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CODES ASSOCIA[redacted]H GENERALIZED POLYGONS

1. Introduction

Throughout, X denotes a *finite* (s,t)-generalized $2m$-gon (P, L), $s, t, m \geq 2$, i.e. a finite linear incidence system such that (i) each element of P, called *points* (respectively each element of L, called *lines*) is incident with exactly $t + 1$ lines (respectively, $s + 1$ points); and (ii) the associated bipartite graph on $P \cup L$ has diameter $2m$ and girth $4m$ ([2, p. 233]). We denote by d the distance on this bipartite graph. By [5], a generalized $2m$-gon with $s, t \geq 2$ exists only for $m = 2, 3, 4$. For more on generalized polygons, see [7], [8], [9] and [10].

Let F be a field. FP denotes the vector space of F-valued functions on P with the inner product defined by $f \cdot g = \Sigma_{x \in P} f(x)g(x)$, $f, g \in FP$. $I_A \in FP$ denotes the indicator function of a subset A of P. $C_F = C_F(X)$ and $\Pi_F = \Pi_F(X)$ denote the vector subspaces of FP generated by $\{I_l : l \in L\}$ and $\{\pi_x = \Sigma\{I_l : x \in l \in L\} : x \in P\}$, respectively. For any subspace M of FP, $M^\perp$ denotes the dual (=orthocomplement) of M in FP with respect to the inner product defined above. We denote the dual incidence system of X by $^*X$ (note that $^*X$ is a finite (t,s)-generalized $2m$-gon) and denote $C_F(^*X)$ and $\Pi_F(^*X)$ by $^*C_F$ and $^*\Pi_F$, respectively. $N = N_{|P| \times |L|}$ denotes the $(0, 1)$-incidence matrix of X with rows and columns indexed by points and lines respectively. Thus C_F, $^*C_F$, Π_F and $^*\Pi_F$ are the column spans over F of N, N', NN' and $N'N$, respectively.

In Section 2, we obtain bounds for the minimum weight of C_F and $C_F^\perp$ for any field F, and, under the assumption that X is regular (see 2.1 and 2.2 below), we describe the words of least weight in C_F and $C_F^\perp$ (Theorem 2.8). It is interesting to note that when X is regular, the supports of minimum-weight words in both C_F and $C_F^\perp$ are independent of the field F. In Section 3, for all fields F except those whose characteristic divides an explicitly given function of the parameters m, s, t, (i) we show that $\dim_F(C_F) = \mathrm{Rank}_Q(N)$ (Theorem 3.6), (ii) determine $C_F \cap C_F^\perp$ (Theorem 3.8) and (iii) show that the minimum-weight words of $C_F^\perp$ generate $C_F^\perp$ for regular X (Theorem 3.10). Though our methods and results are rather elementary, our principal object in this note is to isolate the values of the characteristic of F for which the determination of the dimension and structure of C_F is (perhaps) nontrivial. A beginning has been made in [1, Theorem 4] on one of these nontrivial cases.

2. Minimum-weight words of C_F and $C_F^\perp$

2.1 DEFINITIONS. A subset T of P is a $(1, t)$-subpolygon of X if the incidence system $(T, \{l \cap T : l \in L, |l \cap T| > 1\})$ is a $(1, t)$ generalized $2m$-gon. X is said to

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be *regular* if each pair x, y of points of X with $d(x, y) = 2m$ is contained in a (necessarily unique) $(1, t)$-subpolygon of X.

2.2 EXAMPLES. Among the known generalized polygons, the regular ones are: (i) the (q, q)-generalized 4-gon $W(q) \cong {}^*Q(4, q)$ for q a prime power ([7, pp. 43 and 51]); (ii) the (q^2, q)-generalized 4-gon $H(3, q^2) \cong {}^*Q(5, q)$ with q a prime power ([7, pp. 46 and 51]); (iii) the 'usual' (q, q)-generalized 6-gon associated with the simple group $G_2(q)$ ([8, (2.12), p. 233]), q a prime power; (iv) the (q^3, q) generalized 6-gon associated with the simple group ${}^3D_4(q)$ ([8, (2.12), p. 233]), q a prime power; and (v) the (q^2, q)-generalized 8-gon associated with the simple group ${}^2F_4(q)$, q an odd power of 2 (its regularity follows from the commutation relations in [10] and the transitivity of ${}^2F_4(q)$ on pairs of points at distance 8).

LEMMA 2.3. *Suppose X is regular. Let $x, y \in P$ with $d(x, y) = 2i$ $(0 \leqslant i \leqslant m)$ and $T \subseteq P$ with $|T| = s + 1$. Then,*

(a) *there are exactly s^{m-i} $(1, t)$-subpolygons of X containing both x and y; and*
(b) *T is a line if and only if each pair of distinct points of T is contained in s^{m-1} $(1, t)$-subpolygons of X.*

Proof. Routine.

LEMMA 2.4. *Let $\emptyset \neq S \subseteq P$ be such that no line of X meets S in exactly one point. Then $|S| \geqslant 2(t^m - 1)(t - 1)^{-1}$, and equality holds if and only if S is a $(1, t)$-subpolygon of X.*

Proof. Fix $a \in S$ and define $A_{-1} = \emptyset, A_0 = \{a\}$. For $1 \leqslant p \leqslant m - 1$, construct $A_p \subseteq S$ by choosing exactly one point from each line l such that l is incident with a point in A_{p-1} but not incident with any point in A_{p-2}. Clearly $|A_p| = (t + 1)t^{p-1}$ for $1 \leqslant p \leqslant m - 1$. Now, each of the $(t + 1)t^{m-1}$ lines l – such that l is incident with a point in A_{m-1} but not incident with any point in A_{m-2} – meets $S \setminus \bigcup_{i=0}^{m-1} A_i$, but not necessarily at distinct points. Since at most $t + 1$ of these lines are incident with a point, we have

$$|S| \geqslant \sum_{i=0}^{m-1} |A_i| + t^{m-1} = 2 \cdot (t^m - l)(t - 1)^{-1}$$

and equality holds iff S is a $(1, t)$-subpolygon.

LEMMA 2.5. *Suppose X is regular. Let $\emptyset \neq A \subseteq P$ be such that no $(1, t)$-subpolygon of X meets A in exactly one point. Then $|A| \geqslant s + 1$ and equality holds if and only if A is a line.*

Proof. We fix $x \in A$ and use Lemma 2.3(a) to estimate

$$\alpha = |\{(y, \delta): y \in A, y \neq x, \{x, y\} \subseteq \delta \text{ and } \delta \text{ is a } (1, t)\text{-subpolygon}\}|$$

in two ways to get $s^m \leqslant \alpha \leqslant (|A|-1)s^{m-1}$, whence $|A| \geqslant s+1$. By Lemma 2.3(b), equality holds iff A is a line.

The following Lemma appears to be well known among experts (see, for example, [8, p. 241] for the case $m = 3$).

LEMMA 2.6. *Let T be a $(1,t)$-subpolygon of X and let A and B be the two equivalence clases in T under the equivalence relation $x \sim y$ if and only if $d(x, y)$ is a multiple of 4 ($x, y \in T$). Then the incidence system (A, B), with collinearity in X as the incidence, is a (t,t)-generalized m-gon. In consequence, $I_A - I_B \in C_F^{\perp}$.*

Proof. Routine.

2.7 NOTATION. We denote the word $I_A - I_B$ of Lemma 2.6 by w_T. Clearly w_T is determined by the $(1,t)$-subpolygon T only up to sign.

THEOREM 2.8. *Let F be any field. Then:*

(a) *the minimum weight of $C_F^{\perp}$ is at least $2(t^m-1)(t-1)^{-1}$ and any word of $C_F^{\perp}$ of weight $2(t^m-1)(t-1)^{-1}$ is of the form $\lambda \cdot \omega_T$ for some $0 \neq \lambda \in F$ and some $(1,t)$-subpolygon T of X; in particular, equality holds if X is regular;*

(b) *the minimum weight of C_F is at most $s+1$; and if X is regular then equality holds and any word of C_F of weight $s+1$ is of the form $\lambda \cdot I_l$ for some $0 \neq \lambda \in F$ and some line l of X.*

Proof. Note that if A and S are the supports of a nonzero word of C_F and of $C_F^{\perp}$, respectively, then $|A \cap S| \neq 1$. Hence, by Lemma 2.6, A and S satisfy the hypothesis of Lemma 2.5 and Lemma 2.4, respectively. Hence the result follows from Lemmas 2.4 and 2.5.

3. DIMENSION OF C_F

3.1 NOTATION. For $0 \leqslant i \leqslant m$, A_i denotes the (0, 1)-adjacency matrix of the relation $R_i = \{(x, y) \in P \times P : d(x, y) = 2i\}$. $(P, R_i : 0 \leqslant i \leqslant m)$ is a P-polynomial scheme (Proposition 1.1 in [2, p. 190]). Let V_i be the uniquely determined rational polynomial of degree i such that $V_i(A_1) = A_i$, $0 \leqslant i \leqslant m$. Define $f_m(s,t)$ to be equal to 1 if $m = 1$, $s+t$ if $m = 2$, s^2+st+t^2 if $m = 3$ and $(s+t)(s^2+t^2)$ if $m = 4$. For $2 \leqslant m \leqslant 4$, define $F_m(s,t)$ by

$$F_m(s,t) = |P| \cdot (s+1)^{-1}[1 + st \cdot f_{m-1}(s,t)(f_m(s,t))^{-1}].$$

In the omnibus lemma below, we collect details about the above-mentioned scheme which will be needed for our later arguments.

LEMMA 3.2. (a) *The eigenvalues of A_1 are:*

(i) *$s(t+1)$, $s-1$ and $-t-1$ with the corresponding multiplicities*

1, $st(s+1)(t+1)/(s+t)$ *and* $s^2(st+1)/(s+t)$
if $m = 2$;

(ii) $s(t+1)$, $s-1+(st)^{1/2}$, $s-1-(st)^{1/2}$ *and* $-t-1$ *with the corresponding multiplicities*
1, $st(s+1)(t+1)(st+(st)^{1/2}+1)/2(s+t+(st)^{1/2})$,
$st(s+1)(t+1)(st-(st)^{1/2}+1)/2(s+t-(st)^{1/2})$ *and*
$s^3(s^2t^2+st+1)/(s^2+st+t^2)$
if $m = 3$; *and*

(iii) $s(t+1)$, $s-1$, $s-1+(2st)^{1/2}$, $s-1-(2st)^{1/2}$ *and* $-t-1$ *with the corresponding multiplicities*
1, $st(s+1)(t+1)(s^2t^2+1)/2(s+t)$,
$st(s+1)(t+1)(st+1)(st+1+(2st)^{1/2})/4(s+t+(2st)^{1/2})$,
$st(s+1)(t+1)(st+1)(st+1-(2st)^{1/2})/4(s+t-(2st)^{1/2})$ *and*
$s^4(st+1)(s^2t^2+1)/(s+t)(s^2+t^2)$
if $m = 4$.

(b) $\text{Rank}_Q(N) = F_m(s,t)$.

(c) *Let* $E = \Sigma_{i=0}^{m}(-1)^i s^{m-i} A_i$. *Then* $EN = 0$ *and the eigenvalues of* E *are* 0 *and* $(s+1)f_m(s,t)$ *with the corresponding multiplicities* $F_m(s,t)$ *and* $|P| - F_m(s,t)$.

Proof. (a) The eigenvalues of A_1 with their multiplicities are computed using Theorem 1.3 in [2, p. 197]. Here, the polynomials V_i are given by $V_i = \alpha_{i,m} W_i$ $(0 \leqslant i \leqslant m)$, where $\alpha_{i,m} = 1$ if $i < m$ and $\alpha_{i,m} = (t+1)^{-1}$ if $i = m$; the W_i's are as follow:

$$W_0(Y) = 1, \quad W_1(Y) = Y, \quad W_2(Y) = Y^2 - (s-1)Y - s(t+1),$$
$$W_3(Y) = Y^3 - 2(s-1)Y^2 + (s^2 - 2st - 3s + 1)Y + s(s-1)(t+1),$$
$$W_4(Y) = Y^4 - 3(s-1)Y^3 + (3s^2 - 3st - 7s + 3)Y^2 - (s-1)(s^2 - 4st - 4s + 1)Y + s(t+1)(st - s^2 + 2s - 1).$$

(b) Since $NN' = A_1 + (t+1)I$ and $\text{Rank}_Q(N) = \text{Rank}_Q(NN')$, (b) is immediate from (a).

(c) The verification that $EN = 0$ is routine. Put $G = \Sigma_{i=0}^{m}(-1)^i s^{m-i} V_i$. From (a) and the expressions for V_i given above one sees that $G(\lambda) = 0$ for each eigenvalue $\lambda \neq -t-1$ of A_1, and $G(-t-1) = (s+1)f_m(s,t)$. Since $E = G(A_1)$, (c) follows from (a) and (b).

LEMMA 3.3 $\dim(C_F \cap C_F^\perp) = \dim {}^*C_F - \dim {}^*\Pi_F = \text{Rank}_F(N') - \text{Rank}_F(N'N)$, *and dually,* $\dim({}^*C_F \cap {}^*C_F^\perp) = \dim C_F - \dim \Pi_F = \text{Rank}_F(N) - \text{Rank}_F(NN')$.

Proof. Let τ be any anti-isomorphism from X to $^*X$. Let $t: FP \to {}^*C_F$ be the linear map defined by $t(I_{\{x\}}) = I_{\tau(x)}$, $x \in P$. Clearly the kernel of t is $C_F^\perp$ and $t(C_F) = {}^*\Pi_F$. Hence we have

$$\dim(C_F \cap C_F^\perp) = \dim C_F - \dim {}^*\Pi_F = \dim {}^*C_F - \dim {}^*\Pi_F.$$

LEMMA 3.4. *For $l \in L$ and i odd ($1 \leqslant i \leqslant 2m$), let $\Delta_i(l) = \{x \in P: d(x, l) = i\}$. Then $I_{\Delta_i}(l) \in C_F$. In consequence, $I_P \in C_F$.*

Proof. Since $I_P = \Sigma\{I_{\Delta_i}(l): 1 \leqslant i \leqslant 2m, i \text{ odd}\}$ for any fixed $l \in L$, the second assertion follows from the first. The first follows by induction on i since $\Delta_1(l) = l$ and

$$(t+1)\sum_{j=1}^{k} I_{\Delta_{2j-1}}(l) + I_{\Delta_{2k+1}}(l) = \sum\{I_e: e \in L \text{ and } d(e, l) \leqslant 2k\} \in C_F,$$

for $0 < k < m$ and $l \in L$.

PROPOSITION 3.5. *Let A be a square matrix of order v with integer entries such that all the eigenvalues of A are integers. Let p be a prime. Assume that either* (a) *or* (b) *stated below holds:*

(a) *p is strictly larger than the number of distinct eigenvalues of A which are multiples of p.*
(b) *A is symmetric with constant row sum k; p divides k and p does not divide v; further, p equals the number of distinct eigenvalues of A which are multiples of p.*

Then the p-rank of A is greater than or equal to the sum of the multiplicities of those eigenvalues of A which are not multiples of p. In consequence, if none of the nonzero eigenvalues of A is a multiple of p then the p-rank of A equals the Q-rank of A.

Proof. Let λ_i, $1 \leqslant i \leqslant r$, be the distinct eigenvalues of A with the corresponding multiplicities $\mu_i (i \leqslant i \leqslant r)$. Let us say $p | \lambda_i$ for $1 \leqslant i \leqslant q$ and $p \nmid \lambda_i$ for $q + 1 \leqslant i \leqslant r$. Put $\mu = \Sigma_{i=1}^{r} \mu_i$.

If (a) holds, then $p > q$ and so we can choose an integer n such that $n \neq \lambda_i/p \pmod p$ for $1 \leqslant i \leqslant q$. Put $B = A - npI$. Then $p^{\mu+1} \nmid \det B$. Hence by the Smith normal form argument (see [6, p. 57] for example), we get $\operatorname{rank}_p(A) = \operatorname{rank}_p(B) \geqslant v - \mu$.

If (b) holds, then A commutes with the all-one matrix J, and k is one of the eigenvalues $\lambda_i (1 \leqslant i \leqslant q)$, say $k = \lambda_1$, corresponding to the all-one eigenvector. Since $q = p$ and $p \nmid v$, we can choose integers n' and n such that $\lambda_1/p + n'v = \lambda_2/p \pmod p$ and $n \neq \lambda_i/p \pmod p$ for $2 \leqslant i \leqslant q$. Put $B = A + n'pJ - npI$. Then again $p^{\mu+1} \nmid \det B$, and hence the result follows as before. If p does not divide any of the nonzero eigenvalues of A, then (a) holds and hence

$\text{Rank}_p(A) \geq \text{Rank}_Q(A)$. Since $\text{Rank}_p(A) \leq \text{Rank}_Q(A)$ for any integer matrix A, the last statement follows.

THEOREM 3.6. *Let F be a field of characteristic p. Then* $\dim_F(C_F) \leq F_m(s,t)$ *and equality holds if p does not divide $f_m(s,t)$.*

Proof. If $p = 0$, then this is Lemma 3.2(b). So let p be a prime. Since $\dim_F(C_F) = \text{Rank}_p(N) \leq \text{Rank}_Q(N)$, in view of Lemma 3.2(b), we need only prove the statement about equality. So let $p \nmid f_m(s,t)$. Since $NN' = A_1 + (t+1)I$, we know all the eigenvalues of NN' by Lemma 3.2(a), and they are all integers (since the multiplicities in Lemma 3.2(a) are integers, st is a perfect square when $m = 3$ and $2st$ is a perfect square when $m = 4$). $f_m(s,t)$ is the product of the distinct eigenvalues of NN' other than 0 and $(s+1)(t+1)$.

Case 1. $p \nmid (s+1)(t+1)$. Since $p \nmid f_m(s,t)$ by hypothesis, in this case p does not divide any nonzero eigenvalues of NN'. Hence, by Proposition 3.5,

$$\text{Rank}_p(N) \geq \text{Rank}_p(NN') = \text{Rank}_Q(NN') = \text{Rank}_Q(N) = F_m(s,t).$$

Case 2. $p \mid (s+1)(t+1)$. Without loss of generality we can assume that $p \mid t+1$. (Otherwise apply the following argument to ${}^*X$ and note that $\dim C_F = \dim {}^*C_F$.) Hence $I_L \in {}^*C_F^{\perp}$, and so by Lemma 3.4, $I_L \in {}^*C_F \cap {}^*C_F^{\perp}$. Hence, by Lemma 3.3,

(1) $\qquad \dim C_F = \text{Rank}_p(N) \geq \text{Rank}_p(NN') + 1.$

Since (a) and (b) of Proposition 3.5 hold for $A = NN'$ when $p \neq 2$ and when $p = 2$, respectively, Proposition 3.5 yields:

(2) $\qquad \text{Rank}_p(NN') \geq F_m(s,t) - 1.$

Combining (1) and (2), we are done.

3.7 *Examples.* By [3, p. 553], [4, p. 398] and [9, p. 309], the inequality in Theorem 3.6 holds with equality for $p = 2$ and $X = W(2)$, $W(3)$, $Q(5,2)$ and the $(2,2)$-generalized 6-gons, although 2 divides $f_2(2,2) = 4$, $f_2(3,3) = f_2(2,4) = 6$ and $f_3(2,2) = 12$. On the other hand, by Theorem 4 in [1], the equality does not hold for $p = 2$ and $X = W(q)$ when $q > 2$ is a power of 2.

THEOREM 3.8. *Let F be a field of characteristic p not dividing $f_m(s,t)$. Then,*

(a) *if p divides $s+1$, $C_F \cap C_F^{\perp} = \langle I_p \rangle$ and $C_F + C_F^{\perp} = \langle I_p \rangle^{\perp}$; and*
(b) *if p does not divide $s+1$, $C_F \cap C_F^{\perp} = \langle 0 \rangle$ and $C_F \oplus C_F^{\perp} = FP$.*

Proof. (a) By the argument in the proof of Case 2 of Theorem 3.6, $p \mid s+1$ implies $I_p \in C_F \cap C_F^{\perp}$. On the other hand, Lemma 3.3 and (2) in the proof of Theorem 3.6 (applied to ${}^*X$) yield:

$$\dim(C_F \cap C_F^{\perp}) = \text{Rank}_p(N') - \text{Rank}_p(NN') \leq 1.$$

Hence (a) follows.

(b) Let $p \nmid s+1$. Let $C_F(E)$ be the column span over F of the matrix E in Lemma 3.2(c). Since $EN = 0$, $C_F(E) \subseteq C_F^\perp$. By Lemma 3.2(c), Proposition 3.5 and Theorem 3.6, we get $E^2 = \alpha E$, where $\alpha = (s+1)f_m(s,t)$, and $\dim C_F(E) = \text{Rank}_Q(E) = \dim C_F^\perp$. Hence $C_F(E) = C_F^\perp$. Hence $x \in C_F^\perp \Rightarrow x = Ey$ for some $y \in FP \Rightarrow Ex = E^2 y = \alpha Ey = \alpha x$. On the other hand, since $EN = 0$, $x \in C_F \Rightarrow Ex = 0$. So $x \in C_F \cap C_F^\perp$ implies $\alpha x = 0$. Since $\alpha \neq 0$ in F, it follows that $C_F \cap C_F^\perp = \langle 0 \rangle$.

3.9 REMARKS. The proof of Theorem 3.8(b) shows that when char(F) does not divide $(s+1)f_m(s,t)$, the columns of the matrix E form a set of generators of $C_F^\perp$.

THEOREM 3.10. *Let X be regular and suppose the characteristic p of the field F does not divide $(s+1)f_m(s,t)$. Then $C_F^\perp$ is generated by the set of minimum weight words in $C_F^\perp$.*

Proof. Let A_F be the subspace of FP generated by the set $\{w_T : T$ is a $(1,t)$-subpolygon of $X\}$. In view of Theorem 2.8(a), we have to show that $A_F = C_F^\perp$. Clearly, $A_F \subseteq C_F^\perp$. So it suffices to show, in view of the remarks in 3.9 above, that $\dim A_F \geqslant \text{Rank}_p(E)$.

Let M be the matrix whose rows are indexed by the points of X, whose columns are indexed by the $(1,t)$-subpolygons of X, and whose (x, T)th entry is $w_T(x)$. Thus A_F is the column space of M, and so $\dim(A_F) = \text{Rank}_F(M)$. Using Lemma 2.3, it is easy to check that $MM' = E$. Hence, $\dim(A_F) = \text{Rank}_F(M) \geqslant \text{Rank}_F(E)$. So we are done.

REFERENCES

1. Bagchi, B. and Sastry, N. S. N., 'Even Order Inversive Planes, Generalized Quadrangles and Codes', *Geom. Dedicata* **22** (1987), 137–147.
2. Bannai E. and Ito, T., *Algebraic Combinatorics*, I: *Association Schemes*, Benjamin-Cummings, London, Amsterdam, Sydney, Tokyo, 1984.
3. Brooke, P. L. H., 'On Matrix Representations and Codes Associated with the Simple Group of Order 29920', *J. Algebra* **91** (1984), 537–566.
4. Brooke, P. L. H., 'On the Steiner System S(2, 4, 28) and Codes associated with the Simple Group of Order 6048', *J. Algebra* **97** (1985), 376–406.
5. Feit, W. and Higman, G., 'The Nonexistence of Certain Generalized Polygons', *J. Algebra* **1** (1964), 114–131.
6. Lander, E. S., *Symmetic Designs: An Algebraic Approach*, Cambridge Univ. Press, Cambridge, 1983.
7. Payne, S. E. and Thas, J. A., *Finite Generalized Quadrangles*, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1984.
8. Ronan, M., 'A Geometric Characterization of Moufang Hexagons', *Invent. Math.* **57**, (1980), 227–262.
9. Sastry, N. S. N., 'Codes and Generalized Polygons', in *Combinatorics and Applications* (eds K.

S. Vijayan and N. M. Singhi), Indian Statistical Institute Publication, Calcutta, 1984, pp. 303–310.
10. Tits, J., 'Moufang Octagons and Ree Groups', *Amer. J. Math.* 105, (1983), 539–694.

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