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ON THE ROBUSTNESS OF THE LRT WITH RESPECT TO SPECIFICATION ERRORS IN A LINEAR MODEL

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SUMMARY. We consider the linear model $(Y, X\beta, \sigma^2 I)$ and a set of estimable parametric functionals $A\beta$. In this paper, we consider alternative linear models which differ from $(Y, X\beta, \sigma^2 I)$ in the dispersion of the observations or expectation or both and obtain necessary and sufficient conditions for the F-test under $(Y, X\beta, \sigma^2 I$ for testing $H_0 : A\beta = 0$ to be valid under the alternative model also.

1. Introduction

The triplet $(Y, X\beta, \sigma^2 V)$ will denote a linear model with $E(Y) = X\beta$ and $D(Y) = \sigma^2 V$, where Y is an $n\times 1$ random vector, X is an $n\times m$ known matrix (the design matrix), β is an $m\times 1$ vector of unknown parameters, V is an $n\times n$ positive definite matrix and $\sigma^2 > 0$ is an unknown parameter. We assume that Y has a multivariate normal distribution. Let $A\beta$ be a set of estimable parametric functionals. If L_0 and L respectively denote the likelihood ratio test statistics for testing $H_0 : A\beta = 0$ under $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$, then the F-tests are given by the critical regions $L_0 > F$ and $L > F$ respectively, where F is a constant. Ghosh and Sinha (1980) took V to be the intraclass covariance matrix and obtained necessary and sufficient conditions for $L = L_0$. Later, Khatri (1981) developed a general solution to this problem, applicable to any form of V, positive definite. Sinha and Mukhopadhyay (1980) considered another specified covariance structure and obtained necessary and sufficient conditions for the equality of L and L_0. All these authors have furnished necessary and sufficient conditions under which the LRT statistics retains the same form under various structural forms of V, p.d. However, it is easy to observe that even if L and L_0 are different, but it is known that $L-L_0 \geqslant 0$ (or $\leqslant 0$) with probability 1, then the rejection (or acceptance) of H_0 under $(Y, X\beta, \sigma^2 I)$ will imply its rejection (respectively acceptance) under $(Y, X\beta, \sigma^2 V)$ also. Assuming that the Best Linear Unbiased Estimator (BLUE) of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ is also its BLUE under $(Y, X\beta, \sigma^2 V)$, we obtain in Section 2 necessary and sufficient conditions under which $L-L_0 \geqslant 0$ or $L-L_0 \leqslant 0$. For $L = L_0$, the result derived

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by Khatri (1981) follows as a corollary. As an example, we consider the linear model ($\boldsymbol{Y}$, $\boldsymbol{X\beta}$, $\sigma^2\boldsymbol{V}$), with $\boldsymbol{V} = (1-\rho)\boldsymbol{I}_n+\rho\boldsymbol{1}_n\boldsymbol{1}_n'$, where $\rho \in \left(-\frac{1}{n-1}, 1\right)$ and $\boldsymbol{1}_n$ is the $n\times 1$ column vector with each element equal to unity and provide a very simple proof to a result of Ghosh and Sinha (1980). Assuming that the column space of $\boldsymbol{X}$ contains $\boldsymbol{1}_n$, for testing $H_0 : \boldsymbol{A\beta} = \boldsymbol{0}$, we arrive at the following interesting conclusions, when $L \neq L_0$

(i) $L-L_0 > 0$ if and only if $\rho < 0$,

(ii) $L-L_0 < 0$ if and only if $\rho > 0$.

In Section 3, we consider the linear models ($\boldsymbol{Y}$, $\boldsymbol{X}_1\boldsymbol{\beta}$, $\sigma^2\boldsymbol{I}$) and ($\boldsymbol{Y}$, $\boldsymbol{X\beta}$, $\sigma^2\boldsymbol{I}$) and in Section 4, we consider ($\boldsymbol{Y}$, $\boldsymbol{X}_1\boldsymbol{\beta}$, $\sigma^2\boldsymbol{I}$) and ($\boldsymbol{Y}$, $\boldsymbol{X\beta}$, $\sigma^2\boldsymbol{V}$). In both cases, we obtain necessary and sufficient conditions under which the LRT statistic under the alternative model for testing $H_0 : \boldsymbol{A\beta} = \boldsymbol{0}$ is the same as the LRT statistic for testing H_0 under ($\boldsymbol{Y}$, $\boldsymbol{X}_1\boldsymbol{\beta}$, $\sigma^2\boldsymbol{I}$). Here $\boldsymbol{A\beta}$ is estimable under both the models.

For a matrix $\boldsymbol{B}$, $\mathcal{M}(\boldsymbol{B})$ and $R(\boldsymbol{B})$ denotes the column space and rank of $\boldsymbol{B}$ respectively. $\boldsymbol{B}^-$ denotes any matrix satisfying $\boldsymbol{BB}^-\boldsymbol{B} = \boldsymbol{B}$. For any p.d. matrix $\boldsymbol{N}$, $\boldsymbol{P}_{\boldsymbol{B},\boldsymbol{N}}$ denotes $\boldsymbol{B}(\boldsymbol{B}'\boldsymbol{NB})^-\boldsymbol{B}'\boldsymbol{N}$ and $\boldsymbol{P}_{\boldsymbol{B}}$ stands for $\boldsymbol{P}_{\boldsymbol{B},\boldsymbol{I}}$

2. Specification errors in the dispersion matrix

Let $R(\boldsymbol{X}) = r \leqslant m$ and $R(\boldsymbol{A}) = k$. Under the hypothesis $H_0 : \boldsymbol{A\beta} = \boldsymbol{0}$, the model can be rewritten as ($\boldsymbol{Y}$, $\boldsymbol{X}_0\boldsymbol{\beta}_0$, $\sigma^2\boldsymbol{V}$), where $\boldsymbol{X}_0 = \boldsymbol{X}(\boldsymbol{I}-\boldsymbol{A}^-\boldsymbol{A})$ is an $n\times m$ matrix of rank $r-k$ and $\boldsymbol{\beta}_0$ is an unknown $m\times 1$ vector. After simplifications, the LRT statistics for testing H_0 under ($\boldsymbol{Y}$, $\boldsymbol{X\beta}$, $\sigma^2\boldsymbol{I}$) and ($\boldsymbol{Y}$, $\boldsymbol{X\beta}$, $\sigma^2\boldsymbol{V}$) can be written respectively as,

$$L_0 = \frac{\boldsymbol{Y}'(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_0})\boldsymbol{Y}}{\boldsymbol{Y}'(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}})\boldsymbol{Y}}$$

and

$$L = \frac{\boldsymbol{Y}'\boldsymbol{V}^{-1}(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_0,\boldsymbol{V}^{-1}})\boldsymbol{Y}}{\boldsymbol{Y}'\boldsymbol{V}^{-1}(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X},\boldsymbol{V}^{-1}})\boldsymbol{Y}}$$

Let $\boldsymbol{Z}$ be a matrix of order $n\times n-r$ and $\boldsymbol{Z}_0 = (\boldsymbol{Z} : \boldsymbol{Z}_1)$ be a matrix of order $n\times(n-r+k)$ satisfying $\boldsymbol{Z}'\boldsymbol{X} = \boldsymbol{0}$, $\boldsymbol{Z}_0'\boldsymbol{X}_0 = \boldsymbol{0}$, $\boldsymbol{Z}'\boldsymbol{Z} = \boldsymbol{I}_{n-r}$ and $\boldsymbol{Z}_0'\boldsymbol{Z}_0 = \boldsymbol{I}_{n-r+k}$. With these notations, we state

Lemma 2.1 : $\mathcal{M}(A') = \mathcal{M}(X'Z_0) = \mathcal{M}(X'Z_1)$.

The lemma can be easily established by showing that A and $Z_0'X$ have the same null spaces.

It is known that the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$ if and only if

$$V = I + X\Lambda_1 X' + Z\Lambda_2 Z' + X\Lambda_4 Z' + Z\Lambda_4' X',$$

where Λ_1, Λ_2 and Λ_4 are arbitrary except that $A\Lambda_4 Z' = 0$ and V is p.d. (Rao and Mitra, 1971 page 159) V can be equivalently represented as

$$V = I + X\Lambda_1 X' + Z\Lambda_2 Z' + X_0\Lambda_3 Z' + Z\Lambda_3' X_0' \qquad \ldots \quad (2.1)$$

where Λ_1, Λ_2 and Λ_3 are arbitrary subject to the condition that V is p.d. It can be verified that if a p.d. (or a n.n.d.) matrix V admits the representation (2.1), then the matrices $X\Lambda_1 X'$ and Λ_2 are symmetric and unique.

The following lemma gives further necessary and sufficient conditions for the representation (2.1) to hold.

Lemma 2.2 : *The BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$, or equivalently, the representation* (2.1) *holds, if and only if anyone of the following equivalent condition holds :*

(i) $Z'VZ_1 = 0$,

(ii) $P_X V^{-1}(I - P_{X_0, V^{-1}})$ *is symmetric,*

(iii) $(I - P_{X_0, V^{-1}})(I - P_{X, V^{-1}})'$ *is symmetric,*

(iv) *There exists an orthogonal matrix* T *such that* $T'(I-P_X)T$, $T'(I-P_{X_0})T$, $T'V^{-1}(I-P_{X, V^{-1}})T$ *and* $T'V^{-1}(I-P_{X_0, V^{-1}})T$ *are diagonal matrices.*

Proof : (i) $A(X'X)^- X'Y$ is the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 V)$ if and only if

$$A(X'X)^- X'VZ = 0$$

$$\Longleftrightarrow Z_1'X(X'X)^- X'VZ = 0, \text{ using Lemma 2.1}$$

$$\Longleftrightarrow Z_1'VZ = 0, \text{ since } \mathcal{M}(Z_1) \subset \mathcal{M}(X)$$

(ii) $P_X V^{-1}(I-P_{X_0, V^{-1}})$ is symmetric if and only if

$$ZZ'Z_0(Z_0'VZ_0)^{-1}Z_0' = Z_0(Z_0'VZ_0)^{-1}Z_0'ZZ'$$

$$\iff Z_0'VZZ'Z_0 = Z_0'ZZ'VZ_0$$

$$\iff \begin{pmatrix} Z'VZ & 0 \\ Z_1'VZ & 0 \end{pmatrix} = \begin{pmatrix} Z'VZ & Z'VZ_1 \\ 0 & 0 \end{pmatrix}$$

$$\iff Z'VZ_1 = 0.$$

This proves the equivalence of (i) and (ii).

(iii) $(I-P_{X_0, V^{-1}})(I-P_{X, V^{-1}})'$ is symmetric if and only if

$$Z_0(Z_0'VZ_0)^{-1}Z_0'Z(Z'VZ)^{-1}Z' = Z(Z'VZ)^{-1}Z'Z_0(Z_0'VZ_0)^{-1}Z_0'$$

$$\iff Z_0(Z_0'VZ_0)^{-1}Z_0'Z = Z(Z'VZ)^{-1}$$

$$\iff Z_0'Z = Z_0'VZ(Z'VZ)^{-1}$$

$$\iff \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} Z'VZ \\ Z_1'VZ \end{pmatrix}(Z'VZ)^{-1}$$

$$\iff Z'VZ_1 = 0.$$

The equivalence of (i) and (iii) is thus established.

The equivalent conditions (ii) and (iii) imply that the matrices $I-P_X$, $I-P_{X_0}$, $V^{-1}(I-P_{X, V^{-1}})$ and $V^{-1}(I-P_{X_0, V^{-1}})$ commute pairwise, which is necessary and sufficient for the existence of an orthogonal matrix, which diagonalises them simultaneously (see Rao and Mitra, 1971, p. 124). The proof of Lemma 2.2 is now complete.

Corollary 2.1 : *The BLUE of every estimable parametric functional under* $(Y, X\beta, \sigma^2 I)$ *is its BLUE under* $(Y, X\beta, \sigma^2 V)$ *if and only if anyone of the following equivalent conditions holds* :

(i) $X'VZ = 0$ (Rao, 1967) ;

(ii) VP_X *is symmetric* ;

(iii) $P_{X, V^{-1}}$ *is symmetric.*

Lemma 2.2(i) enables us to prove the following interesting result.

Lemma 2.3 : *If the LRT statistics for testing $H_0 : A\beta = 0$ are the same under $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$, then the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$ also.*

Proof :

$$\frac{Y'V^{-1}(I-P_{X_0, V^{-1}})Y}{Y'V^{-1}(I-P_{X, V^{-1}})Y} = \frac{Y'(I-P_{X_0})Y}{Y'(I-P_X)Y}$$

$$\Longleftrightarrow Y'Z_0(Z_0'VZ_0)^{-1}Z_0'YY'ZZ'Y$$

$$= Y'Z(Z'VZ)^{-1}Z'YY'Z_0Z_0'Y \;\forall\; Y.$$

Putting $Y = VZ\theta$, we get

$$\theta'Z'VZZ'VZ\theta = \theta'Z'VZ_0Z_0'VZ\theta \;\forall\; \theta$$

$$\Longleftrightarrow Z'VZ_1 = 0.$$

Lemma 2.3 stands proved, in view of Lemma 2.2(i).

Remark 2.1 : The interesting observation made in Lemma 2.3 is implicit in the main result derived by Khatri (1980), even though this fact is not stated in his paper.

Now consider the linear models $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$, where V has the representation (2.1), or equivalently V satisfies the conditions in Lemma 2.2. Let T be the orthogonal matrix which reduces $I-P_X$, $I-P_{X_0}$, $V^{-1}(I-P_{X, V^{-1}})$ and $V^{-1}(I-P_{X_0, V^{-1}})$ simultaneously to diagonal forms. The columns of T are the common eigenvectors of these four matrices. It can be verified that each column of T belongs to $\mathcal{M}(Z)$, $\mathcal{M}(Z_1)$ or $\mathcal{M}(X_0)$. If necessary, rearrange the columns of T such that first $n-r$ columns belong to $\mathcal{M}(Z)$, the next k columns belong to $\mathcal{M}(Z_1)$ and the last $r-k$ columns belong to $\mathcal{M}(X_0)$. Let λ_i and λ_{0i} $(i = 1, 2, \ldots, n-r)$ denote the non-zero eigenvalues of $V^{-1}(I-P_{X, V^{-1}})$ and $V^{-1}(I-P_{X_0, V^{-1}})$ respectively corresponding to same eigenvector belonging to $\mathcal{M}(Z)$ and let λ_{0i} $(i = n-r+1, \ldots, n-r+k)$ denote the nonzero eigenvalues of $V^{-1}(I-P_{X_0, V^{-1}})$ corresponding to the eigenvectors belonging to $\mathcal{M}(Z_1)$. Note that the number of nonzero eigenvalues of $V^{-1}(I-P_{X, V^{-1}})$ and $V^{-1}(I-P_{X_0, V^{-1}})$ are respectively

$n-r$ and $n-r+k$, their ranks. Let $T'Y = t = (t_1, t_2, \ldots, t_n)'$. Then we have

$$L_0 = \frac{t'T'(I-P_{X_0})Tt}{t'T'(I-P_X)Tt} = \frac{\sum_{i=1}^{n-r+k} t_i^2}{\sum_{i=1}^{n-r} t_i^2}.$$

Similarly,

$$L = \frac{\sum_{i=1}^{n-r+k} \lambda_{0i} t_i^2}{\sum_{i=1}^{n-r} \lambda_i t_i^2}.$$

Thus we have proved

Lemma 2.4 : *Let L, L_0, λ_{0i}, λ_i and t_i be as defined above. Then*

$$L-L_0 = \frac{\sum_{i=1}^{n-r+k} \sum_{j=1}^{n-r} (\lambda_{0i}-\lambda_j) t_i^2 t_j^2}{\sum_{i=1}^{n-r} \sum_{j=1}^{n-r} \lambda_i t_i^2 t_j^2}.$$

Using Lemma 2.4, it can be easily established that

$$L-L_0 \geqslant 0 \quad \text{with probability 1 if and only if}$$

$$\lambda_{0i} \geqslant \lambda_i \quad \text{for } i = 1, 2, \ldots, n-r$$

and

$$\lambda_{0i} \geqslant \lambda_j \quad \text{for } i = n-r+1, \ldots, n-r+k$$

$$j = 1, 2, \ldots, n-r.$$

Since V is assumed to have the representation (2.1) the condition (i) in Lemma 2.2 holds and hence

$$Z_0(Z_0'VZ_0)^{-1}Z_0'ZZ' = Z(Z'VZ)^{-1}Z'$$

$$\Longleftrightarrow V^{-1}(I-P_{X_0, V^{-1}})(I-P_X^*) = V^{-1}(I-P_{X, V^{-1}})$$

$$\Longleftrightarrow \lambda_{0i} = \lambda_i \quad \text{for } i = 1, 2, \ldots, n-r.$$

Using this observation and Lemma 2.4, we have

Lemma 2.5 : $L-L_0 \geqslant 0$ *if and only if*

$$\lambda_{0i} \geqslant \lambda_{0j}, \quad i = n-r+1, \ldots, n-r+k$$

$$j = 1, 2, \ldots, n-r.$$

Next, we shall derive conditions on $\boldsymbol{V}$ such that the eigenvalues λ_{0i} ($i = 1, 2, \ldots, n-r+k$) satisfy the condition stated in Lemma 2.5. Using the representation (2.1) for $\boldsymbol{V}$ and recalling that $\boldsymbol{Z}_0 = (\boldsymbol{Z} : \boldsymbol{Z}_1)$ satisfies $\boldsymbol{Z}_0'\boldsymbol{Z}_0 = \boldsymbol{I}$, we get

$$\boldsymbol{Z}_0'\boldsymbol{V}\boldsymbol{Z}_0 = \begin{pmatrix} \boldsymbol{I}+\boldsymbol{\Lambda}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}+\boldsymbol{Z}_1'\boldsymbol{X}\boldsymbol{\Lambda}_1\boldsymbol{X}'\boldsymbol{Z}_1 \end{pmatrix}.$$

Hence,

$$\boldsymbol{Z}_0(\boldsymbol{Z}_0'\boldsymbol{V}\boldsymbol{Z}_0)^{-1}\boldsymbol{Z}_0' = (\boldsymbol{Z} : \boldsymbol{Z}_1) \begin{pmatrix} (\boldsymbol{I}+\boldsymbol{\Lambda}_2)^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{I}+\boldsymbol{Z}_1'\boldsymbol{X}\boldsymbol{\Lambda}_1\boldsymbol{X}'\boldsymbol{Z}_1)^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{Z}' \\ \boldsymbol{Z}_1' \end{pmatrix}$$

$$= \boldsymbol{Z}(\boldsymbol{I}+\boldsymbol{\Lambda}_2)^{-1}\boldsymbol{Z}'+\boldsymbol{Z}_1(\boldsymbol{I}+\boldsymbol{Z}_1'\boldsymbol{X}\boldsymbol{\Lambda}_1\boldsymbol{X}'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'. \qquad \ldots \quad (2.2)$$

For $i = 1, 2, \ldots, n-r$, λ_{0i} are the eigenvalues of $\boldsymbol{Z}_0(\boldsymbol{Z}_0'\boldsymbol{V}\boldsymbol{Z}_0)^{-1}\boldsymbol{Z}_0'$ corresponding to eigenvectors belonging to $\mathscr{M}(\boldsymbol{Z})$ and for $i = n-r+1, \ldots, n-r+k$, λ_{0i} are the eigenvalues corresponding to eigenvectors belonging to $\mathscr{M}(\boldsymbol{Z}_1)$. Using this, and the fact that $\boldsymbol{Z}$ and $\boldsymbol{Z}_1$ are chosen to satisfy $\boldsymbol{Z}'\boldsymbol{Z} = \boldsymbol{I}_{n-r}$ and $\boldsymbol{Z}_1'\boldsymbol{Z}_1 = \boldsymbol{I}_k$, it follows from (2.2) that λ_{0i} ($i = 1, 2, \ldots, n-r$) are the eigenvalues of $(\boldsymbol{I}+\boldsymbol{\Lambda}_2)^{-1}$ and λ_{0i} ($i = n-r+1, \ldots, n-r+k$) are the eigenvalues of $(\boldsymbol{I}+\boldsymbol{Z}_1'\boldsymbol{X}\boldsymbol{\Lambda}_1\boldsymbol{X}'\boldsymbol{Z}_1)^{-1}$. Hence it follows that $\lambda_{0i} \geqslant \lambda_{0j}$ for $i = n-r+1, \ldots, n-r+k$ and $j = 1, 2, \ldots, n-r$ if and only if the minimum eigenvalue of $\boldsymbol{\Lambda}_2$ is greater than or equal to the maximum eigenvalue of $\boldsymbol{Z}_1'\boldsymbol{X}\boldsymbol{\Lambda}_1\boldsymbol{X}'\boldsymbol{Z}_1$. Thus we have proved

Theorem 2.1 : *Let L_0 and L respectively denote the LRT statistics for testing $H_0 : \boldsymbol{A\beta} = \boldsymbol{0}$ under $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{I})$ and $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{V})$, where $\boldsymbol{V}$ has the representation (2.1) and $\mathscr{M}(\boldsymbol{A}') \subset \mathscr{M}(\boldsymbol{X}')$. Then $L-L_0 \geqslant 0$ (or $\leqslant 0$) with probability 1 if and only if the minimum (maximum) eigenvalue of $\boldsymbol{\Lambda}_2$ is greater than or equal to (less than or equal to) the maximum (respectively minimum) eigenvalue of $\boldsymbol{Z}_1'\boldsymbol{X}\boldsymbol{\Lambda}_1\boldsymbol{X}'\boldsymbol{Z}_1$. Under this condition, the rejection (or acceptance) of H_0 under $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{I})$ will imply its rejection (respectively acceptance) under $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{V})$ also.*

In Theorem 2.1, we have assumed that $\boldsymbol{V}$ admits the representation (2.1). However, if we are interested in conditions under which $L = L_0$, then this assumption always holds, in view of Lemma 2.3. Thus we have also proved

Theorem 2.2 : *Consider the linear models $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{I})$ and $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{V})$, where $\boldsymbol{V}$ is p.d. Let $\boldsymbol{A}$, $\boldsymbol{X}_0$, $\boldsymbol{Z}$ and $\boldsymbol{Z}_1$ be as defined before. Then, for testing*

$H_0 : \boldsymbol{A\beta} = \boldsymbol{0}$, the LRT under $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{V})$ is the same as the LRT under $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{I})$ if and only if $\boldsymbol{V}$ admits the representation

$$\boldsymbol{V} = \boldsymbol{I} + \boldsymbol{X\Lambda}_1\boldsymbol{X}' + (S-1)\boldsymbol{ZZ}' + \boldsymbol{X}_0\boldsymbol{\Lambda}_2\boldsymbol{Z}' + \boldsymbol{Z\Lambda}_2'\boldsymbol{X}_0',$$

where $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ are arbitrary and S is an arbitrary positive real number subject to the conditions (i) *$\boldsymbol{V}$ is p.d. and* (ii) $\boldsymbol{Z}_1'\boldsymbol{X\Lambda}_1\boldsymbol{X}'\boldsymbol{Z}_1 = (S-1)\boldsymbol{I}_k$.

Corollary 2.2 : *The condition on $\boldsymbol{V}$ given in Theorem* 2.2 *is equivalent to anyone of the following equivalent conditions.*

(i) $(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_0})\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_0}) = a(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_0})$ *for some* $a > 0$

(ii) $\boldsymbol{V}^{-1}(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_0, \boldsymbol{V}^{-1}}) = a(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}_0})$ *for some* $a > 0$

(iii) $$\begin{pmatrix} \boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}} \\ \boldsymbol{LP}_{\boldsymbol{X}} \end{pmatrix} (\boldsymbol{V}-a\boldsymbol{I})(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{X}} : \boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{L}') = 0, \text{ for some } a > 0$$

where $\boldsymbol{L}$ is such that $\boldsymbol{LX} = \boldsymbol{A}$.

The equivalence of the condition (i) or (ii) in the Corollary with the condition stated in Theorem 2.2 can be easily established and the equivalence of the condition (iii) in the Corollary with the one stated in Theorem 2.2 can be proved in a straightforward manner, appealing to Lemma 2.1. Corollary 2.2(iii) is the result obtained by Khatri (1980). From Corollary 2.2(ii), we have

Corollary 2.3 : *For testing $H_0 : \boldsymbol{A\beta} = \boldsymbol{0}$, if L_0 and L denote the LRT statistics under $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{I})$ and $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{V})$, as defined before, then L_0 and L are the same if and only if the numerator of L is proportional to be numerator of L_0.*

Remark 2.2 : Theorem 2.1 and Theorem 2.2 have been proved without assuming that the matrix $\boldsymbol{A}$ is of full row rank.

Example : Let $n > 1$ and let $\boldsymbol{V}$ be the intraclass covariance matrix

$$\boldsymbol{V} = (1-\rho)\boldsymbol{I}_n + \rho\boldsymbol{1}_n\boldsymbol{1}_n', \; -\frac{1}{n-1} < \rho < 1,$$

where $\boldsymbol{1}_n$ is the $n \times 1$ column vector with each element equal to 1. With $\boldsymbol{V}$ defined like this, we consider the linear model $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{V})$, where $\boldsymbol{Y}$ has a multivariate normal distribution. Suppose we want to test the hypothesis $H_0 : A\beta = 0$, where $\mathcal{M}(A') \subset \mathcal{M}(X')$. Let X_0 be defined as before. For the intraclass covariance matrix V, Ghosh and Sinha (1980) proved

Theorem 2.3 : *The LRT statistic L for testing H_0 under $(Y, X\beta, \sigma^2 V)$ has the same value for all $\rho \in \left(-\frac{1}{n-1}, 1\right)$ if and only if $1_n \in \mathcal{M}(X_0)$.*

It is an easy matter to deduce Theorem 2.3 from condition (i) in Corollary 2.2. We present here an extremely simple alternate proof of Theorem 2.3, which, we feel, is of independent interest.

Proof of Theorem 3.1 : Let Z and Z_0 be as defined in the beginning of this section. For $\rho = 0$, if L_0 denotes the LRT statistic for testing H_0, then

$$L_0 = \frac{Y'Z_0Z_0'Y}{Y'ZZ'Y}$$

and

$$L = \frac{Y'Z_0(Z_0'VZ_0)^{-1}Z_0'Y}{Y'Z(Z'VZ)^{-1}Z'Y}$$

$$= \frac{Y'Z_0\left[I_{n-r+k} - \dfrac{\frac{\rho}{1-\rho}Z_0'1_n1_n'Z_0}{1+\frac{1}{1-\rho}1_n'Z_0Z_0'1_n}\right]Z_0'Y}{Y'Z\left[I_{n-r} - \dfrac{\frac{\rho}{1-\rho}Z'1_n1_n'Z}{1+\frac{1}{1-\rho}1_n'ZZ'1_n}\right]Z'Y}$$

where $r = R(X)$ and $k = R(A)$.

$L = L_0$ for all $\rho \in \left(-\frac{1}{n-1}, 1\right)$ if and only if

$$\frac{Y'Z_0Z_0'1_n1_n'Z_0Z_0'Y}{1+\frac{1}{1-\rho}1_n'Z_0Z'1_n} \cdot Y'ZZ'Y$$

$$= \frac{Y'ZZ'1_n1_n'ZZ'Y}{1+\frac{1}{1-\rho}1_n'ZZ'1_n} \cdot Y'Z_0Z_0'Y \qquad \ldots (2.3)$$

$$\forall\, Y \text{ and } \forall\, \rho \in \left(-\frac{1}{n-1}, 1\right).$$

Sufficiency of the condition $\mathbf{1}_n \epsilon \mathcal{M}(X_0)$ is now obvious. To prove its necessity, notice that if $\mathbf{1}_n'Z = 0$, then from (2.3), we get $\mathbf{1}_n'Z_0 = 0$. This proves the theorem. So we proceed under the assumption that $\mathbf{1}_n'Z \neq 0$. For $i = 1, 2, \ldots, n-r$, let ξ_i denote the columns of Z. Then for atleast one i, $\mathbf{1}_n'ZZ'\xi_i \neq 0$. For this i, putting $Y = \xi_i$ in (2.3), we get

$$\mathbf{1}_n'Z_0Z_0'\mathbf{1}_n = \mathbf{1}_n'ZZ'\mathbf{1}_n$$
$$\iff Z_1'\mathbf{1}_n = 0. \qquad \ldots \quad (2.4)$$

Using (2.4), (2.3) simplifies to

$$(Y'ZZ'\mathbf{1}_n)^2Y'ZZ'Y = (Y'ZZ'\mathbf{1}_n)^2Y'Z_0Z_0'Y \quad \forall \; Y,$$
$$\iff Z'\mathbf{1}_n = 0. \qquad \ldots \quad (2.5)$$

(2.4) and (2.5) together imply $Z_0'\mathbf{1}_n = 0$ and this completes the proof of Theorem 2.3.

Now, we assume that the design matrix X satisfies the condition $\mathbf{1}_n \epsilon \mathcal{M}(X)$. Then, it is easy to verify that $X'VZ = 0$, and hence, using a result of Rao (1967), we see that the BLUE of every estimable parametric functional under $(Y, X\beta, \sigma^2 I)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$. If we also have the condition $\mathbf{1}_n \epsilon \mathcal{M}(X_0)$, then Theorem 2.3 applies. So we assume that $\mathbf{1}_n \notin \mathcal{M}(X_0)$, and we shall examine the applicability of Theorem 2.1 in this setup.

Observe that V can be written as

$$V = I_n + \rho(\mathbf{1}_n\mathbf{1}_n' - X(X'X)^-X'] - \rho ZZ'.$$

Comparing with (2.1) and using the assumption $\mathbf{1}_n \epsilon \mathcal{M}(X)$, we get

$$X\Lambda_1X' = \rho[\mathbf{1}_n\mathbf{1}_n' - X(X'X)^-X']$$
$$\Lambda_2 = -\rho I_{n-r} \text{ and } X_0\Lambda_3 = 0.$$

Hence $Z_1'X\Lambda_1X'Z_1 = \rho(Z_1'\mathbf{1}_n\mathbf{1}_n'Z_1 - I_k)$. Since $\mathbf{1}_n \notin \mathcal{M}(X_0)$, $Z_1'\mathbf{1}_n \neq 0$. If μ denotes the positive eigenvalue of $Z_1'\mathbf{1}_n\mathbf{1}_n'Z_1$, then the eigenvalues of $Z_1'X\Lambda_1X'Z_1$ are $\rho(\mu-1)$ of multiplicity one and $-\rho$ of multiplicity $k-1$. The eigenvalues of Λ_2 are each equal to $-\rho$. Now, applying Theorem 2,1, we get

(i) $L - L_0 > 0$ if and only if $\rho < 0$

(ii) $L - L_0 < 0$ if and only if $\rho > 0$

(iii) $L = L_0$ if and only if $\rho = 0$ (i.e., $V = I$).

Thus, when $\mathbf{1}_n \notin \mathcal{M}(X_0)$, we see that if $\rho < 0$ (or $\rho > 0$) then the rejection (or acceptance) of H_0 under $(Y, X\beta, \sigma^2 I)$ implies its rejection (respectively acceptance) under $(Y, X\beta, \sigma^2 V)$ also.

3. Specification errors in the design matrix

In this section we consider two alternative linear models $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 I)$ which differ in the design matrices and not in the dispersion of observations and obtain conditions on X such that the LRT statistic under $(Y, X\beta, \sigma^2 I)$ for testing a hypothesis $H_0 : A\beta = 0$ is same as the LRT statistic for testing H_0 under $(Y, X_1\beta, \sigma^2 I)$. Here $A\beta$ is a parametric functional estimable under both the models. The LRT statistics for testing H_0 under $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 I)$ are respectively given by

$$L_1 = \frac{Y'(I - P_{X_1(I-A^-A)})Y}{Y'(I-P_{X_1})Y}$$

and

$$L = \frac{Y'(I-P_{X(I-A^-A)})Y}{Y'(I-P_X)Y}$$

L_1 is defined for $Y \notin \mathcal{M}(X_1)$ and L is defined for $Y \notin \mathcal{M}(X)$. Hence, for the equality of L and L_1 to be meaningful, we should have $\mathcal{M}(X) = \mathcal{M}(X_1)$. We now prove

Theorem 3.1 : *Consider the linear models $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 I)$, where $\mathcal{M}(X) = \mathcal{M}(X_1)$ and a hypothesis $H_0 : A\beta = 0$, where $\mathcal{M}(A') \subseteq \mathcal{M}(X_1')$. Then, for testing H_0, the LRT statistic under $(Y, X\beta, \sigma^2 I)$ is same as the LRT statistic under $(Y, X_1\beta, \sigma^2 I)$ if and only if $\mathcal{M}(X(I-A^-A)) = \mathcal{M}(X_1(I-A^-A))$.*

Proof : Consider L_1 and L as defined before. Since $\mathcal{M}(X) = \mathcal{M}(X_1)$, $P_X = P_{X_1}$ and hence it follows that $L = L_1$ if and only if

$$P_{X(I-A^-A)} = P_{X_1(I-A^-A)}$$

$$\Longleftrightarrow \mathcal{M}(X(I-A^-A)) = \mathcal{M}((X_1(I-A^-A)).$$

We shall now give a characterisation of matrix X satisfying,

$$\mathcal{M}(A') \subseteq \mathcal{M}(X') = \mathcal{M}(X_1') \text{ and } \mathcal{M}(X(I-A^-A)) = \mathcal{M}(X_1(I-A^-A)).$$

We need

Lemma 3.1 : *Let* $\boldsymbol{A}$ *be a given matrix and let* $\boldsymbol{E}_1$ *and* $\boldsymbol{E}_2$ *be such that the columns of* $\boldsymbol{E}_1'$ *form an orthonormal basis for* $\mathcal{M}(\boldsymbol{A}')$ *and those of* $\boldsymbol{E}_2'$ *form an orthonormal basis for the orthogonal complement of* $\mathcal{M}(\boldsymbol{A}')$. *Then, matrices* $\boldsymbol{X}$ *satisfying* $\mathcal{M}(\boldsymbol{A}') \subseteq \mathcal{M}(\boldsymbol{X}')$ *are given by*

$$\boldsymbol{X} = (\boldsymbol{S}_1 \ \boldsymbol{S}_2) \begin{pmatrix} \boldsymbol{E}_1 \\ \boldsymbol{E}_2 \end{pmatrix}, \text{ where } \boldsymbol{S}_1 \text{ and } \boldsymbol{S}_2 \text{ are arbitrary}$$

except that $\boldsymbol{S}_1$ *is a matrix of full column rank and* $\mathcal{M}(\boldsymbol{S}_1) \cap \mathcal{M}(\boldsymbol{S}_2) = \{0\}$.

Proof : We can write

$$\boldsymbol{X}' = (\boldsymbol{E}_1' \ \boldsymbol{E}_2') \begin{pmatrix} \boldsymbol{S}_1' \\ \boldsymbol{S}_2' \end{pmatrix} \text{ for some } \boldsymbol{S}_1 \text{ and } \boldsymbol{S}_2.$$

$\mathcal{M}(\boldsymbol{A}') \subseteq \mathcal{M}(\boldsymbol{X}') \Longleftrightarrow$ there exists a matrix $\boldsymbol{B}$ satisfying

$$(\boldsymbol{E}_1' \ \boldsymbol{E}_2') \begin{pmatrix} \boldsymbol{S}_1' \\ \boldsymbol{S}_2' \end{pmatrix} \boldsymbol{B} = \boldsymbol{E}_1'$$

$$\Longleftrightarrow \begin{pmatrix} \boldsymbol{S}_1' \\ \boldsymbol{S}_2' \end{pmatrix} \boldsymbol{B} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{pmatrix}$$

$\Longleftrightarrow \boldsymbol{S}_1$ is of full column rank and $\mathcal{M}(\boldsymbol{S}_1) \cap \mathcal{M}(\boldsymbol{S}_2) = \{0\}$.
This completes the proof of Lemma 3.1.

Given $\boldsymbol{X}_1$ with $\mathcal{M}(\boldsymbol{A}') \subseteq \mathcal{M}(\boldsymbol{X}_1')$, we now proceed to characterise matrices X that satisfy $\mathcal{M}(\boldsymbol{X}) = \mathcal{M}(\boldsymbol{X}_1)$ and $\mathcal{M}(\boldsymbol{X}(\boldsymbol{I}-\boldsymbol{A}^-\boldsymbol{A})) = \mathcal{M}(\boldsymbol{X}_1(\boldsymbol{I}-\boldsymbol{A}^-\boldsymbol{A}))$.

Let $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ be as defined in Lemma 3.1 and write

$$\boldsymbol{X}_1 = (\boldsymbol{S}_{11} \ \boldsymbol{S}_{21}) \begin{pmatrix} \boldsymbol{E}_1 \\ \boldsymbol{E}_2 \end{pmatrix} \text{ and } \boldsymbol{X} = (\boldsymbol{S}_1 \ \boldsymbol{S}_2) \begin{pmatrix} \boldsymbol{E}_1 \\ \boldsymbol{E}_2 \end{pmatrix},$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are as defined in Lemma 3.1 and $\boldsymbol{S}_{11}$ and $\boldsymbol{S}_{21}$ satisfy identical conditions. From the proof of Lemma 3.1, we see that we have to characterise $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ satisfying $\boldsymbol{S}_1'\boldsymbol{R} = \boldsymbol{I}$, $\boldsymbol{S}_2'\boldsymbol{R} = \boldsymbol{0}$ for some $\boldsymbol{R}$ and $\mathcal{M}(\boldsymbol{S}_2) = \mathcal{M}(\boldsymbol{S}_{21})$ and $\mathcal{M}(\boldsymbol{S}_1 : \boldsymbol{S}_2) = \mathcal{M}(\boldsymbol{S}_{11} : \boldsymbol{S}_{21})$. Let $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3$ be such that the columns of $\boldsymbol{F}_2$ form an orthonormal basis for $\mathcal{M}(\boldsymbol{S}_{21})$ and those of $(\boldsymbol{F}_1 : \boldsymbol{F}_2)$ form an orthonormal basis for $\mathcal{M}(\boldsymbol{S}_{11} : \boldsymbol{S}_{21})$, and $(\boldsymbol{F}_1 \ \boldsymbol{F}_2 \ \boldsymbol{F}_3)$ is an orthogonal matrix. Choose

S_2 any matrix satisfying $\mathscr{M}(S_2) = \mathscr{M}(S_{21})$. Then $S_2'R = 0 \Longrightarrow R = F_1K_1 + F_2K_2$ for some K_1 and K_2. Since we want $\mathscr{M}(S_1) \subset \mathscr{M}(S_{11} \;\; S_{21})$, let $S_1 = F_1M_1 + F_2M_2$. Then $S_1'R = I \Longrightarrow M_1'K_1 = I$. Hence S_1 is any matrix of the form $S_1 = F_1M_1 + F_2M_2$ where M_1 is any nonsingular matrix and M_2 is arbitrary. It is easy to see that S_1 so chosen satisfies $\mathscr{M}(S_1 . S_2) = \mathscr{M}(S_{11} S_{21})$.

Remark 3.1 : It is not true that equality of the LRT statistics for testing $H_0 : A\beta = 0$ under the models $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 I)$ implies the equality of the BLUES of $A\beta$ under both the models. However, if the BLUE of $A\beta$ under $(Y, X_1\beta, \sigma^2 I)$ is unbiased for $A\beta$ under $(Y, X\beta, \sigma^2 I)$, then equality of the LRT statistics for testing H_0 implies equality of the BLUES.

4. Specification errors in the design and dispersion matrices

Consider the linear models $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$, which differ both in the expectation and the dispersion of the observations. Here, V is a p.d. matrix. We are interested in testing the hypothesis $H_0 : A\beta = 0$, where $A\beta$ is estimable under both the models. We prove

Theorem 4.1 : *Consider the linear models $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$, where $\mathscr{M}(X) = \mathscr{M}(X_1)$ and V is positive definite and a hypothesis $H_0 : A\beta = 0$, $\mathscr{M}(A')$ being a subspace of $\mathscr{M}(X_1')$. Then, for testing H_0, the LRT statistics under $(Y, X\beta, \sigma^2 V)$ is same as the LRT statistic under $(Y, X_1\beta, \sigma^2 I)$ if and only if*

(i) $\mathscr{M}(X(I-A^-A)) = \mathscr{M}(X_1(I-A^-A))$

and

(ii) $(I-P_{X_{10}})V(I-P_{X_{10}}) = a(I-P_{X_{10}})$ *for some* $a > 0$

where $X_{10} = X_1(I-A^-A)$.

Proof: Let $X_{10} = X_1(I-A^-A)$ and let $X_0 = X(I-A^-A)$. Also let $W_0 = (W : W_1)$ and $Z_0 = (Z : Z_1)$ be matrices satisfying $W_0'X_{10} = 0$, $W'X_1 = 0$, $Z_0'X_0 = 0$, $Z'X = 0$, $Z_0'Z_0 = I$ and $W_0'W_0 = I$. Since $\mathscr{M}(X) = \mathscr{M}(X_1)$, we take $Z = W$. Then the LRT statistic for testing H_0 under $(Y, X\beta, \sigma^2 V)$ is same as that under $(Y, X_1\beta, \sigma I)$ iff

$$\frac{Y'Z_0(Z_0'VZ_0)^{-1}Z_0'Y}{Y'W(W'VW)^{-1}W'Y} = \frac{Y'W_0W_0'Y}{Y'WW'Y} \quad \dots \quad (4.1)$$

Since $Z_0'VZ_0 = \begin{pmatrix} W'VW & W'VZ_1 \\ Z_1'VW & Z_1'VZ_1 \end{pmatrix}$ the submatrix appearing in the top left hand corner of $(Z_0'VZ_0)^{-1}$ is

$$(W'VW)^{-1}+(W'VW)^{-1}W'VZ_1[Z_1'VZ_1-Z_1'VW(W'VW)^{-1}W'VZ_1]^{-1} Z_1'VW(W'VW)^{-1}.$$

Hence putting $Y = W\theta$, and observing that $Z_1'W = 0$, we get from (4.1), $W'VZ_1 = 0$. Hence (4.1) simplifies to

$$\frac{Y'Z_1(Z_1'VZ_0)^{-1}Z_1'Y}{Y'W(W'VW)^{-1}W'Y} = \frac{Y'W_1W_1'Y}{Y'WW'Y}. \quad \dots \quad (4.2)$$

Putting $Y = W\theta_1+W_2\theta_2$ in (4.2), where W_2 is such that $(W : W_1 : W_2)$ is an orthogonal matrix, we get

$Z_1'W_2 = 0$, which, together with $Z_1'W = 0$ shows that $\mathcal{M}(Z_1) \subset \mathcal{M}(W_1)$. Similarly one can show that $\mathcal{M}(W_1) \subset \mathcal{M}(Z_1)$. Thus

$$\begin{aligned} &\mathcal{M}(W_1) = \mathcal{M}(Z_1) \\ \Longleftrightarrow\ & W_1W_1' = Z_1Z_1' \\ \Longleftrightarrow\ & WW'+W_1W_1' = ZZ'+Z_1Z_1' \\ \Longleftrightarrow\ & \mathcal{M}(X(I-A^-A) = \mathcal{M}(X_1(I-A^-A)). \end{aligned}$$

Hence (4.1) can be written as

$$\frac{Y'W_0(W_0'VW_0)^{-1}W_0'Y}{Y'W(W'VW)^{-1}W'Y} = \frac{Y'W_0W_0'Y}{Y'WW'Y}$$

which is equivalent to the condition (ii) given in the theorem, in view of Theorem 2.2 and Corollary 2.2. This completes the proof of the theorem.

Remark 4.1 : Equivalent conditions on V can be derived as in the case of Theorem 2.2 and Corollary 2.2.

Remark 4.2 : The observation made in remark 3.1 is valid here also.

References

Ghosh, M. and Sinha, Bimal Kumar (1980) : On the robustness of least squares procedures in regression models. *Jour. Mult. Anal.*, **10**, 332-342.

Khatri, C. G. (1981) : A study of F-tests under dependent model. *Sankhyā*, Ser. A., **43**, 107-110.

Rao, C. R. (1967) : Least squares theory using an estimated dispersion matrix and its application to measurement of signals. *Proc. Fifth Berkely Symposium on Math-Stat and Prob.*, **1**, 355-372.

Rao, C. R. and Mitra, S. K. (1971) : *Generalised Inverse of Matrices and its Applications*, John Wiley and Sons., New York.

Sinha, Bikas Kumar and Mukhopadhyay, B. B. (1980) : On a characterisation of regression models under a specified covariance structure for the robustness of the usual LRT. *Tech. Report* No. 26/80, Stat-Math Division, Indian Statistical Institute, Calcutta.

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