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D-OPTIMAL STATISTICAL DESIGNS WITH RESTRICTED STRING PROPERTY

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ABSTRACT

This paper applies Fréchet derivatives to derive asymptotically D-optimal statistical designs where the design matrix is a (0,1)-matrix having exactly one run [of length at most k(< p, the number of parameters)] of 1's in each row. These asymptotic results have been utilized in dealing with the more intractable design problem with a finite number of observations. The problem of E-optimality has also been considered.

1. INTRODUCTION

Fulkerson and Gross (1965) and Ryser (1969) considered the combinatorial properties of matrices with elements 0 or 1 having exactly one run of 1's in each row. A statistical design where the design matrix has the above property is called a design with string property. This terminology is due to Sinha and Saha (1983) who indicated applications of such

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designs in a number of fields, particularly in biometry and optics, cited several references in this regard (e.g. Marshal and Comisarow (1975), Sloane and Harwit (1976), Harwit and Sloane (1979)) and derived some optimality results. Further results, both asymptotic and exact, were obtained by Mukerjee and Saha Ray(1983).

Basically, Sinha and Saha (1983) and Mukerjee and Saha Ray (1983) considered the standard linear model

$$\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}, \quad E(\underset{\sim}{\varepsilon}) = \underset{\sim}{0}, \quad \mathrm{Disp}(\underset{\sim}{\varepsilon}) = \sigma^2 I, \qquad (1.1)$$

(where $\underset{\sim}{Y}$ is the observational vector, X is the design matrix and $\underset{\sim}{\beta}(p \times 1)$ is the vector of parameters) and developed optimal designs with string property for estimating $\underset{\sim}{\beta}$. These authors did not impose any restriction on the maximum number of 1's in any row of X. In many practical situations, however, such restrictions become relevant. For example, consider the problem of measuring the consecutive distances between (p+1) objects fixed along a line. Since in such a situation the measurement of the distance between any two objects along the line automatically takes account of the intermediate objects, the resulting design matrix X has string property. If further the length of the measuring instrument (say, a tape) be less than the distance between the two objects at the two extreme ends then there will be a natural constraint on the maximum number of 1's in each row of X.

Under the linear model (1.1), the present paper aims at developing D-optimal designs (for estimating $\underset{\sim}{\beta}$) with string property subject to the restriction that the number of 1's in each row of X should not exceed a given positive integer $k(\leq p$, the case of strict inequality being the main interest of this paper). First Fréchet derivatives are applied to obtain asymptotic optimality results which are helpful in dealing with the more intractable design problem with a finite number of observations. Next, some exact optimal designs have been derived.

For optimality results in a different context with restriction on the number of 1's but without string property, we refer to Dey and Gupta (1977) and Swamy (1980). The designs considered in this paper have also a close link with spring balance weighing designs (see Raghavarao (1971), Banerjee (1975) for a comprehensive list of references).

2. PRELIMINARIES

Let $S = \{(u,v) : 1 \le u \le v \le p,\ v-u \le k-1\}$. For $(u,v) \in S$, let $\underset{\sim}{h}_{uv}$ be a $(p \times 1)$ vector with elements 0 and 1 having exactly one run of 1's starting at the uth and ending at the vth (both inclusive) positions. With p parameters, string property and the restriction that no row should contain more than k 1's, each row of the design matrix must be the transpose of one of these vectors. Let X, the design space, be the set of the vectors $\underset{\sim}{h}_{uv}((u,v) \in S)$.

Following Silvey (1980, pp.15), let H be the class of probability distributions on the Borel sets of X. Any $\eta \in H$ will be called a design measure. Since X is finite, any such η defines a discrete distribution over X assigning a mass, say π_{uv} at $\underset{\sim}{h}_{uv}((u,v) \in S)$. For $\eta \in H$, define the $(p \times p)$ information matrix $M(\eta) = E(\underset{\sim}{x}\,\underset{\sim}{x}')$, $\underset{\sim}{x}$ being a $(p \times 1)$ random vector with distribution η. Denoting the (i,j)th element of $M(\eta)$ by $m_{ij}(\eta)$, it can be checked that

$$m_{ij}(\eta) = \Sigma\, \pi_{uv},\quad 1 \le i \le j \le p, \tag{2.1}$$

where the summation extends over $u \le i$, $v \ge j$, $v-u \le k-1$.

Let $M = \{M(\eta) : \eta \in H\}$ and ϕ be a real valued function defined on the class of $(p \times p)$ symmetric matrices and bounded above on $\mathcal{M}$. Then the problem is to determine η^* to maximize $\phi[M(\eta)]$ over H. Such an η^* will be called ϕ-optimal. The following theorem (Silvey (1980, pp.19)) will be used in the next section :

Theorem 2.1. If ϕ is concave on $\mathcal{M}$ and differentiable at $M(\eta^*)$, then η^* is ϕ-optimal if and only if $F_\phi(M(\eta^*), \underset{\sim}{x}\,\underset{\sim}{x}') \leq 0$ for each $x \in \mathcal{X}$, where $F_\phi(M(\eta^*), \underset{\sim}{x}\,\underset{\sim}{x}')$ is the Fréchet derivative of ϕ at $M(\eta^*)$ in the direction of $\underset{\sim}{x}\,\underset{\sim}{x}'$.

3. ASYMPTOTIC D-OPTIMAL DESIGNS

For D-optimality $\phi[M(\eta)] = \log\det M(\eta)$ and if $M(\eta)$ be nonsingular it can be seen that (cf. Silvey (1980, pp.21))

$$F_\phi(M(\eta), \underset{\sim}{x}\,\underset{\sim}{x}') = \underset{\sim}{x}'[M(\eta)]^{-1}\underset{\sim}{x} - p. \qquad (3.1)$$

Before presenting the main result of this section given in Theorem 3.1 below, some notations are introduced. Let $t = \min(k-1, p-k)$. Clearly $t \geq 0$. If $t = 0$, which happens if $k = 1$ or p, define $S_1 = S$. If $t > 0$, denote by S_0 the empty set and define

$$S_r = \{(u,v) \in S - \bigcup_{j=0}^{r-1} S_j;\ u = r \text{ or } v = p-r+1$$
$$\text{or } v - u = k - r\},\quad 1 \leq r \leq t,\quad S_{t+1} = S - \bigcup_{r=0}^{t} S_r.$$

For any $m \geq 1$ and any real numbers $a_1, \ldots, a_m$, let $L(a_1, \ldots, a_m)$ be the $(m \times m)$ symmetric matrix

$$L(a_1,\ldots,a_m) = \begin{bmatrix} a_1 & a_2 & \cdots & a_{m-1} & a_m \\ a_2 & a_1 & \cdots & a_{m-2} & a_{m-1} \\ & \vdots & & & \\ a_{m-1} & a_{m-2} & \cdots & a_1 & a_2 \\ a_m & a_{m-1} & \cdots & a_2 & a_1 \end{bmatrix}$$

For any $s(0 \le s < p/2)$ and any real numbers $a_1, a_2, \ldots, a_{p-2s}$, let $V_p(s; a_1, a_2, \ldots, a_{p-2s})$ be a $(p \times p)$ matrix of the form

$$V_p(s;a_1,\ldots,a_{p-2s}) = \begin{bmatrix} 0^{(s\times s)} & 0^{(s\times p-2s)} & 0^{(s\times s)} \\ 0^{(p-2s\times s)} & L(a_1,\ldots,a_{p-2s}) & 0^{(p-2s\times s)} \\ 0^{(s\times s)} & 0^{(s\times p-2s)} & 0^{(s\times s)} \end{bmatrix} \qquad (3.2)$$

Theorem 3.1 The design measure η^* given by the following probability distribution over x

$$\pi^*_{uv} = 2(k-r+1)/[pk(k+1)] \text{ if } (u,v) \in S_r, 1 \le r \le t+1,$$

(where t and $S_r(1 \le r \le t+1)$ are as defined above) is D-optimal within H.

Proof. It is easy to check that $\pi^*_{uv} > 0((u,v) \in S)$ $\sum_{(u,v)\in S} \pi^*_{uv} = 1$. From (2.1) it follows, after some simplification, that for η^* defined as above

$$M(\eta^*) = [2/\{pk(k+1)\}] \sum_{w=0}^{t} (k-w) V_p(w; k-w, k-w-1, \ldots, 1, \underbrace{0,\ldots,0}_{(p-k-w) \text{ times}})$$

and consequently (as can be verified by actual multiplication)

$$[M(\eta^*)]^{-1} = (p/2) V_p(0; 2, -1, \underbrace{0,\ldots,0}_{(k-2) \text{ times}}, g_1, \ldots, g_{p-k}), \qquad (3.3)$$

where $g_1 = k^{-1}$ and $g_2, \ldots, g_{p-k}$ are unique roots of the equations

$$kg_i + (k-1)g_{i-1} + \ldots + (k-i+1)g_1 = 0 \quad (2 \le i \le k-1)$$

$$kg_j + (k-1)g_{j-1} + \ldots + 2g_{j+2-k} + g_{j+1-k} = 0 \quad (k \le j \le p-k)$$

in the case $k-1 < p-k$, or of the equations

$$kg_i + (k-1)g_{i-1} + \ldots + (k-i+1)g_1 = 0 \quad (2 \le i \le p-k)$$

in the case $k-1 \ge p-k$.

From (3.2),(3.3), it is immediate that $\underset{\sim}{x}'[M(\eta^*)]^{-1}\underset{\sim}{x} = p$ for each $\underset{\sim}{x} \in \mathcal{X}$ whence, by (3.1) and Theorem 2.1, the result follows.

Remark. In particular if $k = p$ (i.e. there is no restriction on the maximum number of 1's on any row of the design matrix) then $t = 0$. Consequently $S_1=S$ and by Theorem 3.1 for the D-optimal design measure η^*, $\pi^*_{uv} = 2/[p(p+1)]$ for each $(u,v) \in S$ so that the result obtained by Mukerjee and Saha Ray (1983) in the unrestricted set up follows as a special case.

Example 3.1. Let $p=4$, $k=2$. Then $t=1$, $S_1=\{(1,1),(1,2),(2,3),(3,4),(4,4)\}$, $S_2=\{(2,2),(3,3)\}$ and, by Theorem 3.1, the D-optimal design measure η^* is given by the probability distribution $\pi^*_{uv} = 1/6$ for $(u,v) \in S_1$ and $= 1/12$ for $(u,v) \in S_2$.

4. DISCUSSION

From a practical point of view, the object of developing the asymptotic theory is to help with the more intractable n observation design problem. With a finite number of observations n, let C_n be the class of n observation statistical designs for p parameters satisfying the string property and the further restriction that the number of 1's in each row of the design matrix X should not exceed $k(< p)$. For $(u,v) \in S$, let n_{uv} be the number of times the transpose of $\underset{\sim}{h}_{uv}$ occurs as a row of X. Then, using the notations of the preceding section, the following exact optimality result follows from Theorem 3.1.

Theorem 4.1. If n be a multiple of $pk(k+1)/2$, then the design matrix for which

$$n_{uv} = 2n(k-r+1)/[pk(k+1)] \text{ if } (u,v)\in S_r,\ 1 \le r \le t+1,$$

is D-optimal within C_n.

Example 4.1. Let $n = 12$, $p = 4$, $k = 2$. Then n is a multiple of $pk(k+1)/2$ and, by Theorem 4.1 and Example 3.1, the exact D-optimal design matrix with 12 observations is given by $n_{uv} = 2$ for $(u,v) \in S_1$ and $= 1$ for $(u,v) \in S_2$.

For general n,p,k even if n is not a multiple of $pk(k+1)/2$, starting from the asymptotic optimality results one can construct n observation designs which are very close to optimality specially when n is not small (see e.g. Fedorov (1972, Ch. 3), Silvey (1980, pp. 37)). This is of particular relevance when the available resources allow a fairly large number of observations and the interest lies in taking these observations efficiently.

Example 4.2. Let $n = 19$, $p = 4$, $k = 2$. Then suppose the probability distribution corresponding to the D-optimal design measure η^*, as shown in Example 3.1, is approximated following the simple rule of taking n_{uv} as $n\,\pi^*_{uv}((u,v) \in S)$ rounded off to the nearest integer. The resulting design measure is $\hat{\eta}^*$ for which $\pi_{uv} = 3/19$ for $(u,v) \in S_1$ and $= 2/19$ for $(u,v) \in S_2$. With $[\det M(\hat{\eta}^*)/\det M(\eta^*)]^{1/p}$ taken as a measure of D-efficiency, the D-efficiency of $\hat{\eta}^*$ turns out to be .9980 which may be considered very high especially in view of the fact that with $n = 19$ observations the asymptotically D-optimal design measure η^* can never be realised.

Before concluding, some observations may be made regarding optimal designs in the present context with respect to criteria other than D-optimality. In the absence of any restriction on the number of 1's in any row of the design matrix, it was shown in Mukerjee and Saha Ray (1983) that the E-optimal design measure $\bar{\eta}$ with string property is given by the probability distribution $\pi_{uu} = p^{-1}(1 \leq u \leq p)$ and $\pi_{uv} = 0$ for every other (u,v). Since under $\bar{\eta}$ each row of the design matrix contains exactly one 1, even under the restricted set up $\bar{\eta}$ belongs to the relevant class of design measures H(as considered in section 2) and is E-optimal in H. Clearly with a finite number of observations n this $\bar{\eta}$ can be realised if n be a multiple of p. As for A-optimality it has been shown in Mukerjee and Saha Ray (1983) that even in the unrestricted set up the A-optimal design measure is rather involved and is given in terms of trigonometric functions. It is felt that in the restricted set up considered in this paper the A-optimal design measure will be still more complicated. The authors intend to take up the problem in a subsequent communication.

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