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# BOOTSTRAPPING STATISTICS WITH LINEAR COMBINATIONS OF CHI-SQUARES AS WEAK LIMIT

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*SUMMARY.* Chandra and Ghosh (1979) consider a class of statistics and obtain Edgeworth expansions with chi-square as the leading term. Very little seems to be known about the statistics which are asymptotically distributed as linear combinations of chi-squares. In this paper we study bootstrap approximation to a class of such statistics.

## 1. Introduction

Let $\mu \in R^k$ and let $H$ be a thrice continuously differentiable function on an open subset $S$ of $R^k$ containing $\mu$. Let $l(y)$ denote the vector of first partial derivatives of $H$ at $y$ in $S$ and $L(y)$ denote the matrix of second partial derivatives of $H$ at $y$ in $S$. Let $\{Z_n\}$ be a sequence of i.i.d. random vectors in $R^k$ with mean $\mu$ and nonsingular dispersion $\Sigma$. If $l(\mu) = 0$ and $L(\mu)$ is non-null, then it can be shown that $n(H(\bar{Z})-H(\mu))$ is asymptotically distributed as linear combinations of chi-squares. Here $\bar{Z}$ denote the sample mean of $Z_1, \ldots, Z_n$. Chandra (1980) and Chandra and Ghosh (1979) have obtained Edgeworth expansions for distributions of such statistics under some conditions. These expansions are known only when $L(\mu)$ is positive semidefinite (or negative semidefinite.) Not much seems to be known about the asymptotic distribution when $L(\mu)$ has both positive and negative eigen-values. In this paper, we shall show that a modification of Efron's (1979) resampling procedure called "bootstrap" would give a good approximation for the distribution of $n(H(\bar{Z})-H(\mu))$.

Let $F_n$ denote the empirical distribution of $Z_1, \ldots, Z_n$ and $Y_1, \ldots, Y_n$ denote a sequence of i.i.d. random variables from $F_n$. Let $\Sigma_n$ denote the dispersion of $Y_1$. Clearly the mean of $Y_1$ is $\bar{Z}$. Let $\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i$. Under

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some conditions, it is shown in Babu and Singh (1984) that, if $l(\mu) \neq 0$ then uniformly for all real $u$,

$$P(\sqrt{n}(H(\bar{Y})-H(\bar{Z})) < u\sqrt{l'(\bar{Z})\Sigma_n l(\bar{Z})})$$

$$= P(\sqrt{n}(H(\bar{Z})-H(\mu)) < u\sqrt{l'(\mu)\Sigma l(\mu)})+o(n^{-1/2}).$$

The result is proved by approximating $\sqrt{n}(H(\bar{Z})-H(\mu))$ by

$$\sqrt{n}l'(\mu)(\bar{Z}-\mu)+\frac{1}{\sqrt{n}}\ [n(\bar{Z}-\mu)'L(\mu)(\bar{Z}-\mu)].$$

Note that the asymptotic distribution of the first term $\sqrt{n}l'(\mu)(\bar{Z}-\mu)$ is Gaussian. If $l(\mu) = 0$ and $L(\mu)$ is non-null, then $n(H(\bar{Z})-H(\mu))$ can be approximated by $n(\bar{Z}-\mu)'L(\mu)(\bar{Z}-\mu)$.

If, further, $\Sigma = I$ and $L(\mu)$ is diagonal with the diagonal elements either zero or 1, then $n(\bar{Z}-\mu)'L(\mu)(\bar{Z}-\mu)$ has asymptotically chi-square distribution with degrees of freedom = rank of $L(\mu)$. If $l(\mu) = 0$, this suggests the possibility of closeness of distributions of $n(H(\bar{Z})-H(\mu))$ and the bootstrap statistic $n(H(\bar{Y})-H(\bar{Z}))$. The following example shows that this is false.

*Example* 1 : Let $\{Z_n\}$ be a sequence of i.i.d. standard normal variables. Let $H(v) = v^2$. Since $\sqrt{n}\bar{Z}$ has standard normal distribution, $nH(\bar{Z}) = n(\bar{Z})^2$ has chi-square distribution with one degree of freedom. But if $\bar{Z} > 0$, we have for any $u \geqslant 0$,

$$P(n(H(\bar{Y})-H(\bar{Z})) < u) \geqslant P(n(H(\bar{Y})-H(\bar{Z})) < 0)$$

$$= P(\sqrt{n}|\bar{Y}| < \sqrt{n}|\bar{Z}|)$$

$$\geqslant P(0 > \sqrt{n}(\bar{Y}-\bar{Z}) > -2\sqrt{n}\bar{Z}).$$

Since a.s. $\sqrt{n}(\bar{Y}-\bar{Z})$ is asymptotically normally distributed and $\limsup_{n\to\infty} \sqrt{n}\bar{Z} = \infty$, it follows a.s., that

$$\limsup P(n(H(\bar{Y})-H(\bar{Z})) < u) \geqslant \tfrac{1}{2}$$

for all $u > 0$. So the distribution of $n(H(\bar{Y})-H(\bar{Z}))$ cannot converge to the chi-square distribution with one degree of freedom. So the bootstrap approximation fails.

One possible reason for this is that in general $nl(\bar{Z})$ is very large, even when $l(\mu) = 0$. This suggests that, a computable modification of the bootstrap statistic might give a good approximation. This is the content of the following theorem.

## 2. Main result

Let for any $y = (y_1, \ldots, y_k) \in R^k$, $\|y\| = \left(\sum_{i=1}^{k} y_i^2\right)^{1/2}$.

Theorem : *Let $\mu$, $H$, $\{Z_i\}$, $\{Y_i\}$ be as defined in the introduction. Suppose $L(\mu)$ is non-null and $E\|Z_1\|^4 < \infty$. Let $a_n = (n/\log\log n)^{-1/2}$,*

$$t^* = n[H(\bar{Y}) - H(\bar{Z}) - l'(\bar{Z})(\bar{Y} - \bar{Z})]$$

$$t = n[H(\bar{Z}) - H(\mu) - l'(\mu)(\bar{Z} - \mu)]$$

*and $u_0 > 0$. Then a.s.*

$$P(t^* < u) = P(t < u) + O(a_n) \qquad \ldots \quad (1)$$

*uniformly for all $|u| \geqslant u_0$. If $L(\mu)$ has at least two non-zero eigen-values of same sign, then* (1) *holds uniformly for all $u$, a.s.*

*Remark* 1 : If $l(\mu) = 0$, the theorem gives an approximation for the distribution of $n(H(\bar{Z}) - H(\mu))$. The error term $O(a_n)$ in (1) cannot be improved even if $t^*$ is replaced by $b_n t^*$, where $b_n = b_n(Z_1, \ldots, Z_n) \to 1$ a.s. This can be seen from example 2 given at the end of this section. If the rank of $L(\mu) \leqslant 2$, it follows from the proof that (1) holds uniformly in $u$ if $O(a_n)$ is replaced by $O(\sqrt{a_n})$. We require the following lemma.

Lemma : *Let $a > 1$. We have uniformly for all $c$, $d$ and $0 < b < a$, that*

$$\int_{c - b a_n |x|^3 < x^2 < d + b a_n |x|^3} \phi(x) dx \leqslant \int_{c - a^2 a_n < x^2 < d + a^2 a_n} \phi(x) dx + O(a_n), \quad \ldots \quad (2)$$

*where $\phi$ is the density of the standard normal distribution on the line.*

*Proof of the lemma :* By changing the variable $w = x - \frac{b}{2} a_n x|x|$, we get that

$$\int_{x^2 < d + b a_n |x|^3} \phi(x) dx \leqslant \int_{w^2 < d + a^2 a_n, |w| < \log n} \phi(w)(1 + O(|w|^3 a_n))$$

$$\times (1 + O(a_n |w|) dw + O(n^{-1})$$

$$\leqslant \int_{w^2 < d + a^2 a_n} \phi(w) dw + O(a_n). \qquad \ldots \quad (3)$$

Similarly, by changing the variable $z = x + \frac{b}{2} a_n x|x|$ we obtain that

$$\int_{x^2 + b a_n |x|^3 < c} \phi(x) dx \geqslant \int_{z^2 < c - a^2 a_n} \phi(z) dz - O(a_n). \qquad \ldots \quad (4)$$

Now (2) follows from (3) and (4).

*Proof of the theorem:* For any symmetric positive definite matrix $B$, let $\phi_B$ denote the normal density with dispersion $B$. For any $y = (y_1, \ldots, y_k)$ and $1 \leqslant j \leqslant k$, let $f_{j,n}(y) = a_n\left(1+\sum_{i=j}^{k} |x_i|^3\right)$ and $f_n(y) = f_{1,n}(y)$. Recall that a.s. for sufficiently large $n$, the dispersion $\Sigma_n$ of $Y_1$ is positive definite. Note that $\Sigma_n$ is the sample dispersion matrix of $Z_1, \ldots, Z_n$. By the law of iterated logarithm the $(i, j)$-th elements of $L(\bar{Z})-L(\mu)$ and $\Sigma_n-\Sigma$ are $O(a_n)$ a.s. for $1 \leqslant i, j \leqslant k$. So

$$\int |\phi_{\Sigma_n}(x)-\phi_\Sigma(x)|\,dx = O(a_n). \qquad \ldots (5)$$

From now on let us write $L$ for $L(\mu)$. As the fourth moments of $\sqrt{n}\,|\bar{Y}-\bar{Z}|$ are bounded a.s., it follows from Theorem 1 of Sweeting (1977), that uniformly in $u$,

$$P(t^* < u) = \int_{y'L(\bar{z})y+O(f_n(y))<u;\ \|y\|<\log n} \phi_{\Sigma_n}(x)dx + \int_{A_n} \phi_{\Sigma_n}(x)dx$$

$$+P(\sqrt{n}\|\bar{Y}-\bar{Z}\| \geqslant \log n)+O(n^{-1/2}), \qquad \ldots (6)$$

where $\sqrt{n}\delta_n = O(1)$ and

$A_n = \{x \in R^k : \|x-y\| \leqslant \delta_n$, for some $y$ with $\|y\| < \log n$

and $n[H(\bar{Z}+yn^{-1/2})-H(\bar{Z})]-\sqrt{n}y'l(\bar{Z}) = u\}$.

Note that $A_n \subset \{y \in R^k : \|y\| \leqslant O(\log n)$ and $y'Ly-u = O(f_n(y))\}$.

Another application of Theorem 1 of Sweeting (1977) yields that

$$P(\sqrt{n}\|\bar{Y}-\bar{Z}\| > \log n) \leqslant \int_{\|y\|>\log n} \phi_{\Sigma_n}(y)dy+O(n^{-1/2})$$

$$+\int_{B_n} \phi_{\Sigma_n}(y)dy$$

$$= O(n^{-1/2}), \qquad \ldots (7)$$

where $B_n = \{y :$ for some $x$ with $\|x-u\| = O(n^{-1/2})$ and $\|x\| = \log n\}$. From (5), (6) and (7), we have

$$P(t^* < u) = \int_{y'Ly<u} \phi_\Sigma(y)dy+c_n+O(a_n),$$

where

$$c_n = \int_{y'Ly-u=O(f_n(y))} \phi_\Sigma(y)dy.$$

By using a similar argument for $t$, we obtain uniformly in $u$, a.s.,

$$P(t^* < u)-P(t < u) = O(a_n)+O(c_n).$$

Since $L$ is symmetric and $\Sigma$ is positive definite, there exist a non-singular matrix $A$ and a diagonal matrix $D$ with diagonal elements $e_1, \ldots, e_k$ such that $A'\Sigma^{-1}A$ is identity matrix and $A'LA = D$. We have a.s. uniformly in $u$

$$P(t^* < u) - P(t < u) = O(a_n) + O(d_n),$$

where

$$d_n = \int_{y'Dy - u = O(f_n(y))} \phi_I(y)dy.$$

To complete the proof it is enough to show that $d_n = O(a_n)$ uniformly for all $|u| > u_0$, and $d_n = O(a_n)$ uniformly in $u$, if $L$ has at least two non-zero eigen-values of same sign. We consider several cases.

*Case* 1. $L$ has only one non-zero eigen-value. In this case all but one $e_i$ are zeroes. Without loss of generality we assume that $e_1 > 0$ and $e_j = 0$ for $j \neq 1$. By the lemma we have for any $u_0 > 0$,

$$d_n = O\Big[\int_{e_1 y_1^2 - u = o(f_{2,n}(y))} \phi_I(y)dy\Big] + O(a_n)$$

$$= O(a_n) + O\Big[\int \Big|\sqrt{\frac{u}{e_1} + O(f_{2,n}(y))} - \sqrt{\frac{u}{e_1}}\Big| \Big(\exp\Big(-\frac{1}{2}\sum_{j=2}^{k} y_j^2\Big)\Big) dy_2 \ldots dy_k\Big].$$

$$= O(a_n) + O\Big(\int f_{2,n}(y) \Big(\exp\Big(-\frac{1}{2}\sum_{j=2}^{k} y_j^2\Big)\Big) dy_2 \ldots dy_k\Big) = O(a_n)$$

uniformly in $|u| > u_0$.

*Case* 2. The rank of $L$ is 2 and $L$ has one positive and one negative eigen-value. Clearly, all but two $e_i$ are zeroes and these two non-zero $e_i$ are of different sign. Without loss of generality assume that $e_1 > 0$, $e_2 < 0$ and $e_j = 0$ for $j \neq 1$ or 2. By applying the lemma twice we get that

$$d_n = O\Big[\int_{e_1 y_1^2 + e_2 y_2^2 - u = o(f_{3,n}(y))} \phi_I(y)dy\Big] + O(a_n). \qquad \ldots (8)$$

If $u > u_0$, then for any real number $y_2$, $u - e_2 y_2^2 > u > u_0$. From (8), it follows as in case 1, that $d_n = O(a_n)$ uniformly in $u > u_0$. If $u < -u_0$, then for any real $y_1$, $-u + e_1 y_1^2 > u_0$. So by a similar argument, as above, we have $d_n = O(a_n)$ uniformly for $u < -u_0$. Putting these together we obtain $d_n = O(a_n)$ uniformly for $|u| > u_0$.

*Case* 3. The rank of $L > 2$. In this case rank of $D > 2$. So at least two of the non-zero $e_i$ are of same sign without loss of generality assume that $e_1 > 0$, $e_2 > 0$ using the lemma twice we obtain

$$d_n = O(a_n) + O\left[\int\limits_{e_1y_1^2+e_2y_2^2-\left(u-\sum_{j=3}^{k} e_j y_j^2\right)-O(f_{3,n}(y))} \phi_I(y)dy\right]. \quad \dots \quad (9)$$

On changing the variables $y_1 = r\sin\theta$, $y_2 = r\cos\theta$, $r > 0$, $0 \leqslant \theta < 2\pi$, we get, uniformly for all $c, d$ that

$$\int\limits_{c<e_1y_1^2+e_2y_2^2<d} e^{-\frac{1}{2}(y_1^2+y_2^2)} dy_1\, dy_2 = \int_0^{2\pi} d\theta \int\limits_{c<r^2(e_1\sin^2\theta+e_2\cos^2\theta)<d} re^{-\frac{1}{2}r^2} dr$$

$$= O\left[(d-c) \sup_{0\leqslant\theta<2\pi} (e_1\sin^2\theta + e_2\cos^2\theta)^{-1}\right]$$

$$= O[(d-c)/\min(e_1, e_2)]. \quad \dots \quad (10)$$

It follows now from (9) and (10) that $d_n = O(a_n)$ uniformly in $u$.

*Case* 4. The rank of $L = 2$ and both the eigen values of $L$ are of same sign. In this case rank of $D = 2$ and both the non-zero $e_i$ are of same sign. Similar arguments as in Case 3 yield that $d_n = O(a_n)$ uniformly in $u$.

This completes the proof of the theorem.

*Remark* 2. Instead of taking the empirical distribution as an estimate of the distribution of $Z_1$, any consistent estimate $G_n$ of distribution of $Z_1$ based on $Z_1, \ldots, Z_n$ can be used. In this case $Y_1, \ldots, Y_n$ would be i.i.d. random vectors from $G_n$. The theorem still holds if $\mu_{n,\beta} \to \mu_\beta$ for all $|\beta| \leqslant 4$ and if the error term $O(a_n)$ is replaced by

$$O\left[\frac{1}{\sqrt{n}} + \sum_{0\leqslant|\beta|\leqslant 2} |\mu_{n,\beta} - \mu_\beta|\right],$$

where $\beta = (\beta_1, \ldots, \beta_k)$, $|\beta| = \sum_{i=1}^{k} \beta_i$, $\beta_i$ are non-negative integers, $\mu_{n,\beta} = E_{G_n}(Y_1^\beta)$ and $\mu_\beta = E(Z_1^\beta)$. Here for any $y = (y_1, \ldots, y_k)$, $y^\beta = \prod_{i=1}^{k} y_i^{\beta_i}$.

The following example shows that the error term in the theorem cannot be improved.

*Example 2* : Let $H(a, b) = (a+b^2)^2$. Let the distribution of $Z_1$ be the bivariate standard normal. Note that the distribution of $n(H(\bar{Z})-H(0))$ is same as that of $\left(X+\frac{Y^2}{\sqrt{n}}\right)^2$, where $X$ and $Y$ are independent standard variables. It is not difficult to show, uniformly in $u$, that

$$P(t < u)+P((X+n^{-\frac{1}{2}}Y^2)^2 < u) = P(X^2 < u)+O(n^{-1}).$$

Observe that the dispersion matrix $\Sigma_n$ of $Y_1$ is

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z})(Z_i-\bar{Z})',$$

and by the law of iterated logarithm $\sigma_i = 1+O(a_n)$, $i = 1, 2$ and $\rho = \sigma_{12}/\sigma_1\sigma_2 = O(a_n)$. Using Theorem 1 of Sweeting (1977) as in the proof of the theorem we obtain, uniformly in $u > 0$, that

$$P(t^* < u\sigma_1^2) = P\Big(\Big[(\bar{Y}_1-\bar{Z}_1)\sigma_1^{-1}+\frac{1}{\sqrt{n}}(\bar{Y}_2-\bar{Z}_2)^2\sigma_2^{-2}+2\bar{Z}_2(\bar{Y}_2-\bar{Z}_2)\sigma_2^{-1}\Big]^2$$

$$+2\bar{Z}_1(\bar{Y}_2-\bar{Z}_2)\sigma_2^{-1}+O((\log n)^3n^{-1}) < u\Big)+O(n^{-1})$$

$$= \frac{1}{2\pi}\int_{D_n} e^{-\frac{1}{2}(w^2+v^2-2\sigma_{12}wv)}\,dw\,dv$$

$$+O\Big(\int_{E_n} e^{-\frac{1}{2}(w^2+v^2-2\sigma_{12}wv)}\,dwdv+O(n^{-1}),$$

where $\bar{Y}_i$ and $\bar{Z}_i$ are respectively the sample means of $i$-th coordinates of $\{Y_j\}$ and $\{Z_j\}$;

$$D_n = \left\{(w, v) : \left(w+\frac{v^2}{\sqrt{n}}+2\bar{Z}_2v\right)^2+2v^2\bar{Z}_1 < u\right\}$$

and

$$E_n = \left\{y = (w, v) : \left(w+\frac{v^2}{\sqrt{n}}+2\bar{Z}_2v\right)^2+2v^2\bar{Z}_1-u = O((1+|y|^2)/\sqrt{n})\right\}.$$

If we make the change of variable $s = w+\frac{v^2}{\sqrt{n}}+2\bar{Z}_2v$, and use the arguments of proof of the theorem we obtain uniformly for all $u \geqslant 0{\cdot}5$ that

$$P(t^* < u\sigma_1^2) = \frac{1}{2\pi}\int_{s^2+2\bar{Z}_1v<u} e^{-\frac{1}{2}(s^2+v^2)}\Big[1+\frac{sv^2}{\sqrt{n}}+2\bar{Z}_2vs+\sigma_{12}vs\Big]\,drds+O(n^{-1})$$

$$= \frac{1}{2\pi}\int_{s^2+2\bar{Z}_1v^2<u} e^{-\frac{1}{2}(s^2+v^2)}\,ds\,dv+o(\tau_n).$$

The last equality holds because $se^{-\frac{1}{2}s^2}$ is an odd function of $s$. Note that if $h_n \to 0$, then we have uniformly in $u > 0{\cdot}5$, that

$$\int_u^{u+h_n} x^{-\frac{1}{2}} e^{-x/2}\, dx = e^{-u/2} \int_u^{u+h_n} x^{-\frac{1}{2}}(1+O(h_n))dx$$

$$= h_n u^{-\frac{1}{2}} e^{-u/2} + O(h_n^2). \qquad \ldots \quad (11)$$

By (11) it follows that uniformly for $u > 0{\cdot}5$

$$\sqrt{2\pi}P(t^* < u\sigma_1^2) = \int_0^u x^{-\frac{1}{2}} e^{-x/2}\, dx - 2\bar{Z}_1 u^{-\frac{1}{2}} e^{-u/2} + o(a_n). \qquad \ldots \quad (12)$$

Now let $b_n = b_n(Z_1, \ldots, Z_n) \to 1$ a.s. Let $t_n = b_n \sigma_1^{-2} - 1$. Then by (11) and (12) we have uniformly in $1 < u < 2$,

$$2\sqrt{2\pi}P(t^* < ub_n) = \int_0^u x^{-\frac{1}{2}} e^{-x/2} dx + t_n \sqrt{u} e^{-u/2} + O(t_n^2)$$

$$-2\bar{Z}_1[u(1+t_n)]^{-\frac{1}{2}} e^{-u/2} e^{-\frac{1}{2}ut_n} + o(a_n)$$

$$= 2\sqrt{2\pi}P(t < u) + \sqrt{u} e^{-u/2}[t_n - 2\bar{Z}_1 u^{-1}] + o(a_n)$$

$$+ o(t_n)$$

since $\lim\sup \bar{Z}_1 a_n^{-1} > 0$, $a_n^{-1}[ut_n - 2\bar{Z}_1]$ can never tend to zero for all $u \in (1, 2)$.

This shows that the error term in the theorem cannot be improved even if we replace $t^*$ by $t^*/b_n$ for some $b_n = b_n(Z_1, \ldots, Z_n) \to 1$.

### 3. Concluding remarks

In almost all the results on the bootstrap method (Babu and Singh, 1983 ; 1984) the distribution of the population is assumed to have at least the second moment. This condition used mainly in approximating the bootstrap distribution by Gaussian distributions. Suppose $\{X_n\}$ is a sequence of i.i.d. random variables, $F_n$ is the empirical distribution of $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ is a sequence of i.i.d. random variables from $F_n$. Suppose $X_1$ has finite mean $\mu$. Since $\bar{Y} - \bar{X}$ and $\bar{X} - \mu$ are well defined, it is natural to enquire whether the distribution of $\bar{X} - \mu$ is close to that of $\bar{Y} - \bar{X}$ uniformly. The following example shows that in general it is false.

*Example* 3 : Let the characteristic function of $X_1$ be $e^{-|t|^\alpha}$, $1 < \alpha < 2$. It is well known that $E|X_1| < \infty$, $EX_1^2 = \infty$ and $EX_1 = 0$. Clearly $P(\bar{X} < xn^{-1+1/\alpha}) = P(X_1 < x)$ for all $n$. For any $s_n$ and $x$, $nP(Y_1 - s_n < xn^{1/\alpha})$

is an integer. So this cannot converge to $c|x|^{-\alpha}$ as $n \to \infty$ for all $x < 0$; $c > 0$ is a constant. Hence by Theorem 4 (see Kolmogorov and Gnedenko, 1954, page 124), for any sequence $s_n$, $P(\bar{Y} - s_n < xn^{-1+1/\alpha}) \nrightarrow P(X_1 < x)$. Thus sup $|P(\bar{Y} - \bar{X} < u) - P(\bar{X} < u)| \nrightarrow 0$.

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