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# $D^{-1}$-PARTIALLY EFFICIENCY-BALANCED DESIGNS WITH AT MOST TWO EFFICIENCY CLASSES

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## ABSTRACT

A wide class of block designs admitting a simple analysis has been considered. The statistical properties of such designs have been indicated and the problems relating to their characterization and construction have been investigated.

## 1. INTRODUCTION AND PRELIMINARIES

Consider a block design with the usual $v \times b$ incidence matrix N. Let $r_1, \cdots, r_v$ be the replication numbers and $k_1, \cdots, k_b$ be the block sizes. Let

$$R = \text{diag}\{r_1, \cdots, r_v\}, \quad K = \text{diag}\{k_1, \cdots, k_b\},$$
$$R^{1/2} = \text{diag}\{r_1^{1/2}, \cdots, r_v^{1/2}\}, \quad R^{-1/2} = (R^{1/2})^{-1},$$
$$\underline{1}^{(v\times 1)} = (1, \cdots, 1)', \quad J = \underline{1}\,\underline{1}', \quad C = R - NK^{-1}N'.$$

In the context of a block design, the concepts of various kinds of balancing, e.g., variance-balance, efficiency-balance, partially efficiency-balance, etc., are well-known (cf. Puri and Nigam (1977), Caliński, Ceranka and Mejza (1980), Kageyama (1980), Puri and Kageyama (1985)). Recently, Das and Ghosh (1985) considered generalized efficiency-balanced (GEB) designs which are the same as $D^{-1}$-balanced designs introduced by Caliński (1977). This paper considers $D^{-1}$-partially efficiency-balanced designs. It is seen that $D^{-1}$-partially efficiency-balanced designs with at most two efficiency classes admit a very simple analysis. The problems relating to the characterization and construction of such designs are investigated.

In the following, for any diagonal matrix D with diagonal elements $d_1,\cdots,d_v$, all positive, we shall write $D^{1/2} = \text{diag}\{d_1^{1/2},\cdots,d_v^{1/2}\}$ and $D^{-1/2} = (D^{1/2})^{-1}$.

DEFINITION 1. Given a diagonal matrix D with diagonal elements all positive, a block design is called a $D^{-1}$-partially efficiency-balanced design with m efficiency classes, or simply a $D^{-1}$-PEB(m) design, if the matrix $D^{-1/2}CD^{-1/2}$ has exactly m distinct positive eigenvalues.

Note that the matrix $D^{-1/2}CD^{-1/2}$ (= W, say) is a generalization of the matrix F used by Pearce, Caliński and Marshall (1974). Clearly for any positive definite D, every connected block design is a $D^{-1}$-PEB(m) design with some $m \leq v-1$ (cf. Caliński and Ceranka (1974)). Hence, in this paper we consider only $D^{-1}$-PEB(m) designs with $m \leq 2$. From a practical point of view, such designs have nice statistical properties, as it will be explained below. For a $D^{-1}$-PEB(m) design with $m \leq 2$ a spectral decomposition of W yields

$$W = \alpha_1 A_1 + \alpha_2 A_2 \qquad (1)$$

where $\alpha_1,\alpha_2$ are possibly distinct positive eigenvalues of W, $A_i = \sum_{j=1}^{\rho_i} \underline{p}_{ij}\underline{p}_{ij}'$ , $\rho_i$ is the multiplicity of $\alpha_i$ and $\underline{p}_{i1},\cdots,\underline{p}_{i\rho_i}$ are the corresponding orthonormal eigenvectors of W (i=1,2). Hence, writing $n = \sum_{i=1}^{v} r_i$ and $d_0 = \text{tr}(D)$, it can be seen that for i = 1,2, the efficiency factor for every treatment parametric contrast in the $i$th efficiency class, defined as the class spanned by the contrasts $D^{1/2}\underline{p}_{ij}$ $(1 \leq j \leq \rho_i)$, equals $(d_0/n)\alpha_i$ when the efficiency is measured relative to an orthogonal design having a replication vector $(n/d_0)D\underline{1}$ (see Pearce (1970)). Furthermore, the best linear unbiased estimator of every $D^{-1}$-normalized contrast from the $i$th class has a variance $\sigma^2/\alpha_i$, $i = 1,2$, where $\sigma^2$ is the error variance (cf. Caliński (1977)). An application of $D^{-1}$-PEB(1) designs for orthonormal contrasts has been discussed by Gupta (1987), with two examples.

The class of $D^{-1}$-PEB(m) designs with varying D but $m \leq 2$ is fairly large and includes all variance-balanced (with D = I), efficiency-balanced (with D = R), partially efficiency-balanced ($m \leq 2$; with D = R), connected two-associate partially balanced incomplete block (PBIB) (with D = I), and generalized efficiency balanced (of Das and Ghosh (1985), i.e. with any D) designs. Also all C-designs (cf. Saha (1976)) belong to this class (with D = R). In the next section, a characterization for $D^{-1}$-PEB(m) designs with $m \leq 2$ is derived. It will be seen that for such designs one can obtain a g-inverse of the C-matrix without an explicit determination of $\alpha_1$, $\alpha_2$, $A_1$ and $A_2$. The resulting computational simplicity is likely to be helpful to an applied statistician. The constructional aspects are taken up in the last section.

Throughout the present paper, we shall deal only with connected $D^{-1}$-PEB(m) designs.

## 2. SOME RESULTS ON $D^{-1}$-PEB(m) DESIGNS WITH $m \leq 2$

THEOREM 1. For a positive definite matrix $D = \text{diag}\{d_1,\cdots,d_v\}$ a block design is a $D^{-1}$-PEB(m) design with $m \leq 2$ if and only if there exist constants $\gamma,\delta$ ($\delta > 0$) such that

$$W^2 - \gamma W + \delta(I - d_0^{-1}\underline{d}^{1/2}\underline{d}^{1/2\prime}) = 0 \qquad (2)$$

where $W = D^{-1/2}CD^{-1/2}$, $d_0 = \text{tr}(D)$, $\underline{d}^{1/2} = (d_1^{1/2},\cdots,d_v^{1/2})'$.

PROOF. If. Obviously $d_0^{-1/2}\underline{d}^{1/2}$ is a normalized eigenvector of W corresponding to the eigenvalue 0. Let $\underline{\xi}$ ($\neq \underline{0}$) be any eigenvector of W which is orthogonal to $d_0^{-1/2}\underline{d}^{1/2}$. Let $\lambda$ be the eigenvalue corresponding to the eigenvector $\underline{\xi}$. Then postmultiplying both sides of (2) by $\underline{\xi}$ we have

$$(\lambda^2 - \gamma\lambda + \delta)\underline{\xi} = \underline{0} \quad \text{which implies} \quad \lambda^2 - \gamma\lambda + \delta = 0.$$

This shows that the non-negative definite(n.n.d.) matrix W can have at most two distinct non-zero, i.e., positive, eigenvalues. Furthermore, if $\delta > 0$ then both the roots of $\lambda^2 - \gamma\lambda + \delta = 0$ are positive

($\gamma$ and $\delta$ must be non-negative as W is n.n.d.; $\delta > 0$ implies then $\gamma > 0$), and it follows that every eigenvector of W which is orthogonal to $d_0^{-1/2}\underline{d}^{1/2}$ corresponds to a positive eigenvalue. Hence rank(W) = v-1, i.e., the design is connected. This proves the 'if' part.

Only if. Suppose the design is a $D^{-1}$-PEB(m) design with $m \leq 2$. Then the spectral decomposition of $D^{-1/2}CD^{-1/2}$ (= W) yields

$$W = \alpha_1 A_1 + \alpha_2 A_2,$$

where $\alpha_1$ and $\alpha_2$ are positive not necessarily distinct and where $A_0 = d_0^{-1}\underline{d}^{1/2}\underline{d}^{1/2\prime}$, $A_1$, $A_2$ are symmetric idempotent and such that

$$A_0 + A_1 + A_2 = I, \quad A_0A_1 = A_0A_2 = A_1A_2 = O, \tag{3}$$

with $d_0$ and $\underline{d}^{1/2}$ being defined as before. Hence we obtain

$$W = \alpha_1(I-A_0-A_2) + \alpha_2A_2, \quad \text{i.e.,} \quad W - \alpha_1(I-A_0) = (\alpha_2-\alpha_1)A_2,$$

from which squaring both sides and after some simplification using (3) one gets $W^2 - \gamma W + \delta(I-A_0) = O$, where $\gamma = \alpha_1+\alpha_2$ and $\delta = \alpha_1\alpha_2$. Note that $\alpha_1, \alpha_2 > 0$ and hence $\delta > 0$. This proves the 'only if' part and completes the proof of the theorem. □

The next theorem presents a compact formula for a g-inverse of the C-matrix in a $D^{-1}$-PEB(m) design with $m \leq 2$.

THEOREM 2. Consider a $D^{-1}$-PEB(m) design with $m \leq 2$. Let $\gamma, \delta$ ($\delta > 0$) be constants such that

$$W^2 - \gamma W + \delta(I - d_0^{-1}\underline{d}^{1/2}\underline{d}^{1/2\prime}) = O. \tag{4}$$

Then a g-inverse of C is given by

$$\bar{C} = D^{-1/2}[\tfrac{\gamma}{\delta}(I - d_0^{-1}\underline{d}^{1/2}\underline{d}^{1/2\prime}) - \tfrac{1}{\delta}W]D^{-1/2}$$

$$= \tfrac{\gamma}{\delta}(D^{-1} - d_0^{-1}J) - \tfrac{1}{\delta}D^{-1}CD^{-1},$$

where W is as defined in Theorem 1.

PROOF. Let $\bar{W} = (\gamma/\delta)(I - d_0^{-1}\underline{d}^{1/2}\underline{d}^{1/2\prime}) - (1/\delta)W$. By (4), and the fact that $W\underline{d}^{1/2} = \underline{0}$, one has by direct multiplication,

$$W\bar{W} = I - d_0^{-1}\underline{d}^{1/2}\underline{d}^{1/2\prime} \implies W\bar{W}W = W \implies C\bar{C}C = C,$$

which completes the proof. Incidentally, it may be noted that $\bar{C}$, as given above, equals $D^{-1/2}(\alpha_1^{-1}A_1 + \alpha_2^{-1}A_2)D^{-1/2}$, where $\alpha_1$, $\alpha_2$, $A_1$, $A_2$ are as in (1). □

Theorems 1 and 2 make the task of analyzing a $D^{-1}$-PEB(m) design with $m \leq 2$ rather simple. Once D is known, it is not necessary to have a spectral decomposition of $W = D^{-1/2}CD^{-1/2}$ and Theorem 2 is enough to yield an explicit expression for a g-inverse of the C-matrix of the design. In particular, all connected PBIB designs with two associate classes are $D^{-1}$-PEB(2) designs (later, we shall see that even PBIB designs with more than two associate classes can be $D^{-1}$-PEB(2) designs) with D = I. For such designs (i.e. with D = I), by Theorem 1, there exist constants $\gamma, \delta$ $(\delta > 0)$ such that

$$C^2 - \gamma C + \delta(I - v^{-1}J) = O,$$

the determination of $\gamma$ and $\delta$ being straightforward (one has just to solve two equations in two unknowns). Then a g-inverse of the C-matrix follows easily from Theorem 2. This makes the task of analyzing such a design very simple as one does not have to take care of the details of the association scheme at all.

Caliński (1971) and Saha (1976) considered C-designs which have the merit of admitting a simple analysis. Unfortunately, however, the class of C-designs is not very large (in particular, there are many two-associate PBIB designs which are not C-designs). In this regard, the concept of $D^{-1}$-PEB(m) designs with $m \leq 2$ is helpful since it extends the class of C-designs to a much wider class while retaining a simplicity in the analysis.

## 3. SOME CONSTRUCTION PROCEDURES

In this section, we shall investigate the construction procedures for and combinatorial properties of some $D^{-1}$-PEB(m) designs with $m \leq 2$. Now, $D^{-1}$-PEB(1) designs are precisely the same as the generalized efficiency balanced designs considered recently by Das and Ghosh (1985) and Kageyama and Mukerjee (1986). We shall, therefore, give special attention to $D^{-1}$-PEB(2) designs. Along with each method of construction we shall specify D and indicate $\alpha_1$, $\alpha_2$, $A_1$, $A_2$ (see (1)) to give an idea about the statistical properties of the resulting designs in the sense described in Section 1. Note that many of these methods of construction may be employed for the construction of $D^{-1}$-PEB(m) designs for general m, although we do not consider that here.

3.1. The method of truncation. Along the line of Cheng (1981), we have the following definition: Let d be a block design with constant block size k ($\geq 3$). Then for any k' ($2 \leq k' < k$) the $k'$th truncation of d is a design obtained from d by replacing each block of d by a set of $\binom{k}{k'}$ blocks considering all possible selection of k' treatments from the k treatments in a block.

THEOREM 3. Every truncation of a binary $D^{-1}$-PEB(2) design having a constant block size k ($\geq 3$) and the other parameters v, b, $r_i$ ($i=1, \cdots, v$) is a $D^{-1}$-PEB(2) design with parameters $v^* = v$, $b^* = b\binom{k}{k'}$, $r_i^* = r_i\binom{k-1}{k'-1}$ and $k^* = k'$.

PROOF. Let $d_1$ be a binary $D^{-1}$-PEB(2) design having a constant block size k ($\geq 3$) and $d_2$ be a $k'$th truncation of $d_1$ ($2 \leq k' < k$). Then after some algebra, it may be seen (cf. Cheng (1981) for the case $k' = 2$) that $C_2 = \alpha C_1$, where $C_1$, $C_2$ are the C-matrices of $d_1$, $d_2$, respectively, and $\alpha = \binom{k-2}{k'-2}(k/k')$. Hence the result follows immediately from Definition 1. Note that the matrices $A_1$, $A_2$ in $d_2$ will be the same as those in $d_1$ while $\alpha_1$, $\alpha_2$ in $d_2$ will be $\alpha$ times those in $d_1$. □

The above theorem is helpful in constructing new $D^{-1}$-PEB(2) designs simply by considering the truncations of a known binary $D^{-1}$-PEB(2) design having a constant block size. It is easy to extend Theorem 3 to $D^{-1}$-PEB(m) designs in general, i.e. with $m \geq 1$.

3.2. The method of supplementation. Let d* be a connected block design involving v treatments, b blocks and having a common replication number r and a constant block size k. Let C* be the C-matrix of d*. Assume that C* has at most two distinct positive eigenvalues, say $\alpha_1^*$ and $\alpha_2^*$, with respective multiplicities $g_1$ and $g_2$ ($g_1+g_2=v-1$). For $i = 1,2$, let $L_i$ be a $v\times g_i$ matrix such that the columns of $L_i$ represent a complete set of orthonormal eigenvectors corresponding to the eigenvalue $\alpha_i^*$ of C*.

To each block of d* add one new treatment which is applied p ($\geq 1$) times in each block. The resulting design, say d**, involving v+1 treatments will be called a supplemented design (cf. Caliński and Ceranka (1974), Puri, Nigam and Narain (1977)) obtained from d*. In the following, for any positive integers a, a', $I_a$ denotes the $a\times a$ identity matrix, $\underline{1}_a$ denotes the a 1×vector of 1's and $J_{aa'} = \underline{1}_a\underline{1}_{a'}'$.

THEOREM 4. If $\alpha_1^* \neq \alpha_2^*$ then the supplemented design d** is a $D^{-1}$-PEB(2) design with

$$D = \begin{bmatrix} I_v & \underline{0} \\ \underline{0}' & h \end{bmatrix}, \quad \alpha_i = (rp + k\alpha_i^*)/(k + p),\ i = 1,2,$$

$$A_1 = \begin{bmatrix} L_1L_1' & \underline{0} \\ \underline{0}' & 0 \end{bmatrix} + \frac{h}{v(v+h)} \begin{bmatrix} J_{vv} & -vh^{-1/2}\underline{1}_v \\ -vh^{-1/2}\underline{1}_v' & v^2h^{-1} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} L_2L_2' & \underline{0} \\ \underline{0}' & 0 \end{bmatrix},$$

where $h = pb/\alpha_1^*$.

PROOF. Clearly d** is connected. Let C** be the C-matrix of d**. Then with D and h as above, it may be seen that

$$D^{-1/2}C^{**}D^{-1/2} = \begin{bmatrix} \frac{rp}{k+p} I_v + \frac{k}{k+p} C^* & -\frac{prh^{-1/2}}{k+p}\underline{1}_v \\ -\frac{prh^{-1/2}}{k+p}\underline{1}_v' & \frac{pbkh^{-1}}{k+p} \end{bmatrix}.$$

Postmultiplication of the above by $(L_1', \underline{0})'$, $\{h/(v(v+h))\}^{1/2}(\underline{1}_v', -vh^{-1/2})'$ and $(L_2', \underline{0})'$ shows that the distinct positive eigenvalues of $D^{-1/2}C^{**}D^{-1/2}$ are $(rp+k\alpha_i^*)/(k+p)$, i=1,2, with multiplicities $g_1+1$ and $g_2$, respectively. Hence the result follows. □

REMARK. Taking $h = pb/\alpha_2^*$ in D, it is easy to construct a dual version of Theorem 4. By Theorem 4, one may construct non-equireplicate $D^{-1}$-PEB(2) designs by supplementation of connected PBIB designs, with two associate classes, in particular. Incidentally, in the set-up of Theorem 4, if $\alpha_1^*$ and $\alpha_2^*$ are equal (e.g., if d* be a balanced incomplete block (BIB) design), then the supplemented design d** becomes a $D^{-1}$-PEB(1) design with D, h and $\alpha_1$ as before.

In Theorem 4, we considered only one additional treatment. If t (> 1) treatments are added to each block of d*, then the resulting design is not necessarily a $D^{-1}$-PEB(2) design. However, we can derive simple sufficient conditions under which the resulting supplemented design is a $D^{-1}$-PEB(2) design. This is considered in Theorem 5 below. Here to each block of d* we add t (> 1) new treatments each new treatment being applied p ($\geq$ 1) times in each block. The resulting supplemented design, say d***, now involves v+t treatments.

The following notation will be helpful. Let $M_1 = (L_1', O_{g_1\times t})'$, $M_2 = (L_2', O_{g_2\times t})'$, $M_3 = (O_{t-1\times v}, L')'$, $Q_i = M_i M_i'$ $(i=1,2,3)$, where L is a $t\times t-1$ matrix such that the columns of L form an orthogonal basis of the orthocomplement of the space spanned by $\underline{1}_t$ in the t-dimensional Euclidian space. Also, for any $h > 0$, let $\underline{s}(h)$ represent the normalized column vector $\{v+v^2(ht)^{-1}\}^{1/2}(\underline{1}_v', -vh^{-1/2}t^{-1}\underline{1}_t')'$ and $S(h) = \underline{s}(h)\underline{s}(h)'$.

THEOREM 5. (i) If $\alpha_1^* = \alpha_2^*$ but $r \neq \alpha_1^*$, then the supplemented design d*** is a $D^{-1}$-PEB(2) design with

(a) $D = \begin{bmatrix} I_v & O \\ O & hI_t \end{bmatrix}$, $\alpha_1 = (rpt+k\alpha_1^*)/(k+pt)$, $\alpha_2 = \alpha_1^*$, $A_1 = Q_1+Q_2+S(h)$, $A_2 = Q_3$, where $h = pb/\alpha_1^*$,

or with

(b) $D = \begin{bmatrix} I_v & O \\ O & hI_t \end{bmatrix}$, $\alpha_1 = (rpt+k\alpha_1^*)/(k+pt)$, $\alpha_2 = r$, $A_1 = Q_1+Q_2$, $A_2 = Q_3+S(h)$, where $h = pb/r$.

(ii) If $\alpha_1^* \neq \alpha_2^*$ but $r = \alpha_1^*$, then d*** is a $D^{-1}$-PEB(2) design with

$D = \begin{bmatrix} I_v & O \\ O & hI_t \end{bmatrix}$, $\alpha_1 = r$, $\alpha_2 = (rpt+k\alpha_2^*)/(k+pt)$, $A_1 = Q_1+Q_3+S(h)$, $A_2 = Q_2$, where $h = pb/\alpha_1^*$.

(iii) If $r$, $\alpha_1^*$, $\alpha_2^*$ are all unequal, but $pt = k(\alpha_1^*-\alpha_2^*)/(r-\alpha_1^*)$, then d*** is a $D^{-1}$-PEB(2) design with

$D = \begin{bmatrix} I_v & O \\ O & hI_t \end{bmatrix}$, $\alpha_1 = (rpt+k\alpha_1^*)/(k+pt)$, $\alpha_2 = \alpha_1^*$, $A_1 = Q_1+S(h)$, $A_2 = Q_2+Q_3$, where $h = pb/\alpha_1^*$.

PROOF. Clearly d*** is connected. Let C*** be the C-matrix of d***. With $D = \begin{bmatrix} I_v & O \\ O & hI_t \end{bmatrix}$, $h > 0$,

it may be seen that

$$D^{-1/2}C^{***}D^{-1/2} = \begin{bmatrix} \frac{rpt}{k+pt} I_v + \frac{k}{k+pt} C^* & -\frac{prh^{-1/2}}{k+pt} J_{vt} \\ -\frac{prh^{-1/2}}{k+pt} J_{tv} & h^{-1}(pbI_t - \frac{p^2b}{k+pt} J_{tt}) \end{bmatrix}$$

Postmultiplication of the above by $M_1$, $M_2$, $M_3$ and $\underline{s}(h)$ shows that the positive eigenvalues of $D^{-1/2}C^{***}D^{-1/2}$ are, say,

$$\rho_1 = (rpt+k\alpha_1^*)/(k+pt),\quad \rho_2 = (rpt+k\alpha_2^*)/(k+pt),\quad \rho_3 = h^{-1}pb,$$
$$\rho_4 = rp(t+h^{-1}v)/(k+pt)$$

with multiplicities $g_1$, $g_2$, $t-1$ and $1$, respectively.

Under the set-up of (i), if $h = pb/\alpha_1^*$, then $\rho_1 = \rho_2 = \rho_4 = (rpt+k\alpha_1^*)/(k+pt)$, $\rho_3 = \alpha_1^*$ and the common value of $\rho_1$, $\rho_2$, $\rho_4$ differs from $\rho_3$. This proves (i-a). The proofs of (i-b), (ii) and (iii) can similarly be worked out by noting that under the stated choices of h, among $\rho_1$, $\rho_2$, $\rho_3$, $\rho_4$ exactly two are distinct. □

The sufficient conditions in Theorem 5 have a wide coverage. In particular, the following corollaries hold:

COROLLARY 5.1. If d* is a BIB design then d*** is a $D^{-1}$-PEB(2) design for every $t(\geq 2)$, $p(\geq 1)$, where D, $\alpha_1$, $\alpha_2$, $A_1$, $A_2$ are as in Theorem 5(i), (a) or (b).

PROOF. Follows from Theorem 5(i). □

COROLLARY 5.2. If d* is a singular or semi-regular connected group divisible (GD) design, then d*** is a $D^{-1}$-PEB(2) design for every t $(\geq 2)$, $p(\geq 1)$, where D, $\alpha_1$, $\alpha_2$, $A_1$, $A_2$ are as in Theorem 5(ii).

PROOF. Follows from Theorem 5(ii). □

COROLLARY 5.3. If d* is a C-design (cf. Caliński (1971), Saha (1976)), then d*** is a $D^{-1}$-PEB(2) design for every $t(\geq 2)$, $p(\geq 1)$, where D, $\alpha_1$, $\alpha_2$, $A_1$, $A_2$ are as in Theorem 5(ii).

PROOF. Follows from Theorem 5(ii) noting that if d* is a C-design then the matrix C* has at most one positive eigenvalue different from r. □

COROLLARY 5.4. If d* is a connected regular GD design then d*** is a $D^{-1}$-PEB(2) design with D, $\alpha_1$, $\alpha_2$, $A_1$, $A_2$ as in Theorem 5(iii) provided either $pt = nk(\lambda_1-\lambda_2)/(r-\lambda_1)$ or $pt = nk(\lambda_2-\lambda_1)/(rk-\lambda_2 v)$, where n represents the number of treatments per group in d* and $\lambda_1$, $\lambda_2$ are the usual concurrence parameters in d*.

PROOF. This is an immediate consequence of Theorem 5(iii). □

REMARK. In many situations, given a regular GD design we can choose p,t $(t\geq 2)$ satisfying the conditions of Corollary 5.4. For example, consider a regular GD design with parameters $v = b = 6$, $r = k = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$, $m = 3$, $n = 2$ (R 42 in Clatworthy (1973)) for which $nk(\lambda_1-\lambda_2)/(r-\lambda_1) = 6$ and the first condition in Corollary 5.4 holds taking (p,t) = (1,6), (2,3) or (3,2). Similarly, the second condition in Corollary 5.4 holds with some $t(\geq 2)$ for every regular GD design satisfying $\lambda_2 = \lambda_1+1$ and $rk-\lambda_2 v = 1$ (recently, such regular designs have been characterized by Bhagwandas, Kageyama and Mukerjee (1986)).

3.3. The method of reinforcement. This is analogous to the method of supplementation. Let $d^*$ be as in subsection 3.2. Take $t(\geq 1)$ new treatments. Apply each of these new treatments $p(\geq 0)$ times in each block of $d^*$. Take $g(\geq 1)$ additional blocks. In each of these additional blocks apply each of the $v$ treatments in $d^*$ $u_1$ times and each of the new $t$ treatments $u_2$ times. Here $u_1 \geq 0$, $u_2 \geq 0$, $(u_1,u_2) \neq (0,0)$ (for otherwise, there is no new block at all) and $(p,u_1) \neq (0,0)$ (for otherwise, the resulting design is disconnected). The resulting design, say $\bar{d}$, will be called a design obtained through a reinforcement of $d_0$. The following result may be proved proceeding along the line of the proof of Theorem 4.

THEOREM 6. If $t = 1$ and $\alpha_1^* \neq \alpha_2^*$, then the design $\bar{d}$ is a $D^{-1}$-PEB(2) design with

$$D = \begin{bmatrix} I_v & \underline{0} \\ \underline{0}' & h \end{bmatrix}, \quad \alpha_i = \{(rp+k\alpha_i^*)/(k+p)\} + u_1 g, \quad i = 1,2,$$

$$A_1 = \begin{bmatrix} L_1L_1' & \underline{0} \\ \underline{0}' & 0 \end{bmatrix} + \frac{h}{v(v+h)} \begin{bmatrix} J_{vv} & -vh^{-1/2}\underline{1}_v \\ -vh^{-1/2}\underline{1}_v' & v^2h^{-1} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} L_2L_2' & \underline{0} \\ \underline{0}' & 0 \end{bmatrix},$$

where

$$h = v\left(\frac{rp}{k+p} + \frac{u_1u_2g}{u_1v+u_2}\right) \Big/ \left(\frac{k\alpha_1^*}{k+p} + \frac{vu_1^2g}{u_1v+u_2}\right).$$

REMARK. By Theorem 6, one can construct $D^{-1}$-PEB(2) designs, which are not necessarily equireplicate or proper, by reinforcement of connected two-associate PBIB designs in particular. In the set-up of Theorem 6, if $\alpha_1^* = \alpha_2^*$, the reinforced design $\bar{d}$ becomes a $D^{-1}$-PEB(1) design with $D$, $h$ and $\alpha_1$ as in the theorem.

THEOREM 7. Let $t \geq 2$ and $Q_1$, $Q_2$, $Q_3$, $S(h)$ be as in subsection 3.2. Then

(i) If $\alpha_1^* = \alpha_2^*$, then $\bar{d}$ is a $D^{-1}$-PEB(m) design, $m \leq 2$, with

$$D = \begin{bmatrix} I_v & 0 \\ 0 & hI_t \end{bmatrix}, \quad \alpha_1 = (rpt+k\alpha_1^*)/(k+pt) + u_1g, \quad \alpha_2 = h^{-1}(pb+u_2g),$$
$$A_1 = Q_1+Q_2+S(h), \quad A_2 = Q_3,$$

where $$h = v\left(\frac{rp}{k+pt} + \frac{u_1u_2g}{u_1v+u_2t}\right) \Big/ \left(\frac{k\alpha_1^*}{k+pt} + \frac{vu_1^2g}{u_1v+u_2t}\right).$$

Here if $\alpha_1 = \alpha_2$, then $\bar{d}$ is a $D^{-1}$-PEB(1) design and if $\alpha_1 \neq \alpha_2$, then $\bar{d}$ is a $D^{-1}$-PEB(2) design.

(ii) If $\alpha_1^* \neq \alpha_2^*$ but $r = \alpha_1^*$ and $ru_2 = pbu_1$, then $\bar{d}$ is a $D^{-1}$-PEB(2) design with

$$D = \begin{bmatrix} I_v & O \\ O & hI_t \end{bmatrix}, \quad \alpha_1 = r+u_1g,\ \alpha_2 = (rpt+k\alpha_2^*)/(k+pt) + u_1g, \quad A_1 = Q_1+Q_3+S(h),\quad A_2 = Q_2,$$

where $h = (pb+u_2g)/(r+u_1g)$.

PROOF. Clearly $\bar{d}$ is connected. Let $\tilde{C}$ be the C-matrix of $\bar{d}$. Then with $D = \begin{bmatrix} I_v & O \\ O & hI_t \end{bmatrix}$, $h > 0$, it may be seen, as in the proof of Theorem 5, that the positive eigenvalues of $D^{-1/2}\tilde{C}D^{-1/2}$ are given by, say,

$$\bar{\rho}_1 = (rpt+k\alpha_1^*)/(k+pt) + u_1g,\ \bar{\rho}_2 = (rpt+k\alpha_2^*)/(k+pt) + u_1g,$$
$$\bar{\rho}_3 = h^{-1}(pb+u_2g), \tag{5}$$
$$\bar{\rho}_4 = \{(rp/(k+pt)) + (u_1u_2g/(u_1v+u_2t))\}(t+h^{-1}v),$$

the corresponding orthonormal eigenvectors being given by the columns of $M_1$, $M_2$, $M_3$ and $\underline{s}(h)$, respectively. The rest of the proof is similar to that of Theorem 5. □

COROLLARY 7.1. If $d^*$ is a BIB design then, for every $t(\geq 2)$ and every $p$, $n$, $u_1$, $u_2$, $\bar{d}$ is a $D^{-1}$-PEB(m) design with $m \leq 2$, where $D$, $\alpha_1$, $\alpha_2$, $A_1$, $A_2$ are as in Theorem 7(i).

COROLLARY 7.2. If $d^*$ is a singular or semi-regular connected GD design then, for every $t(\geq 2)$, $\bar{d}$ is a $D^{-1}$-PEB(2) design, provided $ru_2 = pbu_1$, where $D$, $\alpha_1$, $\alpha_2$, $A_1$, $A_2$ are as in Theorem 7(ii).

COROLLARY 7.3. If $d^*$ is a C-design then, for every $t(\geq 2)$, $\bar{d}$ is a $D^{-1}$-PEB(2) design, provided $ru_2 = pbu_1$, where $D$, $\alpha_1$, $\alpha_2$, $A_1$, $A_2$ are as in Theorem 7(ii).

REMARK. In the present setting, it is difficult to have nice analogues of (iii) in Theorem 5, since the corresponding expressions become somewhat involved. However, for a given $d^*$, the relation (5) may be employed for a numerical determination of $p$, $t$, $u_1$, $u_2$, $g$ for which $\bar{d}$ becomes a $D^{-1}$-PEB(m) design ($m \leq 2$) for a suitably chosen $D$. This is illustrated in the following example where we consider the reinforcement of a regular GD design.

EXAMPLE. Let d* be a regular GD design with the usual parameters $v = 6$, $b = 18$, $r = 12$, $k = 4$, $\lambda_1 = 8$, $\lambda_2 = 7$, $m = 3$, $n = 2$. Then $\alpha_1^* = 11$, $\alpha_2^* = 21/2$ and there are many ways of selecting p, t, $u_1$, $u_2$, g for which h may be so chosen that among $\bar{\rho}_1$, $\bar{\rho}_2$, $\bar{\rho}_3$, $\bar{\rho}_4$ at most two are distinct. For example, if $p = 1$, $t = 8$, $u_1 = 0$, $u_2 = 1$, $g = 2$, then with $h = 12/7$, it may be seen from (5) that $\bar{\rho}_1 = \bar{\rho}_3$ ($= 35/3$) and $\bar{\rho}_2 = \bar{\rho}_4$ ($= 23/2$). Hence the resulting design is a $D^{-1}$-PEB(2) design with $D = \begin{bmatrix} I_v & 0 \\ 0 & hI_t \end{bmatrix}$, $\alpha_1 = 35/3$, $\alpha_2 = 23/2$, $A_1 = Q_1+Q_3$, $A_2 = Q_2+S(h)$, where $h = 12/7$.

3.4. Some other methods. In the preceding subsections we have described some general methods for the construction of $D^{-1}$-PEB(m) designs with $m \leq 2$. The resulting designs are not necessarily proper or equireplicate. In this subsection, we describe a few more methods which are less general but may have some utility in particular practical situations.

First we consider a method of generalized block complementation. With notations as in Section 1, let d be an $R^{-1}$-PEB(m) design with $m \leq 2$, where $R = \text{diag}\{r_1,\cdots,r_v\}$, the elements of $\underline{r} = (r_1,\cdots,r_v)'$ being the replication numbers in d (note that d is then only a PEB(m) design with $m \leq 2$). Let d have b blocks, a constant block size k and an incidence matrix N. Let $\hat{d}$ be a block design with incidence matrix

$$\hat{N} = f\,\underline{r}\underline{1}_b' - N, \tag{6}$$

where the constant f is so chosen that the elements of $\hat{N}$ are non-negative integers.

THEOREM 8. Let $\hat{d}$ be connected. Then it is a $D^{-1}$-PEB(m) design with $m \leq 2$, for $D = (fb-1)R$.

PROOF. By (6), it may be seen that the C-matrix of $\hat{d}$ is given by, say,

$$\hat{C} = (fb-1)R - \{(f^2b-2f)/(fb-1)\}k^{-1}\underline{r}\,\underline{r}' - (fb-1)^{-1}(R-C),$$

where C is the C-matrix of d. Hence with $D = (fb-1)R$,

$$D^{-1/2}\hat{C}D^{-1/2} = I - \{(f^2b-2f)/(fb-1)^2\}k^{-1}\underline{r}^{1/2}\underline{r}^{1/2\,\prime} - (fb-1)^{-2}(I - R^{-1/2}CR^{-1/2}), \tag{7}$$

where $\underline{r}^{1/2} = (r_1^{1/2},\cdots,r_v^{1/2})'$. Since d is an $R^{-1}$-PEB(m) design with $m \leq 2$, the matrix $R^{-1/2}CR^{-1/2}$ has at most two distinct positive eigenvalues the corresponding eigenvectors being orthogonal to $\underline{r}^{1/2}$ which corresponds to the eigenvalue 0 of $R^{-1/2}CR^{-1/2}$. Hence it follows from (7) that $\underline{r}^{1/2}$ is an eigenvector corresponding to the eigenvalue 0 of $D^{-1/2}\hat{C}D^{-1/2}$ and also that $D^{-1/2}\hat{C}D^{-1/2}$ can have at most two distinct positive eigenvalues. Hence the result follows. □

REMARK. Observe that the diagonal elements of $(fb-1)R$ are the replication numbers in $\hat{d}$. Hence $\hat{d}$ is in fact a PEB(m) design with $m \leq 2$. From (7), it is easy to obtain $\alpha_1$, $\alpha_2$, $A_1$, $A_2$ for $\hat{d}$ from those for d. In Theorem 8, it is necessary to assume the connectedness of $\hat{d}$ because examples may be given to demonstrate that the complement (in the sense of (6)) of an $R^{-1}$-PEB(m) design, $m \leq 2$, is not necessarily connected.

Next we consider a method of Kronecker product and obtain the following result.

THEOREM 9. Let N be the incidence matrix of an equireplicate C-design and $d_x$ be a design with incidence matrix $N \otimes \underline{1}_u$. Then (i) $d_x$ is a C-design and (ii) if, in addition, the design given by N is binary and proper, then every truncation of $d_x$ is a $D^{-1}$-PEB(m) design with $m \leq 2$, where $D = I$.

PROOF. Let v be the number of treatments, r be the common replication number and C be the C-matrix of the design given by N. Then the C-matrix of $d_x$ is given by, say,

$$C_x = r\, I_v \otimes (I_u - u^{-1}J_{uu}) + u^{-1}C \otimes J_{uu}.$$

By the definition of a C-design, the matrix C has at most one positive eigenvalue different from r, and hence it follows that the same holds for the matrix $C_x$. Furthermore, $d_x$ is also equireplicate with the common replication number r. This proves (i). The proof of (ii) follows along the line of the proof of Theorem 3. □

Although the above result is simple, it has interesting applications. In particular, it may be employed to construct PBIB designs, with more than two associate classes, which are $D^{-1}$-PEB(2) designs. For example, let N represent a connected singular or semi-regular GD design, which is a C-design. Now, if $d_x$ is formed as in Theorem 9, then $d_x$, or any truncation thereof, will be a $D^{-1}$-PEB(2) design with $D = I$. Note that in general, $d_x$ and its truncations are PBIB designs with three associate classes. Thus we get PBIB designs with more than two associate classes which are $D^{-1}$-PEB(2) designs with D = I.

Note that, as mentioned in Section 1, we restricted ourselves to the case $m \leq 2$ in $D^{-1}$-PEB(m) designs from a practical point of view. Some development on $D^{-1}$-PEB(m) designs for more large value of m may be made purely from a mathematical curiosity. However, this is not taken into account here.

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