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On game-theoretic methods in the theory of Souslin sets

by

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1. Introduction. In this note, we shall use the methods of Blackwell [1] to prove the Coreduction Principle (stated below) for Souslin sets in certain topological spaces and also establish a result on the constituents (defined below) of a Souslin set.

Let Y be a topological space. A subset A of Y is said to be a *Souslin set* if there exists a system $\{A_{n_1n_2...n_k}\}$, indexed by all finite sequences of natural numbers, of closed subsets of Y such that

$$A = \bigcup_{(n_k)} \bigcap_{k=1}^{\infty} A_{n_1n_2...n_k}$$

where the union extends over all sequences of natural numbers.

A subset A of Y is said to be a *bi-Souslin set* if both A and $Y-A$ are Souslin sets.

An alternative way of describing Souslin sets is through sieves. Denote by Q the set of all rationals in the open interval $(0, 1)$, and label the elements of Q as $r_1, r_2, \ldots$ (we shall hold fixed throughout the paper this particular labelling of the elements of Q). Any system $\{W_r, r \in Q\}$, indexed by the elements of Q, of subsets of Y will be called a *sieve*. By *the set sifted by the sieve* $\{W_r, r \in Q\}$ is meant the set of all $y \in Y$ such that there is a sequence $\{r_{n_k}\}$ (possibly depending on y) of elements of Q such that $r_{n_1} > r_{n_2} > \ldots$ and $y \in W_{r_{n_k}}$ for all $k \geqslant 1$. The alternative way of describing Souslin sets is this: A is a Souslin subset of Y if and only if there is a sieve $\{W_r, r \in Q\}$ of closed subsets of Y such that A is the set *sifted* by $\{W_r, r \in Q\}$ (cf. Theorems 9 and 10 in [5], p. 25).

Let A be a Souslin subset of Y and let $\{W_r, r \in Q\}$ be a sieve such that A is the set sifted by $\{W_r, r \in Q\}$. For each ordinal $\alpha < \omega_1$ (= the first uncountable ordinal), let A_α be the set of all $y \in Y$ such that the set $\{r \in Q\colon y \in W_r\}$, when equipped with the usual order on the rationals, is of ordinal type α. The sets $\{A_\alpha\colon \alpha < \omega_1\}$ are called the *constituents* of the Souslin set A relative to the sieve $\{W_r, r \in Q\}$.

The aim of this paper is to prove by game-theoretic methods the following theorems.

THEOREM 1 (COREDUCTION PRINCIPLE). *Let Y be a topological space in which every open set is a Souslin set. If A, B are Souslin sets in Y, then there exist Souslin sets E, F in Y such that $A \subset E$, $B \subset F$, $A \cap B = E \cap F$ and $E \cup F = Y$.*

The classical analogue of Theorem 1 (that is, with Y a Polish space and A, B analytic subsets of Y) was established by Kuratowski [3]. Blackwell [1] used game-theoretic methods to prove the classical result. We shall imitate Blackwell's methods to prove Theorem 1.

THEOREM 2. *Let Y be a topological space in which every open set is a Souslin set. Let A be a Souslin set in Y and let $\{W_r, r \in Q\}$ be any sieve of closed subsets of Y such that A is the set sifted by $\{W_r, r \in Q\}$. Then the constituents of A (relative to $\{W_r, r \in Q\}$) are bi-Souslin subsets of Y.*

Theorem 2 was proved by methods quite different from ours by Rogers and Willmott (see corollary to Theorem 12 in [5], p. 30).

In the next section, we build up the machinery needed to prove Theorems 1 and 2.

2. Sieves and games. Let Y be a topological space and let $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ be two sieves of subsets of Y. Following Blackwell [1], we associate with each $y \in Y$ a two-person game $G(y)$ as follows: Players I and II choose rationals from Q alternately, player I being the first to make a choice, each choice being made with complete information about previous choices of both players. A play $\pi = (r_{m_1}, r_{n_1}, r_{m_2}, r_{n_2}, \ldots)$ is a *win for player* I in $G(y)$ if there is a natural number k such that $r_{m_1} > r_{m_2} > \ldots \ldots > r_{m_k}$, $y \in W_{r_{m_i}}$, $i = 1, 2, \ldots, k$, $r_{n_1} > r_{n_2} > \ldots > r_{n_{k-1}}$, $y \in Z_{r_{n_i}}$, $i = 1, 2, \ldots, k-1$, and either $r_{n_k} \geqslant r_{n_{k-1}}$ or $y \notin Z_{r_{n_k}}$. The play π is a *win for player* II in $G(y)$ if there is a $k \geqslant 1$ such that $r_{m_1} > r_{m_2} > \ldots > r_{m_{k-1}}$, $y \in W_{r_{m_i}}$, $i = 1, 2, \ldots, k-1$, $r_{n_1} > r_{n_2} > \ldots > r_{n_{k-1}}$, $y \in Z_{r_{n_i}}$, $i = 1, 2, \ldots \ldots, k-1$, and either $r_{m_k} \geqslant r_{m_{k-1}}$ or $y \notin W_{r_{m_k}}$. Finally the play π ends in a *draw* in $G(y)$ if for every $k \geqslant 1$, $r_{m_k} > r_{m_{k+1}}$, $y \in W_{r_{m_k}}$, $r_{n_k} > r_{n_{k+1}}$, and $y \in Z_{r_{n_k}}$.

Thus, each player at each stage tries to produce a rational $r \in Q$ which is strictly smaller than his previous choices and such that $y \in W_r$ or $y \in Z_r$ according as whether player I plays or player II plays. The first player to fail in this loses in the game $G(y)$. If neither player fails, it is a draw.

Let P_1 be the collection of all finite sequences of elements of Q (including the empty sequence, which we denote by e) of even length, let P_2 be the collection of all finite sequences of elements of Q of odd length, and let $P = P_1 \cup P_2$. By a *strategy* (in any of the games $G(y)$)

for player I (II) is meant a function from P_1 (P_2) to Q. Denote the set of all strategies for players I and II by Φ and Ψ, respectively; that is, $\Phi = Q^{P_1}$ and $\Psi = Q^{P_2}$. Equip Φ and Ψ with the product of discrete topologies on Q. Since P_1 and P_2 are countably infinite, we note that Φ and Ψ are homeomorphic to N^N, where N is the set of all natural numbers and N^N is equipped with the product of discrete topologies on N.

A strategy φ for player I and a strategy ψ for player II uniquely determine a play $(r_{m_1}, r_{n_1}, r_{m_2}, r_{n_2}, \ldots)$ as follows:

$$r_{m_1} = \varphi(e),$$

$$r_{n_k} = \psi(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}}, r_{m_k}), \quad k \geqslant 1,$$

and

$$r_{m_{k+1}} = \varphi(r_{m_1}, r_{n_1}, \ldots, r_{m_k}, r_{n_k}), \quad k \geqslant 1.$$

We shall denote the play determined by player I using the strategy φ and player II using the strategy ψ by $\langle\varphi, \psi\rangle$. We say that $\varphi^* \in \Phi$ is a *winning* strategy in $G(y)$ for player I if for every $\psi \in \Psi$, the play $\langle\varphi^*, \psi\rangle$ is a win for player I in $G(y)$. Call a strategy $\varphi^* \in \Phi$ a *drawing* strategy for player I in $G(y)$ if for every $\psi \in \Psi$, the play $\langle\varphi^*, \psi\rangle$ is a win for player I in $G(y)$ or the play $\langle\varphi^*, \psi\rangle$ ends in a draw in $G(y)$. Analogous definitions apply to winning and drawing strategies for player II.

We now prove a lemma which will be used in the sequel.

LEMMA. *Let Y be a topological space in which every open set is a Souslin set. Let $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ be two sieves of closed subsets of Y. Define:*

$$E = \{y \in Y\colon \textit{player I has a drawing strategy in } G(y)\}$$

and

$$F = \{y \in Y\colon \textit{player II has a drawing strategy in } G(y)\}.$$

Then E and F are Souslin subsets of Y.

(Here, of course, $G(y)$, $y \in Y$, are the games associated, as above, with the sieves $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ of the lemma).

Proof. We shall prove that E is a Souslin set. An analogous proof works for F.

Let $H = \{(y, \varphi) \in Y \times \Phi\colon \varphi$ is a drawing strategy for player I in $G(y)\}$. Observe that E is the projection of H to Y. Thus, if we can prove that H is a Souslin subset of $Y \times \Phi$, it will follow by a result of Rogers and Willmott [4] that E is a Souslin set in Y. In fact, we shall now show that H is bi-Souslin in $Y \times \Phi$.

With each sequence $(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}}) \in P_1$ (when $k = 1$, the sequence $(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}})$ is to be interpreted as the empty sequence), we associate sets $K(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}})$, $L(r_{m_1}, r_{n_1}, \ldots$ $\ldots, r_{m_{k-1}}, r_{n_{k-1}})$ and $M(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}})$ as follows:

$$K(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}})$$
$$= [\bigcap_{i=1}^{k-1} W_{r_{m_i}} \cap \bigcap_{i=1}^{k-1} Z_{r_{n_i}}] \times [\bigcap_{i=1}^{k-1} \{\varphi \in \Phi:\ \varphi(r_{m_1}, r_{n_1}, \ldots, r_{m_{i-1}}, r_{n_{i-1}}) = r_{m_i}\}] \text{ if } k > 1$$
$$= Y \times \Phi \quad \text{if } k = 1 .$$

$$L(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}})$$
$$= \bigcup_{r \in Q} [W_r^c \times \{\varphi \in \Phi:\ \varphi(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}}) = r\}], \quad k \geqslant 1 .$$

$$M(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}})$$
$$= \bigcup_{r \in Q(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}})} [Y \times \{\varphi \in \Phi:\ \varphi(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}}) = r\}]$$

where

$$Q(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}}) = \{r \in Q:\ r \geqslant r_{m_{k-1}}\} \quad \text{if } k > 1$$
$$= \emptyset \quad \text{if } k = 1 .$$

(union over the empty set is to be interpreted as the empty set). It is easy to see that the sets

$$K(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}}),$$
$$L(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}}),$$
$$M(r_{m_1}, r_{n_1}, \ldots, r_{m_{k-1}}, r_{n_{k-1}})$$

are all bi-Souslin in $Y \times \Phi$. Finally, note that

$$H^c = \bigcup_{s \in \overline{P}_1} [K(s) \cap [L(s) \cup M(s)]]$$

where

$$\overline{P}_1 = \bigcup_{k=1}^{\infty} [\{(r_{m_1}, r_{n_1}, \ldots, r_{m_k}, r_{n_k}) \in P_1:\ r_{m_i} > r_{m_{i+1}}, r_{n_i} > r_{n_{i+1}},$$
$$i = 1, 2, \ldots, k-1\}] \cup \{e\} .$$

Since $\overline{P}_1$ is countable, it follows that H^c is bi-Souslin in $Y \times \Phi$, and so H is bi-Souslin in $Y \times \Phi$. This completes the proof of the lemma.

3. Proof of theorems.

Proof of Theorem 1. Let $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ be sieves of closed subsets of Y such that A, B are, respectively, the sets sifted by $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$. Let $\{A_\alpha, \alpha < \omega_1\}$ and $\{B_\beta, \beta < \omega_1\}$ be the constituents of A, B with respect to the sieves $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$, respectively. For each $y \in Y$, let $G(y)$ be the game associated with the sieves $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ as in Section 2. Let E and F be the sets defined in the lemma of Section 2. We shall prove that the sets E, F have the required properties.

First, by the lemma of Section 2, E and F are Souslin subsets of Y. Next, we note that

$$E = A \cup \Big[\bigcup_{\alpha<\omega_1} \big(A_\alpha \cap \bigcup_{\beta<\alpha} B_\beta\big)\Big] \tag{1}$$

and

$$F = B \cup \Big[\bigcup_{\beta<\omega_1} \big(B_\beta \cap \bigcup_{\alpha\leqslant\beta} A_\alpha\big)\Big]. \tag{2}$$

To see this, let $y \in A$. Then there exists a sequence $\{r_{m_k^*}\}$ of elements of Q such that for every $k \geqslant 1$, $r_{m_k^*} > r_{m_{k+1}^*}$ and $y \in W_{r_{m_k^*}}$. Now consider a strategy φ^* for player I defined by:

$$\varphi^*(r_{i_1}, r_{i_1'}, \ldots, r_{i_{k-1}}, r_{i_{k-1}'}) = r_{m_k^*}.$$

It is easy to see that φ^* is a drawing strategy for player I in the game $G(y)$, so $y \in E$. Next suppose that $y \in A_\alpha \cap B_\beta$, where $\beta < \alpha < \omega_1$. Set $H_1 = \{r \in Q\colon y \in W_r\}$ and $H_2 = \{r \in Q\colon y \in Z_r\}$. Then H_1 and H_2 are of ordinal types α and β, respectively. Since $\beta < \alpha$, there is a similarity mapping (that is, a one-to-one and order-preserving mapping) g which takes H_2 onto a proper segment of H_1. Choose an element $r^* \in H_1 - g(H_2)$ and define a strategy $\varphi^{*\prime}$ for player I (in the game $G(y)$) as follows:

$$\varphi^{*\prime}(e) = r^*$$

and

$$\begin{aligned} \varphi^{*\prime}(r_{i_1}, r_{i_1'}, \ldots, r_{i_k}, r_{i_k'}) &= g(r_{i_k'}) && \text{if } r_{i_k'} \in H_2 \\ &= r' && \text{if } r_{i_k'} \in Q - H_2, \end{aligned}$$

where r' is a fixed but arbitrary element of Q. It is not difficult to see that $\varphi^{*\prime}$ is a winning strategy for player I in the game $G(y)$, so $y \in E$. Thus $E \supset A \cup [\bigcup_{\alpha<\omega_1} (A_\alpha \cap \bigcup_{\beta<\alpha} B_\beta)]$. Conversely, suppose

$$y \notin A \cup \Big[\bigcup_{\alpha<\omega_1} \big(A_\alpha \cap \bigcup_{\beta<\alpha} B_\beta\big)\Big].$$

We distinguish two cases.

Case 1. $y \in B$. As $y \notin A$, it follows that $y \in A_\alpha$ for some $\alpha < \omega_1$. As H_1 is well-ordered while H_2 is not, it is clear that player II has a winning strategy in $G(y)$. Indeed, the set H_2 contains a strictly decreasing sequence $\{r_{n_k^*}\}$ so that the strategy ψ^* for player II defined by

$$\psi^*(r_{i_1}, r_{i_1'}, \ldots, r_{i_{k-1}}, r_{i_{k-1}'}, r_{i_k}) = r_{n_k^*}$$

wins for player II in the game $G(y)$. Hence $y \notin E$.

Case 2. $y \notin B$. It now follows that $y \in A_\alpha \cap B_\beta$ where $\alpha \leqslant \beta < \omega_1$. Hence there is a similarity mapping g' from H_1 onto a segment of H_2.

Define a strategy $\psi^{*\prime}$ for player II as follows:

$$\begin{aligned}\psi^{*\prime}(r_{t_1}, r_{t_1'}, \ldots, r_{t_{k-1}}, r_{t_{k-1}'}, r_{t_k}) &= g'(r_{t_k}) \quad \text{if } r_{t_k} \in H_1, \\ &= r' \quad \text{if } r_{t_k} \in Q - H_1,\end{aligned}$$

where r' is a fixed but arbitrary element of Q. It is clear that $\psi^{*\prime}$ is a winning strategy for player II in the game $G(y)$, so that $y \notin E$. We have thus proved that

$$E \subset A \cup [\bigcup_{\alpha<\omega_1} (A_\alpha \cap \bigcup_{\beta<\alpha} B_\beta)],$$

from which equation (1) follows. Equation (2) follows analogously.

It is now straightforward to derive from equations (1)–(2) that $A \subset E$, $B \subset F$, $A \cap B = E \cap F$ and $E \cup F = Y$. This completes the proof of Theorem 1.

Proof of Theorem 2. Fix an ordinal $\alpha_0 < \omega_1$ and choose a subset T of Q so that T is of ordinal type α_0. Define $Z_r = Y$ if $r \in T$ and $Z_r = \emptyset$ if $r \notin T$. If B is the set sifted by $\{Z_r, r \in Q\}$, then plainly $B = \emptyset$. Moreover, $B_\beta = \emptyset$ if $\beta \neq \alpha_0$ and $\beta < \omega_1$ and $B_\beta = Y$ if $\beta = \alpha_0$, where $\{B_\beta, \beta < \omega_1\}$ are the constituents of B relative to the sieve $\{Z_r, r \in Q\}$. Let $\{A_\alpha, \alpha < \omega_1\}$ be the constituents of A relative to the sieve $\{W_r, r \in Q\}$. For each $y \in Y$, let $G(y)$ be the game associated with the sieves $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ as in Section 2. Let E, F be the sets defined in the lemma of Section 2.

By the lemma of Section 2, E and F are Souslin subsets of Y. Moreover, the proof of Theorem 1 shows that

$$E = A \cup \bigcup_{\alpha > \alpha_0} A_\alpha$$

and

$$F = \bigcup_{\alpha \leqslant \alpha_0} A_\alpha .$$

It follows that $\bigcup_{\alpha \leqslant \alpha_0} A_\alpha$ is a bi-Souslin subset of Y, since $E \cup F = Y$ and $E \cap F = \emptyset$. As α_0 was arbitrary, we have proved that for every ordinal $\delta < \omega_1$, $\bigcup_{\beta \leqslant \delta} A_\beta$ is bi-Souslin. Consequently,

$$A_\alpha = \bigcup_{\beta \leqslant \alpha} A_\beta - \bigcup_{\delta < \alpha} (\bigcup_{\beta \leqslant \delta} A_\beta)$$

is a bi-Souslin subset of Y. This completes the proof of Theorem 2.

Remark 1. Theorem 2 can be proved by means of classical methods as follows. Let D be the Cantor set, which we shall think of as a countable product of copies of the two—element set $\{0, 1\}$. Define a sieve $\{P_r, r \in Q\}$ of closed subsets of D as follows: $P_{r_n} = \{t \in D: t_n = 1\}$ where t_n denotes the nth coordinate of t. Let G be the set sifted by $\{P_r, r \in Q\}$

and let $\{G_\alpha, \alpha < \omega_1\}$ be the constituents of G relative to $\{P_r, r \in Q\}$. Then it is known that the sets G_α are Borel subsets of D (see [2], p. 272). Now consider the characteristic function (in the sense of Marczewski) of the sieve $\{W_r, r \in Q\}$, that is, $f(y) = \sum_{n=1}^{\infty} \frac{2}{3^n} I_{W_{r_n}}(y)$, $y \in Y$, where $I_{W_{r_n}}$ is the indicator of the set W_{r_n}. It is easy to verify that the function f is measurable between the spaces (Y, S) and (D, B), where S is the σ-algebra of bi-Souslin subsets of Y and B the σ-algebra of Borel subsets of D. Moreover, for each $\alpha < \omega_1$, $A_\alpha = f^{-1}(G_\alpha)$ (cf. [2], p. 408). Consequently each A_α is bi-Souslin in Y.

Remark 2. It is true that Theorem 1 can also be obtained by imitating Kuratowski's method in [3]. But this involves suitably modifying the sieves $\{W_r, r \in Q\}$ and $\{Z_r, r \in Q\}$ with which we started and then the sets E and F are no longer as naturally related to the orginial sieves as in our proof.

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