# CONSTRUCTION OF SOME GENERAL AND TWO LEVEL PARTIALLY BALANCED ARRAYS

*By* A. C. MUKHOPADHYAY

*Indian Statistical Institute*

*SUMMARY*. A method is given for obtaining PBA's of strength 3 from equireplicate pairwise balanced arrays. It is proved that the existence of a PBA of strength $2t$ with 2 levels implies the existence of a PBA with 2 levels of strength $2t+1$, for any positive integer $t$. A simple method of obtaining two level PBA's of strength 2 and 3 from the incidence matrices of BIBD's is also discussed.

## 1. Definitions and Notation

'Partially balanced array' (PBA) was first introduced by Chakravarty (1956) as a substitute for 'Orthogonal array' (OA), both serving the purpose of fractional replicates of factorial experiments. It is shown by Chakravarty (1956) that by considering a PBA in place of an OA it may be possible to reduce the size without sacrificing the essential orthogonality properties of the estimates of the factorial effects satisfied by the corresponding OA, although estimation may be a bit more complicated in PBA. The definition of a PBA as given by Chakravarty (1956; 1961) is as follows :

> A PBA $[N, k, s, d]$ of strength $d$, size $N$ with $k$ constraints or factors and $s$ levels for each factor is a subset of $N$ treatment combinations from an $s^k$ factorial experiment with the property that for any group of $d$ factors ($d \leqslant k$), a combination of the levels of $d$ factors $(i_1, i_2, \ldots i_d)$ $(i_j = 0, 1, 2, \ldots, s-1, \forall\, i_j)$ occurs $\lambda_{i_1, i_2 \ldots i_d} (>0)$ times, where $\lambda_{i_1, i_2 \ldots i_d}$ remains the same for all permutations of a given set $(i_1, i_2, \ldots, i_d)$ and for any group of $d$ factors. If in the definition of PBA, $\lambda_{i_1 \ldots i_d}$ is a constant, not depending on the levels $i_1, i_2, \ldots, i_d$, the resulting array is an OA.

For the present paper a few more definitions are given.

An equireplicate pairwise balanced design (Bose *et al*, 1960) with parameters ($v$; $k_1, k_2 \ldots, k_m$; $b_1, b_2 \ldots, b_m$; $r, \lambda$) is an arrangement of $v$ varieties in blocks of $m$ different sizes $k_1, k_2 \ldots, k_m$, there being $b_i$ blocks of size $k_i$, $\sum_{i=1}^{m} b_i = b$ satisfying the condition that (i) no block contains a single variety more than once, (ii) each variety occurs in $r$ blocks and (iii) any two varieties occur together in $\lambda$ blocks.

An equireplicate triple-wise balanced design is an equireplicate pairwise balanced design where any three varieties occur together in a constant number, say $\mu$, of blocks and may be represented as the design ($v$; $k_1 \ldots, k_m$; $b_1 \ldots, b_m$; $r, \lambda, \mu$).

## 2. PBA's of strength 3

Lemma 2.1 : *The existence of an equireplicate pairwise balanced design with parameters* ($v$; $k_1, \ldots, k_m$; $b_1, \ldots, b_m$; $r$, $\lambda$) *implies the existence of another pairwise balanced design* *with parameters* $(v; v-k_1, v-k_2, \ldots, v-k_m; b_1, \ldots, b_m; b-r, b-2r+\lambda)$ *and the two sets of blocks given by the two designs together constitute a triply balanced design of* $v$ *varieties in* $2b$ *blocks with each pair of varieties occurring together in* $b-2r+2\lambda$ *blocks and each triplet occurring in* $b-3r+3\lambda$ *blocks.*

*Proof :* The second design is obtained from the first as follows. To each block of the first design, there corresponds a block of the second design consisting of all the varieties absent in the former. The parameters of the second design are obvious and hence the number of times a pair of varieties occur together in the combined design is $b-2r+2\lambda$. Now, let $\mu$ be the number of times a given triplet of varieties occurs together in the first design. Then, the number of times the triplet occurs in the second design is $b-3r+3\lambda-\mu$. So, in the combined design the triplet occurs $b-3r+3\lambda$ times.

Theorem 2.1 : *The existence of an equireplicate triple-wise balanced design with parameters* $(v; k_1, \ldots, k_m; b_1, \ldots, b_m; r, \lambda, \mu)$ *and* OA's $(pk_i^3, t_i, k_i, 3)$, $i = 1, 2, \ldots, m$ *implies the existence of* PBA $(p \sum_{i=1}^{m} k_i^3 b_i, t, v, 3)$, *where* $t = \min (t_1, t_2 \ldots, t_m)$ *and* $\lambda_{iii} = pr, \forall i$; $\lambda_{iij} = p\lambda, \lambda_{ijj} = p\lambda, i \neq j, \forall i, j$; $\lambda_{ijk} = p\mu$, $i \neq j \neq k, \forall i, j, k$.

*Proof :* The proof is similar to that of Theorem 4.1 of Chakravarty (1950).

Theorem 2.2 : *The existence of an equireplicate pairwise balanced design with parameters* $(v; k_1, \ldots, k_m; b_1, \ldots, b_m; r, \lambda)$, OA's$(p k_i^3, t_i, k_i; 3)$ *and* OA's $[p(v-k_i)^3, t_i', (v-k_i), 3]$. $i = 1, 2, \ldots, m$ *implies the existence of a* PBA $\left[ p \sum_{i=1}^{m} \left\{k_i^3+(v-k_i)^3\right\}, t, v, 3\right]$, *where* $t = \min (t_1, \ldots, t_m, t_1'; \ldots, t_m')$ *and* $\lambda_{iii} = p.b., \forall i$; $\lambda_{iij} = p(b-2r+2\lambda)$, $\lambda_{ijj} = p(b-2r+2\lambda)$, $i \neq j$, $\forall i, j$, $\lambda_{ijk} = p(b-3r+3\lambda)$, $i \neq j \neq k. \forall i, j, k$.

*Proof :* The result follows from Lemma 2.1.

Corollary 2.1: *The existence of a* BIBD $(v, b, r, k, \lambda)$, OA $[pk^3, t_1, k, 3]$ *and* OA $[p(v-k)^3, t_2, (v-k), 3]$ *implies the existence of a* PBA $[pb\{k^3+(v-k)^3\}, t, v, 3]$ *with* $\lambda$ *parameters same as in Theorem 2.2.*

## 3. Two Level PBA's

Let $x$ and $y$ be two $t\times 1$ column vectors with $x' = (x_1, x_2, \ldots, x_t)$, $y' = (y_1, y_2, \ldots, y_t)$ where $x_i \neq y_i$ and $x_i$'s and $y_i$'s are elements in the class of residues mod 2, $\forall i$. Here, $y$ may be called the complement of $x$.

From the definition of a PBA, in any $t\times N$ submatrix of the matrix of a PBA $[N, k, 2, t]$, the number of times a $t\times 1$ column vector $x$ occurs depends only on the number of 1's in the vector and not on thier arrangement.

Lemma 2.1 : *Let in a* $t\times N$ *matrix with elements in the class of residues mod 2, which is* PBA *of strength* $t-1$, *a* $t\times 1$ *column vector* $x$ *occurs* $\mu$ *times. Then its complement* $y$ *occurs* $c+(-1)^t\mu$ *times, where* $c$ *depends only on the number of* 1's *in* $x$.

*Proof:* The number of times $y$ occurs is evidently $c+(-1)^t\mu$, where

$$c = N - \sum_{i_1=1}^{t} N_{i_1} + \sum_{\substack{i_1, i_2=1 \\ i_1 < i_2}}^{t} N_{i_1 i_2} + \ldots + (-1)^{t-1} \sum_{\substack{i_1, i_2, \ldots, i_{t-1}=1 \\ i_1 < i_2 \ldots < i_{t-1}}}^{t} N_{i_1 i_2 \ldots i_{t-1}}$$

and

$$N_{i_1 i_2 \ldots i_k}, \; i_1 < i_2 < \ldots, < i_k, \quad k = 1, 2, \ldots t-1 \qquad \ldots \quad (3.1)$$

denotes the number of columns in the matrix with $i_j$-th element same as that of $x$, viz. $x_{i_j}$, $\forall i_j$. The matrix being a PBA of strenth $(t-1)$, $N_{i_1, i_2, \ldots, i_k}$ in (3.1) depends only on the number of $x_{i_j}$'s equal to 1, $\forall i_j$. Thus $c$ depends only on the number of 1's in $x$.

Theorem 3.1 : *The existence of a* PBA$[N, k, 2, 2p]$ *implies the existence of a* PBA $[2N, k, 2, 2p+1]$, *for any positive integer* $p$.

*Proof:* Let $A_1$, a $k \times N$ matrix give the PBA $[N, k, 2, 2p]$ with 2 levels 0 and 1. Let $A_2$ be another $k \times N$ matrix, obtained from $A_1$ by writing 1 for each 0 and 0 for each 1 in $A_1$. $A = [A_1 : A_2]$ gives the required PBA of strength $2p+1 = t$, the number of times a $t \times 1$ column vector $x$ occurs in any $t \times 2N$ submatrix of $A$ is, from Lemma 2.1, $c+(-1)^t\mu + \mu = c$ i.e. depends only on the number of 1's in $x$.

Corollary 3.1 : *The existence of a* PBA $[N, k, 2, 2]$ *with*

$$\begin{aligned} \lambda_{x_1 x_2} &= \lambda_1, \quad x_1 = 0, \quad x_2 = 0 \\ &= \lambda_2, \quad x_1 = 1, \quad x_2 = 1 \\ &= \lambda_3, \; \textit{one of } x_1 \textit{ and } x_2\textit{'is 1 and the other 0,} \end{aligned}$$

*implies the existence of a* PBA $[2N, k, 2, 3]$ *with*

$$\begin{aligned} \lambda_{x_1 x_2 x_3} &= N - 3\lambda_3, & & x_1 = x_2 = x_3 = 0 \\ & & & \textit{or } x_1 = x_2 = x_3 = 1 \\ &= N - \lambda_1 - \lambda_2 - \lambda_3, & & \textit{one of } x\textit{'s is 1, others 0} \\ & & & \textit{or one of } x\textit{'s is 0, others 1.} \end{aligned}$$

From Theorem 2.1 of Chakravarty (1961) it is known that the incidence matrix of a BIBD with parameters $v, b, r, k, \lambda$ gives a PBA$[b, v, 2, 2]$ with

$$\begin{aligned} \lambda_{x_1 x_2} &= \lambda & & x_1 = x_2 = 1 \\ &= b - 2r + \lambda, & & x_1 = x_2 = 0 \\ &= r - \lambda, & & \text{one of } x\text{'s is 1, other 0.} \end{aligned}$$

From this PBA of strength 2, a PBA of strength 3 can be derived. In general, the existence of a tactical configuration $\lambda-\mu-\nu$ of $m$ elements implies the existence of a PBA of strength $\mu+1$ with 2 levels under the conditions of Theorem 2.1 of Chakravarty (1961) provided $\mu$ is an even integer.

REFERENCES

BOSE, R. C., SHRIKHANDE, S. S. and PARKER, E. T. (1960) : Further results on the construction of mutually orthogonal latin squares and the falsity of Euler's conjecture. *Canad. J. Math*, **12**., 189-203.

CHAKRAVARTY, I. M. (1956) : Fractional replication in asymmetrical factorial designs and partially balanced arrays. *Sankhyā*, **17**, 143-164.

——— (1961) : On some method of construction of PBA's. *Ann. Math. Stat.*, **32**, 1181-1185.

*Paper received : April, 1970.*