BALANCED ORTHOGONAL DESIGNS AND THEIR APPLICATION IN THE CONSTRUCTION OF SOME BIB AND GROUP DIVISIBLE DESIGNS

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SUMMARY. Some systematic study has been made for the construction of BOD's and their application for the construction of BIB, PBIB and SRGD.

1. Introduction

In the paper (1966a) the author gives a method of construction of group divisible family of PBIB designs using row-orthogonal matrices with the elements ± 1, 0. These matrices should satisfy that when -1's of them are replaced by 1's, they become the incidence matrices of BIB designs. Such matrices, we call here, as balanced orthogonal designs (BOD). In Section 3 of this paper an attempt is made to study them in a systematic way; also, construction of some such designs is presented. In Section 4.1 is given a method of construction of a series of BIB designs using BOD's. In Section 4.2 BOD's have been utilised for constructing some PBIB designs. In Section 4.3 construction of some group divisible designs, which are not available in the tables of PBIB designs of Bose, Clatworthy and Shrikhande (1954), are included. Though some of the results included in this paper have already been published they are included in this paper for the sake of completeness.

2. Some properties of balanced orthogonal designs

Definition 2.1 : A matrix of order $v \times b$ with (± 1, 0) is said to be balanced orthogonal design (BOD) if it satisfies the following conditions :

(i) inner product of any two rows of it is zero;

(ii) when -1's of it are changed to 1's, the resulting matrix becomes the incidence matrix of a BIB design.

Let X be a BOD of order $v \times b$. Let N be the incidence matrix of the BIB design obtained from X by replacing -1's by 1's in X. Let (v, b, r, k, λ) be the parameters of the BIBD.

Therefore $XX' = rI_v$ where X' is the transpose of X and I_v is the identity matrix of order v. When $b = v = r$ the BOD becomes the Hadamard matrix of order v.

Some necessary conditions for the existence of BOD. From the author's paper (1966b) and when $v = b$ the following result is true.

Editorial Note : This is the second of the two posthumous publications of Dr. M. B. Rao. The paper is published with a few revisions by Dr. B. Adhikari.

Theorem 2.1 : *When v is odd, a necessary condition for the existence of BOD is that r should be perfect square.*

Example :

	$v=b$	$r=k$	λ
(i)	11	5	2
(ii)	15	8	4
(iii)	23	12	6

These examples violate the necessary condition in Theorem 2.1 and hence BOD's with these parameters are not possible.

Theorem 2.2 : *A necessary condition for the existence of BOD is that λ should be even.*

Proof : Without loss of generality we can take that the first row of X consists of no negative units. Then the theorem is immediate because of the definition of BOD.

Example :

	v	b	r	k	λ
(i)	5	10	6	3	3
(ii)	4	6	3	2	1

Therefore by Theorem 2.2 we have that the BOD's with these parameters are non-existent.

Theorem 2.3 : *A necessary condition for the existence of BOD is that when $k=3 \bmod 4$, b should be $0 \bmod 4$ and when $k-1$ is 4 times an odd integer, b should be even.*

Proof : Let N_1 and N_2 be the matrices obtained from X such that

$$N_1+N_2 = N \text{ and } N_1-N_2 = X. \quad \ldots \quad (2.1)$$

Let $k_1, k_2, \ldots, k_b$ be the column totals of N_1. We have

$$N_1N_1'+N_2N_2' = \frac{1}{2}(NN'+XX') \quad \ldots \quad (2.2)$$

$$= \frac{1}{2}[(r-\lambda)I_v+\lambda_{vv}+rI_v] \quad \ldots \quad (2.3)$$

when E_{mn} is the matrix of order $m \times n$ with positive unit elements everywhere. Consider

$$E_{1v}(N_1\ N_2)(N_1\ N_2)'E_{v1} = E_{1v}(N_1\ N_1'+N_2N_2')E_{v1}$$

$$= \frac{1}{2}E_{1v}[(2r-\lambda)I_v+\lambda E_{vv}]E_{v1}. \quad \ldots \quad (2.4)$$

The LHS of (2.4) is $\sum_{j=1}^{b} k_j^2+\sum_{j=1}^{b}(k-k_j)^2$. The RHS of (2.4) is $\frac{bk(k+1)}{4}$;

therefore

$$\sum_{j=1}^{b} (k-k_j)\, k_j = \frac{bk(k-1)}{4} \qquad \ldots \quad (2.5)$$

when $k \equiv 3 \bmod 4$, the LHS of (2.5) is even, and $k(k-1)$ is 2 mod 4. Hence from (2.5) we must have that $b \equiv 0 \bmod 4$ when $\frac{k-1}{4}$ is an odd intotger and for (1.5) to be true we should have that b is even.

The following examples violate the necessary condition in Theorem 2.3 and hence BOD's with these parameters are not possible.

	v	b	r	k	λ
(i)	6	10	5	3	2
(ii)	7	14	6	3	2
(iii)	11	11	5	5	2
(iv)	15	30	15	7	6

3. Construction of some BOD's

Theorem 3.1 : *There always exists BOD with the parameters* $(v,\ tv(v-1),\ 2t(v-1),\ 2,\ 2t)$.

Proof : It is sufficient, if we can prove it for the design $(v,\ v(v-1),\ 2(v-1),\ 2,\ 2)$. Let N be the incidence matrix of the BIBD with the parameters $\left(v, \binom{v}{2}, v-1, 2, 1\right)$. Every column of N contains only 2 units. Replace one of them by -1. Let this resulting matrix be X_1. Hence $X = (X_1\ N)$ is the required design.

Example :

$$X = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Though the following theorem is not new, since the proof of it is different from the proof given in Rao and Das (1969), it is presonted here.

Theorem 3.2 : *There always exists BOD with the parameters* $(s^2+s+1,\ s^2,\ s^2-s)$ *where s is odd prime power.*

For proving this we use the two lemmas of Williamson (1944) given below.

Definition 3.1 : A square symmetric matrix of order s with ± 1 in the off-diagonal and zeros in the diagonal is called S_s matrix if

$$S_s S_s' = sI_s - E_{ss}. \qquad \ldots \quad (3.1)$$

Lemma 3.1 : *When $s = p^n \equiv 1$ mod 4 where p is odd prime power and n, a positive integer, there always exists S_s matrix.*

Definition 3.2 : A square skew-symmetric matrix of order s with ± 1's in the non-diagonal and zeros in the diagonal is called Σ_s matrix if

$$\Sigma_s \Sigma_s' = sI_s - E_{ss}. \qquad \dots \quad (3.2)$$

Lemma 3.2 : *When $s = p^n \equiv 3$ mod 4 where p is odd prime and n positive integer, then there always exists Σ_s matrix.*

Proof of Theorem 3.2 : We know that when $s = p^n$, semi regular group divisible (SRGD) design with the following parameters $v^* = s^2 = b^*$, $r^* = s = k^*$, $\lambda_1^* = 0$, $\lambda_2^* = 1$, $m^* = n^* = s$ always exists. Let s be odd prime power. Construct SRGD design with the above parameters. Let N be the incidence matrix of this design where

$$NN' = N'N = \begin{bmatrix} s\,I_s & E_{ss} & \dots & E_{ss} \\ E_{ss} & s\,I_s & \dots & E_{ss} \\ \vdots & \vdots & \vdots & \vdots \\ E_{ss} & E_{ss} & \dots & s\,I_s \end{bmatrix} \qquad \dots \quad (3.3)$$

Since this design is SRGD, N can be partitioned into sub-matrices as

$$N = \begin{bmatrix} N_{11} & N_{12} & \dots & N_{1s} \\ N_{21} & N_{22} & \dots & N_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ N_{s1} & N_{s2} & \dots & N_{ss} \end{bmatrix} \qquad \dots \quad (3.4)$$

where N_{ij} is of order $s \times s$ and consists only one unit in every column and row.

By the Lemmas 3.1 and 3.2 we can have either S_s or Σ_s.

Let

$$X = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1s} \\ X_{21} & X_{22} & \dots & X_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ X_{s1} & X_{s2} & \dots & X_{ss} \end{bmatrix} \qquad \dots \quad (3.5)$$

obtained from N, by replacing N_{ij} by X_{ij} where X_{ij} is obtained in the following way. If f-th row and g-th column of N_{ij} contains unit, replace f-th row of N_{ij} by g-th row of S_s (or Σ_s); $f, g = 1, 2, \ldots, s$

Evidently we have that

$$\sum_{j=1}^{s} X_{ij}X'_{ij} = s^2 I_s - s E_{ss} \qquad \ldots \quad (3.6)$$

and

$$\sum_{j=1}^{s} X_{ij}X'_{i'j} = 0 \qquad i, i' (i \neq i') = 1, 2, \ldots, s. \qquad \ldots \quad (3.7)$$

Let

$$S_s = ((\sigma_{ij})) \quad [\text{use } \Sigma_s \text{ if } s \equiv 3 \bmod 4] \qquad \ldots \quad (3.8)$$

Then X, the following defined matrix, is our BOD.

$$X = \begin{bmatrix} X_{11} & X_{12} & \ldots & X_{1s} & \sigma_{11}E_{s1} & \sigma_{12}E_{s1} & \ldots & \sigma_{1s}E_{s1} & E_{s1} \\ X_{21} & X_{22} & \ldots & X_{2s} & \sigma_{21}E_{s1} & \sigma_{22}E_{s1} & \ldots & \sigma_{2s}E_{s1} & E_{s1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{s1} & X_{s2} & \ldots & X_{ss} & \sigma_{s1}E_{s1} & \sigma_{s2}E_{s1} & \ldots & \sigma_{ss}E_{s1} & E_{s1} \\ \sigma_{11}E_{1s} & \sigma_{12}E_{1s} & \ldots & \sigma_{1s}E_{1s} & 1 & 1 & \ldots & 1 & 0 \\ \sigma_{21}E_{1s} & \sigma_{22}E_{1s} & \ldots & \sigma_{2s}E_{1s} & 1 & 1 & \ldots & 1 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \sigma_{s1}E_{1s} & \sigma_{s2}E_{1s} & \ldots & \sigma_{ss}E_{1s} & 1 & 1 & \ldots & 1 & 0 \\ E_{1s} & E_{1s} & \ldots & E_{1s} & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \qquad \ldots \quad (3.9)$$

4. Some applications of BOD's

4.1. *Construction of some BIB's using BOD's.*

Lemma 4.1 : *Let N be the incidence matrix of the binary incomplete block design with $v = 2k$, $b = 2r$. Let N^* be the incidence matrix of the complement of the design. Let x, y be the column vectors consisting ± 1's and zeros such that $x'y = 0$. If we replace 1's, -1's, 0's in x, y by N, N^*, $0_{v,b}$ (null matrix) then the inner product becomes $\frac{rt}{2} E_{vv}$ where t is the number of non-zero elements common between the vectors x and y.*

Proof: Use the results $NN' = N^*N^{*\prime}$ and $N^*N' = NN^{*\prime}$.

Theorem 4.1 : *Existence of BOD $(s^2+s+1,\ s^2,\ s^2-s)$ and the BIBD with parameters $\left(s+1,\ 2s,\ s,\ \frac{s+1}{2},\ \frac{s-1}{2}\right)$ implies the existence of BIBD with the parameters*

$$\left[(s+1)(s^2+s+1),\ 2s(s^2+s+1),\ s^2,\ \frac{s^2(s+1)}{2},\ \frac{s^2(s-1)}{2}\right]. \qquad \ldots \quad (4.1)$$

Proof : Let X be the BOD with the above parameters and N_1 be the incidence matrix of the BIB design $\left(s+1,\ 2s,\ s,\ \frac{s+1}{2},\ \frac{s-1}{2}\right)$. Let N_1^* be its complement. Substitute N_1, N_1^*, $0_{s+1,2s}$ (null-matrix, of order $s+1\times 2s$) in the place of 1^s, -1^s, 0^s respectively in X. We get the incidence matrix of the required design. The proof of it is simple. Let x_i', x_j' be the i-th and j-th rows of X. It is evident that after replacing $+1$, -1, 0^s by N_1, N_1^*, $0_{s+1,2s}$ in x_i, x_j $(i, j = 1, 2, \ldots, s^2+s+1)$ we have

$$x_i' x_i \text{ gives } s^2\left[\frac{s+1}{2}\, I_{s+1} + \frac{s-1}{2}\, E_{s+1,\,s+1}\right] \qquad \ldots \quad (4.2)$$

and by Lemma 4.1

$$x_i' x_j (i \neq j) \text{ gives } \frac{s(s^2-s)}{2}\, E_{s+1,\,s+1}. \qquad \ldots \quad (4.3)$$

From (4.2) and (4.3) we can easily see that the incidence matrix obtained from X, when we replace 1, -1, 0^s by N_1, N_1^*, $0_{s+1,2s}$ is incidence matrix of the required BIB design.

We know that, when s is a power of odd prime, BIBD with the parameters $\left(s+1, 2s, s, \frac{s+1}{2}, \frac{s-1}{2}\right)$ exists and by Theorem 3.2 the BOD $(s^2+s+1,\ s^2,\ s^2-s)$ exists. Hence when s is power of odd prime there always exists BIBD with the parameters given in (4.1). Below are given several values of X or X^* which may be utilised for constructing some BIB designs.

1. $$X^*(4, 4, 4) = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

2. $$X(6, 5, 4) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

$$3.\quad X^*(7,4,2)=\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$$4.\quad X(8,7,6)=\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

$$5.\quad X^*(5,10,8,4,6)=\begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 0 & 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 & -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 1 & -1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 & 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 & 1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

6. $X(9, 24, 8, 3, 2) =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	−1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
3	1	0	−1	0	0	0	0	0	−1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
4	0	1	0	−1	0	0	0	0	0	1	0	0	0	0	1	−1	0	0	0	1	1	−1	0	0
5	0	0	1	0	−1	0	0	0	0	0	1	−1	0	0	0	0	1	0	0	1	−1	0	1	0
6	0	0	0	0	0	1	−1	0	1	0	0	0	−1	0	0	0	0	1	0	0	1	1	1	0
7	0	0	0	1	0	0	−1	0	0	1	0	0	0	−1	0	0	1	0	−1	0	0	0	−1	1
8	0	0	0	0	−1	0	0	−1	0	0	−1	0	1	0	1	0	0	0	−1	0	0	1	0	−1
9	0	0	0	0	0	1	0	1	0	0	0	1	0	−1	0	1	0	−1	0	1	0	0	0	−1

Designs 5 and 6 are obtained by trial.

4.2. *Construction of PBIBD's based on BOD's*

Theorem 4.2 : *The existence of BOD (v, b, r, k, λ) implies the existence of PBIB design with the following parameters*

$$v^* = 2v,\ b^* = 2b,\ r^* = r,\ k^* = k,\ \lambda_1^* = \frac{\lambda}{2} = \lambda_3^*,\ \lambda_2^* = 0$$

$$n_1^* = v-1,\ n_2^* = 1,\ n_3^* = v-1. \qquad \ldots \quad (4.4)$$

$$P_1^* = \begin{pmatrix} v-2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & v-2 \end{pmatrix} \quad P_2^* = \begin{pmatrix} 0 & 0 & v-1 \\ 0 & 0 & 0 \\ v-1 & 0 & 0 \end{pmatrix} \quad P_3^* = \begin{pmatrix} 0 & 1 & v-1 \\ 1 & 0 & 0 \\ v-2 & 0 & 0 \end{pmatrix}$$

Proof : Let X be the BOD. Let N_1, N_2 be the matrices obtained from X and $-X$ by replacing -1's by zeros respectively. Then

$$M = \begin{pmatrix} N_1 & N_2 \\ N_2 & N_1 \end{pmatrix} \qquad \ldots \quad (4.5)$$

is the incidence matrix of the required design. See Theorem 2.3 for the proof.

This result can be easily generalised.

4.3. *Construction of GD designs for BOD's.*

Definition : A BOD becomes balanced matrix if N_1, N_2, defined in Theorem 4.2, are the incidence matrices of BOD designs. We call this BOD as BOD(•).

Balanced matrices were defined in Shah (1959).

The following theorem is from the author's paper (1966).

Theorem 4.3 : *Existence of BOD* $(v_1, b_1, r_1, k, \lambda_1)$ *and the BIBD with the parameters* $(v_2 = 2k_2,\ b_2 = 2r_2,\ \lambda_2)$ *implies the existence of group divisible design with the parameters*

$$v^* = 2v_1k_2,\ b^* = 2b_1r_2,\ r^* = r_1r_2,\ k^* = k_1k_2,\ \lambda_1^* = r_1\lambda_2,$$

$$\lambda_2^* = \frac{r_2\lambda_1}{2},\quad m^* = v_1,\ n = v_2. \qquad \dots\ (4.6)$$

Theorem 4.4 : *Existence of BOD*$(\bullet)$ $(v_1, b_1, r_1, k_1, \lambda_1)$ *and the BIBD's* $(v_2, r_2, k_2, \lambda_2)$ *implies the existence group divisible design with the parameters*

$$v^* = v_1v_2,\ b^* = b_1b_2,\ r^* = r_{11}r_2 + r_{12}(b_2 - r_2),\ k^* = r_{11}r_2 + r_{12}(v_2 - K_2)$$

$$\lambda_1^* = r_{11}\lambda_2 + r_{12}(b_2 - 2r_2 + \lambda_2),\ \lambda_2^* = \lambda_{11}r_2 + \lambda_{12}(b_2 - r_2),\ m^* = v_1,\ n^* = v_2 \qquad \dots\ (4.7)$$

where $(v_1, b_1, r_{11}, k_{11}, \lambda_{11})$, $(v_1, b_1, r_{12}, k_{12}, \lambda_{12})$ *are the parameters of the BIBD's obtainable from the BOD*$(\bullet)$.

The proof of it is similar to the proof of Theorem 4.1 given by the author (1966a).

The following are some group divisible designs which are not available in the tables of two associate partially balanced designs of Bose, Clatworthy and Shrikhande (1954).

sl. no.	v	b	r	k	m	n	λ_1	λ_2	Reference
1	12	12	5	5	4	3	1	2	Th.4.4, $X^*(4,4,4)$, $N(3, 1, 0,)$
2	12	12	7	7	4	3	3	4	complement of the design 1
3	12	12	5	5	6	2	0	2	Th. 4. 3, $X(6, 5, 4)$, $N(2, 1, 0)$
4	12	24	10	5	4	3	2	4	repeat the design 1 once again
5	12	24	10	5	6	2	0	4	repeat the design 3
6	16	16	7	7	8	2	0	3	Th. 4. 3, $X(8, 7, 6)$, $N(2, 1, 0)$
7	20	20	7	7	4	5	3	2	Th. 3. 4, $X^*(4, 4, 4)$, $N(5, 1, 0)$
8	21	21	7	7	7	3	3	2	Th. 3. 4, $X^*(7. 4. 2)$, $N(3, 2, 1)$
9	24	24	8	8	4	6	4	2	Th. 4. 4, $X^*(4, 4, 4)$, $N(6, 1, 0)$
10	28	28	6	6	7	4	2	1	Th. 4. 4, $X^*(7,4, 2)$, $N(4, 1, 0)$
11	28	28	9	9	4	7	5	2	Th. 4. 4, $X^*(4, 4, 4)$, $N(7, 1, 0)$
12	28	28	10	10	7	4	6	3	Th. 4. 4, $X^*(7. 4, 2)$, $N(4, 3, 2)$
13	32	32	10	10	4	8	6	2	Th. 4. 4, $X^*(4, 4, 4)$, $N(8, 1, 0)$
14	35	35	7	7	7	5	3	1	Th. 3. 4, $X^*(7, 4, 2)$ $N(5, 1, 0)$
15	42	42	8	8	7	6	4	1	Th. 4. 4, $X^*(7, 4, 2)$, $N(6, 1, 0)$
16	49	49	9	9	7	7	5	1	Th. 4. 4, $X^*(7, 4, 2)$, $N(7, 1, 0)$
17	56	56	10	10	7	8	6	1	Th. 4. 4, $X^*(7, 4, 2)$, $N(8, 1, 0)$

In the above table $X^*(v, K, \lambda)$ means *BOD* $(\bullet)$ with the parameters (v, K, λ), $X(v, K, \lambda)$ means *BOD* (v, K, λ) and $N(v, K, \lambda)$ means the incidence matrix of the *BIBD* with the parameters $(v, K; \lambda)$.

REFERENCES

BOSE, R. C., CONNOR, W. S. (1952): Combinatorial properties of group divisible incomplete block designs. *Ann. Math. Stat.*, 23, 367-385.

BOSE, R. C., CLATWORTHY, W. H., SHRIKHANDE, S. S. (1954): Table of partially balanced designs with two associate classes, North Carolina Agricultural Experiment Station, *Technical Bulletin* No. 107.

RAO, M. B. (1966a): Group divisible family of PBIB designs. *Jour. Ind. Stat.*, 4, 14-28.

——— (1966b). Weighing designs when *n* is odd. *Ann. Math. Stat.*, 37, 1371-1381.

RAO, S. V. P., DAS, M. N. (1969): Incomplete weighing designs. *Jour. Ind. Soc. Agri. Stat.*, 21, 67-79.

SHAH, B. V. (1959): On generalisation of the Kronecker product designs. *Ann. Math. Stat.*, 30, 48-54.

WILLIAMSON, J. (1944): Hadamard's determinant theorem and the sum of 4 squares. *Duke. Math. Jour.*, 11, 65-82.

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