Find the conditional density of Y given $X = x$.
(c) Find the joint density of (Z, Y), where $Z = \frac{X}{Y}$. Find the conditional density of Z given $Y = y$, $y > 0$. Compute $E(X| Y = y)$.
[5+3+(6+3+3)]

4. Let the random variables (X, Y) have a joint density given by :
$$f(x, y) = \frac{1}{2\pi}\frac{1}{\sqrt{(1+x^2+y^2)^3}} , \quad -\infty < x, y < \infty$$
(a) Find the marginal densities of X and Y. Find the conditional density of $X\big|Y = y$.
(b) Make the polar transformation $(X, Y) \mapsto (R, \Theta)$ where
$$X = R\,Cos\Theta \text{ and } Y = R\,Sin\Theta , 0 < R < \infty, 0 < \Theta < 2\pi.$$
Show that R and Θ are independent random variables , where R has the density $g(r) = r(\sqrt{(1+r^2)^{-3}}), r > 0$ and Θ is uniformly distributed on $(0, 2\pi)$.
[(5+2)+8]

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5. Let (X, Y) have bivariate Normal $N_2\big((0,0), 1, 1, \rho\big)$ – distribution, $-1 < \rho < 1$.
(a) Let $U = \frac{X-\rho Y}{\sqrt{(1-\rho^2)}}$, and $V = Y$. Show that U and V are independent $N(0,1)$ random variables.
(b) Using (a) or otherwise show that $P(X > 0, Y > 0) = \frac{1}{4} + \frac{\arcsin \rho}{2\pi}$.

[6+9]

6. Let $f(x)$ be any univariate probability density function such that $f(x) = 0$ for all $x \leq 0$.
Let $g(x, y)$ be a function defined as follows:

$$g(x,y) = \begin{cases} \frac{f(x+y)}{x+y} & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Prove that $g(x, y)$ is always a bivariate density.
(b) Let the expectation and the variance of a random variable having the density $f(x)$ be μ and σ^2 respectively. Compute the variance-covariance matrix corresponding to the bivariate density $g(x, y)$ in terms of μ and σ^2.

[7+8]

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