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BOUNDS ON THE NUMBER OF CONSTRAINTS FOR BALANCED ARRAYS OF STRENGTH t

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Abstract: We describe a method, based on non-negative definiteness of moment matrices, for deriving upper bounds on the number of constraints in balanced arrays of strength t, involving two or more symbols. It is seen that the method covers, in particular, those due to Rafter and Seiden (1974) and Chopra (1982, 1983).

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1. Introduction

Balanced arrays (B-arrays) are useful to various combinatorial areas of design of experiments. In particular, in recent years extensive work has been done, mainly by the schools of J. Srivastava and S. Yamamoto, on B-arrays – see e.g., Chakravarti (1956, 1961), Chopra (1975a,b, 1982, 1983), Chopra and Srivastava (1973a,b, 1974, 1975), Kuwada (1979, 1980, 1981, 1982), Kuwada and Nishii (1979), Longyear (1984), Rafter (1971), Rafter and Seiden (1974), Shirakura (1976), Shirakura and Kuwada (1975), Srivastava (1970, 1972), Srivastava and Chopra (1971a,b,c, 1973, 1974), Srivastava and Ghosh (1977), Srivastava and Wijetunga (1981), Yamamoto, Shirakura and Kuwada (1975, 1976), Yamamoto, Kuwada and Yuan (1985). For an excellent review, we refer to Srivastava (1978).

For ease of reference, we recall the definition of a B-array. A B-array of strength t with two symbols, m constraints, N runs and index set $\{\mu_0, \mu_1, \ldots, \mu_t\}$ is an $m \times N$ matrix B whose elements are the two symbols (0 and 1, say) such that in every $t \times N$ submatrix B_0 of B, every t-vector (i.e., a vector with t elements) α of weight i ($i = 0, 1, \ldots, t$; the weight of α is the number of 1's in it) appears as a column of B_0 exactly μ_i times. The constants m, N, t and μ_i ($i = 0, 1, \ldots, t$) are called the parameters of the B-array, and we denote it by BA$[m, N, 2, t; \mu_0, \mu_1, \ldots, \mu_t]$. Note that $N = \sum_{j=0}^{t} \binom{t}{j} \mu_j$. The definition of a B-array with s (≥ 2) symbols is presented later in Section 3.

Unfortunately, B-arrays may not exist for an arbitrary set of parameter values. To construct such arrays with the maximum possible number of constraints is an important problem both in statistical design of experiments and combinatorial mathematics (for example, when a B-array is interpreted as a fractional factorial plan, with its rows identified with the factors, an investigation of the maximum possible number of constraints may lead to a solution to the relevant packing problem). This problem for some B-arrays with two symbols has been considered by Rafter and Seiden (1974) and Chopra (1982, 1983). In this paper, we demonstrate the application of non-negative definiteness (n.n.d.-ness) of moment matrices in the derivation of bounds for the number of constraints.

Essentially, for fixed values of the other parameters we derive here upper bounds for m, the number of constraints. This means that if the value of m exceeds the upper bound then the corresponding B-array is non-existent. In this sense, the results presented in this paper may as well be interpreted as those on non-existence. It may be remarked that possibly the most important single tool, available in the literature, for checking such non-existence is based on the theory of Diophantine equations. The pioneering work involving the application of Diophantine equations for examining the existence of B-arrays is due to Srivastava (1972) and for further references one may see Longyear (1984). In fact, the methods developed in this paper are also based on Diophantine equations (for example, the equation (2.1) presented later for the two-symbol situation is just another version of the equation (2.4) in Srivastava and Chopra (1973)). The main new point in this paper is, however, that while the previous authors making explicit use of Diophantine equations employ discrete and number-theoretic arguments for checking the existence of solutions to such equations, we apply moment inequalities, based on the n.n.d.-ness of certain matrices, for the same purpose. The moment method, in a sense, supplements the discrete analysis, and a combination of the two may become a very powerful tool through a fuller utilization of the technique of Diophantine equations. An example in this connection has been presented later in Section 2. Moreover, it appears that despite its theoretical elegance, the discrete analysis may become somewhat involved when B-arrays with more than two symbols are considered. In such situations one may first apply the method of moments (see Section 3) to attain a considerable reduction of the problem and then the discrete analysis may be employed to achieve possible further improvements.

2. Bounds in the two-symbol case

Let B be a $\mathrm{BA}[m,N,2,t;\mu_0,\mu_1,\ldots,\mu_t]$, for $t\geq 2$. Let k_i be the number of unities in the i-th column or run for $i=1,2,\ldots,N$. Since a B-array of strength t is also a B-array of strength less than t, it is obvious (cf. Chopra (1983)) that

$$\sum_{i=1}^{N}\binom{k_i}{j}=\binom{m}{j}\sum_{l=0}^{t-j}\binom{t-j}{l}\mu_{l+j},\quad j=1,2,\ldots,t, \tag{2.1}$$

whence it is possible to express $\sum_{i=1}^{N} k_i^j$ $(j=1,2,\ldots,t)$ as linear combinations of $\mu_0,\mu_1,\ldots,\mu_t$.

For given $\mu_0,\mu_1,\ldots,\mu_t$, it may be checked whether the expressions for $\sum_{i=1}^{N} k_i^j$ $(j=1,2,\ldots,t)$ satisfy the different well-known moment inequalities or not and non-existence results may possibly be obtained. In particular, one may note that for every positive integer v, the matrices

$$W_{2v}=\begin{bmatrix} N & \sum k_i & \cdots & \sum k_i^{v} \\ \sum k_i & \sum k_i^2 & \cdots & \sum k_i^{v+1} \\ \vdots & \vdots & & \vdots \\ \sum k_i^{v} & \sum k_i^{v+1} & \cdots & \sum k_i^{2v} \end{bmatrix}$$

and

$$W_{2v+1}=\begin{bmatrix} \sum k_i & \sum k_i^2 & \cdots & \sum k_i^{v+1} \\ \sum k_i^2 & \sum k_i^3 & \cdots & \sum k_i^{v+2} \\ \vdots & \vdots & & \vdots \\ \sum k_i^{v+1} & \sum k_i^{v+2} & \cdots & \sum k_i^{2v+1} \end{bmatrix}$$

where the summations extend over $i=1,2,\ldots,N$, are n.n.d. This follows by observing the n.n.d.-ness of the quadratic forms

$$\prod_{i=1}^{N}(\xi_0+\xi_1k_i+\cdots+\xi_vk_i^{v})^2 \quad\text{and}\quad \sum_{i=1}^{N}(\tau_0k_i^{1/2}+\tau_1k_i^{3/2}+\cdots+\tau_vk_i^{(2v+1)/2})^2,$$

in variables $\xi_0,\xi_1,\ldots,\xi_v$ and $\tau_0,\tau_1,\ldots,\tau_v$ respectively. For even t $(=2v)$, the inequalities arising from the n.n.d.-ness of W_{2v} and W_{2v-1} may be employed to derive upper bounds for m or to prove non-existence results. For odd t $(=2v+1)$, the inequalities arising from the n.n.d.-ness of W_{2v+1} and W_{2v} may be employed for the same purpose. Theorems 3.2 in Rafter and Seiden (1974), 2.1 and 2.2 in Chopra (1982) or 2.1 in Chopra (1983), concerning upper bounds for m, follow from such n.n.d.-ness. It may be remarked that whereas these authors make elegant applications of particular inequalities implicit in the n.n.d.-ness of the moment matrix we propose to make an explicit use of such n.n.d.-ness in its entirety.

A comparison between the results obtained from the n.n.d.-ness of moment matrices as indicated above and those obtained from similar n.n.d.-ness of information matrices arising out of balanced arrays (see e.g., Srivastava and Chopra (1971b), Yamamoto, Shirakura and Kuwada (1976), Kuwada (1981), Yamamoto, Kuwada and Yuan (1985)) shows that none of these methods is uniformly superior to the other. The results based on the information matrix approach, as reported in the literature, mostly relate to arrays of even strength. Therefore, in applying them to arrays of odd strength ($t=2v+1$) one has to interpret the array as one of strength $2v$ and hence considerable information is lost. On the other hand, for $t=2v+1$, the inequalities arising from the n.n.d.-ness of W_{2v+1} make an explicit use of the fact that the array is of strength $2v+1$. Hence for odd t, the moment method appears to be superior to the information matrix approach. An examination of the simple case $t=3$ provides plenty of examples in support of this observation.

On the other hand, for even t $(=2v)$, it may be seen that there exists a matrix L_v such that $W_{2v}=L_v \mathscr{I} L_v'$, where $\mathscr{I}$ is the information matrix arising from the B-array interpreting the array as a resolution $(2v+1)$ plan. Hence the inequalities obtained from the n.n.d.-ness of $\mathscr{I}$ can never be weaker than those obtained from W_{2v}. Although it is hard to prove the same theoretically for the inequalities obtained from W_{2v-1}, numerical studies make us believe that for even t, the moment method, based on both W_{2v} and W_{2v-1}, will be inferior to the information matrix approach. Anyway, for even t (≥ 6) and general s, results based on the information matrix are not yet available in the literature and are hard to obtain. Hence in such situations concerning even t, the moment method, because of its considerable simplicity, may have some merit at least as an ad hoc procedure.

The following example illustrates how the application of n.n.d.-ness may improve upon the previously obtained results based on simpler moment inequalities and also how, combined with a discrete analysis, it may lead to an almost saturated exploitation of the technique of Diophantine equations.

Example. Consider the problem posed in Srivastava and Chopra (1973; Theorem 7.3) regarding the existence of a B-array with $t=4$, $m=8$, $\mu_2=5$, $N=66$. From their Theorem 7.3, it follows that for the existence of such an array it is necessary that $\mu_1+\mu_3=8$, $\mu_0+\mu_4=4$. This, together with (2.1), yields

$$\sum k_i = 8(23+2\mu_3+\mu_4), \qquad \sum k_i^2 = 8(58+16\mu_3+8\mu_4),$$
$$\sum k_i^3 = 8(128+86\mu_3+64\mu_4), \qquad \sum k_i^4 = 8(268+352\mu_3+512\mu_4).$$

Interchanging the roles of 0's and 1's, it is clear that if a $\mathrm{BA}[m,N,2,4;\mu_i=y_i$ $(i=0,1,\ldots,4)]$ is non-existent, then a $\mathrm{BA}[m,N,2,4;\mu_i=y_{4-i}$ $(i=0,1,\ldots,4)]$ is also non-existent. This observation, together with the n.n.d.-ness of W_3 and W_4 implies that all choices of μ_3 and μ_4, except possibly $\mu_3=3$, $\mu_4=1$; $\mu_3=5$, $\mu_4=3$; $\mu_3=4$, $\mu_4=2$, are impossible. A reduction to this extent cannot be achieved through simpler moment inequalities; for example, the inequality $(\sum k_i)(\sum k_i^3)\geq(\sum k_i^2)^2$, employed by Chopra (1983), cannot eliminate the case $\mu_3=3$, $\mu_4=1$, whereas a consideration of n.n.d.-ness is successful in doing so. Hence if $t=4$, $m=8$, $N=66$,

$\mu_2=5$, then it is necessary that the other index parameters must be of one of the following forms: (i) $\mu_0=3$, $\mu_1=5$, $\mu_3=3$, $\mu_4=1$; (ii) $\mu_0=1$, $\mu_1=3$, $\mu_3=5$, $\mu_4=3$; (iii) $\mu_0=2$, $\mu_1=\mu_3=4$, $\mu_4=2$. It may be checked that inequalities based on the information matrix approach (see Theorem 3.2 in Srivastava and Chopra (1971b)) fail to eliminate any of these three cases. Furthermore, it may be readily verified that in each of these three situations the fundamental Diophantine equations (see (2.4) in Srivastava and Chopra (1973)) admit integral-valued solutions implying thereby that the technique of single Diophantine equations cannot be utilized in achieving a further reduction of the problem. Thus the present approach, coupled with the discrete analysis, leads to a quick and complete exploitation of single Diophantine equations for the problem under consideration.

3. Bounds in the s-symbol case

In the s-symbol case, the algebraic expressions become involved and, therefore, it becomes cumbersome to express the bounds in a compact form. Even if the algebraic expressions for the bounds are given, such expressions are likely to become difficult to comprehend. Hence, in the s-symbol case, instead of deriving algebraic expressions for bounds, we suggest some methods using which the bounds may be calculated in any particular situation. These methods are obtained by generalizing the procedure mentioned in Section 2. It may be noted that the multi-symbol situation has been effectively taken into account through the consideration of essentially multivariate inequalities like those obtained from n.n.d.-ness of a dispersion matrix.

A B-array of strength t with s symbols $\{0,1,\dots,s-1\}$, m constraints, N runs and index set $\{\mu_{i_1\cdots i_{s-1}}:(i_1,\dots,i_{s-1})\in T^{(t)}\}$, where

$$T^{(t)}=\{(i_1,\dots,i_{s-1}):0\le i_1,\dots,i_{s-1}\le t;\, i_1+\cdots+i_{s-1}\le t\},$$

is an $m\times N$ matrix B whose elements are 0, 1, ..., or $s-1$ such that in every $t\times N$ submatrix B_0 of B every t-vector with i_1 1's, i_2 2's, ..., i_{s-1} $(s-1)$'s appears as a column of B_0 $\mu_{i_1i_2\cdots i_{s-1}}$ times for $(i_1,\dots,i_{s-1})\in T^{(t)}$. [This definition slightly differs from the standard one, but is equivalent to the standard one and consistent with that in Section 2.] Such a B-array is denoted by BA$[m,N,s,t;\mu_{i_1\cdots i_{s-1}},(i_1,\dots,i_{s-1})\in T^{(t)}]$. Clearly

$$N=\sum_{(i_1,\dots,i_{s-1})\in T^{(t)}}\frac{t!}{i_1!\,i_2!\cdots i_{s-1}!\,(t-i_1-\cdots-i_{s-1})!}\mu_{i_1\cdots i_{s-1}}.$$

Let B be a B-array as defined above. A B-array of strength t is known to be also a B-array of strength w $(<t)$, i.e., for $w<t$, B is a BA$[m,N,s,w;\mu^{(w)}_{i_1\cdots i_{s-1}},(i_1,\dots,i_{s-1})\in T^{(w)}]$, where

$$\mu^{(w)}_{i_1\cdots i_{s-1}}=\sum_{(\alpha_1,\dots,\alpha_{s-1})\in T^{(t-w)}}\frac{(t-w)!}{\alpha_1!\cdots\alpha_{s-1}!\,(t-w-\alpha_1-\cdots-\alpha_{s-1})!}\times\mu_{i_1+\alpha_1,\dots,i_{s-1}+\alpha_{s-1}}. \quad (3.1)$$

In particular, the special cases $w=1,2$ (for $t\ge 2$), $w=1,2,3$ (for $t\ge 3$) will be helpful for our purpose.

For $u=1,2,\dots,N$; $j=1,2,\dots,s-1$, let k_{ju} denote the number of times the symbol j appears in the u-th column of B. Then for each fixed u, $k_{ju}\ge 0$, $1\le j\le s-1$; $\sum_{j=1}^{s-1} k_{ju}\le m$. Let for $1\le j\le s-1$; $r=1,2,3$, e_{jr} be an $(s-1)$-component row vector with r at the j-th position and 0 at every other position and for $1\le j<l\le s-1$, f_{jl} be an $(s-1)$-component row vector with 1 at the j-th and l-th positions and 0 at every other position. Then the following are easily verified (for $t\ge 3$):

$$\sum_{u=1}^{N}\binom{k_{ju}}{r}=\binom{m}{r}\mu^{(r)}_{e_{jr}},\quad 1\le j\le s-1;\ r=1,2,3,$$
$$\sum_{u=1}^{N}k_{ju}k_{lu}=m(m-1)\,\mu^{(2)}_{f_{jl}},\quad 1\le j<l\le s-1, \tag{3.2}$$

where, by (3.1), the right-hand sides of the expressions in (3.2) can be expressed as linear combinations of $\mu_{i_1\cdots i_{s-1}}$'s. [For $t=2$, the second relation in (3.2) holds and the first relation holds for $r=1,2$.]

At this stage it is possible to start from known inequalities in descriptive statistics connecting moments (both univariate and multivariate) of different orders and employ (3.1) and (3.2) for getting bounds on m for given $m,N,s,t,\mu_{i_1\cdots i_{s-1}}((i_1,\dots,i_{s-1})\in T^{(t)})$. In fact, scores of results will follow and here we present only a few of them.

Let $t\ge 2$. The first relation in (3.2) with $r=1,2$ gives

$$\sum_{u=1}^{N}k_{ju}=m\mu^{(1)}_{e_{j1}},\qquad \sum_{u=1}^{N}k_{ju}(k_{ju}-1)=m(m-1)\,\mu^{(2)}_{e_{j2}},$$
$$\sum_{u=1}^{N}k_{ju}^2=m(m-1)\,\mu^{(2)}_{e_{j2}}+m\mu^{(1)}_{e_{j1}}, \tag{3.3}$$
$$\phi_{jj}=m(m-1)\,\mu^{(2)}_{e_{j2}}+m\mu^{(1)}_{e_{j1}}-[m\mu^{(1)}_{e_{j1}}]^2/N,\quad 1\le j\le s-1,$$

where

$$\phi_{jl}=\sum_{u=1}^{N}(k_{ju}-\bar{k}_j)(k_{lu}-\bar{k}_l),\quad 1\le j\le l\le s-1,$$

$$\bar{k}_j=N^{-1}\sum_{u=1}^{N}k_{ju},\quad 1\le j\le s-1.$$

Similarly, from the second relation in (3.2),

$$\phi_{jl}=m(m-1)\,\mu^{(2)}_{f_{jl}}-[(m\mu^{(1)}_{e_{j1}})(m\mu^{(1)}_{e_{l1}})/N],\quad 1\le j<l\le s-1. \tag{3.4}$$

Defining, for $j<l$, $\phi_{lj}=\phi_{jl}$ and $\Phi((s-1)\times(s-1))=((\phi_{jl}))$, clearly, Φ is non-negative definite (n.n.d.) and hence all its principal minors are non-negative. By (3.1), (3.3) and (3.4), this observation leads to a large number of inequalities involving $m,s,t,\mu_{i_1\cdots i_{s-1}}((i_1,\dots,i_{s-1})\in T^{(t)})$, and for given s, t and $\mu_{i_1\cdots i_{s-1}}$'s, these inequalities may be utilised in obtaining feasible ranges or bounds for m. In particular, the following inequalities may be considered:

$$\phi_{jj} \geq 0, \text{ i.e., } \sum_{u=1}^{N} k_{ju}^2 \geq \left(\sum_{u=1}^{N} k_{ju}\right)^2 \Big/ N, \quad 1 \leq j \leq s-1,$$
$$\phi_{jj}\phi_{ll} \geq (\phi_{jl})^2, \quad 1 \leq j < l \leq s-1. \tag{3.5}$$

These ideas will later be explained with examples.

For $t \geq 3$, further results will follow considering the third order moments. Then the first relation in (3.2) with $r=3$ gives

$$\sum_{u=1}^{N} k_{ju}(k_{ju}-1)(k_{ju}-2) = m(m-1)(m-2)\mu_{e_{j3}}^{(3)}, \quad 1 \leq j \leq s-1,$$

which, together with (3.3), yields

$$\sum_{u=1}^{N} k_{ju}^3 = m(m-1)(m-2)\mu_{e_{j3}}^{(3)} + 3m(m-1)\mu_{e_{j2}}^{(2)} + m\mu_{e_{j1}}^{(1)}. \tag{3.6}$$

Since

$$\sum_{u=1}^{N} k_{ju}^2 \sum_{u=1}^{N} k_{ju} \geq \left(\sum_{u=1}^{N} k_{ju}^2\right)^2, \tag{3.7}$$

applying (3.1), (3.3) and (3.6), one may again proceed to find bounds on m for given s, t, $\mu_{i_1 \cdots i_{s-1}}$'s.

In fact, for values of t greater than 3 one may consider inequalities of the type (3.7) involving still higher order moments or even Liapounoff type inequalities combining all possible inequalities of the form (3.7). Furthermore, inequalities involving higher order multivariate moments may as well be considered. Basically, each inequality in descriptive statistics involving univariate or multivariate moments up to order t has the potential of dictating a feasible range for m, and the intersection of all such feasible ranges is likely to yield sharp bounds for m. Although keeping notations general and proceeding algebraically the task seems formidable, in any given context, for numerically specified s, t, $\mu_{i_1 \cdots i_{s-1}}$'s, it is usually a routine matter to follow the above procedure and that is possibly the most important thing for all practical purposes. Actually, often good bounds for m may be obtained from inequalities involving only the relatively lower order moments.

Example. Let $t=2$. Consider a BA$[m, N, s, 2; \mu_{00\cdots0}, \mu_{e_{jr}}$ $(1 \leq j \leq s-1;\ r=1,2)$, $\mu_{f_{jl}}$ $(1 \leq j < l \leq s-1)]$. Trivially, $\mu_{i_1 \cdots i_{s-1}}^{(2)} = \mu_{i_1 \cdots i_{s-1}}$ and clearly

$$N = \mu_{00\cdots0} + \sum_{j=1}^{s-1} \mu_{e_{j2}} + 2\left[\sum_{j=1}^{s-1} \mu_{e_{j1}} + \sum_{j<l=1}^{s-1}\sum \mu_{f_{jl}}\right],$$
$$\mu_{00\cdots0}^{(1)} = \mu_{00\cdots0} + \sum_{j=1}^{s-1} \mu_{e_{j1}}, \tag{3.8}$$
$$\mu_{e_{j1}}^{(1)} = \mu_{e_{j1}} + \mu_{e_{j2}} + \sum_{l=j+1}^{s-1} \mu_{f_{jl}} + \sum_{u=1}^{j-1} \mu_{f_{uj}}, \quad 1 \leq j \leq s-1.$$

Since, by (3.3) and (3.4),

$$\phi_{jj} = m(m-1)\mu_{e_{j2}} + m\mu^{(1)}_{e_{j1}} - [m\mu^{(1)}_{e_{j1}}]^2/N, \quad 1\le j\le s-1,$$
$$\phi_{jl} = m(m-1)\mu_{f_{jl}} - [(m\mu^{(1)}_{e_{j1}})(m\mu^{(1)}_{e_{l1}})]/N, \quad 1\le j<l\le s-1, \tag{3.9}$$

inequalities arising from non-negative definiteness of $\Phi = ((\phi_{jl}))$ (or in particular, those in (3.5)) may, together with (3.8), be employed to obtain possible bounds on m for given s, $\mu_{i_1\cdots i_{s-1}}$'s. In particular let $\mu_{00\cdots 0} = \mu_{e_{j2}} = \mu$ $(1\le j\le s-1)$ and $\mu_{e_{jl}} = \mu_{f_{jl}} = \lambda$ $(1\le j\le s-1,\ 1\le j<l\le s-1)$. Then by (3.8), $N = s\mu + s(s-1)\lambda$, $\mu^{(1)}_{e_{j1}} = (s-1)\lambda + \mu$ $(1\le j\le s-1)$, and hence

$$\phi_{jj} = m(s-1)[\lambda - m(\lambda-\mu)/s], \quad 1\le j\le s-1,$$
$$\phi_{jl} = -m[\lambda - m(\lambda-\mu)/s], \qquad 1\le j<l\le s-1.$$

Hence

$$\Phi = ((\phi_{jl})) = m[\lambda - m(\lambda-\mu)/s]\{sI_{s-1} - J_{s-1}\},$$

where I_{s-1} is the identity matrix of order $s-1$ and J_{s-1} is a square matrix of order $s-1$ having all elements unity. Since Φ is n.n.d., it follows that $\lambda - m(\lambda-\mu)/s \ge 0$, and hence

$$m \le \lambda s/(\lambda-\mu), \quad \text{provided } \lambda>\mu. \tag{3.10}$$

In particular, if $\mu = 0$, (3.10) yields $m\le s$, which is attainable if s is a prime or a prime power (cf. Nair and Rao (1948)).

Example. In the above example, the second inequality in (3.5) was not used explicitly. The present example makes an explicit use of this inequality. Consider a BA$[m, N, 3, 2;\ \mu_{00}=2,\ \mu_{20}=\mu_{02}=3,\ \mu_{10}=\mu_{01}=2,\ \mu_{11}=4]$. Then $N=24$. By (3.8), (3.9),

$$\phi_{11} = \phi_{22} = 3m(m-1) + 9m - (9m)^2/24 = m(6-3m/8),$$
$$\phi_{12} = 4m(m-1) - (9m)^2/24 = m(5m/8-4).$$

Hence the first inequality in (3.5) gives $m\le 16$, while the second inequality in (3.5) gives, on simplification, $(2+m/4)(10-m)\ge 0$, i.e., $m\le 10$. Hence one gets $m\le 10$ ultimately.

In fact, it is interesting to see that this bound is sharper than those one can obtain through the existing results in the two-symbol case through suitable collapsing of symbols. Thus if one merges the symbols 1 and 2 (to get a symbol, say 1), then for the resulting two-symbol balanced array of strength two, $\mu_0=2$, $\mu_1=4$, $\mu_2=14$, so that $\mu_1^2<\mu_0\mu_2$ and Theorem 3.2 of Rafter and Seiden (1974) cannot be applied. Again if one merges the symbols 0 and 2 (to get a symbol, say 0), then for the resulting two-symbol BA, $\mu_0=9$, $\mu_1=6$, $\mu_2=3$. Hence $\mu_1^2-\mu_0\mu_2>0$ and an application of Theorem 3.2 of Rafter and Seiden yields $m\le \mu_1(\mu_0+2\mu_1+\mu_2)/(\mu_1^2-\mu_0\mu_2)=16$. Similarly, merging the symbols 0 and 1 of the original three-symbol BA yields

$m \leq 16$. Thus, application of the existing results for two-symbol arrays, through a collapsing of symbols, yields $m \leq 16$, which is weaker than the bound obtained by our methods. This is expected since our methods are essentially based on multivariate inequalities which are stronger than the univariate ones.

Similarly for $t \geq 3$, examples may be constructed to illustrate applications of inequalities involving higher order moments.

4. Concluding remarks

It is well-known that the incidence matrix of a $t-(v,k,\lambda_t)$ design is a BA$[v,b,2,t;\mu_0,\mu_1,\ldots,\mu_t]$ with $\mu_j = \sum_{i=0}^{t-j}(-1)^i\binom{t-j}{i}\lambda_{j+i}$ for $j=0,1,\ldots,t$. It may be easily verified that for a B-array derivable from a t-design as above, equality holds in many of the inequalities [in particular, in those line $\sum k_i^2 \geq (\sum k_i)^2/N$ and $(\sum k_i^3)(\sum k_i) \geq (\sum k_i^2)^2$ and so on] considered in Section 2. Hence such B-arrays attain the relevant upper bounds on the number of constraints. For further details, we refer to Saha, Mukerjee and Kageyama (1983).

In the case of s-symbol B-arrays, $s>2$, an analogous criterion for equality could be: $\Phi=0$, a null matrix. One can easily check that the s-symbol B-arrays having $\Phi=0$ are the incidence matrices of some proper and equireplicated block designs that can be considered as s-ary versions of t-designs.

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