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ON SPECTRA OF NON-SINGULAR TRANSFORMATIONS AND FLOWS

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SUMMARY. In this note we prove that the spectrum of the unitary operator associated with a non-singular transformation on a continuous measure space is the whole unit circle.

In this note we prove that the spectrum of the unitary operator associated with a non-singular transformation on a continuous measure space is the whole unit circle, thus, in particular, answering the question, raised by Feldman (1974, p. 391) of describing those closed subsets of the unit circle which appear as spectrum of the unitary operator associated with an ergodic measure preserving transformation. An analogous result for an ergodic non-singular flow is also proved. These results are rather natural consequences of appropriate Rokhlin-Kakutani theorems and we shall recall them below as we need.

1. Let $(X, \mathcal{B}, m)$ denote unit interval X, Borel σ-algebra and Lebesgue measure m. Let $T: X \to X$ be an invertible non-singular ergodic transformation, i.e., an ergodic automorphism of X (null sets preserving but not necessarily measure preserving). Let ϕ be a measurable function on X, complex valued and of absolute value one. Define unitary operator U on $L_2(X, \mathcal{B}, m)$ by

$$(Uf)(x) = \phi(x) f(Tx) \left(\frac{dm_T}{dm}(x)\right)^{1/2}, \quad x \in X$$

where m_T is the measure defined by $m_T(B) = m(TB)$.

We now show that the spectrum of U is the whole unit circle. To this end it is enough to show that given any λ in the unit circle and $\varepsilon > 0$ there is a function f in $L_2(X, \mathcal{B}, m)$ of unit norm such that $\| U_f - \lambda f \| < \varepsilon$. Now Friedman (1970, p. 108) proves the following Rokhlin-Kakutani theorem for non-singular transformations.

If T is an ergodic automorphism on $(X, \mathcal{B}, m)$ then given $\delta > 0$ and a positive integer n there exists a measurable set A such that $A, TA, \ldots T^{n-1} A$ are disjoint and $m(X - \bigcup_{k=0}^{n-1} T^k A) < \delta$.

Let us write C for the set $X - \dot{\bigcup}_{k=0}^{n-1} T^k A$. We choose A corresponding to δ and n where $\delta < \frac{\varepsilon^2}{4}$ and $\frac{1}{n} < \frac{\varepsilon^2}{4}$.

Our definition of f is dictated by the requirement that Uf should equal λf on a large set. We set f equal to a constant a on A, which will be chosen presently. Inductively define for

$$1 \leq k \leq n-1,\ f(x) = \lambda\bar{\phi}(T^{-1}x)\left(\frac{dm}{dm_T}(T^{-1}x)\right)^{\frac{1}{2}} f(T^{-1}x),\ x \in T^k A$$

and finally set f equal to 1 on C, the rest of X. ($\bar{\phi}$ denotes the complex conjugate of ϕ). It is easy to see that

$$\int_{T^k A} |f|^2\, dm = \int_{T^{k-1}A} |f|^2\, dm.$$

Hence $\|f\|^2 = na^2 m(A) + m(C)$. Now choose

$$a = +\left(\frac{1-m(C)}{nm(A)}\right)^{\frac{1}{2}} \text{ so that } \|f\|^2 = 1.$$

For this f note that

$$(Uf)(x) = \lambda f(x) \text{ for } x \in \bigcup_{k=0}^{n-2} T^k A \qquad \ldots \quad (1)$$

and

$$\int_{T^k A} |f|^2\, dm \leq \frac{1}{n},$$

hence in particular

$$\int_A |f|^2\, dm \leq \frac{1}{n},\ \int_{T^{n-1}A} |f|^2\, dm \leq \frac{1}{n}. \qquad (2)$$

Finally using (1) and (2) we get

$$\|Uf - \lambda f\|^2 = \int_{T^{n-1}A \cup C} |Uf - \lambda f|^2\, dm$$

$$\leq 2\left(\frac{1}{n} + m(C)\right) < \varepsilon^2.$$

Hence $\|Uf - \lambda f\| < \varepsilon$ and we are done.

2. Let T_t, $t \in R$ the real numbers, be a group of ergodic non-singular transformations on $(X, \mathcal{B}, m)$ such that for each $A \in \mathcal{B}$, $t \to m(T_t A)$ is continuous. We call T_t, $t \in R$, an ergodic flow. Let U_t, $t \in R$ be a group of unitary operators on $L_2(X, \mathcal{B}, m)$ defined by

$$(U_t f)(x) = a(t, x) f(T_t x) \left(\frac{dm_t}{dm}(x)\right)^{\frac{1}{2}}, \quad x \in X \qquad \ldots \quad (3)$$

where a is a measurable function on $R \times X$ of absolute value 1 and satisfying

$$a(t_1 + t_2, x) = a(t_1, x)\, a(t_2, T_{t_1} x). \qquad \ldots \quad (4)$$

This U_t, $t \in R$, is a strongly continuous group of unitary operators. By Stones theorem we have

$$U_t = \int_R e^{it\lambda}\, dE(\lambda), \quad t \in R.$$

Theorem : *Support of E is the whole real line, where support means the smallest closed subset of R complement of which has E measure zero.*

The theorem will be proved if we show that given any real λ, positive real K and $\varepsilon > 0$ there exists f in $L^2(X, \mathcal{B}, m)$ such that $\|f\| = 1$ and

$$|(U_t f, f) - e^{it\lambda}| < \varepsilon \text{ for } 0 \leqslant t \leqslant K \qquad \ldots \quad (5)$$

where $(U_t f, f)$ denotes the inner product between $U_t f$ and f. For this the n means that we can find a sequence f_n of functions in $L^2(X, \mathcal{B}, m)$ of norm one such that the sequence of measures m_n on the real line defined by $m_n(A) = (E(A) f_n, f_n)$ converges weakly to unit mass at λ, which in turn means that every neighbourhood of λ has non-trivial E measure. We will prove (5) in Section 4 after recalling in Section 3 recent extensions of Ambrose-Kakutani theorem and Rokhlin-Kakutani theorem to (non-singular) flows.

3. Given a flow S_t, $t \in R$ on a measure space (Y, τ, n) where $n(Y) = 1$, we say that $S_t, t \in R$ is isomorphic to $T_t, t \in R$, if there exists a one-one bimeasurable map $\phi : X \to Y$ which preserves null sets and satisfies $T_t = \phi^{-1} S_t \phi$, $t \in R$. If we form the unitary group V_t, $t \in R$ on $L^2(Y, \tau, n)$ by

$$(V_t f)(y) = b(t, y) f(S_t y) \left(\frac{dn_t}{dn}(y)\right)^{\frac{1}{2}}, \quad y \in Y \qquad \ldots \quad (6)$$

where

$$b(t, y) = a(t, \phi^{-1}(y)) \text{ and } a \text{ is as in (4),} \qquad \dots \quad (7)$$

then

$$V_t,\ t \in R \text{ and } U_t,\ t \in R \text{ are unitarily equivalent.}$$

Indeed if $l = m \circ \phi^{-1}$

and

$$(Q.f)(y) = f(\phi^{-1}y) \left(\frac{dl}{dm} (y) \right)^{\frac{1}{2}} \qquad \dots \quad (8)$$

then Q is an invertible isometry from $L_2(X, \mathcal{B}, m)$ to $L_2(Y, \tau, n)$ such that

$$Q^{-1}V_tQ = U_t, \quad t \in R.$$

Consequently the associated spectral measures are equivalent (unitarily).

Let $(X_1, \mathcal{B}_1, m_1)$ be a measure space isomorphic to unit interval endowed with Borel σ-algebra and Lebesgue measure except that $m_1(X_1)$ although finite need not be one. Let $S : X_1 \rightarrow X_1$ be an ergodic automorphism and let F be a Borel function on X_1 non-negative and such that $\int_{X_1} F\, dm_1 = 1$. Give R the Borel σ-algebra and Lebesque measure. Give $X_1 \times R$ the product σ-algebra and product measure. Restrict these to the portion under the graph of F, i.e., to the set

$$Y = \{(x, t) : 0 \leqslant t < F(x)\}.$$

Let (Y, τ, n) be this new measure space. Define on Y a flow S_t, $t \in R$, as follows :

Each point (x, u) moves straight up at unit speed until it hits $(x, F(x))$. It then goes to $(Sx, 0)$ and continues to move up at unit speed and so on up to time t. Point thus reached at time $t > 0$ is defined to be $S_t(x, u)$. $S_t(x, u)$ for $t < 0$ is obtained by moving downward with unit speed until it hits $(x, 0)$. It then moves to $(S^{-1}x, F(S^{-1}x))$ and starts moving down with unit speed and so on. Point thus reached at time $|t|$ is $S_t(x, u)$. The flow S_t, $t \in R$, thus defined is called a flow built under the function F. $(X_1, \mathcal{B}_1, m_1)$ is called the base space and S the base transformation. Krengel (1969) and Dani (1976, p. 129) have proved Ambrose-Kakutani theorem for non-singular flows according to which every ergodic non-singular flow is isomorphic to a flow built under a function. Now Ornstein (1974, p. 63) has proved

Rokhlin-Kakutani theorem for measure preserving flows and essentially the same proof combined with result of Krengel and Dani yields the following version of Rokhlin-Kakutani theorem for non-singular flows.

Let T_t, $t \in R$, be an ergodic non-singular flow and $\varepsilon > 0$, $N > 0$ be given. Then T_t, $t \in R$ is isomorphic to a flow built under a function F with a base space $(X_1, \mathcal{B}_1, m_1)$ where $F = N$ on a set

$$\bar{X} \subseteq X_1,\ m(\bar{X}) > (1-\varepsilon)m(X_1)$$

and

$$F < N \text{ on the rest of } X_1,$$

i.e., on $X_1 - \bar{X}$.

4. Equipped with this theorem we can now prove (5). Let λ real and $K > 0$, $\varepsilon > 0$ be given. Let N be a positive integer so large that $\frac{1}{N} < \varepsilon/4$. Replace the flow T_t, $t \in R$, by an isomorphic flow S_t, $t \in R$, built under a function F on a base space $(X_1, \mathcal{B}_1, m_1)$ such that

$$F = NK \text{ on a set } \bar{X} \subseteq X_1 \text{ of measure} > \left(1-\frac{\varepsilon}{4NK}\right) m_1(X_1)$$

and

$$F \leqslant NK \text{ on the rest of } X_1.$$

Since total integral of F is one, if N is large $m_1(X_1)$ would be less than one and we assume this to be the case.

Let (Y, τ, n) denote the space on which S_t, $t \in R$, acts and let $Z = \bar{X} \times [0, NK)$. Then $n(Y-Z) < \varepsilon/4$. We also note that

$$\frac{dn_t}{dn}(x, u) = 1 \text{ if } (x, u) \in Z \text{ and } u+t < NK. \qquad \dots \quad (9)$$

Let $b(t, (x, u))$ be the transplant of the function a appearing (3) to the space (Y, τ, n) (cf.(7)). Then b satisfies the cocycle identity :

$$b(t+s, (x, u)) = b(t, (x, u))\, b(s, S_t(x, u)) \qquad \dots \quad (10)$$

because a satisfies (4).

Now define

$$g(x, u) = \begin{cases} e^{iu\lambda}\, \bar{b}(u, (x, 0)), & (x, u) \in Z \\ 1 & \text{otherwise} \end{cases}$$

where

$\bar{b}$ denotes complex conjugate of b. Clearly $|g| = 1$ hence $\|g\| = 1$.

If V_t, $t \in R$, be as in (6) then using cocycle identity (10) of b and (9) we get

$$(V_t g)(x, u)\, \bar{g}\,(x, u) = e^{it\lambda}, \text{ for } 0 \leqslant t \leqslant K,$$

$$x \in \bar{X},\ 0 \leqslant u < NK - t.$$

A calculation also shows that for $0 \leqslant t \leqslant K$

$$\int_B V_t g \,.\, \bar{g}\, dn < \frac{\epsilon}{2}$$

where

$$B = Y - \bar{X} \times [0, NK - t).$$

Thus we have for $0 \leqslant t \leqslant K$

$$(V_t g, g) = e^{it\lambda}\, n(\bar{X} \times [0, NK - t)) + \int_B V_t g \,.\, \bar{g}\, dn.$$

Since

$$n(\bar{X} \times [0, NK - t)) \geqslant n(Z) - \frac{1}{N} > 1 - \frac{\epsilon}{2},$$

we see that

$$|(V_t g, g) - e^{it\lambda}| < \epsilon \quad 0 \leqslant t \leqslant K.$$

Finally if $f = Q^{-1} g$, where Q is as in (8) then it is clear that

$$|(U_t f, f) - e^{it\lambda}| < \epsilon; \quad 0 \leqslant t \leqslant K$$

and the proof is finished.

Remark 1 : Rokhlin-Kakutani theorem is valid for aperiodic transformation and flows, hence the results of this paper are valid for such transformations and flows.

Remark 2 : If the flow T_t, $t \in R$ is measure preserving then the isomorphic flow built under a function can also be chosen to be measure preserving and so also the function which establishes the isomorphism. Under such situation, since g is of absolute value one, so will be f. Thus if T_t, $t \in R$, is measure preserving then given any λ and $\varepsilon > 0$ we can choose a function f of absolute value one such that the measure $(E(\cdot)f, f)$ puts mass bigger than $1-\varepsilon$ in a pre-assigned neighbourhood of λ. This result is descriptive as against the analytic one due to Helson (1976, Theorem 1) where it is shown that for a certain flow on Bohr group there is a function q of absolute value one such that

$$\int_{-\infty}^{\infty} e^{itu} (E(du)\, q, q), \quad \text{for all real } t.$$

Remark 3 : We could have made U_t, $t \in R$, act on $L^2_H (X, \mathcal{B}, m)$, i.e., L^2 space of function taking values in a separable complex Hilbert Space H. The cocycle a would then be unitary operator valued. It is clear that spectrum of such a U_t, $t \in R$, would also be whole of R.

Remark 4 : Assume that T is measure preserving automorphism on $(X, \mathcal{B}, m)$ and let $Uf = f \circ T$, $f \in L^2(X, \mathcal{B}, m)$. Let us say that a function ϕ on X is ε-eigenfunction with eigen value λ if $\| U\phi - \lambda\phi \| < \varepsilon$. It is clear from our construction that for each λ of absolute value one we can find an ε-eigenfunction ϕ_λ such that $|\phi_\lambda| = 1$ and $\phi_{\lambda_1 . \lambda_2} = \phi_{\lambda_1} . \phi_{\lambda_2}$ for all λ_1, λ_2 in the circle group.

Remark 5 : Every spectral measure E on the unit circle has associated with it a complete invariant which determines E up to unitary equivalence. When E acts on a separable Hilbert Space this invariant is a sequence of mutually singular measure classes together with an increasing sequence of cardinal numbers $\leqslant \aleph_0$. What is the subclass of these invariants which come from spectral measure of unitary operators arising from measure preserving transformations ? Answer to this question does not seem to be known. We refer the reader to the paper of Chacon (1970) and the bibliography therein for some literature on the topic.

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