FUNDAMENTA
MATHEMATICAE
XLVI (1958)

On a theorem of F. Riesz concerning the form of linear functionals

by

V. S. Varadarajan (Calcutta)

1. Let X be a locally compact Hausdorff space and $L(X)$ the space of continuous functions on X with compact supports. A famous theorem of functional analysis asserts that the only non-negative linear functionals on $L(X)$ are of the form $\int g d\mu$ where μ is a regular Borel measure on X. This theorem was first proved when $X = [0, 1]$ by F. Riesz and the general case was treated by Kakutani, who used the theory of contents [6] (also [5], p. 216-247). Pettis [10] obtained the same general form by deducing it from an extension theorem for measures. In all these cases the measure obtained has as domain the σ-ring of Borel subsets of X. Edwards [4] extended this domain to the minimal σ-field containing all open subsets of X and proved that regularity still persists for a wide subfamily.

In this paper we discuss the above theorem (to which we shall refer as the Riesz theorem in conformity with the accepted usage) from another point of view. Our aim is (i) to obtain the Riesz theorem by methods structurally more direct than the classical ones and (ii) to investigate the part played by the assumptions of local compactness and Hausdorffness in the validity of the theorem.

As far as (i) is concerned, our method of proof can be explained as follows. In order to integrate a function f, it is enough to know the values of the integrating measure μ on the sets of the form $f^{-1}(B)$ where B is an arbitrary Borel set on the reals not containing the origin. Consequently if μ is defined on the minimal σ-ring containing all such sets $f^{-1}(B)$ (with $f \in L(X)$ and B arbitrary), μ can be used to integrate every $f \in L(X)$. S_X is the σ-ring of Baire sets of X. Thus the natural form of the representation theorem should involve only a measure over S_X. If we are given a measure over S_X, the problem of extending it regularly to the σ-ring of Borel subsets of X is an entirely different question and can indeed be solved under general conditions [9]. Another agreeable feature of S_X is that any measure on S_X which is finite for compact sets in S_X is regular. Thus, the natural method of proving the Riesz theorem at least in the compact case is to represent a compact Hausdorff space in such a way as to simplify its Baire sets structurally. We represent any compact Hausdorff space as a continuous image of a closed subset of a product of two point spaces ([7], p. 166). We identify the Baire sets of this product space (proposition 2.3) and obtain the required Baire measure by first forming it for finite dimensional subsets and then extending it by the well-known consistency theorem of Kolmogorov ([8], p. 29). The locally compact case is then deduced from the compact case.

(ii) is discussed in § 4, where it is shown that the Riesz theorem is valid in general topological spaces without either of the assumptions of local compactness and Hausdorffness. This in itself may not be much, but it does point out that the classical restrictions are only to ensure that $L(X)$ is wide enough to make the theorem interesting. An interesting question about ideals of the space of continuous functions on a compact space is raised.

Throughout this paper, unless explicitly stated otherwise, X is a topological space. $C(X)$ is the space of all bounded continuous functions on X and $L(X)$ is the space of all continuous functions on X vanishing outside compact sets (functions with compact supports). $L(X) \subset C(X)$, $= C(X)$ if and only if X is compact. All functions considered in this paper are real-valued. $C(X)$ and $L(X)$ are vector lattices over the reals with the lattice operations $\max(f, g)$ and $\min(f, g)$. For f, g in $C(X)$ we write $f \leqslant g$ to mean $f(x) \leqslant g(x)$ for all $x \in X$. 0, when no confusion arises, denotes the function identically zero.

If E is a vector lattice, any linear map of E into the reals is called a *linear functional*. A linear functional on E is called *bounded* if it maps bounded subsets of E into bounded sets of reals. (A subset A of E is called *bounded* if there are elements $\alpha, \beta \in E$ such that $\alpha \leqslant x \leqslant \beta$ for all $x \in A$.) A linear functional Λ on E is called *non-negative* if $\Lambda(a) \geqslant 0$ whenever $a \geqslant 0$. A non-negative linear functional is bounded. If Λ is a bounded linear functional, there are non-negative linear functionals Λ^+, Λ^- such that (a) $\Lambda = \Lambda^+ - \Lambda^-$, (b) if $\Lambda = \Lambda_1 - \Lambda_2$ for non-negative linear functionals Λ_1, Λ_2, then $\Lambda_1 - \Lambda^+$ and $\Lambda_2 - \Lambda^-$ are non-negative. For $x \geqslant 0$

$$\Lambda^+(x) = \sup_{0 \leqslant y \leqslant x} \Lambda(y), \quad \Lambda^-(x) = -\inf_{0 \leqslant y \leqslant x} \Lambda(y).$$

These facts are well known and also easy to prove ([2], p. 245).

For any topological space X, $C(X)$ is a Banach space with $\|f\| = \sup_{x \in X} |f(x)|$. When $E = C(X)$ is considered as a vector lattice, a linear functional on E is bounded if and only if it is bounded in the usual sense as a linear functional over the Banach space $C(X)$ and therefore continuous in the topology of uniform convergence. In this paper, all the linear functionals considered will be on $C(X)$ or on its subsets which are also vector lattices. If $M \subset C(X)$ is a linear manifold of $C(X)$ which is closed under the lattice operations, and Λ a bounded linear functional on M, Λ is called *σ-smooth* if the relations $f_n \in M$, $f_n(x) \downarrow 0$ for each $x \in X$ (in symbols: $f_n \downarrow 0$) imply $\Lambda(f_n) \to 0$.

PROPOSITION 1.1. *Let X be any topological space. Then any bounded linear functional on $L(X)$ is σ-smooth.*

Proof. It is enough to prove the proposition for non-negative linear functionals. Let $\{f_n\} \in L(X)$ and $f_n \downarrow 0$. By Dini's theorem, $f_n \downarrow 0$ uniformly. Thus if $\varepsilon_n = \sup_{x \in X} f_n(x)$, $\varepsilon_n \to 0$. Since $f_1^{1/2} \in L(X)$ and since $f_n \leqslant \varepsilon_n^{1/2} f_1^{1/2}$, we have $\Lambda(f_n) \leqslant \varepsilon_n^{1/2} \Lambda(f_1^{1/2}) \to 0$ as $n \to \infty$. This proves the result.

When X is a locally compact Hausdorff space, S_X denotes the σ-ring generated by the compact G_δ-s of X. Sets of S_X are called the *Baire sets of X*. S_X is the smallest σ-ring with respect to which all functions of $L(X)$ are measurable. If X is a compact Hausdorff space, $X \in S_X$ and S_X becomes a σ-field. A Baire measure is a measure μ on S_X which is finite for the compact G_δ-s. Any Baire measure μ is regular, i. e. $\mu(A) = \sup\{\mu(K)$: $K \subset A$, $K \in S_X$ and K compact$\} = \inf\{\mu(U)$: $U \supset A$, $U \in S_X$ and U open$\}$. These facts are to be found in Halmos' book on measure theory [5], p. 217-247. Hereafter, this book will be referred to as H. Regarding Baire measures, we immediately have a uniqueness proposition.

PROPOSITION 1.2. *If m, m' are two Baire measures such that $\int g\,dm = \int g\,dm'$ for all $g \in L(X)$ where X is a locally compact Hausdorff space, then $m = m'$.*

Proof. It is enough to show that $m = m'$ on the compact G_δ-s of X. For, if this is shown, then the compact G_δ-s forming a lattice (H, p. 25, ex. 2) and the two measures being finite and equal on this lattice, it will follow that they will be equal on the σ-ring generated by the lattice, i. e. on S_X (H, p. 188, ex. 3a). This will then show that $m = m'$.

Now if K is any compact G_δ, there exists a sequence $\{f_n\}$ in $L^+(X)$ such that $f_n \downarrow \chi_K$. Then we have

$$m(K) = \int \chi_K\,dm = \lim_{n\to\infty} \int f_n\,dm = \lim_{n\to\infty} \int f_n\,dm' = \int \chi_K\,dm' = m'(K).$$

Since K is arbitrary, the result follows.

We conclude this section with two propositions which are needed further on.

PROPOSITION 1.3. *Let X be a locally compact Hausdorff space and K a compact subset. Suppose that f is a function defined on K, non-negative and continuous. Then there exists an $f^* \in L(X)$ such that $f^* \geqslant 0$ and $f^* = f$ on K.*

If $g \geqslant 0$ is any continuous function of X such that $g \geqslant f$ on K, f^* can be chosen to satisfy the inequality $0 \leqslant f^* \leqslant g$.

Proof. Let V be an open set with compact closure containing K and let X^* be the one-point compactification of X. Then K and X^*-V are disjoint closed subsets of X^*. The function f_1 defined on $K \cup (X^*-V)$, which is f on K and 0 on X^*-V, is $\geqslant 0$ and continuous; since X^* is normal it can be extended to a non-negative continuous function f_1^* on X^*. If f^* is its restriction to X, $f^* \geqslant 0$, $= f$ on K. Since $f^*(x) = 0$ for $x \in X-V$, $f^* \in L(X)$. This proves the first part.

For the second part it is enough to note that $\min(f^*, g)$ is a function of $L(X)$ having the required properties.

PROPOSITION 1.4. *Let X be a locally compact Hausdorff space and K a compact subset. The Baire sets of the compact Hausdorff space K are precisely the intersections of the Baire sets of X with K.*

Proof. In view of a standard result (H, p. 25) it is enough to prove that compact G_δ-s of K are precisely the intersections with K of the compact G_δ-s of X. Obviously if K_1 is a compact G_δ of X, $K_1 \cap K$ is a compact G_δ of K. Suppose now that $K_1 \subset K$ is a compact G_δ of K. There is then a function $f \geqslant 0$, defined and continuous on the set K, such that $K_1 = \{x: f(x) = 1\}$. Let $f^* \geqslant 0$ and $\in L(X)$ be some extension of f. Then $K_1 = K_1^* \cap K$ where $K_1^* = \{x: f^*(x) = 1\}$. Since obviously K_1^* is a compact G_δ of X, the result follows.

Remarks. 1. If K is itself a compact G_δ of X, it follows easily that the Baire sets of the compact Hausdorff space K are precisely those subsets of K which are Baire sets of X. In symbols, $S_K = \{A: A \subset K, A \in S_X\}$.

2. In this case, S_K has a crucial "ideal" property: the relations $A \in S_K$, $B \in S_X$, $B \subset A$ imply that $B \in S_K$.

2. In this section, we discuss the Riesz theorem when X is a compact Hausdorff space. For convenience, we say that a *topological space has property* R if it is a compact Hausdorff space and every non-negative linear functional on $C(X)$ has an integral representation with an integrating Baire measure. It follows that if X has property R, any bounded linear functional on $C(X)$ has an integral representation with an integrating signed Baire measure. If φ is a signed Baire measure and $\Lambda(g) = \int_X g\,d\varphi$ for $g \in C(X)$, Λ is a non-negative linear functional on $C(X)$ if and only if φ is a measure. This can easily be shown by using the regularity of φ, φ^+ and φ^-.

PROPOSITION 2.1. *If X has property* R *and the Hausdorff space Y is a continuous image of X, then Y has property* R.

Proof. Y is certainly a compact Hausdorff space. Let t be a continuous map of X onto Y. For any compact G_δ K of Y, $t^{-1}(K)$ is a compact G_δ of X and hence $t^{-1}(S_Y) \subset S_X$.

For any function f on Y, $f[t]$ is a function on X and if h is a function on X expressed as $h^*[t]$, where h^* is a function on Y, h^* is uniquely determined by h (since t maps X onto Y). $h \in C(X)$ if and only if $h^* \in C(Y)$. Define $L = \{h\colon h \in C(X), = h^*[t] \text{ for } h^* \in C(Y)\}$. L is a linear manifold $C(X)$. For a given non-negative linear functional Λ on $C(Y)$ we set $\Lambda^*(h) = \Lambda(h^*)$ for all $h \in L$. Λ^* is a non-negative linear functional on L and is even bounded (in the Banach space sense) since L contains constants.

By the Hahn-Banach theorem ([1], p. 27), the bounded linear functional Λ^* on L can be extended as a bounded linear functional to $C(X)$. Since X has property R, there is an integral representation for this extension. We thus obtain a signed Baire measure φ such that $\Lambda^*(h) = \int h\,d\varphi$ for all $h \in L$.

Since $t^{-1}(S_Y) \subset S_X$, t induces a signed measure φ_t on S_Y with the property that $\varphi_t(A) = \varphi\{t^{-1}(A)\}$ for all $A \in S_Y$. It then follows that $\Lambda(h^*) = \Lambda^*(h) = \int h\,d\varphi = \int h^*\,d\varphi_t$ for all $h^* \in C(Y)$. This shows that Λ has an integral representation. Since Λ is non-negative, φ_t is actually a measure. This completes the proof that Y has property R.

Proposition 2.2. *If X has property* R *and Y is a closed subspace of X, then Y has property* R.

Proof. Y is evidently compact Hausdorff space. Let Λ be a non-negative linear functional on $C(Y)$. For any $f \in C(X)$, its restriction f_Y to Y is in $C(Y)$. Define Λ_1 by setting $\Lambda_1(f) = \Lambda(f_Y)$ for $f \in C(X)$. Λ_1 is a non-negative linear functional on $C(X)$. Since X has property R, there is a Baire measure m such that $\Lambda_1(f) = \int f\,dm$ for all $f \in C(X)$.

Firstly $m_*(X-Y) = 0$. For if $m_*(X-Y) > 0$, there is a Baire subset A of $X-Y$ such that $m(A) > 0$. Since m is regular, we can get a compact Baire set $K \subset A$ such $m(K) > 0$. Let f be a continuous function on X such that $0 \leqslant f \leqslant 1$, $f = 0$ on Y and 1 on K. Then, $\int f\,dm \geqslant \int_K 1\,dm = m(K) > 0$ while $\Lambda_1(f) = \Lambda(f_Y) = 0$. This contradiction shows that $m_*(X-Y) = 0$.

So Y is a thick subset of the measure space (X, S_X, m) (H, p. 75). We can therefore obtain a measure m_0 on the class of intersections of S_X with Y, which is precisely S_Y. We now show that for any $f \in C(X)$, $\int f\,dm = \int f_Y\,dm_0$. Since for $A \in S_X$, $A \cap Y \in S_Y$ and $m(A) = m_0(A \cap Y)$, the relation $\int g\,dm = \int g_Y\,dm_0$ is valid for all $g = \chi_A$ with $A \in S_X$. It is thus valid for all S_X-measurable step functions, and hence for all bounded S_X-measurable functions by approximation. It is thus valid for all $g \in C(X)$.

We thus have $\int f_Y dm_0 = \int f dm = \Lambda_1(f) = \Lambda(f_Y)$ for all f_Y. Since X is a compact Hausdorff space, it is normal and hence any $h \in C(Y)$ is f_Y for some $f \in C(X)$. This shows that $\Lambda(h) = \int h\, dm_0$ and completes the proof that Y has property R.

In the next proposition we discuss the structure of Baire sets of some special types of compact Hausdorff spaces.

Proposition 2.3. *Let X_a $(a \in I)$ be compact Hausdorff spaces and $S_a = S_{X_a}$. Let $X = \prod_{a \in I} X_a$ and S be the product σ-field generated by the S_a. Then X is a compact Hausdorff space and $S = S_X$.*

Proof. X is evidently a Hausdorff space. It is compact by a famous theorem of Tychonoff. Since S is the smallest σ-field generated by finite dimensional cylinder subsets of X with compact G_δ-s as bases, $S \subset S_X$. We complete the proof by showing that $S_X \subset S$. It is enough to show that every compact G_δ of X is in S. Let K be a compact G_δ of X. We write $K = \bigcap_n G_n$ where each G_n is open in X.

Fix the integer n. For any $x \in K$, $x \in G_n$ and hence there is a finite dimensional cylinder open set H_x such that $x \in H_x \subset G_n$. Since the class of open Baire subsets of any compact Hausdorff space is a base for its topology, we can suppose that the base of the cylinder set H_x is an open Baire subset. Thus $H_x \in S$. $(H_x)_{x \in K}$ is a covering of K and since K is compact, we can find $x_1, \ldots, x_k \in K$ such that $K \subset \bigcup H_{x_i} \subset G_n$. Write $H_n = \bigcup_i H_{x_i}$. Then $H_n \in S$ (since each $H_{x_i} \in S$) and $K \subset H_n \subset G_n$.

Since $K = \bigcap_n G_n$ we must have $K = \bigcap_n H_n$. It then follows that $K \in S$ since $H_n \in S$ for each n. This completes the proof.

Now let X_a $(a \in I)$ be compact Hausdorff spaces and let $S_a = S_{X_a}$. For any $F \subset I$ we define $X_F = \prod_{a \in F} X_a$ and $S_F = S_{X_F}$. S_F is the product σ-field generated by $\{S_a\colon a \in F\}$. For any pair F, G such that $F \subset G \subset I$, there is a map of S_F into S_G that takes $A \in S_F$ into the set $A_G \in S_G$ which is a cylinder subset of X_G with base A in X_F. For two measures m, n respectively defined on S_F and S_G, we write $m < n$ if $m(A) = n(A_G)$ for all $A \in S_F$. Suppose now that for each finite $F \subset I$, m_F is a Baire measure on X_F such that $m_{F_1} < m_{F_2}$ whenever F_1 and F_2 are two finite sets such that $F_1 \subset F_2 \subset I$. The Kolmogorov consistency theorem (cf. H, p. 212 for a proof when each X_a is the unit interval $[0, 1]$; the modifications required in that proof to yield this more general version are easy to see) asserts that there is a unique Baire measure m on X $(= X_I)$ such that $m_F < m$ for all F.

Proposition 2.4. *If for each finite $F \subset I$, X_F has property* R, *then X $(= X_I)$ has property* R.

Proof. X is the set of all functions x on I such that $x(a) \in X_a$ for each $a \in I$. For any $J \subset I$ we write as x_J the restriction of x to J. x_J is a point of X_J. For any $f \in C(X_F)$ we define f_F on X by setting $f_F(x) = f(x_F)$ for each $x \in X$. $f_F \in C(X)$ and if $C_F = \{f_F\colon f \in C(X_F)\}$, C_F is a linear manifold of $C(X)$. Let $C = \bigcup_{F:\, F \text{ finite}} C_F$. C is dense in $C(X)$ in the topology of uniform convergence ([3], p. 57).

Now let A be a non-negative linear functional on $C(X)$. Defining A_F on $C(X_F)$ by setting $A_F(f) = A(f_F)$ for all $f \in C(X_F)$, we obtain a non-negative linear functional A_F on $C(X_F)$. Since F is finite, X_F has property R and hence there is a Baire measure m_F on X_F such that $A_F(f) = \int f\, dm_F$ for all $f \in C(X_F)$. This Baire measure m_F is uniquely determined (proposition 1.2) and hence if $F_1 \subset F_2$, $m_{F_1} < m_{F_2}$. By the Kolmogorov theorem, there is a unique Baire measure m on X such that $m_F < m$ for each finite $F \subset I$. Define $A^*(g) = \int g\, dm$ for all $g \in C(X)$.

It is easily seen that $A = A^*$ on C. Since C is dense in $C(X)$ in the topology of uniform convergence and since A and A^* are continuous in this topology, it follows that $A = A^*$ on $C(X)$. This proves that A has the integral representation and hence that X has property R.

THEOREM (F. Riesz). *Any compact Hausdorff space has property* R.

Proof. It is well known that any compact Hausdorff space is the continuous image of a closed subset of a product of spaces X_a $(a \in I)$ where for each a, X_a is a space consisting of the two points 0 and 1 with the discrete topology. In view of propositions 2.1-2.4, it is enough to prove that X_F has property R for each finite $F \subset I$. This is trivial since X_F is finite.

3. In this section we discuss the Riesz theorem in the locally compact Hausdorff case. Throughout this section, K denotes (with or without suffixes) non-empty compact G_δ-s of X. For any K, S_K denotes the σ-ring of subsets of K which are Baire sets of X. If $K_1 \subset K_2$, then $S_{K_1} \subset S_{K_2}$ and $\bigcup_K S_K$ is a ring. The σ-ring generated by this ring is S_X.

We derive the integral representation of an arbitrary non-negative linear functional on $L(X)$ by first forming measures over S_K and then extending these to S_X. We prove a preliminary proposition.

PROPOSITION 3.1. *For each K let m_K denote a finite measure on S_K, and for $K_1 \subset K_2$ let $m_{K_1} = m_{K_2}$ on S_{K_1}. There exists one and only one Baire measure m on S_X such that $m = m_K$ on S_K.*

Proof. Define $T = \bigcup_K S_K$. T is a ring. For $A \in T$, define $m(A) = m_K(A)$ if $A \in S_K$. If A is empty, $m(A) = 0$. If A is non-empty and $\in S_{K_1}$ and S_{K_2}, then A is a non-empty Baire subset of K_1 and K_2. Consequently, $K_1 \cap K_2$

is a non-empty compact G_δ and $A \in S_{K_1 \cap K_2}$. This shows that $m_{K_1}(A) = m_{K_1 \cap K_2}(A) = m_{K_2}(A)$ and proves that m is well defined on T. If $A, B \in T$ then $A \in S_{K_1}$ and $B \in S_{K_2}$ and hence A, B, $A \cup B$ all belong to $S_{K_1 \cup K_2}$. This shows that m is additive on T. Lastly let $\{A_n\}$ be a sequence in T such that $A_n \downarrow \emptyset$. $A_1 \in S_{K_1}$ and hence $A_n \in S_{K_1}$ for $n \geqslant 1$ (proposition 1.4 remark 2). Since $m_{K_1}(A_n) \to 0$ and $m = m_{K_1}$ on S_{K_1}, it follows that $m(A_n) \to 0$. Thus m is a measure on T. It can then be uniquely extended to the σ-ring generated by T, i.e. to S_X. Since each compact G_δ belongs to T and since m is finite on T, it follows that m is a Baire measure.

Remark. If C is a compact set there is a compact G_δ K containing C (H, p. 218). For any $f \in L(X)$ vanishing outside C we have $\int f dm = \int_K f_K dm_K$ where f_K is the restriction of f to K.

Now let Λ be an arbitrary non-negative linear functional on $L(X)$ and $K \subset X$. For any $f \geqslant 0$ defined and continuous on K, we can find a decreasing sequence $\{f_n\}$ in $L(X)^+$ such that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} f(x) & \text{if} \quad x \in K, \\ 0 & \text{if} \quad x \notin K. \end{cases}$$

Define $\Lambda_K(f) = \lim_{n\to\infty} \Lambda(f_n)$. Since $\Lambda(f_n)$ is decreasing, this limit surely exists and it is perfectly straight-forward to show that this limit is independent of the sequence $\{f_n\}$. Further, any f defined and continuous on K is capable of being written as $f_1 - f_2$ where f_1, f_2 are $\geqslant 0$ and continuous on K. Define $\Lambda_K(f)$ as $\Lambda_K(f_1) - \Lambda_K(f_2)$. It is easy to verify that Λ_K is a non-negative linear functional on $C(K)$. Let m_K be the Baire measure on S_K such that $\Lambda_K(f) = \int_K f dm_K$ for all $f \in C(K)$. Our aim now is to show that the $\{m_K\}_{K \subset X}$ satisfy the conditions of proposition 3.1. By definition $\Lambda_K(f_K) = \Lambda(f)$ whenever $f \in L(X)$ and vanishes outside K.

Proposition 3.2. *Let $K_1 \subset K_2$ and f_1, f_2 be two functions such that $f_1 \in C(K_1)^+$, $f_2 \in C(K_2)^+$ and $0 \leqslant f_1 \leqslant f_2$ on K_1. Then $\Lambda_{K_1}(f_1) \leqslant \Lambda_{K_2}(f_2)$.*

Proof. Let $\{u_n\}$ and $\{v_n\}$ be two decreasing sequences in $L(X)^+$ such that

$$\lim_{n\to\infty} u_n(x) = \begin{cases} f_1(x) & \text{if} \quad x \in K_1, \\ 0 & \text{if} \quad x \notin K_1; \end{cases} \qquad \lim_{n\to\infty} v_n(x) = \begin{cases} f_2(x) & \text{if} \quad x \in K_2, \\ 0 & \text{if} \quad x \notin K_2. \end{cases}$$

Then

$$\Lambda_{K_1}(f_1) = \lim_{n\to\infty} \Lambda(u_n) \quad \text{and} \quad \Lambda_{K_2}(f_2) = \lim_{n\to\infty} \Lambda(v_n).$$

If $w_n = \max(u_n, v_n)$, then $w_n \in L(X)^+$ and $\{w_n\}$ is decreasing. Further $\lim_{n\to\infty} w_n = \lim_{n\to\infty} v_n$. Consequently $\Lambda_{K_2}(f_2) = \lim_{n\to\infty} \Lambda(w_n)$. But since $u_n \leqslant w_n$ for all n, $\Lambda(u_n) \leqslant \Lambda(w_n)$ for all n.

This proves the result.

PROPOSITION 3.3. *If $K_1 \subset K_2$, then $m_{K_1} = m_{K_2}$ on S_{K_1}.*

Proof. It suffices (proposition 1.2) to show that for any $g \in C(K_1)^+$,

$$\int_{K_1} g\,dm_{K_1} = \int_{K_1} g\,dm_{K_2}.$$

Let $\{g_n\}$ be a decreasing sequence in $L(X)^+$ such that

$$\lim_{n\to\infty} g_n(x) = \begin{cases} g(x) & \text{if} \quad x \in K_1, \\ 0 & \text{if} \quad x \notin K_1. \end{cases}$$

Let g'_n be the restriction of g_n to K_2. Then, from proposition 3.2 we deduce that for all n

$$\int_{K_1} g\,dm_{K_1} \leqslant \int_{K_2} g'_n\,dm_{K_2} \leqslant \Lambda(g_n).$$

As $n\to\infty$, the right extreme term tends to

$$\Lambda_{K_1}(g) = \int_{K_1} g\,dm_{K_1}$$

while the middle term tends to

$$\int_{K_2} \chi_{K_1}\cdot(\lim_{n\to\infty} g'_n)\,dm_{K_2} = \int_{K_1} g\,dm_{K_2}$$

(monotone convergence theorem). This proves the proposition.

THEOREM (F. Riesz). *Every non-negativeli near functional on $L(X)$ has an integral representation.*

Proof. Let Λ be a non-negative linear functional on $L(X)$. Form the measures $\{m_K\}$ and using propositions 3.1 and 3.3 we obtain the unique Baire measure m on S_X. It is then obvious that $\Lambda(f) = \int f\,dm$ for all $f \in L(X)$. This completes the proof.

4. The purpose of the remarks of this section is to investigate the effect of the assumption on X in the classical form of the Riesz theorem. The conclusion reached can be paraphrased to the effect that the theorem is valid without any restriction. Thus it turns out that the restrictions of Hausdorffness and local compactness are only to have a sufficiently wide $L(X)$.

Let X be a topological space. A subset $L \subset C(X)$ is called an *ideal* if (i) L is linear, (ii) $f \in L$, $g \in C(X)$ and $|g| \leqslant |f|$ imply that $g \in L$. If L is an ideal and $f, g \in L$, then $|f|$, $\max(f, g)$ and $\min(f, g)$ are all in L. $S(L)$ is defined as the minimal σ-ring with respect to which all functions of L are measurable. A non-negative linear functional Λ on an ideal L is called an *integral* if there is a unique measure m on $S(L)$ such that $\Lambda(h) = \int h\,dm$ for all $h \in L$. For any ideal L, let $N_L = \bigcap_{f \in L} Z(f)$ where $Z(f) = \{x: f(x) = 0\}$. N_L is a closed (possibly empty) subset of X. Let L_N be those functions of $C(X)$ which vanish outside compact subsets of $X - N_L$. Let $\bar{L}$ denote the closure of L in $C(X)$.

PROPOSITION 4.1. *Let X be a compact Hausdorff space and L an ideal of $C(X)$. Then $L_N \subset L \subset \bar{L}$ and L_N is dense in L (hence in $\bar{L}$).*

Proof. We first show that $L_N \subset L$. Let $f \in L_N$. We show that $f \in L$. We can assume $f \geqslant 0$. Let K be a compact subset of $X - N_L$ such that f vanishes outside K. For each $x \in K$, since $x \in X - N_L$, there is a $g \in L^+$ such that $g(x) > 0$ and hence we can find $g_x \in L^+$ such that $g_x(x) > f(x)$. We can thus find an open set G_x containing x such that $g_x(y) > f(y)$ for all $y \in G_x$. $\{G_x\}_{x \in K}$ is a covering of K from which a finite subcovering $\{G_{x_i}\}$ is extracted. If $h = \max(g_{x_1}, \dots, g_{x_k})$, then $h > f$ on K. Since $f = 0$ outside K and $h \geqslant 0$, it follows that $0 \leqslant f \leqslant h$. Since $h \in L$ and L is an ideal, $f \in L$.

We now show that L_N is dense in L. In fact, we prove that for any $f \geqslant 0$ vanishing on N_L, there is an increasing sequence $\{f_n\}$ in L_N^+ such that $f_n \uparrow f$ uniformly. Let $f \geqslant 0, = 0$ on N_L and n be any integer. Now define $K_n = \{x: f(x) \geqslant 1/n\}$. $K_n \subset X - N_L$ and K_n is compact. K_n being a compact subset of $X - N_L$, we can have a function g_n continuous on $X - N_L$, $= f$ on K_n, vanishing outside a compact subset of $X - N_L$ and $0 \leqslant g_n \leqslant f$ on $X - N_L$ (proposition 1.3). If we define g_n as 0 on N_L, $g_n \in L_N$ and $0 \leqslant g_n \leqslant f$. Further, $0 \leqslant f(x) - g_n(x) \leqslant 1/n$ for all x. If we now define $f_n = \max(g_1, \dots, g_n)$, $\{f_n\}$ is an increasing sequence in L_N^+ increasing uniformly to f. This completes the proof.

COROLLARY. *We have at once $\bar{L} = \{f: f \in C(X), = 0$ on $N_L\}$. Thus if L is closed, L consists of all continuous functions vanishing on N_L.*

We can now rephrase the Riesz theorem as follows.

PROPOSITION 4.2. *Let X be a compact Hausdorff space and L an ideal of $C(X)$. Then any non-negative σ-smooth linear functional on L is an integral.*

Proof. Let Λ be an arbitrary σ-smooth non-negative linear functional on L. Λ is a non-negative linear functional on L_N and hence for a unique Baire measure m on $X - N_L$, $\Lambda(h) = \int h\,dm$ for all $h \in L_N$. Since L_N is dense in L, the minimal σ-rings induced by L_N and L coincide and hence m is defined on $S(L)$. We now show that $\Lambda(h) = \int h\,dm$

for all $h \in L$. Let $h \in L$. We can suppose that $h \geqslant 0$. There is then a sequence $\{f_n\}$ in L_N^+ increasing to h. Since is σ-smooth, $\Lambda(f_n) \uparrow \Lambda(h)$. But $\Lambda(f_n) = \int f_n dm$ for all n. Hence, $\int f_n dm \uparrow \Lambda(h)$. This shows that $\int h dm < \infty$ and is equal to $\lim\limits_{n\to\infty} \int f_n dm$. It follows that $\Lambda(h) = \int h dm$ and the proof is complete.

We next show that the restriction of Hausdorffness is unnecessary.

PROPOSITION 4.3. *Let X be any compact space and L an ideal of $C(X)$. Then every non-negative σ-smooth linear functional on L is an integral.*

Proof. For $x, y \in X$, we write $x \sim y$ if $g(x) = g(y)$, for all $g \in L$. $\sim$ is an equivalence relation and let Y be the space of equivalence classes furnished with the quotient topology. Let t be the canonical map of X onto Y. t is continuous and hence Y is compact.

For a function f on X which is constant over the equivalence classes (all functions of L are of this form), there is a unique function f^* on Y such that $f = f^*[t]$. $f \in C(X)$ if and only if $f^* \in C(Y)$. Let $L^* = \{f^*: f \in L\}$ and define Λ^* on L^* by setting $\Lambda^*(f^*) = \Lambda(f)$. Λ^* is a σ-smooth non-negative linear functional on L^* and L^* is an ideal of $C(Y)$.

We note that Y is a Hausdorff space. If $y_1, y_2 \in Y$ and $y_1 \neq y_2$, $t^{-1}(y_1)$, and $t^{-1}(y_2)$ are different equivalence classes of X. Hence there is a $g \in L$ taking different values on $t^{-1}(y_1)$ and $t^{-1}(y_2)$ and hence g^* takes different values at y_1 and y_2. This proves that Y is a Hausdorff space.

By proposition 4.2, there is a unique measure m^* on $S(L^*)$ such that $\Lambda^*(f^*) = \int f^* dm^*$ for all $f^* \in L^*$.

t maps X onto Y and it is easy to see that $t^{-1}(S(L^*)) = S(L)$. Therefore t and t^{-1} considered as set transformations of $S(L)$ onto $S(L^*)$ and vice versa preserve countable unions and countable intersections. Therefore there exists a unique measure m on $S(L)$ such that $m^*(A) = m(t^{-1}(A))$ for all $A \in S(L^*)$. For such an m, $\int g dm = \int g^* dm^*$ for all $g \in L$ (H, p. 163).

This completes the proof.

THEOREM. *Every non-negative linear functional on the space $L(X)$ (of continuous functions vanishing outside compact subsets of a topological space X) is an integral.*

Proof. All non-negative linear functionals on $L(X)$ are σ-smooth. We deduce the theorem from proposition 4.3. Let X^* be the one-point compactification of X. Any $f \in L(X)$ can be continuously extended to an f^* on X^* by prescribing for it the value 0 at ∞. Let $L^* = \{f^*: f \in L(X)\}$. L^* is an ideal and $S(L) = S(L^*)$. The theorem then follows from proposition 4.3 since X^* is compact.

Remarks. Let X be a compact Hausdorff space and L an ideal of $C(X)$. An interesting question arises: when can we say that all non-negative linear functionals on L are σ-smooth? Modifying proposition 1.1, we can say that if $f \in L$ implies $f^{1/2} \in L$, then L has this property. For example (by proposition 4.1, corollary) it turns out that all closed ideals L have this property. What is the general characterization of such ideals? We are unable to answer this question.

Acknowledgement. The writer's sincere thanks are due to Prof. R. E. Edwards of Birkbeck College, London for going through a part of the manuscript before it was submitted for publication and also for some useful comments.

References

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications 1948.

[3] N. Bourbaki, *Éléments de Mathématique, Première Partie, Livre 3, Topologie Générale, Ch. X, Actualités Scientifiques et industrielles*, Paris 1949, Hermannand Cie, No 1084.

[4] R. E. Edwards, *A theory of Radon Measures on locally compact spaces*, Acta Math. 89 (1953), p. 133-164.

[5] P. R. Halmos, *Measure Theory*, New York 1950.

[6] S. Kakutani, *Concrete representations of abstract (M) spaces*, Ann. of Math. (2) 42 (1941), p. 994-1024.

[7] J. L. Kelley, *General Topology*, New York 1955.

[8] A. Kolmogorov, *Foundations of Probability*, New York 1950.

[9] J. Maryk, *Borel and Baire measures*, Czechoslovak Math. J. 82 (2) (1957), p. 248-253.

[10] B. J. Pettis, *On the extension of measures*, Ann. of Math. 54 (1951), p. 186-197.

INDIAN STATISTICAL INSTITUTE, CALCUTTA

Reçu par la Rédaction le 4. 3. 1958