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EXISTENCE OF UNBIASED ESTIMATES IN SEQUENTIAL BINOMIAL EXPERIMENTS

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SUMMARY. For sequential Bernoulli trials, a necessary condition for a parametric function to be unbiasedly estimable is that it be continuous. Depending on the existence of the moments of the corresponding stopping time and the estimator, these functions are differentiable upto a given order. We also study the implications of these results to the problem of estimating min $(p, 1-p)$ unbiasedly.

1. Introduction

Let (X_i), $i \geqslant 1$ be a sequence of i.i.d. Bernoulli r.v.'s with $P(X_i = 1) = p$ and $P(X_i = 0) = 1-p$, $0 < p < 1$. Basu (1975) posed the problem of generating an event with probability p^{α}, $\alpha > 0$. Banerjee and Sinha (1979) pursued this problem and were led to the problem of estimability of $g(p) = \min(p, 1-p)$.

Singh (1964) had proved that if (X_i), $i \geqslant 1$ are i.i.d. observations from some parametric family G_θ, $\theta \in \Omega$ and there is a fixed sample size estimator which estimates θ unbiasedly, then a sufficient condition for a parametric function $g(\theta)$ to be estimable on the basis of a random sample size N, is that it be in Baire class 0 or 1.

This result is clearly applicable to the Bernoulli situation. However, Singh's sequential plans are not true sequential plans since he assumes N to be independent of the entire sequence (X_i), $i \geqslant 1$.

In the Bernoulli case, we consider the class of all true sequential plans (with no external randomization). By quite simple arguments, we are able to show that if a parametric function is (unbiasedly) estimable, it is necessarily continuous. Higher order smoothness can be proved under existence of moments of the sample size and the estimator.

The original motivation for writing this paper was to see whether there are sequential plans allowing the unbiased estimability of the function $g(p) = \min(p, 1-p)$. As a consequence of our main result, this is not possible if we restrict our attention to proper or bounded estimators and with expected sample size finite.

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2. The main results

Let (X_i), $i \geqslant 1$ be a sequence of i.i.d. variables with $P(X_i = 1) = p$, $P(X_i = 0) = 1-p$, $0 < p < 1$. Any realization of this sequence can be viewed as a path in the first quadrant of $\boldsymbol{R}^2$. Starting from the origin, at the i-th trial ($i \geqslant 1$), we move one step to the right if $X_i = 0$, and one step above if $X_i = 1$. A non-randomised stopping rule tells us whether to stop or continue sampling after we have reached a given point—this depends on the path traced upto that point. Hence if T denotes the point where we stop, then $P_p(T = (x, y)) = K(x, y)\, p^x(1-p)^y$ where K is an integer with $0 \leqslant K(x, y) \leqslant \binom{x+y}{y}$. Only closed plans are relevant and so we assume $P_p(T<\infty)=1$, or in other words, $\sum\limits_{x, y \geqslant 0} K(x, y)p^x(1-p)^y = 1$ for all p, $0<p<1$. If $T=(x, y)$, the random sample size N_T is given by $N_T = x+y$. An estimator e is a function defined on the possible points (x, y) of T. A parametric function $g(p)$ is said to be (unbiasedly) estimable if there exists a stopping rule T and an estimator e such that

$$\sum_{x, y} |e(x, y)|\, P_p(T = (x, y)) < \infty \text{ for } 0 < p < 1$$

and

$$\sum_{x, y} e(x, y)P_p(T = (x, y)) = g(p) \text{ for } 0 < p < 1.$$

Theorem 1 : *If $g(p)$ is unbiasedly estimable, then g is continuous.*

Proof : Let e^+ and e^- be the positive and negative parts of e and let

$$g_+(p) = \Sigma\, e^+(x, y)P_p(T = (x, y))$$

$$g_-(p) = \Sigma\, e^-(x, y)P_p(T = (x, y)).$$

Then $g_+(p)$ and $g_-(p)$ exist and $g(p) = g_+(p)-g_-(p)$. It is enough to prove that $g^+(p)$ is continuous on any subset $[a, b]$ of $(0, 1)$.

Define $$g_+^n(p) = \sum_{x+y \leqslant n} e^+(x, y)P_p(T = (x, y)).$$

On $[a, b]$ each g_+^n is continuous (in fact a polynomial in p) and $g_+^n \uparrow g_+$ pointwise. Hence by Dini's theorem (page 248, Apostol, 1974), this convergence is uniform, which in turn shows that g_+ is continuous on $[a, b]$. □

Remarks : (1) As a consequence, if for a stopping time T, $E_p(N_T) < \infty$ for $0 < p < 1$, then $p \to E_p(N_T)$ is continuous.

(2) We believe that continuity of $g(p)$ is sufficient for its estimability but we have not been able to prove it.

The following theorem strengthens continuity to differentiability under stronger conditions.

Theorem 2 : *Suppose g is unbiasedly estimable by an estimator e and a stopping rule T such that $E_p|eN_T| < \infty$. Then g is continuously differentiable.*

Proof: Fix $[a, b] \subset (0, 1)$ and define g_+, g_+^n as before. It is enough to show that g_+ is continuously differentiable on (a, b)

$$(g_+^n(p))' = \sum_{x+y \leqslant n} \left(\frac{x}{p} - \frac{p}{1-p}\right) e^+(x, y) P_p(T = (x, y)).$$

Define $$R_n(p) = \sum_{x+y \geqslant n+1} \left(\frac{x}{p} - \frac{y}{1-p}\right) e^+(x, y) P_p(T = (x, y)).$$

Note that by the assumption that $E_p|eN_T| < \infty$, $R_n(p)$ exists and further,

$$|R_n(p)| \leqslant cE_p(N_T\, e^+ I(T \geqslant n+1)) \downarrow 0 \text{ for very fixed } p$$

by dominated convergence theorem.

By Theorem 1, $E_p(N_T\, e^+ I(T \geqslant n+1))$ is continuous in p and hence the above convergence is uniform.

By dominated convergence theorem, as $n \to \infty$,

$$(g_+^n(p))' \to f(p) = \sum_{x,y \geqslant 0} \left(\frac{x}{p} - \frac{y}{1-p}\right) e^+(x, y) P_p(T = (x, y))$$

and as shown above, this convergence is uniform.

On the other hand, $g_+^n(p) \to g_+(p)$ for every fixed p. This $g_+'(p)$ exists and $g_+'(p) = f(p)$ (see Theorem 9.13 of Apostol (1974)). The continuity of $g_+'(p)$ follows easily (e.g. by Theorem 1). □

Remarks : (3) Exactly the same arguments show that if $E_p|eN_T^k| < \infty$ for some integer $k \geqslant 1$, then $g(p)$ is k times continuously differentiable. Hence if $E_p(N_T^2) < \infty$ then $p \to E_p(N_T)$ is continuously differentiable.

(4) We conjecture that if $E_p(|e|\exp(\alpha N_T)) < \infty$ for some $\alpha > 0$, then $g(p)$ is real analytic.

3. Estimation of $g(p) = \min(p, 1-p)$

Since $g(p)$ lies between 0 and 1/2, it is realistic to restrict attention to only bounded estimators. If we restrict ourselves further to stopping rules which have finite expected stopping times, then $g(p)$ is not unbiasedly estimable. This follows immediately from Theorem 2.

We give below examples of a large class of stopping rules which allow us to estimate $g(p)$ (with possibly proper estimators).

Define $\eta = p(1-p)$. Then it follows easily that

$$\min(p, 1-p) = \frac{1-(1-4\eta)^{1/2}}{2}$$

$$= \sum_{x=1}^{\infty} \frac{(2x-2)!}{(x-1)!x!} p^x(1-p)^x.$$

(This relation also follows from the theory of random walks). Hence for any stopping rule for which (at least) the points (x, x) are boundary points, we can estimate $g(p)$ by

$$e(x, y) = \frac{(2x-2)\,!}{(x-1)\,!x!\,K(x, y)} \quad \text{if } x = y \text{ and } T = (x, x)$$

$$= 0 \qquad\qquad \text{otherwise.}$$

However, it is easy to see that any such estimator is highly improper since $K(x, x) \leqslant \binom{2x}{x}$.

We have not been able to get a plan with a proper estimator of $g(p)$. Note that any such estimator has to be positive at some points (x, y), $x \neq y$. Further, by Theorem 2, $g(p)$ is not unbiasedly estimable by a bounded estimator with finite expected stopping time for all p. This remains true for any other $g(p)$ which is nondifferentiable.

This leads us to the problem of characterizing all plans which yield (proper or improper) estimators of $g(p)$. It is easy to see that a plan which includes as stopping points, the points $\{(1, 1), (x+1, x), (x, x+1)\ ;\ x = 2, 3, \ldots\}$ enables to estimate $g(p)$. This is because for any x, $p^{x+1} q^x + p^x q^{x+1} = p^x q^x$. This argument can be pushed further. For instance a plan including $\{(x, x), (x_0+2, x_0), (x_0, x_0+2)\ ;\ x \neq x_0,\ x = 1, 2, \ldots\}$ as stopping points, yields an unbiased estimate of $g(p)$. These facts make the problem of characterization difficult to solve.

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