# AN ALGORITHM FOR COMPUTING THE VON NEUMANN BALANCED GROWTH PATH

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THIS NOTE DESCRIBES a computational procedure for finding the maximal balanced growth path in the von Neumann model—more precisely, in its version given in Gale [2][2]—by means of parametric linear programming (LP). The LP problem corresponds to a two-period version of the von Neumann model with the growth rate treated as a fixed parameter and the total return to the activities in the second period as the maximand, where the rates of return are simply taken over from an arbitrary normalization rule used in defining the von Neumann balanced growth path. Next, the growth rate is varied in such a way as to generate a sequence of LP problems with their solutions converging to a limit. The desired balanced growth path is obtained from the solution of the limiting LP problem.[3] Thus, the basic idea behind our algorithm is simple and intuitive: so far as one is interested only in the balanced growth solution of an optimizing dynamic model (with single-period lags and constant coefficients), one can truncate its horizon to two periods only. The optimum solution of the two-period model, for an appropriate objective function and an appropriate set of boundary conditions (both unknown), will always yield the optimum balanced growth path of the original model. In full generality, a computational procedure based on this principle will have to search for both sets of unknowns. In the present case, it turns out that all the requisite variations in the boundary conditions for this search can be confined to a single coefficient, the growth rate, while the objective function can be specified, once and for all, solely with regard to an arbitrary normalization rule.

That was by way of introduction. Let us now define the problem in formal terms. The von Neumann balanced growth path consists of a scalar, $\alpha$ (the expansion factor), and a semi-positive $n$-vector, $x$ (the intensity vector), such that $\alpha$ is a maximum subject to:

$$(1) \qquad (B - \alpha A)x \geqq 0$$

where $A$ and $B$ are respectively the input and output matrices (both of order $m \times n$) of the model (rows of $A$ and $B$ correspond to commodities and columns to activities). Following Kemeny, Morgenstern, and Thompson [4] and Gale [1], we shall assume each row of $B$ and each column of $A$ to be a semi-positive vector (of appropriate dimensions), meaning, respectively, that each commodity is producible by some activity and that each activity uses some commodity as input. It is known that, under these assumptions, a solution to the problem exists, and has $\alpha > 0$. Note that in the solution $x$ is determined at most up to positive scalar multiplication, and hence can be normalized in any fashion. We shall do this by requiring:

$$(2) \qquad ex = 1$$

where $e$ is the $n$-vector with all its components equal to unity. (There is nothing special about this; any positive vector can replace $e$ in (2), provided that the same is done wherever $e$ occurs below.) We shall denote the set of $(n + 1)$ vectors, $(\alpha, x) \geqq 0$, satisfying (1) and (2) by $N$, and its subset for which $\alpha$ takes on the maximum value, by $N^*$. Our problem, then, is to find an $(\alpha, x) \in N^*$. The maximum value of $\alpha$ will be denoted by $\alpha^*$.

[1] We wish to thank Messrs. B. C. Sanyal and M. Dey for computational assistance at various stages of the work.

[2] The reader is referred to Gale [2, pp. 310–316] for the details of the model. Much of our terminology and notation is also taken from this source.

[3] Recently, Hamburger, Thompson, and Weil (HTW) [3] proposed an algorithm for the same problem, which also consists of solving a convergent sequence of LP problems. Our method of constructing the sequence is broadly similar to theirs (some differences will be noted later), but in other respects the two algorithms are completely different.

Our LP problem is the following: find two $n$-vectors, $z$ and $y$, $z \geqslant 0$, such that $ey$ is a maximum subject to:

(3) $Bz - Ay \geqslant 0,$

(4) $\alpha z - y \leqslant 0,$ and

(5) $ez \leqslant 1,$

where $\alpha \geqslant 0$ is taken as a parameter. For a given $\alpha$, we shall denote the problem by $P_\alpha$ and its maximum objective function value by $V(\alpha)$. One easily verifies the correspondence of $P_\alpha$ to its earlier specification by interpreting $z$ and $y$ as the intensity vectors of a von Neumann model for any two successive periods.

To begin the analysis, we immediately find that for all $\alpha$, (i) $P_\alpha$ has $(z, y) = (0, 0)$ as a feasible solution, and (ii) by (3), (5), and the assumptions on $A$ and $B$ stated earlier, $ey$ is bounded above on the set of feasible solutions of $P_\alpha$. It follows that $V(\alpha)$ is a well-defined non-negative number for all $\alpha$. Also, $V(\alpha)$ is nonincreasing in $\alpha$ since any $(z, y)$ which satisfies (3)–(5) for a given $\alpha$ also satisfies (3)–(5) for a smaller $\alpha$. Our next result can be stated as a lemma.

LEMMA: *For any $\alpha \geqslant 0$, either $V(\alpha) > 0$, implying $V(\alpha) \geqslant \alpha^* \geqslant \alpha$, or $V(\alpha) = 0$, implying $\alpha > \alpha^*$.*

PROOF: We have already seen that either $V(\alpha) > 0$ or $V(\alpha) = 0$ for any $\alpha$. Let $V(\alpha) > 0$ and $(z, y)$ be the optimum solution of $P_\alpha$. Then $ez = 1$ from (5) and $V(\alpha) > 0$, while $Bz \geqslant Ay \geqslant \alpha Az$ by (3) and (4). Hence $(\alpha, z) \in N$, so that $\alpha \leqslant \alpha^*$. However, for all $(\alpha, x) \in N$, it is easily seen that $(z, y) = (x, \alpha x)$ is feasible for $P_\alpha$, and hence $V(\alpha) \geqslant \alpha ex = \alpha$ by (2), i.e., $V(\alpha) \geqslant \alpha$ for all $\alpha \leqslant \alpha^*$. Since $V(\alpha)$ is non-increasing, one must have $V(\alpha) \geqslant \alpha^* \geqslant \alpha$ whenever $V(\alpha) > 0$. It follows that if $V(\alpha) = 0$, then $\alpha > \alpha^*$, for otherwise, $V(\alpha) \geqslant \alpha^* > 0$, a contradiction. *Q.E.D.*

As a corollary, we note that if $V(\alpha) = \alpha$, then $\alpha = \alpha^*$, although the converse of this is not necessarily true. However, regardless of whether $V(\alpha^*) = \alpha^*$ or $V(\alpha^*) > \alpha^*$, if $(z, y)$ is the optimum solution of $P_{\alpha^*}$, then $(\alpha^*, z) \in N^*$. Hence our task boils down to finding $\alpha^*$ whence an $x$, satisfying $(\alpha^*, x) \in N^*$, is obtained simply by solving $P_{\alpha^*}$. We shall do this by constructing a sequence, $\{\alpha_k\}$, where (i) each $\alpha_k$ lies between a least upper bound, $U_k$, and a greatest lower bound, $L_k$, on $\alpha^*$, obtained from the values of $\alpha_i$ and $V(\alpha_i)$ for $i \leqslant k$; and (ii) $U_k$ and $L_k$ approach a common limit as $k \to \infty$. Specifically, we work through the sequential relation:

(6) $$\alpha_{k+1} = (U_k + L_k)/2,$$

where

(7) $$L_k = \begin{cases} \alpha_k & \text{if } V(\alpha_k) > 0, \\ L_{k-1} & \text{if } V(\alpha_k) = 0, \end{cases}$$

and

(8) $$U_k = \begin{cases} \min\,[V(\alpha_k), U_{k-1}] & \text{if } V(\alpha_k) > 0, \\ \alpha_k & \text{if } V(\alpha_k) = 0. \end{cases}$$

To initiate the algorithm, choose $\alpha_0$ small enough ($\alpha_0 = 0$ will always do) such that $V(\alpha_0) > 0$, and put $U_0 = V(\alpha_0)$, $L_0 = \alpha_0$. The algorithm continues so long as $(U_k - L_k)$ is larger than some preassigned number indicating the degree of approximation sought. Our main result is the following theorem:

THEOREM: *The sequence $\{\alpha_k\}$ converges to $\alpha^*$.*

PROOF: It follows from the lemma and the definitions of $U_k$ and $L_k$ that $U_k \geqslant \alpha^* \geqslant L_k$, and hence from (6) that $U_k \geqslant \alpha_{k+1} \geqslant L_k$ at each $k$. Hence all that is to be proved is that $U_k$ and $L_k$ approach a common limit. From (6)–(8) it is seen that

$$U_{k+1} - L_{k+1} \leqslant (U_k - L_k)/2$$

(with equality except if $U_k > V(\alpha_{k+1}) > 0$). Thus, $(U_k - L_k)$ *decreases* so long as it is positive, and hence must converge to zero. *Q.E.D.*

This completes our stated job. We shall now briefly compare the present algorithm with that proposed in [3] (see footnote 3), and then turn to the complementary problem of finding the von Neumann interest rate and price vector. The HTW algorithm describes an iterative procedure to find an $\alpha$ such that the value of the matrix game, defined by $(B - \alpha A)$ as the pay-off matrix, is zero. The game is solved by LP and, as in our case, $\alpha$ for the next iteration is set at the midpoint of an upper and a lower bound on $\alpha^*$, discovered from the previous iterations. However, unlike ours, the HTW algorithm changes only one of the bounds at each iteration and replaces it by $\alpha$, and consequently reduces the range, within which $\alpha^*$ must lie, by half. In contrast, we sometimes change both the bounds simultaneously—and reduce the range by *more* than half—and at other times follow HTW in reducing the range by half. Hence the *rate of convergence* is expected to be faster in the present algorithm,[4] although this does not necessarily imply a greater overall computational efficiency. To be fair, we note that our LP problem has more constraints and variables than that of HTW and that, on the related question of finding the von Neumann prices, our algorithm will generally require additional computations (indicated below), while the HTW algorithm solves for the von Neumann activity levels and prices simultaneously.

Let us now turn to the von Neumann price problem: find a scalar, $\beta$ (the interest factor), and a semi-positive $m$-vector, $p$ (the price vector), such that $\beta$ is a minimum subject to:

(9) $p(B - \beta A) \leqslant 0.$

It is again known that a solution to this problem exists and satisfies $0 < \beta \leqslant \alpha^*$. Since $\beta > 0$, the problem is equivalent to one of maximizing $\theta$ $(= 1/\beta)$ subject to: $(A' - \theta B')p' \geqslant 0$ (the primes denote transposes). This is formally the same as the earlier growth problem. Hence if we replace $\alpha$, $A$, and $B$ in (3)–(5) by $\theta$, $B'$, and $A'$ respectively, then we can use our algorithm to find $\beta$ and $p$. Note that once $\alpha^*$ is known, one can always initiate this algorithm by setting $\theta = 1/\alpha^* \leqslant 1/\beta$, so that $V(\theta) > 0$. In fact, if $\alpha^* = \beta$, then $p$ is obtained in the very first step of the algorithm, and no further iteration is necessary.

However, a more interesting question in this connection is whether $(\beta, p)$ can be computed directly from the solution of the dual LP problem associated with $P_\alpha$, while solving for an $(\alpha^*, x) \in N^*$, rather than by an independent run of the algorithm. The answer, in general, is *no*, and the reason is not far to seek. The objective function of $P_\alpha$ assigns an arbitrary rate of return to each activity in the second period, bearing no relation to the rates of return under the von Neumann prices, i.e., to $pB$. Clearly, these objective function coefficients will influence the optimal values of dual variables corresponding to $P_{\alpha^*}$. The latter, consequently, will give us a solution of the von Neumann price problem only under rather special and arbitrary conditions.

These conditions are easily discovered. Let $u$ (an $m$-vector) and $w$ (a scalar) be the optimum dual variables associated with (3) and (5) respectively for $\alpha = \alpha^*$. It is then seen from the dual constraints that:

(10) $u(B - wA) \leqslant 0$, $u$ semi-positive,

and from the duality theorem of LP that:

(11) $u(B - \alpha^* A)z = \alpha^* - w$.

where $z$ comes from the optimum solution of $P_{\alpha^*}$ (and hence $(\alpha^*, z) \in N^*$). It follows that $(w, u)$ is a solution of the von Neumann price problem if and only if $w = \alpha^* = \beta$, where the two equalities have to be treated independently, for $w = V(\alpha^*) \geqslant \alpha^*$ and $\alpha^* \geqslant \beta$.

On an interpretive level, the condition $\alpha^* = \beta$ can be directly inferred from the fact that each activity shows a *positive* return in the objective function of $P_\alpha$. Hence so must they under the

[4] Professor Weil has since pointed out to us that the rate of convergence in both algorithms can be improved by adopting the Newton-Raphson method of iteration instead of (6) which is common to both.

von Neumann prices—i.e., $pB$ must be positive—if we are to calculate $p$ from the optimum solution of the dual to $P_{\alpha^*}$. It is well known that if $pB$ is positive, then $\alpha^* = \beta$. The other condition, $V(\alpha^*) = \alpha^*$, is, so far as we can see, a reflection of the arbitrariness of the objective function of $P_\alpha$ itself. It is, however, interesting to see that if this condition alone is satisfied, then the right hand side of (11) is zero. In other words, $(z, u)$ in this case is an element of the so-called von Neumann facet.

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