CALCULUS ON POISSON MANIFOLDS

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ABSTRACT

We prove the existence of a Lie bracket on the space of 1-forms on a Poisson manifold. This gives rise to a calculus of skewsymmetric contravariant tensors dual to Cartan's calculus of forms.

Introduction

The concept of a Poisson structure is currently of much interest and is being studied by Lichnerowicz, Weinstein and others ([**5**, **10**, **8**, **4**]). A Poisson structure is a Lie bracket $\{\ ,\ \}$ on the multiplicative algebra N of smooth functions on a manifold M satisfying the additional condition $\{fg, h\} = f\{g, h\} + g\{f, h\}$ for $f, g, h \in N$. N together with $\{\ ,\ \}$ is called a Poisson algebra.

If (M, ω) is a symplectic manifold then $\{f, g\} = \omega(X_f, X_g)$ (where X_f is the hamiltonian vector field corresponding to f) defines a Poisson structure on M. However, this is not the only way in which a Poisson structure arises. In general the existence of a Poisson structure is equivalent to the existence of a skewsymmetric contravariant 2-tensor G on M satisfying $[G, G] = 0$ (where $[\ ,\ ]$ denotes the alternating Schouten product [**6**, **7**]). If G is as above and P is any skewsymmetric contravariant tensor on M then $\partial P = [G, P]$ defines a cohomology operator ∂ [**5**]. The purpose of this article is to show that this operator ∂ is part of an entire calculus of skewsymmetric contravariant tensors which is dual to the Cartan calculus of forms [**1**]. The key to this calculus is the possibility of defining a Lie bracket on the space of 1-forms on a Poisson manifold. (This bracket coincides in the symplectic case with the bracket defined in [**1**]). Since Cartan's calculus involving the operators L_X, i_X and d [**1**] is ultimately due to the existence of the Lie bracket of vector fields, it is possible to use the Lie bracket on 1-forms to define a calculus of skewsymmetric contravariant tensors in terms of operators L_α, i_α and ∂ where α is a 1-form.

1. *Definitions and notation*

Throughout this article M denotes a Hausdorff, second countable and connected C^∞-manifold and N denotes the algebra of C^∞-functions on M. As usual a p-form on M means a covariant alternating p-tensor on M. We shall refer to a contravariant alternating p-tensor on M as a p-field on M. Thus a 1-field is just a vector field. Let $\chi(M)$ and $\chi^*(M)$ denote the spaces of vector fields and 1-forms on M respectively. The space of all p-forms and the space of all p-fields on M will be denoted by $\Omega^p(M)$ and

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$\Phi_p(M)$ respectively (so $\Omega^1(M) = \chi^*(M)$ and $\Phi_1(M) = \chi(M)$). If $X \in \chi(M)$ and $\alpha \in \chi^*(M)$ then the contraction of α with X is written as either $X(\alpha)$ or $\alpha(X)$.

Choose and fix $G \in \Phi_2(M)$. Define a map $G^\bullet : \chi^*(M) \to \chi(M)$ by $G^\bullet(\alpha)(\beta) = G(\alpha, \beta)$ for $\alpha, \beta \in \chi^*(M)$. For convenience we shall often write $G^\bullet(\alpha)$ as $\alpha^\bullet$.

Finally, let us write $\{f, g\} = G(df, dg)$ and $X_f = -(df)^\bullet$ for $f, g \in N$. Then X_f is called the hamiltonian vector field corresponding to f.

PROPOSITION 1.1. *Let $G \in \Phi_2(M)$. Then the following are equivalent.*
(a) *Jacobi's identity holds for* $\{\ ,\ \}$.
(b) $X_{\{f,g\}} = -[X_f, X_g]$ *for* $f, g \in N$.
(c) $(N, \{\ ,\ \})$ *is a Poisson algebra.*

DEFINITION 1.2. If G satisfies any of the above conditions then the pair (M, G) is called a *Poisson manifold*. Also, G is called a Poisson structure on M. In general any unexplained notation will be that of [1].

NOTATION. Throughout this article (M, G) denotes a Poisson manifold unless otherwise specified; α, β, γ denote 1-forms on M; X, Y denote vector fields on M and f, g, h, denote C^∞-functions on M.

2. *The Lie bracket of 1-forms on a Poisson manifold*

Let (M, G) be a Poisson manifold.

DEFINITION 2.1. If $\alpha, \beta \in \chi^*(M)$ then let $\{\alpha, \beta\} = L_{\alpha^\bullet}\beta - L_{\beta^\bullet}\alpha - d(G(\alpha, \beta))$.

LEMMA 2.2. $\{f\alpha, g\beta\} = fg\{\alpha, \beta\} + f(L_{\alpha^\bullet} g)\beta - g(L_{\beta^\bullet} f)\alpha$.

PROPOSITION 2.3. $\{\alpha, \beta\}^\bullet = [\alpha^\bullet, \beta^\bullet]$ *where* $[\ ,\ ]$ *denotes the Lie bracket of vector fields on M.*

Proof.
$$\begin{aligned}
\{\alpha,\beta\}^\bullet(df) &= -\{\alpha,\beta\}(df)^\bullet \\
&= -[(L_{\alpha^\bullet}\beta)((df)^\bullet) - (L_{\beta^\bullet}\alpha)((df)^\bullet) - (d(G(\alpha,\beta)))(df)^\bullet)] \\
&= -[\alpha^\bullet(\beta(df)^\bullet) - \beta(L_{\alpha^\bullet}(df)^\bullet) - \beta^\bullet(\alpha(df)^\bullet) \\
&\quad + \alpha(L_{\beta^\bullet}(df)^\bullet) - (d(G(\alpha,\beta)))(df)^\bullet] \\
&= \alpha^\bullet(\beta^\bullet(df)) - \beta^\bullet(\alpha^\bullet(df)) + \beta([\alpha^\bullet,(df)^\bullet]) \\
&\quad - \alpha([\beta^\bullet,(df)^\bullet]) + (df)^\bullet(G(\alpha,\beta)) \\
&= [\alpha^\bullet,\beta^\bullet](df) + A(\alpha,\beta),
\end{aligned}$$
where
$$\begin{aligned}
A(\alpha,\beta) &= \beta([\alpha^\bullet,(df)^\bullet]) - \alpha([\beta^\bullet,(df)^\bullet]) + (df)^\bullet(G(\alpha,\beta)) \\
&= \alpha([\beta^\bullet, X_f]) - \beta([\alpha^\bullet, X_f]) - X_f(G(\alpha,\beta)).
\end{aligned}$$

It is easy to check that A is a 2-field. To prove that $A = 0$, it is sufficient to prove that A is zero on exact forms, that is, that $A(dg, dh) = 0$ for $g, h \in N$. But
$$A(dg, dh) = \{g, \{h, f\}\} + \{h, \{f, g\}\} + \{f, \{g, h\}\} = 0$$
(by Proposition 1.1).

We will now prove a lemma which will be useful later.

LEMMA 2.4. *Let G be any 2-field on a smooth manifold M. Define*

$$H(\alpha,\beta,\gamma) = \alpha^\#(G(\beta,\gamma)) + \beta^\#(G(\gamma,\alpha)) + \gamma^\#(G(\alpha,\beta)) - (G(\{\alpha,\beta\},\gamma) + G(\{\beta,\gamma\},\alpha) + G(\{\gamma,\alpha\},\beta)).$$

Then H is a 3-field on M. If G is a Poisson structure on M, then $H = 0$.

Proof. That H is a tensor follows from (2.2) and the N-linearity of G.

Now let G be a Poisson structure. It is enough to show that $H(df,dg,dh) = 0$. But

$$H(df,dg,dh) = 2(\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}) = 0,$$

(since $d\{f,g\} = \{df,dg\}$).

PROPOSITION 2.5. *Let (M,G) be a Poisson manifold. Then (2.1) defines a Lie bracket on $\chi^*(M)$.*

Proof. By the above lemma, it is sufficient to prove that

$$\{\{\alpha,\beta\},\gamma\} + \{\{\beta,\gamma\},\alpha\} + \{\{\gamma,\alpha\},\beta\} = d(H(\alpha,\beta,\gamma)). \qquad (*)$$

Consider

$$\begin{aligned}\{\{\alpha,\beta\},\gamma\} &= L_{\{\alpha,\beta\}^\#}\gamma - L_{\gamma^\#}\{\alpha,\beta\} - d(G(\{\alpha,\beta\},\gamma)) \\ &= L_{\alpha^\#}L_{\beta^\#}\gamma - L_{\beta^\#}L_{\alpha^\#}\gamma - L_{\gamma^\#}L_{\alpha^\#}\beta + L_{\gamma^\#}L_{\beta^\#}\alpha \\ &\quad + L_{\gamma^\#}(d(G(\alpha,\beta))) - d(G(\{\alpha,\beta\},\gamma))\end{aligned}$$

by (2.3) and $L_{[X,Y]} = L_X L_Y - L_Y L_X$. Writing the remaining terms and using $L_X d = dL_X$ one can see that (*) holds.

We observe that $\#: \chi^*(M) \to \chi(M)$ is a Lie algebra homomorphism by Proposition 2.3. So $\#$ is a representation of the Lie algebra $\chi^*(M)$ on N. Hence using the methods of Chevalley and Eilenberg ([3, p. 115]) one can define a cohomology operator ∂ on $\Phi(M) = \sum_{p>0}\Phi_p(M)$. This operator coincides with $[G,\ ]$ defined by Lichnerowicz [5]. The formula for ∂ is given by

$$\partial P(\alpha_0,\ldots,\alpha_p) = \sum_{i=0}^{p}(-1)^i\alpha_i^\#(P(\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_p)) + \sum_{i<j}(-1)^{i+j}P(\{\alpha_i,\alpha_j\},\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\hat{\alpha}_j,\ldots,\alpha_p)$$

where $P\in\Phi_p(M)$. Its cohomology is called *Poisson cohomology*.

In the course of the proof of $\partial^2 = 0$ one comes across the operators $L_\alpha: \Phi_p(M) \to \Phi_p(M)$ given by

$$(L_\alpha P)(\alpha_1,\ldots,\alpha_p) = \alpha^\#(P(\alpha_1,\ldots,\alpha_p)) + \sum_{i=1}^{p}(-1)^i P(\{\alpha,\alpha_i\},\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_p)$$

and $i_\alpha: \Phi_p(M) \to \Phi_{p-1}(M)$ given by

$$(i_\alpha P)(\alpha_1,\ldots,\alpha_{p-1}) = P(\alpha,\alpha_1,\ldots,\alpha_{p-1})$$

for $P\in\Phi_p(M)$ satisfying the relations

$$i_{\{\alpha,\beta\}} = L_\alpha i_\beta - i_\beta L_\alpha,$$

$$L_{\{\alpha,\beta\}} = L_\alpha L_\beta - L_\beta L_\alpha,$$

$$L_\alpha = i_\alpha\partial + \partial i_\alpha \quad\text{and}\quad L_\alpha\partial = \partial L_\alpha.$$

Thus one can develop a dual calculus to Cartan's calculus on differential forms [1].

Brylinski [2] defined a canonical complex on Poisson manifolds

$$\longrightarrow \Omega^{n+1}(M) \xrightarrow{\delta} \Omega^{n}(M) \xrightarrow{\delta} \Omega^{n-1}(M) \longrightarrow$$

where δ is given by the formula:

$$\delta(f_0\, df_1 \wedge \ldots \wedge df_n) = \sum_{i=1}^{n} (-1)^{i+1} \{f_0, f_i\} \wedge df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_n$$
$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} f_0\, d\{f_i, f_j\}\, df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge \widehat{df_j} \wedge \ldots \wedge df_n.$$

Its homology is called the *canonical homology* of the Poisson manifold M by Brylinski.

We now deduce the relationship between ∂ and δ. Recall that there exists a duality map $\langle\ ,\ \rangle: \Omega^p(M) \times \Phi_p(M) \to C^\infty(M)$ such that $\langle \alpha_1 \wedge \ldots \wedge \alpha_p, X_1 \wedge \ldots \wedge X_p \rangle = \det(\alpha_i(X_j))$. If $P \in \Phi_{p-1}(M)$ and $\omega \in \Omega^p(M)$ define $i_p \omega \in \chi^*(M)$ by $(i_p\omega)(X) = \langle \omega, P \wedge X \rangle$ for all $X \in \chi(M)$. For $\omega = f_0\, df_1 \wedge \ldots \wedge df_p$ it is not difficult to check that

$$i_p \omega = (-1)^p \sum_{i=1}^{p} (-1)^i f_0\, P(df_1, \ldots, \widehat{df_i}, \ldots, df_p)\, df_i.$$

It then follows easily that $\langle \omega, \partial P \rangle - \langle \delta\omega, P \rangle = (-1)^p \delta(i_p \omega)$ for $\omega \in \Omega^p(M)$ and $P \in \Phi_{p-1}(M)$. Consequently if $\omega \in \Omega^p(M)$ and $Q \in \Phi_p(M)$ such that $\delta\omega = 0$ and $\partial Q = 0$ then $\langle [\omega], [Q] \rangle$ is defined up to a function of the form $\delta(\alpha)$ with $\alpha \in \chi^*(M)$. Hence there is a well-defined map from (the notation is obvious) $H^p_\delta \times H^p_\partial$ into H^0_δ.

3. *Lie algebras attached to a Poisson manifold*

In this section we first define certain Lie algebras of vector fields on a Poisson manifold which were introduced and studied by Lichnerowicz [5]. Next we show that it is possible to define certain Lie algebras of 1-forms in an analogous manner.

Let (M, G) be a Poisson manifold. An element f of N is called a central function or a Casimir function [10] if $\partial f = 0$. Equivalent conditions are (i) $\{f, g\} = 0$ for all $g \in N$, (ii) f is constant on the leaves of the canonical symplectic foliation due to R, the range of $G^\#$. Let $\mathscr{A}$ denote the set of all central functions on M. Then $\mathscr{A}$ is a multiplicative subalgebra of N and is a Lie ideal in $(N, \{\ ,\ \})$. Since $-[X_f, X_g] = X_{\{f,g\}}$, $\mathscr{A}$ is precisely the kernel of the map $f \to X_f$ so that $N/\mathscr{A}$ is anti-isomorphic as a Lie algebra to χ_H, the Lie algebra of hamiltonian vector fields on M. Consider $\chi_C = \{X \in \chi(M) \mid \partial X = 0\}$, $\chi_{LH} = \{X \in \chi(M) \mid$ for x in M, there exists a neighbourhood U such that $X \mid U$ is hamiltonian$\}$,

$$\mathscr{R} = \{X \in \chi(M) \mid \exists\, \alpha \in \chi^*(M) \ni \alpha^\# = X\}, \quad \chi_{\text{Conf}} = \{X \in \chi(M) \mid \exists\, f \in \mathscr{A} \ni L_X G = fG\}.$$

It is easy to see the following: (1) $\chi_H \subset \chi_{LH} \subset \chi_C \subset \chi_{\text{Conf}}$, (2) $\chi_{LH} \subset \mathscr{R}$, (3) each of $\chi_H, \chi_{LH}, \chi_C, \chi_{\text{Conf}}$ and $\mathscr{R}$ is an $\mathscr{A}$-module and a Lie subalgebra of $\chi(M)$, (4) χ_H and χ_{LH} are Lie ideals in χ_{Conf}, (5) if (M, G) is a Poisson manifold of constant rank then $\chi_{LH} = \chi_C \cap \mathscr{R}$.

We will now consider $\chi^*(M)$, the Lie algebra of 1-forms. Let $\mathscr{E}$ be the space of exact 1-forms, and $\mathscr{C}$ the space of closed 1-forms. Call a 1-form α hamiltonian if $\alpha^\#$

is hamiltonian and let $\chi_H^\bullet$ denote the set of all hamiltonian 1-forms. Define similarly $\chi_{LH}^\bullet$, $\chi_C^\bullet$ and $\chi_{Cont}^\bullet$. Finally let $\mathscr{Z}$ denote the set of all 1-forms α such that $\alpha^\bullet = 0$.

PROPOSITION 3.1. (a) *The spaces* $\mathscr{E}$, $\mathscr{C}$, $\chi_H^\bullet$, $\chi_{LH}^\bullet$, $\chi_C^\bullet$ *and* $\chi_{Cont}^\bullet$ *are all Lie subalgebras of* $\chi^\bullet(M)$.

(b) *Except for* $\mathscr{E}$ *and* $\mathscr{C}$ *all the rest of the spaces considered in* (a) *are* $\mathscr{A}$*-modules. In fact,* $\mathscr{Z}$ *is an N-module.*

(c) *The space* $\mathscr{E}$ *is a Lie ideal in* $\mathscr{C}$; $\mathscr{Z}$ *is a Lie ideal in* $\chi^\bullet(M)$; $\chi_H^\bullet$ *and* $\chi_{LH}^\bullet$ *are Lie ideals in* $\chi_{Cont}^\bullet$.

(d) (i) $\chi_H^\bullet = \mathscr{E} + \mathscr{Z}$

(ii) $\chi_{LH}^\bullet = \{\alpha \in \chi^\bullet(M) \mid \alpha \text{ is locally in } \mathscr{E} + \mathscr{Z}\}$,

(iii) $\chi_C^\bullet = \{\alpha \in \chi^\bullet(M) \mid L_\alpha G = 0\}$,

(iv) $\chi_{Cont}^\bullet = \{\alpha \in \chi^\bullet(M) \mid L_\alpha G = fG \text{ for } f \in \mathscr{A}\}$.

Proof. We prove (d).(i) If $\alpha \in \mathscr{E} + \mathscr{Z}$ then $\alpha^\bullet = (df)^\bullet$ for some $f \in N$ so that α is hamiltonian. Conversely if α is hamiltonian then there is an $f \in N$, such that $\alpha^\bullet = (df)^\bullet \Rightarrow \alpha^\bullet - (df)^\bullet = 0$. (ii) follows from (i). (iii) Since $L_\alpha G = i_\alpha \partial G + \partial i_\alpha G = \partial \alpha^\bullet$, $L_\alpha G = 0$ if and only if $\partial\alpha^\bullet = 0$, that is if and only if $\alpha^\bullet$ is canonical. (iv) follows from the fact that $L_\alpha G = \partial\alpha^\bullet = -[G, \alpha^\bullet] = L_{\alpha^\bullet} G$.

NOTE. If $\chi_H^\bullet$, $\chi_{LH}^\bullet$, $\chi_C^\bullet$ and $\chi_{Cont}^\bullet$ are defined by the equations in (d) above then we may prove (a), (b) and (c) directly by using our calculus of p-fields.

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References

1. R. ABRAHAM and J E. MARSDEN, *Foundations of mechanics* (Benjamin/Cummings Publishing Company, 1978)
2. J. L. BRYLINSKI, 'A differential complex for Poisson manifolds', IHES Preprint, March 1986.
3. C. CHEVALLEY and S. EILENBERG, 'Cohomology theory of Lie groups and Lie algebras', *Trans. Amer. Math. Soc* 63 (1948) 85–124.
4. R. JOST, 'Poisson brackets (an unpedagogical lecture)', *Rev. Modern Phys* 36 (1964) 572–579.
5. A. LICHNEROWICZ, 'Les variétés de Poisson et leurs algèbres de Lie associées', *J. Differential Geom.* 12 (1977) 253-300.
6. A. NIJENHUIS, 'Jacobi-type identities for bilinear differential concomitants of tensor fields I', *Indag Math.* 17 (1955) 390–403.
7. J. A. SCHOUTEN, 'Über differentiаl Komitanten zweier kontravariant Grossen', *Proc. Kon. Ned. Akad. Wet. Amsterdam* 43 (1940) 449–452.
8. W M. TULCZYJEW, 'Poisson brackets and canonical manifolds', *Bull Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* 22 (1974) 931–935.
9. W. M. TULCZYJEW, 'The graded Lie algebra of multi-vector fields and the generalized Lie derivative of forms', *Bull. Acad Polon. Sci. Ser Sci. Math. Astronom. Phys.* 22 (1974) 937–942.
10. A. WEINSTEIN, 'The local structure of Poisson manifolds', *J. Differential Geom.* 18 (1983) 523–557

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