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On the decomposition of Haar measure in compact groups

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0. Introduction. The behaviour of singularity under convolution has always been an interesting question. In particular, it may be asked whether the convolution of singular measures is necessarily singular. However, Salem ([5]) has constructed examples of singular measures whose iterates are absolutely continuous. In this paper we examine this question in another direction. The main theorem of this paper asserts that the Haar measure on any infinite compact abelian group can always be written as the convolution of two singular measures. It is also proved that in any non-discrete locally compact abelian group there are singular measures whose convolution is absolutely continuous.

1. Background. Throughout this paper, with the exception of the last section, we shall be dealing with compact abelian groups. For any locally compact abelian group G, we use the symbol λ_G to denote the Haar measure of G. If G is compact, λ_G is always normalized to have $\lambda_G(G) = 1$.

If G is any compact abelian group, let $\mathfrak{B}_0$ and $\mathfrak{B}$ denote the σ-field of Baire and Borel subsets respectively. If G is metric $\mathfrak{B}_0 = \mathfrak{B}$ and in any case $\mathfrak{B}_0$ is the smallest σ-field with respect to which all continuous functions are measurable. The term measure will be used to denote probability measures on $\mathfrak{B}_0$. Since every measure on $\mathfrak{B}_0$ has a regular unique extension to $\mathfrak{B}$, we may regard the measure as defined on $\mathfrak{B}$ itself and assume its regularity whenever it is necessary. We will have occasion to use the Riesz theorem. This asserts that if L is any non-negative linear functional on $C(G)$ (the space of continuous functions on G) with $L(1) = 1$, then there exists a unique maesure p such that $L(x) = \int_G x\,dp$ for all $x \in C(G)$.

A measure p on a compact group G is called *singular* if $p([x]) = 0$ for all $x \in G$ and if there exists a set A such that $p(A) = 1$ and $\lambda_G(A) = 0$. Singularity is a special case of orthogonality. Measures p and q on G are said to be *orthogonal* if there exists a set A with $p(A) = q(G-A) = 0$; in this case we write $p \perp q$. With this notation, p is singular if and only $p([x]) = 0$ for all $x \in G$ and if $p \perp \lambda_G$. A measure p is said to be *absolutely* *continuous*, $p \prec \lambda_G$, in symbols, if $p(A) = 0$ for every set A for which $\lambda(A) = 0$.

If p and q are two measures on G, their convolution, denoted by $p * q$, is the measure whose value for every set $A \in \mathfrak{B}_0$ is given by

$$(p * q)(A) = \int p(A - x)\, dq(x) .$$

If the group G is abelian, it follows that $p * q = q * p$. If H and H' are two compact groups, θ a homomorphism [1] of H into H' and p a measure on H, the measure $p\theta^{-1}$ on H' is defined by setting $p\theta^{-1}(A) = p[\theta^{-1}(A')]$ for every Baire set $A' \subset H'$. If θ is onto H' then $\lambda_H \theta^{-1} = \lambda_{H'}$. If p and q are any two measures on H, $(p * q)\theta^{-1} = P\theta^{-1} * q\theta^{-1}$.

2. Preliminary lemmas. Our method depends upon first factorizing Haar measure into singular measures on certain simple groups and then passing on to more complicated groups. To this end, several technical devices are utilized. We summarize, in this section, those that are essential for our purposes.

LEMMA 2.1 (Weil). *Let G be a compact group, T a subgroup and $H = G/T$. Then, for every measure p on H, there exists a unique measure $\hat{p}$ on G, satisfying the relation*

$$\int_G f\, d\hat{p} = \int_H dp \int_T f(x+t)\, d\lambda_T(t)$$

for every continuous function f on G.

Before proceeding to prove this proposition we make a few remarks on its meaning. If f is continuous on G, $\int_T f(x+t)\, d\lambda_T(t)$ is a continuous function on G (the variable being x) and moreover this function is constant on the cosets of T. It can therefore be regarded as a continuous function on H and can be integrated with respect to p.

Indeed, for any $f \in C(G)$ we define $L(f)$ as $\int_H dp \int_T f(x+t)\, d\lambda_T(t)$. L is a non-negative linear functional on $C(G)$ with $L(1) = 1$ and hence there exists a unique measure $\hat{p}$ such that $L(f) = \int_G f\, d\hat{p}$ for all $f \in C(G)$.

LEMMA 2.2. *Let the set up be as in the preceding lemma and let θ be the canonical homomorphism $G \to H$. Then, the correspondence $p \to \hat{p}$ has the properties:*

(i) $p = \hat{p}\theta^{-1}$;

(ii) $\lambda_G = \hat{\lambda}_H$;

(iii) $\widehat{(p * q)} = \hat{p} * \hat{q}$;

(iv) *if p is singular on H, $\hat{p}$ is singular on G;*

(v) *if p is absolutely continuous on H, $\hat{p}$ is absolutely continuous on G.*

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[1] By homomorphism we always understand continuous homomorphisms.

The proofs are straightforward and are omitted.

Before proceeding to the next lemma, we introduce a few definitions. Let G be a compact group and S_n a descending sequence of subgroups. We denote by H_n the quotient group G/S_n and by τ_n the canonical homomorphism $G \to G/S_n$. It is clear that $H_n = H_{n+1}/(S_n/S_{n+1})$ and hence H_n is a quotient group of H_{n+1}. We denote by θ_n the canonical homomorphism $H_{n+1} \xrightarrow{\theta_n} H_n$. A sequence $\{p_n\}$ of measures, with p_n defined on H_n, is called *consistent* if $p_n = p_{n+1}\theta_n^{-1}$ for all n. A consistent sequence $\{p_n\}$ is said to *extend to a measure p on G* if there exists a measure p on G such that $p_n = p\tau_n^{-1}$ for all n. Finally, a sequence of subgroups S_n is called *small* if S_n is decending and $\bigcap_n S_n = \{e\}$.

LEMMA 2.3. *Let G be a compact group, S_n a small sequence of subgroups and $\{p_n\}$ a consistent sequence of measures. Then, p_n extends to a uniquely defined measure p on G.*

Let A denote the set of all continuous functions which are constant on the cosets of at least one S_n. A is a subalgebra of $C(G)$. Since $\bigcap_n S_n = \{e\}$, it follows easily that A separates points of G. Hence, by the Stone-Weierstrass theorem A is dense in $C(G)$.

Let L_n denote the linear functional $\int \cdot dp_n$ on $C(H_n)$. If $f \in A$, then f can be regarded as a continuous function on some H_n. More precisely, for each $f \in A$, there exists an integer n and a $g \in C(H_n)$ such that $f = g \circ \theta_n$. Define $L(f) = L_n(g)$. The consistency of p_n implies that L is well defined on A. It is further non-negative and hence bounded on A. It can therefore be extended (since A is dense in $C(G)$) as a unique bounded linear functional on $C(G)$. Further this extension is obviously non-negative and hence there exists a unique measure p on G such that $L(f) = \int f dp$ for all $f \in C(G)$. It is clear that p is the unique measure to which the p_n extend.

LEMMA 2.4. *With notations as in lemma 2.3, p is the Haar measure on G if and only if p_n is the Haar measure on H_n for each n.*

The only if part is trivial. As to the if part, note that A is an invariant collection of functions and that L on A is an invariant linear functional if L_n on $C(H_n)$ is invariant for each n. It is then clear that the (unique) extension of L to $C(G)$ is also invariant and hence the corresponding measure is the Haar measure on G.

LEMMA 2.5. *With notations as in lemma 2.3, suppose that*

(1) $\sup_{x \in H_n} p_n([x]) \to 0$ *as* $n \to \infty$,

(2) *there exists a sequence C_n of sets, $C_n \subset H_n$, such that $p_n(C_n) = 1$ for all n while $\lambda_{H_n}(C_n) \to 0$.*

Then, p is singular.

By definition of p, $p_n = p\tau_n^{-1}$ for all n and hence for any $x \in G$, $p([x]) \leqslant p_n[\tau_n(x)] \leqslant \sup_{y \in H_n} p_n([y]) \to 0$ as $n \to \infty$, so that $p([x]) = 0$. Further if $\tilde{C}_n = \tau_n^{-1}(C_n)$, and $C = \bigcap_n \tilde{C}_n$, $p(C) = 1$ since $p(\tilde{C}_n) = 1$ for all n, while $\lambda_G(C) = 0$ since $\lambda_G(C) \leqslant \lambda_G(\tilde{C}_n) = \lambda_{H_n}(C_n) \to 0$. This proves that p is singular.

3. Special groups. We shall now prove that Haar measure can be written as the convolution of two singular measures for some special groups G. We use the symbol K ambiguously to denote either the multiplicative group of complex numbers of modulus unity or the additive group of reals t with $0 \leqslant t < 1$ with addition carried out modulo 1.

THEOREM 3.1. *There exist singular measures μ and ν on K such that $\mu * \nu = \lambda_K$.*

Let H be the additive group of integers $0, 1, 2$ and 3, with group addition carried out modulo 4. Let H_0 be the infinite direct product of H with itself. Under the product topology, H_0 is a compact group and λ_{H_0} is the infinite direct product of λ_H on H. Let μ_H and ν_H be measures on H which assign masses $\frac{1}{2}$ each to the points 0, 1 and 0, 2. It is easily verified that $\lambda_H = \mu_H * \nu_H$. Let μ_{H_0} and ν_{H_0} be the infinite direct products of μ_H and ν_H respectively. It is clear that $\lambda_{H_0} = \mu_{H_0} * \nu_{H_0}$ and that λ_{H_0}, μ_{H_0} and ν_{H_0} all vanish for single point sets. Let $C = \{0, 1\}$ and $D = \{0, 2\}$. If we define $C_0 = \prod C$ and $D_0 = \prod D$, then $\mu_{H_0}(C_0) = \nu_{H_0}(D_0) = 1$ and $\lambda_{H_0}(C_0) = \lambda_{H_0}(D_0) = \prod \frac{1}{2} = 0$. This shows that λ_{H_0} is orthogonal to both μ_{H_0} and ν_{H_0}.

Consider now the map θ of H_0 into $[0, 1]$ which sends the vector $(h_1, h_2, \ldots)$ of H_0 into $\sum h_n \cdot 4^{-n}$. Even though this is *not* a homomorphism of H_0 into K, it is clear that for $x \in C_0$ and $y \in D_0$, $\theta(x+y) = \theta(x) \oplus \theta(y)$ where $\oplus$ denotes addition in K. If therefore $\mu = \mu_{H_0}\theta^{-1}$ and $\nu = \nu_{H_0}\theta^{-1}$ then $\mu * \nu = \lambda_{H_0}\theta^{-1}$. Now it can be proved by elementary arguments that $\lambda_{H_0}\theta^{-1}$ is Lebesgue measure on $[0, 1]$ and hence it follows that $\mu * \nu = \lambda_K$.

Since θ is one-one except at a countable set of points of H_0 (over which λ_{H_0}, μ_{H_0} and ν_{H_0} all vanish), it follows that μ and ν are both orthogonal to λ_K and vanish for all single point sets. In other words it follows that μ and ν are singular. This proves the theorem.

The above construction gives at the same time singular measures on the additive group of reals whose convolution is absolutely continuous.

THEOREM 3.2. *There are singular measures μ and ν on the real line R whose convolution is absolutely continuous.*

With the same notation as in the proof of the preceding theorem, we observe that $\mu = \mu_{H_0}\theta^{-1}$ is concentrated in $A = [0, \frac{1}{3}]$ and $\nu = \nu_{H_0}\theta^{-1}$ is concentrated in $B = [0, \frac{2}{3}]$. But then for $x \in A$ and $y \in B$, $x + y = x \oplus y$ where $+$ denotes ordinary addition and $\oplus$ denotes addition in K. This shows that $\mu * \nu$ is Lebesgue measure in $[0, 1]$, $*$ denoting convolution of measures on the additive group R. This proves the theorem.

The circle group is the simplest of compact connected groups. At the other extreme we shall now examine certain compact totally disconnected groups. It is well known that a compact group G is totally disconnected if and only if its character group is a torsion group ([4]). In a certain sense, the simplest infinite torsion group is the group $Z(p^\infty)$ of all numbers m/p^n, p being a prime, the group operation being addition carried out modulo 1. In the rest of the section, X denotes the discrete group $Z(p^\infty)$ and G the (compact) character group of X. For any integer n, X_n will denote the subgroup of X consisting of all numbers of the form m/p^n. $X_1 \subset X_2 \subset \ldots$ and these are all the subgroups of X. Another description of X_n is obtained by regarding it as the additive group of integers $0, 1, 2, \ldots, p^n - 1$, addition carried out modulo p^n.

The Pontrjagin duality theory enables us to view G and X in a perfectly symmetric manner. To this end we introduce the function $(.\,,.)$ such that for fixed $x \in X$, $(.\,, x)$ is the character on G represented by x and for fixed $g \in G$, $(g\,,.)$ is the character on X represented by g. We define T_n as the annihilator of X_n, i. e. $T_n = \{g\colon (g, x) = 1 \text{ for all } x \in X_n\}$. T_n is a subgroup of G and $T_{n+1} \subset T_n$. Since $X_n \uparrow X$, $\bigcap_n T_n = \{e\}$ so that T_n is a small sequence. Further, from duality theory it follows that G/T_n and X_n are character groups of each other. X_n being a finite group, its character group is isomorphic to itself and hence G/T_n is isomorphic to X_n.

THEOREM 3.3. *There are singular measures p and q on G such that $\lambda_G = p * q$.*

We write $S_n = T_{2^n}$ and $H_n = G/S_n$. We denote by τ_n and θ_n respectively the canonical homorphisms $G \to G/S_n$ and $H_{n+1} \to H_n$. Our method of proof consists in building up suitable consistent sequences of measures on the H_n. H_n is isomorphic to X_{2^n} as we noted above.

We first observe that every element of H_n can be written as $\sum_{i=0}^{2^n-1} r_i p^i$ with $0 \leqslant r \leqslant p-1$ for all i. This representation moreover is unique. Let C_n be the set of all points in whose representation $r_1 = r_3 = \ldots = r_{2^n-1} = 0$ and let D_n be the set of points in whose representation $r_0 = r_2 = \ldots = r_{2^n-2} = 0$. C_n and D_n each contain exactly $p^{2^{n-1}}$ points, $C_n \cap D_n = [0]$, and $C_n + D_n = H_n$. Let p_n be the measure with masses $1/p^{2^{n-1}}$ at the points of C_n (with zero masses at others) and q_n the measure defined likewise over D_n. It is easy to verify that $p_n * q_n = \lambda_{H_n}$.

We shall now verify that p_n and q_n are consistent sequences. Choose and fix the integer n and consider the groups H_{n+1} and H_n. From the special nature of the groups, it follows easily that the kernel of the homomorphism $\theta_n\colon H_{n+1} \to H_n$ consists of the p^{2^n} points $0, p^{2^n}, 2p^{2^n}, \ldots, (p^{2^n} - 1)p^{2^n}$

If $x \in H_{n+1}$ has the representation $\sum_{i=0}^{2^{n+1}-1} r_i p^i$, it is then obvious that $\theta_n(x)$ has the representation (in H_n) $\sum_{i=0}^{2^n-1} r_i p^i$. Consequently C_{n+1} is mapped into C_n and D_{n+1} mapped into D_n. It is then not difficult to verify that $p_n = p_{n+1}\theta_n^{-1}$ and $q_n = q_{n+1}\theta_n^{-1}$.

The consistency of the sequences p_n and q_n implies that they extend to measures p and q on G. Since λ_{H_n} extends to λ_G and since $p_n * q_n = \lambda_{H_n}$ for all n, it follows that $p * q = \lambda_G$. Since

$$\sup_{x \in H_n} p_n([x]) = \sup_{x \in H_n} q_n([x]) = \frac{1}{p^{2^n}} \to 0$$

and since, by lemma 2.4,

$$\lambda_{H_n}(C_n) = \lambda_{H_n}(D_n) = \frac{1}{p^{2^{n-1}}} \to 0\,,$$

lemma 2.5 implies that p and q are singular. This completes the proof of the theorem.

4. The main theorem. In this section we shall prove the main theorem of this paper. Before proceeding to its actual proof, it is convenient to obtain a few preliminary propositions which clarify the relation between the general compact abelian group and the special groups considered in the preceding section.

For the group-theoretic terms employed in the following lemmas, such as divisible group, reduced group etc., we refer to [2]. Two sub-groups of a group are said to be independent when identity is their only common element.

LEMMA 4.1. *If X is an infinite torsion group which is reduced, then we can find two infinite sub-groups of X which are independent.*

X is a direct sum of primary groups. If the number of terms is infinite, the assertion is obvious. Otherwise one term at least is infinite. Since X is reduced, so are its sub-groups and hence we may (and do) suppose that X is a primary group which is reduced. We can then assert the existence of a cyclic group C_1 which is a direct summand of X ([2]). We may thus write $X = C_1 \oplus D_1$ where D_1 is infinite, primary and reduced. The argument applies to D_1 also and hence we can write $D_1 = C_2 \oplus D_2$ where D_2 is infinite, primary and reduced. Since X is infinite, this procedure can go on ad infinitum. Let X_0 be the sub-group generated by $C_1, C_3, \ldots, C_{2k+1}, \ldots$ and X_1 the sub-group generated by $C_2, C_4, \ldots, C_{2k}, \ldots$ Then X_0 and X_1 are independent infinite sub-groups of X.

LEMMA 4.2. *An infinite torsion group X can be represented as $Z(p^\infty) \oplus F$, with F a finite group, if and only if it does not have two independent infinite sub-groups.*

If $X = Z(p^\infty) \oplus F$, it is clear that every infinite sub-group of X must contain $Z(p^\infty)$ and hence X does not possess two infinite, independent sub-groups. Conversely let X be a torsion group not possessing two infinite independent sub-groups. Since X is infinite, X cannot be reduced (Lemma 4.1). Let D be the maximal divisible sub-group of X. We can then write X as $D \oplus F$ where F is reduced. Lemma 4.1 once again applies to prove that F is finite. Now D is a divisible torsion group and is hence a direct sum of $Z(p^\infty)$'s ([2]). The non-existence of infinite independent sub-groups of X then implies that D must be $Z(p^\infty)$ for some prime p. This proves the lemma.

LEMMA 4.3. *An infinite compact abelian group G necessarily satisfies one of the following relations:*

(1) *It has a non-trivial component of the identity or equivalently there exists a sub-group T such that G/T is the circle group K.*

(2) *It is totally disconnected and has a sub-group T such that G/T can be written as $H_1 \oplus H_2$ where H_1 and H_2 are infinite compact groups.*

(3) *It is representable as $G_0 \oplus F$ where F is finite and G_0 is the character group of $Z(p^\infty)$.*

Let X be the character group of G and let T be a sub-group of G and X_0 the annihilator of T in X. Then G/T and X_0 are character group of each other. Consequently G/T is isomorphic to K if and only if X_0 is isomorphic to the integer group, i. e. if and only if X_0 is the cyclic group generated by an element x_0 of infinite order. Now it is well known that X contains elements of infinite order if and only if G has a non-trivial component of the identity. This shows that group with non-trivial components of identity are precisely those which admit K as a factor group.

If G is totally disconnected, X is a torsion group. Then lemmas 4.1 and 4.2 imply that either X has two infinite independent sub-groups or X is of the form $Z(p^\infty) \oplus F$ where F is finite. If X has infinite sub-groups X_1 and X_2 which are independent, X' is the group $X_1 \oplus X_2$ and T the annihilator of X' in G, it follows that G/T has $X_1 \oplus X_2$ for its character group and hence decomposes as $H_1 \oplus H_2$ where H_1 and H_2 are respectively the infinite compact groups which are character groups of X_1 and X_2.

Finally, if X is of the form $Z(p^\infty) \oplus F$, G evidently satisfies relation (3). This proves the lemma.

THEOREM 4.1. *If G is any infinite compact abelian group, there are singular measures p and q such that $\lambda_G = p * q$.*

Suppose that G has a sub-group T such that G/T is isomorphic to K. By theorem 3.1 there are singular measures μ and ν on K such that $\lambda_K = \mu * \nu$. Construct now measures $\hat{\mu}$ and $\hat{\nu}$ on G via the method described in lemmas 2.1 and 2.2. If $p = \hat{\mu}$ and $q = \hat{\nu}$, then it follows from these lemmas the p and q are singular and $p * q = \lambda_G$.

Suppose now that G has a sub-group T such that G/T can be written as $H_1 \oplus H_2$ where H_1 and H_2 are infinite compact groups. Let λ_1 and λ_2 be respectively the Haar measures on H_1 and H_2 considered as measures on $H_1 \oplus H_2 = H$. Since H_1 and H_2 are infinite, $\lambda_H(H_1) = \lambda_H(H_2) = 0$ so that λ_1 and λ_2 are singular. It can further be proved easily that $\lambda_H = \lambda_1 * \lambda_2$. If we now define $p = \hat{\lambda}_1$ and $q = \hat{\lambda}_2$ (vide lemmas 2 1 and 2 2), it is clear that p and q are singular and $\lambda_G = p * q$.

There remains, in view of lemma 4.3 only the case when G can be written as $G_0 \oplus F$ where F is finite and G_0 the character group of $Z(p^\infty)$. By virtue of the construction envisaged in lemma 2.2 and 2.2 it suffices (theorem 3.3) the theorem for G_0. This however has already been done to prove so that the proof of the theorem is complete.

5. Locally compact groups. We shall, in this section, make a few remarks concerning locally compact abelian groups. We prove a simple preliminary proposition.

LEMMA 5.1. *Let G be a locally compact abelian group and H an open Baire sub-group. If there are measures p and q on H such that* (i) *p and q are singular on H,* (ii) *$p * q$ is absolutely continuous on H, then the same property holds for G also.*

Indeed, define p' and q' on G by putting p and q to be zero outside H. Since $p' * q' \prec \lambda_H$ and since $\lambda_H \prec \lambda_G$, it is obvious that $p' * q' \prec \lambda_G$.

THEOREM 5.1. *If G is any non-discrete locally compact abelian group, there exist singular measures p and q on G such that $p * q \prec \lambda$.*

It is a well-known theorem of Pontrjagin that G has an open Baire sub-group ([2]) H which can be written in the form $C \oplus V$ where C is compact and V a vector group. In view of lemma 5.1 it suffices to assume that G itself can be written as $C \oplus V$ and this we shall do.

If C is finite, then G being non-discrete, the vector component must be present and hence V is an *open* vector sub-group of C, say of dimension m. In view of lemma 5.1 again it suffices to construct measures on V. Now V can be written as $\oplus R_i$ where the direct sum is m-fold and R_i is the real line for each i. By theorem 3.2, there are singular measures p_i and q_i on R_i such that $p_i * q_i$ is absolutely continuous on R_i. If p and q are the direct products of p_i and q_i respectively, it is obvious that p and q are singular and $p * q$ is absolutely continuous.

If C is infinite theorem 4.1 disposes of the case when V is absent. On the other hand, if V is present, the above construction together with

([2]) For instance the sub-group generated by a compact neighbourhood of the identity is one such.

theorem 4.1 yields singular measures on both C and V whose convolutions are absolutely continuous. By taking direct products we then have singular measures p and q on $G = C \oplus V$ such that $p * q < \lambda_G$. This completes the proof of the theorem.

References

[1] Paul R. Halmos, *Measure theory*, New York, 1950.

[2] Irving Kaplansky, *Infinite abelian groups*, Michigan University Press 1953.

[3] L. H. Loomis, *Introduction to abstract harmonic analysis*, New York 1955.

[4] L. S. Pontrjagin, *Topological groups*, Princeton 1946.

[5] R. Salem, *On sets of multiplicity for trigonometric series*, Amer. Jour. Math. 64 (1942), pp. 531-538.

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