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OPTIMAL REPEATED MEASUREMENTS DESIGNS UNDER INTERACTION

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Abstract: The robustness of some optimality results on repeated measurements designs is investigated when the underlying model is allowed to be non-additive incorporating an interaction due to the direct and residual effects of treatments. The procedure involves the checking of some orthogonality conditions and the calculus for factorial arrangements is applied for this purpose. Some new constructions of optimal repeated measurements designs have also been considered.

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1. Introduction and preliminaries

In repeated measurements designs (RMD's) each experimental unit is exposed to a number of treatments applied sequentially over periods. For a general review of such designs, including a discussion on practical applications and a comprehensive bibliography upto that stage, reference is made to Hedayat and Afsarinejad (1975). The pioneering work in the area of optimal RMD's is due to Hedayat and Afsarinejad (1978) and further significant contributions, covering the optimality and constructional aspects, were made by Cheng and Wu (1980), Magda (1980), Constantine and Hedayat (1982) and Kunert (1983, 1984 a, b); for an excellent review of the literature on optimal RMD's see Hedayat (1981). Applying a fundamental tool due to Kiefer (1975), many of these authors considered the problem of universal optimality under fixed effects additive linear models incorporating direct and first order residual effects of treatments apart from effects due to units and periods.

In some situations, however, an interaction due to the direct and first order residual effects of treatments is likely to be present. John and Quenouille (1977, pp. 211–214) present a practical example on grass yield where such interaction turns out to be significant. Interesting results on the problems of construction and analysis under such non-additive models were obtained among others by Patterson (1968, 1970) and Kershner and Federer (1981) (see also the discussion by Federer following Hedayat (1981)). Patterson (1973) considered some orthogonality conditions in this context. The present work primarily investigates how far the optimality results in Cheng and Wu (1980) and Magda (1980) remain robust when the direct versus residual effects interaction is taken into account. Some new constructions of optimal RMD's are also presented.

Following the standard notations and definitions (vide Hedayat and Afsarinejad (1978), Cheng and Wu (1980) and Magda (1980)), an RMD with n experimental units, p periods $0, 1, \ldots, p-1$ and t treatments $0, 1, \ldots, t-1$ will be abbreviated by RMD(t, n, p) and the class of all such designs will be denoted by $\Omega_{t,n,p}$. Let $d(i, j)$ be the treatment assigned by an RMD d in the i-th period to the j-th unit and Y_{ij} the response under $d(i, j)$. The observations are assumed to be homoscedastic and uncorrelated. The underlying model is called circular if in each unit the residuals in the initial period are incurred from the last period. Otherwise, i.e. if there is no residual effect in the first period, the model is called non-circular.

Taking the direct versus residual effect interaction into account, the circular model is given by

$$E(Y_{ij}) = \mu + \alpha_i + \beta_j + \xi_{d(i,j)d(i-1,j)} \quad (0 \le i \le p-1;\ 1 \le j \le n) \tag{1.1}$$

where $i-1$ is reduced mod p, and the unknown constants μ, α_i, β_j represent respectively the general mean, the i-th period effect and the j-th unit effect. Also, the unknown constant $\xi_{h_1h_2}$ $(0 \le h_1, h_2 \le t-1)$ represents the effect produced when the treatment h_1 is applied in the current period with the treatment h_2 being applied in the immediately preceding period.

For the non-circular model, $E(Y_{ij})$ is as in (1.1) for $1 \le i \le p-1$; $1 \le j \le n$, while for $i = 0$,

$$E(Y_{0j}) = \mu + \alpha_i + \beta_j + \tau_{d(0,j)} \quad (1 \le j \le n) \tag{1.2a}$$

where

$$\tau_{h_1} = t^{-1} \sum_{h_2=0}^{t-1} \xi_{h_1h_2} \quad (0 \le h_1 \le t-1). \tag{1.2b}$$

A design will be called uniform if in each period the same number of units is assigned to each treatment and on each unit each treatment appears in the same number of periods. Under the non-circular model an RMD is called strongly balanced if the collection of ordered pairs $\{d(i-1, j), d(i, j)\}$, $1 \le i \le p-1$; $1 \le j \le n$, contains each ordered pair of treatments, distinct or not, the same number, say λ, of times; under the circular model an RMD is called strongly balanced if the same holds considering ordered pairs $\{d(i-1, j), d(i, j)\}$, $0 \le i \le p-1$; $1 \le j \le n$. A strongly balanced uniform RMD(t, n, p) will be abbreviated by SBURMD(t, n, p).

2. Application of the calculus for factorial arrangements

Since this paper takes into account the interaction between the direct and first order residual effects of treatments, it appears convenient to apply the calculus for factorial arrangements, introduced by Kurkjian and Zelen (1962) and strengthened further among others by Zelen and Federer (1964), in the subsequent development. Consider the $t^2=v$ treatment combinations (h_1,h_2), $0\le h_1,h_2\le t-1$, such that the first (second) member of each combination represents the treatment contributing a direct (first order residual) effect to an experimental unit. The direct and first order residual effects of treatments may then be looked upon as the main effects of factors, say, F_1 and F_2 (each at t levels) respectively, while their interaction is given by the interaction F_1F_2.

For any positive integer a, let I_a be the $a\times a$ identity matrix, $\mathbf{1}_a$ be an $a\times 1$ vector with all elements unity and $E_a=\mathbf{1}_a\mathbf{1}_a'$. Define the $v\times 1$ vector

$$\boldsymbol{\xi}=(\xi_{00},\xi_{01},\ldots,\xi_{0t-1},\ldots,\xi_{t-10},\xi_{t-11},\ldots,\xi_{t-1t-1})'.$$

Then by (1.1), (1.2), for a design $d\in\Omega_{t,n,p}$, the coefficient matrix of the reduced normal equations for $\boldsymbol{\xi}$, under both the circular and the non-circular models, is of the form

$$C_d^{(v\times v)}=V_d-n^{-1}N_dN_d'-p^{-1}M_dM_d'+(np)^{-1}(N_d\mathbf{1}_p)(N_d\mathbf{1}_p)', \tag{2.1}$$

where

$$V_d=\sum_{i=0}^{p-1}\sum_{j=0}^{n}\lambda_{ij}\lambda_{ij}', \tag{2.2a}$$

$$M_d^{(v\times n)}=\left(\sum_{i=0}^{p-1}\lambda_{i1},\ldots,\sum_{i=0}^{p-1}\lambda_{in}\right),\qquad N_d^{(v\times p)}=\left(\sum_{j=1}^{n}\lambda_{0j},\ldots,\sum_{j=1}^{n}\lambda_{p-1j}\right); \tag{2.2b}$$

$$\lambda_{ij}=e_{d(i,j)}\otimes e_{d(i-1,j)}\quad(0\le i\le p-1;\ 1\le j\le n) \tag{2.3}$$

for the circular model;

$$\lambda_{ij}=e_{d(i,j)}\otimes e_{d(i-1,j)}\quad(1\le i\le p-1;\ 1\le j\le n)$$

$$\lambda_{0j}=t^{-1}e_{d(0,j)}\otimes\mathbf{1}_t\quad(1\le j\le n) \tag{2.4}$$

for the non-circular model; e_h is a $t\times 1$ vector with 1 in the h-th position and zero elsewhere and $\otimes$ denotes Kronecker product.

Note that typical contrasts belonging to main effect F_1, main effect F_2 and interaction F_1F_2 are respectively of the form $(w_1\otimes\mathbf{1}_t)'\boldsymbol{\xi}$, $(\mathbf{1}_t\otimes w_2)'\boldsymbol{\xi}$, $(w_1\otimes w_2)'\boldsymbol{\xi}$, where w_1,w_2 are any $t\times 1$ non-null vectors satisfying $w_1'\mathbf{1}_t=w_2'\mathbf{1}_t=0$. Defining $Z_1=I_t\otimes E_t$, $Z_2=E_t\otimes I_t$, one has the following lemma from Mukerjee (1980) which will be helpful in the sequel.

Lemma 2.1. *In a design d, the best linear unbiased estimators of contrasts belonging to main effect F_1 (F_2) are orthogonal to those of contrasts belonging to main effect F_2 (F_1) and interaction F_1F_2 if and only if Z_1C_d (Z_2C_d) is symmetric.*

3. Optimality results under a non-circular model

Throughout this section, the underlying model is non-circular. The aim is to examine the robustness of the main results in section 3 of Cheng and Wu (1980) and to develop some further results. Let d^* be an SBURMD(t, n, p). Cheng and Wu (1980) proved the universal optimality of d^* over $\Omega_{t,n,p}$ for the estimation of direct as well as first order residual effects. The next result establishes the robustness of their findings for the direct effects under a non-additive setting.

Theorem 3.1. *Under a non-additive model, d^* is universally optimal over $\Omega_{t,n,p}$ for the estimation of direct effects.*

Proof. In view of Theorem 3.1 of Cheng and Wu (1980), it is enough to show that in d^*, under the non-additive model, contrasts belonging to main effect F_1 are estimable orthogonally to those belonging to main effect F_2 and interaction F_1F_2. Hence by Lemma 2.1, one has to establish that $Z_1C_{d^*}$ is symmetric. Let $1 = 1_t \otimes 1_t$, $E = E_t \otimes E_t$. Then by (2.3), (2.4) and the definition of SBURMD(t, n, p), it follows that for d^*,

$$\sum_{j=1}^{n} \lambda_{0j}\lambda_{0j}' = nt^{-3}(I_t \otimes E_t), \qquad \sum_{i=1}^{p-1}\sum_{j=1}^{n} \lambda_{ij}\lambda_{ij}' = n(p-1)t^{-2}(I_t \otimes I_t), \tag{3.1a}$$

$$\sum_{j=1}^{n} \lambda_{0j} = nt^{-2}1, \qquad \sum_{i=1}^{p-1}\sum_{j=1}^{n} \lambda_{ij} = n(p-1)t^{-2}1, \tag{3.1b}$$

$$Z_1\left(\sum_{j=1}^{n} \lambda_{ij}\right) = nt^{-1}1 \quad (0 \le i \le p-1),$$

$$Z_1\left(\sum_{i=0}^{p-1} \lambda_{ij}\right) = pt^{-1}1 \quad (1 \le j \le n). \tag{3.1c}$$

Hence by (2.2),

$$V_{d^*} = nt^{-3}(I_t \otimes E_t) + n(p-1)t^{-2}(I_t \otimes I_t), \tag{3.2a}$$

$$N_{d^*}1_p = npt^{-2}1, \quad Z_1N_{d^*}N_{d^*}' = n^2pt^{-3}E, \quad Z_1M_{d^*}M_{d^*}' = np^2t^{-3}E. \tag{3.2b}$$

It is now clear from (2.1) that $Z_1C_{d^*}$ is symmetric. □

As in the above theorem, an SBURMD d^* is universally optimal over $\Omega_{t,n,p}$ for the residual effects under the non-additive model provided d^* allows orthogonal estimation of the residual effects contrasts, i.e., by Lemma 2.1, provided $Z_2C_{d^*}$ is symmetric. Unlike in the case of direct effects, however, not all SBURMD's satisfy this criterion as the following example illustrates.

Example 3.1. Consider the designs d_1^*, d_2^*, each an SBURMD(2, 4, 6):

d_1^*: (units across, periods down)

0	0	1	1
0	1	0	1
0	1	1	0
1	1	0	0
1	0	1	0
1	0	0	1

d_2^*: (units across, periods down)

1	0	0	1
0	0	1	1
0	0	1	1
1	1	0	0
0	1	1	0
1	1	0	0

If one computes $C_{d_1^*}$, $C_{d_2^*}$ by (2.1), then an application of Lemma 2.1 shows that while d_2^* allows estimation of the residual effects orthogonally to direct effects and direct versus residual effect interaction, d_1^* does not. In fact, a direct computation shows d_1^* to be inferior to d_2^* in so far as the estimation of the (single) residual effect contrast is concerned.

In view of the above example, the problem of identifying those SBURMD's d^* which allow orthogonal estimation of the residual effects contrasts becomes non-trivial. Essentially, this calls for a combinatorial characterization of the commutativity of Z_2 and C_{d^*}. In general, it appears that such a characterization may become too involved to be helpful in actual construction and hence one has to look for simpler sufficient conditions. In the special case $n=t^2$, $p=2t$, Patterson (1973) considered sufficient conditions in this regard. A more general set of sufficient conditions with a very wide coverage is presented below.

For any $d\in\Omega_{t,n,p}$, let S_{dh} be the set of units receiving the treatment h $(0\le h\le t-1)$ in the last period. Then the following holds.

Theorem 3.2. *Under a non-additive model, an* SBURMD(t,n,p) d^* *allows orthogonal estimation of the residual effects contrasts and hence becomes universally optimal over* $\Omega_{t,n,p}$ *for the residual effects if* (i) *for each* h,h' $(0\le h,h'\le t-1)$, *there are exactly* nt^{-2} *units receiving the treatments* h *and* h' *in the initial and the last periods respectively and* (ii) *for each* h $(0\le h\le t-1)$, *in the collection of ordered pairs* $\{d^*(i-1,j), d^*(i,j)\}$, $1\le i\le p-1$; $j\in S_{d^*h}$ *each ordered pair* (h,h_2) $(0\le h_2\le t-1)$ *occurs the same number (say* v_1) *of times while each ordered pair* (h_1,h_2) $(0\le h_1,h_2\le t-1$; $h_1\ne h)$ *occurs the same number (say* v_2) *of times.*

If d^* satisfies the condition (ii) above then by recalling the definition of an SBURMD, one may count in two ways the number of times each treatment appears in S_{d^*h} to get $v_1=n(p-t)t^{-3}$, $v_2=npt^{-3}$. Theorem 3.2, which has been proved in the Appendix, is seen to cover almost all situations where an SBURMD may exist. Note that an SBURMD(t, n, p) exists only if $t^2|n$ and $t|p$ (with $p>t$ for obvious reasons). Now if $t^2|n$ and pt^{-1} is even, then it may be checked that the SBURMD's constructed through Theorem 3.2 of Cheng and Wu (1980) satisfy the conditions of Theorem 3.2 of this paper and are hence universally optimal for the residual effects under the non-additive model. It may be remarked that in particular if $n=t^2$ and $p=2t$ then this finding also follows from the sufficient conditions in Patterson (1973).

Turning to the situation where $t^2|n$ and pt^{-1} is odd, let $pt^{-1}=2m+1$ $(m\geq 1)$ and consider the following method of construction which is successful for $t\neq 6$. First let $t\neq 2,6$. Then a pair of mutually orthogonal latin squares, Q_1 and Q_2 with entries $0,1,\ldots,t-1$, of order t exists. Let q_{uh} be the h-th column of Q_u, g_h be a $t\times 1$ vector with all elements equal to h, $G_h=(q_{1h}, q_{2h}, g_h)$ $(0\leq h\leq t-1;\ u=1,2)$ and $G=(G_0, G_1,\ldots,G_{t-1})$. If $t=2$, let

$$G=\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

For $t\neq 6$, define

$$B_0=\begin{bmatrix} 0 & 1 & \cdots & t-1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}'$$

and for $1\leq h\leq t-1$, let B_h be obtained by adding h (mod t) to each element of B_0. Let $B=(B_0, B_1,\ldots,B_{t-1})$ and define the $t\times p$ array $A_0=(G, B,\ldots, B)$, where the array B is repeated $m-1$ times. Let A_h be obtained by adding h (mod t) to each element of A_0. Then the $p\times t^2$ array $A=(A_0', A_1',\ldots,A_{t-1}')$, with columns and rows identified with units and periods respectively, is seen to be an SBURMD(t, t^2, p) satisfying the conditions of Theorem 3.2. An SBURMD(t, n, p) satisfying the same conditions is obtained considering nt^{-2} copies of A. Evidently, such a design is universally optimal over $\Omega_{t,n,p}$ for the residual effects under the non-additive model.

In the above, which is essentially a method of differences, the choice of G for $t=2$ has been made by trial and error and the design d_2^* in Example 3.1 serves as an illustration. Yet another example is presented below.

Example 3.2. Let $t=3$, $n=9$, $p=15$. One may take

$$Q_1=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix},\qquad Q_2=\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix},$$

$$B=\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 1 & 0 & 2 \\ 2 & 0 & 0 & 1 & 1 & 2 \end{bmatrix},$$
$$\underbrace{}_{B_0}\quad \underbrace{}_{B_0+1}\quad \underbrace{}_{B_0+2}$$

$$G=\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 0 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 2 \end{bmatrix}.$$

Then $A_0=(G,B)$. Form the arrays A_1, A_2 by adding 1 and 2 (mod 3) respectively to each element of A_0. Now the 15×9 array $A=(A_0', A_1', A_2')$ gives an SBURMD(3, 9, 15) satisfying the conditions of Theorem 3.2.

By Theorems 3.1, 3.2 and the discussion above, under a non-circular non-additive model an SBURMD(t, n, p) which is universally optimal over $\Omega_{t,n,p}$ for both the direct and the residual effects exists whenever $t^2|n$, $t|p$ $(p>t)$ except when $t=6$ and p is an odd multiple of 6. Derivation of such an optimal design for $t=6$ and p an odd multiple of 6 is left as an open problem.

Remark. If, however, one ignores the conditions of Theorem 3.2 then as indicated below an SBURMD(t, n, p) exists even when $t=6$ and p is an odd multiple of 6 provided $t^2|n$. Let $p=(2m+1)6$ $(m\geq 1)$. Define the 36×1 vector

$$\delta=(0,1,2,3,4,5,1,2,3,4,5,0,\ldots,5,0,1,2,3,4)'$$

and the 36×2 matrix

$$\Delta=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 5 & 5 & 5 & 5 & 5 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots & 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}'.$$

Construct the $36\times(2m+1)$ array $L_0=(\Delta, \Delta, \ldots, \Delta, \delta)$, where the array Δ is repeated m times. Let L_h be obtained by adding h (mod 6) to each element of L_0 $(0\leq h\leq 5)$. Then the $p\times 36$ array $L=(L_0, L_1, \ldots, L_5)'$, with columns and rows interpreted as before, gives an SBURMD(6, 36, $(2m+1)6$). An SBURMD(6, n, $(2m+1)6$) is obtained considering $n/36$ copies of L. Although the design does not satisfy the conditions of Theorem 3.2 (and hence nothing can be said about its optimality for the residual effects under a non-additive model), the findings in this remark are important since they, together with the preceding discussion and Theorem 3.2 of Cheng and Wu (1980), establish that the conditions $t^2|n$, $t|p$ $(p>t)$ are not only necessary but also sufficient for the existence of an SBURMD(t, n, p).

Before concluding this section, the robustness of another result in Cheng and Wu (1980) will be examined. Let d_0 be a strongly balanced RMD(t, n, p) which is uniform on the periods and is uniform on the units in the first $p-1$ periods. In their Theorem 3.3, Cheng and Wu (1980) show that such a design is universally optimal over $\Omega_{t,n,p}$ for both the direct and the first-order residual effects under an additive model. When the model is non-additive it may be seen along the line of Theorem 3.1 that their result for the residual effects remains robust. Simple examples may, however, be cited to demonstrate the non-robustness of the corresponding result for the direct effects.

4. Optimality results and constructions under a circular model

Throughout this section the underlying model is circular and let $\bar{d}$ be an SBURMD(t, n, p) under such a model. Under an additive set-up, Magda (1980, Theorem 3.1) proved the universal optimality of $\bar{d}$ over $\Omega_{t,n,p}$ for both direct and residual effects. The next result proves the robustness of his findings under a non-additive model.

Theorem 4.1. *Under a non-additive model, $\bar{d}$ is universally optimal over $\Omega_{t,n,p}$ for the estimation of direct as well as residual effects.*

The proof is similar to that of Theorem 3.1 and may be worked out by checking that $Z_1 C_{\bar{d}}$ and $Z_2 C_{\bar{d}}$ are both symmetric.

Turning to the problem of construction, note that for the existence of an SBURMD(t, n, p) in a circular setting, it is necessary that $t|n$, $t|p$ $(p>t)$.

Theorem 4.2. *Under the circular model, if $t|n$ and pt^{-1} is an even integer then an SBURMD(t, n, p) exists.*

Proof. First let t be even and define the $2t\times 1$ vector

$$\phi_0=(0, t-1, 1, t-2, \ldots, t-1, 0)'.$$

Observe that each of $0, 1, \ldots, t-1$ occurs twice in ϕ_0 and also among the differences $\{f_1-f_0, f_2-f_1, \ldots, f_{2t-1}-f_{2t-2}, f_0-f_{2t-1}\}$ (mod t), f_u being the u-th element of ϕ_0 $(0\le u\le 2t-1)$. Hence if ϕ_h be obtained by adding h (mod t) to each element of ϕ_0 $(1\le h\le t-1)$, then the $2t\times t$ array $[\phi_0, \phi_1, \ldots, \phi_{t-1}]$, with columns and rows identified with units and periods respectively, gives an SBURMD$(t, t, 2t)$. An SBURMD(t, n, p) is obtained taking nt^{-1} and $\frac{1}{2}pt^{-1}$ copies of this $2t\times t$ array along the directions of the units and periods respectively. The proof for odd t follows in a similar manner starting from the $2t\times 1$ vector

$$(0, 1, t-1, 2, t-2, \ldots, t-2, 2, t-1, 1, 0)'$$

instead of ϕ_0. □

Example 4.1. The designs $\bar{d}_1, \bar{d}_2$ in this example, constructed by the above method, represent an SBURMD(4, 4, 8) and an SBURMD(5, 5, 10) respectively.

$\bar{d}_1$:

0	1	2	3
3	0	1	2
1	2	3	0
2	3	0	1
2	3	0	1
1	2	3	0
3	0	1	2
0	1	2	3

$\bar{d}_2$:

0	1	2	3	4
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1
3	4	0	1	2
3	4	0	1	2
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
0	1	2	3	4

It has recently been shown by Roy (1985) that when $t|n$ and pt^{-1} is an odd integer, an SBURMD(t, n, p) exists provided $t=0, 1$ or 3 (mod 4); however, such a design may be non-existent if $t=2$ (mod 4), e.g., as a complete enumeration reveals, an SBURMD(2, 2, 6) is non-existent.

5. Concluding remarks

Restricting to a subclass of $\Omega_{t,n,p}$ consisting of the equireplicate designs, Cheng and Wu (1980, Theorems 3.4, 3.5) and Magda (1980, Theorem 3.2) proved some further optimality results on SBURMD's in terms of minimization of the variance of the best linear unbiased estimator of every contrast belonging to the direct or the residual effects. These results remain robust under the non-additive model whenever the relevant orthogonality properties, as in Sections 3 and 4, hold.

Hedayat and Afsarinejad (1978), Cheng and Wu (1980) and Magda (1980) also derived universal optimality results on uniform RMD's which are balanced in the sense that each treatment never precedes itself but precedes each other treatment the same number of times. With notations as in Section 2 under a non-additive model this means that the treatment combinations (h, h) $(0 \le h \le t-1)$ never appear in such a design so that not all contrasts belonging to direct or residual effects remain estimable. Therefore, the optimality results on balanced uniform RMD's become non-robust.

As a final remark, in the present work the underlying model was non-additive but the emphasis was on the optimal estimation of the direct or residual effects, i.e. the main effects, contrasts. If interest lies also in the optimal estimation of the interaction contrasts then some other, possibly larger, designs should be tried. It is intended to take up this problem in future.

Appendix. Proof of Theorem 3.2

By (2.2), (3.1), (3.2) and the definition of an SBURMD, it is not difficult to see that for every SBURMD d^* the matrix Z_2 commutes with V_{d^*}, $N_{d^*}N'_{d^*}$ and $(N_{d^*}1_p)(N_{d^*}1_p)'$. Hence by (2.4), Lemma 2.1 and the discussion preceding Theorem 3.2, it remains to show that $Z_2M_{d^*}M'_{d^*}$ is symmetric when d^* satisfies the conditions of Theorem 3.2. From (2.2) note that

$$M_{d^*}M'_{d^*} = \sum_{j=1}^{n}\left(\sum_{i=0}^{p-1}\lambda_{ij}\right)\left(\sum_{i=0}^{p-1}\lambda_{ij}\right)'. \tag{A.1}$$

As d^* is an SBURMD(t, n, p), by (2.3), (2.4), for each j $(1 \le j \le n)$,

$$Z_2\left(\sum_{i=0}^{p-1}\lambda_{ij}\right) = 1_t \otimes \left[t^{-1}1_t + \sum_{i=1}^{p-1} e_{d^*(i-1,j)}\right]$$

$$=1_t\otimes\{(p+1)t^{-1}1_t-e_{d^*(p-1,j)}\}.$$

This together with (A.1) yields

$$Z_2M_{d^*}M'_{d^*}=(p+1)t^{-1}(1_t\otimes 1_t)\left(\sum_{j=1}^{n}\sum_{i=0}^{p-1}\lambda_{ij}\right)' - \sum_{j=1}^{n}(1_t\otimes e_{d^*(p-1,j)})\left(\sum_{i=0}^{p-1}\lambda_{ij}\right)'.$$

While by (3.1) the first term in $Z_2M_{d^*}M'_{d^*}$ is symmetric, by (2.3), (2.4), the second term equals

$$\begin{aligned}&t^{-1}\sum_{j=1}^{n}(1_te'_{d^*(0,j)})\otimes(e_{d^*(p-1,j)}1'_t)\\&\quad+\sum_{j=1}^{n}(1_t\otimes e_{d^*(p-1,j)})\left(\sum_{i=1}^{p-1}e'_{d^*(i,j)}\otimes e'_{d^*(i-1,j)}\right)\\&=nt^{-3}(E_t\otimes E_t)+\sum_{h=0}^{t-1}(1_t\otimes e_h)\left(\sum_{j\in S_{d^*h}}\sum_{i=1}^{p-1}e'_{d^*(i,j)}\otimes e'_{d^*(i-1,j)}\right)\\&=nt^{-3}(E_t\otimes E_t)+\sum_{h=0}^{t-1}(1_t\otimes e_h)[1'_t\otimes\{v_21'_t-(v_2-v_1)e'_h\}]\\&=(nt^{-3}+v_2)(E_t\otimes E_t)-(v_2-v_1)(E_t\otimes I_t),\end{aligned}\tag{A.2}$$

applying the conditions (i) and (ii). Since (A.2) is symmetric the result follows.

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References

Cheng, C.S. and C.F. Wu (1980). Balanced repeated measurements designs. *Ann. Statist.* 8, 1272–1283.

Constantine, G. and A. Hedayat (1982). A construction of repeated measurements designs with balance for residual effects. *J. Statist. Plann. Inference* 6, 153–164.

John, J.A. and M.H. Quenouille (1977). *Experiments: Design and Analysis*, 2nd ed. Charles Griffin, London.

Hedayat, A. (1981). Repeated measurements designs, IV: Recent advances (with discussion). *Bull. Int. Statist. Inst.* 49 Book 1, 591–610.

Hedayat, A. and K. Afsarinejad (1975). Repeated measurements designs, I. In: J.N. Srivastava, Ed., *A Survey of Statistical Designs and Linear Models.* North-Holland, Amsterdam, 229–242.

Hedayat, A. and K. Afsarinejad (1978). Repeated measurements designs, II. *Ann. Statist.* 6, 619–628.

Kershner, R.P. and W.T. Federer (1981). Two treatment cross over designs for estimating a variety of effects. *J. Amer. Statist. Assoc.* 76, 612–619.

Kiefer, J. (1975). Construction and optimality of generalized Youden designs. In: J.N. Srivastava, Ed., *A Survey of Statistical Designs and Linear Models*. North-Holland, Amsterdam, 333–353.

Kunert, J. (1983). Optimal design and refinement of the linear model with applications to repeated measurements designs. *Ann. Statist.* 11, 247–257.

Kunert, J. (1984a). Optimality of balanced uniform repeated measurements designs. *Ann. Statist.* 12, 1006–1017.

Kunert, J. (1984b). Designs balanced for circular residual effects. *Comm. Statist. Theory Meth.* 13, 2665–2671.

Kurkjian, B. and M. Zelen (1962). A calculus for factorial arrangements. *Ann. Math. Statist.* 33, 600–619.

Magda, C. (1980). Circular balanced repeated measurements designs. *Comm. Statist. Theory Meth. A* 9, 1901–1918.

Mukerjee, R. (1980). Further results on the analysis of factorial experiments. *Calcutta Statist. Assoc. Bull.* 29, 1–26.

Patterson, H.D. (1968). Serial factorial design. *Biometrika* 55, 67–81.

Patterson, H.D. (1970). Non-additivity in change-over designs for a quantitative factor at four levels. *Biometrika* 57, 537–549.

Patterson, H.D. (1973). Quenouille's change over designs. *Biometrika* 60, 33–46.

Roy, B.K. (1985). Construction of strongly balanced uniform repeated measurements designs. Indian Statist. Inst. Tech. Rep. ASC/85/10.

Zelen, M. and W.T. Federer (1964). Applications of the calculus for factorial arrangements, II. Designs with two-way elimination of heterogeneity. *Ann. Math. Statist.* 35, 658–672.