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CARDINALITIES OF BANACH SPACES

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1. Introduction. In the theory of Boolean algebras, the following result is well known. A complete Boolean algebra of a given infinite cardinal number m exists if and only if $m = m^{\aleph_0}$. In this note a similar theorem is proved.

THEOREM. *A Banach space of a given cardinal number m exists if and only if* $m = m^{\aleph_0}$. In this note, we assume all Banach spaces to be nontrivial.

2. LEMMA 1. *Let B be a Banach space. Given* $0 < \epsilon < 1$, *there exists a set* $A \subset B$ *with the following properties*:

(i) *Closed linear manifold spanned by A*, $\overline{\text{sp}}(A) = B$.

(ii) x *in A implies* $\|x\| = 1$ *and* x, y *in A implies* $\|x - y\| > \epsilon$.

PROOF. Apply Zorn's lemma for the collection of sets satisfying (ii) and then use Lemma 3 of ([1, pp. 578]).

LEMMA 2. *If B is a Banach space, then its cardinal number*, $|B| = m^{\aleph_0}$ *for some cardinal number m.*

PROOF. Let $(2/3) < \epsilon < 1$. Let A be a subset of B given by Lemma 1. *Case (i)* $|A| = m \geqslant c$, where c is the cardinality of the continuum. Obviously, $|B| \leqslant m^{\aleph_0}$. Since for every infinite cardinal number m, $m\aleph_0 = m$, we can write $A = A_1 \cup A_2 \cup \ldots$ such that A_i's are disjoint and $|A_i| = m$ for every i.

Let $B_i = \frac{1}{2^{2i}} A_i$. If x_i belongs to B_i, then $\sum_i x_i$ is convergent. Further, if x_i, y_i belong to B_i and $\sum_i x_i = \sum_i y_i$, then $x_i = y_i$ for every i. For, if n is the first natural number for which $x_n \neq y_n$, then $\frac{\epsilon}{2^{2n}} < \| x_n - y_n \| =$

$\| \sum_{i \geqslant n+1} x_i - \sum_{i \geqslant n+1} y_i \| \leqslant (2/2^{2n+2})\,(4/3)$. This implies $\epsilon < \frac{2}{3}$ giving rise to a contradiction. Consequently, $|B| \geqslant m^{\aleph_0}$.

Case (ii) $|A| < c$. Let D be the linear manifold spanned by A. Since $\bar{D} = B$ and every point in B is a limit of a sequence of elements from D, we have

$c \leqslant |B| \leqslant |D|^{\aleph_0} =$ number of sequences from $D = |A|^{\aleph_0} \leqslant c^{\aleph_0} = c$.

$|B| \geqslant c$ follows from the fact that every Banach space contains a copy of the real line. Thus we have $|B| = c^{\aleph_0} = c$.

Lemma 3. *If m is a cardinal number such that $m = m^{\aleph_0}$, then there exists a Banach space B with the property $|B| = m$*

Proof. Let H be any Hilbert space of dimension m. Applying a technique used in the proof of Lemma 2 to an orthonormal basis of H, we can conclude that $|H| = m^{\aleph_0} = m$.

Theorem. *A Banach space of a given cardinal number m exists if and only if $m = m^{\aleph_0}$.*

3. Remarks. 1. The proof of the analogous theorem for Hilbert spaces is quite simple if one works with its orthonormal basis.

2. From the theorem, we conclude if $m = c + 2^c + 2^{2^c} + \ldots,$ then there is no Banach space of cardinal number m. For, by König's theorem (see [2, pp. 45]), we can easily verify that $m \neq m^{\aleph_0}$.

3. From the results proved above, the following result follows. A Banach algebra of a given cardinal number m exists if and only if $m = m^{\aleph_0}$. If B is a Banach algebra and $|B| = m$, then $m^{\aleph_0} = m$. For, every Banach algebra is a Banach space. If $m = m^{\aleph_0}$, take any Banach space B of cardinality m and make it a Banach algebra by defining product $x.y = 0$ for every x, y in B. Alternatively, take any abelian group G of

cardinality m and equip G with discrete topology. Then G becomes a locally compact Abelian topological group and let μ be Haar measure on G. Then $L_1(G, \mu)$ is a Banach algebra of cardinality $m^{\aleph_0} = m$.

4. It has been brought to the authors' attention that the main theorem of this paper follows from the results of [3] and also from the Lemma 3·1 of . [4]. But this theorem is not explicitly stated and deduced in these papers. Moreover, the proof given here is simple, direct and functional analytic.

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