A NOTE ON λ AND (r, λ) SYSTEMS

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SUMMARY. For the λ- and (r, λ) systems defined by Mullin and Stanton (1966) with v varieties, b blocks and average replication per variety $\bar{r}$ [$\bar{r} = r$ for a (r, λ)-system], it is proved that $b \geqslant \frac{v\bar{r}}{k_0}$ where $k_0 = \frac{\lambda(v-1)}{\bar{r}} + 1$, the equality implying that the system is a BIBD. It is also proved that any such system with $\lambda(v-1) \geqslant \bar{r}(\bar{r}-1)$ is a symmetrical BIBD if $b = v$. A counter example to conjecture 1 in the reference is also provided.

1. Preliminary definitions and notations

The definitions of λ and (r, λ) systems occur in Mullin and Stanton (1966). λ systems do not include (r, λ) systems, because blocks consisting of a single element are not permitted in the former, while they are permitted in the latter. So, deleting the condition L_2 of λ systems in Mullin and Stanton (1966), let us define $(n*\lambda)$ systems as follows:

A collection of b subsets (called blocks) of a set V of v varieties is said to form a $(n*\lambda)$ system when the following axioms are satisfied :

I : every pair of varieties occurs in precisely λ blocks ;

II : sum of the block sizes, giving the total number of points in the system is n.

Associated with every $(n*\lambda)$ system there is a sequence of non-negative integers $B = (b_1, b_2, b_3 \ldots)$, where b_i is the number of blocks containing exactly i varieties, $i \geqslant 1$, b_i's being all zero after a certain stage. Also associated is a sequence of non-negative integers $(r_1, r_2, \ldots, r_v)$ where r_i is the number of blocks which contain i-th variety (also called the replication of i-th variety), $r_i \geqslant \lambda$, $i = 1, 2, \ldots, v$, the inequality being strict for at least one i.

Obviously for a $(n*\lambda)$ system with v varieties,

$$\sum_{i=1}^{\infty} i b_i = \sum_{i=1}^{v} r_i = n$$

and

$$\sum_{i=1}^{\infty} \binom{i}{2} b_i = \lambda . \binom{v}{2}.$$

A $(n*\lambda)$ system becomes an (r, λ) system of Stanton and Mullin (1966) when $r = r_1 = r_2 \ldots = r_v$

For a $(n*\lambda)$-system let us define average replication of the varieties as

as

$$\bar{r} = \frac{\sum_{i=1}^{v} r_i}{v} = \frac{n}{v} .$$

2. Main results

Theorem 2.1 : *In a (n, λ) system with v varieties, the total number of blocks b satisfies the inequality $b \geqslant \frac{n}{k_0}$, where $k_0 = \frac{\lambda(v-1)}{\bar{r}}+1$. The equality implies the system is a* BIBD *with parameters $v, b, r = \bar{r}, k = k_0, \lambda$.*

Proof : For a (n, λ) system with v varieties

$$\sum_{i=1}^{\infty} b_i = b \qquad \ldots \quad (2.1)$$

$$\sum_{i=1}^{\infty} ib_i = n = v\bar{r} \qquad \ldots \quad (2.2)$$

$$\sum_{i=1}^{\infty} \binom{i}{2} b_i = \lambda . \binom{v}{2}. \qquad \ldots \quad (2.3)$$

From (2.2) and (2.3),

$$\sum_{i=1}^{\infty} i^2 b_i = \lambda v(v-1)+v\bar{r}$$

$$= v\bar{r}(k_0-1)+v\bar{r}, \text{ from the given expression for } k_0.$$

$$= v\bar{r}k_0. \qquad \ldots \quad (2.4)$$

By Cauchy-Schwartz inequality,

$$\left(\sum_{i=1}^{\infty} b_i\right)\left(\sum_{i=1}^{\infty} i^2 b_i\right) \geqslant \left(\sum_{i=1}^{\infty} ib_i\right)^2,$$

which on simplification gives

$$b \geqslant \frac{v\bar{r}}{k_0}. \qquad \ldots \quad (2.5)$$

Equality in (2.5) implies $\frac{i\sqrt{b_i}}{\sqrt{b_i}} = i$ is constant for all $i \geqslant 1$, which is impossible unless i takes only one value, say k. In that case $b_i = 0$ for all $i \neq k$ and $b_k = b$. i.e., $kb_k = v\bar{r}$ and $k^2b_k = v\bar{r}k_0$. Hence $k = k_0$. This implies that k_0 must be a positive integer. The resulting (n, λ) system with v varieties is such that there are b blocks, each of same size k_0 and each pair of varieties occurs together in precisely λ blocks. Then by Theorem 2 of Mullin and Stanton (1960), the system is a BIBD with replication for each variety $\bar{r}$. This implies again that $\bar{r}$ is a positive integer.

It is to be noted that we did not assume $\bar{r}$ and k_0 to be positive integers, but the equality in (2.5) implies that they are so.

Corollary : *Non-existence of a* BIBD *with parameters v, b, r and $\lambda \Longrightarrow$ the non-existence of a (n, λ) system, n being given by vr with v varieties and b blocks and in particular, the non-existence of a (r, λ) system with v varieties and b blocks.*

Following Mullin and Stanton (1966), we can define a $(n*\lambda)$ system to be elliptic, parabolic or hyperbolic according as the expression $\lambda(v-1)-\bar{r}(\bar{r}-1)$ is negative, zero or positive.

Theorem 2.2 : *A non-hyperbolic $(n*\lambda)$ system with v varieties and b blocks is a symmetrical* BIBD *if $b = v$.*

Proof : As the $(n*\lambda)$ system is non-hyperbolic,

$$\bar{r}(\bar{r}-1) \geqslant \lambda(v-1). \quad \ldots \quad (2.6)$$

Defining

$$k_0 = \frac{\lambda(v-1)}{\bar{r}}+1,$$

$$\bar{r}(k_0-1) = \lambda(v-1) \leqslant \bar{r}(\bar{r}-1)$$

$$\therefore \qquad k_0 \leqslant \bar{r}. \quad \ldots \quad (2.7)$$

Again, the result (2.5) with $b = v$ implies

$$k_0 \geqslant \bar{r}. \quad \ldots \quad (2.8)$$

From (2.7) and (2.8), $k_0 = \bar{r}$. This implies equality in (2.5). So, the system is BIBD by Theorem 2.1 and it is symmetrical because $b = v$.

3. Conjecture by Mullin and Stanton

A counterexample is provided to conjecture 1 in Mullin and Stanton (1966), in the following lines. The conjecture states :

'For $\lambda \leqslant 2$(and perhaps all λ), $\lambda(v-1) = r(r-1)$ implies $v = b$ if the corresponding design is irreducible'.

Here by 'design' is meant an (r, λ)-system. In Mullin and Stanton (1966), a design has been termed irreducible if it contains neither a complete block consisting of all v varieties nor a set of v single element blocks whose union is V.

The following counter example disproves the conjecture for $\lambda = 2$. The example gives an irreducible (r, λ)-system with $r = 4$, $\lambda = 2$ and $v = 7$, so that $\lambda(v-1) = r(r-1)$, but $b = 8$.

Blocks in the system are :

(1 2 3 4),

(1 2 5 6),

(1 3 5 7),

(1 4 6 7),

(2 3 4 5 6 7), (27), (36), (45).

Similar counter examples can be easily provided for (r, λ) systems with $\lambda > 2$. Hence, it can be asserted that the condition $\lambda(v-1) = r(r-1)$ is sufficient for an irreducible (r, λ) system to be a symmetrical BIBD only when $\lambda = 1$.

REFERENCE

MULLIN, R. C. and STANTON, R. G. (1966) : Inductive methods for BIBD's. *Ann. Math. Stat.*, 37, 1348-1354.

Paper received : October, 1971.