ESTIMABILITY-CONSISTENCY AND ITS EQUIVALENCE WITH REGULARITY IN FACTORIAL EXPERIMENTS

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Abstract. The notion of estimability-consistency in factorial experiments is introduced and its equivalence with the concept of regularity has been proved. A connexion between partial estimability-consistency and partial regularity has also been indicated.

1. Introduction and preliminaries.

The notion of efficiency-consistency in factorial designs, as introduced by Lewis and Dean (1985), has received considerable attention in recent years. Lewis and Dean (1985) established that orthogonal factorial structure implies efficiency-consistency while Mukerjee and Dean (1986) proved the converse and derived some further results. Some additional results were obtained by Gupta (1986). This paper considers an analogous concept, namely, that of efficiency-consistency and proves its equivalence with the concept of regularity (Mukerjee (1979), Chauhan and Dean (1986)) in factorial designs. The result may be of interest from a practical viewpoint in the sense that it provides a simple and intuitively appealing interpretation for the somewhat abstract phenomenon of regularity.

Throughout the paper, the fixed effects intrablock model with independent homoscedastic errors and no block-treatment interaction is assumed. Consider a possibly disconnected $m_1 \times m_2 \times \ldots \times m_n$ factorial block design, d, whose n-digit treatment labels represent the treatment combinations. Neither the block sizes nor the replication numbers in d are assumed to be equal. Let T be the set of non-null binary vectors $x = x_1 x_2 \ldots x_n$ ($x_i = 0$ or 1; $i = 1, \ldots, n$) and a^x denote the interaction (by 'interaction' we mean either a main effect or an interaction) among those factors for which $x_i = 1$, $i = 1, \ldots, n$. Let A be the intrablock matrix of d and V^x be the estimable space corresponding to a^x. The row space of any matrix B will be denoted by $R(B)$.

Definition 1.1: (Mukerjee, 1979) An n-factor design, d, is regular provided $R(A) \equiv \oplus V^x$, where $\oplus$ denotes direct sum over $x \in T$.

While a connected design is always regular, the same cannot be said about disconnected designs. As seen in Mukerjee (1979), in irregular disconnected designs estimable contrasts belonging to interactions do not span the space of all estimable contrasts and, therefore, such designs are wasteful in the sense that they achieve estimability of unimportant contrasts (that is, contrasts belonging to none of the interactions) at the cost of the important ones. Thus the concept of regularity plays a crucial role in dealing with disconnected factorial designs. For further discussion on the practical relevance of regularity with examples, see Mukerjee (1979).

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2. Estimability-consistency.

The notations in this section are essentially along the line of Dean and Lewis (1983) and Lewis and Dean (1985). Let $v = m_1 m_2 \ldots m_n$ and $\underline{\tau}$ be the $v \times 1$ vector of factorial treatment effects in d arranged lexicographically. For $i = 1, \ldots, n$, let $\underline{1}_i$ be the $m_i \times 1$ vector $(1, 1, \ldots \beta, 1)'$, I_i be the $m_i \times m_i$ identity matrix and $J_i = \underline{1}_i \underline{1}'_i$. For $x \in T$, define

$$W^x = W_1^{x_1} \otimes W_2^{x_2} \otimes \ldots \otimes W_n^{x_n}, \quad S^x = S_1^{x_1} \otimes S_2^{x_2} \otimes \ldots \otimes S_n^{x_n}, \tag{2.1}$$

where $\otimes$ denotes Kronecker product and for $i = 1, \ldots, n$,

$$\left.\begin{aligned} W_i^{x_i} &= I_i - m_i^{-1} J_i \ \text{ if } x_i = 1 \\ &= m_i^{-1} J_i \qquad \text{ if } x_i = 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} S_i^{x_i} &= I_i \ \text{ if } x_i = 1 \\ &= \underline{1}'_i \ \text{ if } x_i = 0 \end{aligned}\right\}.$$

Note that the rows of W^x span the space of all contrasts, not necessarily estimable, belonging to a^x in d.

For $x \in T$, let d_x be the design obtained from d by deleting the ith digit from treatment labels for all i such that $x_i = 0$, $i = 1, \ldots, n$. Let $v_x = \Pi m_i^{x_i}$ and let $\underline{\tau}_x$ denote the vector of v_x factorial treatment effects in d_x. Note that the rows of $W^x S^{x'}$ span the space of all contrasts belonging to a^x in d_x. For each $x \in T$, one can define a correspondence between the contrasts belonging to a^x in d and d_x such that for $\underline{c} \neq \underline{0}$ the contrast $\underline{c}' W^x \underline{\tau}$ belonging to a^x in d corresponds to the contrast $\underline{c}' W^x S^{x'} \underline{\tau}_x$ belonging to a^x in d_x and conversely.

Lemma 2.1. *Let $\underline{c}' W^x \underline{\tau}$ be a contrast belonging to a^x in d. Then the estimability of $\underline{c}' W^x \underline{\tau}$ in d implies the estimability of the corresponding contrast $\underline{c}' W^x S^{x'} \underline{\tau}_x$ in d_x.*

The proof of Lemma 2.1 is not hard and hence omitted here. Examples, however, show that the converse of Lemma 2.1 is not generally true. Thus let d be a disconnected 2^3 design in two blocks $\{000, 001, 010, 100\}, \{110, 101, 011, 111\}$. Then the contrast representing the main effect of the first factor is not estimable in d although the corresponding contrast is easily seen to be estimable in d_{100}. A factorial design where the converse of Lemma 2.1 also holds will be called estimability-consistent. More formally, we have the following definition:

Definition 2.1: An n-factor design, d, is estimability-consistent provided for each $x \in T$, every contrast belonging to a^x in d is estimable in d if and only if the corresponding contrast belonging to a^x in d_x is estimable in d_x.

We are now in a position to state the main result of this paper which has been proved in the next section.

Theorem 2.1. *An n-factor design, d, is estimability-consistent if and only if it is regular.*

Clearly, the notion of estimability-consistency is much simpler than that of regularity. Hence Theorem 2.1 gives a simple interpretation for the rather involved concept of regularity in terms of estimability-consistency .

3. Proof of Theorem 2.1.

Lemma 3.1. *An n-factor design, d, is regular if and only if*

$$R(AW^x) \subseteq R(A), \quad \forall x \in T, \tag{3.1}$$

as usual A being the intrablock matrix of the design.

Proof. Only if: Suppose d is regular. For each $x \in T$, since $V^x \subseteq R(W^x)$, there exists a matrix L_x such that $V^x = R(L_x W^x)$. Also $V^x \subseteq R(A)$, so that

$$R(L_x W^x) \subseteq R(A), \quad \forall x \in T. \tag{3.2}$$

Since the design is regular, by Definition 1.1, there exist matrices $H_x (x \in T)$ such that $A = \sum_{x \in T} H_x L_x W^x$. Observing that $W^x W^y = W^x$ if $x = y$, $= 0$ if $x \neq y$, it follows that $AW^x = H_x L_x W^x$. Hence $R(AW^x) \subseteq R(L_x W^x)$ and by (3.2), the 'only if' part of the lemma follows.

If: Let (3.1) hold. Then obviously,

$$R(AW^x) \subseteq R(W^x) \cap R(A) = V^x, \tag{3.3}$$

by the definition of V^x. Let I be the $v \times v$ identity matrix and J be the $v \times v$ matrix with all elements unity. Since $\sum_{x \in T} W^x = I - v^{-1} J$, and $AJ = 0$, one obtains $A = \sum_{x \in T} AW^x$, which, together with (3.3) implies that $R(A) \subseteq \oplus V^x$, where as before $\oplus$ denotes direct sum over $x \in T$. But from Mukerjee (1979), for every design $R(A) \supseteq \oplus V^x$. Therefore, $R(A) \equiv \oplus V^x$ and the 'if' part of the lemma follows.

Lemma 3.2. *An n-factor design, d, is regular if and only if* $R(AZ^x) \subseteq R(A)$, $\forall x \in T$, *where* $Z^x = S^{x'} S^x$.

Proof: This follows from Lemma 3.1, observing that for each x, Z^x is a linear combination of J and W^y for $y \in T$ and that for each x, W^x is a linear combination of J and Z^y for $y \in T$.

Proof of Theorem 2.1. Sufficiency: Let the design be regular. Then by Lemma 3.2,

$$R(AZ^x) \subseteq R(A), \quad \forall x \in T. \tag{3.4}$$

To show that d is estimability-consistent consider the subdesign d_x for any $x \in T$. In view of Lemma 2.1, it will be enough to show that the estimability of the contrast $\underline{c}'W^x S^{x'} \underline{\tau}_x$ ($\underline{c} \neq \underline{0}$) belonging to α^x in d_x implies the estimability of the corresponding contrast $\underline{c}'W^x \underline{\tau}$ in d. Since the intrablock matrix of d_x is given by $S^x A S^{x'}$, the estimability of $\underline{c}'W^x S^{x'} \underline{\tau}_x$ in d_x implies that

$$\underline{c}'W^x S^{x'} = \underline{u}' S^x A S^{x'},$$

for some vector $\underline{u}$. Postmultiplying both sides by S^x and noting that by (2.1), $W^x S^{x'} S^x = (v/v_x) W^x$, one obtains $\underline{c}'W^x = (v_x/v)\underline{u}' \; S^x A Z^x$. Hence $\underline{c}'W^x \in R(AZ^x)$. Therefore by (3.4), $\underline{c}'W^x \underline{\tau}$ is estimable in d and the sufficiency is proven.

Necessity: This will be proved by induction along the line of Mukerjee and Dean (1986). Let the design be estimability-consistent. For $u = 1, \ldots, n$, define $T_u = \{x : x \in T, x$ contains exactly u unit digits$\}$.

Take any $x \in T_1$. From (2.1), it is then easy to see that $Z^x = q_x W^x + p_x J$, where q_x and p_x are non-zero constants. Since $AJ = 0$, one gets

$$AZ^x = q_x A W^x. \tag{3.5}$$

Postmultiplying both sides by $S^{x'}$ and noting that

$$Z^x S^{x'} = (v/v_x) S^{x'}, \tag{3.6}$$

it follows that $(v/v_x) A S^{x'} = q_x A W^x S^{x'}$. Since $q_x \neq 0$ and A is non-negative definite, this implies that $R(AW^x S^{x'}) \equiv R(AS^{x'}) \equiv R(S^x A S^{x'})$. Hence recalling that the design is estimability-consistent and that $S^x A S^{x'}$ is the intrablock matrix of d_x, one obtains $R(AW^x) \subseteq R(A)$ so that by (3.5), $R(AZ^x) \subseteq R(A)$, this being true for all $x \in T_1$.

To apply the method of induction, suppose $R(AZ^x) \subseteq R(A)$, $\forall x \in T_1 \cup T_2 \cup \ldots \cup T_g$ ($1 \leq g < n$) and consider $x \in T_{g+1}$. Defining $T(x) = \{y : y \in T, y \neq x, y_i \leq x_i, i = 1, \ldots, n\}$, it may be seen from (2.1) that

$$Z^x = f_x W^x + \sum_{y \in T(x)} f_y Z^y + k_x J,$$

where $f_x (\neq 0)$, f_y ($y \in T(x)$), k_x are constants. Hence analogously to (3.5),

$$AZ^x = f_x A W^x + \sum_{y \in T(x)} f_y A Z^y. \tag{3.7}$$

Clearly, $T(x) \subset T_1 \cup T_2 \cup \ldots \cup T_g$ and hence by induction hypothesis, $R(AZ^y) \subseteq R(A)$, $\forall y \in T(x)$. Hence there exist matrices G_y such that

$$f_y A Z^y = G_y A, \quad \forall y \in T(x). \tag{3.8}$$

Postmultiplying both sides of (3.7) by $S^{x'}$ and using (3.6), (3.8), $f_x AW^x S^{x'} = (v/v_x)AS^{x'} - \sum_{y \in T(x)} G_y AS^{x'} = \{(v/v_x)I - \sum_{y \in T(x)} G_y\}AS^{x'}$. Since $f_x \neq 0$, the above yields $R(AW^x S^{x'}) \subseteq R(AS^{x'})$. But A is non-negative definite and hence of the form $A = KK'$ for some matrix K.

Consequently, $R(AS^{x'}) \equiv R(S^x AS^{x'})$ and it follows that

$$R(AW^x S^{x'}) \subseteq R(S^x AS^{x'}).$$

Hence recalling that the design is estimability-consistent, it follows that $R(AW^x) \subseteq R(A)$, which, together with (3.7), (3.8), implies that $R(AZ^x) \subseteq R(A)$, this being true for all $x \in T_{g+1}$. Thus by induction, $R(AZ^x) \subseteq R(A)$, $\forall x \in T$, and by Lemma 3.2 the design is regular.

4. Partial estimability-consistency.

Mukerjee and Dean (1986) proved certain equivalence theorems connecting partial efficiency-consistency and partial orthogonal factorial structure. The analogues of some of their results can be proved in the present context.

Definition 4.1: An n-factor design, d, is partially estimability-consistent of order t provided for every $x \in T_1 \cup T_2 \cup \ldots \cup T_t$, each contrast belonging to a^x in d is estimable in d if and only if the corresponding contrast belonging to a^x in d_x is estimable in d_x.

The following may be proved along the line of Theorem 2.1.

Theorem 4.1. *An n-factor design, d, is partially estimability-consistent of order t ($\leq n$) if and only if it is regular of order t.*

In the above, the definition of regularity of order t is as in Mukerjee (1980) where some discussion on the implications of such regularity is also available. It is hard to carry out the equivalence relations as in Theorem 2.1 and Theorem 4.1 any further. For example, one may define partial estimability-consistency and partial regularity with respect to individual interactions (along the line of Chauhan and Dean (1986) who considered orthogonal factorial structure with respect to individual interactions), but it is believed that they will no more be equivalent.

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