INT. J. SYSTEMS SCI., 1980, VOL. 11, NO. 12, 1435–1445

Recognition and fuzzy description of sides and symmetries of figures by computer

B. B. CHAUDHURI† and D. DUTTA MAJUMDER†

The detection of sides and corners is useful and important in the shape perception of visual picture patterns. Similarly, the recognition of symmetry in pictures is helpful in reducing the storage space, in retrieving a missing portion, or in correcting the orientation of its shape. In the present paper, algorithms are found to detect the sides and symmetry of simply closed two-dimensional man-made and machine-generated outlines. A description based on the fuzzy set theoretic approach is also developed for imperfect non-geometric figures and results of the execution of the algorithms are presented.

1. Introduction

Shape is the primary and one of the most important features of an object in computer as well as in human visual learning processes, pattern recognition and scene analysis. The perception of shape can be viewed as a collective understanding of properties like size, form, symmetry and orientation. Some of these properties may not have precise mathematical definitions. Yet, in learning and recognition of figures by computers, different approaches based on graph theory, contextual automata and topological description of outline have been proposed in recent years (Fu 1974, Gonzalez and Wintz 1977).

Attenave (1954) first focused attention on the importance of angles in determining the perceived shape of outlines. He found that the information in a contour was concentrated at points having high curvature and discontinuity. This discovery suggests a correspondence between discontinuity and uncertainty in connection with the information content.

Since angles are local maxima of curvature, any model of shape perception should include a component that detects angles. Computer models for angle detection have been suggested by different authors such as Rosenberg (1972), Johnston and Rosenfeld (1973), Pavlidis and Horowitz (1973) and Davis (1977 a, b). Most of the methods analyse patterns by a process of hierarchical decomposition. The top levels of hierarchy contain a coarse description of the shape, while the lower levels reflect the finer structure of the contour.

Another property that plays an important role in the shape understanding process is the symmetry of outline. If a shape is symmetric, then it can be represented economically, because one need only store a description of half the figure and the ' axis ' of symmetry. Further economy may be possible if the halves also possess an axis of symmetry. Again, if a portion of the figure is missing, symmetry provides an important clue to complete the figure. This may be useful for digital picture processing where continuous data is unavailable. The symmetry of shape can also help in human and computer perception

---

Received 14 July 1980.

† Electronics and Communication Sciences Unit, Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700035, India.

of the orientation of shapes. The notion of top or bottom can be clarified from the symmetry of the figure.

A figure, in general, may not have perfectly straight sides or sharp corners. Also, it may not be perfectly symmetric about an axis. The representation of such a figure at each level of hierarchy is associated with fuzziness and can best be accounted by a grade of membership assignment of the respective sides, corners or symmetry. The membership values allow a translation of linguistic hedges like ' slightly symmetric ', ' nearly n-sided ', ' not very sharp corners ' from a qualitative into a quantitative scale, and may be useful in man–machine interaction and decision processes.

In the present paper, simple methods of recognition and description of sides, corners and symmetry of a simply closed outline have been presented. The recognition of sides is based on the application of Hough (1962) transform technique on the outline of the figure while symmetry is found from the correlation measure of curvature at equidistant points from the axis of symmetry in question. The definition of grade of membership and subsequent description is based on the fuzzy set theory (Zadeh 1963) which has found wide applications in pattern analysis and machine intelligence problems (Negoita 1973, Albin 1975, Pal and Dutta Majumder 1977). The method has been tested here on different sets of man-made and machine-generated figures. The machine generation is based on the finite polar Fourier description of a continuous function. The results of the execution of the algorithms are presented in § 5.

2. Recognition of sides

Consider an outline of a figure to be represented by a set of n points given by $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. If the figure is closed, $x_{n+1} = x_i$ and $y_{n+1} = y_i$. The colinear points may be grouped by Hough transform, as follows.

2.1. *Hough transform*

The normal form of a straight line is

$$x \cos \theta + y \sin \theta = r \tag{1}$$

where r is the normal distance of the origin from the line and θ is the angle the normal makes with the positive x-axis. If we restrict θ to the interval $[0, \pi]$ then the normal parameters of the line r, θ are unique and are represented by a point in the r–θ plane.

Given the set $\{x_i, y_i\}_{i=1}^{n}$, let (x_i, y_i) be transformed into the sinusoidal curves in the r–θ plane

$$r = x_i \cos \theta + y_i \sin \theta \tag{2}$$

Then it can be shown that the curves corresponding to collinear figure points have a common point of intersection. This point in the r–θ plane, say r_0, θ_0 defines the line passing through the collinear points (Fig. 1). Thus, the problem of detecting collinear points can be converted to the problem of finding concurrent curves.

A dual property of the point to curve transformation can also be established. Furthermore, points corresponding to second order curves in two and three dimensions can be found in a similar manner. The theory and applications of higher order and higher dimensional Hough transforms will be reported elsewhere.

In the execution of Hough transforms to find the sides of a closed figure, it is necessary to quantize θ with a tolerance $\pm \Delta\theta$ so that successive points within $\pm \Delta\theta$ can be grouped to form the side. This is so because a real figure may be associated with noise and other degradation effects, and also because a continuous value of θ cannot be accommodated in a digital computer.

Although the Hough transform is computationally efficient in finding the sides, cornerity information is lost in the process. For the description of sidedness and cornerity, a fuzzy set theoretic approach, described in § 4, may be used.

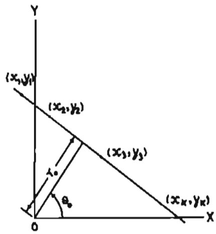

Figure 1. Principle of Hough transform.

3. Recognition of symmetry

A symmetric outline with its axis of symmetry is shown in Fig. 2. An observation on this ideal figure shows that the axis (*a*) divides the enclosed area and (*b*) the perimeter in equal parts in which (*c*) the curvatures at equidistant points along the curve are equal. The sign convention assumed here is to assign positive curvature if the incremental angle is positive in the anti-clockwise sense. If a figure is antisymmetric, the property (*c*) is modified with a negative sign at one side of the equality.

A careful study shows that one of the properties (*a*) and (*b*) is redundant if the other and (*c*) is satisfied. Since property (*c*) is most important, a discussion about the curvature with the sign convention is given in Fig. 2.

3.1. *Curvature analysis*

The curvature at any point is the rate of change, if it exists, of angular direction of the curve with respect to the length of the arc at the point, i.e.

$$C \triangleq \operatorname*{Lt}_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} \tag{3}$$

where ϕ and s denote, respectively, the angle and arc length.

Given the set of n points

$$\{x_i, y_i\}_{i=1}^{n} \quad \text{where } x_{n+i} = x_i \text{ and } y_{n+i} = y_i$$

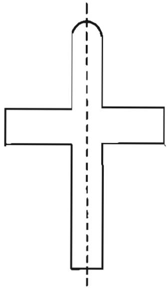

Figure 2. A symmetric figure (- - - - axis of symmetry).

$y_i - y_{i-1}$	$x_i - x_{i-1}$	$y_{i+1} - y_i$	$x_{i+1} - x_i$	Curvature
+	+	+	+	$(\theta_2 - \theta_1)/\Delta s$
			−	$[\pi - (\theta_1 + \theta_2)]/\Delta s$
		−	−	$(\theta_1 - \theta_2)[\pi - (\theta_2 - \theta_1)]/[(\theta_2 - \theta_1)\Delta s]$
			+	$-(\theta_1 + \theta_2)/\Delta s$
	−	−	+	$(\theta_2 - \theta_1)[\pi - (\theta_2 - \theta_1)]/[(\theta_2 - \theta_1)\Delta s]$
			−	$(\theta_1 + \theta_2)/\Delta s$
		+	−	$(\theta_1 - \theta_2)/\Delta s$
			+	$(\theta_1 + \theta_2 - \pi)/\Delta s$
−	+	+	+	$(\theta_1 + \theta_2)/\Delta s$
			−	$(\theta_2 - \theta_1)[\pi - (\theta_2 - \theta_1)]/[(\theta_2 - \theta_1)\Delta s]$
		−	−	$(\theta_1 + \theta_2 - \pi)/\Delta s$
			+	$(\theta_1 - \theta_2)/\Delta s$
	−	−	+	$(\pi - \theta_1 - \theta_2)/\Delta s$
			−	$(\theta_2 - \theta_1)/\Delta s$
		+	−	$(\theta_1 + \theta_2)/\Delta s$
			+	$(\theta_1 - \theta_2)[\pi - \theta_2 - \theta_1]/[(\theta_2 - \theta_1)\Delta s]$

Table 1. Rules to generate local curvature of a closed outline.

Let

$$[(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2]^{1/2} = \Delta s/2 \tag{4}$$

where Δs is constant and

$$\left.\begin{aligned} \theta_1 &= \text{abs}\,[\tan^{-1}(y_i - y_{i-1})/(x_i - x_{i-1})] \\ \theta_2 &= \text{abs}\,[\tan^{-1}(y_{i+1} - y_i)/(x_{i+1} - x_i)] \end{aligned}\right\} \tag{5}$$

Then the sixteen rules given in Table 1 are sufficient to find the curvature and its sign convention uniquely. Sixteen different rules are necessary since the tringonometric functions are doubly degenerate in cartesian quadrants and vector information is necessary to get the sign convention. If however, angular increment per unit arc length at a point is known, the sign can be inserted from the respective co-ordinate transformation and rotation as follows

$$\left.\begin{aligned} x' &= x - x_{i-1} \\ y' &= y - y_{i-1} \end{aligned}\right\} \tag{7}$$

and

$$\left.\begin{aligned} X' &= x' \cos\theta - y' \sin\theta \\ Y' &= x' \sin\theta + y' \cos\theta \end{aligned}\right\} \tag{8}$$

where θ is the angle between x-axis and the line joining the $(i-1)$ and ith points. Because of two possible values of θ, we choose one for which x'_i is positive. If the corresponding y'_i is positive then the sense of curvature is also positive.

3.2. *Determination of lateral symmetry*

Consider the closed outline $\{x_i, y_i\}_{i=1}^n$ with curvatures $\{C_k\}_{i=1}^n$. If n is even, take correlation between pairs of points i, $n+2-i$, for $1 < i < n/2 - 1$

$$R_{j,n/2+j} = \sum_{i=2}^{n/2} C_i C_{n+2-i} \tag{9}$$

This is the correlation measure about an axis passing through points $[x_j, y_j]$ and $[x_{n/2+j}, y_{n/2+j}]$. We wish to find the axis about which the correlation is maximum. To do this, the axis is rotated keeping the perimeter on either side of the axis equal and the correlation is taken at each step of $j = j+1$. Formally, to choose the best j, the sequence i of points is re-ordered such that for each j,

$$i = i - j - 1 \quad \text{if } i > j \quad \text{and} \quad i = n + i - j \quad \text{if } i < j \tag{10}$$

and the values of $R_{j,n/2+j}$ for all $1 < j < n/2$ are calculated. The best axes of symmetry pass through the points $[x_j, y_j]$ and $[x_{n/2+j}, y_{n/2+j}]$, corresponding to $[R_{j,n/2+j}]_{\max}$.

There may be more than one value of j for which R is maximum. Such curves have more than one axis of symmetry.

If n is odd, one point interpolated midway between $[x_{(n+1)/2}, y_{(n+1)/2}]$ and $[x_{(n-1)/2}, y_{(n-1)/2}]$ at every step of j may be designated as $n/2$. After reordering the remaining $i > (n+1)/2$ points as $i = i+1$, the programme may be executed as above. The interpolation is necessary to provide an even distribution of points about the axis so that correlation everywhere is defined.

For lateral antisymmetry, the value of j for which $R_{j,n/2}$ is a minimum is chosen. The axis of antisymmetry pass through the corresponding pair of points $[x_i, y_i]$ and $[x_{n/2+j}, y_{n/2+j}]$.

It is interesting to note that figures having many axes of symmetry are circular in nature. For example, an ideal circle has an infinite number of axes of symmetry or antisymmetry.

4. Fuzzy description of outlines

When the outline is described in terms of straight lines, i.e. sides and corners, the description should be hierarchical in nature. At the finest level of the hierarchy, two neighbouring points of the outline define a side and two neighbouring sides define a corner. But, as seen from Fig. 3, a compact, yet

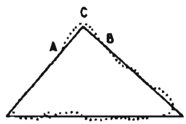

Figure 3. A man-made triangle and its Hough transform approximation.

meaningful description of this figure is to call it ' nearly ' triangular, where the level of hierarchy is coarse, and irregular curves have been approximated by straight lines. The straight line approximation has been described earlier by Hough transform. To translate the linguistic hedge ' nearly ' it is necessary to know the grade of membership of the figure as a triangle. This grade of membership measures the degree to which the figure may be assigned as triangle which, again, depends on the grades of linearity of the sides and the sharpness of the corners. The mathematical framework that appears suitable to characterize this membership function is the fuzzy set theory which defines :

A fuzzy set A in space of points $X = \{x\}$ is a class of events with a continuum grade of membership and is characterized by a membership function $\mu_A(x)$ which associates with each point in X a real number in the interval (0, 1) with the value of $\mu_A(x)$ at x representing the grade of membership of x in A. Formally, a fuzzy set A with its finite number of supports $x_1, x_2, \ldots, x_n$ is defined as

$$A = \{\mu_A(x_i), x_i\} \tag{11}$$

As discussed above, the set of closed outlines is a fuzzy set with respect to properties like sidedness, cornerity or symmetry. The membership of a side may be given as

$$\mu_{s1} = \left(1 - \frac{\theta}{\pi F_{d\,s1}}\right)^{F_{e\,s1}} \tag{12}$$

where θ is the tolerance accounted for by grouping the set of successive points by Hough transform, F_e and F_d are the respective exponential and denominational fuzzifier (Pal and Dutta Majumder 1978) where $F_d > 1$, $F_e > 0$. These quantities have the effect of changing the membership values and allow flexibility in describing the set in linguistic hedges. For example, change of F_e from 1 to 2 changes the membership of 'straight side' to that of 'very straight side'.

As an illustrative example, the linear approximation of a closed outline obtained through Hough transform is shown in Fig. 3. A quantization of 20° with a tolerance of $\pm 5°$ has been used here. Also, the straight lines corresponding to the longest sequence of successive points have been drawn. The membership of the sides of the figure, with $F_e = 1$ and $F_d = 1{\cdot}2$ is 0·78.

As stated earlier, Hough transform does not account for the cornerity, i.e. the sharpness of a corner. Consider the corner region AB in Fig. 3. We define membership of the corner as

$$\mu_c = \left(1 - \frac{\min |\text{Curve length AB}, \overline{AC} + \overline{BC}|}{F_{d\,c} \max |\text{Curve length AB}, \overline{AC} + \overline{BC}|}\right)^{F_{e\,c}} \tag{13}$$

where $\overline{AC}$ means linear distance between A and C.

Similarly, the membership of symmetry of a figure is defined as

$$\mu_{sy} = \left(1 - \frac{4 \max\limits_{j} R_{j,n/2+j}}{F_{d\,sy} \sum\limits_{i=1}^{n} C_i^2}\right)^{F_{e\,sy}} \tag{14}$$

To find the grade of the n-sided symmetric figure, it is convenient to use the fuzzy max–min rule of inference, given by

$$\mu_{fig} = \max \{ \min_i \mu_{sl}(i), \ \min_i \mu_c(i), \mu_{sy} \} \tag{15}$$

where $i = 1, 2, 3, \ldots, n$ and $\mu_x(i)$ denotes the membership of the property x of the ith side or corner.

To describe the quality of the outline, a decision boundary may be fixed for μ_{fig}. For example, Fig. 3 may be called a 'good triangle' if $\mu_{fig} > 0{\cdot}5$ for the figure. Goodness of finer qualities like sides and corners of the figure are reflected in μ_{sl} and μ_c, respectively.

This effect may be used to choose the level of hierarchy in describing a figure, especially in a man–machine interactive device. For a complex figure, however, a fuzzy syntactic approach (Tamura and Tanaka 1973) may be necessary to complete the description.

5. Results and discussions

Both man-made and machine-generated data was used to study the effectiveness of the algorithms described above. For man-made figures, several persons of junior school background and different age were asked to draw some simply closed figures. A few of the figures, as shown in Fig. 4 were chosen to find the sides, corners and symmetry.

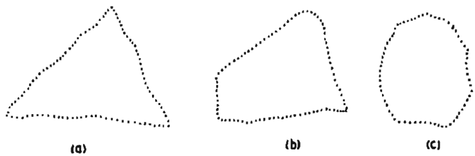

Figure 4. Some man-made simply closed figures.

The machine-generation of the figures is based on the expansion of a periodic real function in polar Fourier series (Zusne 1970, Shepard and Cermak 1972).

$$r(\theta)=\sum_{n=1}^{N}\alpha_n\cos(n\theta+\gamma_n) \qquad (16)$$

To ensure positive values of $r(\theta)$, the above equation is rewritten as

$$r(\theta)=\sum_{n=1}^{N}\alpha_n\exp\cos(n\theta+\gamma_n) \qquad (17)$$

which, for different values of N, α_n and γ_n defines different star-shaped curves

Curves drawn using eqn. (17) are given in Fig. 5 with α_n and γ_n as in Table 2.

n	Fourier coefficients of Fig. 5 (a)		(b)		(c)	
	α_n	γ_n	α_n	γ_n	α_n	γ_n
1	1·0	0·524	1·0	0·00	1·0	0·00
2	1·0	1·047	1·0	1·570	1·0	3·140
3	1·0	1·570	1·0	0·00	1·5	0·314
4	1·0	2·094	1·0	0·00	1·0	1·870
5			1·0	0·0	1·5	0·750
6			1·0	0·314		
7			1·0	0·524		

Table 2. Coefficients for polar Fourier generation of closed figures.

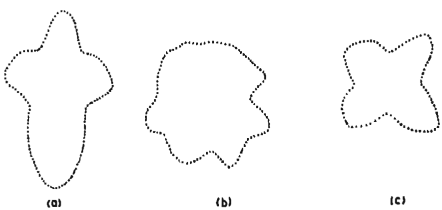

Figure 5. Simply closed figures generated by polar Fourier series approach.

Curves shown in Fig. 4 and Fig. 5 have been used to test the algorithms for the detection of sides, corners and symmetry. The linear approximations using Hough transform are shown in Fig. 6. The transform has been obtained with 10° quantization and ± 2° tolerance and the longest sequence of successive points within the tolerance has been chosen to define each side. It is apparent that the number of sides defining each figure agree with human judgment. It should be noted that, with finer tolerance and hence at a finer level of hierarchy, the number of sides defining the figure will increase.

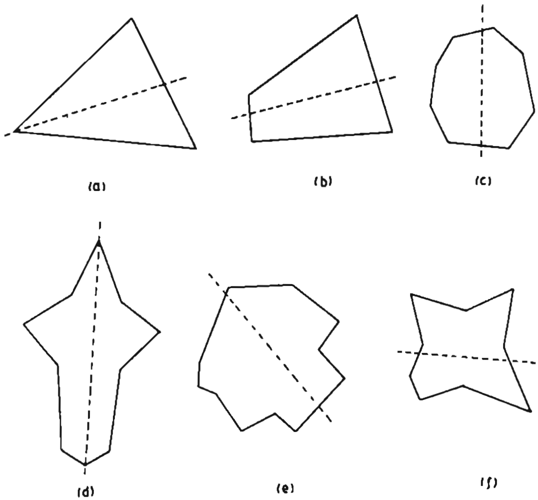

Figure 6. Hough transform approximation and best axes of symmetry of the figures in Fig. 4 and Fig. 5 (- - - - best axis of symmetry).

The result of using the sixteen rules given in Table 1 to find the curvature of the curve of Fig. 4 (*a*) is shown in Fig. 7 where the arrow in Fig. 4 (*a*) shows the starting point. The curvature has been normalized assuming $\Delta s = 1$ and $C_{max} = 2\pi$. The curve clearly shows the points of inflation of the outline by zero cross-overs and sharp corners by peaks. The correlations taken through eqn. (9) are equivalent to folding the curve about an axis dividing the perimeter in equal halves and testing the similarity between them. The best axes for all the outlines of Fig. 4 and Fig. 5 are also shown in Fig. 6.

The membership of sides, corners and symmetry of Fig. 5, obtained using eqns. (12)–(15) are given in Table 3. The hierarchical level is the same as above. Using these membership values and assuming the decision level $\mu_p > 0{\cdot}5$ as the 'goodness' of the property p of the figures, a linguistic description of each figure is also presented in Table 3. Here the hedge 'very' is used when $\mu_p^2 > 0{\cdot}5$, and 'not' means $1 - \mu_p$. Also 'many' is used when the number

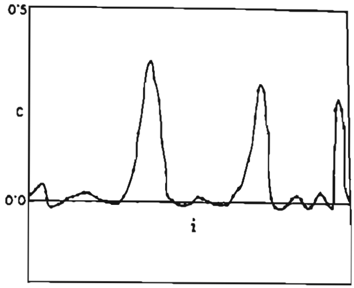

Figure 7. Local curvature of Fig. 4 (*a*) using Table 1.

Figure	$\mu_{si\ min}$	$\mu_{c\ min}$	μ_{sy}	μ_{fig}	Linguistic description
5 (*a*)	0·783	0·657	0·981	0·981	'Very' symmetric, sharp cornered but not 'very' sharp cornered, 'many' sided figure.
5 (*b*)	0·679	0·413	0·560	0·523	Symmetric but not 'very' symmetric, not sharp cornered 'many' sided figure.
5 (*c*)	0·679	0·614	0·471	0·414	Not symmetric, sharp cornered, but not 'very' sharp cornered, many sided figure.

Table 3. Fuzzy membership of properties of Fig. 5 and linguistic description of the figures (with $F_e=1$, $F_d=1{\cdot}2$).

exceeds eight. μ_{fig} reflects the overall performance of the present approximation. It is apparent that the descriptions agree with human judgment within tolerable limit. However, the detailed comparative study of human judgment and fuzzy membership assignment is a subject of further research.

6. Conclusions

The recognition and description of two-dimensional simply closed outlines using Hough transform, curvature correlation and the fuzzy set theoretic approach has been developed and implemented on man-made and machine-generated figures. The description is limited to the simple curve where the corners do not form a blob structure that prevails the shape of the figure over a domain. However, if the level of quantization is fine, linear approximation is a good estimation of the shape inside and outside the blob.

The work may be useful in computer vision and scene analysis and may be extended to the recognition and description of three-dimensional figures provided suitable multiple projections are presented and graphical relationship among the projections is known.

ACKNOWLEDGMENT

Valuable help rendered by Shri S. K. Chakraborty and Mrs. S. De Bhowmick is gratefully acknowledged.

REFERENCES

ALBIN, M., 1975, Ph.D. Thesis, Dept. of Mathematics, University of California, Berkeley.

ATTENAVE, F., 1954, *Psych. Rev.*, **61**, 183.

CHAUDHURI, B. B., and DUTTA MAJUMDER, D., 1979, *Proc. Comp. Soc. India, Bangalore*, December (in the press).

DAVIS, L. A., 1977 a, *I.E.E.E. Trans. Comput.*, **26**, 236; 1977 b, *I.E.E.E. Trans. Syst. Man Cybernet.*, **7**, 204.

FU, K. S., 1974, *Syntactic Methods in Pattern Recognition* (New York: Academic Press).

GONZALEZ, R. C., and WINTZ, P., 1977, *Digital Image Processing* (London: Addison-Wesley).

HOUGH, P. V. C., 1962, U.S. Patent No. 3069654.

JOHNSTON, E., and ROSENFELD, A., 1973, *I.E.E.E. Trans. Comput.*, **22**, 875.

NEGOITA, C. C., 1973, *Inf. Sci.*, **5**, 279.

PAL, S. K., and DUTTA MAJUMDER, D., 1977, *I.E.E.E. Trans. Syst. Man Cybernet.*, **7**, 625; 1978, *Ibid.*, **8**, 302.

PAVLIDIS, T., and HOROWITZ, S., 1973, *Proc. I.J.C.P.R.*, **1**, 396.

ROSENBERG, B., 1972, *Comput. Graphics Image Processing*, **1**, 183.

SHEPARD, R., and CERMAK, G., 1972, *Cog. Psych.*, **4**, 351.

TAMURA, S., and TANAKA, K., 1973, *I.E.E.E. Trans. Syst. Man Cybernet.*, **3**, 98.

ZADEH, L. A., 1963, *Inf. Control*, **8**, 338.

ZUSNE, L., 1970, *Visual Perception of Form* (New York: Academic Press).